A conjecture about the unitary dual of a reductive Lie group

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The Plancherel decomposition and tempered representations

$G_{\mathbb{R}}$ reductive matrix group over $\mathbb{R}$ e.g., $G_{\mathbb{R}} = SL(n,\mathbb{R})$

$K_{\mathbb{R}} \subset G_{\mathbb{R}}$ a maximal compact subgroup e.g., $K_{\mathbb{R}} = SO(n)$

$\widehat{G_{\mathbb{R}}}$ the unitary dual of $G_{\mathbb{R}}$

i.e., set of isomorphism classes of irreducible unitary representations

$\widehat{G_{\mathbb{R}}} \ni \iota \mapsto (\pi_\iota, V_\iota)$

Harish Chandra:

$L^2(G_{\mathbb{R}}) \simeq \int_{\widehat{G}_{\mathbb{R}}} V_\iota \widehat{\otimes} V_\iota^* \, d\mu(\iota)$ $\quad$ ($d\mu$ is Plancherel measure)

$\{\, \iota \in \widehat{G_{\mathbb{R}}} \mid \mu(\iota) > 0 \,\}$ is the *discrete series*

$\iota \in \widehat{G_{\mathbb{R}}}$ is *tempered* if $\iota \in \text{support}(\mu)$

Roughly speaking, tempered representations are unitarily induced from discrete series representations of smaller reductive groups

The tempered irreducible representations are well understood

Non-tempered irreducible unitary representations do occur as building blocks in the decomposition of other function spaces, such as $L^2(\Gamma\backslash G_{\mathbb{R}})$, where $\Gamma \subset G_{\mathbb{R}}$ is a discrete subgroup

There are many results on irreducible unitary representations (many due to Barbasch, Vogan, their students and collaborators), but a general picture has not yet emerged

The main problem: Understand the irreducible unitary representations, including the non-tempered representations, of a general reductive group $G_{\mathbb{R}}$

The most common approach to the problem involves looking at all irreducible – not necessarily unitary – representations, and then trying to determine which among them "can be made unitary"

Passage to the infinitesimal representation

G = complexification of $G_{\mathbb{R}}$, K = complexification of $K_{\mathbb{R}}$
$\mathfrak{g}_{\mathbb{R}}$ = Lie algebra of $G_{\mathbb{R}}$, $\mathfrak{k}_{\mathbb{R}}$ = Lie algebra of $K_{\mathbb{R}}$
$\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$ = Lie algebra of G, $\mathfrak{k} = \mathbb{C} \otimes \mathfrak{k}_{\mathbb{R}}$ = Lie algebra of K

$$\begin{array}{ccc} K & \subset & G \\ \cup & & \cup \\ K_{\mathbb{R}} & \subset & G_{\mathbb{R}} \end{array} \quad \text{e.g.,} \quad \begin{array}{ccc} SO(n,\mathbb{C}) & \subset & SL(n,\mathbb{C}) \\ \cup & & \cup \\ SO(n) & \subset & SL(n,\mathbb{R}) \end{array} \quad \begin{array}{c}\text{and on the}\\ \text{level of}\\ \text{Lie algebras,}\end{array} \quad \begin{array}{ccc} \mathfrak{k} & \subset & \mathfrak{g} \\ \cup & & \cup \\ \mathfrak{k}_{\mathbb{R}} & \subset & \mathfrak{g}_{\mathbb{R}} \end{array}$$

(π, V) an admissible representation of finite length of $G_{\mathbb{R}}$ on a complete, locally convex Hausdorff topological vector space

"finite length" means: every chain of closed, $G_{\mathbb{R}}$-invariant subspaces $0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots \subset V$ has finite length

"admissible" means: $\pi|_{K_R} \simeq \bigoplus_{\mu \in \widehat{K_{\mathbb{R}}}} n_\mu \cdot \mu$, $n_\mu < \infty$, i.e., π breaks up as a direct sum of irreducible representation of $K_{\mathbb{R}}$, each occurring only *finitely often* – automatic for irreducible unitary representations

