

QUASI-EQUIVARIANT $\mathcal{D}$ -MODULES, EQUIVARIANT DERIVED CATEGORY, AND REPRESENTATIONS OF REDUCTIVE LIE GROUPS

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§1 Introduction.

In this note, we describe proofs of certain conjectures on functorial, geometric constructions of representations of a reductive Lie group $G_{\mathbb{R}}$. Our methods have applications beyond the conjectures themselves: unified proofs of the basic properties of the maximal and minimal globalizations of Harish-Chandra modules, and a criterion which insures that the solutions of a $G_{\mathbb{R}}$ -invariant system of linear differential equations constitute a representation of finite length.

Let G be a connected, reductive, complex algebraic group, and $G_{\mathbb{R}}$ a real form of G – i.e., a closed subgroup of the complex Lie group G^{an} , whose Lie algebra $\mathfrak{g}_{\mathbb{R}}$ lies as real form in the Lie algebra $\mathfrak{g}$ of G . We fix a maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$; $K_{\mathbb{R}}$ is the group of real points of some algebraic subgroup $K \subset G$. As subgroups of G , $G_{\mathbb{R}}$ and K operate on the flag variety X of $\mathfrak{g}$. For each linear form λ on the universal (i.e., equipped with a positive root system) Cartan algebra, we consider the bounded equivariant derived categories $D_{G_{\mathbb{R}}, \lambda}^b(X)$, $D_{K, \lambda}^b(X)$ of constructible sheaves of $\mathbb{C}$ -vector spaces on X with twist λ , and the twisted sheaf of holomorphic functions $\mathcal{O}_X(\lambda)$. In [K2], one of us conjectured:

- a) There exists a natural equivalence of categories $\Phi : D_{K, \lambda}^b(X) \xrightarrow{\sim} D_{G_{\mathbb{R}}, \lambda}^b(X)$;
- b) the Ext-groups $\text{Ext}^p(\mathcal{S}, \mathcal{O}_X(\lambda))$, for $\mathcal{S} \in D_{G_{\mathbb{R}}, \lambda}^b(X)$, carry natural Fréchet topologies and continuous linear $G_{\mathbb{R}}$ -actions;
- c) the cohomology groups $H^q(X, \mathcal{S} \otimes \mathcal{O}_X(-\lambda))$, for $\mathcal{S} \in D_{G_{\mathbb{R}}, \lambda}^b(X)$, have natural DNF topologies and continuous linear $G_{\mathbb{R}}$ -actions;
- (1.1) d) the resulting representations of $G_{\mathbb{R}}$ on $\text{Ext}^p(\mathcal{S}, \mathcal{O}_X(\lambda))$ and $H^q(X, \mathcal{S} \otimes \mathcal{O}_X(-\lambda))$ are admissible, of finite length;
- e) $\text{Ext}^p(\mathcal{S}, \mathcal{O}_X(\lambda))$ and $H^{d-p}(X, \mathcal{S} \otimes \mathcal{O}_X(-\lambda))$, with $d = \dim_{\mathbb{C}} X$, are each other's strong duals;
- f) if $\mathcal{M} \in D_{K, \lambda}^b(X)$ is the image of a holonomic $(\mathcal{D}_{-\lambda}, K)$ -module $\mathfrak{M}$ under the Riemann-Hilbert correspondence, then $H^p(X, \mathfrak{M})$ coincides with the dual of the Harish-Chandra module of $K_{\mathbb{R}}$ -finite vectors in $\text{Ext}^{d-p}(\Phi \mathcal{M}, \mathcal{O}_X(\lambda))$.

The first of these conjectures was established by Mirković-Uzawa-Vilonen [MUV]. In this

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note, we sketch proofs of (b-f).

In effect, (1.1b-f) may be viewed as counterparts, on the level of group representations, of the Beilinson-Bernstein construction of Harish-Chandra modules. Our positive answer to the conjectures has already been used, in [SV1,2], to establish other, related conjectures. A general discussion of these matters can be found in [K1,K4].

Special cases of (1.1b-f) had been worked out previously. If $j : D \hookrightarrow X$ is the inclusion of a $G_{\mathbb{R}}$ -orbit D , and $\mathcal{S} = j_! \mathcal{E}$ the proper direct image of a $G_{\mathbb{R}}$ -equivariant twisted local system $\mathcal{E}$ on the orbit, then $\mathrm{Ext}^p(j_! \mathcal{E}, \mathcal{O}_X(\lambda))$ can be identified with an appropriately defined local cohomology group along D , and $H^q(X, j_! \mathcal{E} \otimes \mathcal{O}_X(-\lambda))$ with compactly supported cohomology of an appropriate sheaf on D . These representations are produced and studied in [S1,HT,SW], and the remaining assertions (1.1e,f), in these special cases, then follow from results in [HMSW,S2].

It turns out that $\mathrm{Ext}^p(\mathcal{S}, \mathcal{O}_X(\lambda))$ and $H^q(X, \mathcal{S} \otimes \mathcal{O}_X(-\lambda))$ are, respectively, the maximal and minimal globalizations, in the sense of [S2], of their underlying Harish-Chandra modules. It is not difficult to construct the maximal and minimal globalization functors. However, two crucial properties – topological exactness and the explicit characterization of the maximal and minimal globalizations of principal series representations – are not so obvious. The arguments outlined in [S2] depend on relatively subtle lower bounds for the matrix coefficients of Harish-Chandra modules. Our proofs of the conjectures (1.1b-f) not only depend on these properties of the maximal and minimal globalization functors, they also imply them. As a consequence, we obtain alternate, more satisfactory proofs of the results announced in [S2].

The first named author observed that the explicit characterization of the maximal globalization of principal series representations is but a special instance of a quite general phenomenon. Representations of a reductive Lie group $G_{\mathbb{R}}$ often arise as spaces of solutions of a system of $G_{\mathbb{R}}$ -invariant, linear differential equations on a manifold on which $G_{\mathbb{R}}$ acts. Intuitively, one expects the space of solutions of such a system to be a representation of finite length only if the manifold is “small” and the system of differential equations “strong”. Theorem (7.6) below gives a precise sufficient condition. In particular, the theorem covers the following situation. Let Z be a smooth, quasi-projective variety over $\mathbb{C}$, with an algebraic action of the complexified group G , $Z_{\mathbb{R}}$ a $G_{\mathbb{R}}$ -invariant real form of Z , and $\mu_Z : T^*Z \rightarrow \mathfrak{g}^*$ the moment map. We consider a G -invariant linear system of differential equations on Z , with algebraic coefficients, and let $\mathfrak{M}$ denote the $\mathcal{D}_Z$ -module defined by the system. The space of hyperfunction solutions of the restricted system on $Z_{\mathbb{R}}$ has a natural Fréchet topology, constitutes a $G_{\mathbb{R}}$ -representation of finite length, and coincides with the maximal globalization of its space of $K_{\mathbb{R}}$ -finite vectors, provided the following two conditions are satisfied:

- a) $\mathfrak{M}$ is annihilated by an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g})$ (= center of $\mathcal{U}(\mathfrak{g})$);
- (1.2) b) the characteristic variety $\mathrm{Ch}(\mathfrak{M})$ intersects $\mu_Z^{-1}(\mathfrak{b}^{\perp})$ in a Lagrangian subvariety of T^*Z , for every Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.

Condition b) holds vacuously if a Borel subgroup B of G acts on Z with only finitely many orbits – for example, if Z is a flag variety, or a quotient of G by a maximal unipotent subgroup U , or a complexified symmetric space G/K . In the case of a flag variety, even the

“empty” system of differential equations satisfies a). In particular, theorem (7.6) applies to the principal series, to Whittaker models [GW,Mo], and to Helgason’s conjecture for affine symmetric spaces [He,KMOT].

We exhibit the representations (1.1b,c) as cohomology groups of objects in a derived category of representations. The construction of this derived category, in section three below, is of some independent interest. To make the connection between the Beilinson-Bernstein construction and $G_{\mathbb{R}}$ -representations, we set up a correspondence between $G_{\mathbb{R}}$ -sheaves on X and certain $\mathcal{D}$ -modules on the complexified symmetric space G/K . Our arguments, and also the statement of theorem (7.6), involve the notion of quasi-equivariant $\mathcal{D}$ -module, which was introduced, independently, in [K3] and – under the names “weakly equivariant $\mathcal{D}$ -module” and “weak $(\mathcal{D}_X, G)$ -module” – in [BBM] and [BB3]; we shall elaborate upon it in section four. This, too, has implications beyond the proofs outlined in this note. We intend to publish our results in more detailed form, including complete proofs, in the future.

§2 Minimal and Maximal Globalization.

We consider the category $\mathcal{R}(G_{\mathbb{R}})$ of $G_{\mathbb{R}}$ -representations – by definition, its objects are complete, locally convex, Hausdorff topological vector spaces with continuous, linear $G_{\mathbb{R}}$ -action, such that the resulting representation is admissible¹, of finite length; morphisms in $\mathcal{R}(G_{\mathbb{R}})$ are continuous, linear, $G_{\mathbb{R}}$ -invariant maps. For $V \in \mathcal{R}(G_{\mathbb{R}})$,

$$(2.1) \quad \begin{aligned} V_{K_{\mathbb{R}}-f\text{ini}} &= \text{linear span of the finite dimensional,} \\ &\quad K_{\mathbb{R}}\text{-invariant subspaces of } V \end{aligned}$$

is dense in V and consists entirely of C^{∞} vectors [HC3]. Both $\mathfrak{g}$ and K operate on $V_{K_{\mathbb{R}}-f\text{ini}}$ – the former by differentiation of the $G_{\mathbb{R}}$ -action, the latter by complexification of the $K_{\mathbb{R}}$ -action – and the two operations are compatible. Thus $V_{K_{\mathbb{R}}-f\text{ini}}$ becomes an algebraic $(\mathfrak{g}, K)$ -module. On the infinitesimal level, the hypotheses of admissibility and finite length on the representation V imply that $V_{K_{\mathbb{R}}-f\text{ini}}$ is in fact a Harish-Chandra module: an algebraic $(\mathfrak{g}, K)$ -module, finitely generated over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, with finite K -multiplicities [HC1,2].

We denote the category of Harish-Chandra modules by $\mathcal{HC}(\mathfrak{g}, K)$. It is a full subcategory of $\text{Mod}(\mathfrak{g}, K)$, the category of algebraic $(\mathfrak{g}, K)$ -modules and $(\mathfrak{g}, K)$ -invariant linear maps. The passage from V to $V_{K_{\mathbb{R}}-f\text{ini}}$ defines a functor

$$(2.2) \quad HC : \mathcal{R}(G_{\mathbb{R}}) \longrightarrow \mathcal{HC}(\mathfrak{g}, K).$$

This functor is faithful, exact, and assigns irreducible Harish-Chandra modules to irreducible representations [HC1-3]. For $M \in \mathcal{HC}(\mathfrak{g}, K)$, the K -finite subspace of the algebraic dual,

$$(2.3) \quad M' = (M^*)_{K-f\text{ini}},$$

¹i.e., in the restriction to $K_{\mathbb{R}}$, each irreducible representation of $K_{\mathbb{R}}$ occurs only finitely often.

is another Harish-Chandra module, the Harish-Chandra module dual to M . We write V' for the continuous dual of a topological vector space V , and topologize V' with the strong dual topology. The natural $G_{\mathbb{R}}$ -action on the dual V' of some $V \in \mathcal{R}(G_{\mathbb{R}})$ need not be continuous; when it *is* continuous, then the resulting representation has finite length, is admissible, and

$$(2.4) \quad HC(V') \simeq (HC(V))',$$

i.e., duality of Harish-Chandra modules corresponds to duality of representations.

