CHARACTERISTIC CYCLES OF CONSTRUCTIBLE SHEAVES

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1. Introduction.

In his paper [K], Kashiwara introduced the notion of characteristic cycle for complexes of constructible sheaves on manifolds: let X be a real analytic manifold, and $\mathcal{F}$ a complex of sheaves of $\mathbb{C}$-vector spaces on X, whose cohomology is constructible with respect to a subanalytic stratification; the characteristic cycle $\mathrm{CC}(\mathcal{F})$ is a subanalytic, Lagrangian cycle (with infinite support, and with values in the orientation sheaf of X) in the cotangent bundle T^*X. The definition of $\mathrm{CC}(\mathcal{F})$ is Morse-theoretic. Heuristically, $\mathrm{CC}(\mathcal{F})$ encodes the infinitesimal change of the Euler characteristic of the stalks of $\mathcal{F}$ along the various directions in X. It tends to be difficult in practice to calculate $\mathrm{CC}(\mathcal{F})$ explicitly for all but the simplest complexes $\mathcal{F}$; on the other hand, the characteristic cycle construction has good functorial properties.

The behavior of $\mathrm{CC}(\mathcal{F})$ with respect to the operations of proper direct image, Verdier duality, and non-characteristic inverse image of $\mathcal{F}$ is well understood [KS]. In this paper, we describe the effect of the operation of direct image by an open embedding. Combining our result with those that were previously known, we obtain descriptions of $\mathrm{CC}(Rf_*\mathcal{F})$ and $\mathrm{CC}(f^*\mathcal{F})$ – analogous to those in [KS] – for arbitrary morphisms $f : X \to Y$ in the semi-algebraic category, and complexes $\mathcal{F}$ with semi-algebraically constructible cohomology. In effect, this provides an axiomatic characterization of the functor CC, at least in the semi-algebraic context. Our arguments do apply more generally in the subanalytic case, but because statements become quite convoluted, we shall not strive for the greatest degree of generality.

As a concrete application, we consider the case of the flag manifold X of a complex semisimple Lie algebra $\mathfrak{g}$. Here the Weyl group W of $\mathfrak{g}$ operates, via intertwining functors [BB1,BB2], on the K-group of the bounded derived category $\mathrm{D}^b(X)$ of semi-algebraically constructible sheaves. Since the characteristic cycle construction descends to the level of the K-group, W operates also on $\mathrm{CC}(\mathrm{D}^b(X))$; we remark parenthetically that $\mathrm{CC}(\mathrm{D}^b(X))$ coincides with the group of $\mathbb{R}^+$-invariant, semi-algebraic Lagrangian cycles on T^*X [KS]. There is also a natural, geometrically defined action of W on the group of all semi-algebraic Lagrangian cycles on T^*X, due to Rossmann [R], who extends an earlier construction of Kazhdan-Lusztig [KL]. We use our results on characteristic cycles to show that these two Weyl group actions coincide on $\mathrm{CC}(\mathrm{D}^b(X))$. For Lagrangian cycles supported on

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the union of the conormal bundles of Schubert cells, this statement was first proved by Kashiwara-Tanisaki [KT].

We shall take up further applications – both of our main technical result and of the result on Weyl group actions – in future papers: an analogue of the Kirillov character formula in the semisimple case, and an affirmative answer to a conjecture of Barbasch and Vogan. These future applications have been announced already [SV1,SV2], as have been the contents of this paper [SV1].

Kashiwara and Kashiwara-Schapira [K,KS] define characteristic cycles for subanalytically constructible sheaves. For their purposes and ours, it is not so much the analyticity that matters – rather, subanalytic sets enjoy finiteness and hereditary properties crucial for various arguments. Recent work in model theory [DMM,W] has produced geometric categories larger than the subanalytic one, with many of the same finiteness and hereditary properties [DM]. We explain, at the end of our paper, how the characteristic cycle construction and most results about them carry over to these more general geometric settings. That is important for us: our proof of the Barbasch-Vogan conjecture forces us to go outside the subanalytic category.

Several of our arguments involve delicate questions of sign. Our point of view, and also our intended applications, force us to make the signs explicit, and not to work up to certain universally determined choices of signs, as is often done. The elaboration of signs accounts, in part, for the length of this paper.

As for the organization of our paper, we recall the characteristic cycle construction and its basic properties in section 2, in a form convenient for our purposes. Families of cycles and limits of such families are the subject matter of section 3. We have kept the discussion quite general – we consider arbitrary Whitney stratifiable cycles, not only semi-algebraic or subanalytic cycles – and this additional degree of generality makes the section considerably longer than it would otherwise be. Sections 4 and 5 contain the formal statement and the proof of our main technical result, the open embedding theorem. Our statement formally resembles a result of Ginsburg [G2] in the complex algebraic category; the precise relationship between his result and ours will be described in section 4. Section 6 contains our axiomatic characterization of the functor CC in the semi-algebraic context and geometric formulas for the effect, on characteristic cycles, of the pushforward and the inverse image under arbitrary morphisms. In the end, this gives geometric descriptions of the effect of *all* the standard operations in the derived category. To simplify the discussion, we shall assume throughout that the base manifold is oriented; we briefly comment on the more general, non-orientable case in section 6. Our result on Weyl group actions occupies sections 7, 8, and 9: the two actions are reviewed in sections 7 and 8, respectively, and the fact that they coincide is established in section 9. Section 10, finally, discusses generalizations of the results in sections 1-6 to the geometric categories described in [DM].

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2. The characteristic cycle construction.

As in the introduction, we let X denote a real analytic manifold of dimension n, though in later chapters we shall assume more specifically that X is semi-algebraic. For convenience we also suppose that X is oriented. This will have the effect of making the characteristic cycles into oriented cycles, with infinite supports. Our orientation assumption gives meaning, in particular, to the fundamental cycle $[X]$, as an oriented Lagrangian cycle, with infinite support, in the symplectic manifold T^*X.

Initially we fix a subanalytic Whitney stratification $\mathcal{S}$ of X, and we consider a bounded complex $\mathcal{F}^{\boldsymbol{\cdot}}$ whose cohomology is constructible with respect to $\mathcal{S}$ – in other words, the cohomology sheaves $\mathcal{H}^p(\mathcal{F}^{\boldsymbol{\cdot}})$, restricted to any one of the strata, are locally constant of finite rank. The characteristic cycle $\mathrm{CC}(\mathcal{F}^{\boldsymbol{\cdot}})$ will turn out to be a Lagrangian cycle in T^*X, whose support is contained in

$$T^*_{\mathcal{S}}X \;=\; \bigcup_{S\in\mathcal{S}} T^*_S X\,; \tag{2.1}$$

here $T^*_S X$ stands for the conormal bundle of the submanifold $S \subset X$. We note that

$$\bigcup_{S\in\mathcal{S}} \big(T^*_S X - \bigcup_{R\neq S} \overline{T^*_R X}\big)$$

is open, smooth, and dense in $T^*_{\mathcal{S}}X$, with subanalytic complement ($\overline{T^*_R X}$ = closure of $T^*_R X$ in T^*X). We express this set as the union of its connected components

$$\begin{gathered}\bigcup_{S\in\mathcal{S}} \big(T^*_S X - \bigcup_{R\neq S} \overline{T^*_R X}\big) \;=\; \bigcup_{\alpha\in I} \Lambda_\alpha \quad \text{(disjoint union)},\\ \text{with } \Lambda_\alpha \text{ connected and } \Lambda_\alpha \subset T^*_{S_\alpha}X \text{ for some } S_\alpha \in \mathcal{S}\,.\end{gathered} \tag{2.2}$$

To describe a Lagrangian chain, with infinite support, carried by $T^*_{\mathcal{S}}X$, is equivalent to assigning an orientation and an integer to each component Λ_α; such a chain is a cycle if the appropriate linear combinations of the integer coefficients vanish. We shall describe $\mathrm{CC}(\mathcal{F}^{\boldsymbol{\cdot}})$ in terms of such data. The fact that our data satisfy the cycle condition, and that the cycle thus defined coincides with the characteristic cycle in the sense of Kashiwara [K] will then follow from known results [K,KS]; cf. (4.22) and the discussion below it.

First we specify the orientation of the various components Λ_α by orienting the $T^*_S X$. Around any point $p \in S$, we choose local coordinates $(x_1, \ldots, x_n)$ on X, such that S is locally given by the equations $x_{k+1} = \cdots = x_n = 0$, and such that $dx_1 \wedge \cdots \wedge dx_n$ is a positive covector, with respect to the given orientation on X. Let $(\xi_1, \ldots, \xi_n)$ be the fibre coordinates on T^*X dual to the frame $dx_1, \ldots, dx_n$. Then, near points in the fibre $T^*_p X$ over p, $T^*_S X$ is characterized by the equations $x_{k+1} = \cdots = x_n = \xi_1 = \cdots = \xi_k = 0$. Thus

$$(-1)^{n-k} dx_1 \wedge \cdots \wedge dx_k \wedge d\xi_{k+1} \wedge \cdots \wedge d\xi_n \tag{2.3}$$

is a nonzero covector of top degree. One checks that the sign of this covector does not depend of any of the choices we have made – whether or not S itself is orientable. In effect, then, the covector (2.3) puts a definite orientation on $T^*_S X$.

Next, we assign an integer m_α to each Λ_α. We fix a point $(p,\xi) \in \Lambda_\alpha$; here p is a point in the stratum S_α whose conormal bundle contains Λ_α, and ξ a conormal vector to S_α at p. It is then possible to choose a C^∞ function ϕ on some open neighborhood of p in X, with the following properties:

$$\begin{aligned} &\text{a) } \ \phi(p) = 0 \ \text{ and } \ d\phi_p = \xi\,; \\ &\text{b) } \ d\phi \ \text{ is transverse to } \ \Lambda_\alpha \ \text{at} \ (p,\xi)\,; \\ &\text{c) } \ \text{the Hessian of } \ \phi|S_\alpha \ \text{ is positive definite at } \ p\,. \end{aligned} \tag{2.4}$$

The existence of such a function – even one that is required to be real analytic – can be verified either geometrically or by explicit calculation in local coordinates. In terms of our choices, we define the multiplicity m_α as the Euler characteristic of the stalk, at p, of the local cohomology sheaf of $\mathcal{F}^{\cdot}$ along the set $\{x \in X \mid \phi(x) \geq 0\}$,

$$m_\alpha = \chi(R\Gamma_{\{\phi\geq 0\}}(\mathcal{F}^{\cdot})_p\,)\,. \tag{2.5}$$

We now appeal to theorem 9.5.6 and formulas (9.4.10), (9.5.18) in [KS], to conclude: the integers m_α describe the characteristic cycle $\mathrm{CC}(\mathcal{F}^{\cdot})$ in the sense of Kashiwara; in particular, m_α does not depend on the choices of (p,ξ) and ϕ, and the chain defined by the m_α satisfies the cycle condition. In effect, the formula (2.5) becomes a theorem in the functorial approach of [KS]. For us, it is convenient to take (2.5) as the definition of CC because of its more directly geometric flavor.

What we shall actually use to compute $\mathrm{CC}(\mathcal{F}^{\cdot})$ in particular situations is a Morse-theoretic version of (2.5). We retain our choices of (p,ξ) and of the function ϕ. By lemma 3.5.1 in [GM], if we fix a C^∞ Riemannian metric on X near p, then for every sufficiently small open ball B centered at p and every stratum $S \in \mathcal{S}$,

$$\begin{aligned} &\text{a) the boundary } \partial B \text{ is transverse to } S\,; \\ &\text{b) the restriction of } \phi \text{ to } B\cap S \text{ has no critical points with critical} \\ &\quad\ \text{value 0, except for } S = S_\alpha \text{ and the value 0 at } p\,; \\ &\text{c) the restriction of } \phi \text{ to } \partial B\cap S \text{ has no critical points with} \\ &\quad\ \text{critical value } 0\,. \end{aligned} \tag{2.6}$$

As is pointed out in [GM, p. 82], the verification of (2.6) can be simplified considerably when ϕ and the metric are real analytic, as we may assume without loss of generality. We apply Goresky-MacPherson's stratified Morse theory to the pair

$$(\,B\cap\phi^{-1}(-\epsilon,\epsilon)\ ,\ B\cap\phi^{-1}(-\epsilon,0)\,)\,,$$

with B "sufficiently small" in the sense of (2.6) and ϵ positive and sufficiently small in relation to B. Then, in the exact cohomology sequence

$$\begin{aligned} \cdots \to \mathrm{H}^q_{\{\phi\geq 0\}}(B\cap\phi^{-1}(-\epsilon,\epsilon),\mathcal{F}^{\cdot}) \to \mathrm{H}^q(B\cap\phi^{-1}(-\epsilon,\epsilon),\mathcal{F}^{\cdot}) \to \\ \to \mathrm{H}^q(B\cap\phi^{-1}(-\epsilon,0),\mathcal{F}^{\cdot}) \to \mathrm{H}^{q+1}_{\{\phi\geq 0\}}(B\cap\phi^{-1}(-\epsilon,\epsilon),\mathcal{F}^{\cdot}) \to \dots\ , \end{aligned}$$

the terms $\mathrm{H}^q(B \cap \phi^{-1}(-\epsilon, \epsilon), \mathcal{F}^{\boldsymbol{\cdot}})$ and $\mathrm{H}^q(B \cap \phi^{-1}(-\epsilon, 0), \mathcal{F}^{\boldsymbol{\cdot}})$ remain stable when the radius of B, and correspondingly ϵ, tend towards 0 – see Proposition 3.5.3 in [GM]. Consequently,

$$
\begin{aligned}
m_\alpha \; &= \; \chi(\, \mathrm{H}^*_{\{\phi \geq 0\}}(B \cap \phi^{-1}(-\epsilon, \epsilon)\, ,\, \mathcal{F}^{\boldsymbol{\cdot}})\,) \\
&= \; \chi(\, \mathrm{H}^*(B \cap \phi^{-1}(-\epsilon, \epsilon), B \cap \phi^{-1}(-\epsilon, 0)\, ;\, \mathcal{F}^{\boldsymbol{\cdot}})\,)\, ,
\end{aligned}
\tag{2.7}
$$

again for B as in (2.6) and $\epsilon > 0$ small. The second equality in (2.7), we note, really amounts to the definition of relative cohomology.

We started out assuming that the cohomology sheaves of the complex $\mathcal{F}^{\boldsymbol{\cdot}}$ are constructible with respect to a particular subanalytic Whitney stratification $\mathcal{S}$. Any two such stratifications have a common Whitney refinement, and passing from the original stratification $\mathcal{S}$ to a Whitney refinement does not alter $\mathrm{CC}(\mathcal{F}^{\boldsymbol{\cdot}})$. This is implicit in the approach taken by [KS], but also follows from the geometric description (2.7): the Euler characteristics m_α vanish for any newly introduced stratum S_α because even the relative cohomology groups in (2.7) vanish. Also, every subanalytic stratification has a subanalytic Whitney refinement. In effect, then, we have made sense of $\mathrm{CC}(\mathcal{F}^{\boldsymbol{\cdot}})$ whenever the cohomology of $\mathcal{F}^{\boldsymbol{\cdot}}$ is constructible with respect to some arbitrary subanalytic stratification. The integers remain unchanged when $\mathcal{F}^{\boldsymbol{\cdot}}$ is replaced by a quasi-isomorphic complex, so $\mathrm{CC}(\mathcal{F}^{\boldsymbol{\cdot}})$ depends only on the class of $\mathcal{F}^{\boldsymbol{\cdot}}$ in $\mathrm{D}^b(X)$, the bounded derived category of subanalytically constructible complexes. Since they were defined as Euler characteristics, the integers m_α behave additively in distinguished triangles. Consequently the definition of CC descends even to the K-group of $\mathrm{D}^b(X)$,

$$
\mathrm{CC} \,:\, \mathrm{K}(\mathrm{D}^b(X)) \;\longrightarrow\; \mathcal{L}^+(X)\,; \tag{2.8}
$$

here $\mathcal{L}^+(X)$ denotes the group of integral, $\mathbb{R}^+$-invariant, subanalytic, Lagrangian cycles (with infinite support) in T^*X. In fact, (2.8) turns out to be an isomorphism [KS], but this will not be important for our purposes.

Let us describe the behavior of the characteristic cycle construction with respect to proper direct image and normally nonsingular pullback of sheaves. We consider a real analytic map $f : X \to Y$ between oriented real analytic manifolds. The induced map df between the cotangent spaces fits into the commutative diagram of bundle maps,

$$
\begin{array}{ccccc}
T^*X & \xleftarrow{\;df\;} & X \times_Y T^*Y & \xrightarrow{\;\tau\;} & T^*Y \\
{\scriptstyle \pi_X}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_Y} \\
X & = & X & \xrightarrow{\;f\;} & Y\;.
\end{array}
\tag{2.9}
$$

We write n for the dimension of X, m for the dimension of Y. By assumption, the space X has been oriented. This orientation induces an orientation on the fibres of T^*X. Locally near some $p \in X$, T^*X can be identified with the product $X \times T^*_pX$, and the product decomposition induces an orientation on T^*X. We note that this orientation on T^*X changes by the sign $(-1)^n$ when the order of the factors X,

$T^*_p X$ is reversed, but is independent of the chosen orientation of X. We use the same conventions to orient T^*Y and $X \times_Y T^*Y$, in the latter case by making the local identification $X \times_Y T^*Y \simeq X \times T^*_{f(p)}Y$.

When f is proper, the diagram (2.9) induces a morphism

$$f_* : \mathcal{L}^+(X) \longrightarrow \mathcal{L}^+(Y) \tag{2.10}$$

as follows. Let $C \in \mathcal{L}^+(X)$ be a particular Lagrangian cycle, and let $\Lambda \subset T^*X$ denote the support of C. Then

$$C \in \mathrm{H}^{inf}_n(\Lambda, \mathbb{Z}) \simeq \mathrm{H}^n_\Lambda(T^*X, \mathbb{Z})\,; \tag{2.11}$$

here $\mathrm{H}^{inf}_*(\dots)$ denotes homology with infinite support (Borel-Moore homology), and $\mathrm{H}^*_\Lambda(\dots)$ local cohomology along Λ. The isomorphism in (2.11) is given by Poincaré duality

$$\begin{aligned} &\mathrm{H}^*_\Lambda(T^*X, \mathbb{Z}) \xrightarrow{\ \sim\ } \mathrm{H}^{inf}_*(\Lambda, \mathbb{Z})\,, \\ &\alpha \longmapsto (-1)^{\deg(\alpha)}\alpha \cap [T^*X]\,, \end{aligned} \tag{2.12}$$

i.e., $(-1)^{\deg(\alpha)}$ times the cap product with the fundamental class of T^*X, which is given meaning by the orientation of T^*X. There are a number of sign conventions in the definition of cap product; ours agrees with that of [Sp].

We digress briefly to make the sign in (2.11) completely explicit. We shall use the language of differential forms, though one could equally well work with simplices. Let M be an oriented manifold – in our situation, $M = T^*X$ – and $S \subset M$ an oriented submanifold. We pick local coordinates $x_1, \dots, x_{n-c}$ on S compatibly with the orientation on S, and normal coordinates $y_1, \dots, y_c$ for $S \subset M$, such that the coordinate system $x_1, \dots, x_{n-c}, y_1, \dots, y_c$ is compatible with the orientation of X. In terms of these local coordinates, classes in $\mathrm{H}^c_S(M, \mathbb{C})$ can be represented by differential forms, with coefficients which are hyperfunctions on M supported on S. We put a sign on the Poincaré duality map

$$\mathrm{H}^c_S(M, \mathbb{C}) \to \mathrm{H}^{inf}_{n-c}(S, \mathbb{C}) \tag{2.13a}$$

as follows. If $\alpha \in \mathrm{H}^c_S(M, \mathbb{C})$ is represented by $\omega = f dy_1 \wedge \dots \wedge dy_c$, where f is a locally constant function on S, we send (locally) α to the oriented cycle S, multiplied by the constant f. The resulting sign for the map (2.13a) pins down the sign for the integral map

$$\mathrm{H}^c_S(M, \mathbb{Z}) \to \mathrm{H}^{inf}_{n-c}(S, \mathbb{Z})\,, \tag{2.13b}$$

and hence also in (2.11): since only the local orientation is at issue, no information is lost by going to complex coefficients.

Analogously to (2.11) we define the Poincaré duality map

$$\mathrm{H}^n_{df^{-1}(\Lambda)}(X \times_Y T^*X, \mathbb{Z}) \xrightarrow{\ \sim\ } \mathrm{H}^{inf}_m(df^{-1}(\Lambda), \mathbb{Z})\,. \tag{2.14}$$

The composition, right to left, of the inverse of Poincaré duality (2.12), pullback

$$\mathrm{H}^n_\Lambda(T^*X,\mathbb{Z}) \xrightarrow{df^*} \mathrm{H}^n_{df^{-1}(\Lambda)}(X\times_Y T^*X,\mathbb{Z}),$$

and Poincaré duality (2.14) defines the operation of "pullback of cycles", i.e., the Gysin map

$$df^* \ : \ \mathrm{H}^{inf}_n(\Lambda,\mathbb{Z}) \longrightarrow \mathrm{H}^{inf}_m(df^{-1}(\Lambda),\mathbb{Z}). \tag{2.15}$$

Our map (2.10) is the composition

$$f_* \ = \ \tau_*\circ df^* \tag{2.16}$$

of the Gysin map (2.15) and the proper pushforward

$$\mathrm{H}^{inf}_m(df^{-1}(\Lambda),\mathbb{Z}) \xrightarrow{\tau_*} \mathrm{H}^{inf}_m(\tau(df^{-1}(\Lambda)),\mathbb{Z}).$$

Implicit in this description of f_* is the assertion that $\tau(df^{-1}(\Lambda))$ is Lagrangian in T^*Y – this follows from the fact that the proper map f can be stratified, as in [GM, §§I.1.6-7]. Alternatively, one can argue as in [KS, Proposition 8.3.11].

The preceding description of f_* amounts to a concrete reformulation of a construction of Kashiwara-Schapira [KS], who then prove the following statement: for any proper real analytic map $f:X\to Y$ and any $\mathcal{F}\in \mathrm{D}^b(X)$,

$$\mathrm{CC}(Rf_*\mathcal{F}) \ = \ f_*(\mathrm{CC}(\mathcal{F})) \tag{2.17}$$

[KS, Proposition 9.4.2].

