

CHARACTER FORMULAS AND LOCALIZATION OF INTEGRALS

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In these notes, I shall describe two character formulas for semisimple Lie groups. Both are of interest by themselves, but the potential connections between the two formulas raise some intriguing questions. The formulas represent joint work with Kari Vilonen; full details will appear elsewhere.

The formulas and their relation are well understood in the case of a compact group. For motivation, I shall start out with a discussion of the compact case. Thus I consider K , a connected, compact Lie group, and $T \subset K$, a maximal torus. I write $\mathfrak{k}_{\mathbb{R}}$, $\mathfrak{t}_{\mathbb{R}}$ for the Lie algebras of K , T , and $\mathfrak{k}$, $\mathfrak{t}$ for their complexified Lie algebras.

Let π be an irreducible unitary representation of K . Because of the compactness of K , π must be finite dimensional. The *Weyl character formula* describes the character Θ_{π} as a function of T , and thus – since the maximal torus meets every conjugacy class – globally as function on all of K . To recall the formula, I use the exponential map to identify the torus T with the quotient of its Lie algebra by the *unit lattice*,

$$(1a) \quad T \simeq \mathfrak{t}_{\mathbb{R}}/L, \quad \text{with } L = \text{unit lattice} =_{\text{def}} \{x \in \mathfrak{t}_{\mathbb{R}} \mid \exp(x) = \text{id}\}.$$

Dually, the character group $\widehat{T}$ is isomorphic to the *weight lattice*,

$$(1b) \quad \widehat{T} \simeq \Lambda = \text{weight lattice} =_{\text{def}} \{\lambda \in i\mathfrak{t}_{\mathbb{R}}^* \mid \langle \lambda, L \rangle \subset 2\pi i\mathbb{Z}\}.$$

Under this isomorphism, $\lambda \in \Lambda$ corresponds to $e^{\lambda} \in \widehat{T}$, where $e^{\lambda}(\exp x) = e^{\langle \lambda, x \rangle}$ for $x \in \mathfrak{t}_{\mathbb{R}}$. The *Weyl group* $W = N_K(T)/T$ operates on $i\mathfrak{t}_{\mathbb{R}}^*$. This action preserves a positive definite inner product $(\cdot, \cdot)$ on $i\mathfrak{t}_{\mathbb{R}}^*$. It also preserves the weight lattice and the *root system* Φ , i.e., the set of non-zero $\alpha \in \Lambda$ such that there exist $x \neq 0$ in $\mathfrak{k}$ on which T acts according to the character e^{α} . The Weyl character formula can be stated as follows:

$$(2) \quad \Theta_{\pi}|_T = \sum_{w \in W} \frac{e^{w\lambda}}{\prod_{\alpha \in \Phi, (w(\lambda+\rho), \alpha) > 0} (1 - e^{-\alpha})}.$$

Here $\lambda = \lambda(\pi)$ is a particular element of the weight lattice, the *highest weight* of the representation π , and ρ a certain small vector in $i\mathfrak{t}_{\mathbb{R}}^*$, independent of π , which will be described presently.

The *root hyperplanes* $\{\mu \in i\mathfrak{t}_{\mathbb{R}}^* \mid (\alpha, \mu) = 0\}$, $\alpha \in \Phi$, divide $i\mathfrak{t}_{\mathbb{R}}^*$ into a number of convex cones, the *Weyl chambers*, which are permuted simply transitively by W .

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Customarily, one singles out one of the Weyl chambers and calls it the *dominant chamber*. When K happens to be simply connected, ρ can be characterized as the shortest element of Λ lying in the interior of the dominant Weyl chamber. In general, the definition of ρ can be reduced to the case of a simply connected group: some finite covering of any connected, compact Lie group splits into the product of a torus and a simply connected compact group. To complete the statement of the Weyl character formula, one only needs to add that an element λ of the weight lattice Λ arises as highest weight of an irreducible representation precisely when it is dominant, i.e., when it lies in the (closed) dominant Weyl chamber.

The Weyl character formula (2) is a beautiful, explicit formula – in some sense, one could hardly ask for more. Still, it has inspired attempts to explain it, or at least interpret it, in geometric terms.