By a *globalization* of a Harish-Chandra module M , we shall mean a $G_{\mathbb{R}}$ -representation $V \in \mathcal{R}(G_{\mathbb{R}})$ such that $M = HC(V)$. Every Harish-Chandra module can be globalized [P,C1], and this fact makes it a relatively simple matter to construct functorial globalizations, i.e., right inverses to the functor (2.2). In the next definition, we view $C^{\infty}(G_{\mathbb{R}})$ as left $(\mathfrak{g}, K_{\mathbb{R}})$ -module via right translation, composed with the canonical antiautomorphism of $\mathfrak{g}$, and as $G_{\mathbb{R}}$ -module via left translation. Every $M \in \mathcal{HC}(\mathfrak{g}, K)$ has a countable vector space basis, so

$$(2.5) \quad MG(M) = \text{Hom}_{(\mathfrak{g}, K_{\mathbb{R}})}(M', C^{\infty}(G_{\mathbb{R}}))$$

inherits a Fréchet topology and a continuous, linear $G_{\mathbb{R}}$ -action from $C^{\infty}(G_{\mathbb{R}})$. The space $MG(M)$ remains unchanged if one replaces $C^{\infty}(G_{\mathbb{R}})$ by the space of real analytic functions $C^{\omega}(G_{\mathbb{R}})$, or by the space of distributions $C^{-\infty}(G_{\mathbb{R}})$:

2.6 Lemma. *The inclusions $C^{\omega}(G_{\mathbb{R}}) \hookrightarrow C^{\infty}(G_{\mathbb{R}}) \hookrightarrow C^{-\infty}(G_{\mathbb{R}})$ induce topological isomorphisms*

$$\text{Hom}_{(\mathfrak{g}, K_{\mathbb{R}})}(M', C^{\omega}(G_{\mathbb{R}})) \simeq \text{Hom}_{(\mathfrak{g}, K_{\mathbb{R}})}(M', C^{\infty}(G_{\mathbb{R}})) \simeq \text{Hom}_{(\mathfrak{g}, K_{\mathbb{R}})}(M', C^{-\infty}(G_{\mathbb{R}})).$$

One can deduce this from the regularity theorems for elliptic $\mathcal{D}$ -modules by embedding $MG(M)$ into $\text{Hom}_{\mathcal{D}_G}(\mathcal{D}_G \otimes_{\mathcal{U}(\mathfrak{g})} M', C^{\infty}(G_{\mathbb{R}}))$ – in fact, the two spaces coincide if $G_{\mathbb{R}}$ is connected – and noting that the $\mathcal{D}_G$ -module $\mathcal{D}_G \otimes_{\mathcal{U}(\mathfrak{g})} M'$ is elliptic on $G_{\mathbb{R}}$, in the sense that its characteristic variety intersects the conormal bundle $T_{G_{\mathbb{R}}}^*G$ only along the zero section.

The lemma and the mere existence of globalizations imply that the action of $G_{\mathbb{R}}$ on $MG(M)$ defines an admissible representation of finite length, which globalizes the Harish-Chandra module M . By construction, MG is a functor, and thus a right inverse to HC .

There is a dual construction, as follows. We now regard $C_0^{\infty}(G_{\mathbb{R}})$, the space of compactly supported, C^{∞} functions, as right $(\mathfrak{g}, K_{\mathbb{R}})$ -module via right translation. As a right module, it can be tensored, simultaneously over $\mathcal{U}(\mathfrak{g})$ and the group ring $\mathbb{C}[K_{\mathbb{R}}]$, with the left $(\mathfrak{g}, K_{\mathbb{R}})$ -module M . This tensor product is naturally a quotient of the tensor product over $\mathbb{C}$, and therefore inherits a topology. We define

$$(2.7) \quad mg(M) = \text{largest separated quotient of } C_0^{\infty}(G_{\mathbb{R}}) \otimes_{(\mathfrak{g}, K_{\mathbb{R}})} M.$$

We shall see later, as a consequence of our arguments, that the topology on the tensor product is Hausdorff, so the phrase “largest separated quotient” turns out to be unnecessary in the end. The left translation action of $G_{\mathbb{R}}$ on $C_0^{\infty}(G_{\mathbb{R}})$ induces a continuous, linear action on $mg(M)$, and the resulting representation is another functorial globalization of the Harish-Chandra module M .

2.8 Theorem. *The functors mg , MG are, respectively, left and right adjoint to HC . They are each other's dual, i.e., $(mg(M))' \simeq MG(M')$ and $(MG(M))' \simeq mg(M')$. Both mg and MG are topologically exact functors.*

The first assertion can be rephrased as follows. For $V \in \mathcal{R}(G_{\mathbb{R}})$ and $M = HC(V)$, there exist functorial morphisms

$$(2.9) \quad \begin{aligned} mg(M) &\xrightarrow{\alpha} V, \quad V \xrightarrow{\beta} MG(M), \quad \text{such that} \\ HC(\alpha) \text{ and } HC(\beta) &\text{ induce the identity on } M = HC(V). \end{aligned}$$

Both α and β are injective, since the functor HC is faithful. Thus any globalization of a Harish-Chandra module M lies sandwiched in between $mg(M)$ and $MG(M)$, and that is the reason for calling the two functors the *minimal globalization* and *maximal globalization*. The duality between the two globalizations follows formally from (2.6) and the closed graph theorem of L. Schwartz: a surjective morphism in the category of locally convex, Hausdorff topological vector spaces, from a Suslin space to an inductive limit of Banach spaces, is necessarily open; moreover, $C_0^\infty(G_{\mathbb{R}})$ has the Suslin property [Sz,T].

In slightly different, but equivalent form, theorem 2.8 and the companion statements (2.12, 2.13) below were announced in [S2]. Except for (2.12) and topological exactness, they can be readily inferred from known results. Topological exactness and (2.12) are more delicate; the arguments outlined in [S2] deduce them from certain lower bounds on the matrix coefficients of Harish-Chandra modules. In this note, they will become natural consequences of our proofs of the conjectures (1.1b-f).

The minimal globalization functor first appears in the work of Litvinov-Zhelobenko and Prichepionok [P]. Two other functorial globalizations were constructed by Casselman-Wallach [C2,W]. Although their terminology differs, we shall refer to these two globalizations as the C^∞ and distribution globalization, since they relate to C^∞ functions and distributions in the same way the minimal and maximal globalizations relate to real analytic functions and hyperfunctions – cf. (2.12).

We choose a minimal parabolic subgroup $P_{\mathbb{R}}$ of $G_{\mathbb{R}}$, and let P denote its complexification. The variety Y of G -conjugates of $\mathfrak{p} = Lie(P)$ is a generalized flag variety, which contains the $G_{\mathbb{R}}$ -orbit of $\mathfrak{p}$ as real form:

$$(2.10) \quad Y_{\mathbb{R}} = G_{\mathbb{R}}/P_{\mathbb{R}} \hookrightarrow G/P = Y.$$

Each irreducible, finite dimensional $P_{\mathbb{R}}$ -module E associates an irreducible, $G_{\mathbb{R}}$ -equivariant, real analytic vector bundle $\mathbf{E} \rightarrow Y_{\mathbb{R}}$ to the principal bundle $G_{\mathbb{R}} \rightarrow G_{\mathbb{R}}/P_{\mathbb{R}} = Y_{\mathbb{R}}$. The $K_{\mathbb{R}}$ -finite sections of $\mathbf{E}$ constitute a Harish-Chandra module,

$$(2.11) \quad \text{Ind}_{(\mathfrak{p}, K_{\mathbb{R}} \cap P_{\mathbb{R}})}^{(\mathfrak{g}, K)}(E) = C^\omega(Y_{\mathbb{R}}, \mathbf{E})_{K_{\mathbb{R}}-fini}.$$

Collectively, Harish-Chandra modules of this type make up the principal series. Both $C^\omega(Y_{\mathbb{R}}, \mathbf{E})$, the space of real analytic sections, and $C^{-\omega}(Y_{\mathbb{R}}, \mathbf{E})$, the space of hyperfunction sections, have natural Hausdorff topologies – the latter because $Y_{\mathbb{R}}$ is compact – and continuous $G_{\mathbb{R}}$ -actions. The resulting representations globalize the Harish-Chandra module (2.11). Hence, by (2.9), they contain its minimal globalization, and in turn are contained in its maximal globalization.

2.12 Theorem. *The natural inclusions induce topological isomorphisms*

$$mg\left(\mathrm{Ind}_{(\mathfrak{p}, K_{\mathbb{R}} \cap P_{\mathbb{R}})}^{(\mathfrak{g}, K)}(E)\right) \xrightarrow{\sim} C^{\omega}(Y_{\mathbb{R}}, \mathbf{E}), \quad C^{-\omega}(Y_{\mathbb{R}}, \mathbf{E}) \xrightarrow{\sim} MG\left(\mathrm{Ind}_{(\mathfrak{p}, K_{\mathbb{R}} \cap P_{\mathbb{R}})}^{(\mathfrak{g}, K)}(E)\right).$$

This statement implies Helgason's conjecture; conversely, it was inspired by the original proof of Helgason's conjecture [KMOT], and by Casselman-Wallach's construction of canonical globalization. In effect, Casselman-Wallach take the analogue of (2.12) as the point of departure of their definition: the spaces of C^{∞} and distribution sections of $\mathbf{E}$ constitute, respectively, the C^{∞} globalization and the distribution globalization of the Harish-Chandra module (2.11), but it then takes considerable effort to show that these are concrete instances of functorial constructions [C2, W].

Now let $V \in \mathcal{R}(G_{\mathbb{R}})$ be a Banach representation. The space of analytic vectors $V^{\omega} \subset V$ has a natural inductive limit topology and a natural, continuous $G_{\mathbb{R}}$ -action. If the Banach topology on V is reflexive, the dual representation is continuous also [HC1], so it makes sense to define the space of hyperfunction vectors, $V^{-\omega} =_{\mathrm{def}} ((V')^{\omega})'$. This space contains V , and has a natural Fréchet topology and continuous $G_{\mathbb{R}}$ -action. Both V^{ω} and $V^{-\omega}$ globalize the Harish-Chandra module $HC(V)$, and thus lie between the minimal and the maximal globalization.

2.13 Theorem. *The natural inclusion $mg(HC(V)) \hookrightarrow V^{\omega}$ is a topological isomorphism. Dually, if V is a reflexive Banach space, $V^{-\omega} \hookrightarrow MG(HC(V))$ is a topological isomorphism.*

This formally implies (2.12), since $C^{\omega}(Y_{\mathbb{R}}, \mathbf{E})$ and $C^{-\omega}(Y_{\mathbb{R}}, \mathbf{E})$ can be identified with the spaces of, respectively, analytic and hyperfunction vectors for any of the reflexive Banach $G_{\mathbb{R}}$ -modules $L^p(Y_{\mathbb{R}}, \mathbf{E})$, $1 < p < \infty$. Logically, (2.13) follows from (2.12), in conjunction with [W, §5.8] and the main theorem in [HS].