Then

$$\mathrm{HC}(V) =_{\mathrm{def}} \{ v \in V \mid v \in W \subset V,\ \dim(W) < \infty,\ \pi(K_{\mathbb{R}})W \subset W \}$$

is dense in V and has the following properties:

a) $\mathrm{HC}(V)$ is a $(\mathfrak{g}, K)$-module, i.e., both $\mathfrak{g}$ and K act on $\mathrm{HC}(V)$, and the two actions are compatible

b) every $v \in \mathrm{HC}(V)$ lies in a finite dimensional K-invariant subspace

c) $\mathrm{HC}(V)$ is finitely generated over $U(\mathfrak{g})$

The properties a) - c) are the defining properties of a *Harish-Chandra module*, and

$$\begin{array}{c} \{\text{admissible } G_{\mathbb{R}}\text{-representations of finite length}\} \\ \Big\downarrow \mathrm{HC} \\ \{\text{Harish-Chandra modules}\} \end{array}$$

is an exact, faithful, covariant functor; it is far from being one-to-one, but has functorial, topologically exact inverses

If (π, V) is a unitary representation, HC(V) carries a $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-invariant inner product – terminology: HC(V) has the structure of *unitary Harish-Chandra module*; conversely, every unitary Harish-Chandra module arises from a unitary $G_{\mathbb{R}}$-representation

A more general notion: *hermitian Harish-Chandra module*, i.e., a Harish-Chandra modules equipped with a non-trivial $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-invariant hermitian, but possibly *indefinite*, inner product

Schur's lemma applies in the context of Harish-Chandra modules, hence: the hermitian form on an irreducible hermitian Harish-Chandra module is unique up to scaling

Conclusion:

$$\begin{array}{c}\{\text{irreducible Harish-Chandra modules}\}\\ \cup\\ \{\text{irreducible hermitian Harish-Chandra modules}\}\\ \cup\\ \{\text{irreducible unitary Harish-Chandra modules}\}\\ \mathrm{HC}\ \big\uparrow\wr\\ \{\text{irreducible unitary } G_{\mathbb{R}}\text{-representations}\}\end{array}$$

The classification of irreducible Harish-Chandra modules is known, and it is easy to decide which of them carry a hermitian structure

This reduces the **main problem** to the

Equivalent problem: determine which irreducible hermitian Harish-Chandra modules have a positive definite inner product

The signature character

Let M be an arbitrary Harish-Chandra module; under the action of K – or equivalently, of $K_{\mathbb{R}}$ – M breaks up with finite multiplicities:

$$M \;=\; \bigoplus\nolimits_{\mu\in\widehat{K_{\mathbb{R}}}} n_\mu\cdot\mu$$

By definition, the formal sum

$$\tau_M \;=\; \sum\nolimits_{\mu\in\widehat{K_{\mathbb{R}}}} n_\mu\,\chi_\mu \qquad \left(\chi_\mu = \text{character of } \mu\right)$$

is the <u>K-*character of* M</u>

Similarly, one can define the *signature character* (Vogan) of a hermitian Harish-Chandra module M,

$$\sigma_M \;=\; \sum\nolimits_{\mu\in\widehat{K_{\mathbb{R}}}} (p_\mu - q_\mu)\,\chi_\mu\,, \qquad \text{with}$$

$p_\mu =$ number of copies of μ in M on which (,) is positive
$q_\mu =$ number of copies of μ in M on which (,) is negative

If M is an irreducible hermitian Harish-Chandra module, the hermitian form is non-degenerate on each K-type μ, hence $p_\mu + q_\mu = n_\mu$

Essentially as a tautology, an irreducible hermitian Harish-Chandra module M is unitary if and only if $\sigma_M = \tau_M$

Each irreducible Harish-Chandra module M can be realized as the unique submodule of a so-called standard Harish-Chandra module

Standard Harish-Chandra modules occur in families $\{ M_\lambda \mid \lambda \in \mathbb{C}^n \}$ parameterized by Euclidean spaces; M_λ is irreducible for generic values of λ, and the K-character τ_{M_λ} is independent of λ

In any such family, the modules M_λ parameterized by $\lambda \in i\mathbb{R}^n$ are unitarily induced, tempered unitary Harish-Chandra modules; in addition, the M_λ are hermitian for values of λ in a finite number of real subspaces of the parameter space $\mathbb{C}^n$

If M is an ireducible, hermitian Harish-Chandra, realized as the unique submodule of a standard Harish-Chandra module M_λ, then M_λ is also hermitian

The signature character σ_{M_λ} for λ in the hermitian range is constant away from the reduction locus; along the reduction locus σ_{M_λ} has jumps

To solve the main problem, it would suffice to:

a) understand how σ_{M_λ} jumps at all reduction points, and

b) keep track of the accumulated sign changes as more and more reduction points are crossed as one moves away from the unitary range $\lambda \in i\mathbb{R}^n$

In specific cases, this can be done, but in general it appears to be a very difficult problem