Now the case of normally nonsingular pullback. We fix a real analytic map $f:X\to Y$ as before, and an object $\mathcal{G}\in\mathrm{D}^b(Y)$. We shall call f normally nonsingular with respect to $\mathcal{G}$ if there exists a Whitney stratification $\mathcal{S}$ of Y, such that

$$\begin{aligned} &\text{a) } \mathcal{G} \text{ is constructible with respect to } \mathcal{S}\,;\\ &\text{b) } f \text{ is transverse to all the strata of } \mathcal{S}\,. \end{aligned} \tag{2.18}$$

Explicitly, b) means that, for each stratum $S\in\mathcal{S}$, each point $s\in S$, and each $x\in f^{-1}(s)$, $f_*(T_xX)+T_sS = T_sY$. We fix a particular stratification $\mathcal{S}$ with these properties. Then, for $C\in\mathrm{H}^{inf}_m(T^*_{\mathcal{S}}Y,\mathbb{Z})$, we can define $f^*(C)\in\mathrm{H}^{inf}_n(T^*_{f^{-1}(\mathcal{S})}X,\mathbb{Z})$ by the equation

$$f^*(C) \ = \ df_*(\tau^*C)\,; \tag{2.19}$$

here τ^* denotes the Gysin map corresponding to τ. Under the hypotheses just mentioned,

$$\mathrm{CC}(f^*\mathcal{G}) \ = \ f^*(\mathrm{CC}(\mathcal{G}))\,. \tag{2.20}$$

In effect, this is [KS, Proposition 9.4.3], except for our requirement of normal nonsingularity, which is not quite as general as Kashiwara-Schapira's non-characteristic assumption. We should point out that (2.20) follows directly from the definition of the characteristic cycle given here.

The definitions of both f_*, as in (2.16), and f^*, as in (2.19), involve Gysin maps. Our hypothesis of normal nonsingularity ensures that the Gysin map τ^* in (2.19) is literally the pullback of cycles. On the other hand, the effect of the Gysin map df^* in (2.16) on a particular cycle C is less transparent: typically df will not be transverse to C, so to compute $df^*(C)$ in geometric terms it is necessary – loosely speaking – to deform C slightly. We shall make this precise in the next section.

3. Families of cycles.

Our objective in this section is to make sense of the limit of a one parameter family of cycles. As in the previous section, the cycles we work with are true geometric cycles – not cycles up to some notion of equivalence, as in the definition of homology. We fix a connected, oriented, smooth manifold M of dimension d and a particular Whitney stratification $\mathcal{T}$, for example the stratification induced by a triangulation of M. We do not require the stratification to be smooth, but the strata must be at least C^1, of course, to give meaning to the Whitney conditions. In the setting of section 2, T^*X will play the role of M and the Λ_α will be the highest dimensional non-open strata for $\mathcal{T}$. In assuming only that M is a smooth manifold, we are departing temporarily from the real analytic context: in future applications we need to consider families of cycles which are not necessarily subanalytic.

Let us define the notion of a k-cycle, with infinite support, subordinate to the stratification $\mathcal{T}$. Refining $\mathcal{T}$ if necessary, we shall assume that all strata are orientable. We enumerate the connected components of the k-dimensional strata as $\Lambda_{k,\alpha}$, with α ranging over the index set A_k, and we write $\Lambda_k(\mathcal{T})$ for the union of the closures of the $\Lambda_{k,\alpha}$; in other words, $\Lambda_k(\mathcal{T})$ denotes the "k-skeleton of $\mathcal{T}$", i.e, the union of the strata of dimension $\leq k$. To simplify the discussion, we put a definite orientation on each of the $\Lambda_{k,\alpha}$. Then every integral linear combination

$$(3.1) \qquad \sum_{\alpha\in A} m_\alpha \, \Lambda_{k,\alpha} \, , \qquad m_\alpha \in \mathbb{Z}$$

defines a k-chain, with infinite support, carried by $\Lambda_k(\mathcal{T})$. The boundary of such a chain is a well defined $(k-1)$-chain (the $\Lambda_{k,\alpha}$ constitute a locally finite family!). This gives us the notion of k-cycle subordinate to $\mathcal{T}$. As a matter of notation, we write $\mathrm{C}_k^{inf}(M,\mathcal{T})$ for the group of k-chains,

$$(3.2) \qquad \partial \ : \ \mathrm{C}_k^{inf}(M,\mathcal{T}) \ \longrightarrow \ \mathrm{C}_{k-1}^{inf}(M,\mathcal{T})$$

for the boundary operator, and

$$(3.3) \qquad \mathrm{Z}_k^{inf}(M,\mathcal{T}) \ = \ \mathrm{Ker}\{\, \partial : \mathrm{C}_k^{inf}(M,\mathcal{T}) \longrightarrow \mathrm{C}_{k-1}^{inf}(M,\mathcal{T})\,\}$$

for the group of k-cycles subordinate to $\mathcal{T}$. The subspace $\Lambda_k(\mathcal{T}) \subset M$ contains no strata of dimension $> k$, so

$$(3.4) \qquad \mathrm{Z}_k^{inf}(M,\mathcal{T}) \ = \ \mathrm{H}_k^{inf}(\Lambda_k(\mathcal{T}),\mathbb{Z})\,.$$

On the other hand,

$$(3.5) \qquad \mathrm{H}_k^{inf}(\Lambda_k(\mathcal{T}),\mathbb{Z}) \ \cong \ \mathrm{H}^{d-k}(M, M-\Lambda_k(\mathcal{T});\mathbb{Z}) \ \cong \ \mathrm{H}^{d-k}_{\Lambda_k(\mathcal{T})}(M,\mathbb{Z})$$

by Poincaré duality and the definition of local cohomology; for the explicit choice of sign, see (2.13). This gives us yet another description of the group of k-cycles,

$$(3.6) \qquad \mathrm{Z}_k^{inf}(M,\mathcal{T}) \ \cong \ \mathrm{H}^{d-k}_{\Lambda_k(\mathcal{T})}(M,\mathbb{Z})\,,$$

as a local cohomology group.

Whenever a second Whitney stratification $\mathcal{T}'$ refines the original stratification, one obtains an inclusion

$$\mathrm{C}_*^{inf}(M,\mathcal{T}) \;\hookrightarrow\; \mathrm{C}_*^{inf}(M,\mathcal{T}')$$

which commutes with the operation of taking boundary, hence

$$\mathrm{Z}_k^{inf}(M,\mathcal{T}) \;\hookrightarrow\; \mathrm{Z}_k^{inf}(M,\mathcal{T}')\,. \tag{3.7}$$

If M were a real analytic manifold, and if we were dealing only with cycles subordinate to subanalytic stratifications, we could use (3.7) to give meaning to the notion of a cycle without reference to a particular stratification: a subanalytic k-cycle would then be an element of the direct limit of $\mathrm{Z}_k^{inf}(M,\mathcal{T})$, as $\mathcal{T}$ runs through all possible subanalytic Whitney stratifications of M. Since we have departed, temporarily, from the real analytic context, we can no longer assume that any two Whitney stratifications have a common refinement. Thus, instead of (3.7), we need to appeal to (3.6) to free our definition of a cycle from the dependence on the stratification $\mathcal{T}$. For any particular

$$C \;\in\; \mathrm{Z}_k^{inf}(M,\mathcal{T}) \;\cong\; \mathrm{H}^{d-k}_{\Lambda_k(\mathcal{T})}(M,\mathbb{Z})\,, \tag{3.8}$$

we may replace the closed subset $\Lambda_k(\mathcal{T}) \subset M$ by the support $|C|$ of C. Then $C \in \mathrm{H}^{d-k}_{|C|}(M,\mathbb{Z})$ has meaning independently of the choice of $\mathcal{T}$.

Let us adopt this as our working definition of a k-cycle: a pair $(|C|, C)$, consisting of a Whitney stratifiable, closed subset $|C| \subset M$ of dimension $\leq k$, and a cohomology class $C \in \mathrm{H}^{d-k}_{|C|}(M,\mathbb{Z})$ whose support is all of $|C|$. To avoid convoluted notation, we shall use the symbol C not only for the cohomology class in $\mathrm{H}^{d-k}_{|C|}(M,\mathbb{Z})$ that defines it, but also as shorthand for the pair $(|C|, C)$. Though we allow the support $|C|$ to have dimension $< k$ for formal reasons, this possibility arises only if $|C| = \emptyset$ and $C = 0$, since $\mathrm{H}^{d-k}_{|C|}(M,\mathbb{Z}) = 0$ if $\dim|C| < k$. The k-cycles in our present sense constitute an abelian group $\mathrm{Z}_k^{inf}(M)$. As was remarked earlier, for each Whitney stratification $\mathcal{T}$ of M, there is a natural homomorphism

$$\mathrm{Z}_k^{inf}(M,\mathcal{T}) \longrightarrow \mathrm{Z}_k^{inf}(M)\,. \tag{3.9}$$

Any particular k-cycle C lies in the image of this homomorphism, provided only that $|C| \subset M$ is a Whitney stratified subspace with respect to $\mathcal{T}$.

Now we turn to the notion of a family of k-cycles in M, parametrized by an open interval $I \subset \mathbb{R}$. By definition, such a family consists of a $(k+1)$-cycle C_I in $I \times M$, such that

$$\begin{aligned} &|C_I| \cap (\{s\}\times M) \;\text{ is a Whitney stratifiable subset} \\ &\text{of } \{s\}\times M \text{ of dimension} \leq k, \text{ for each } s \in I\,. \end{aligned} \tag{3.10}$$

Via the restriction map

$$\mathrm{H}^{d-k}_{|C_I|}(I\times M,\mathbb{Z}) \;\longrightarrow\; \mathrm{H}^{d-k}_{|C_I|\cap(\{s\}\times M)}(\{s\}\times M,\mathbb{Z}),$$

C_I maps to a local cohomology class on $\{s\} \times M$ in degree $d-k$, which is greater than or equal to the codimension of $|C_I| \cap (\{s\} \times M)$ in $\{s\} \times M$. The interpretation of this cohomology class as a k-cycle allows us to replace $|C_I| \cap (\{s\} \times M)$ by the support of the cohomology class. Let us denote this support, transferred to M via the identification $\{s\} \times M \cong M$, by $|C_s|$, and the corresponding local cohomology class by C_s. Thus, for each $s \in I$, the passage

$$C_I \longmapsto C_s \in \mathrm{H}^{d-k}_{|C_s|}(M, \mathbb{Z}) \tag{3.11}$$

associates to C_I the *specialization* $C_s \in \mathrm{Z}_k^{inf}(M)$ at s. Our identification of cycles with local cohomology classes involves a choice of orientation for the ambient space. For $I \times M$, we choose the product orientation, in the given order, with $I \subset \mathbb{R}$ oriented positively.

To define the notion of limit of a family of cycles as the parameter s tends to a limit, say $s \to 0^+$, we specialize the choice of the open interval I to

$$I \;=\; (0, b)\,, \tag{3.12a}$$

for some constant $b > 0$, and set

$$J \;=\; [0, b)\,. \tag{3.12b}$$

We consider a family of k-cycles C_I, subject to the following condition: the closure $\overline{|C_I|} \subset J \times M$ admits a Whitney stratification such that

$$\overline{|C_I|} \cap (\{0\} \times M) \;\text{ is a stratified subset of }\; \overline{|C_I|}\,. \tag{3.13a}$$

To give meaning to the notion of Whitney stratification of a subset of the manifold with boundary $J \times M$, we embed the latter in $\mathbb{R} \times M$. Our hypothesis (3.13a) implies that

$$\overline{|C_I|} \cap (\{0\} \times M) \;\text{ has dimension } \leq k\,. \tag{3.13b}$$

Indeed, any stratum in $\overline{|C_I|} \cap (\{0\} \times M)$ of dimension $> k$ would have to lie in the boundary of a stratum in $|C_I|$ of dimension $> (k+1)$, but there are no such strata.

We shall argue presently that the inclusion $(I \times M, |C_I|) \hookrightarrow (J \times M, \overline{|C_I|})$ induces an isomorphism

$$\mathrm{H}^*_{\overline{|C_I|}}(J \times M, \mathbb{Z}) \;\xrightarrow{\sim}\; \mathrm{H}^*_{|C_I|}(I \times M, \mathbb{Z})\,. \tag{3.14}$$

Via this isomorphism and specialization at 0 – i.e., restriction from $J \times M$ to $\{0\} \times M$, cutting down the support of the local cohomology class, if necessary, and making the identification $\{0\} \times M \cong M$ – the local cohomology class $C_I \in \mathrm{H}^{d-k}_{|C_I|}(I \times M, \mathbb{Z})$ determines a class

$$C_0 \in \mathrm{H}^{d-k}_{|C_0|}(M, \mathbb{Z}), \tag{3.15}$$

in other words, a k-cycle $C_0 \in \mathrm{Z}_k^{inf}(M)$. Except for the use of the isomorphism (3.14), this process of specializing at 0 is entirely analogous to specialization at any non-zero s. By definition, C_0 is the limit of the family C_I :

$$C_0 = \lim_{s \to 0^+} C_s\,. \tag{3.16}$$

We should remark that this is a very weak notion of limit, reflecting the weak hypothesis (3.10) in the definition of a family of cycles. In due course, we shall impose stronger hypotheses, and that will result in a correspondingly stronger notion of limit.

We still need to verify (3.14). Applying base change to the Cartesian square

$$\begin{tikzcd} |C_I| \arrow[r, "\tilde{j}"] \arrow[d, "\tilde{i}"'] & \overline{|C_I|} \arrow[d, "i"] \\ I \times M \arrow[r, "j"'] & J \times M\,, \end{tikzcd}$$

we get $R\tilde{j}_*\tilde{i}^!\mathbb{Z}_{I\times M} \cong i^! Rj_*\mathbb{Z}_{I\times M}$. Since $Rj_*\mathbb{Z}_{I\times M} \cong \mathbb{Z}_{J\times M}$, the previous isomorphism reduces to $R\tilde{j}_*\tilde{i}^!\mathbb{Z}_{I\times M} \cong i^!\mathbb{Z}_{J\times M}$. Taking global cohomology on both sides, we obtain (3.14).

Our assumption (3.13) implies, in particular, the triangulability of the pair $(\overline{|C_I|}\,,\ \overline{|C_I|} \cap (\{0\} \times M))$. Thus we may view C_I not only as a $(k+1)$-cycle in $I \times M$, but also as a $(k+1)$-chain in $J \times M$. As such, it has a boundary ∂C_I, which is necessarily supported in $\{0\} \times M \cong M$. We shall tacitly regard ∂C_I as a k-cycle in M.

3.17 Proposition. $\lim_{s\to 0^+} C_s \;=\; -\,\partial C_I$.

Proof. Intuitively, this is a fairly simple matter. A proof making precise one's geometric intuition would be straightforward if we could argue in the subanalytic context. Our weaker hypotheses, and our need to pin down signs precisely, makes the argument considerably more involved.

We fix a Whitney stratification $\mathcal{T}$ of $\overline{|C_I|}$ as in (3.13), and let $\mathcal{T}_0$ denote the induced stratification on $\{0\} \times M \cong M$. By construction, both C_0 and ∂C_I are k-cycles in $\{0\} \times M \cong M$, subordinate to $\mathcal{T}_0$. For each k-dimensional stratum S_0 of $\mathcal{T}_0$ we need to equate the multiplicities with which S_0 occurs in C_0 and $-\partial C_I$. If a particular $(k+1)$-dimensional stratum S for $\mathcal{T}$ contains S_0 in its boundary, then near each point of S_0, it is finite union of branches – submanifolds of $I \times M$, whose closures in $J \times M$ are manifolds with boundary, each containing S_0 and contained in $S \cup S_0$. That, in effect, is the essence of a Whitney stratification.

To equate the multiplicities with which S_0 occurs in C_0 and $-\partial C_I$, we express the $(k+1)$-cycle C_I, locally near a point $p \in S_0$, as a linear combination of branches of $(k+1)$-dimensional strata S as above. We shall equate multiplicities one branch at a time. To simplify the notation, we now let S denote a particular local branch, and $\bar{S}$ the closure of the local branch, i.e., $\bar{S} = S \cup S_0$. We also replace the ambient manifold $J \times M$ by its intersection with a small ball around p, and we replace $\{0\} \times M$ by its intersection with the same ball. We denote these intersections by

$\tilde{M}$ and M_0, respectively; then $(\tilde{M}, M_0)$ is a manifold with boundary. Restricting our situation to an even smaller ball around p, we can arrange that $\bar{S}$ is closed in $\tilde{M}$, that $(\bar{S}, S_0)$ is still a manifold with boundary, and that both $\bar{S}$ and S_0 are connected. One justifies these assertions – in the topological category, but not necessarily differentiably – by triangulating $J \times M$ compatibly with $\tilde{\mathcal{T}}$. One additional piece of notation: $\tilde{M}^+ = \tilde{M} \cap (I \times M)$. We recall that $\tilde{M}$, M_0, $\bar{S}$, and S_0 have dimension $d+1$, d, $k+1$, and k, respectively. In terms of our previous notation, we had used the given orientation of M to orient $\mathbb{R} \times M$, so that $I \times M$ "lies on the positive side" of $\{0\} \times M$, i.e., $\partial(I \times M) = -(\{0\} \times M)$, thought of as geometric chains. Translated into our present notation, this means $\partial \tilde{M}^+ = -M_0$.

The assertion to be proved involves the manifold with boundary $(\bar{S}, S_0)$ lying as a closed subspace in the manifold with boundary $(\tilde{M}, M_0)$. Our hypotheses do not seem to imply that the former is a submanifold with boundary of the latter. From the technical point of view, that is what makes our argument complicated.

Our choices of orientations for $\tilde{M}$ and M_0 amount to explicit identifications between the dualizing sheaves and constant sheaves

$$\mathbb{Z}_{M_0}[d] \;\cong\; \mathbb{D}_{M_0}, \qquad \mathbb{Z}_{\tilde{M}}[d+1] \;\cong\; \mathbb{D}_{\tilde{M}}. \tag{3.18}$$

The various inclusions among the spaces S_0, S, $\bar{S}$, M_0, $\tilde{M}^+$, $\tilde{M}$, fit into two Cartesian squares

$$\begin{array}{ccccc} \tilde{M}^+ & \xrightarrow{\tilde{j}} & \tilde{M} & \xleftarrow{\tilde{i}} & M_0 \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} & & \uparrow{\scriptstyle i_0} \\ S & \xrightarrow{j} & \bar{S} & \xleftarrow{k} & S_0 \end{array} \;. \tag{3.19}$$

By base change and (3.18),

$$\begin{aligned} &\text{a)}\;\; \bar{i}^!\mathbb{Z}_{\tilde{M}} \;\cong\; Rj_*i^!\mathbb{Z}_{\tilde{M}^+} \;\cong\; Rj_*\mathbb{D}_S[-(d+1)], \\ &\text{b)}\;\; \bar{i}^!\mathbb{Z}_{\tilde{M}} \to \bar{i}^!\tilde{i}_*\tilde{i}^*\mathbb{Z}_{\tilde{M}} \cong k_*i_0^!\mathbb{Z}_{M_0} \cong k_*\mathbb{D}_{S_0}[-d]. \end{aligned} \tag{3.20}$$

There is a distinguished triangle

$$k_*\mathbb{D}_{S_0} \to \mathbb{D}_{\bar{S}} \to Rj_*\mathbb{D}_S \tag{3.21}$$

associated to the bottom row of (3.19). The definition of cohomology and the connecting homomorphism of the above triangle give the two top horizontal arrows in the diagram

$$\begin{array}{ccccc} \mathrm{H}^{-(k+1)}(S\,,\mathbb{D}_S) & \xrightarrow{\sim} & \mathrm{H}^{-(k+1)}(\bar{S}\,, Rj_*\mathbb{D}_S) & \xrightarrow{\delta} & \mathrm{H}^{-k}(S_0\,,\mathbb{D}_{S_0}) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathrm{H}^{d-k}_S(\tilde{M}^+\,,\mathbb{Z}_{\tilde{M}^+}) & \xrightarrow{\sim} & \mathrm{H}^{d-k}_{\bar{S}}(\tilde{M}\,,\mathbb{Z}_{\tilde{M}}) & \longrightarrow & \mathrm{H}^{d-k}_{S_0}(M_0\,,\mathbb{Z}_{M_0}) \end{array} \;. \tag{3.22}$$

The definition of local cohomology and (3.20a) determine the three vertical isomorphisms. The first horizontal arrow is the canonical isomorphism (3.14), rewritten in terms of our present notation, and the left square commutes by definition – cf. the proof of (3.14). The second arrow in the bottom row is restriction of local cohomology. By (3.20b) and the definition of triangle (3.21), the right square commutes also. The composition of the two bottom arrows is precisely specialization of cycles, i.e., the map which associates the limit at 0 to a family of cycles, rewritten in our present notation.

Let us reinterpret the top row in (3.22). We extend $\tilde{M}$ across its boundary M_0; This gives us a manifold U, containing M_0 as a closed submanifold, which separates U into two components – the one "on the positive side" is $\tilde{M}^+$. We now argue as in the case of the diagram (3.22), but with U taking the place of the ambient space $\tilde{M}$; technically this is simpler than the previous case, since U is a manifold, not a manifold with boundary. We conclude the commutativity of the top two squares in the following diagram:

$$
\begin{tikzcd}
\mathrm{H}^{-(k+1)}(S\,,\,\mathbb{D}_S) \arrow[r, "\delta"] \arrow[d, "\cong"] & \mathrm{H}^{-k}(S_0\,,\,\mathbb{D}_{S_0}) \arrow[r, equal] \arrow[d, "\cong"] & \mathrm{H}^{-k}(S_0\,,\,\mathbb{D}_{S_0}) \arrow[d, "\cong"] \\
\mathrm{H}^{d-k}_S(U\,,\,\mathbb{Z}_U) \arrow[r, "\delta"] \arrow[d, "\cong"'] & \mathrm{H}^{d+1-k}_{S_0}(U\,,\,\mathbb{Z}_U) \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}^{d-k}_{S_0}(M_0\,,\,\mathbb{Z}_{M_0}) \arrow[d, "\cong"'] \\
\mathrm{H}^{inf}_{k+1}(S\,,\,\mathbb{Z}) \arrow[r, "\partial"] & \mathrm{H}^{inf}_k(S_0\,,\,\mathbb{Z}) \arrow[r, equal] & \mathrm{H}^{inf}_k(S_0\,,\,\mathbb{Z})\,.
\end{tikzcd}
\tag{3.23}
$$

Also, the coboundary map in the top row coincides, by construction, with the coboundary map in (3.22) composed with the canonical isomorphism – in short, the top rows in the two diagrams, composed left to right, coincide. The three vertical arrows in the bottom squares are given by Poincaré duality, as in (2.13). The bottom left square commutes, and the second bottom square *anticommutes*, i.e., commutes when one negative sign is introduced. These are simple local calculations; the signs can be pinned down by using the explicit description in (2.13).

The assertion of the proposition follows: as was remarked before, the bottom row in (3.22) expresses the passage from a family of cycles to the limit at 0, the bottom row of (3.23) expresses the process of taking the boundary, the top rows in the two diagrams coincide, and the anticommutativity in the second diagram accounts for the factor of -1.