The Atiyah-Bott fixed point formula [1] provides one such geometric interpretation. It depends on the Borel-Weil theorem [4,17], whose statement I need to recall. I fix a particular dominant weight λ . The splitting $\mathfrak{k}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \oplus [\mathfrak{t}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}]$ induces a dual splitting of the vector space $i\mathfrak{k}_{\mathbb{R}}^*$, which allows me to think of λ and ρ as lying in $i\mathfrak{k}_{\mathbb{R}}^*$, and to define

$$(3) \quad \Omega_{\lambda+\rho} = K\text{-orbit of } \lambda + \rho \text{ in } i\mathfrak{k}_{\mathbb{R}}^*.$$

As homogeneous space, $\Omega_{\lambda+\rho}$ is isomorphic to the quotient space K/T , and thus to the *flag variety* of $\mathfrak{k}$. Since λ exponentiates to a character e^λ of the isotropy subgroup T of K at $\lambda + \rho$, it determines a K -equivariant line bundle

$$(4) \quad \mathcal{L}_\lambda \longrightarrow \Omega_{\lambda+\rho}.$$

Except for the multiplication by $i = \sqrt{-1}$, $\Omega_{\lambda+\rho}$ is a coadjoint orbit of K – an orbit in the dual of the Lie algebra; as such, it carries a canonical symplectic structure. An *complex polarization* for $\Omega_{\lambda+\rho}$ is a K -invariant pseudo-Kähler structure compatible with the intrinsic symplectic structure. Complex polarizations exist; they correspond naturally and bijectively to the Weyl chambers. Once a polarization is chosen, the line bundle $\mathcal{L}_\lambda$ has a unique structure of K -equivariant *holomorphic* line bundle. The polarization corresponding to the dominant Weyl chamber is the unique *positive polarization* – positive in the sense that the pseudo-Kähler structure is positive definite, i.e., a true Kähler structure.

Equip $\Omega_{\lambda+\rho}$ with the positive polarization, and let $\mathcal{O}(\mathcal{L}_\lambda)$ denote the sheaf of holomorphic sections of the line bundle $\mathcal{L}_\lambda$. Since the action of K preserves the complex structures of the space $\Omega_{\lambda+\rho}$ and of the line bundle $\mathcal{L}_\lambda$, it induces an action also on the cohomology groups of the sheaf $\mathcal{O}(\mathcal{L}_\lambda)$. The Borel-Weil theorem describes the resulting representations of K :

$$(5) \quad \begin{aligned} H^p(\Omega_{\lambda+\rho}, \mathcal{O}(\mathcal{L}_\lambda)) &= 0 \quad \text{if } p \neq 0, \\ H^0(\Omega_{\lambda+\rho}, \mathcal{O}(\mathcal{L}_\lambda)) &\text{ is non-zero, irreducible, and has highest weight } \lambda, \end{aligned}$$

provided λ is dominant and the polarization positive, as has been assumed. Bott's generalized Borel-Weil theorem [4,11] describes the cohomology groups attached to an arbitrary invariant polarization and every $\lambda \in \Lambda$; the precise statement need not concern us here.

The maximal torus T operates on $\Omega_{\lambda+\rho}$ with isolated fixed points, namely the various W -translates of the base point $\lambda + \rho$. The Atiyah-Bott fixed point formula, applied to the T -action on $\Omega_{\lambda+\rho}$ and the holomorphic line bundle $\mathcal{L}_\lambda$, expresses the alternating sum of the T -characters of the cohomology groups $H^p(\Omega_{\lambda+\rho}, \mathcal{O}(\mathcal{L}_\lambda))$ as a sum of terms corresponding to the fixed points; the term corresponding to any one of the fixed points $w(\lambda + \rho)$, $w \in W$, turns out to be the summand in the formula (2) indexed by w . In this way, the Weyl character formula becomes a formal consequence of the fixed point formula, as was observed already by Atiyah and Bott.