The new ingredient: Saito's theory of mixed Hodge modules

X smooth quasiprojective algebraic variety over $\mathbb{C}$
$\mathcal{D}_X$ sheaf of linear differential operators, with algebraic coefficients

Saito: There exists a category MHM(X), the category of mixed Hodge modules on X, consisting of a regular holonomic sheaf $\mathcal{M}$ of $\mathcal{D}_X$-modules, equipped with two filtrations,

the (finite, increasing) weight filtration $W_\ell \mathcal{M}$ by $\mathcal{D}_X$-submodules, and

the (infinite) Hodge filtration $F_p \mathcal{M}$ by coherent $\mathcal{O}_X$-submodules,
$0 \subset F_0 \mathcal{M} \subset F_1 \mathcal{M} \subset F_2 \mathcal{M} \subset \ldots \subset F_p \mathcal{M} \subset F_{p+1} \mathcal{M} \subset \ldots \subset \mathcal{M}$,
the latter a "good filtration" in the sense of $\mathcal{D}$-modules; the two filtrations must satisfy certain compatibility conditions

Most importantly, the category MHM(X) is preserved by all the standard direct and inverse image functors for $\mathcal{D}$-modules induced by morphisms between smooth algebraic varieties

Two special cases are easy to describe:

a) MHM({point}) $=$ {category of mixed Hodge structures}, and

b) $\mathcal{O}_X(\mathrm{M}) \in \mathrm{MHM}(X)$, for any variation of Hodge structure $\mathbf{M}$ on X when $\mathcal{O}_X(\mathrm{M})$ is equipped with the trivial weight filtration $0 = W_{-1} \subset W_0 = \mathcal{O}_X(\mathrm{M})$, and with the Hodge filtration $F_p\mathcal{O}(\mathrm{M})$ induced by the Hodge filtration of the variation of Hodge structure $\mathbf{M}$

In even moderately more complicated situations, the filtrations are very difficult to compute explicitly

How does this apply to the unitarity problem?

With $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, G, K, $\mathfrak{g}$ as before, let

X = flag variety of $\mathfrak{g}$ $=_{\text{def}}$ variety of all Borel subalgebras of $\mathfrak{g}$;

X is a complex projective variety, acted upon transitively by the algebraic group G, and is universal with respect to these properties

For example, in the special case of $G_{\mathbb{R}} = SL(n,\mathbb{R})$, $G = SL(n,\mathbb{C})$, X is the variety of all "complete flags" in $\mathbb{C}^n$, i.e.,

$$X = \{ F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n \mid \dim F_k = k \}$$

Back to the general case: the Beilinson-Bernstein construction realizes every Harish-Chandra module M as $M = H^0(X, \mathcal{M})$, the space of global sections of a sheaf $\mathcal{S}$ of coherent $(\mathcal{D}_X, K)$-module $\mathcal{M}$ – i.e., of a coherent sheaf of $\mathcal{D}_X$-modules, equipped with a compatible action of the group K.

The sheaves $\mathcal{M}$ that come up in this construction have canonical Hodge and weight fitrations $\{F_p M\}$, $\{W_\ell \mathcal{M}\}$ which turn them into mixed Hodge modules

Via the global section functor, each Harish-Chandra module $M = H^0(X, \mathcal{M})$ inherits two filtrations from the mixed Hodge module $\mathcal{M}$:

the weight filtration $W_\ell M$, by sub-Harish-Chandra-modules, and

the Hodge filtration, by finite dimensional, K-invariant subspaces $0 \subset F_0 M \subset F_1 M \subset F_2 M \subset \ldots \subset F_p M \subset F_{p+1} M \subset \ldots \subset M$,

the latter a "good filtration" in the sense of $U(\mathfrak{g})$-modules

The weight filtration $W_\ell M$ coincides with the *socle filtration* of M (Beilinson-Bernstein)

The Hodge filtration of a Harish-Chandra modules M is:

a) *canonical* and *functorial*

b) difficult to compute, even in simple cases

c) without obvious algebraic interpretation

d) trivial, i.e., $F_0 M = M$, in the special case of a finite dimensional, irreducible M

Conjecture: An irreducible, hermitian Harish-Chandra module M is unitary if and only if

i) the hermitian form is positive definite on $F_0 M$, and

ii) it is positive definite asymptotically,

asymptotically in the sense that $\sigma_M = \tau_M - 2\,\tau_U$, where τ_U is the K-character of a small (in the sense of the Gelfand-Kirillov dimension) K-invariant subspace $U \subset M$