Until now, we have required only that the sets $|C_I| \cap (\{s\} \times M)$ be stratifiable, but not necessarily in a way which is compatible with a stratification of $|C_I|$. If we impose the stronger condition

$$
\begin{aligned}
&\text{for each } s \in I, \text{ there exists a Whitney stratification of } |C_I|, \\
&\text{such that } |C_I| \cap (\{s\} \times M) \text{ is a Whitney stratified subset of } |C_I|,
\end{aligned}
\tag{3.24}
$$

the specializations C_s, $s \in I$, are naturally homologous to the limit C_0, provided the limit exists:

3.25 Proposition. *Suppose the family C_I satisfies both (3.13) and (3.24). Then, for $s \in I$,*

$$C_s - C_0 \;=\; \partial C_{(0,s)}\,,$$

where $C_{(0,s)}$ denotes the restriction of C_I to $(0,s)\times M$.

Proof. This comes down to a second application of (3.17), in the much simpler situation when the family extends across the endpoint in the parametrizing interval. Since s is now the right endpoint, the fiber over s must be counted with the opposite sign.

Let us use the preceding discussion to describe, in geometric terms, the operation (2.10) – i.e., taking the direct image $f_*(C_0)$ of a subanalytic Lagrangian cycle C_0 under a proper real analytic map $f : X \to Y$. As in section 2, n and m are the dimensions of the real analytic manifolds X and Y, and $\mathcal{L}^+(X)$ denotes the group of integral, $\mathbb{R}^+$-invariant, subanalytic Lagrangian cycles in T^*X. For any particular $C_0 \in \mathcal{L}^+(X)$, there exists a subanalytic Whitney stratification $\mathcal{S}$, such that the support $|C_0|$ of C_0 is contained in the Lagrangian subvariety $T^*_{\mathcal{S}}X$ (recall the definition (2.1)); for the existence of such a stratification, see [KS, 8.3.22].

To describe the cycle $f_*(C_0)$ geometrically, we start out by deforming C_0, so that the deformed cycle is in general position with respect to df. This will be made precise in the lemma stated below. MacPherson has pointed out to us that the lemma follows readily from standard techniques; unfortunately there does not appear to be a statement in the literature that would imply it directly. In concrete situations the existence of such a deformation will be fairly obvious. In particular, in all our eventual applications, we shall work with explicitly given deformations, and therefore need not appeal to the lemma. We omit the proof since it would take us too far afield.

3.26 Lemma. *There exists a subanalytic family C_I of $n-$cycles, such that*

a) *C_0 is the limit, as s tends to zero from above, of the family C_I;*
b) *the map $df : X \times_Y T^*Y \to T^*X$ is transverse to $|C_s| \subset T^*X$, for every $s \in I$.*

We remark that the family C_I automatically satisfies (3.13) and (3.24), because we have returned to the subanalytic context.

Because of the transversality condition, for each $s \in I$, the geometric inverse image $df^{-1}(C_s)$ of C_s is well defined as an m-cycle in $X \times_Y T^*Y$: the set theoretic inverse image $df^{-1}(S)$ of any stratum S in the support of C_s is a submanifold of $X \times_Y T^*Y$, and df identifies the normal spaces of $S \subset T^*X$ with those of $df^{-1}(S) \subset X \times_Y T^*Y$; in particular, an orientation of S determines an orientation[1] of $df^{-1}(S)$. Thus $df^{-1}(C_s)$ is well defined at least as an m-chain. The fact that it is an m-cycle, and not just a chain, follows from our interpretation of the map $C_s \mapsto df^{-1}(C_s)$ a s Gysin map; cf. (2.15). Since $\tau : X \times_Y T^*Y \to T^*Y$ is proper and analytic, the geometric image $\tau_*(df^{-1}(C_s))$ of of the subanalytic cycle $df^{-1}(C_s)$ becomes a well defined m-cycle in T^*Y.

[1]More precisely, a co-orientation of S determines a co-orientation of $df^{-1}(S)$. However, co-oriented submanifolds of an oriented manifold are canonically oriented – see the discussion on signs in section 2: symbolically, (orientation of the submanifold)$\wedge$(co-orientation of the submanifold) = (ambient orientation).

3.27 Proposition. *Let C_I be a subanalytic family of n-cycles which satisfies the conditions a), b) in the statement of lemma 3.26. Then $\tau_*(df^{-1}(C_s))$, $s \in I$, constitutes a family of m-cycles in our sense, and the limit of this family, as s tends to 0 from above, coincides with the direct image of $C_0 = \lim_{s\to 0^+} C_s$ under the morphism $f_* : \mathcal{L}^+(X) \to \mathcal{L}^+(Y)$, i.e.,*

$$\lim_{s\to 0^+} \tau_*(df^{-1}(C_s)) = f_*(C_0)\,.$$

Proof. The operation f_* in (2.16) was defined in terms of the top row of the commutative diagram (2.9). This top row, combined with the inclusions $I \subset J \supset \{0\}$, results in the following diagram:

$$\begin{tikzcd}
I \times T^*X \arrow[d, "j"'] & I \times X \times_Y T^*Y \arrow[l, "1_I \times df"'] \arrow[r, "1_I \times \tau"] \arrow[d, "j"'] & I \times T^*Y \arrow[d, "j"'] \\
J \times T^*X & J \times X \times_Y T^*Y \arrow[l, "1_J \times df"'] \arrow[r, "1_J \times \tau"] & J \times T^*Y \\
T^*X \arrow[u, "i"] & X \times_Y T^*Y \arrow[l, "df"'] \arrow[r, "\tau"] \arrow[u, "i"] & T^*Y \arrow[u, "i"]\,.
\end{tikzcd} \tag{3.28}$$

We let C_J denote the image of $C_I \in \mathrm{H}^n_{|C|}(I \times T^*X, \mathbb{Z})$ in $\mathrm{H}^n_{|\bar{C}|}(J \times T^*X, \mathbb{Z})$ via the isomorphism (3.14); its support is contained in $|\overline{C_I}|$, the closure of $|C_I|$. By assumption, the cycle C_0 is the limit of the family C_I – in other words, the negative of the boundary of C_J. As discussed earlier, C_J can be regarded as an $(n+1)$-chain in $J \times T^*X$. To simplify the notation, we let $|D_I|$, $|\overline{D_I}|$, and $|D_0|$ denote the inverse images of $|C_I|$, $|\overline{C_I}|$, and $|C_0|$ under, respectively, $1_I \times df$, $1_J \times df$, and df. Analogously, $|E_I|$, $|\overline{E_I}|$, and $|E_0|$ shall denote the images of $|D_I|$, $|\overline{D_I}|$, and $|D_0|$ under the proper analytic maps $1_I \times \tau$, $1_J \times \tau$, and τ. We claim: $\dim |\overline{E_I}| \le m+1$. Indeed, the discussion following lemma 3.26 shows that each $\tau_*(df^{-1}(C_s))$, $s \in I$, is a subanalytic m-cycle; since the subanalytic set $|E_I|$ is contained in the union of the carriers of these cycles, both $|E_I|$ and its closure have dimension at most $m+1$. We shall argue that there is a natural commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n_{|C_I|}(I \times T^*X, \mathbb{Z}) \arrow[r, "(1_I \times df)^*"] & \mathrm{H}^n_{|D_I|}(I \times X \times_Y T^*Y, \mathbb{Z}) \arrow[r] & \mathrm{H}^m_{|E_I|}(I \times T^*Y, \mathbb{Z}) \\
\mathrm{H}^n_{|\overline{C_I}|}(J \times T^*X, \mathbb{Z}) \arrow[r, "(1_J \times df)^*"] \arrow[u, "j^*", "\simeq"'] \arrow[d, "i^*"'] & \mathrm{H}^n_{|\overline{D_I}|}(J \times X \times_Y T^*Y, \mathbb{Z}) \arrow[r] \arrow[u, "j^*", "\simeq"'] \arrow[d, "i^*"'] & \mathrm{H}^m_{|\overline{E_I}|}(J \times T^*Y, \mathbb{Z}) \arrow[u, "j^*", "\simeq"'] \arrow[d, "i^*"'] \\
\mathrm{H}^n_{|C_0|}(T^*X, \mathbb{Z}) \arrow[r, "df^*"] & \mathrm{H}^n_{|D_0|}(X \times_Y T^*Y, \mathbb{Z}) \arrow[r] & \mathrm{H}^m_{|E_0|}(T^*Y, \mathbb{Z})
\end{tikzcd}$$

in cohomology. To see this, note that the left two squares commute for purely formal reasons. The three horizontal arrows on the right denote push-forwards of cycles – in the case of the top and bottom arrows – or chains, in the case of the middle one. In each instance we reinterpret the domain and the target as groups of cycles or chains via Poincaré duality. The two maps j^* in the top right square signify restriction to open subsets, either restriction of chains or restriction of local cohomology classes, since the identification between chains and local cohomology is local. We conclude that the top right square commutes: restriction to a full inverse image of an open subset in the target commutes with push-forward of cycles. We argue similarly in the case of the right bottom square. Here the two arrows i^* can be interpreted as the negatives of the boundary maps; cf. (3.17). This square, then, is commutative because the operations of push-forward and taking the boundary commute.

We complete the argument as follows. Let E_I denote the image of C_I in the group $\mathrm{H}^m_{|E_I|}(I \times T^*Y, \mathbb{Z})$; since $|E_I|$ is subanalytic, of dimension at most $m+1$, E_I constitutes a family of m-cycles in our sense. The commutativity of the above diagram implies $\lim_{s \to 0^+} E_s = f_*(C_0)$. It still remains to be shown that $E_s = f_*(C_s) = \tau_*(df^{-1}(C_s))$ for $s \in I$. But this follows from the preceding equality, since specialization of the family at any $s \in I$ can be viewed as a particular case of taking a limit.

4. Open embeddings.

In this section we describe the effect on characteristic cycles of the operation of direct image under an open embedding. We shall work in the subanalytic category; for a general discussion of this category, including a list of primary references see, for example, [BM,GM,KS]. As a general convention, which differs from that of some authors, we shall call a map subanalytic if it is continuous and has a subanalytic graph. The properties of the subanalytic category that we use are all satisfied in the semi-algebraic category also, and are usually easier to verify in the latter. Thus all our statements apply equally in the semi-algebraic context.

We consider a real analytic manifold X and an embedding $j : U \hookrightarrow X$ of an open subanalytic subset U. Let f be a real valued, subanalytic, C^1-function, defined on a neighborhood W of ∂U, such that

$$\text{(4.1)} \qquad \begin{aligned} &\text{a) the boundary } \partial U \text{ is the zero set of } f; \\ &\text{b) } f \text{ is positive on } W \cap U; \end{aligned}$$

the differential df is then necessarily subanalytic also. Bierstone, Milman, and Pawlucki have shown that such a function f exists [BMP]. For the semi-algebraic analogue of this statement, see [Sh, Proposition I.4.5]. In the applications we have in mind, one usually works with a very concrete choice for f. The last ingredient of our result is a bounded complex of sheaves $\mathcal{F}$ on U, whose cohomology is constructible with respect to a subanalytic Whitney stratification of the pair $(X, \partial U)$. These hypotheses imply that $Rj_*\mathcal{F}$ and $Rj_!\mathcal{F}$ also have subanalytically constructible cohomology. Thus we can speak of the characteristic cycles $\mathrm{CC}(\mathcal{F})$, $\mathrm{CC}(Rj_*\mathcal{F})$, and $\mathrm{CC}(Rj_!\mathcal{F})$.

4.2 Theorem. *Under the hypotheses just stated,*

$$\mathrm{CC}(Rj_*\mathcal{F}) = \lim_{s \to 0+} (\mathrm{CC}(\mathcal{F}) + s\frac{df}{f}), \qquad \mathrm{CC}(Rj_!\mathcal{F}) = \lim_{s \to 0+} (\mathrm{CC}(\mathcal{F}) - s\frac{df}{f}).$$

In the statement of the theorem, we regard $\mathrm{CC}(\mathcal{F}) + s\frac{df}{f}$ and $\mathrm{CC}(\mathcal{F}) - s\frac{df}{f}$ as cycles on $\mathbb{R}^+ \times T^*U$. The theorem asserts, in effect, that these constitute families of cycles in T^*X, whose limits as $s \to 0^+$ exist and equal $\mathrm{CC}(Rj_*\mathcal{F})$, $\mathrm{CC}(Rj_!\mathcal{F})$.

The open mapping theorem 4.2 resembles theorem 3.2 of [G2], but there are two important differences. First of all, Ginsburg's statement and proof involve complex algebraic, regular holonomic $\mathcal{D}$-modules, whereas we must deal directly with real constructible sheaves. Secondly, Ginsburg's theorem applies to complements of principal divisors, not arbitrary Zariski open subsets, since he works with a complex algebraic defining function f for the complement. Indeed, his result for a particular algebraic defining function f follows form our theorem using the defining function $|f|^2$.

Proof of 4.2. Let us argue first that $\mathrm{CC}(\mathcal{F}) + s\frac{df}{f}$ and $\mathrm{CC}(\mathcal{F}) - s\frac{df}{f}$ are families of cycles. For simplicity we consider the former; the argument for the latter is completely analogous. We set $C = \mathrm{CC}(\mathcal{F})$. By assumption the support $|C|$ is a subanalytic subset of T^*X, which is contained in T^*U. We claim:

$$(4.3) \quad |C_I| \;=_{\mathrm{def}}\; \{\, |C| + s\frac{df}{f} \mid s \in \mathbb{R}^+\} \;\text{ is a subanalytic subset of }\; \mathbb{R} \times T^*X\,.$$

Indeed, $|C_I|$ consists of all pairs $(x,\xi) \in T^*X$ such that $|C|$ contains the cotangent vector $(x, f(x)\xi - sdf_x)$, the map $(x,\xi) \mapsto (x, f(x)\xi - sdf_x)$ is subanalytic, and inverse images of subanalytic sets under subanalytic maps are subanalytic.

Topologically, the pair $(\mathbb{R}^+ \times T^*U\,,\, |C_I|)$ is the product of $\mathbb{R}^+$ with $(T^*U\,,\,|C|)$. Thus $C \in \mathrm{H}^n_{|C|}(T^*U,\mathbb{Z})$, with $n = \dim X$, determines a class

$$(4.4) \qquad C_I \in \mathrm{H}^n_{|C_I|}(\mathbb{R}^+ \times T^*U, \mathbb{Z})$$

via the Künneth isomorphism $\mathrm{H}^n_{|C|}(T^*U,\mathbb{Z}) \cong \mathrm{H}^n_{|C_I|}(\mathbb{R}^+ \times T^*U,\mathbb{Z})$. The local cohomology class C_I now gives meaning to $\mathrm{CC}(\mathcal{F}) + s\frac{df}{f}$ as a family of cycles in T^*X. The conditions (3.10), (3.13) in the definition of family and limit follow from the fact that subanalytic sets can be Whitney stratified.

Let us collect the information we have about the limit C_0 of the family C_I. Since the function f is strictly positive at points in U, the restriction of C_0 to T^*U coincides with the the original cycle C. We set

$$(4.5) \qquad C_\partial \;=\; C_0 - C\,;$$

then the support $|C_\partial|$ of the n-chain C_∂ is contained in the inverse image of ∂U in T^*X. As in the case of any n-chain, $|C_\partial|$ has dimension exactly n, or else is empty. Rescaling the parameter s of the family C_I has the same effect as scaling each of its members C_s, so C_0 is $\mathbb{R}^+$-invariant. We deduce

$$(4.6a) \qquad |C_\partial| \;\text{ is } \mathbb{R}^+\text{-invariant.}$$

A simple calculation in local coordinates shows that $T^*_Z U + s\frac{df}{f}$, for each fixed $s \in \mathbb{R}^+$ and each locally closed submanifold $Z \subset U$, is Lagrangian in T^*U. It follows that the carriers $|C_s|$ of the members C_s of the family C_I are Lagrangian. By Whitney's condition A, the tangent space to $|C_\partial|$ at any regular point is contained in the limit of a sequence of tangent spaces to the $|C_s|$ at regular points, hence

$$|C_\partial| \text{ is Lagrangian,} \tag{4.6b}$$

unless it is empty. Appealing to [KS, 8.3.10], for example, we can draw the following conclusion from (4.6a,b):

(4.6c) $|C_\partial|$ is contained in the union of the conormal bundles of a locally finite family of smooth, locally closed, subanalytic subsets of ∂U.

We recall that the complex of sheaves $\mathcal{F}$ has cohomology constructible with respect to a subanalytic Whitney stratification of the pair $(X, \partial U)$. We refine this stratification on ∂U subanalytically to a Whitney stratification – of X – so that the latter constitutes also a refinement of the locally finite family in (4.6c). To summarize, we have constructed a subanalytic Whitney stratification $\mathcal{S}$ of $(X, \partial U)$ such that

$$T^*_{\mathcal{S}}X \text{ carries } \mathrm{CC}(Rj_*\mathcal{F}),\ C_0, \text{ and } C_\partial. \tag{4.7}$$

At this point, to show that $\mathrm{CC}(Rj_*\mathcal{F}) = C_0$, it suffices to equate the multiplicities of C_∂ and $\mathrm{CC}(Rj_*\mathcal{F})$ at generic points of $T^*_{\mathcal{S}\cap\partial U}X$.

We now construct a sequence of subanalytic stratifications in T^*X as follows. First we stratify $|C_\partial|$ as a subset of the closure of $|C_I|$, as in (3.13a). Next, we stratify $T^*_{\mathcal{S}}X$ so that all strata project into strata of $\mathcal{S}$ of X. Finally, we choose a common Whitney refinement of these two stratifications. We let $\mathcal{T}$ denote the restriction of this Whitney stratification to $T^*_{\mathcal{S}\cap\partial U}X$. Then $\mathcal{T}$ has the following two crucial properties: it is fine enough to carry the limit of our family of cycles, and each open stratum lies in the conormal bundle of a stratum in ∂U. To complete the proof of the theorem, we shall show that each top dimensional stratum of $\mathcal{T}$ occurs with the same multiplicity in C_∂ and $\mathrm{CC}(Rj_*\mathcal{F})$.

We fix a top dimensional stratum Λ_α of $\mathcal{T}$ and a point $(p,\xi) \in \Lambda_\alpha$; here p is a point in the stratum $S_\alpha \subset \partial U$ of $\mathcal{S}$ over which Λ_α lies, and $\xi \in T^*_pX$. We choose a locally defined Morse function ϕ on an open neighborhood of p in X, as in (2.4). In addition to (2.4) we may and shall assume that ϕ is an algebraic function in terms of a local real analytic coordinate system Σ around p. In the following, when we talk of a ball centered at p, we shall mean a ball defined by the Euclidean metric relative to the local coordinate system Σ. Arguing as in section 2, if B is a sufficiently small open ball with center p, and with $\epsilon > 0$ sufficiently small in relation to B, we can infer (2.6) and (2.7) – the latter with $Rj_*\mathcal{F}$ in place of $\mathcal{F}$. In particular, the multiplicity m_α of Λ_α in $\mathrm{CC}(Rj_*\mathcal{F})$ is

$$m_\alpha \;=\; \chi(\,\mathrm{H}^*(B\cap\phi^{-1}(-\epsilon,\epsilon), B\cap\phi^{-1}(-\epsilon,0)\,;\,Rj_*\mathcal{F}\ )\,. \tag{4.8}$$

It will be convenient to rephrase (4.8) slightly. Because of (2.6), ϕ has no critical points (in the stratified sense) on $\bar{B}\cap\phi^{-1}[-\epsilon,\epsilon]$, except at p. Thus, by [GM, I.3.2],

$$\phi \text{ defines a fibration of } \bar{B}\cap\phi^{-1}[-\epsilon,\epsilon], \text{ away from the fibre over } 0\,; \tag{4.9}$$

here "fibration" is to be taken in the stratified sense. Hence, by excision and local triviality,

$$m_\alpha \;=\; \chi(\,\mathrm{H}^*(B\cap\phi^{-1}(-\epsilon,\epsilon'), B\cap\phi^{-1}(-\epsilon,-\epsilon')\,;\,Rj_*\mathcal{F}\;)\,, \tag{4.10}$$

for any ϵ' between 0 and ϵ, and then

$$m_\alpha \;=\; \chi(\,\mathrm{H}^*(\bar B\cap U\cap\phi^{-1}(-\epsilon,\epsilon'), \bar B\cap U\cap\phi^{-1}(-\epsilon,-\epsilon')\,;\,\mathcal{F}\;)\,, \tag{4.11}$$

first by definition of the operation of direct image, and secondly, because ∂B is transverse to the stratification $\mathcal{S}$.

We now compute the right hand side of (4.11) by Morse theory on $B\cap U$, with $\phi - s\log f$ as Morse function, with $s>0$ small in relation to B and ϵ'. We shrink B, if necessary, so that

$$f<1 \text{ on the closure of } B, \tag{4.12}$$

and define

$$\bar B(\alpha,\beta) = \bar B\cap\phi^{-1}(\alpha,\beta)\,. \tag{4.13}$$

By construction the strata of $\mathcal{S}$ are transverse to ∂B, so $\mathcal{S}$ induces a Whitney $\mathcal{C}$-stratification of $\bar B$. In view of (4.9), this stratification is transverse to the level sets $\phi^{-1} = \pm\epsilon$, and therefore in turn induces a stratification $\mathcal{S}_\epsilon$ of the closure of $B(-\epsilon,\epsilon)$.

4.14 Proposition. *Let $N\subset B(-\epsilon,\epsilon)$ be a neighborhood of p, δ a positive constant, and $\epsilon'<\epsilon$ a positive constant. Then there exists an $\eta>0$, such that for all $0<s<\eta$ the following two statements hold:*
a) for any stratum S of $\mathcal{S}_\epsilon$ not on the boundary of $B(-\epsilon,\epsilon)$, the critical points of $\phi - s\log f$ on $S\cap U$ are non-degenerate, lie in N, and have critical values in the interval $(-\delta,\delta)$; and
b) for any stratum S of $\mathcal{S}_\epsilon$ lying on the boundary of $B(-\epsilon,\epsilon)$, the critical values of $\phi - s\log f$ on $S\cap U$ lie outside of the interval $[-\epsilon',\epsilon']$.

This, in the technical sense, is the crux of our proof of theorem (4.2). We shall complete the proof of the theorem assuming the proposition, and shall verify the proposition in the next section. To simplify the notation we set $\phi_s = \phi - s\log f$. As one consequence of (4.14),

4.15 Lemma. $\mathrm{H}^*(\bar B(-\epsilon,\epsilon)\cap U\cap\{\phi_s<\epsilon'\}, \bar B(-\epsilon,\epsilon)\cap U\cap\{\phi_s<-\epsilon'\};\mathcal{F})$ *does not depend on s, for $\delta<\epsilon'<\epsilon$ and $0<s<\eta$, and is canonically isomorphic to* $\mathrm{H}^*(\bar B\cap U\cap\phi^{-1}(-\epsilon,\epsilon'), \bar B\cap U\cap\phi^{-1}(-\epsilon,-\epsilon')\,;\,\mathcal{F}\;)$.