Kirillov's formula provides a totally different expression for the irreducible characters of K . By definition,

$$(6) \quad \theta_\pi = (\det \exp_*)^{1/2} \exp^* \Theta_\pi$$

is the *character on the Lie algebra* of the irreducible representation π . Except for the factor $(\det \exp_*)^{1/2}$ – which has the effect of relating invariant differential operators on the group and on the Lie algebra – this is the pull-back of the global character Θ_π to the Lie algebra. It is a smooth, bounded function, but is not square integrable. The exponential map of the connected, compact Lie group K is surjective and generically non-singular, so θ_π completely determines Θ_π . It will be convenient to define the Fourier transform $\hat{\phi}$ of a test function $\phi \in C_c^\infty(\mathfrak{k}_\mathbb{R})$ without the customary factor of $i = \sqrt{-1}$ in the exponent, as a function on $i\mathfrak{k}_\mathbb{R}^*$:

$$(7) \quad \hat{\phi}(\zeta) = \int_{\mathfrak{k}_\mathbb{R}} \phi(x) e^{\langle \zeta, x \rangle} dx.$$

The Fourier transform of θ_π exists as distribution, in the weak sense; with the convention (7), it is given by the relation

$$(8) \quad \hat{\theta}_\pi = \text{integration over } \Omega_{\lambda+\rho}$$

(integration relative to the measure induced by the canonical symplectic structure). Kirillov calls this the “universal formula” for irreducible characters [10]; he established it not only in the compact case, but more importantly, in the nilpotent case where he used it to describe the unitary dual.

One may picture the Weyl character formula, its interpretation in terms of fixed points, and Kirillov's formula for $\hat{\theta}_\pi$ as the vertices of a triangle. One side of the triangle – the connection between the Weyl character formula and its fixed point interpretation – is filled in by the Atiyah-Bott fixed point formula. Kirillov's proof of (8) provides the second side, by direct computation. Berline-Vergne posed, and later answered the question of whether one can fill in the third side of the triangle: an identification of the right hand side of (8) with the sum of terms attached to the fixed points of the action of T on $\Omega_{\lambda+\rho}$, without the “detour” via the Weyl character formula.

The argument of Berline-Vergne involves the K -equivariant cohomology groups $H_K^*(\Omega_{\lambda+\rho}, \mathbb{C})$. These were defined, for any manifold with a differentiable K -action,

by Cartan, who also expressed this cohomology in terms of a complex of differentiable forms,

$$(9a) \quad H_K^*(\Omega_{\lambda+\rho}, \mathbb{C}) \simeq H^*(\{\mathcal{A}(\Omega_{\lambda+\rho}) \otimes S(\mathfrak{k}^*)\}^K);$$

here $\mathcal{A}(\Omega_{\lambda+\rho})$ denotes the deRham complex, $S(\mathfrak{k}^*)$ the symmetric algebra of $\mathfrak{k}^*$, and $\{\dots\}^K$ the space of K -invariants, relative to the translation action on $\mathcal{A}(\Omega_{\lambda+\rho})$ and the conjugation action on $S(\mathfrak{k}^*)$. The differential

$$(9b) \quad d : \{\mathcal{A}(\Omega_{\lambda+\rho}) \otimes S(\mathfrak{k}^*)\}^K \longrightarrow \{\mathcal{A}(\Omega_{\lambda+\rho}) \otimes S(\mathfrak{k}^*)\}^K$$

is the sum of the deRham differential and a term involving the infinitesimal action of $\mathfrak{k}$ as a Lie algebra of vector fields and the multiplication map $\mathfrak{k}^* \otimes S(\mathfrak{k}^*) \rightarrow S(\mathfrak{k}^*)$. It should be remarked that the equivariant cohomology can be defined whether or not K is compact. The description (9), on the other hand, depends crucially on the compactness of K . In the case of a one point space,

$$(10) \quad H_K^*(\{\text{pt}\}, \mathbb{C}) \simeq \{S(\mathfrak{k}^*)\}^K,$$

since the differential reduces to zero. The analogue of (10) for the maximal torus T simplifies because T is abelian: $H_T^*(\{\text{pt}\}, \mathbb{C}) \simeq S(\mathfrak{t}^*)$.