§3 A Derived Category of Representations.

In this section we introduce a derived category of group representations. Much of the formalism, though in a different setting, is already known – cf. [BBD, L]: while the category of topological vector spaces fails to be abelian, it is an exact category, and that suffices for our purposes. However, to keep the discussion short, we shall stay within the context of topological vector spaces, even though our arguments apply more generally in the setting of exact categories. We are indebted to J.-P. Schneiders for helpful conversations about the matters treated below.

For the moment, H will denote an arbitrary Hausdorff topological group. Eventually, $G_{\mathbb{R}}$ will play the role of H , but H can also be the trivial group $\{e\}$, for example, in which case the derived category of representations reduces to the derived category of topological vector spaces.

Let $\mathcal{TV}$ be the category of locally convex, Hausdorff topological vector spaces and continuous linear maps, and $\mathcal{F}$ an additive full subcategory of $\mathcal{TV}$. We consider the category $\mathcal{F}_H$ whose objects are vector spaces belonging to $\mathcal{F}$, together with a continuous, linear action of H ; its morphisms are H -invariant, continuous linear maps. Next we form the category $C^b(\mathcal{F}_H)$ of bounded complexes in $\mathcal{F}_H$, and the quotient category $K^b(\mathcal{F}_H)$, which has the same objects as $C^b(\mathcal{F}_H)$, but with homotopic morphisms identified. We call

a complex (X, d_X) in $C^b(\mathcal{F}_H)$ *exact* if, for each n , $d_X^{n-1} : X^{n-1} \rightarrow \text{Ker } d_X^n$ is an open, surjective map, relative to the subspace topology on $\text{Ker } d_X^n$.

3.1 Lemma. *Let $f : X \rightarrow Y$, $g : Y \rightarrow X$ be morphisms in $C^b(\mathcal{F}_H)$, and let $M(f)$ denote the mapping cone of f .*

a) If $g \circ f$ is homotopic to id_X , and if Y is exact, then so is X .

b) If X and Y are exact, then $M(f)$ is exact also.

Because of a), the notion of exactness descends from $C^b(\mathcal{F}_H)$ to $K^b(\mathcal{F}_H)$: if two complexes are isomorphic in $K^b(\mathcal{F}_H)$, then either both or neither are exact. Statement b) implies that $\mathcal{N}(\mathcal{F}_H)$, the full subcategory of $K^b(\mathcal{F}_H)$ consisting of exact complexes, is a null-system in the terminology of [KSa, §1.6]. Thus one can “divide” $K^b(\mathcal{F}_H)$ by $\mathcal{N}(\mathcal{F}_H)$. We let $D^b(\mathcal{F}_H)$ denote the resulting triangulated category; this is our (bounded) derived category of H -representations. We call a morphism f in $K^b(\mathcal{F}_H)$ a *quasi-isomorphism* if it has an exact mapping cone. In that case, f induces an isomorphism in the category $D^b(\mathcal{F}_H)$. A complex X in $K^b(\mathcal{F}_H)$ becomes zero in $D^b(\mathcal{F}_H)$ precisely when it is quasi-isomorphic to zero, i.e., when it is exact.

The cohomology groups of an exact complex vanish. Consequently, the cohomology functors on $C^b(\mathcal{F}_H)$ determine functors on the level of $D^b(\mathcal{F}_H)$; they take values in the category $\mathcal{V}$ of vector spaces without topology:

$$(3.2) \quad H^n : D^b(\mathcal{F}_H) \longrightarrow \mathcal{V}.$$

From now on, we assume that the category $\mathcal{F}$ is hereditary, in the following sense:

$$(3.3) \quad \begin{array}{l} \text{if } V \text{ belongs to } \mathcal{F}, \text{ and if } W \subset V \text{ is a closed subspace,} \\ \text{then } W \text{ and } V/W \text{ also belong to } \mathcal{F}. \end{array}$$

If in addition the open mapping theorem holds in the category $\mathcal{F}$ – for example, if $\mathcal{F}$ is the category of Fréchet spaces – then exactness of complexes reduces to the vanishing of cohomology, and a morphism in $D^b(\mathcal{F}_H)$ is an isomorphism if and only if it induces isomorphisms on the level of cohomology.

The hypothesis (3.3) allows us to introduce two pairs of truncation functors $q_{\tau \leq n}$, $q_{\tau \geq n}$ and $s_{\tau \leq n}$, $s_{\tau \geq n}$ on the category $C^b(\mathcal{F}_H)$, as follows:

$$(3.4) \quad \begin{aligned} q_{\tau \leq n}(X) &= \dots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow \text{Ker } d_X^n \rightarrow 0 \rightarrow \dots \\ q_{\tau \geq n}(X) &= \dots \rightarrow 0 \rightarrow \text{Ker } d_X^{n-1} \rightarrow X^{n-1} \rightarrow X^n \rightarrow \dots \\ s_{\tau \leq n}(X) &= \dots \rightarrow X^{n-1} \rightarrow X^n \rightarrow \overline{\text{Im } d_X^n} \rightarrow 0 \rightarrow \dots \\ s_{\tau \geq n}(X) &= \dots \rightarrow 0 \rightarrow X^n / \overline{\text{Im } d_X^{n-1}} \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \dots, \end{aligned}$$

where $\overline{\text{Im } d_X^n}$ denotes the closure of $\text{Im } d_X^n$, equipped with the subspace topology inherited from X^{n+1} . Both are compatible with homotopy and quasi-isomorphism, and hence define

truncation functors on $D^b(\mathcal{F}_H)$. We set

$$\begin{aligned}
 {}^q D^{\leq 0}(\mathcal{F}_H) &= \{X \in D^b(\mathcal{F}_H) ; {}^q \tau^{\leq 0} X \rightarrow X \text{ is an isomorphism} \} , \\
 {}^q D^{\geq 0}(\mathcal{F}_H) &= \{X \in D^b(\mathcal{F}_H) ; X \rightarrow {}^q \tau^{\geq 0} X \text{ is an isomorphism} \} , \\
 {}^s D^{\leq 0}(\mathcal{F}_H) &= \{X \in D^b(\mathcal{F}_H) ; {}^s \tau^{\leq 0} X \rightarrow X \text{ is an isomorphism} \} , \\
 {}^s D^{\geq 0}(\mathcal{F}_H) &= \{X \in D^b(\mathcal{F}_H) ; X \rightarrow {}^s \tau^{\geq 0} X \text{ is an isomorphism} \} .
 \end{aligned}
 \tag{3.5}$$

3.6 Theorem. *Both $({}^q D^{\leq 0}(\mathcal{F}_H), {}^q D^{\geq 0}(\mathcal{F}_H))$ and $({}^s D^{\leq 0}(\mathcal{F}_H), {}^s D^{\geq 0}(\mathcal{F}_H))$ are t-structures on $D^b(\mathcal{F}_H)$.*

We let $\mathcal{Q}(\mathcal{F}_H) = {}^q D^{\leq 0}(\mathcal{F}_H) \cap {}^q D^{\geq 0}(\mathcal{F}_H)$, $\mathcal{S}(\mathcal{F}_H) = {}^s D^{\leq 0}(\mathcal{F}_H) \cap {}^s D^{\geq 0}(\mathcal{F}_H)$ denote the hearts of these two t-structures. By construction, they are abelian categories. Each object of $\mathcal{Q}(\mathcal{F}_H)$ is isomorphic to the cokernel of a morphism f in $\mathcal{F}_H$ – viewed as morphism in $\mathcal{Q}(\mathcal{F}_H)$ – such that f is set-theoretically injective. This is our reason for calling the first t-structure the “quotient” structure. Dually, every object in the heart $\mathcal{S}(\mathcal{F}_H)$ of the “sub” t-structure is isomorphic to the kernel of a morphism in $\mathcal{F}_H$ with dense image. One can show that the bounded derived categories built from $\mathcal{Q}(\mathcal{F}_H)$ or $\mathcal{S}(\mathcal{F}_H)$ are equivalent to $D^b(\mathcal{F}_H)$, but this is not crucial for our purposes.

The functors

$$\begin{aligned}
 {}^q H^n(X) &= ({}^q \tau^{\leq n} \circ {}^q \tau^{\geq n})X , \\
 {}^s H^n(X) &= ({}^s \tau^{\leq n} \circ {}^s \tau^{\geq n})X
 \end{aligned}
 \tag{3.7}$$

take $D^b(\mathcal{F}_H)$ to the hearts of the two t-structures. We note:

$$\begin{aligned}
 H^0({}^q H^n(X)) &= H^n(X) , \\
 H^k({}^q H^n(X)) &= 0 \text{ if } k \neq 0 .
 \end{aligned}
 \tag{3.8a}$$

On the other hand,

$$\begin{aligned}
 H^0({}^s H^n(X)) &= \text{Ker } d_X^n / \overline{\text{Im } d_X^{n-1}} , \\
 H^1({}^s H^n(X)) &= \overline{\text{Im } d_X^n} / \text{Im } d_X^n , \\
 H^k({}^s H^n(X)) &= 0 \text{ if } k \neq 0, 1 .
 \end{aligned}
 \tag{3.8b}$$

For $X \in D^b(\mathcal{F}_H)$ and $n \in \mathbb{Z}$, the following conditions are equivalent:

$$\begin{aligned}
 &a) \text{ Im } d_X^{n-1} \text{ is closed in } X_n , \text{ and } d_X^{n-1} \text{ is an open map onto its image;} \\
 &b) {}^q H^n(X) \text{ is isomorphic to an object in } \mathcal{F}_H , \\
 &c) {}^s H^{n-1}(X) \text{ is isomorphic to an object in } \mathcal{F}_H .
 \end{aligned}
 \tag{3.9}$$

The natural functor $\psi : \mathcal{F}_H \rightarrow D^b(\mathcal{F}_H)$ is fully faithful. Hence, if the conditions (3.9) are satisfied, ${}^q H^n(X)$ and ${}^s H^{n-1}(X)$ are well defined objects in $\mathcal{F}_H$. If these conditions hold for every n , we say that X is *strict*; in that case, ${}^q H^n(X) = {}^s H^n(X)$ for all n .

Finally, let us suppose that $H = G_{\mathbb{R}}$ is a reductive Lie group. We shall say that $X \in D^b(\mathcal{F}_{G_{\mathbb{R}}})$ has the property MG – respectively, the property mg – if X is strict and all its cohomology groups can be realized as maximal – respectively, minimal – globalizations of Harish-Chandra modules.

§4 Quasi-equivariant $\mathcal{D}$ -modules.