Proof. By the five lemma it suffices to argue separately for the cohomology groups of $\mathcal{F}$ on the sets $\bar B(-\epsilon,\epsilon)\cap U\cap\{\phi_s<\epsilon'\}$ and $\bar B(-\epsilon,\epsilon)\cap U\cap\{\phi_s<-\epsilon'\}$. Each of these families of sets are increasing as s tends to 0 from above. Moreover, according to (4.14), the hypersurfaces $\{\phi_s=\pm\epsilon'\}$ are transverse to the stratification $\mathcal{S}$ on $\bar B(-\epsilon,\epsilon)\cap U$. At this point we appeal to the non-characteristic deformation lemma of Kashiwara-Schapira [KS, Proposition 2.7.2], with $\overline{B(-\epsilon,\epsilon)}\cap U$ playing the role of the total space. This gives us the conclusion of the lemma when we replace $\bar{B}(-\epsilon,\epsilon)$ by $\overline{B(-\epsilon,\epsilon)}$. But

$$\begin{aligned}\mathrm{H}^*(\overline{B(-\epsilon,\epsilon)}\cap U\cap\{\phi_s<\epsilon'\},\overline{B(-\epsilon,\epsilon)}\cap U\cap\{\phi_s<-\epsilon'\};\mathcal{F}) \quad &\cong \\ \mathrm{H}^*(\bar{B}(-\epsilon,\epsilon)\cap U\cap\{\phi_s<\epsilon'\},\bar{B}(-\epsilon,\epsilon)\cap U\cap\{\phi_s<-\epsilon'\};\mathcal{F})\,,&\end{aligned} \tag{4.16}$$

because $\{\phi=\pm\epsilon\}$ is transverse to the stratification which $\mathcal{S}$ induces on $\bar{B}$.

Before completing the formal proof of theorem (4.2), let us pause briefly and summarize the main idea. We need to equate the multiplicity a top dimensional stratum lying over the boundary of U in $\mathrm{CC}(Rj_*\mathcal{F})$ with its multiplicity in the limit of $\mathrm{CC}(\mathcal{F})+s\,d\log f$. Formula (4.11) expresses the former multiplicity as an Euler characteristic. We shall reinterpret this Euler characteristic as the intersection number, locally near the point in question, of $\mathrm{CC}(\mathcal{F})$ and the cycle $d\phi$. Analogously, the multiplicity of the stratum in the limit cycle is (locally) the intersection number of $\mathrm{CC}(\mathcal{F})+s\,d\log f$ with $d\phi$ – equivalently, the intersection number of $\mathrm{CC}(\mathcal{F})$ with $d\phi_s$. The latter intersection number can again be expressed as an Euler characteristic. Lemma (4.15) equates the two Euler characteristics, and therefore the two multiplicities.

The interpretation of the Euler characteristic (4.11) as an intersection number is a particular instance of Kashiwara's index formula [K,KS]. Let us recall the index formula in the form most convenient for our purposes – indeed, in this form it is a fairly direct consequence of the proper pushforward formula and our formalism of families of cycles. We consider a real analytic manifold M, a real analytic function $\psi: M\to\mathbb{R}$, and a bounded complex of sheaves $\mathcal{G}$ with subanalytically constructible cohomology. Then, if $\mathcal{G}$ has compact support (when viewed as an object in the derived category),

$$\chi(\mathcal{G})=(pt)_*d\psi^*(\mathrm{CC}(\mathcal{G}))\,; \tag{4.17}$$

here $pt: M\to\{pt\}$ denotes the collapsing map. In effect, the right hand side of (4.17) is the intersection number between $\mathrm{CC}(\mathcal{G})$ and the Lagrangian cycle $\{d\psi\}$ in T^*M – which makes sense as a finite quantity because of the proper support hypothesis. By scaling, this cycle can be deformed to the 0-section in T^*M, so the right hand side of (4.17) is equal also to the intersection number between $\mathrm{CC}(\mathcal{G})$ and the 0-section. By the proper pushforward theorem, the latter intersection number describes the "characteristic cycle" of $(pt)_*\mathcal{G}$, in other words, the Euler characteristic of $\mathcal{G}$.

Still in the setting of (4.17), let us suppose now that ψ is a Morse function in the stratified sense with respect to $\mathcal{G}$. We now drop the hypothesis of compact support on $\mathcal{G}$, and require instead that

$$\psi \text{ is proper on the support of } \mathcal{G}\,. \tag{4.18}$$

Then ψ can have only finitely many critical points, hence only finitely many critical values. Let α, β be two regular values, with $\alpha<\beta$. Since ψ is Morse, the cycle $d\psi$ meets $\mathrm{CC}(\mathcal{G})$ only over the critical points, and the intersections are transverse. Thus $d\psi^*(\mathrm{CC}(\mathcal{G}|_{\psi^{-1}(\alpha,\beta)}))$ is well defined as a (finite) zero cycle. In effect, the integer $(pt)_*d\psi^*(\mathrm{CC}(\mathcal{G}|_{\psi^{-1}(\alpha,\beta)}))$ is the intersection number between the cycles $d\psi$ and $\mathrm{CC}(\mathcal{G})$ over $\{\alpha<\psi<\beta\}$. The following statement can be extracted from [K] or [KS].

4.19 Lemma. *Under the hypotheses just stated,* $\chi(\mathrm{H}^*(\{\psi < \beta\}, \{\psi < \alpha\}; \mathcal{G})) = (pt)_* d\psi^*(\mathrm{CC}(\mathcal{G}|_{\psi^{-1}(\alpha,\beta)}))$.

Proof. We use the open inclusions

$$\psi^{-1}(\alpha, \beta) \xrightarrow{j_\beta} \psi^{-1}(\alpha, \beta] \xrightarrow{j_\alpha} \psi^{-1}[\alpha, \beta]$$

to truncate the sheaf $\mathcal{G}$. The truncated sheaf

$$\mathcal{G}_{\alpha,\beta} \;=\; (j_\alpha)_!(Rj_\beta)_*(\mathcal{G}|_{\psi^{-1}(\alpha,\beta)}) \tag{4.20}$$

has compact support because of (4.18). Essentially by definition, the left hand side of the identity to be proved coincides with the Euler characteristic of $\mathcal{G}_{\alpha,\beta}$. The lemma will follow from (4.17), once we have shown that the cycles $d\psi$ and $\mathrm{CC}(\mathcal{G}_{\alpha,\beta})$ intersect only over the interior of $\psi^{-1}[\alpha, \beta]$, where $\mathrm{CC}(\mathcal{G}_{\alpha,\beta})$ agrees with $\mathrm{CC}(\mathcal{G})$. To see this, we fix a point p in $\psi^{-1}(\alpha)$ or $\psi^{-1}(\beta)$. Locally near p, M is a product in the stratified sense, $M \simeq \mathbb{R} \times N$, with $\psi = t$ (= coordinate function on $\mathbb{R}$). Correspondingly and again locally, $\mathrm{CC}(\mathcal{G}_{\alpha,\beta})$ splits into a product of $\mathrm{CC}(\mathcal{G}|_N)$ with the characteristic variety of $\mathbb{C}_{\alpha,\beta}$, the sheaf on $\mathbb{R}$ defined by the truncation process (4.20) applied to the constant sheaf. This, finally, reduces the problem to the case of $M = \mathbb{R}$ and $\psi = t$. The crux of the matter is that the characteristic variety of $\mathbb{C}_{\alpha,\beta}$ consists of the zero section over (α, β) and the negative half lines spanned by dt over α and β, and thus does not meet the cycle dt.

We apply the lemma with $M = U$, with $\psi = \phi_s$ for s small, and $\mathcal{G} \;=\; Ri_*i^*\mathcal{F}$, where $i : \overline{B(-\epsilon, \epsilon)} \cap U \hookrightarrow U$. Note that ϕ_s is indeed proper on the support of $\mathcal{G}$ – ϕ_s tends to infinity on the boundary of U. We conclude

$$\begin{aligned}&(pt)_* d\phi_s^*(\mathrm{CC}(\mathcal{G}|_{\phi_s^{-1}(-\epsilon',\epsilon')})) = \\ &\qquad \chi(\mathrm{H}^*(\overline{B(-\epsilon, \epsilon)} \cap U \cap \{\phi_s < \epsilon'\}, \overline{B(-\epsilon, \epsilon)} \cap U \cap \{\phi_s < -\epsilon'\}; \mathcal{F}))\,,\end{aligned}$$

if ϵ' and s are chosen as in (4.14) and (4.15). In view of (4.11) and (4.15,16), this gives us

$$m_\alpha = (pt)_* d\phi_s^*(\mathrm{CC}(\mathcal{G}|_{\phi_s^{-1}(-\epsilon',\epsilon')}))\,.$$

In effect, the right hand side of this equality is the intersection number between the cycles $d\phi_s$ and $\mathrm{CC}(\mathcal{G}|_{\phi_s^{-1}(-\epsilon',\epsilon')})$; by (4.14), all intersections between these two cycles – including all potential intersections over $\partial B(-\epsilon, \epsilon)$, where $\mathcal{F}$ was truncated to produce $\mathcal{G}$ – occur over points in N. This allows us to replace the sheaf $\mathcal{G}|_{\phi_s^{-1}(-\epsilon',\epsilon')}$ by $\mathcal{F}|_N$, provided we correspondingly restrict ϕ_s to $N \cap U$:

$$m_\alpha = (pt)_*(d\phi_s|_{N\cap U})^*(\mathrm{CC}(\mathcal{F}|_N))\,.$$

Intersecting the cycles $d\phi_s$ and $\mathrm{CC}(\mathcal{F})$ is equivalent to intersecting $d\phi$ and $\mathrm{CC}(\mathcal{F}) + s\, d\mathrm{log} f$, hence finally

$$m_\alpha = (pt)_*(d\phi|_N)^*(\mathrm{CC}(\mathcal{F}) + s\, d\mathrm{log} f)\,, \tag{4.21}$$

for all sufficiently small $s > 0$.

Because of the original choice of the Morse function ϕ, the cycle $d\phi$ is a normal slice to the stratum Λ_α at (p,ξ). According to (4.21), then, this normal slice meets the cycle $\mathrm{CC}(\mathcal{F}) + s\, d\log f$ locally near (p,ξ) with multiplicity m_α – in other words, with the same multiplicity with which Λ_α appears in $\mathrm{CC}(Rj_*\mathcal{F})$. The same argument that led to (4.21) allows us to interpret this latter multiplicity with the intersection number, locally near (p,ξ), between $d\phi$ and $\mathrm{CC}(Rj_*\mathcal{F})$:

$$m_\alpha = (pt)_*(d\phi|_N)^*(\mathrm{CC}(Rj_*\mathcal{F}))\,. \tag{4.22}$$

Indeed, it is this expression for m_α which shows that our definition of the characteristic cycle agrees with that in [KS]. Geometrically, the identity obtained by combining (4.21) and (4.22),

$$(pt)_*(d\phi|_N)^*(\mathrm{CC}(Rj_*\mathcal{F})) = (pt)_*(d\phi|_N)^*(\mathrm{CC}(\mathcal{F}) + s\, d\log f)\,, \tag{4.23}$$

asserts that the normal slice $d\phi$ to the top dimensional stratum Λ_α intersects, locally near (p,ξ), the cycles $\mathrm{CC}(Rj_*\mathcal{F})$ and $\mathrm{CC}(\mathcal{F}) + s\, d\log f$ with the same multiplicity. This is true for all sufficiently small $s > 0$, for any top dimensional boundary stratum Λ_α and any generic point $(p,\xi) \in \Lambda_\alpha$; The particular sign convention for transverse intersections of cycles does not matter, since it affects both sides of (4.23) in the same way. We conclude that $\mathrm{CC}(Rj_*\mathcal{F})$ is the limit of $\mathrm{CC}(\mathcal{F}) + s\, d\log f$ as s tends to 0, as asserted by theorem 4.2.

5. Proof of Proposition 4.14.

Let us recall the setting of the proposition. Shrinking the ambient manifold X if necessary, we may assume that the boundary ∂U of the subanalytic open subset $U \subset X$ is the zero set of a globally defined C^1-subanalytic function $f : X \to \mathbb{R}$; moreover, $0 < f < 1$ on U. We work with two subanalytic Whitney stratifications: a stratification $\mathcal{S}$ of the pair $(X, \partial U)$, and a stratification $\mathcal{T}$ of $T^*_{\mathcal{S}\cap\partial U}X$. The two stratifications satisfy three crucial properties. First, $\mathcal{S}$ is sufficiently fine on ∂U: symbolically,

$$\lim_{s\to 0^+}(T^*_{\mathcal{S}}U + s\frac{df}{f}) \ \subset\ T^*_{\mathcal{S}}X\,. \tag{5.1a}$$

To make this precise, we recall that the closure, in $\mathbb{R}\times T^*X$, of the subanalytic set $\{\, T^*_{\mathcal{S}}U + s\, d\log f \mid s > 0\}$ is also subanalytic. This closure intersects the set $\{s = 0\}$ in a subset of $T^*_{\mathcal{S}}X$ – that is the assertion (5.1a). Secondly,

(5.1b) $\mathcal{T}$ is the restriction to $T^*_{\mathcal{S}\cap\partial U}X$ of a Whitney stratification of the closure of $\{\, T^*_{\mathcal{S}}U + s\, d\log f \mid s > 0\}$.

Lastly, $\mathcal{S}$ and $\mathcal{T}$ are compatible, i.e.,

(5.1c) each stratum of $\mathcal{T}$ lies over a stratum of $\mathcal{S}$.

These three assertions summarize the discussion around (4.7).

We work near a generic point (p,ξ) in $T^*_{\mathcal{S}\cap\partial U}X$. In other words, (p,ξ) lies on an n-dimensional stratum Λ_α of $\mathcal{T}$, where $n = \dim X$ as before. This stratum Λ_α is contained in the conormal bundle of S_α, the stratum $S_\alpha \subset \partial U$ on which p lies. We let ϕ be a real algebraic[2] Morse function as in (2.4). By assumption, $\{d\phi\}$ meets $T^*_{\mathcal{S}}X$ transversly at (p,ξ), so every sufficiently small ball B centered at p satisfies (2.6). We fix a particular ball B which is small in this sense, and shrink it further if necessary, so that

$$d\phi(\bar{B}) \cap T^*_{\mathcal{S}}X \;=\; \{(p,\xi)\}\,. \tag{5.2}$$

The constant $\epsilon > 0$ was chosen to ensure that

$$\phi \text{ has no critical points on } \bar{B}\cap\phi^{-1}[-\epsilon,\epsilon] \text{ except at } p\,; \tag{5.3}$$

here, as usual, "critical point" is to be taken in the stratified sense. We also fix the neighborhood N of p, $N \subset B(-\epsilon,\epsilon)$, and the constants $\delta > 0$ and ϵ', $0 < \epsilon' < \epsilon$. The two parts of the proposition will be proved separately. We begin with an auxiliary statement.

5.4 Lemma. *For $\eta > 0$ sufficiently small, the two sets $\mathbb{R}\times d\phi(\bar{B})$ and $\{(s,\zeta) \mid 0 < s < \eta\,,\ \zeta \in T^*_{\mathcal{S}}U + s\,d\log f\,\}$ intersect transversly. Moreover, the intersection consists of a finite number of smooth subanalytic curves $\gamma_i(s)$, $1 \le i \le N$, parametrized by s, $0 < s < \eta$, with $\lim_{s\to 0^+}\gamma_i(s) = (p,\xi)$ for all i.*

Proof. Because of (5.1a) and (5.2), the intersection

$$(\mathbb{R}\times d\phi(\bar{B})) \cap \overline{\{(s,\zeta) \mid s > 0,\ \ \zeta \in T^*_{\mathcal{S}}U + s\,d\log f\,\}}$$

meets $\{s = 0\}$ only at (p,ξ), and there the intersection is transverse. By compactness of $d\phi(\bar{B})$ and the Whitney property, for every sufficiently small $\eta > 0$, the intersection is therefore transverse over $\{0 \le s \le \eta\}$ – of dimension one, as can be seen by counting. When this compact, one dimensional set is projected to the s-axis, there can be only finitely many critical points. Thus, restricting η further if necessary,

$$(\mathbb{R}\times d\phi(B)) \cap \{(s,\zeta) \mid 0 < s < \eta\,,\ \zeta \in T^*_{\mathcal{S}}U + s\,d\log f\,\}$$

consists of finitely many subanalytic curves parametrized by s. As s decreases to 0, some of these curves will tend to the point (p,ξ). Because of (5.2), all the other curves must reach the boundary at some strictly positive value of the parameter s. Shrinking η once more, we can eliminate all of these latter curves.

The critical points of $\phi_s = \phi - s\log f$ on a stratum S correspond precisely to the points of intersection of $\{d\phi\}$ and $\{\,T^*_{\mathcal{S}}U + s\,d\log f\,\}$, and non-degeneracy of the critical points corresponds to the transversality of the intersection. The curves mentioned in the lemma, projected down to B, have p as limit as $s \to 0^+$, hence

[2] relative to a local coordinate system at p

can be made to lie entirely in the neighborhood N of p, provided that η is made sufficiently small. This implies part a) of proposition (4.14) once we know:

$$\phi_s \to 0 \quad \text{as} \quad s \to 0^+ \text{ along } \gamma_i \tag{5.5}$$

for $1 \leq i \leq N$. In this statement, we regard ϕ_s as a function on $\mathbb{R} \times T^*X$ in the natural way. Subanalytic curves can be parametrized (locally) by a real analytic parameter [L] – [BM] is a convenient reference. Let us then parametrize γ_i real analytically by a parameter t, $0 \leq t < \tau$, so that $t = 0$ corresponds to the point $(0, p, \xi) \in \mathbb{R} \times T^*X$. At this point, s and ϕ become real analytic functions of t, and f is at least C^1; moreover, all three of these functions tend to 0 as t goes to 0. The criticality of ϕ_s along γ_i translates into the differential equation

$$\frac{d\phi}{dt} \;=\; s\,\frac{d}{dt}\log f\,. \tag{5.6}$$

Thus $\log f$ has a real meromorphic t-derivative at $t = 0$,

$$\frac{d}{dt}\log f \;=\; a\,t^k + \dots\,, \quad \text{with} \;\; a \neq 0\,,\; k \in \mathbb{Z}\,.$$

On the other hand, $s = bt^l + \dots$, with $b > 0$ and $l \geq 1$. Since ϕ is regular at $t = 0$, (5.6) forces $l + k \geq 0$. We conclude that $s \log f$ tends to 0 as $t \to 0$. Hence so does ϕ_s, as asserted in (5.5).

We turn to the proof of part b). We are free, of course, to make a separate choice of η. For $s > 0$ sufficiently small, then, we must show that

$$\phi_s \;:\; U \cap \partial B(-\epsilon, \epsilon) \cap \phi_s^{-1}[-\epsilon', \epsilon'] \;\to\; [-\epsilon', \epsilon'] \quad \text{has no critical points.} \tag{5.7}$$

Since $0 < f < 1$, the set $\partial B(-\epsilon, \epsilon) \cap \phi_s^{-1}[-\epsilon', \epsilon']$ does not meet $\{\phi = \epsilon\}$, hence

$$\partial B(-\epsilon, \epsilon) \cap \phi_s^{-1}[-\epsilon', \epsilon'] \quad \subset \quad (\bar{B} \cap \{\phi = -\epsilon\}) \;\cup\; (\partial B \cap \{-\epsilon < \phi < \epsilon\})\,. \tag{5.8}$$

First, let us exclude all potential critical points on $U \cap \bar{B} \cap \{\phi = -\epsilon\}$. Any such critical point would have to correspond to a critical value $-\epsilon + \tilde{\epsilon}$, with $\tilde{\epsilon} \geq \epsilon - \epsilon' > 0$. On the set in question, $\phi_s = -\epsilon - s \log f$. In effect, we must exclude critical points of $\log f$ with critical values $\frac{-\tilde{\epsilon}}{s}$, with small $s > 0$. Since $\tilde{\epsilon}$ is bounded away from 0, this amounts to excluding critical points of f on $\bar{B} \cap \{\phi = -\epsilon\}$ with small positive critical values. Here smallness of critical values is measured by the constant η which bounds s from above. The set V of critical values of f on $\bar{B} \cap \{\phi = -\epsilon\}$ is the image of the subanalytic set $\{df = 0\}$ under the proper map f, hence is subanalytic. If V is discrete we are done. Otherwise, by the curve selection lemma (see [BM] for example), there exists a curve in $U \cap \{df = 0\}$ along which f tends to 0. Impossible: f is strictly positive on U!

The remaining task is to exclude critical points on $\partial B \cap \{-\epsilon < \phi < \epsilon\}$. Let us argue that the containment (5.1a) implies

$$\lim_{s \to 0^+} (T^*_{\mathcal{S} \cap \partial B} U + s\frac{df}{f}) \;\subset\; T^*_{\mathcal{S} \cap \partial B} X\,. \tag{5.9}$$

By construction, each stratum $S \in \mathcal{S}$ meets ∂B transversly. Thus, at any point $x \in S$, the conormal space of $S \cap \partial B$ is the direct sum of the conormal space of S and L_x, the conormal line to ∂B at x. We consider a convergent sequence

$$(s_n, x_n, l_n + \xi_n + s_n \frac{df}{f}(x_n)) \in T^*_{\mathcal{S}\cap\partial B} U + s\frac{df}{f}\,, \tag{5.10}$$

with $s_n \to 0$, $x_n \in S$, $l_n \in L_{x_n}$, and ξ_n conormal to S at x_n. We may as well assume that the l_n have a limit at least in the projective sense. This limit is conormal to ∂B, and therefore linearly disjoint from the conormal space of the stratum on which $x = \lim x_n$ lies. Thus l_n has a limit l in the usual sense, so

$$(s_n, x_n, \xi_n + s_n \frac{df}{f}(x_n))$$

converges, necessarily to a point $(x, \zeta) \in T^*_{\mathcal{S}} X$ because of (5.1a). But then $(x, l+\zeta) \in T^*_{\mathcal{S}\cap\partial B} X$, proving (5.9).

Our choice of ϕ and ϵ implies, in particular, that $d\phi$ does not meet $T^*_{\mathcal{S}\cap\partial B} X$ over the compact set $\partial B \cap \{-\epsilon \le \phi \le \epsilon\}$. Thus (5.9) makes it impossible for $d\phi$ to intersect $T^*_{\mathcal{S}\cap\partial B} U + s\frac{df}{f}$ over $\partial B \cap \{-\epsilon \le \phi \le \epsilon\}$ for arbitrarily small values of s. Equivalently, ϕ_s cannot have critical points on $\partial B \cap \{-\epsilon < \phi < \epsilon\}$ for small s. This completes the verification of (5.7), and hence of proposition 4.14.