Integration of the deRham component over the space $\Omega_{\lambda+\rho}$ and restriction from K to T induce a sequence of morphisms

$$(11) \quad H_K^*(\Omega_{\lambda+\rho}, \mathbb{C}) \xrightarrow{\int} H_K^*(\{\text{pt}\}, \mathbb{C}) \xrightarrow{\text{res}_T^K} H_T^*(\{\text{pt}\}, \mathbb{C}) \simeq S(\mathfrak{t}^*).$$

Just as in the absolute case, it is possible to define a K -equivariant index of the line bundle $\mathcal{L}_\lambda$ and to prove an equivariant Riemann-Roch theorem. However, in the equivariant setting, the index formally involves cohomology classes in all degrees, so its image in the equivariant cohomology of a point lies in the completion of $\{S(\mathfrak{k}^*)\}^K$, i.e., in the space of K -invariant formal power series on $\mathfrak{k}$. Berline-Vergne [2] identify the image of the equivariant index in the completion of $\{S(\mathfrak{k}^*)\}^K$ with θ_π and deduce the Kirillov's formula. The crux of the matter is the process of localization – the expression of the composite of the homomorphisms (11) in terms of contributions of the various fixed points of the action of T .

Berline-Vergne localization applies to any Hamiltonian torus action, with isolated fixed points, on a compact symplectic manifold. Their ideas have been generalized by Witten [19] and Jeffrey-Kirwan [5], who describe the image of equivariant cohomology classes on the Marsden-Weinstein reduction when it exists.

Let me turn to the noncompact case. I consider a connected, linear, semisimple Lie group G and a maximal compact subgroup $K \subset G$. Any two maximal compact subgroups are conjugate, so the particular choice of K does not matter. By a *representation* of G , I shall mean a continuous representation on a complete, locally convex Hausdorff topological vector space. A representation π is said to have *finite length* if every increasing chain of closed, invariant subspaces breaks off after finitely many steps. One calls π *admissible* if its restriction to K contains any irreducible representation of K at most finitely often. Admissibility is automatic for irreducible

unitary representations. Loosely speaking, the admissible representations of finite length constitute the smallest universe of representations closed under all standard constructions, containing at least the irreducible unitary representations. They are the natural objects of study: on the one hand, admissible representations of finite length provide the most appropriate context for harmonic analysis on G and its quotient spaces; on the other, inadmissible irreducible representations are truly exotic – the first explicit example is relatively recent [18].

An admissible representation of finite length π has a character Θ_π , not in the naive sense since π will not be finite dimensional in general, but in a suitable weak sense: it is a conjugation invariant distribution on G , and the distributions $D \Theta_\pi$, with D ranging over the space of bi-invariant linear differential operators, span a finite dimensional space. An irreducible admissible representation π is determined by its character up to *infinitesimal equivalence*, in other words, up to isomorphism of the underlying infinitesimal representation. For irreducible unitary representations, infinitesimal equivalence forces unitary equivalence, so the character of an irreducible unitary representation π completely determines π as unitary representation. These facts are all due to Harish-Chandra, as is the *regularity theorem*. This theorem asserts that the character Θ_π of any admissible π of finite length is (integration against) a locally L^1 function, which is even real analytic on G_{rs} , the set of regular semisimple elements in G . The complement of G_{rs} is a proper real analytic subvariety, so one may think of Θ_π as a “real meromorphic function”, with singularities only along the complement of G_{rs} . Every $g \in G_{\text{rs}}$ lies in a (unique) Cartan subgroup, of which there are only finitely many conjugacy classes. In particular, Θ_π is fully determined by its restriction to finitely many Cartan subgroups, one from each conjugacy class.

The Weyl character formula, its interpretation in terms of fixed points, and Kirillov’s formula all have analogues in the present setting. It will simplify the discussion to restrict attention to characters Θ_π of representations with *trivial infinitesimal character*, meaning representations whose character is annihilated by all bi-invariant linear differential operators with zero constant term. All interesting phenomena show up already in this special case. Everything that will be said carries over to the case of an arbitrary admissible representation of finite length, though the statements and the proofs become somewhat more complicated.