We recall the definition of a quasi-equivariant $\mathcal{D}$ -module [K3,BB3,BBM]. Let G be an algebraic group with Lie algebra $\mathfrak{g}$, and X a smooth algebraic variety with an algebraic G -action – or algebraic G -manifold for short. We write μ for the action morphism, p for the projection from $G \times X$ to X ,

$$(4.1) \quad \mu : G \times X \rightarrow X, \quad \mu(g, x) = gx, \quad p : G \times X \rightarrow X, \quad p(g, x) = x,$$

and we consider the three maps

$$(4.2) \quad \begin{aligned} q_j : G \times G \times X &\rightarrow G \times X, \quad 1 \leq j \leq 3, \\ q_1(g_1, g_2, x) &= (g_1, g_2 x), \quad q_2(g_1, g_2, x) = (g_1 g_2, x), \quad q_3(g_1, g_2, x) = (g_2, x). \end{aligned}$$

Then $\mu \circ q_1 = \mu \circ q_2$, $p \circ q_2 = p \circ q_3$, and $\mu \circ q_3 = p \circ q_1$.

As usual, $\mathcal{D}_X$ will refer to the sheaf of linear, algebraic differential operators on X . We let $\mathcal{O}_G \boxtimes \mathcal{D}_X$ denote the subalgebra $\mathcal{O}_{G \times X} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{D}_X$ of $\mathcal{D}_{G \times X}$. A quasi- G -equivariant $\mathcal{D}_X$ -module, by definition, is a $\mathcal{D}_X$ -module $\mathfrak{M}$, together with the datum of an $\mathcal{O}_G \boxtimes \mathcal{D}_X$ -linear isomorphism $\beta : \mu^* \mathfrak{M} \xrightarrow{\sim} p^* \mathfrak{M}$, such that the diagram

$$(4.3) \quad \begin{array}{ccc} q_2^* \mu^* \mathfrak{M} & \xrightarrow{q_2^* \beta} & q_2^* p^* \mathfrak{M} \\ \wr \parallel & & \wr \parallel \\ q_1^* \mu^* \mathfrak{M} & \xrightarrow{q_1^* \beta} q_1^* p^* \mathfrak{M} \simeq q_3^* \mu^* \mathfrak{M} & \xrightarrow{q_3^* \beta} q_3^* p^* \mathfrak{M} \end{array}$$

commutes. If β is linear even over $\mathcal{D}_{G \times X}$, this reduces to the usual definition of a G -equivariant $\mathcal{D}_X$ -module.

The definition of quasi- G -equivariance can be loosely paraphrased as follows. For $g \in G$, let $\mu_g : X \rightarrow X$ be translation by g . The datum of β consists of a family of isomorphisms of $\mathcal{D}_X$ -modules $\beta_g : \mu_g^* \mathfrak{M} \xrightarrow{\sim} \mathfrak{M}$, depending algebraically on g , and multiplicative in the variable g .

As example, we mention the sheaf of sections $\mathcal{O}(\mathbf{E})$ of $(\mathbf{E}, \nabla_{\mathbf{E}})$, a G -equivariant algebraic vector bundle $\mathbf{E}$ with a G -equivariant, algebraic, flat connection $\nabla_{\mathbf{E}}$. Then $\mathcal{D}_X$ acts on $\mathcal{O}(\mathbf{E})$ via the flat connection, and the resulting $\mathcal{D}_X$ -module is quasi-equivariant. The Lie algebra $\mathfrak{g}$ of G acts on sections of the G -equivariant vector bundle $\mathbf{E}$ by infinitesimal translation; on the other hand, each $A \in \mathfrak{g}$ determines a vector field on X , again by

infinitesimal translation, and as such also operates on sections via the connection. When these two actions of $\mathfrak{g}$ coincide, the $\mathcal{D}_X$ -module $\mathcal{O}(\mathbf{E})$ is G -equivariant, and one calls the sheaf of flat sections a G -equivariant local system.

If G acts transitively on X , equivariant vector bundles with flat, equivariant connection are the only examples of quasi- G -equivariant $\mathcal{D}_X$ -modules coherent over $\mathcal{O}_X$: the category of quasi- G -equivariant, $\mathcal{O}_X$ -coherent $\mathcal{D}_X$ -modules coincides with the category of pairs $(\mathbf{E}, \nabla_{\mathbf{E}})$, consisting of a G -equivariant algebraic vector bundle $\mathbf{E}$ with a G -equivariant, algebraic, flat connection $\nabla_{\mathbf{E}}$, and flat, G -equivariant bundle maps between them. The latter category, in turn, can be identified with the category of finite dimensional, algebraic $(\mathfrak{g}, H)$ -modules, where $H \subset G$ denotes the isotropy subgroup at some reference point $x_0 \in X$. The preceding statements remain correct even without the hypothesis of coherence, provided one allows for algebraic vector bundles of possibly infinite rank, and possibly infinite dimensional algebraic $(\mathfrak{g}, H)$ -modules [K3]: if $H \subset G$ is a closed subgroup, there exists a natural equivalence between $\text{Mod}_G(\mathcal{D}_{G/H})$, the category of quasi-coherent, quasi- G -equivariant $\mathcal{D}_{G/H}$ -modules and G -equivariant morphisms, and $\text{Mod}(\mathfrak{g}, H)$, the category of algebraic $(\mathfrak{g}, H)$ -modules and $(\mathfrak{g}, H)$ -invariant linear maps,

$$(4.4) \quad \phi : \text{Mod}_G(\mathcal{D}_{G/H}) \xrightarrow{\sim} \text{Mod}(\mathfrak{g}, H), \quad \phi(\mathfrak{M}) = \mathfrak{M} / \mathfrak{I}_{eH} \mathfrak{M}.$$

Here $\mathfrak{I}_{eH}$ denotes the ideal sheaf of the point eH ; the isotropy group H operates on the “geometric fiber” $\mathfrak{M} / \mathfrak{I}_{eH} \mathfrak{M}$ via the equivariant $\mathcal{O}_X$ -module structure, and $\mathcal{U}(\mathfrak{g})$ via an action which we shall now describe.

We return to the general case of a quasi- G -equivariant $\mathcal{D}_X$ -module $\mathfrak{M}$. Again, the Lie algebra $\mathfrak{g}$ acts on $\mathfrak{M}$ in two ways. First, via $\mathfrak{g} \rightarrow \Gamma(\Theta_X) \hookrightarrow \Gamma(\mathcal{D}_X)$ and the $\mathcal{D}_X$ -module structure, and secondly, through differentiation of the G -action when we regard $\mathfrak{M}$ as a G -equivariant $\mathcal{O}_X$ -module. We denote the former action by $\alpha_{\mathcal{D}}$, the latter by α_t , and we set $\gamma = \alpha_t - \alpha_{\mathcal{D}}$. Then

$$(4.5) \quad \begin{aligned} \gamma : \mathfrak{g} &\longrightarrow \text{End}_{\mathcal{D}_X}(\mathfrak{M}), \text{ and} \\ \gamma &\text{ is a Lie algebra homomorphism.} \end{aligned}$$

Thus γ turns $\mathfrak{M}$ into a $(\mathcal{D}_X, \mathcal{U}(\mathfrak{g}))$ -bimodule. The quasi- G -equivariant $\mathcal{D}_X$ -module $\mathfrak{M}$ is G -equivariant as $\mathcal{D}_X$ -module precisely when $\gamma \equiv 0$.

In our discussion of quasi-equivariance so far, no particular assumptions were imposed on the algebraic group G and the G -manifold X . From now on, however, we shall require G to be an affine algebraic group – which will be the case if G is reductive, as was assumed in the introduction – and X a quasi-projective variety. These hypotheses will enable us to define direct and inverse images, under G -equivariant morphisms, of objects in the derived category of quasi- G -equivariant $\mathcal{D}_X$ -modules. A crucial tool is the following result of Sumihiro [Su]:

4.6 Proposition. *Under the hypotheses just stated,*

- a) X has a G -equivariant, open embedding into a projective G -manifold $\overline{X}$;
- b) the G -equivariant projective completion $\overline{X}$ admits an ample, G -equivariant line bundle.

As before, we let $\text{Mod}_G(\mathcal{D}_X)$ denote the category of quasi-coherent, quasi- G -equivariant $\mathcal{D}_X$ -modules and G -equivariant morphisms between them; $\text{Mod}_G^{coh}(\mathcal{D}_X)$ shall denote the full subcategory of $\mathcal{D}_X$ -coherent modules. The analogues of locally free $\mathcal{D}_X$ -modules in the quasi-equivariant setting are modules of the form $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathfrak{F}$, for some coherent, locally free, G -equivariant $\mathcal{O}_X$ -module $\mathfrak{F}$. We write $\text{Mod}_G^{lf}(\mathcal{D}_X)$ for the full subcategory of $\text{Mod}_G(\mathcal{D}_X)$ consisting of such locally free objects. As a consequence of Sumihiro's result, one can show:

4.7 Lemma. *Every $\mathfrak{M} \in \text{Mod}_G^{coh}(\mathcal{D}_X)$ has a finite left resolution by objects in $\text{Mod}_G^{lf}(\mathcal{D}_X)$.*

Standard arguments in homological algebra imply that

$$(4.8) \quad \text{every } \mathfrak{M} \in \text{Mod}_G(\mathcal{D}_X) \text{ can be embedded into an injective object.}$$

Injective objects in $\text{Mod}_G(\mathcal{D}_X)$ are injective also as objects of $\text{Mod}_G(\mathcal{O}_X)$, the category of G -equivariant $\mathcal{O}_X$ -modules. In this latter category, direct image under a G -equivariant morphism of quasi-projective G -manifolds, as well as the global Ext functors behave as they do in the non-equivariant case.

We now consider the bounded derived categories $D_G^b(\mathcal{D}_X)$, $D_{G,coh}^b(\mathcal{D}_X)$ built from the abelian categories $\text{Mod}_G(\mathcal{D}_X)$ and $\text{Mod}_G^{coh}(\mathcal{D}_X)$. Alternatively, the latter derived category can be described as the full subcategory of $D_G^b(\mathcal{D}_X)$ consisting of complexes whose cohomology lies in $\text{Mod}_G^{coh}(\mathcal{D}_X)$.

In view of the discussion in the preceding paragraphs, direct and inverse image functors can be defined on the bounded derived categories, as follows. Let $f : X \rightarrow Y$ be a G -equivariant morphism between the quasi-projective G -manifolds X, Y . Then

$$(4.9) \quad f^* : \mathfrak{N} \longmapsto \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathfrak{N} \simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathfrak{N}$$

defines a right exact functor from $\text{Mod}_G(\mathcal{D}_Y)$ to $\text{Mod}_G(\mathcal{D}_X)$, which can be left derived in the bounded derived category:

$$(4.10) \quad Lf^* : D_G^b(\mathcal{D}_Y) \longrightarrow D_G^b(\mathcal{D}_X).$$

The usual definition of direct image,

$$(4.11) \quad \mathfrak{M} \longmapsto Rf_*(\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes}_{\mathcal{D}_X} \mathfrak{M}),$$

has meaning also in the G -equivariant case, if properly interpreted: $\mathcal{D}_{Y \leftarrow X}$ has a deRham-type resolution by $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$ -bimodules which are flat over $\mathcal{D}_X$; if $\mathcal{D}_{Y \leftarrow X}$ is replaced by this resolution, and if $\mathfrak{M}$ is represented by an injective complex, then (4.11) makes sense in the bounded derived category. We use the customary notation

$$(4.12) \quad \int_f : D_G^b(\mathcal{D}_X) \longrightarrow D_G^b(\mathcal{D}_Y)$$

for this $\mathcal{D}$ -module direct image functor.