6. Calculus of characteristic cycles.

In this section we describe how certain results of Kashiwara and Kashiwara-Schapira, in conjunction with our theorem 4.2, provide a complete set of rules for computing – in principle – the characteristic cycle of any constructible sheaf. The analogy we have in mind is ordinary homology: typically one calculates the homology of a space not by going back to the technical definition, but rather using axioms, such as exactness, normalization, excision. Characteristic cycles, it turns out, are determined by their behavior under the operation of direct image of sheaves by open embeddings of manifolds, behavior with respect to distinguished triangles, a normalizing axiom, and the fact that the characteristic cycle is a local invariant. The functorial operations of direct and inverse image, with or without proper support, on objects in the derived category have geometric counterparts on the level of characteristic cycles. For proper direct images and non-characteristic pullbacks this is done in [KS], as we explained earlier. Theorem 4.2 makes it possible to describe these functorial operations on characteristic cycles in general. Everything we do is predicated on knowing that the characteristic cycle construction is well-defined. One may wonder if the rules and axiomatic properties can be used to give an a priori definition of the characteristic cycle of a constructible sheaf; we do not know the answer to this question.

To keep the discussion as concrete as possible, we restrict ourselves to the semi-algebraic setting, and we also assume that the ambient manifold is oriented. The advantage of the semi-algebraic context is the fact that the operations of direct image and pullback preserve constructibility. When this is not the case, for subanalytically constructible sheaves, constructibility has to be made part of the hypotheses in various statements. The orientability assumption on X is of little consequence, and can be removed using known techniques – see for example [KS].

Unless stated otherwise, in this section "manifold" shall mean a real algebraic manifold, and constructibility shall be taken in the semi-algebraic sense. We shall work inside the bounded derived category $\mathrm{D}^b(X)$ of constructible sheaves on an oriented manifold X. Let us list the properties which determine the characteristic cycle map $\mathrm{CC} : \mathrm{D}^b(X) \to \mathcal{L}^+(X)$ – recall that $\mathcal{L}^+(X)$ denotes the group of $\mathbb{R}^+$-invariant, integral Lagrangian cycles, with infinite support, on T^*X. First of all, *normalization*:

$$\mathrm{CC}(\mathbb{C}_X) \ = \ [X] \ = \ \text{zero section in} \ \ T^*X\,, \tag{6.1a}$$

oriented by the fixed orientation of X. Next there is *additivity* in distinguished triangles:

$$\mathrm{CC} \ \text{descends to a map} \ \mathrm{CC} : \mathrm{K}(\mathrm{D}^b(X)) \ \longrightarrow \ \mathcal{L}^+(X)\,. \tag{6.1b}$$

The *local nature* of the characteristic cycle is embodied in the commutativity of the square

$$\begin{array}{ccc} \mathrm{D}^b(X) & \xrightarrow{\ \mathrm{CC}\ } & \mathcal{L}^+(X) \\ \big\downarrow & & \big\downarrow \\ \mathrm{D}^b(U) & \xrightarrow{\ \mathrm{CC}\ } & \mathcal{L}^+(U) \end{array} \tag{6.1c}$$

for any semi-algebraic open $U \subset X$ – note that restriction to open subsets makes sense for cycles with infinite support. The fourth characterizing property, *extension from open subsets* as stated in theorem 4.2, asserts the commutativity of the square

$$\begin{array}{ccc} \mathrm{D}^b(U) & \xrightarrow{\ \mathrm{CC}\ } & \mathcal{L}^+(U) \\ {\scriptstyle Rj_*}\big\downarrow & & \big\downarrow{\scriptstyle j_*} \\ \mathrm{D}^b(X) & \xrightarrow{\ \mathrm{CC}\ } & \mathcal{L}^+(X)\ ; \end{array} \tag{6.1d}$$

here $j_* : \mathcal{L}^+(U) \to \mathcal{L}^+(X)$ is the limit operation

$$j_*(C) = \lim_{s\to 0^+} (C + s\, d\log f)\,, \quad C \in \mathcal{L}^+(U)\,,$$

which was described in detail in section 4.

To see that the properties (6.1a-d) do determine the characteristic cycle construction, we recall that the K-group $\mathrm{K}(\mathrm{D}^b(X))$ is generated by objects of the form $Ri_*\mathcal{E}$, where $i : M \hookrightarrow X$ is the inclusion of a locally closed, semi-algebraic submanifold, and $\mathcal{E}$ a local system on M. By (6.1b), then, it suffices to consider objects of this particular type. The inclusion $i : M \hookrightarrow X$ factors as the composition of closed embedding $M \hookrightarrow X - \partial M$ and the open embedding $X - \partial M \hookrightarrow X$. The effect of the latter is described by (6.1d), so we may as well suppose that M is closed in X. Because of (6.1c), we can localize the situation, and make M simply connected. This reduces the problem to the case of the constant sheaf $\mathcal{E} = \mathbb{C}_M$ on the closed submanifold M. Let j denote the open embedding $X - M \hookrightarrow X$. In the distinguished triangle

$$i_* i^! \mathbb{C}_X \;\to\; \mathbb{C}_X \;\to\; Rj_* j^* \mathbb{C}_X \,, \tag{6.2}$$

$i^! \mathbb{C}_X = \mathbb{C}_M[\mathrm{codim}_M X]$, so $\mathrm{CC}(i_* \mathbb{C}_M)$ is the difference of two known – by (6.1a) and (6.1d) – characteristic cycles. Concretely, $\mathrm{CC}(i_* \mathbb{C}_M)$ is the cycle $[T^*_M X]$ with the orientation prescribed by (2.3).

The same ideas can be used to describe the effect on characteristic cycles of the four operations of direct and inverse image of sheaves, with or without proper support. The cases of direct image under a proper map and of non-characteristic pullback are done in [KS]; we have already discussed these cases in sections 2 and 3. The effect of Verdier duality on characteristic cycles follows fairly directly from the definition, either Kashiwara's formal definition, or its Morse theoretic reinterpretation, as explained in chapter 2. Specifically,

$$\mathrm{CC}(\mathbb{D}\mathcal{F}) \;=\; a_* \, \mathrm{CC}(\mathcal{F}) \,, \tag{6.3}$$

with $a : T^*X \to T^*X$ denoting the antipodal map. In particular, formulas involving proper direct or inverse images are concretely and immediately equivalent to their counterparts for ordinary direct and inverse images.

To describe $\mathrm{CC}(F^*\mathcal{G})$ for an arbitrary semi-algebraic map $F : X \to Y$ and $\mathcal{G} \in \mathrm{D}^b(Y)$, we use the usual device of factoring F into the composition of the closed embedding $i : X \hookrightarrow X \times Y$, via the graph of F, and the projection $p : X \times Y \to Y$. The formula for $\mathrm{CC}(p^*\mathcal{G})$ is especially simple – it is the product of the zero section in T^*X with $\mathrm{CC}(\mathcal{G})$, properly oriented – since this is a product situation. The case of the closed embedding i can be reduced to that of the open embedding of the complement of the graph of F in $X \times Y$, just as we did in the case of the distinguished triangle (6.2). This process results in an explicit formula, in terms of a limit of cycles. The point is that there are natural choices of a defining function for the graph of F, for example the square of the distance between y and $f(x)$, relative to a real analytic, or even real algebraic, metric on Y.

The description of $\mathrm{CC}(RF_*\mathcal{F})$ is more delicate. We embed X as an open subset in a compact, real algebraic manifold $\bar{X}$, and we factor F into a product of three mappings: the closed embedding $i : X \hookrightarrow X \times Y$ as before, the open inclusion $j : X \times Y \hookrightarrow \bar{X} \times Y$, and the projection $\bar{p} : \bar{X} \times Y \to Y$. The open embedding is dealt with by theorem 4.2, the closed embedding is a very simple case of a proper direct image, and $\bar{p}$ is proper. In principle, this provides a formula for $\mathrm{CC}(RF_*\mathcal{F})$. The formula, we shall argue, can be expressed in terms of a limit, which makes it as explicit as the geometric description (3.27) of $\mathrm{CC}(RF_*\mathcal{F})$ for a proper map F.

Using the same notation as in the last paragraph, we note that $\mathrm{CC}(i_*\mathcal{F})$ is just the inverse image, properly oriented, of $\mathrm{CC}(\mathcal{F})$ under

$$di : T^*(X \times Y)|_{i(X)} \longrightarrow T^*X \,.$$

Let us separate this simple step, and describe $\mathrm{CC}(Rp_*\mathcal{E})$ for $\mathcal{E} \in \mathrm{D}^b(X \times Y)$ – in particular for $\mathcal{E} = i_*\mathcal{F}$.

6.4 Lemma. *Let $\mathcal{S}$ be a semi-algebraic Whitney stratification of $X \times Y$. There exists a C^1 semi-algebraic function $f : \bar{X} \to \mathbb{R}$, which takes strictly positive values on X and vanishes on ∂X, such that $T^*_{\mathcal{S}}(X \times Y) + s\, d\log f$ is transverse, in the stratified sense, to $T^*_X X \times T^*Y$, for every sufficiently small $s > 0$.*

We postpone the proof until the end of this section. Let us apply the lemma with a stratification $\mathcal{S}$ with respect to which $\mathcal{E}$ becomes constructible. By theorem 4.2,

$$\mathrm{CC}(Rj_*\mathcal{E}) = \lim_{s\to 0^+} \left(\mathrm{CC}(\mathcal{E}) + s\frac{df}{f}\right). \tag{6.5}$$

Here f can be any function satisfying the conditions stated in (6.4). These conditions ensure that the family of cycles $\{\mathrm{CC}(\mathcal{E}) + s\, d\log f\}$ satisfies the transversality hypotheses of proposition 3.27. Hence

$$\bar{p}_*(\mathrm{CC}(Rj_*\mathcal{E})) = \lim_{s\to 0^+} \bar{\tau}_* \left((d\bar{p})^{-1}\left(\mathrm{CC}(\mathcal{E}) + s\frac{df}{f}\right)\right), \tag{6.6}$$

where $\bar{\tau} : T^*_{\bar{X}}\bar{X} \times T^*Y \to T^*Y$ is the projection. But $\bar{p} \circ j = p$, so

$$\bar{p}_*(\mathrm{CC}(Rj_*\mathcal{E})) = \mathrm{CC}(R\bar{p}_*Rj_*\mathcal{E}) = \mathrm{CC}(Rp_*\mathcal{E}) \tag{6.7}$$

by Kashiwara-Schapira's proper push-forward theorem (2.17). Away from $s = 0$, the family $\{\mathrm{CC}(\mathcal{E}) + s\, d\log f\}$ lies entirely in $T^*(X \times Y)$; this permits us to leave off the bars on the right hand side of (6.6),

$$\bar{\tau}_* \left((d\bar{p})^{-1}\left(\mathrm{CC}(\mathcal{E}) + s\frac{df}{f}\right)\right) = \tau_* \left((dp)^{-1}\left(\mathrm{CC}(\mathcal{E}) + s\frac{df}{f}\right)\right), \tag{6.8}$$

with $\tau : T^*_X X \times T^*Y \to T^*Y$ denoting the projection. Combining (6.5-8), we obtain the description of $\mathrm{CC}(Rp_*\mathcal{E})$ we want:

6.9 Theorem. $\mathrm{CC}(Rp_*\mathcal{E}) = \lim_{s\to 0^+} \tau_*(dp)^{-1}(\mathrm{CC}(\mathcal{E}) + s\, d\log f)$.

The right hand side of this identity has a simple geometric interpretation. Since $dp : T^*_X X \times T^*Y \to T^*(X \times Y)$ is just an embedding, $(dp)^{-1}(\mathrm{CC}(\mathcal{E}) + s\, d\log f)$ is simply the intersection – transverse because of the hypotheses on f – between $T^*_X X \times T^*Y$ and $\mathrm{CC}(\mathcal{E}) + s\, d\log f$; the intersection is to be oriented as was discussed in section 3.

As has been mentioned already, theorem 6.9 leads to a geometric description of $\mathrm{CC}(RF_*\mathcal{F})$ for an arbitrary semi-algebraic map $F : X \to Y$. Sometimes such a map has a natural, explicit completion; in that case the completion can be used to give a more direct description of $\mathrm{CC}(RF_*\mathcal{F})$, which does not involve factoring F. Let us suppose then that X lies as an open subset in a real algebraic manifold $\bar{X}$, and that F extends to a proper map $\bar{F} : \bar{X} \to Y$. As before, we write $dF : X \times_Y T^*Y \to T^*X$ for the map induced by F and $\tau : X \times_Y T^*Y \to T^*Y$ for the projection.

6.10 Theorem. *Let $f : \bar{X} \to \mathbb{R}$ be a C^1, semi-algebraic function, strictly positive on X and vanishing on ∂X. Suppose that $|\,\mathrm{CC}(\mathcal{F})| + s\, d \log f$ is transverse to dF, for every sufficiently small $s > 0$. Then*

$$\mathrm{CC}(RF_*\mathcal{F}) = \lim_{s \to 0^+} \tau_*(dF)^{-1}(\mathrm{CC}(\mathcal{F}) + s\, d \log f)\,.$$

The proof of this theorem proceeds along exactly the same lines as that of theorem 6.9.

Proof of lemma 6.4. We embed $\bar{X} \hookrightarrow B \subset \mathbb{R}^N$, where B is the open unit ball. We choose a particular defining function f_0 for $\bar{X} - X$, strictly positive on X as usual, and consider the $(N+1)$-parameter family of functions

$$f_{a,b} = (a_1 x_1 + \cdots + a_N x_N + b) f_0\,. \tag{6.11}$$

These will be again be defining equations with the same positivity property, provided the linear function $a_1 x_1 + \cdots + a_N x_N + b$ is strictly positive on the unit ball. We claim: the map

$$\begin{aligned} &T^*_{\mathcal{S}}(X \times Y) \times \mathbb{R} \times \mathbb{R}^{N+1} \to T^*X \times T^*Y\,, \\ &(x, \xi, y, \eta, s, a, b) \mapsto (x, \xi + s\, d_x\, \log\, f_{a,b}, y, \eta, s, a, b)\,. \end{aligned} \tag{6.12}$$

is transverse to $T^*_X X \times T^*Y$ in the stratified sense; here d_x signifies the differential with respect to the X-variables only. This is clear, since $d_x(a_1 x_1 + \cdots + a_N x_N + b)$ spans all of $T^*_x X$ at each point $x \in X$. We will be done as soon as we can argue that transversality for the family – parametrized by (s, a, b) – implies transversality for its generic member. As explained in [GM, pp.39-40], this follows from the stratified version of the method of Abraham and Morse.

7. Intertwining operators and Lagrangian cycles.

As one application of theorem 6.9, we shall show that two seemingly different Weyl group actions on Lagrangian cycles on the cotangent bundle of the flag manifold coincide. This generalizes a result of Kashiwara-Tanisaki [KT], who prove the equality of the two actions for characteristic cycles of Verma modules. In the present section we recall the construction of one of the two actions, via intertwining operators. We shall take up the second action in section 8, and prove that the two actions coincide in section 9.

We consider a complex semisimple Lie algebra $\mathfrak{g}$, and let X denote its flag variety. We write $\tilde{\mathfrak{g}} \to X$ for the tautological bundle of Borels – the subbundle of the trivial bundle $\mathfrak{g} \times X \to X$ whose fibre at $x \in X$ is $\mathfrak{b}_x$, the Borel corresponding to x. The various quotients $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$, as x ranges over X, are canonically isomorphic, so the quotient bundle $\tilde{\mathfrak{g}}/\tilde{\mathfrak{n}}$ of $\tilde{\mathfrak{g}}$ by $\tilde{\mathfrak{n}} =_{\mathrm{def}} [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ is canonically trivial:

$$\tilde{\mathfrak{g}}/\tilde{\mathfrak{n}} \;\simeq\; \mathfrak{h} \times X\,. \tag{7.1}$$

By definition, the fibre $\mathfrak{h}$ of this trivial bundle is the universal Cartan. Its dual $\mathfrak{h}^*$ contains the universal root system Φ of $\mathfrak{g}$. We adopt the convention that $\mathfrak{g}/\mathfrak{b}_x$

is the sum of the positive root spaces when $\mathfrak{h} \simeq \mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$ is identified with a concrete Cartan in $\mathfrak{b}_x$. That defines a universal positive root system Φ^+ in Φ. The Weyl group W of $\mathfrak{g}$ has an incarnation as a reflection group acting on the universal Cartan $\mathfrak{h}$; in particular, W acts on the universal root system Φ.

Let us recall what it means for a point $y \in X$ to lie in position w relative to a given $x \in X$. The intersection $\mathfrak{b}_x \cap \mathfrak{b}_y$ contains a Cartan subalgebra $\mathfrak{c}$; via the canonical isomorphisms $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x] \cong \mathfrak{c} \cong \mathfrak{b}_y/[\mathfrak{b}_y, \mathfrak{b}_y]$, the positive root system for $\mathfrak{h} \cong \mathfrak{b}_y/[\mathfrak{b}_y, \mathfrak{b}_y]$ corresponds to a positive root system Φ_y^+ in Φ, which depends only on x and y, not the particular choice of $\mathfrak{c}$. The unique $w \in W$ which maps Φ^+ to Φ_y^+ specifies the relative position. The subvariety

$$Y_w \;=\; \{\,(x,y) \in X \times X \mid y \text{ is in position } w \text{ relative to } x\,\} \tag{7.2}$$

is an $\mathrm{Aut}(\mathfrak{g})^0$-orbit in $X \times X$, and every orbit is of this form for exactly one $w \in W$.

As in the previous section, $\mathrm{D}^b(X)$ shall denote the bounded derived category of semi-algebraically constructible sheaves on X. We fix $w \in W$ for the moment, and write $p, q : Y_w \to X$ for the projections onto the first and second factor of $X \times X$. Both p and q are fibrations with affine fibers, of dimension $\ell(w)$, the minimal length of an expression of w as a product of simple reflections. One calls

$$I_w \;=\; Rq_*\, p^*[\ell(w)] \;:\; \mathrm{D}^b(X) \to \mathrm{D}^b(X) \tag{7.3a}$$

the intertwining functor corresponding to w. We shall also use the related functor

$$J_w \;=\; Rq_!\, p^*[\ell(w)] \;:\; \mathrm{D}^b(X) \to \mathrm{D}^b(X)\,. \tag{7.3b}$$

The construction of these functors – in the slightly different, but formally analogous setting of $\mathcal{D}$-modules – goes back to [BB1]. Their basic formal properties were proved in an earlier, unpublished version of [BB2], and also in [HMSW].

7.4 Lemma. *I_w is an equivalence of categories with inverse $J_{w^{-1}}$. If $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, the composition $I_{w_1} \circ I_{w_2}$ is naturally equivalent to $I_{w_1 w_2}$.*

Proof. The product formula $I_{w_1} \circ I_{w_2} \cong I_{w_1 w_2}$ follows formally by base change in the diagram obtained by taking the fibre product of the two correspondences defined by Y_{w_1} and Y_{w_2}. For details see, for example, [HMSW]. The same argument establishes also the analogous product formula for the J_w. Thus, to prove $I_w \circ J_{w^{-1}} \cong \mathrm{Id} \cong J_{w^{-1}} \circ I_w$, it suffices to consider the case of a simple reflection s. Both I_s and J_s can be expressed as the sheaf analogues of integral operators with kernels, and their composition can be computed by "composing the kernels". Let us digress briefly to explain this process – details can be found in [KS, §3.6], for example.

Let M_1, M_2 be two topological spaces, which we assume are manifolds, to simplify matters. We write q_1, q_2 for the projections from $M_1 \times M_2$ to the two factors. For each $\mathcal{K} \in \mathrm{D}^b(M_1 \times M_2)$, the operation $\Phi_{\mathcal{K}} : \mathcal{F} \mapsto Rq_{2!}(\mathcal{K} \otimes q_1^*\mathcal{F})$ defines a functor

$$\Phi_{\mathcal{K}} \;:\; \mathrm{D}^b(M_1) \;\longrightarrow\; \mathrm{D}^b(M_2)\,. \tag{7.5}$$

If $\Phi_{\mathcal{L}} : \mathrm{D}^b(M_2) \to \mathrm{D}^b(M_3)$, with $\mathcal{L} \in \mathrm{D}^b(M_2 \times M_3)$ is defined analogously, the composition $\Phi_{\mathcal{L}} \circ \Phi_{\mathcal{K}}$ is naturally equivalent to $\Phi_{\mathcal{L}\circ\mathcal{K}} : \mathrm{D}^b(M_1) \to \mathrm{D}^b(M_3)$, with $\mathcal{L} \circ \mathcal{K} \in \mathrm{D}^b(M_1 \times M_3)$ given by the formula

$$\mathcal{L} \circ \mathcal{K} \; = \; Rq_{13!}({q_{12}}^* \mathcal{K} \otimes {q_{23}}^* \mathcal{L})\,; \tag{7.6}$$

here q_{ij} denotes the projection from $M_1 \times M_2 \times M_3$ to $M_i \times M_j$.

We apply this principle with $M_1 = M_2 = M_3 = X$. The intertwining functors I_s, J_s can be represented as

$$I_s \; = \; \Phi_{\mathcal{K}}[1]\,, \qquad J_s \; = \; \Phi_{\mathcal{L}}[1]\,. \tag{7.7}$$

To describe the kernels $\mathcal{K}$, $\mathcal{L}$, we let j denote the inclusion $Y = Y_s \hookrightarrow X \times X$; then

$$\mathcal{K} \; = \; j_! \mathbb{C}_Y\,, \qquad \mathcal{L} \; = \; Rj_* \mathbb{C}_Y\,. \tag{7.8}$$

We note that the sheaf $\mathcal{L}\circ\mathcal{K}$ on $X \times X$ is supported on the closure $\bar{Y}$ of Y. Similarly, ${q_{12}}^* \mathcal{K} \otimes {q_{23}}^* \mathcal{L}$ is supported on the closure $\bar{Z}$ of $Z = (Y \times X) \cap (X \times Y)$. The map q_{13} induces a $\mathbb{P}^1$-fibration $\pi : \bar{Z} \to \bar{Y}$. At points $y \in Y$, $Z \cap \pi^{-1}(y) \cong \mathbb{C}^*$, and the sheaf $\mathcal{L} \circ \mathcal{K}$ restricted to $\pi^{-1}(y) \cong \mathbb{P}^1$ can be identified with the constant sheaf $\mathbb{C}$ on $\mathbb{C}^*$, extended by zero over one of the two punctures, and by the derived lower $*$ extension over the other. This sheaf on $\mathbb{P}^1$ has no cohomology, hence the stalk of $\mathcal{L} \circ \mathcal{K}$ at points $y \in Y$ reduces to zero. Thus $\mathcal{L} \circ \mathcal{K}$ is supported on $\bar{Y} - Y$, in other words, on the diagonal in $X \times X$. At points y in the diagonal, $Z \cap \pi^{-1}(y) \cong \mathbb{C}$, and the sheaf $\mathcal{L} \circ \mathcal{K}$ restricted to $\pi^{-1}(y) \cong \mathbb{P}^1$ can be identified with the constant sheaf $\mathbb{C}$ on $\mathbb{C}$, extended by zero over the puncture. This latter sheaf has cohomology in degree two only, of dimension one. We conclude that $\mathcal{L} \circ \mathcal{K}$ is the constant sheaf on the diagonal in degree two. Noting the shifts in (7.7), we now see that $J_s \circ I_s$ corresponds to the "identity kernel", i.e., the constant sheaf on the diagonal. Thus $J_s \circ I_s$ is naturally equivalent to the identity functor. The case of the composition $I_s \circ J_s$ is totally analogous.