Thus I consider the character Θ_π of a representation π with trivial infinitesimal character, a Cartan subgroup $H \subset G$, and a connected component C of $H_{\text{rs}} = H \cap G_{\text{rs}}$. Since G was assumed to be linear, it has a complexification $G_{\mathbb{C}}$, and H has a complexification $H_{\mathbb{C}}$ which is a Cartan subgroup of the complex semisimple group $G_{\mathbb{C}}$. The quotient $W_{\mathbb{C}} = N_{G_{\mathbb{C}}}(H)/H_{\mathbb{C}}$ is the *complex Weyl group* associated to H . According to results of Harish-Chandra, there exist integers $n_w = n_w(\pi, C)$ indexed by $w \in W_{\mathbb{C}}$, such that

$$(12) \quad \Theta_\pi|_C = \sum_{w \in W_{\mathbb{C}}} \frac{n_w}{\prod_{\alpha \in \Phi, (w\rho, \alpha) > 0} (1 - e^{-\alpha})};$$

in this formula, Φ and ρ are defined analogously to the corresponding quantities in the compact case. When $G = K$ is compact, “Cartan subgroup” is synonymous with “maximal torus”, the complex Weyl group coincides with the usual Weyl

group, and the only irreducible representation with trivial infinitesimal character is the trivial representation which has highest weight $\lambda = 0$. Thus, in this very special case, the Harish-Chandra character formula (12) reduces to the Weyl character formula (2), with $n_w = 1$ for all $w \in W_{\mathbb{C}}$ and all connected components $C \subset H_{\text{rs}}$. In distinction to the compact case, there exists no simple, completely explicit formula for the n_w in the general.

The description of the n_w in terms of a fixed point formalism depends on a geometric realization of the representation π – in effect, a generalization of the Borel-Weil-Bott theorem. I write $\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}$ for the Lie algebras of G, H, K , and $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ for the complexified Lie algebras. The *flag variety* X of $\mathfrak{g}$ is a complex projective variety with a transitive action of $G_{\mathbb{C}}$, and is universal with respect to these properties: any other complex projective variety Y with transitive $G_{\mathbb{C}}$ -action is an equivariant quotient of X . As homogeneous space, the flag variety can be identified with $G_{\mathbb{C}}/B$, the quotient of $G_{\mathbb{C}}$ by a *Borel subgroup* B , i.e., by a maximal solvable subgroup. When B contains $H_{\mathbb{C}}$, as can be arranged, the identity coset is a particular fixed points for the action of H on X , and the complex Weyl group $W_{\mathbb{C}}$ acts simply transitively on the set of all such fixed points. In the following, I shall view X as a complex manifold, rather than complex algebraic variety, and write $\mathcal{O}$ for the sheaf of holomorphic functions on X .

As subgroup of $G_{\mathbb{C}}$, G acts on X ; this action has a finite number of orbits. Let $\mathcal{F}$ be a *G-equivariant sheaf* of finite dimensional complex vector spaces, for example, the direct image of a G -equivariant local system on a G -orbit $S \subset X$ via the embedding $S \hookrightarrow X$. More generally, I want to allow any finite extension of direct images of G -equivariant local systems – technically speaking, any element $\mathcal{F}$ of the *G-equivariant derived category* $D_G(X)$ defined by Bernstein and Lunts [3]. Then, for every $p \in \mathbb{Z}$,

$$(13) \quad \begin{aligned} \text{Ext}^p(\mathcal{F}, \mathcal{O}) &\text{ carries a natural Fréchet topology and a natural continuous} \\ &G\text{-action; the resulting representation is admissible, of finite length.} \end{aligned}$$

This statement was conjectured by Kashiwara [7], and was subsequently proved by Kashiwara and myself [9]. The proof of (13) also shows that every admissible, finite length representation π , with trivial infinitesimal character, is infinitesimally equivalent to the G -module $\text{Ext}^0(\mathcal{F}, \mathcal{O})$ for some $\mathcal{F} \in D_G(X)$, all of whose other Ext-groups vanish. When $G = K$ is compact, with $\mathcal{F}$ equal to the constant sheaf $\mathbb{C}_X$, (13) reduces to the Borel-Weil realization of the trivial representation.

There exists a natural notion of duality in the equivariant derived category, the *Verdier duality* operator $\mathbb{D} : D_G(X) \rightarrow D_G(X)$. For $\mathcal{F} \in D_G(X)$, I set

$$(14) \quad \Theta(\mathcal{F}) = \sum_p (-1)^p \Theta(\text{Ext}^p(\mathbb{D}\mathcal{F}, \mathcal{O})) ;$$

here $\Theta(\dots)$ denotes the character of the admissible, finite length G -module $\dots$; the substitution of $\mathbb{D}\mathcal{F}$ for $\mathcal{F}$ serves the purpose of making the dependence of the Ext-groups on $\mathcal{F}$ covariant. In view of the remarks about (13), every character Θ_{π} of a representation π with trivial infinitesimal character can be realized as $\Theta(\mathcal{F})$, for some $\mathcal{F} \in D_G(X)$.