If f is smooth, Lf^* sends $D_{coh}^b(\mathcal{D}_Y)$ into $D_{coh}^b(\mathcal{D}_X)$, and if f is projective, $\int_f$ maps $D_{coh}^b(\mathcal{D}_X)$ into $D_{coh}^b(\mathcal{D}_Y)$. If f is both smooth and projective, the two functors are each other's left and right adjoints when restricted to the coherent subcategories, up to a shift in degree:

4.13 Theorem. *Suppose $f : X \longrightarrow Y$ is both smooth and projective, and let $\dim X/Y$ denote the dimension of the fibers. Then, for $\mathfrak{M} \in D_{G,coh}^b(\mathcal{D}_X)$ and $\mathfrak{N} \in D_{G,coh}^b(\mathcal{D}_Y)$,*

$$\begin{aligned} a) \quad & \mathrm{Hom}_{D_G^b(\mathcal{D}_Y)}(\mathfrak{N}, \int_f \mathfrak{M}) \simeq \mathrm{Hom}_{D_G^b(\mathcal{D}_X)}(Lf^* \mathfrak{N}[-\dim X/Y], \mathfrak{M}), \\ b) \quad & \mathrm{Hom}_{D_G^b(\mathcal{D}_X)}(\mathfrak{M}, Lf^* \mathfrak{N}) \simeq \mathrm{Hom}_{D_G^b(\mathcal{D}_Y)}(\int_f \mathfrak{M}[\dim X/Y], \mathfrak{N}), \end{aligned}$$

functorially in $\mathfrak{M}$ and $\mathfrak{N}$.

We shall apply these results about quasi-equivariant $\mathcal{D}$ -modules in the setting of the flag variety X of a reductive group G , and we shall need them also in the twisted case. It is possible to develop them systematically for quasi- G -equivariant modules over a G -equivariant ring of twisted differential operators – for a general discussion of twisting, see [K3]. On the flag variety, twisted sheaves are particularly easy to visualize; in effect, they are monodromic sheaves on the “enhanced flag variety” [BB3]. The extension of the results above to the twisted case on the flag variety² is therefore straightforward, and we shall use this more general case without further elaboration.

§5 Equivariant Derived Category and $G_{\mathbb{R}}$ -modules.

The equivariant derived category was introduced by Bernstein-Lunts [BL]; a concise summary can be found in [MUV]. We now recall the properties that are relevant for our purposes, but only in as much generality as is needed here.

Let X be a real or complex algebraic variety, considered as topological space in its Hausdorff topology, and G a linear, real or complex algebraic group acting on X . We write $\mathrm{Mod}(\mathbb{C}_X)$ for the category of sheaves of $\mathbb{C}$ -vector spaces, $D^b(\mathbb{C}_X)$ for the bounded derived category $D^b(\mathrm{Mod}(\mathbb{C}_X))$; this differs from the notation of the introduction, but is consistent with the conventions in the previous section. In the real algebraic case, $D_{\mathbb{R}-c}^b(\mathbb{C}_X)$ will denote the full subcategory of complexes with $\mathbb{R}$ -algebraically constructible cohomology; similarly, in the complex case, $D_{\mathbb{C}-c}^b(\mathbb{C}_X)$ is the bounded derived category of sheaves with $\mathbb{C}$ -algebraically constructible cohomology – cf. [KSa], where the analytic case is treated; the algebraic case is considerably simpler in most respects.

We choose a tower of connected real, respectively complex algebraic G -manifolds

$$(5.1) \quad V_1 \hookrightarrow V_2 \hookrightarrow \dots \hookrightarrow V_n \hookrightarrow V_{n+1} \hookrightarrow \dots,$$

such that

$$(5.2) \quad \begin{aligned} a) \quad & G \text{ acts freely on each } V_n; \\ b) \quad & \text{for each } p > 0, \quad H^p(V_n, \mathbb{C}) = 0 \quad \text{if } n \gg 0. \end{aligned}$$

For example, a sequence of Stiefel manifolds V_n will do, since the group G is assumed to be linear. Because of (5.2a), the quotients $G \backslash (X \times V_n)$ exist as algebraic varieties. We let

²more precisely, on products of the flag variety with another variety, with the twisting confined to the flag variety factor.

$p_n : X \times V_n \rightarrow X$ denote the projection onto the first factor, $q_n : X \times V_n \rightarrow G \backslash (X \times V_n)$ the quotient map, $i_n : X \times V_n \hookrightarrow X \times V_{n+1}$ the natural inclusion, and $j_n : G \backslash (X \times V_n) \hookrightarrow G \backslash (X \times V_{n+1})$ the map induced by i_n .

Objects of the bounded G -equivariant derived category $D_G^b(\mathbb{C}_X)$ are, by definition, quadruples $\mathcal{S} = (\mathcal{S}_\infty, (\mathcal{S}_n), (\phi_n), (\psi_n))$, with $\mathcal{S}_\infty \in D^b(\mathbb{C}_X)$, $(\mathcal{S}_n)_{1 \leq n < \infty}$ a sequence with $\mathcal{S}_n \in D^b(\mathbb{C}_{G \backslash (X \times V_n)})$, and $(\phi_n)_{1 \leq n < \infty}$, $(\psi_n)_{1 \leq n < \infty}$ sequences of isomorphisms

$$(5.3a) \quad \phi_n : j_n^{-1} \mathcal{S}_{n+1} \xrightarrow{\sim} \mathcal{S}_n, \quad \psi_n : p_n^{-1} \mathcal{S}_\infty \xrightarrow{\sim} q_n^{-1} \mathcal{S}_n,$$

such that, for each n ,

$$(5.3b) \quad \begin{aligned} & \psi_n : p_n^{-1} \mathcal{S}_\infty \xrightarrow{\sim} q_n^{-1} \mathcal{S}_n \text{ coincides with the composition of} \\ & p_n^{-1} \mathcal{S}_\infty \simeq i_n^{-1} p_{n+1}^{-1} \mathcal{S}_\infty \xrightarrow[i_n^{-1} \psi_{n+1}^{-1}]{\sim} i_n^{-1} q_{n+1}^{-1} \mathcal{S}_{n+1} \simeq q_n^{-1} j_{n+1}^{-1} \mathcal{S}_{n+1} \xrightarrow[q_n^{-1} \phi_n]{\sim} q_n^{-1} \mathcal{S}_n. \end{aligned}$$

Morphisms in this category are pairs $(\eta_\infty, (\zeta_n))$, consisting of a single morphism η_∞ and a sequence of morphisms $(\zeta_n)_{1 \leq n < \infty}$ between corresponding objects, which are compatible with the consistency conditions (5.3a). The category $D_G^b(\mathbb{C}_X)$ inherits the structure of triangulated category from $D^b(\mathbb{C}_X)$ and the $D^b(\mathbb{C}_{G \backslash (X \times V_n)})$.

The G -equivariant bounded derived category of $\mathbb{R}$ -algebraically constructible sheaves, $D_{G, \mathbb{R}-c}^b(\mathbb{C}_X)$, is the full subcategory of $D_G^b(\mathbb{C}_X)$ consisting of objects $\mathcal{S}$ whose components $\mathcal{S}_\infty, \mathcal{S}_n$ belong to the appropriate derived categories of constructible sheaves. Analogously, in the complex algebraic case, there is a G -equivariant derived category of $\mathbb{C}$ -algebraically constructible sheaves, $D_{G, \mathbb{C}-c}^b(\mathbb{C}_X)$.

As in the previous section, we shall need to work also with twisted equivariant derived categories. A systematic discussion can be found in [K3]. In the applications we have in mind here, the twisting will take place only on flag varieties. This case is particularly simple, so we shall not go into further detail.

The direct and inverse image functors Rf_* , $Rf^!$, f^{-1} , $f^!$ corresponding to a G -equivariant, algebraic morphism $f : X \rightarrow Y$ exist in the G -equivariant case, and also in the G -equivariant constructible case. They have the same formal properties as in the usual bounded derived category $D^b(\mathbb{C}_X)$.

Now let $H \subset G$ be a closed, algebraic subgroup. The tower $\{V_n\}$ that was used to define the G -equivariant derived category can be used also for H , and thus, via inverse image from the $G \backslash (X \times V_n)$ to the $H \backslash (X \times V_n)$, one obtains the restriction functor

$$(5.4) \quad \text{Res}_H^G : D_G^b(\mathbb{C}_X) \longrightarrow D_H^b(\mathbb{C}_X).$$

If H is normal in G , and if H acts freely on X , $D_G^b(\mathbb{C}_X) \simeq D_{G/H}^b(\mathbb{C}_{H \backslash X})$. Applying this twice, viewing first G and then H as normal subgroup of $G \times H$, and letting $X \times G$ play the role of X , results in equivalence of categories

$$(5.5) \quad \text{Ind}_H^G : D_H^b(\mathbb{C}_X) \xrightarrow{\sim} D_G^b(\mathbb{C}_{X \times (G/H)}),$$

“induction from H to G ”; here G acts diagonally on $X \times (G/H)$. The terminology is appropriate: Let $p : X \times (G/H) \rightarrow X$ denote the projection; then $Rp_* \circ \text{Ind}_H^G$ and $Rp^! \circ \text{Ind}_H^G$

are, respectively, the right and almost left adjoint functors of Res_H^G – almost left adjoint in the sense that the adjointness relation involves a dimension shift and tensoring by the stalk of the orientation sheaf of G/H at the identity coset. Here, again, everything that was said remains valid in the twisted case.

Let us recall the definition of the Matsuki correspondence for sheaves. We adopt the notation of the introduction: X is the flag variety of the complex reductive group G , $G_{\mathbb{R}}$ is a real form of G^{an} , $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ a maximal compact subgroup. We fix a linear function λ on the universal Cartan algebra, and let $D_{K,\lambda}^b(\mathbb{C}_X)$, $D_{G_{\mathbb{R}},\lambda}^b(\mathbb{C}_X)$ denote the equivariant derived categories with twist λ . For equivariant derived categories on products of X with another space, the subscript λ shall refer to the category of sheaves with twist λ along the factor X , but without twist along the other factor. Since K and $G_{\mathbb{R}}$ operate on X with finitely many orbits, constructibility for objects in $D_{K,\lambda}^b(\mathbb{C}_X)$, $D_{G_{\mathbb{R}},\lambda}^b(\mathbb{C}_X)$ comes down to finite dimensionality of the stalks of the cohomology sheaves. The inclusion

$$(5.6) \quad i : S_{\mathbb{R}} =_{def} G_{\mathbb{R}}/K_{\mathbb{R}} \hookrightarrow S =_{def} G/K$$

realizes the Riemannian symmetric space $S_{\mathbb{R}}$ as real form in the affine symmetric space S . Further notation:

$$(5.7) \quad p : X \times S \longrightarrow X, \quad q : X \times S \longrightarrow S$$

are the projections onto the two factors. In [K2], it was conjectured that

$$(5.8) \quad \begin{aligned} \Phi : D_{K,\lambda,\mathbb{C}-c}^b(\mathbb{C}_X) &\longrightarrow D_{G_{\mathbb{R}},\lambda,\mathbb{R}-c}^b(\mathbb{C}_X), \\ \Phi(\mathcal{T}) &= R p_! \left(\left(\text{Res}_{G_{\mathbb{R}}}^G \text{Ind}_K^G(\mathcal{T}) \right) \otimes (q^{-1} i_* i^! \mathbb{C}_S) \right) [2 \dim S] \end{aligned}$$

defines an equivalence of categories. This was proved by Mirković-Uzawa-Vilonen³ [MUV]. Note that $i^! \mathbb{C}_S \simeq \text{or}_{S_{\mathbb{R}}/S}[-\text{codim}_{\mathbb{R}} S_{\mathbb{R}}] \simeq \text{or}_{S_{\mathbb{R}}}[-\dim S]$, since $S_{\mathbb{R}}$ lies as real form in the complex manifold S .