On the level of the K-group $K(\mathrm{D}^b(X))$, the natural equivalences $I_{w_1} \circ I_{w_2} \cong I_{w_1 w_2}$ asserted by the lemma become equalities. Also, if s is a simple reflection,

$$I_s = J_s \;\text{ as operators on } K(\mathrm{D}^b(X))\,. \tag{7.9}$$

To see this, we consider the diagram

$$\begin{tikzcd} X \arrow[d, equal] & \bar{Y}_s \arrow[l, "\bar{p}"'] \arrow[r, "\bar{q}"] & X \arrow[d, equal] \\ X & Y_s \arrow[l, "p"'] \arrow[r, "q"] \arrow[u, "j"] & X \end{tikzcd} \tag{7.10}$$

and the inclusion $i : X \cong \Delta_X \hookrightarrow \bar{Y}_s$. Since $i(X)$ is the complement of Y_s in $\bar{Y}_s$, we get distinguished triangles

$$i_* i^! \mathcal{G} \to \mathcal{G} \to Rj_* j^* \mathcal{G}\,, \qquad j_! j^* \mathcal{G} \to \mathcal{G} \to i_* i^* \mathcal{G} \tag{7.11}$$

corresponding to any $\mathcal{G} \in \mathrm{D}^b(\bar{Y}_s)$, in particular for $\mathcal{G} = \bar{p}^*\mathcal{F}$, with $\mathcal{F} \in \mathrm{D}^b(X)$. In this latter case, $i_* i^! \bar{p}^* \mathcal{F} = i_* \mathcal{F}[-2]$ (p has real fiber dimension 2 !), $Rj_* j^* \bar{p}^* \mathcal{F} = Rj_* p^* \mathcal{F}$, $j_! j^* \bar{p}^* \mathcal{F} = j_! p^* \mathcal{F}$, and $i_* i^* \bar{p}^* \mathcal{F} = i_* \mathcal{F}$. The resulting distinguished triangles translate into the following equalities in the K-group:

$$\bar{p}^* \mathcal{F} = i_* \mathcal{F} + Rj_* p^* \mathcal{F} = i_* \mathcal{F} + j_! p^* \mathcal{F} \quad \text{in } K(\mathrm{D}^b(\bar{Y}_s))\,, \tag{7.12}$$

and hence $Rj_* p^* \mathcal{F} = j_! p^* \mathcal{F}$. At this point, again in the K-group,

$$I_s \mathcal{F} = Rq_* p^* \mathcal{F}[1] = R\bar{q}_* Rj_* p^* \mathcal{F}[1] = R\bar{q}_* j_! p^* \mathcal{F}[1] = Rq_! p^* \mathcal{F}[1] = J_s \mathcal{F}\,, \tag{7.13}$$

as was claimed in (7.9). Combining this assertion with (7.4), we find:

7.14 Proposition. *The intertwining functors I_w, $w \in W$, induce an action of the Weyl group W on $K(\mathrm{D}^b(X))$.*

8. Rossmann's construction.

Rossmann [R], following an earlier idea of Kazhdan-Lusztig [KL], constructs a proper homotopy action of the Weyl group on the cotangent bundle of the flag variety. This action, in turn, determines an action on the Borel-Moore homology of certain subsets of T^*X. In this section we shall show that Rossmann's construction produces, in particular, a Weyl group action on $\mathcal{L}^+(X)$, the group of all semi-algebraic, $\mathbb{R}^+$-invariant Lagrangian cycles on T^*X.

Let us continue with the notation of the previous section. In particular, $\tilde{\mathfrak{g}} \to X$ stands for the tautological bundle of the Borels, and $\mathfrak{h}$ for the universal Cartan. By definition, $\mathfrak{h}$ is canonically isomorphic to each of the quotients $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$, $x \in X$. The tangent space of X at x can be naturally identified with $\mathfrak{g}/\mathfrak{b}_x$. The resulting exact sequences

$$0 \to \mathfrak{h} \to \mathfrak{g}/[\mathfrak{b}_x, \mathfrak{b}_x] \to T_x X \to 0 \tag{8.1}$$

fit together, into the exact sequence of vector bundles

$$0 \to \mathfrak{h} \times X \to (\mathfrak{g} \times X)/\tilde{\mathfrak{n}} \to TX \to 0\,. \tag{8.2}$$

Dually this gives

$$0 \to T^*X \to (\mathfrak{g} \times X/\tilde{\mathfrak{n}})^* \to \mathfrak{h}^* \times X \to 0\,. \tag{8.3}$$

We note that the exact sequences (8.2-3) are equivariant with respect to the group

$$G = \mathrm{Aut}(\mathfrak{g})^0 \tag{8.4}$$

when G is made to act trivially on $\mathfrak{h}$.

Let $q : (\mathfrak{g} \times X/\tilde{\mathfrak{n}})^* \to \mathfrak{h}^*$ denote the second arrow in (8.3), composed with the projection to the factor $\mathfrak{h}^*$ and $p : (\mathfrak{g} \times X/\tilde{\mathfrak{n}})^* \to \mathfrak{g}^*$ the mapping induced by the quotient maps $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{b}_x, \mathfrak{b}_x]$. We claim: for each regular $\lambda \in \mathfrak{h}^*$,

$$\Omega_\lambda = pq^{-1}\lambda \tag{8.5}$$

is a regular, semisimple, coadjoint orbit, i.e., an orbit in $\mathfrak{g}^*$ under the action of G; moreover, the assignment $\lambda \to \Omega_\lambda$ establishes a bijection between the regular W-orbits in $\mathfrak{h}^*$ and the regular, semisimple, coadjoint orbits. To see this, we observe that $q^{-1}\lambda$ intersects $(\mathfrak{g}/[\mathfrak{b}_x, \mathfrak{b}_x])^*$ – i.e., the fiber of $(\mathfrak{g} \times X/\tilde{\mathfrak{n}})^*$ at x – in the set of all linear functions $\phi \in \mathfrak{g}^*$ which vanish on $[\mathfrak{b}_x, \mathfrak{b}_x]$ and coincide with λ on $\mathfrak{h} = \mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$. The isotropy subgroup $B_x \subset G$ acts transitively on this set, precisely because λ was assumed to be regular[3]. Consequently G acts transitively on Ω_λ. Also, if $\mathfrak{t} \subset \mathfrak{b}_x$ is a concrete Cartan, λ corresponds to some $\lambda_\mathfrak{t}$ in $\mathfrak{t}^*$ via the distinguished isomorphism $\mathfrak{t} \cong \mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x] = \mathfrak{h}$. Like any Cartan subalgebra, $\mathfrak{t}$ has a distinguished linear complement in $\mathfrak{g}$, so this $\lambda_\mathfrak{t}$ can be thought of as an element of $\mathfrak{g}^*$. Our earlier description of the fiber of q at x shows that $\lambda_\mathfrak{t} \in \Omega_\lambda$. In particular, Ω_λ is a regular, semisimple, coadjoint orbit, as we had claimed, and the correspondence $\lambda \mapsto \Omega_\lambda$ agrees with the usual enumeration of such orbits in terms of a concrete Cartan subalgebra.

We fix a compact real form $U_\mathbb{R} \subset G$. Then each $x \in X$ is fixed by a unique maximal torus in $U_\mathbb{R}$; the complexified Lie algebra of this maximal torus is a concrete Cartan $\mathfrak{t}_x \subset \mathfrak{b}_x$, hence a distinguished linear complement to $[\mathfrak{b}_x, \mathfrak{b}_x]$ in $\mathfrak{b}_x$. The choice of this complement, as x varies over X, splits the exact sequences (8.2-3) $U_\mathbb{R}$-equivariantly and real algebraically. Thus

$$(\mathfrak{g} \times X/\tilde{\mathfrak{n}})^* \cong (\mathfrak{h}^* \times X) \oplus T^*X \,, \tag{8.6}$$

again $U_\mathbb{R}$-equivariantly and real algebraically. For any fixed $\lambda \in \mathfrak{h}^*$, the section $\{\lambda\} \times X \subset \mathfrak{h}^* \times X$ now determines a map from T^*X to $(\mathfrak{g} \times X/\tilde{\mathfrak{n}})^*$, which we compose with the map $p : (\mathfrak{g} \times X/\tilde{\mathfrak{n}})^* \to \mathfrak{g}^*$. The result is a $U_\mathbb{R}$-equivariant, real algebraic map

$$\mu_\lambda : T^*X \to \mathfrak{g}^*,$$

which takes values in $pq^{-1}\lambda$, whether or not λ is regular. When λ *is* regular,

$$\mu_\lambda : T^*X \xrightarrow{\sim} \Omega_\lambda \,, \tag{8.7}$$

as follows from our earlier description of the fibers of $q^{-1}\lambda$. This is Rossmann's twisted moment map. At the opposite extreme, for $\lambda = 0$, $pq^{-1}(0) = \mathcal{N}^*$ is the nilpotent cone in $\mathfrak{g}^*$, and

$$\mu =_{\mathrm{def}} \mu_0 : T^*X \to \mathcal{N}^* \tag{8.8}$$

is the usual moment map. In this case μ no longer depends on the choice of the compact real form $U_\mathbb{R} \subset G$, and μ is, in fact, G-equivariant and complex algebraic. The difference between the twisted moment map μ_λ and the usual moment map μ can be described explicitly: for $x \in X$ and $\xi \in T^*_x X$,

$$\mu_\lambda(x, \xi) = \mu(x, \xi) + \lambda_x \,; \tag{8.9}$$

here λ_x denotes the image of λ in $\mathfrak{t}^*_x \subset \mathfrak{g}^*$ under the canonical isomorphism $\mathfrak{h} = \mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x] \cong \mathfrak{t}_x$ between the universal Cartan and $\mathfrak{t}_x$, the complexified Lie algebra

[3]One can see this, for example, by identifying $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form and using the standard fact that the B_x-orbit through any regular semisimple $\xi \in \mathfrak{b}_x$ coincides with $\xi + [\mathfrak{b}_x, \mathfrak{b}_x]$.

of the unique maximal torus in $U_{\mathbb{R}}$ which fixes x. To verify (8.9), one can argue as in the remarks below (8.5).

In the discussion that follows, $S \subset \mathcal{N}^*$ will be a closed, $\mathbb{R}^+$-invariant, semi-algebraic subset. Rossmann's construction produces a proper homotopy action of W on the semi-algebraic set $\mu^{-1}(S)$, and therefore a representation of W on the Borel-Moore homology $\mathrm{H}_*^{inf}(S,\mathbb{Z})$ [R]. Since Rossmann's argument is sketchy, we shall describe the action in detail, in our setting – i.e., with the stated hypotheses on S.

8.10 Lemma. *There exists a neighborhood $\mathcal{U}$ of $\mu^{-1}(S)$ in T^*X such that*
a) the inclusion $\mu^{-1}(S) \hookrightarrow \mathcal{U}$ is a proper homotopy equivalence;
b) $\mathcal{U}$ contains $\mu^{-1}\{\, \xi \in \mathcal{N}^* \mid \operatorname{dist}(\xi, S) < \epsilon \,\}$, for some $\epsilon > 0$.

In b), $\operatorname{dist}(\xi, S)$ refers to the distance between ξ and S with respect to a Euclidean metric on $\mathfrak{g}^*$. By a proper homotopy equivalence, we mean an isomorphism in the proper homotopy category: the category of topological spaces and proper maps, with any two maps identified if they are homotopic via a homotopy which itself is a proper map.

Proof of 8.10. We put a $U_{\mathbb{R}}$-invariant – hence real algebraic – hermitian metric on the vector bundle T^*X and let $\mathcal{B}$ denote the bundle of closed unit balls in T^*X. Then the pair $(\mu^{-1}(S) \cap \mathcal{B}, \mu^{-1}(S) \cap \partial\mathcal{B})$ is compact and semi-algebraic, so the triple $(\mathcal{B}, \mu^{-1}(S) \cap \mathcal{B}, \mu^{-1}(S) \cap \partial\mathcal{B})$ can be finitely triangulated. Using standard arguments, one can construct an open neighborhood $\mathcal{U}_{\leq 1}$ of $\mu^{-1}(S)\cap\mathcal{B}$ and a strong deformation retraction

$$(r_{\leq 1}, r_1) : (\mathcal{U}_{\leq 1}, \mathcal{U}_1) \to (\mu^{-1}(S) \cap \mathcal{B}, \mu^{-1}(S) \cap \partial\mathcal{B})\,; \tag{8.11}$$

here $\mathcal{U}_1$ denotes the intersection $\mathcal{U}_{\leq 1} \cap \partial\mathcal{B}$ and r_1 the restriction of $r_{\leq 1}$ to the unit sphere bundle $\partial\mathcal{B}$. Since S was assumed to be $\mathbb{R}^+$-invariant, we can scale out both r_1 and $\mathcal{U}_1$; the result will be a strong deformation retraction $r_{\geq 1} : \mathcal{U}_{\geq 1} \to \mu^{-1}(S)\cap(T^*X - \operatorname{int}\mathcal{B})$ which agrees with r_1 on $\mathcal{U}_1$. Thus we can glue along $\mathcal{U}_1$, and produce a strong deformation retraction $r : \mathcal{U} \to \mu^{-1}(S)$. Since r is $\mathbb{R}^+$-equivariant outside a compact set, it is a proper homotopy inverse of the inclusion, and $\mathcal{U}$ satisfies condition b) in the lemma.

From the definition of the twisted moment map, one sees that the Euclidean distance between $\mu_\lambda(x,\xi)$ and $\xi = \mu(x,\xi)$ can be uniformly bounded in terms of the size of λ. Thus, for $w \in W$ and any sufficiently small regular $\lambda \in \mathfrak{h}^*$,

$$a_{w,\lambda} \ =_{\text{def}} \ (\mu_{w\lambda})^{-1} \circ \mu_\lambda \tag{8.12}$$

maps $\mu^{-1}(S)$ into the neighborhood $\mathcal{U}$ constructed in lemma 8.10. The regular set in $\mathfrak{h}^*$, intersected with any small ball around the origin, is connected. It follows that the proper homotopy class of $a_{w,\lambda} \in \operatorname{Map}(\mu^{-1}(S)\,, \mathcal{U}\,)$ depends only on w, not on λ. The inclusion $\mu^{-1}(S) \hookrightarrow \mathcal{U}$ is an isomorphism in the proper homotopy category, hence $a_{w,\lambda}$ determines a morphism, in the proper homotopy category,

$$\bar{a}_w \,:\, \mu^{-1}(S) \to \mu^{-1}(S)\,. \tag{8.13}$$

The $a_{w,\lambda}$ satisfy the identity

$$a_{w,v\lambda} \circ a_{v,\lambda} \ = \ a_{wv,\lambda}\,, \tag{8.14}$$

hence $\bar{a}_w \circ \bar{a}_v = \bar{a}_{wv}$. This is Rossmann's proper homotopy action:

8.15 Proposition. *The $\bar{a}_w$ define a proper homotopy action of W on $\mu^{-1}(S)$.*

Like Rossmann, we are not so much interested in the homotopy action itself, but rather in the induced action on Borel-Moore homology; for that, of course, it is crucial that the construction takes place in the proper homotopy category.

8.16 Corollary. *The $\bar{a}_w$ induce a representation of W on $\mathrm{H}_*^{inf}(\mu^{-1}(S)\,,\,\mathbb{Z}\,)$.*

We shall see presently – in lemma 8.29 below – that every $\mathbb{R}^+$-invariant Lagrangian cycle C in T^*X has support contained in a semi-algebraic, isotropic subvariety of the form $\mu^{-1}(S)$, with $S \subset \mathcal{N}^*$ closed, semialgebraic, $\mathbb{R}^+$-invariant. In particular, then, C can be regarded as an element of the Borel-Moore homology of $\mu^{-1}(S)$ in degree equal to the real dimension of X; conversely, every element of that Borel-Moore homology group represents some $C \in \mathcal{L}^+(X)$. Once 8.29 has been proved, 8.16 gives the conclusion we want:

8.17 Theorem. *The $\bar{a}_w$ define an action of W on $\mathcal{L}^+(X)$.*

Let us turn to the last remaining ingredient of our construction. The statement we need, lemma 8.29 below, depends on a geometric result which relates the symplectic structure of a nilpotent coadjoint orbit to that of T^*X. Let us state and prove this result first. We recall that $\mathcal{N}^*$ is the union of finitely many G-orbits, and that each of these orbits carries a canonical, G-invariant, complex algebraic symplectic structure. Specifically, if $\mathcal{O} \subset \mathcal{N}^*$ is a G-orbit and ξ a point in $\mathcal{O}$,

$$\omega_{\mathcal{O}}|_\xi\,(Z_1 \cdot \xi\,, Z_2 \cdot \xi) = \xi([Z_1, Z_2])\,, \quad \text{for } Z_1 \cdot \xi\,, Z_2 \cdot \xi \ \in\ \mathfrak{g} \cdot \xi \ \cong\ T_\xi \mathcal{O}\,, \tag{8.18}$$

is a well-defined element of $\wedge^2 T^*_\xi \mathcal{O}$, and $\xi \mapsto \omega_{\mathcal{O}}|_\xi$ a well-defined, nondegenerate, closed, algebraic two form.

8.19 Lemma. *The canonical symplectic form ω on T^*X and the symplectic form $\omega_{\mathcal{O}}$ on a G-orbit $\mathcal{O} \subset \mathcal{N}^*$ are compatible, in the sense that $\mu^*\omega_{\mathcal{O}} = -\omega|_{\mu^{-1}(\mathcal{O})}$ at the smooth points of $\mu^{-1}(\mathcal{O})$.*

Proof. By definition, the symplectic form ω is the exterior derivative $d\alpha$ of the contact form α on T^*X, which is given as follows. We write $\pi : T^*X \to X$ for the projection; then, for $v \in T_{(x,\xi)}T^*X$,

$$\alpha(v) \ = \ \xi(\pi_* v). \tag{8.20}$$

The action of G on X induces an action on T^*X. For $Z \in \mathfrak{g}$ and any space on which G acts – in particular, X, T^*X, and $\mathcal{N}^*$ – we let $\ell(Z)$ denote the vector field "infinitesimal translation by Z". Since G acts transitively on X, $\mathrm{Ker}(\pi_*)$ and $\ell(\mathfrak{g})$ span the tangent space of T^*X at any point (x, ξ). If (x, ξ) is a smooth point of $\mu^{-1}(\mathcal{O})$, with $\mathcal{O}$ the G-orbit through $\xi = \mu(x, \xi)$, $T(\mu^{-1}(\mathcal{O}))$ contains all of $\ell(\mathfrak{g})$, so $T(\mu^{-1}(\mathcal{O}))$ is spanned by $\ell(\mathfrak{g})$ and $\mathrm{Ker}(\pi_*) \cap T(\mu^{-1}(\mathcal{O}))$. Thus, to prove the identity

$$\omega_{\mathcal{O}}(\mu_* v_1, \mu_* v_2) = -\omega(v_1, v_2)\,, \quad \text{with } v_1, v_2 \in T_{(x,\xi)}(\mu^{-1}(\mathcal{O}))\,, \tag{8.21}$$

it suffices to consider three cases: both v_1 and v_2 lie in $\ell(\mathfrak{g})$, both lie in $\mathrm{Ker}(\pi_*)$, or v_1 lies is one of the two spaces and v_2 in the other.

We begin with some preliminary remarks. In view of the definition, ℓ commutes with equivariant maps, for example

$$\mu_* \ell(Z) \;=\; \ell(Z)\,. \tag{8.22a}$$

When we identify T^*X with a subvariety of $X \times \mathcal{N}^*$, as usual, the moment map μ is the projection to the second factor, hence

$$\alpha(\ell(Z)) = \xi(\mu_*(\ell(Z)) = \xi(Z)\,. \tag{8.22b}$$

Since ℓ defines an action, it relates the Lie bracket on $\mathfrak{g}$ to the Lie bracket of vector fields:

$$\ell([Z_1, Z_2]) \;=\; [\ell(Z_1), \ell(Z_2)]\,. \tag{8.22c}$$

First we consider tangent vectors $v_i = \ell(Z_i) \in \ell(\mathfrak{g}) \subset T_{(x,\xi)}(\mu^{-1}(\mathcal{O}))$. In this case

$$\begin{aligned} &\omega(\ell(Z_1), \ell(Z_2)) = d\alpha(\ell(Z_1), \ell(Z_2)) = \\ &\quad \ell(Z_1)\alpha(\ell(Z_2)) - \ell(Z_2)\alpha(\ell(Z_1)) - \alpha(\ell([Z_1, Z_2])) = \\ &\qquad \ell(Z_1)(\xi(Z_2)) - \ell(Z_2)(\xi(Z_1)) - \xi([Z_1, Z_2]) = \\ &\qquad\quad (Z_1 \cdot \xi)(Z_2) - (Z_2 \cdot \xi)(Z_1) - \xi([Z_1, Z_2]) = \\ &\qquad\qquad - \xi([Z_1, Z_2]) + \xi([Z_1, Z_2]) - \xi([Z_1, Z_2]) = \\ &\qquad\qquad\quad - \xi([Z_1, Z_2]) = -\omega_{\mathcal{O}}([Z_1, Z_2]). \end{aligned} \tag{8.23}$$

Here we have used, in sequence, the definition of ω in terms of α, the formula for the exterior derivative in terms of the Lie bracket and (8.22c), the formula (8.22b) for α, the interpretation of the action ℓ on $\mathcal{N}^*$ as the restriction of the coadjoint action, the explicit formula for the coadjoint action, and the definition (8.18).