In the Harish-Chandra character formula (12), each coefficient n_w corresponds to a fixed point for the action of H on X , as follows. I fix a dominant Weyl chamber for the pair $(\mathfrak{g}, \mathfrak{h})$, consistently with the choice of the quantity ρ in (12). The direct sum of the root space corresponding to the negative roots constitutes a maximal nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{g}$, and $B = N_{G_{\mathbb{C}}}(\mathfrak{n})$ is a particular Borel subgroup which contains $H_{\mathbb{C}}$. I identify $X \simeq G_{\mathbb{C}}/B$ as before, and set $x_w = w$ -translate of the identity coset; then $\{x_w \mid w \in W_{\mathbb{C}}\}$ is precisely the set of fixed points for H acting on X . As in (12), I let C denote a connected component of H_{rs} . For $h \in C$, the action of h on the tangent space $T_{x_w}X$ at the fixed point x_w splits that space into the direct sum of, respectively, the contracting and the non-contracting directions, and that splitting depends only on the connected component C , not on h itself. Using the structure theory of $G_{\mathbb{C}}$, one can construct an $H_{\mathbb{C}}$ -invariant, locally closed submanifold $N(x_w, C) \subset X$ passing through the point x_w , whose tangent space at x_w contains precisely the non-contracting directions. The action of H on the triple $(X, N(x_w, C), \{x_w\})$ and the G -equivariant structure of $\mathcal{F}$ induce a linear action of H on $H_{\{x_w\}}^*(N(x_w, C), \mathcal{F})$, the local cohomology of the sheaf $\mathcal{F}$ restricted to the subspace $N(x_w, C)$ relative to the one point subspace $\{x_w\}$. Kashiwara [6] conjectured that

$$(15) \quad \Theta(\mathcal{F})(h) = \sum_{w \in W_{\mathbb{C}}} \frac{\sum_p (-1)^p \text{tr} \left(h^{-1} : H_{\{x_w\}}^p(N(x_w, C), \mathcal{F}) \rightarrow H_{\{x_w\}}^p(N(x_w, C), \mathcal{F}) \right)}{\prod_{\alpha \in \Phi, (w\rho, \alpha) > 0} (1 - e^{-\alpha}(h))},$$

for all $h \in C$. The trace in the numerator is locally constant as function of h , hence the entire numerator depends only on C . In particular, this formula expresses the constants n_w in the formula (12) for the virtual character $\Theta(\mathcal{F})$ as local Lefschetz numbers at the fixed points x_w . A proof of the fixed point formula (15) is sketched in [14], with details to appear in [16].

As illustration of the fixed point formula, let me consider the simplest case, of a compact group $G = K$ and the constant sheaf $\mathbb{C}_X$, which corresponds to the trivial representation π . The Cartan subgroup H is then $H = T$, a maximal torus, whose action at the fixed points cannot have any strictly expanding or contracting directions. In this case, then, the non-contracting manifold $N(x_w, C)$ is all of X . The local cohomology of the constant sheaf relative to a one-point subspace is one dimensional and concentrated in degree equal to the dimension of the ambient space, i.e., in degree $2 \dim_{\mathbb{C}} X$. The identity in $H = T$ must act trivially, of course, but the trace of the action is locally constant as function on the torus T , which is connected. The Lefschetz numbers in the numerator in (15) are therefore all equal to 1, as they must be for the trivial representation.