The key to our proof of (1.1b-f) is the definition of $\text{RHom}_{\mathcal{D}_X}^{top}(\mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_X)$ as an object in the derived category of Fréchet $G_{\mathbb{R}}$ -modules $D^b(\mathcal{F}_{G_{\mathbb{R}}})$, for $\mathfrak{M} \in D_{G,coh}^b(\mathcal{D}_X)$ and $\mathcal{T} \in D_{G_{\mathbb{R}},\mathbb{R}-c}^b(\mathbb{C}_X)$. For this purpose, we consider – slightly more generally than in the previous paragraph – an affine algebraic group G , a quasi-projective G -manifold X , and a real form $G_{\mathbb{R}}$ in G^{an} . Since our construction involves both the algebraic and analytic structure, we shall now make notational distinctions, for example, between the structure sheaves $\mathcal{O}_X$ of the algebraic variety X and $\mathcal{O}_{X^{an}}$ of the complex manifold X^{an} . We write $\mathcal{F}$ for the category of Fréchet spaces and $\mathcal{F}_{G_{\mathbb{R}}}$ for the category of Fréchet $G_{\mathbb{R}}$ -modules.

Let us work backwards to justify and motivate our definition. As before, p_n shall denote the projection from $X \times V_n$ to X , with $\{V_n\}$ as in (5.1). If we disregard the topology and $G_{\mathbb{R}}$ -action for the moment,

$$\text{RHom}_{\mathcal{D}_{X \times V_n}}(L p_n^* \mathfrak{M} \otimes p_n^{-1} \mathcal{T}, \mathcal{O}_{(X \times V_n)^{an}}) \simeq \text{RHom}_{\mathcal{D}_X}(\mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_{X^{an}}) \otimes \text{R}\Gamma(\mathbb{C}_{V_n});$$

³[MUV] establish a slightly different equivalence, but their arguments can also be used to show that (5.8) is an equivalence.

hence, and because of (5.2b),

$$(5.9) \quad \mathrm{RHom}_{\mathcal{D}_X}(\mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_{X^{an}}) \simeq \varprojlim_n \mathrm{RHom}_{\mathcal{D}_{X \times V_n}}(Lp_n^* \mathfrak{M} \otimes p_n^{-1} \mathcal{T}, \mathcal{O}_{(X \times V_n)^{an}}).$$

This, in effect, allows us to replace X by $X \times V_n$; in other words, we may as well assume that G acts freely on X . But then $\mathcal{T}$ is isomorphic to the inverse image $\pi^{-1}\mathcal{S}$ of some $\mathcal{S} \in D_{\mathbb{R}-c}^b(\mathbb{C}_{G_{\mathbb{R}} \backslash X})$ under the quotient map $\pi : X \rightarrow G_{\mathbb{R}} \backslash X$. Like any object in the latter derived category, $\mathcal{S}$ can be represented by a complex whose terms are finite direct sums of sheaves $j_! \mathbb{C}_U$, where j is the inclusion of some open, semi-algebraic subset $U \subset G_{\mathbb{R}} \backslash X$. Thus $\mathcal{T}$ can be similarly represented, but the open subsets in question are then inverse images of open subsets of $G_{\mathbb{R}} \backslash X$, i.e., they are $G_{\mathbb{R}}$ -invariant open subsets of X . We now replace $\mathcal{O}_{X^{an}}$ by the C^∞ Dolbeault complex $\mathcal{A}^{(0,\cdot)}$, to which it is quasi-isomorphic, and $\mathfrak{M}$ by a bounded complex of locally free quasi- G -equivariant $\mathcal{D}_X$ -modules – cf. (4.7). Again neglecting the topology and $G_{\mathbb{R}}$ -action,

$$(5.10) \quad \begin{aligned} \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathfrak{F} \otimes j_! \mathbb{C}_U, \mathcal{A}^{(0,\cdot)}) &\simeq \mathrm{RHom}_{\mathcal{O}_X}(\mathfrak{F} \otimes j_! \mathbb{C}_U, \mathcal{A}^{(0,\cdot)}) \simeq \\ &\simeq \mathrm{R}\Gamma(U; (\mathfrak{F}^*)^{an} \otimes_{\mathcal{O}_{X^{an}}} \mathcal{A}^{(0,\cdot)}) \simeq \Gamma(U; (\mathfrak{F}^*)^{an} \otimes_{\mathcal{O}_{X^{an}}} \mathcal{A}^{(0,\cdot)}), \end{aligned}$$

for every $\mathfrak{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathfrak{F} \in \mathrm{Mod}_G^{lf}(\mathcal{D}_X)$, with $\mathfrak{F} \in \mathrm{Mod}_G^{coh}(\mathcal{O}_X)$ locally free, and every inclusion $j : U \hookrightarrow X$ of a $G_{\mathbb{R}}$ -invariant, open subset U of X .

The complex on the right in (5.10) has a natural Fréchet topology – the C^∞ topology for differential forms – and continuous $G_{\mathbb{R}}$ -action. That, in conjunction with the acyclicity asserted by (5.10), makes it possible to define the functor

$$(5.11) \quad \begin{aligned} D_{G,coh}^b(\mathcal{D}_X) \times D_{G_{\mathbb{R}},\mathbb{R}-c}^b(\mathbb{C}_X) &\longrightarrow D^b(\mathcal{F}_{G_{\mathbb{R}}}), \\ (\mathfrak{M}, \mathcal{T}) &\longmapsto \mathrm{RHom}_{\mathcal{D}_X}^{top}(\mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_{X^{an}}). \end{aligned}$$

Formally, this functor is a projective limit with respect to the tower (5.1), and an inductive limit with respect to the choice of a complex in $\mathrm{Mod}_G^{lf}(\mathcal{D}_X)$ quasi-isomorphic to $\mathfrak{M}$, and choices of particular representatives of $\mathcal{T}$.

The functor (5.11) interchanges the roles of direct and inverse images on its two arguments: let $f : X \rightarrow Y$ be a G -equivariant morphism between algebraic, quasi-projective G -manifolds X, Y ; then

5.12 Theorem. *If f is projective, there exists an isomorphism*

$$\mathrm{RHom}_{\mathcal{D}_X}^{top}(\mathfrak{M} \otimes f_{an}^{-1} \mathcal{T}, \mathcal{O}_{X^{an}}) \simeq \mathrm{RHom}_{\mathcal{D}_Y}^{top}\left(\int_f \mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_{Y^{an}}\right)[- \dim X/Y],$$

functorially in $\mathfrak{M} \in \mathcal{D}_{G,coh}^b(\mathcal{D}_X)$ and $\mathcal{T} \in D_{G_{\mathbb{R}},\mathbb{R}-c}^b(\mathbb{C}_Y)$. If f is smooth,

$$\mathrm{RHom}_{\mathcal{D}_X}^{top}(Lf^* \mathfrak{M} \otimes \mathcal{T}, \mathcal{O}_{X^{an}}) \simeq \mathrm{RHom}_{\mathcal{D}_Y}^{top}(\mathfrak{M} \otimes R(f_{an})_* \mathcal{T}, \mathcal{O}_{Y^{an}})[-2 \dim X/Y],$$

functorially in $\mathfrak{M} \in \mathcal{D}_{G,coh}^b(\mathcal{D}_Y)$ and $\mathcal{T} \in D_{G_{\mathbb{R}},\mathbb{R}-c}^b(\mathbb{C}_X)$.

We end this section with the statement of our main technical result, from which the other results will follow. We return to our earlier hypotheses, with X denoting the flag variety of $\mathfrak{g}$, the Lie algebra of the connected reductive group G . The groups $K_{\mathbb{R}}$ and K have the same meaning as in the introduction, $i : S_{\mathbb{R}} \hookrightarrow S$ is the inclusion (5.6), and p, q are the projections from $X \times S$ to the two factors, as in (5.7). We view X, S as algebraic G -manifolds, and $S_{\mathbb{R}}$ as real analytic manifold.

The twisted version of the covariant Riemann-Hilbert correspondence [K3] establishes an equivalence of categories

$$(5.13) \quad \begin{aligned} DR_X : D_{K-eq,coh}^b(\mathcal{D}_{X,\lambda}) &\xrightarrow{\sim} D_{K,-\lambda,\mathbb{C}-c}^b(\mathbb{C}_X), \\ DR_X(\mathfrak{M}) &= R\mathcal{H}om_{\mathcal{D}_{X,\lambda}}(\mathcal{O}_X(\lambda), \mathfrak{M}), \end{aligned}$$

between $D_{K-eq,coh}^b(\mathcal{D}_{X,\lambda})$, the bounded K -equivariant derived category of coherent $\mathcal{D}_{X,\lambda}$ -modules⁴, and $D_{K,-\lambda,\mathbb{C}-c}^b(\mathbb{C}_X)$, the bounded K -equivariant, $\mathbb{C}$ -constructible derived category with twist $-\lambda$. A word about the twists: $\mathfrak{M}$ is a complex of sheaves on X , i.e., sheaves without twist, over the ring of twisted differential operators $\mathcal{D}_{X,\lambda}$, whereas $\mathcal{O}_X(\lambda)$ is a twisted sheaf, with twist λ , of $\mathcal{D}_{X,\lambda}$ -modules. Thus it makes sense to apply the functor $R\mathcal{H}om$ over $\mathcal{D}_X$ to this pair, and the result will be an object in the derived category with twist opposite to that of $\mathcal{O}_X(\lambda)$, since $R\mathcal{H}om$ is contravariant in the first variable. The Riemann-Hilbert correspondence is compatible with induction: if

$$(5.14) \quad \text{Ind}_K^G : D_{K-eq,coh}^b(\mathcal{D}_{X,\lambda}) \longrightarrow D_{G-eq,coh}^b(\mathcal{D}_{X \times S,\lambda})$$

is defined analogously to the induction functor (5.5), then

$$(5.15) \quad \begin{array}{ccc} D_{K-eq,coh}^b(\mathcal{D}_{X,\lambda}) & \xrightarrow{\text{Ind}_K^G} & D_{G-eq,coh}^b(\mathcal{D}_{X \times S,\lambda}) \\ DR_X \downarrow & & \downarrow DR_{X \times S} \\ D_{K,-\lambda,\mathbb{C}-c}^b(\mathbb{C}_X) & \xrightarrow{\text{Ind}_K^G} & D_{G,-\lambda,\mathbb{C}-c}^b(\mathbb{C}_{X \times S}) \end{array}$$

commutes. The following result is now a consequence, essentially, of theorem 5.12, (5.15), and other functorial properties of the deRham functor:

5.16 Theorem. For $\mathfrak{M} \in D_{G,coh}^b(\mathcal{D}_{X,-\lambda})$, $\mathfrak{L} \in D_{K-eq,coh}^b(\mathcal{D}_{X,\lambda})$, and $\mathcal{L} = DR_X(\mathfrak{L}) \in D_{K,-\lambda,\mathbb{C}-c}^b(\mathbb{C}_X)$,

$$\begin{aligned} R\text{Hom}_{\mathcal{D}_S}^{top} \left(\int_q (Lp^* \mathfrak{M} \otimes_{\mathcal{O}_{X \times S}} \text{Ind}_K^G(\mathfrak{L})) \otimes i_* i^! \mathbb{C}_S, \mathcal{O}_{S^{an}} \right) &\simeq \\ &\simeq R\text{Hom}_{\mathcal{D}_{X,-\lambda}}^{top}(\mathfrak{M} \otimes \Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-\lambda))[\dim X] \end{aligned}$$

⁴coherent K -equivariant $\mathcal{D}_{X,\lambda}$ -modules are necessarily holonomic, since K operates on X with finitely many orbits.

as objects in $D^b(\mathcal{F}_{G_{\mathbb{R}}})$.