Next we consider the case of tangent vectors $v_i \in T_{(x,\xi)}(\mu^{-1}(\mathcal{O}))$, with $v_1 \in \operatorname{Ker}(\pi_*)$ and $v_2 = \ell(Z_2)$, $Z_2 \in \mathfrak{g}$. The kernel of π_* on $T_{(x,\xi)}T^*X$ can be identified with the tangent space of T^*_xX, and hence with T^*_xX. With this interpretation – i.e., $v_1 \in T^*_xX$ – and for any $v_2 \in T_{(x,\xi)}T^*X$,

$$\omega(v_1, v_2) \;=\; v_1(\pi_* v_2)\,; \tag{8.24}$$

this is a general fact about the symplectic structure on T^*X for any manifold X, and can be verified easily by a computation in local coordinates. Thus, in our concrete situation,

$$\omega(v_1, v_2) = v_1(\pi_*\ell(Z_2)) = v_1(\ell(Z_2)) = (\mu v_1)(Z_2) = (\mu_* v_1)(Z_2)\,. \tag{8.25a}$$

Here we have used (8.22a) and interpret the second instance of $\ell(Z_2)$ as a vector in $T_xX \cong \mathfrak{g}/\mathfrak{b}_x$, namely as the image of Z_2 in $\mathfrak{g}/\mathfrak{b}_x$; also, μ is linear on T^*_xX, and thus $\mu v_1 = \mu_* v_1$. Since v_1 was assumed to lie in $T_{(x,\xi)}(\mu^{-1}(\mathcal{O}))$, $\mu_* v_1$ is a tangent vector to $\mathcal{O}$ at ξ, so

$$\mu_* v_1 \;=\; Z_1 \cdot \xi \tag{8.25b}$$

for some $Z_1 \in \mathfrak{g}$. Combining (8.25b,c), and noticing that $\mu_* v_2 = \mu_* \ell(Z_2) = Z_2 \cdot \xi$, we find

$$\begin{aligned}\omega(v_1, v_2) = (Z_1 \cdot \xi)(Z_2) = -\xi[Z_1, Z_2] = \\ -\,\omega_{\mathcal{O}}(Z_1 \cdot \xi, Z_2 \cdot \xi) = -\omega_{\mathcal{O}}(\mu_* v_1, \mu_* v_2)\,.\end{aligned} \tag{8.26}$$

We still need to consider the case of two tangent vectors $v_i \in \mathrm{Ker}\pi_* \cap T_{(x,\xi)}(\mu^{-1}(\mathcal{O}))$. In this case $\omega(v_1, v_2) = 0$. As in the previous argument, we identify $\mathrm{Ker}\pi_* \cong T_x^* X$, and further with $(\mathfrak{g}/\mathfrak{b}_x) = \mathfrak{b}_x^\perp$. It follows that $\mu_*(v_i) \in \mathcal{O} \cap \mathfrak{b}_x^\perp$, and so the equality (8.21), in this particular instance, reduces to:

$$\mathcal{O} \cap \mathfrak{b}_x^\perp \text{ is isotropic in } \mathcal{O}\,. \tag{8.27}$$

In fact, $\mathcal{O} \cap \mathfrak{b}_x^\perp$ is Lagrangian in $\mathcal{O}$. This was proved by Joseph [J], and also follows from a general result of Ginsburg on Hamiltonian actions of solvable algebraic groups [Gi1, Theorem 4.1]. This completes the proof of lemma 8.19.

Lemma 8.19 makes an assertion about $\mu^*\omega_{\mathcal{O}}$ only at the smooth points of $\mu^{-1}(\mathcal{O})$, but has implications at non-smooth points as well. To see this, we Whitney stratify the proper, complex algebraic map $\mu : T^*X \to \mathcal{N}^*$ compatibly with the orbit stratification on $\mathcal{N}^*$. If $\mathcal{O} \subset \mathcal{N}^*$ is a particular orbit and Σ a top dimensional stratum in $\mu^{-1}(\mathcal{O})$, lemma 8.19 implies that the restriction of the symplectic form $-\omega$ to Σ agrees with $(\mu|_\Sigma)^*\omega_{\mathcal{O}}$. On the other hand, if $\Sigma \subset \mu^{-1}(\mathcal{O})$ is a lower dimensional stratum and (x, ξ) a point on Σ, then any tangent vector at (x, ξ) can be expressed as the limit of a sequence of tangent vectors to a top dimensional stratum in $\mu^{-1}(\mathcal{O})$, by Whitney's condition A. Now, using the continuity of μ_*, we conclude:

8.28 Corollary. *For each G-orbit and each stratum $\Sigma \subset \mu^{-1}(\mathcal{O})$, the restriction of $-\omega$ to Σ coincides with $(\mu|_\Sigma)^*\omega_{\mathcal{O}}$.*

Our next lemma, whose proof depends critically on lemma 8.19, establishes the last remaining ingredient of the proof of theorem 8.17.

8.29 Lemma. *For each $C \in \mathcal{L}^+(X)$, there exists a closed, $\mathbb{R}^+$-invariant, semialgebraic subset $S \subset \mathcal{N}^*$, such that $\mu^{-1}(S)$ is isotropic in T^*X and contains the support of C.*

8.30 Remark. The proof will show that the $\mathbb{R}^+$-invariance of the cycle $C \in \mathcal{L}^+(X)$ plays no role. Without the hypothesis of $\mathbb{R}^+$-invariance on C, we cannot expect S to be $\mathbb{R}^+$-invariant, of course.

Proof. Since μ is proper, the μ-image S of the support $|C|$ of C is a closed, semialgebraic, $\mathbb{R}^+$-invariant subset of $\mathcal{N}^*$. We stratify the proper semialgebraic map $\mu : |C| \to S$ compatibly with the stratification of $\mu : T^*X \to \mathcal{N}^*$ which was used in corollary 8.28. Let us consider a particular stratum $T \subset |C|$. We note that the Lagrangian nature of C forces T to be an isotropic submanifold of T^*X; this follows from the same kind of argument as the one preceding corollary 8.28. By the definition of stratified map, $\mu : T \to \mu(T)$ is a submersion. The μ-image of the isotropic submanifold T is contained in some G-orbit $\mathcal{O} \subset \mathcal{N}^*$. Hence, by corollary 8.28, $\mu(T)$ is an isotropic submanifold of $\mathcal{O}$. Also, if $T' \subset \mu^{-1}\mu(T)$ is any stratum then corollary 8.28 shows that T' is isotropic in T^*X. This proves that $\mu^{-1}(S) \subset T^*X$ is isotropic. By definition, $|C| \subset \mu^{-1}(S)$.

9. The equality of the two Weyl group actions.

In the previous two sections we described Weyl group actions on the K-group of the derived category $\mathrm{D}^b(X)$ and on the group of Lagrangian cycles $\mathcal{L}^+(X)$. Recall that the characteristic cycle construction maps the former group isomorphically onto the latter [KS].

9.1 Theorem. *The isomorphism* $\mathrm{CC} : K(\mathrm{D}^b(X)) \to \mathcal{L}^+(X)$ *is equivariant with respect to the two actions of* W.

We begin the proof with a reinterpretation of the W-action on $\mathcal{L}^+(X)$, using the language of families of cycles. Let us fix a particular element $w \in W$, a particular cycle $C \in \mathcal{L}^+(X)$, and a real algebraic curve $\{ \lambda(s) \mid s \in [0,\eta) \}$ in $\mathfrak{h}^*$, such that

$$\lambda(s) \text{ is regular if } s \neq 0\,, \quad \text{and } \lambda(0) = 0\,. \tag{9.2}$$

The maps $a_{w,\lambda}$ defined in (8.12), with $\lambda = \lambda(s)$ and $0 < s < \eta$, constitute a real algebraic, one parameter family of isomorphisms of T^*X. Thus

$$C_s = a_{w,\lambda(s)}(C)\,, \quad 0 < s < \eta\,, \tag{9.3}$$

is a family of cycles C_I in the sense of section 3.

9.4 Lemma. *The w-translate of the cycle C is equal to the limit of the family C_s as $s \to 0^+$.*

Proof. Since the family C_I is semi-algebraic, the closure of $|C_I|$ in $[0,\eta) \times T^*X$ is semialgebraic also; in particular, the family has a limit as $s \to 0^+$. We choose $S \subset \mathcal{N}^*$ and $\mathcal{U} \subset T^*X$ as in lemmas 8.29 and 8.10. Then, for all sufficiently small $s > 0$, $\mathcal{U}$ contains C_s. Also,

$$C \in \mathrm{H}_n^{inf}(\mu^{-1}(S),\mathbb{Z}) \;\cong\; \mathrm{H}_n^{inf}(\mathcal{U},\mathbb{Z})\,, \tag{9.5}$$

with $n = \dim_{\mathbb{R}}(X)$. As element of $\mathrm{H}_n^{inf}(\mathcal{U},\mathbb{Z})$, the w-translate of C is equal to any one of the C_s, with $s > 0$ sufficiently small. We now appeal to proposition 3.25: the limit of the family is Borel-Moore homologous, in $\mathcal{U}$, to C_s. On the other hand, the support of the limit is contained in $\mu^{-1}(S)$ because $\lambda(0) = 0$, so the conclusion of the lemma follows.

Proof of theorem 9.1. Before going into the details of the argument, let us outline the strategy. It suffices to show that CC commutes with the action of any simple reflection s_α. The intertwining operator

$$I \;=_{\mathrm{def}}\; I_{s_\alpha} \;:\; \mathrm{D}^b(X) \to \mathrm{D}^b(X) \tag{9.6}$$

was constructed as the composition of a smooth pullback and a non-proper push-forward, with a shift in degree. On the level of characteristic cycles, the latter operation is described in section 6; it involves the choice of an auxiliary function f. When f is chosen appropriately, it turns out that the effect of I on a characteristic cycle C is given by the same family of cycles as in lemma 9.4.

We recall the definition 7.2 of $Y = Y_{s_\alpha}$ and the definition of the projections $p, q : Y \to X$. With this notation,

$$I \;=\; Rq_* p^*[1]\,. \tag{9.7}$$

The choice of the simple root α determines a generalized flag variety X_α and G-equivariant fibrations

$$\begin{aligned} &X \to X_\alpha \quad \text{with fiber } \mathbb{P}^1\,, \\ &Y \to X_\alpha \quad \text{with fiber } \mathbb{P}^1 \times \mathbb{P}^1 - \Delta\,, \end{aligned} \tag{9.8}$$

($\Delta =$ diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$). Thus Y has a natural smooth G-equivariant completion $\bar{Y}$ which fibers over X_α, obtained by inserting the diagonals,

$$\bar{Y} \to X_\alpha \quad \text{with fiber } \mathbb{P}^1 \times \mathbb{P}^1\,, \tag{9.9}$$

and the projections $p, q : Y \to X$ extend smoothly to projections $\bar{p}, \bar{q} : \bar{Y} \to X$.

To choose a $U_{\mathbb{R}}$-invariant defining function f for the boundary of Y in $\bar{Y}$ is equivalent to choosing an $SU(2)$-invariant defining function for the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, and that in turn is equivalent to choosing a function on $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$ which vanishes precisely at the origin, and is invariant under the rotation group. Our particular choice will be $|z|^2/(1 + |z|^2)$, and we let f denote the resulting $U_{\mathbb{R}}$-invariant defining function for ∂Y. We observe that f is real algebraic.

We shall apply theorem 6.10 to the map q and cycles in T^*Y of the form p^*C, where C is a Lagrangian cycle on T^*X. The transversality hypothesis, we shall see, is satisfied in a very strong sense. Not only is $p^*(C) + s\,d\,\log\,f$ transverse to

$$dq : Y \times_X T^*X \to T^*Y\,, \tag{9.10a}$$

but even $p^*(\{\xi\}) + s\,d\,\log\,f$ is transverse, for any cotangent vector $\xi \in T^*X$ – as ought to be expected, since we must deal with arbitrary Lagrangian cycles in T^*X. Since q is a fibration, the map dq is injective, with image

$$\operatorname{Im} dq = \{\, (y, \eta) \mid y \in Y,\, \eta \in T^*_y Y,\, \eta \perp \operatorname{Ker} q_* \}\,. \tag{9.10b}$$

The following lemma contains, in particular, the transversality assertion that was just made. We fix a real number s, $s > 0$, a point $x \in X$, and a cotangent vector $\xi \in T^*_x X$. Note that $\operatorname{Im} dq \subset T^*Y$ is a smooth algebraic hypersurface, hence a submanifold of real codimension 2, whereas $p^*(\{\xi\}) + s\,d\,\log\,f$, the image of the algebraic curve $p^*(\{\xi\})$ under the diffeomorphism

$$p^*(\{\xi\}) \simeq p^*(\{\xi\}) + s\,d\,\log\,f\,, \quad (y, \eta) \mapsto (y, \eta + s\,d\,\log\,f(y)), \tag{9.11}$$

is a real algebraic 2-manifold in T^*Y.

9.12 Lemma. *The submanifolds* $\operatorname{Im} dq$, $p^*(\{\xi\}) + s\, d \log f(y)$ *intersect transversely, in a single point* (y, η)*, with* $y = y(x, \xi, s) \in p^{-1}(x)$ *and* $\eta = \eta(x, \xi, s) \in (\operatorname{Ker} q_*)^\perp \subset T^*_y Y$. *Let* $\xi' \in T^*_{q(y)} X$ *be the unique cotangent vector such that* $q^* \xi' = \eta$*. The assignment* $(x, \xi) \mapsto (q(y), \xi')$ *defines a real algebraic automorphism of* T^*X*, which coincides with* $a_{s_\alpha, \lambda}$ *whenever* $\lambda \in \mathfrak{h}^*$ *is regular and takes the value* s *on the co-root* $\check\alpha$.

Before proving the lemma, let us argue that it implies theorem 9.1. The projection $p : Y \to X$ is a fibration, and the operation $p^* : \mathcal{L}^+(X) \to \mathcal{L}^+(Y)$ reduces to the geometric pullback of Lagrangian cycles along the fibers of p; cf. (2.19). Lemma 9.12 guarantees that the family of cycles $p^*(C) + s\, d \log f(y)$, $s \in \mathbb{R}^+$, satisfies the transversality hypothesis in theorem 6.10, for any $C \in \mathcal{L}^+(X)$. Thus, up to a fixed sign,

$$C \longmapsto \lim_{s \to 0^+} \tau_* (dq)^{-1} (p^* C + s\, d \log f) \tag{9.13}$$

describes the effect of the intertwining operator I on characteristic cycles. Here $\tau : Y \times_X T^*X \to T^*X$ is the natural projection, and the operation $(dq)^{-1}$ can be re-interpreted as "intersection with $\operatorname{Im} dq$", again up to a universal sign. Since the tranversality condition holds in the very strong sense spelled out in (9.12), the assignment (9.13) is induced by a real algebraic map – a real algebraic diffeomorphism in fact, since it coincides with $a_{s_\alpha, \lambda}$, which is known to be a diffeomorphism. In particular, the operation (9.13) makes sense for any locally defined, oriented submanifold S of T^*X, not just for Lagrangian cycles, and maps any such S diffeomorphically to the oriented submanifold $a_{s_\alpha, \lambda}(S)$, possibly with a reversal of orientation. Whether or not the orientation gets reversed depends only on the dimension of S, for continuity reasons. We now parametrize $\lambda = \lambda(s)$ linearly in s, so that $\langle \lambda(s), \check\alpha \rangle = s$, and appeal to lemma 9.4, to conclude that the two Weyl group actions coincide on any simple reflection s_α, except possibly for a sign factor. To pin down the sign, it suffices to check the effect of the reflection on any particular non-zero cycle in $\mathcal{L}^+(X)$ – for example the zero section $T^*_X X \simeq X$, oriented by the complex structure of X. This is the characteristic cycle of the constant sheaf $\mathbb{C}_X$ in degree zero, and the intertwining operator (9.7) reproduces this sheaf with a shift in degree by one. On the other hand, the diffeomorphism $a_{s_\alpha, \lambda}$, for any regular λ, maps the zero section X to itself, preserves the fibration $X \to X_\alpha$, acts as the identity on X_α and as an anti-holomorphic involution on the fiber $\mathbb{P}^1$, and thus reverses the orientation of the zero section; these properties of $a_{s_\alpha, \lambda}$ can be deduced from the description (8.9) of the twisted moment map. At this point, we have determined the sign that was still in question: both actions, applied to the simple reflection s_α, reverse the sign of $\mathrm{CC}(\mathbb{C}_X) = [X]$. The proof of theorem 9.1 is now complete, except for the verification of lemma 9.12.

Proof of lemma 9.12. We shall first reduce the problem to the special case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $X = \mathbb{P}^1 \simeq \mathbb{C} \cup \{\infty\}$, and then establish the assertion in that special case by an explicit computation.

Let us choose local coordinates (z, w, v) on $\bar{Y}$ so that z and w extend to global algebraic coordinates – with values in $\mathbb{C} \cup \{\infty\}$ – on the $\mathbb{P}^1$ fibers of the fibration (9.9), and $v = (v_j)$ is obtained by pulling back a local algebraic coordinate system from X_α. Then (z,v) and (w,v) provide local coordinate systems on the two copies of X whose product contains $\bar{Y}$. We can arrange further that

$$(9.14a) \qquad (z,v) \mapsto (w,v)\,,\ w = \frac{1}{z}\,, \quad \text{corresponds to } 1_X\,,$$

the identity map between the two factors. In terms of these coordinates,

$$(9.14b) \qquad p : (z,w,v) \mapsto (z,v)\,, \quad q : (z,w,v) \mapsto (w,v)\,.$$

The divisor $\partial Y \subset \bar{Y}$ is given by the equation $p(y) = q(y)$. Because of (9.14b),

$$(9.14c) \qquad wz \neq 1 \ \text{ on } Y\,.$$

Finally, we shall normalize the coordinates (z,v) so that they are centered at the point $x \in X$ in the statement of the lemma:

$$(9.14d) \qquad z(x) \ = \ 0\,, \quad v_j(x) \ = \ 0 \ \text{ for all } j\,.$$

Note that (z,w,v) are holomorphic coordinates, defined and finite on an open set in Y which contains all of $p^{-1}\{x\}$.

We extend the coordinate system on Y to a holomorphic coordinate system (z,w,v,a,b,c) on T^*Y by expressing a generic cotangent vector ζ as the linear combination

$$(9.15) \qquad \zeta \ = \ a\,dz + b\,dw + \sum_j c_j\,dv_j\,.$$

The diffeomorphism $(y,\zeta) \mapsto (y, \zeta + s\,d\log(y))$ of T^*Y then corresponds to the assignment

$$(9.16) \qquad \begin{aligned} &(z,w,v,a,b,c) \ \longmapsto \ (z,w,v,\tilde{a},\tilde{b},\tilde{c})\,, \quad \text{with} \\ &\tilde{a} = a + 2s\frac{\partial \log f}{\partial z}\,,\ \tilde{b} = b + 2s\frac{\partial \log f}{\partial w}\,,\ \tilde{c}_j = c_j + 2s\frac{\partial \log f}{\partial v_j}\,. \end{aligned}$$

Here we are using the $\mathbb{R}$-linear isomorphism $\zeta \mapsto \operatorname{Re}\zeta$ to identify the holomorphic tangent bundle with the real tangent bundle, and that accounts for the factors $2s$ instead of s. Note that (z,v,a,c) and (w,v,b,c) can be viewed as systems of coordinate functions on the cotangent bundles of the two factors in the ambient manifold $X \times X$ of $\bar{Y}$. With $\xi \in T^*_x X$ as in the statement of the lemma, $p^*(\{\xi\})$ is described by the complex equations

$$(9.17) \qquad z = 0\,, \quad v = 0\,, \quad a = a(\xi)\,, \quad b = 0\,, \quad c = c(\xi)\,,$$

and the algebraic hypersurface $\operatorname{Im} dq$ by the one equation

$$(9.18) \qquad a \ = \ 0\,.$$

Finding the intersection of $p^*(\{\xi\})+s\,d\,\log\,f$ and $\operatorname{Im} dq$ thus comes down to solving

$$a(\xi) + 2s\frac{\partial \log f}{\partial z} \;=\; 0 \tag{9.19}$$

for w, with $z = z(x) = 0$ and $v = v(x) = 0$. The uniqueness of the solution and the transversality of the intersection only depend on the values of f on $\{v = 0\}$, i.e., on the fiber $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$ of the fibration $Y \to X_\alpha$ which contains $p^{-1}\{x\}$. In other words, the assertions of uniqueness and tranversality in the lemma reduces to the special case of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $X = \mathbb{P}^1$.

Let us suppose then that we do have a unique intersection. Once the points $y \in p^{-1}(x)$ and $q(y) \in X$ have been found – corresponding to the value for w which solves (9.19) and $v = 0$ – the cotangent vector $\xi' \in T^*_{q(y)}X$ is determined by the equations

$$b = 2s\frac{\partial \log f}{\partial w}(y)\,, \quad c_j = c_j(\xi) + 2s\frac{\partial \log f}{\partial v_j}(y)\,. \tag{9.20}$$

The first of these again depends only on the values of f on $\{v = 0\}$. That can be paraphrased as follows. Let

$$\pi \,:\, X \;\longrightarrow\; X_\alpha \tag{9.21}$$

denote the natural fibration. As we just saw, the point $q(y)$ is contained in $\pi^{-1}(x) \cong \mathbb{P}^1$ and is determined by the $\mathfrak{sl}(2,\mathbb{C})$-analogue of (9.19). We call $T_{q(y)}\pi^{-1}(q(y)) \subset T_{q(y)}X$ the "vertical subspace" of the tangent space to X at $q(y)$; then, according to the first equality in (9.20),

(9.22) the restriction of ξ' to the vertical subspace is completely determined by the $\mathfrak{sl}(2,\mathbb{C})$-analogue of the equations (9.20).

The equations for the c_j can also be restated in geometric terms. The maximal torus in $U_\mathbb{R}$ wich fixes $q(y)$ operates on $T_{q(y)}X \cong \mathfrak{g}/\mathfrak{b}_{q(y)}$ in a multiplicity-free fashion, and hence the vertical subspace of $T_{q(y)}X$ has a unique linear complement invariant under the maximal torus. We call this the "horizontal subspace". Note that is the choice of the compact real form $U_\mathbb{R} \subset G$ which determines the splitting. Of course, no special property about the point $q(y)$ has been used in the definition, so analogous splittings exist at all points of X. Since x and $q(y)$ lie in the same fiber of π, the differential π_* maps the horizontal subspaces at both of these points isomorphically onto $T_{\pi(x)}X_\alpha$, and hence can be used to identify the two horizontal subspaces. In effect, the $c_j(\xi)$ and the $c_j(\xi')$ specify the restrictions of ξ and ξ' to the two horizontal subspaces. Via the natural identification between the two,

(9.23) the restriction of ξ to the horizontal subspace of T^*_xX agrees with the restriction of ξ' to the horizontal subspace of $T^*_{q(y)}X$;

this follows from the transitivity of the $U_\mathbb{R}$-action on X and the $U_\mathbb{R}$-equivariance of the function f.

Let us argue that the action of $a_{s_\alpha,\lambda}$ on T^*X satisfies the statements analogous to (9.22–23). We shall use the notation of section 8. In particular, $\mathfrak{t}_x$ denotes the complexified Lie algebra of the maximal torus in $U_\mathbb{R}$ which fixes x; we identify $\mathfrak{t}_x$ with the universal Cartan via

$$\mathfrak{t}_x \;\cong\; \mathfrak{b}_x/[\mathfrak{b}_x,\mathfrak{b}_x] \;\cong\; \mathfrak{h}\,. \tag{9.24}$$

Correspondingly, the abstract root system gets identified with the concrete root system of the pair $(\mathfrak{g},\mathfrak{t}_x)$. Recall that the notion of positivity was specified so that the weights of $\mathfrak{g}/\mathfrak{b}_x$ become the positive roots. Thus

$$[\mathfrak{b}_x,\mathfrak{b}_x] \;=\; \mathfrak{g}^{-\alpha}\oplus\mathfrak{r}_\alpha\,, \tag{9.25}$$

with $\mathfrak{g}^{-\alpha} = (-\alpha)$-root space of $(\mathfrak{g},\mathfrak{t}_x)$, and $\mathfrak{r}_\alpha$ equal to the span of all the other negative root spaces. The decomposition (9.25) is dual, via the Killing form, to the decomposition of $T_xX \cong \mathfrak{g}/\mathfrak{b}_x$ into the vertical and horizontal subspaces. Both summands in (9.25) are normalized by $\mathfrak{t}_x$, of course, but $\mathfrak{r}_\alpha$ is normalized even by

$$\mathfrak{s}_\alpha \;=_{\text{def}}\; \mathfrak{g}^{\alpha}\oplus\mathfrak{g}^{-\alpha}\oplus[\mathfrak{g}^{\alpha},\mathfrak{g}^{-\alpha}] \;\cong\; \mathfrak{sl}(2,\mathbb{C})\,. \tag{9.26}$$

Let $S_\alpha \subset G$ denote the connected subgroup with Lie algebra $\mathfrak{s}_\alpha$. Then S_α acts transitively on the fiber of $X \to X_\alpha$ containing the point x.