Kirillov's "universal formula" fails in the noncompact semisimple case, as can be seen already for $G = SL(2, \mathbb{R})$ – there are simply not enough coadjoint orbits! A remedy for the failure of Kirillov's formula was suggested by Rossmann [13]. For any admissible representation π of finite length, define θ_{π} , the character on the Lie algebra, by the formula (6) and normalize the Fourier transform $\widehat{\theta}_{\pi}$ as in (7). I continue with the simplifying assumption that π has trivial infinitesimal character. The complex coadjoint orbit

$$(16) \quad \Omega_{\mathbb{C}} = G_{\mathbb{C}}\text{-orbit of } \rho \text{ in } \mathfrak{g}^*$$

intersects $i\mathfrak{g}_{\mathbb{R}}^*$ in a finite number of coadjoint orbits for G . Rossmann's idea is to express $\widehat{\theta}_{\pi}$ as an integral not necessarily over one of these G -coadjoint orbits, but as an integral over a more general type of middle dimensional cycle in the complex coadjoint orbit $\Omega_{\mathbb{C}}$. If – contrary to the standing assumption – π were allowed to have arbitrary infinitesimal character, one would have to work with arbitrary complex coadjoint orbits, rather than the single orbit (16).

To describe the idea concretely, I let $U \subset G_{\mathbb{C}}$ denote the (unique) compact real form which contains K and is invariant under complex conjugation with respect to the real form $G \subset G_{\mathbb{C}}$. Its Lie algebra $\mathfrak{u}_{\mathbb{R}}$ is then a real form in $\mathfrak{g}$. Counting dimensions, one finds that U acts transitively on X ; in fact,

$$(17) \quad X \simeq U\text{-orbit of } \rho \text{ in } i\mathfrak{u}_{\mathbb{R}}^* = \Omega_{\mathbb{C}} \cap i\mathfrak{u}_{\mathbb{R}}^*,$$

as homogeneous space for U . Thus X can be identified with the coadjoint orbit for U passing through ρ . As such, it carries a distinguished symplectic form, which makes the U -action on X Hamiltonian. The resulting moment map

$$(18) \quad m_U : X \longrightarrow i\mathfrak{u}_{\mathbb{R}}^* \subset \mathfrak{g}^*$$

realizes the isomorphism (17); it is U -equivariant and real algebraic. On the other hand, the complex algebraic action of $G_{\mathbb{C}}$ on the flag variety X induces a complex algebraic, Hamiltonian action on the cotangent bundle T^*X – Hamiltonian with respect to the intrinsic complex algebraic, non-degenerate 2-form which T^*X carries as cotangent bundle of an algebraic manifold. By construction, its moment map

$$(19) \quad m : T^*X \longrightarrow \mathfrak{g}^*$$

is $G_{\mathbb{C}}$ -equivariant and complex algebraic. Let $p : T^*X \rightarrow X$ denote the natural projection. Rossmann calls

$$(20a) \quad m_{\rho} =_{\text{def}} m + m_U \circ p : T^*X \longrightarrow \mathfrak{g}^*$$

the *twisted moment map* and shows that

$$(20b) \quad m_{\rho} : T^*X \xrightarrow{\sim} \Omega_{\mathbb{C}},$$

real algebraically and U -equivariantly.

Recall that G acts on X with only finitely many orbits, which stratify X real algebraically. Thus

$$(21) \quad T_G^*X =_{\text{def}} \text{union of the conormal bundles of the } G\text{-orbits}$$

is a real algebraic, Lagrangian subvariety of T^*X ; in particular, it has dimension $2n$, where $n = \dim_{\mathbb{C}} X$. By a “ $2n$ -cycle in T_G^*X with possibly infinite support”, I mean an element of $H_{2n}^{\text{inf}}(T_G^*X, \mathbb{C})$, the *Borel-Moore homology* of T_G^*X in degree $2n$; this is the homology theory built on cycles that are allowed to run off to infinity. Since $2n$ is the dimension of T_G^*X , any $c \in H_{2n}^{\text{inf}}(T_G^*X, \mathbb{C})$ can be regarded as a true geometric cycle, rather than a homology class. I can now state Rossmann's result:

for each admissible π of finite length, subject to our standing assumption of trivial infinitesimal character, there exists a unique $2n$ -cycle $c \in H_{2n}^{\text{inf}}(T_G^*X, \mathbb{C})$, such that

$$(22) \quad \int_{\mathfrak{g}_{\mathbb{R}}} \theta_{\pi} \phi dx = \int_{m_{\rho}(c)} \widehat{\phi} \sigma^n \quad \text{for any } \phi \in C_c^{\infty}(\mathfrak{g}_{\mathbb{R}});$$