On the left in this identity, the (complexes of) sheaves $Lp^* \mathfrak{M}$, $\text{Ind}_K^G(\mathfrak{L})$ are modules over the rings of twisted differential operators $\mathcal{D}_{X \times S, -\lambda}$ and $\mathcal{D}_{X \times S, \lambda}$, respectively, so their tensor product over $\mathcal{O}_{X \times S}$ becomes a module for the ring of (untwisted) differential operators $\mathcal{D}_{X \times S}$ via the “twisted comultiplication”

$$(5.17) \quad \mathcal{D}_{X \times S} \longrightarrow \mathcal{D}_{X \times S, -\lambda} \otimes_{\mathcal{O}_{X \times S}} \mathcal{D}_{X \times S, \lambda}.$$

The direct image of this tensor product is simply G -equivariant “integration over the fibers” of a complex in $D_{G, coh}^b(\mathcal{D}_{X \times S})$. On the right hand side in (5.16), $\mathfrak{M}$ is an untwisted module over the ring of twisted differential operators $\mathcal{D}_{X, -\lambda}$, and the sheaf of $\mathbb{C}_X$ -modules $\Phi(\mathcal{L})$ has twist $-\lambda$, so their tensor product – this time over $\mathbb{C}_X$ – becomes a $\mathcal{D}_{X, -\lambda}$ -module with twist $-\lambda$, i.e., with the same twist as $\mathcal{O}_{X^{an}}(-\lambda)$.

We need to comment on our notational convention concerning twists. To keep the discussion in the introduction brief, we tacitly incorporated the shift by ρ (=one half of the sum of the positive roots), as is customary in representation theory. In the context of $\mathcal{D}$ -modules, this ρ -shift would affect the definition of inverse image. Thus, beginning with the present section, we normalize twists so that $\lambda = 0$ corresponds to the untwisted case.

If one disregards both the topology and $G_{\mathbb{R}}$ -action, one can re-interpret the left hand side of the identification (5.16) as

$$\text{R Hom}_{\mathcal{D}_S} \left(\int_q (Lp^* \mathfrak{M} \otimes_{\mathcal{O}_{X \times S}} \text{Ind}_K^G(\mathfrak{L})), \mathcal{C}_{S_{\mathbb{R}}}^{-\omega} \right),$$

where $\mathcal{C}_{S_{\mathbb{R}}}^{-\omega}$ denotes the sheaf of hyperfunctions on $S_{\mathbb{R}}$. Thus (5.16) amounts to a Poisson transform from $G_{\mathbb{R}}$ -modules, geometrically realized on the symmetric space $S_{\mathbb{R}}$, to the same $G_{\mathbb{R}}$ -modules, but now realized on the flag variety X .

Under suitable ellipticity hypotheses, hyperfunction solutions are necessarily smooth. Concretely, the space $\text{Hom}_{\mathcal{D}_S}(\mathfrak{N}, \mathcal{C}_{S_{\mathbb{R}}}^{\infty})$, for any $\mathfrak{N} \in \text{Mod}_G^{coh}(\mathcal{D}_S)$, has a natural Fréchet topology and continuous $G_{\mathbb{R}}$ -action, as a consequence of (4.7), for example. If $\mathfrak{N}$ is elliptic along $S_{\mathbb{R}}$, in the sense that its characteristic variety $Ch(\mathfrak{N})$ intersects the conormal bundle $T_{S_{\mathbb{R}}}^* S$ of $S_{\mathbb{R}}$ only in the zero section, then this C^{∞} solution space coincides with the space of hyperfunction solutions, as topologized $G_{\mathbb{R}}$ -module:

5.18 Proposition. $\text{Hom}_{\mathcal{D}_S}(\mathfrak{N}, \mathcal{C}_{S_{\mathbb{R}}}^{\infty}) \simeq {}^q H^0(\text{R Hom}_{\mathcal{D}_S}^{top}(\mathfrak{N} \otimes i_* i^! \mathbb{C}_S, \mathcal{O}_{S^{an}})),$ as objects in $\mathcal{Q}(\mathcal{F}_{G_{\mathbb{R}}})$, provided $Ch(\mathfrak{N}) \cap T_{S_{\mathbb{R}}}^* S \subset T_S^* S$.

§6 Proof of the Conjectures.

We now have the machinery in hand to prove (1.1b-f), as well as (2.12) and the exactness of the functors mg , MG . The crux of the matter is to identify both sides in (5.16) explicitly for particular choices of $\mathfrak{M}$ and $\mathfrak{L}$. Throughout this section, the notation of (5.16) shall remain in force.

For any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, we identify the quotient $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ with the universal Cartan algebra by specifying the the set of weights of $\mathfrak{g}/\mathfrak{b}$ as the set of positive roots.

This differs from the convention in [BB1,K3], but has the advantage of making dominant weights correspond to positive line bundles. In particular, if ρ denotes one half of the sum of the positive roots, then

$$(6.1) \quad \mathcal{O}_X(2\rho) \simeq (\Omega_X^d)^{-1} \quad (d = \dim X)$$

is the reciprocal of the canonical sheaf.

The equivalence of categories (4.4) induces an equivalence of derived categories

$$(6.2) \quad \phi : D_G^b(\mathcal{D}_S) \xrightarrow{\sim} D^b(\text{Mod}(\mathfrak{g}, K)).$$

We note that the original definition of ϕ can be re-interpreted as the $\mathcal{D}$ -module inverse image functor (4.9) corresponding to the inclusion $\{eK\} \hookrightarrow G/K = S$. Our next lemma follows from base change in the Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & X \times S \\ \downarrow & & \downarrow \\ \{eK\} & \longrightarrow & S, \end{array}$$

applied to the $\mathcal{D}_S$ -module $Lp^* \mathfrak{M} \otimes_{\mathcal{O}_X} \text{Ind}_K^G(\mathfrak{L})$, and with $\mathcal{D}_{X,-\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1}$ in the role of $\mathfrak{M}$.

6.3 Lemma. *For $\mathfrak{M} = \mathcal{D}_{X,-\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1}$ and $\mathfrak{L} \in D_{K-eg,coh}^b(\mathcal{D}_{X,\lambda})$,*

$$\int_q \left(Lp^* \mathfrak{M} \otimes_{\mathcal{O}_X} \text{Ind}_K^G(\mathfrak{L}) \right) \simeq \phi^{-1}(R\Gamma(\mathfrak{L})).$$

The characteristic variety of any $\mathfrak{N} \in \text{Mod}_G^{coh}(\mathcal{D}_S)$ is G -invariant and intersects $T_{eK}^* S \simeq (\mathfrak{g}/\mathfrak{k})^*$ exactly in the characteristic variety⁵ of the finitely generated $(\mathfrak{g}, K)$ -module $\phi(\mathfrak{N})$. If, moreover, $\mathfrak{N}$ is annihilated by an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g})$ – equivalently, if $\phi(\mathfrak{N})$ is a Harish-Chandra module [HC1] – $Ch(\phi(\mathfrak{N}))$ lies in the nilpotent cone when one identifies $\mathfrak{g}^* \simeq \mathfrak{g}$ via a non-degenerate, Ad -invariant symmetric bilinear form. On the other hand, $T_{S_{\mathbb{R}}}^* S \cap T_{eK}^* S \simeq (\mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}})^*$ consists of semisimple elements, so the assumptions on $\mathfrak{N}$ ensure that $\mathfrak{N}$ satisfies the ellipticity hypothesis of (5.18). Recall the notation (2.3) for the dual of a Harish-Chandra module. At this point, the definition of the $\mathcal{U}(\mathfrak{g})$ -module structure (4.5) and the definition of the functor MG imply:

6.4 Lemma. *Suppose $\mathfrak{N} \in \text{Mod}_G^{coh}(\mathcal{D}_S)$ is annihilated by an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g})$. Then $\phi(\mathfrak{N})$ is a Harish-Chandra module, and*

$$MG(\phi(\mathfrak{N})') \simeq \text{Hom}_{\mathcal{D}_S}(\mathfrak{N}, \mathcal{C}_{S_{\mathbb{R}}}^\infty) \simeq {}^q H^0(R\text{Hom}_{\mathcal{D}_S}^{top}(\mathfrak{N} \otimes i_* i^! \mathbb{C}_S, \mathcal{O}_{S^{an}})),$$

as objects in $\mathcal{Q}(\mathcal{F}_{G_{\mathbb{R}}})$.

We now combine the previous two lemmas with theorem (5.16). We suppose that $\mathfrak{L} \in D_{K-eg,coh}^b(\mathcal{D}_{X,\lambda})$ satisfies the vanishing condition

$$(6.5) \quad H^n(X, \mathfrak{L}) = 0 \quad \text{for } n \neq 0,$$

and set $\mathcal{L} = DR_X(\mathfrak{L})$, as before. Then $H^0(X, \mathfrak{L})$ is a Harish-Chandra module, hence

⁵i.e., the “associated variety” in the terminology of [Vo].

6.6 Corollary. *Under the hypotheses just stated,*

$$MG(H^0(X, \mathcal{L})') \simeq {}^q H^0 \left(R \operatorname{Hom}_{\mathcal{D}_{X, -\lambda}}^{\operatorname{top}} (\mathcal{D}_{X, -\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1} \otimes \Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-\lambda)) [d] \right),$$

as objects in $\mathcal{Q}(\mathcal{F}_{G_{\mathbb{R}}})$.

Disregarding both the topology and $G_{\mathbb{R}}$ -action for the moment, we can make the further identifications

$$\begin{aligned} (6.7) \quad R \operatorname{Hom}_{\mathcal{D}_{X, -\lambda}} (\mathcal{D}_{X, -\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1} \otimes \Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-\lambda)) [d] &\simeq \\ &\simeq R \operatorname{Hom}_{\mathcal{O}_X} ((\Omega_X^d)^{-1} \otimes \Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-\lambda)) [d] \simeq \\ &\simeq R \Gamma(R \mathcal{H}om(\Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-2\rho - \lambda)) [d]); \end{aligned}$$

cf. (6.1). Here we view $\Phi(\mathcal{L})$ as object in $D_{K, -2\rho - \lambda, \mathbb{C}-c}^b(\mathbb{C}_X)$, as we may: 2ρ is an integral weight, and this implies the existence of a canonical isomorphism

$$(6.8) \quad D_{K, -2\rho - \lambda, \mathbb{C}-c}^b(\mathbb{C}_X) \simeq D_{K, -\lambda, \mathbb{C}-c}^b(\mathbb{C}_X).$$

We shall apply (6.7) more specifically in the case of a $\mathcal{D}_{X, \lambda}$ -module $\mathcal{L}$ which corresponds to the Harish-Chandra module (2.11) via the Beilinson-Bernstein equivalence.