In view of (8.9), the relation $a_{s_\alpha,\lambda}(x,\xi) = (\tilde{x},\tilde{\xi})$, which expresses $\tilde{x}\in X$ and $\tilde{\xi}\in T^*_{\tilde{x}}X$ in terms of x and ξ, is equivalent to

$$\lambda_x + \mu(x,\xi) \;=\; (s_\alpha\lambda)_{\tilde{x}} + \mu(\tilde{x},\tilde{\xi})\,. \tag{9.27}$$

It will simplify matters to think of the ordinary moment map μ and the twisted moment map μ_λ as taking values in $\mathfrak{g}$ rather than $\mathfrak{g}^*$, via the identification $\mathfrak{g}^*\cong\mathfrak{g}$ effected by the Killing form:

$$\mu_\lambda\,,\ \mu\,:\,T^*X \longrightarrow \mathfrak{g}\,. \tag{9.28}$$

With this convention, the moment map induces the canonical isomorphism

$$\mu\,:\,T^*_xX \xrightarrow{\ \sim\ } [\mathfrak{b}_x,\mathfrak{b}_x] \tag{9.29}$$

dual to $T_xX\cong\mathfrak{g}/\mathfrak{b}_x$ – not just at our particular choice of x of course, but at all points in X. We now regard ξ as a vector in $[\mathfrak{b}_x,\mathfrak{b}_x]$, and λ_x as a regular element in $\mathfrak{t}_x$. We can write

$$\begin{aligned} \xi \;&=\; \xi_1+\xi_2\,, \qquad \text{with } \xi_1\in\mathfrak{g}^{-\alpha}\,,\ \xi_2\in\mathfrak{r}_\alpha\,,\\ \lambda_x \;&=\; \lambda_1+\lambda_2\,, \quad \text{with } \lambda_1\in[\mathfrak{g}^{\alpha},\mathfrak{g}^{-\alpha}]\,,\ \lambda_2\in\alpha^\perp\,, \end{aligned} \tag{9.30}$$

using (9.25) and the decomposition $\mathfrak{t}_x = [\mathfrak{g}^{\alpha},\mathfrak{g}^{-\alpha}]\oplus\alpha^\perp$. Solving the analogue of (9.27) for $\mathfrak{sl}(2,\mathbb{C})$, we find uniquely determined $\tilde{\xi}_1$, $\tilde{\lambda}_1\in\mathfrak{s}_\alpha$, and some $u\in S_\alpha\cap U_\mathbb{R}$, such that $\tilde{\xi}_1$ is nilpotent and normalized by $\tilde{\lambda}_1$, and

$$\lambda_1+\xi_1 \;=\; \tilde{\lambda}_1+\tilde{\xi}_1\,, \quad \tilde{\lambda}_1 \;=\; -(\operatorname{Ad}u)\lambda_1\,. \tag{9.31}$$

The knowledge of $\tilde{\xi}_1$, $\tilde{\lambda}_1$, u allows us to solve the equation (9.27), with

$$\tilde{x} \ = \ u\,x\,, \quad \tilde{\xi} \ = \tilde{\xi}_1 + \xi_2\,. \tag{9.32}$$

Indeed, S_α normalizes $\mathfrak{r}_\alpha$ and $\alpha^\perp \subset \mathfrak{t}_x$ centralizes $\mathfrak{s}_\alpha$; these facts imply that (9.32) does provide a solution.

The relationship between the solution (9.32) of (9.27) in the general case and the $\mathfrak{sl}(2,\mathbb{C})$ solution (9.31) has exactly the same form as the relationship between the general solution and the $\mathfrak{sl}(2,\mathbb{C})$ solution of (9.19–20). In particular, the relationship between ξ_1 and $\tilde{\xi}_1$ is analogous to (9.22), and the equality of the $\mathfrak{r}_\alpha$-components of ξ and $\tilde{\xi}$ is analogous to (9.23). We conclude that the proof of lemma 9.12 reduces to the case of $\mathfrak{sl}(2,\mathbb{C})$, as had been asserted.

Let us suppose, then, that $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $X = \mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$, $G = \mathrm{Sl}(2,\mathbb{C})$, and $U_\mathbb{R} = \mathrm{SU}(2)$. Strictly speaking, we ought to work with the adjoint groups, to be consistent with our earlier definition of $G = \mathrm{Aut}(\mathfrak{g})^0$, but this distinction is of no consequence. We use coordinates as in (9.14–15), except that now there are no v_j and no c_j. In our present situation,

$$f(z,w) \ = \ \frac{|zw-1|^2}{(1+|z|^2)(1+|w|^2)}\ , \tag{9.33}$$

since this function is SU(2)-invariant and reduces to $|z|^2/(1+|z|^2)$ when $w=\infty$. At $z=0$, which corresponds to x, the equation (9.19) becomes $2sw=a$. In particular, this describes a transverse intersection at a single point. The first, non-vacuous equation in (9.20) becomes $b = -2s\bar{w}/(1+|w|^2)$. Rewriting this information in terms of the usual coordinate $z=1/w$, we find:

$$(0, a\,dz) \ \longmapsto \ \left(\frac{2s}{a}\,,\ \frac{a|a|^2}{|a|^2+4s^2}\,dz \right) \tag{9.34}$$

describes the assignment $(x,\xi) \mapsto (q(y),\xi')$ of lemma 9.12.

Still in the case of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, we identify $\mathfrak{h}^* \cong \mathbb{C}$ so that the positive weights correspond to the positive integers, and we normalize the Killing form so that it agrees with the trace form of the standard representation. Then, with the convention (9.28), for $z \in \mathbb{P}^1 \cong \mathbb{C}$,

$$\lambda_z \ = \ \frac{\lambda}{1+|z|^2} \begin{pmatrix} 1-|z|^2 & -2z \\ -2\bar{z} & |z|^2-1 \end{pmatrix}\ . \tag{9.35}$$

To see this, note that the right hand side is purely imaginary with respect to the compact real form SU(2) whenever λ is real, that it depends SU(2)-equivariantly on z, and that for $z=0$, it fixes $0 \in \mathbb{P}^1$ under the infinitesimal action. The preceding conditions determine λ_z up to a real scaling factor; that factor can be pinned down as follows: the identity matrix corresponds to the weight 2 via the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, and it acts – via infinitesimal translation – as multiplication by 2 on the tangent space $T_0\mathbb{P}^1 \cong \mathfrak{g}/\mathfrak{b}_0 \cong \mathfrak{g}^\alpha$; the positive root α, finally, corresponds to the weight 2. We argue similarly to identify the ordinary moment map as

$$\mu(z,dz) \ = \ \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}\ . \tag{9.36}$$

First of all, the formula has the right equivariance properties, namely equivariance with respect to all of $\mathrm{Sl}(2,\mathbb{C})$, and secondly, it takes the value at $z = 0$ which corresponds to dz under the isomorphism $T_0^*\mathbb{P}^1 \cong [\mathfrak{b}_0, \mathfrak{b}_0]$ dual to $T_0\mathbb{P}^1 \cong \mathfrak{g}/\mathfrak{b}_0$. Combining (9.35–36), we find

$$\mu_\lambda(z, a\,dz) \;=\; \frac{\lambda}{1+|z|^2}\begin{pmatrix} 1-|z|^2 & -2z \\ -2\bar{z} & |z|^2-1 \end{pmatrix} \;+\; a\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}. \tag{9.37}$$

The equation $a_{s_\alpha,\lambda}(0, a\,dz) = (z, c\,dz)$ is equivalent to $\mu_\lambda(0, a\,dz) = \mu_{-\lambda}(z, c\,dz)$, hence to the matrix equation

$$\begin{pmatrix} \lambda & 0 \\ a & \lambda \end{pmatrix} \;=\; \frac{-\lambda}{1+|z|^2}\begin{pmatrix} 1-|z|^2 & -2z \\ -2\bar{z} & |z|^2-1 \end{pmatrix} \;+\; c\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}. \tag{9.38}$$

Simple algebraic manipulations now give the identities $za = 2\lambda$ and $cz(1+|z|^2) = 2\lambda$, and finally

$$a_{s_\alpha,\lambda}(0, a\,dz) \;=\; \left(\frac{2\lambda}{a}\,,\; \frac{a|a|^2}{|a|^2+4|\lambda|^2}\,dz \right). \tag{9.39}$$

This agrees with (9.34) if $\lambda = s \in \mathbb{R}^+$, completing the proof of the lemma.

10. Generalizations.

The characteristic cycle construction depends crucially on the constructibility of the sheaves in question. As soon as a sheaf $\mathcal{F}$ is constructible with respect to some Whitney stratification $\mathcal{S}$, Kashiwara's construction [K1,KS] gives meaning to the characteristic cycle $\mathrm{CC}(\mathcal{F})$, at least as a local cohomology class. If $\mathrm{CC}(\mathcal{F})$ is to be a true cycle in the geometric sense, the Whitney conditions on $\mathcal{S}$ are not enough. This is the reason why Kashiwara, Kashiwara-Schapira – and also we in this paper so far – require constructibility with respect to a stratification $\mathcal{S}$ which is at least subanalytic. The point is that subanalytic stratifications have good hereditary properties; in particular, the conormal bundle of each stratum is again subanalytic. As a consequence, the union of the conormal bundles is also Whitney stratifiable. This latter fact alone ensures that $\mathrm{CC}(\mathcal{F})$ is a geometric cycle.

Recent results of Wilkie [W] and van den Dries-Macintyre-Marker [DMM] can be used to construct geometric categories which are larger than the subanalytic category, but have similar hereditary properties. The characteristic cycle construction, and in particular the results of sections 2-6 work equally in this more general setting. That is important for us: one of our representation theoretic applications [SV2] forces us to go outside the subanalytic setting. Specifically we need a version of theorem 6.10 in one of the larger geometric categories. The purpose of this section is to describe various such extensions of our results on characteristic cycles.

The two papers [DMM,W] are model theoretic in character, and do not contain the general geometric statements which they imply. The transition from the model theoretic results to geometry is made by van den Dries-Miller [DM]. Their axiomatic approach not only provides general statements, but is also remarkably efficient – indeed, even when specialized to the subanalytic setting, this approach can be helpful.

Let us briefly recall the axioms of van den Dries-Miller and some of the important consequences. An analytic-geometric category $\mathcal{C}$ in the sense of [DM] is specified by the datum of a collection of subsets $\mathcal{C}(X)$ – the $\mathcal{C}$-subsets of X – for each real analytic manifold X, subject to the following axioms:

(10.1)
a) $\mathcal{C}(X)$ is a Boolean algebra of subsets of X;
b) if $S \in \mathcal{C}(X)$, then $\mathbb{R} \times S \in \mathcal{C}(\mathbb{R} \times X)$;
c) if $S \in \mathcal{C}(X)$ and if $F : X \to Y$ is a proper real analytic map between real analytic manifolds, then $F(S) \in \mathcal{C}(Y)$;
d) the property of being a $\mathcal{C}$-subset of an ambient manifold X is a local property with respect to the ambient manifold;
e) the boundary of any bounded $\mathcal{C}$-subset of $\mathbb{R}$ is finite.

The objects in the category $\mathcal{C}$ are precisely the pairs (S, X) with $S \in \mathcal{C}(X)$; the morphisms from (S, X) to (T, Y) are precisely the continuous maps $F : S \to T$ whose graphs are $\mathcal{C}$-subsets of $X \times Y$. Like [DM], we shall usually omit reference to the ambient manifold of a $\mathcal{C}$-set. The axioms (10.1) imply that $\mathcal{C}$, as defined above, is indeed a category. All subanalytic subsets of X are automatically $\mathcal{C}$-subsets. Conversely, the subanalytic sets form an analytic-geometric category – the smallest one.

Let us briefly mention another example – the analytic-geometric category which will be important in future applications of our results. In the language of [DM], it is the category $\mathcal{C}$ corresponding to the the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. Loosely speaking, it contains sets definable by expressions in real analytic functions, exponentials and logarithms. Specifically, and more precisely, is contains at least all sets of the following sort. Let $P(X_1, \ldots, X_n)$ be a polynomial in the variables $X_1, \ldots, X_n$, with coefficients which are real analytic functions on $[0, 1]^n$. Then

$$\{(x_1, \ldots, x_n) \mid 0 < x_i \leq 1,\ P(\log x_1, \ldots, \log x_n) = 0\} \tag{10.2}$$

is a $\mathcal{C}$-subset of $\mathbb{R}^n$ (even at the origin!). It is precisely sets of this type which come up in our representation theoretic applications.

As is argued in [DM], the axioms (10.1) have numerous consequences – some fairly obviously, and some not so obviously. We shall list only those we need. To begin with, the closure, interior, and boundary of any $\mathcal{C}$-set are again $\mathcal{C}$-sets. Also, finite products of $\mathcal{C}$-sets are $\mathcal{C}$-sets, and any map

$$F = (F_1, \ldots, F_n) : S \to T_1 \times \cdots \times T_n,$$

where S and the T_i are $\mathcal{C}$-sets, is a $\mathcal{C}$-map if and only if all the F_i are $\mathcal{C}$-maps. The inverse image of a $\mathcal{C}$-subset T of Y under a $\mathcal{C}$-map $F : S \to Y$ is a $\mathcal{C}$-set, provided S is closed in its ambient manifold. One can take derivatives in the category $\mathcal{C}$, in the following sense:

(10.3) the derivative of a $\mathcal{C}$-map, if it exists and is continous, is a $\mathcal{C}$-map.

Now let $S \subset X$ be a $\mathcal{C}$-set, and k a positive integer; then

(10.4) the points at which S is a C^k-submanifold of X constitute a $\mathcal{C}$-set.

Let us mention, parenthetically, that this statement fails for $k = \infty$. Conormal bundles behave well with respect to the notion of $\mathcal{C}$-set:

(10.5) if $S \subset X$ is both a $\mathcal{C}$-subset and a (locally closed) C^k-submanifold, then $T^*_S X$ is a $\mathcal{C}$-subset of T^*X.

Here again, k is an arbitrary finite, positive integer.

Most crucially for our purposes, Whitney stratifications exist in the setting of analytic-geometric categories. We fix a closed $\mathcal{C}$-subset $S \subset X$, a locally finite subcollection $\mathcal{A} \subset \mathcal{C}(X)$, and an integer k, $1 \leq k < \infty$. Then

(10.6a) there exists a C^k-Whitney stratification of S, compatible with $\mathcal{A}$, whose strata are $\mathcal{C}$-subsets of S;

here compatibility means that $A \cap S$, for any $A \in \mathcal{A}$, is a union of strata. With $S \subset X$, $\mathcal{A}$, and k as before, any proper $\mathcal{C}$-map $F : S \to Y$ into a real analytic manifold Y can be C^k-Whitney stratified compatibly with $\mathcal{A}$ and any given locally finite subcollection $\mathcal{B} \subset \mathcal{C}(Y)$. More precisely,

(10.6b) there exists a C^k-Whitney stratifications of S and Y, compatible with $\mathcal{A}$ and $\mathcal{B}$ respectively, whose strata are $\mathcal{C}$-subsets.

The axioms (10.1) and the consequences mentioned so far are enough to define the characteristic cycle $\mathrm{CC}(\mathcal{F})$ for any $\mathcal{F} \in \mathrm{D}^b(X)$ which is constructible with respect to a $\mathcal{C}$-stratification, as a Lagrangian cycle in T^*X subordinate to a $\mathcal{C}$-stratification of T^*X. Indeed, the definition in [KS] does not depend on any particular properties of the subanalytic category beyond those mentioned so far in this section. Our own discussion in section 2 also applies without change, even though we now need to work with C^k-stratifications, with k as large as needed but finite, not $k = \infty$ as in [GM]. Formally, the distinction between finite and infinite k comes up only twice: in the construction of the Morse function ϕ in (2.4), and the existence of the metric with properties (2.6). In the first instance,

(10.7) there exist $\mathcal{C}$-functions ϕ satisfying the conditions (2.4),

even real analytic functions, as can be verified by a calculation in local coordinates. As for (2.6), the existence proof in [GM] is phrased in terms of C^∞ stratifications, but uses only finitely many derivatives, and thus applies also in our situation. Alternatively, the Euclidean metric with respect to any particular local coordinate system will do, provided ϕ is a $\mathcal{C}$-function: one can argue as in [GM, page 82], using the category $\mathcal{C}$ instead of the subanalytic category, and appealing to the finiteness axiom (10.1e).

Most of section 3 – from the beginning through proposition 3.25 – involves general Whitney stratifications. The subanalytic setting comes into play only in lemma 3.26 and proposition 3.27. It appears likely that lemma 3.26 has a counterpart in the setting of an arbitrary analytic-geometric category $\mathcal{C}$. In any case, in concrete applications of proposition 3.27 the existence of a $\mathcal{C}$-family with the properties asserted in 3.26 is usually clear. Once such a family exists, one can argue precisely as in section 3. Let then $C_0 \in \mathcal{L}^+(X)$ be a $\mathcal{C}$-cycle, $f : X \to Y$ a real analytic map and C_I a $\mathcal{C}$-family of cycles satisfying (3.26a,b). Then

$$f_*(C_0) = \lim_{s\to 0^+} \tau_*(df^{-1}(C_s))\,, \tag{10.8}$$

just as in the subanalytic context.

The open embedding theorem 4.2 also carries over. Let $j : U \hookrightarrow X$ be an embedding of an open $\mathcal{C}$-subset into a real analytic manifold X, and f a $\mathcal{C}$-function of class C^1 as in (4.1). Such a function always exists: the argument of Bierstone-Milman-Pawlucki still applies [DM]. Next, suppose that $\mathcal{F} \in \mathrm{D}^b(U)$ is constructible with respect to a $\mathcal{C}$-stratification of the pair $(X, \partial U)$. Then $Rj_*\mathcal{F}$ and $Rj_!\mathcal{F}$ also have $\mathcal{C}$-constructible cohomology, so $\mathrm{CC}(\mathcal{F})$, $\mathrm{CC}(Rj_*\mathcal{F})$, and $\mathrm{CC}(Rj_!\mathcal{F})$ are well-defined Lagrangian $\mathcal{C}$-cycles. In this situation, $\mathrm{CC}(\mathcal{F}) + s\frac{df}{f}$ and $\mathrm{CC}(\mathcal{F}) - s\frac{df}{f}$ are $\mathcal{C}$-families of cycles; we can argue exactly as in section 4, at each point replacing properties of the subanalytic category by their $\mathcal{C}$-analogues. We then get our $\mathcal{C}$-version of the open embedding theorem:

$$\mathrm{CC}(Rj_*\mathcal{F}) = \lim_{s\to 0+}(\mathrm{CC}(\mathcal{F}) + s\frac{df}{f})\,, \quad \mathrm{CC}(Rj_!\mathcal{F}) = \lim_{s\to 0+}(\mathrm{CC}(\mathcal{F}) - s\frac{df}{f})\,. \tag{10.9}$$

There are only two instances in the proof of 4.2, including the proof of the auxiliary proposition 4.14, where statements or arguments need to be modified beyond the simple substitution of the $\mathcal{C}$-counterparts for all subanalytic properties, assertions, etc. The first is entirely superficial: in (4.6c), we should replace "smooth" by "C^k, with k large". The second instance occurs in the verification of (5.5), where we must now argue differently, as we shall explain next.

Let us summarize what we need to show. Considerations preceding (5.5) produce a closed, one dimensional $\mathcal{C}$-set γ in an ambient manifold – specifically $\mathbb{R} \times T^*X$, but that will not matter – and $\mathcal{C}$-functions f, s, ϕ, defined in some neighborhood of a point $p \in \gamma$ such that

$$\begin{aligned} &\text{a)}\ \ f, s, \phi \text{ are at least } C^1; \\ &\text{b)}\ \ f(p) = s(p) = \phi(p) = 0\,; \\ &\text{c)}\ \ 0 < f < 1 \ \text{ and } \ s > 0 \ \text{ on } \ \gamma - \{p\}\,; \\ &\text{d)}\ \ d\phi = s\, d\log f \ \text{ on the } C^1 \text{ points of } \ \gamma - \{p\}\,. \end{aligned} \tag{10.10}$$

Under these hypotheses, we must show

$$(s\, \log f)(q) \to 0, \ \text{ as } \ q \to p \ \text{ on } \ \gamma\,. \tag{10.11}$$

We need to use the fact that one dimensional, closed $\mathcal{C}$-sets are very special:

$$\text{(10.12)} \qquad \begin{aligned} &\text{locally near } p,\ \gamma \text{ is a finite union of branches,} \\ &\text{each of which posses a } C^k \text{ parametrization by } \mathcal{C}\text{-functions}. \end{aligned}$$

We may as well replace γ by one of these branches, and choose a C^k, $\mathcal{C}$-parameter t with p and $\gamma - \{p\}$ corresponding to, respectively, $t = 0$ and $t > 0$. Let us argue by contradiction, and consider the following two alternatives:

$$\text{(10.13)} \qquad \begin{aligned} &\text{a)}\ \ s \log f \text{ is bounded away from } 0\,; \\ &\text{b)}\ \ s \log f \text{ is not bounded away from } 0\,. \end{aligned}$$

In the situation a),

$$\frac{d}{dt}(\log(s\ \log \frac{1}{f})) = \frac{1}{s}\frac{ds}{dt} + \frac{1}{s\ \log f}\frac{d\phi}{dt}$$

differs from $\frac{1}{s}\frac{ds}{dt}$ by a bounded term, which implies $s\ \log \frac{1}{f} \sim cs$, which is incompatible with a). In the situation b), if (10.11) fails, there must exist a sequence $\{t_n\}$ converging to 0, along which the derivative of $s \log f$ tends to $+\infty$. But

$$\frac{d}{dt}(s\ \log\ f) = \frac{d\phi}{dt} + \log\ f\,\frac{ds}{dt}$$

differs from the bounded term $\frac{d\phi}{dt}$ by a negative quantity – cf. (10.10b,c). Contradiction!

This completes the verification of the open embedding theorem in our more general setting. In section 6 we had limited ourselves to the real algebraic situation to simplify the hypotheses of our statements; and had indicated that these statements, with appropriate restrictions, remain valid also in the subanalytic case. Here, too, we may just as well work in a general analytic-geometric category.

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