here σ refers to the intrinsic holomorphic symplectic form on the complex coadjoint orbit $\Omega_{\mathbb{C}}$, and the Fourier transform $\widehat{\phi}$ of the test function ϕ , normalized as in (7), is regarded as a holomorphic function on $\mathfrak{g}^*$. According to the Paley-Wiener theorem, $\widehat{\phi}$ decays rapidly in the imaginary directions, and that makes the integral on the right in (22) converge. Loosely paraphrased, (22) expresses the Fourier transform $\widehat{\theta}_{\pi}$ of the distribution θ_{π} as integration over the cycle $m_{\rho}(c)$ in $\Omega_{\mathbb{C}}$.

Rossmann's argument does not specify the cycle c , except in the special case of a complex group. The linear span of the various θ_{π} lies in the space of *invariant eigendistributions* with trivial infinitesimal character, i.e., conjugation invariant distributions on $\mathfrak{g}_{\mathbb{R}}$ that are annihilated by all conjugation invariant, constant coefficient linear differential operators with zero constant term. Whatever the cycle c , the right hand side of (22) defines such an invariant eigendistribution, and the resulting map from $H_{2n}^{\text{inf}}(T_G^*X, \mathbb{C})$ to the space of invariant eigendistributions is injective. The crux of Rossmann's proof of (22) is the equality of the dimensions of the two vector spaces, which he establishes by calculating each and comparing them.

To make the character formula (22) explicit, it is necessary to describe the cycle c in terms of the representation π . This was done in the announcement [15], with details to appear in [16]. The description of the cycle c involves first of all the geometric realization of the character as $\theta_{\pi} = \theta(\mathcal{F})$ for some $\mathcal{F} \in D_G(X)$, as in (13-14), and secondly, Kashiwara's notion of the *characteristic cycle* of a constructible sheaf: to each such sheaf $\mathcal{F}$ on a real analytic manifold, Kashiwara associates $\text{CC}(\mathcal{F})$, a Lagrangian, $\mathbb{R}^+$ -conical cycle in the cotangent bundle; details of the construction can be found in [8]. In the case at hand, CC defines a $\mathbb{Z}$ -linear map

$$(23) \quad \text{CC} : D_G(X) \longrightarrow H_{2n}^{\text{inf}}(T_G^*X, \mathbb{Z}).$$

With these ingredients, the Fourier transform of $\theta(\mathcal{F})$ is given by the formula

$$(24) \quad \int_{\mathfrak{g}_{\mathbb{R}}} \theta(\mathcal{F}) \phi dx = \frac{1}{(2\pi)^n n!} \int_{m_{\rho}(\text{CC}(\mathcal{F}))} \widehat{\phi} \sigma^n \quad (\phi \in C_c^{\infty}(\mathfrak{g}_{\mathbb{R}})).$$

In what sense can (24) be regarded as an extension of Kirillov's formula to the noncompact semisimple case? The integrand $\widehat{\phi} \sigma^n$ is closed; thus, if we were dealing with ordinary (finite) cycles, the integral would remain unchanged if the cycle $m_{\rho}(\text{CC}(\mathcal{F}))$ were replaced by any homologous cycle in $\Omega_{\mathbb{C}}$. Even without the finiteness condition, one can still deform the cycle without changing the value of the integral, provided the notion of homology is taken in a restricted sense, taking into account the growth of chains at infinity. The characteristic cycles associated to so-called *tempered representations* turn out to be homologous, in this restricted sense, to G -coadjoint orbits. Thus (24) reduces to Kirillov's formula for those representations of G for which it is known to hold [12], namely the tempered representations.

Via the process of going to the Lie algebra and taking the Fourier transform, the integral formula (24) describes the same object as the fixed point formula. The equivalence of the two expressions can be stated in terms of the sheaves $\mathcal{F}$ alone, without any reference to their representation-theoretic significance. The existing proof, on the other hand, depends totally on representation theory. It is natural to ask whether this equivalence can be seen directly, just as in the compact case. A hypothetical argument would most likely involve a version of localization of integrals, in the manner of Berline-Vergne, Witten, and Jeffrey-Kirwan, but for actions of noncompact groups. The technical problems of such an approach appear formidable!

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