We use the notation of (2.10-11). Let $\pi : X \rightarrow Y$ denote the natural projection. Then $D = \pi^{-1}(Y_{\mathbb{R}})$ is the unique closed $G_{\mathbb{R}}$ -orbit in X . It is contained in the unique open K -orbit $Q \subset X$, and the Matsuki correspondence pairs the two orbits D, Q . The highest weight spaces $\mathbf{E}_{\pi(x)}/\mathfrak{n}_x \mathbf{E}_{\pi(x)}$, as x ranges over D , with $\mathfrak{b}_x = \text{stabilizer of } x \text{ in } \mathfrak{g}$ and $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$, constitute the fibers of a $G_{\mathbb{R}}$ -equivariant line bundle $\mathbf{L}$ over D . This line bundle extends to a K -equivariant algebraic line bundle over the K -orbit Q , and we refer to the extension by the same letter $\mathbf{L}$. As is the case with all K -orbits in X , the inclusion of the open orbit $i : Q \hookrightarrow X$ is an affine morphism, so the $\mathcal{D}$ -module direct image of $\mathcal{O}_Q(\mathbf{L})$ coincides with the sheaf direct image

$$(6.9a) \quad \mathcal{L} = i_* \mathcal{O}_Q(\mathbf{L}) \in \operatorname{Mod}_{K-eq}^{\operatorname{coh}}(\mathcal{D}_{X, \lambda}),$$

and the higher (sheaf) direct images vanish; here λ denotes the highest weight of $\mathbf{E}$. Moreover,

$$(6.9b) \quad H^0(X, \mathcal{L}) = \operatorname{Ind}_{(\mathfrak{p}, K_{\mathbb{R}} \cap P_{\mathbb{R}})}^{(\mathfrak{g}, K)}(E), \quad H^n(X, \mathcal{L}) = 0 \text{ if } n \neq 0.$$

Proofs of these assertions can be found in [HMSW], for example.

The Riemann-Hilbert correspondence relates the $\mathcal{D}$ -module direct image to the direct image in the derived category of sheaves of $\mathbb{C}$ -vector spaces, so

$$(6.10a) \quad \mathcal{L} =_{\operatorname{def}} DR_X(\mathcal{L}) \simeq Ri_*(DR_Q(\mathcal{O}_Q(\mathbf{L})))$$

is the direct image of a K -equivariant, twisted local system on Q . The proof of (1.1a) in [MUV] gives a description of $\Phi(\mathcal{L})$ for this particular sheaf $\mathcal{L}$, namely

$$(6.10b) \quad \Phi(\mathcal{L}) \simeq j_* j^{-1} \mathcal{L},$$

with j denoting the inclusion $D \hookrightarrow X$. We now use the fact that $\pi : D \rightarrow Y_{\mathbb{R}}$ is a real analytic fibration, with smooth complex projective fibers, and Bott's description of the cohomology of $\mathcal{O}_{X^{an}}(-2\rho - \lambda)$ along the fibers, to conclude

$$(6.11) \quad \begin{aligned} & H^n(\mathrm{R}\Gamma(\mathrm{R}\mathcal{H}om(\Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-2\rho - \lambda))[d])) \simeq \\ & \simeq \begin{cases} C^{-\omega}(Y_{\mathbb{R}}, \mathbf{E}^* \otimes \wedge^{max} T^*Y) & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, \end{cases} \end{aligned}$$

still without regard to the topology and $G_{\mathbb{R}}$ -action. Formally, the identity (6.11) should involve also the orientation sheaf of $Y_{\mathbb{R}}$, but the connectivity assumption about G implies that $Y_{\mathbb{R}}$ is orientable. In the derived category of Fréchet $G_{\mathbb{R}}$ -modules, vanishing of the ordinary cohomology forces exactness. Hence, in view of (6.6-8), we have shown:

6.12 Proposition. *If $\mathfrak{L}$ satisfies the hypotheses (6.9),*

$$\begin{aligned} {}^q H^n \left(\mathrm{R}\mathrm{Hom}_{\mathcal{D}_{X, -\lambda}}^{top} (\mathcal{D}_{X, -\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1} \otimes \Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-\lambda))[d] \right) & \simeq \\ & \simeq \begin{cases} MG \left(\mathrm{Ind}_{(\mathfrak{p}, K_{\mathbb{R}} \cap P_{\mathbb{R}})}^{(\mathfrak{g}, K)} (E^* \otimes \wedge^{max} (\mathfrak{g}/\mathfrak{p})^*) \right) & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases} \end{aligned}$$

In the next statement, $C^\infty(G_{\mathbb{R}})_{K_{\mathbb{R}}-fini}$ shall denote the space of C^∞ functions on $G_{\mathbb{R}}$ which are $K_{\mathbb{R}}$ -finite under the right action. We regard this space as a left $(\mathfrak{g}, K)$ -module by composing the right action with the canonical anti-automorphism of $\mathfrak{g}$.

6.13 Theorem. *For every Harish-Chandra module M and every $n \neq 0$,*

$$\mathrm{Ext}_{(\mathfrak{g}, K)}^n(M, C^\infty(G_{\mathbb{R}})_{K_{\mathbb{R}}-fini}) \simeq \mathrm{Ext}_{\mathcal{D}_S}^n(\phi^{-1}M \otimes i_* i^! \mathbb{C}_S, \mathcal{O}_{S^{an}}) = 0.$$

The isomorphism between the two Ext groups is formal. For any member M of the principal series, the vanishing of the higher Ext groups follows from (5.16) and (6.12). It suffices to prove the vanishing for irreducible Harish-Chandra modules. That can be done by downward induction on n – irreducible Harish-Chandra modules can be realized as submodules of modules belonging to the principal series [C1, BB2]; for large n , vanishing follows from the finiteness of the global dimension of $\mathcal{D}_S$, or alternatively, from the analogous finiteness statement in the category $\mathrm{Mod}(\mathfrak{g}, K)$.

Theorem 6.13 implies the exactness of the functor MG , and by duality, also of mg . Indeed, we may replace $C^\infty(G_{\mathbb{R}})$ by $C^\infty(G_{\mathbb{R}})_{K_{\mathbb{R}}-fini}$ in the definition (2.5), and $(\mathfrak{g}, K_{\mathbb{R}})$ -invariance by $(\mathfrak{g}, K)$ -invariance – the image of a $K_{\mathbb{R}}$ -finite vector under a $K_{\mathbb{R}}$ -invariant linear map is necessarily $K_{\mathbb{R}}$ -finite also – so the theorem applies directly. The vanishing of the higher Ext groups means, in particular, that $\mathrm{R}\mathrm{Hom}_{(\mathfrak{g}, K)}(M, C^\infty(G_{\mathbb{R}})_{K_{\mathbb{R}}-fini}) \in D^b(\mathcal{F}_{G_{\mathbb{R}}})$ is strict. That, in turn, insures that the induced topology on $C_0^\infty(G_{\mathbb{R}}) \otimes_{(\mathfrak{g}, K_{\mathbb{R}})} M$ is Hausdorff, so the phrase “largest separated quotient” in the definition (2.7) becomes unnecessary.

The isomorphisms (6.7) justify the following definition. For $\mathcal{S} \in D_{G_{\mathbb{R}}, \lambda, \mathbb{R}-c}^b(\mathbb{C}_X)$, we regard $\mathrm{RHom}(\mathcal{S}, \mathcal{O}_{X^{an}}(\lambda))$ as object in the derived category of Fréchet $G_{\mathbb{R}}$ -modules via the identification

$$(6.14) \quad \mathrm{RHom}(\mathcal{S}, \mathcal{O}_{X^{an}}(\lambda)) \simeq \mathrm{RHom}_{\mathcal{D}_{X, \lambda}}^{\mathrm{top}}(\mathcal{D}_{\mathcal{D}_{X, \lambda}} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1} \otimes \mathcal{S}, \mathcal{O}_{X^{an}}(\lambda + 2\rho)).$$

According to (5.16), (6.6) and (6.13), this complex has the property MG and satisfies (1.1b,d,f)⁶, at least if $\mathcal{L} = (DR_X)^{-1}\Phi^{-1}(\mathcal{S}) \in D_{K-eg, coh}^b(\mathcal{D}_{X, -\lambda-2\rho})$ has non-zero cohomology in only one degree; the general case follows by means of standard techniques in representation theory [SW]. The definition of $\mathrm{R}\Gamma(\mathcal{S} \otimes \mathcal{O}_{X^{an}}(-\lambda - 2\rho))$ as object in the derived category of DNF $G_{\mathbb{R}}$ -modules is analogous and formally dual to the preceding case. The duality between the two globalization functors then implies the property mg and the assertions (1c,d,e) for the latter complex.

Until now, we have used (6.7) and (6.11) only to establish the vanishing of the higher Ext groups in (6.12). A careful examination of this chain of isomorphisms shows that the topology and $G_{\mathbb{R}}$ -actions are preserved, i.e.,

$$(6.15) \quad {}^q\mathrm{H}^0\left(\mathrm{RHom}_{\mathcal{D}_{X, -\lambda}}^{\mathrm{top}}(\mathcal{D}_{X, -\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1} \otimes \Phi(\mathcal{L}), \mathcal{O}_{X^{an}}(-\lambda))[d]\right) \simeq C^{-\omega}(Y_{\mathbb{R}}, \mathbf{E}^* \otimes \wedge^{\mathrm{max}} T^*Y),$$

as objects in $\mathcal{Q}(\mathcal{F}_{G_{\mathbb{R}}})$, if $\mathcal{L}$ is chosen as in (6.9). That, in conjunction with (6.6), implies the second half of theorem 2.12 – the other half follows by duality.

§7 Invariant Systems of Differential Equations.

In this section, Z shall denote a quasi-projective G -manifold, $Z_{\mathbb{R}}$ a $G_{\mathbb{R}}$ -invariant real form, $i : Z_{\mathbb{R}} \hookrightarrow Z$ the inclusion map, and $\mathfrak{M}$ a coherent, quasi- G -equivariant $\mathcal{D}_Z$ -module. We regard $\mathrm{RHom}_{\mathcal{D}_Z}(\mathfrak{M}, \mathcal{C}_{Z_{\mathbb{R}}}^{-\omega})$ as an object in the derived category of Fréchet $G_{\mathbb{R}}$ -modules $D^b(\mathcal{F}_{G_{\mathbb{R}}})$ by making the identification

$$(7.1) \quad \mathrm{RHom}_{\mathcal{D}_Z}^{\mathrm{top}}(\mathfrak{M}, \mathcal{C}_{Z_{\mathbb{R}}}^{-\omega}) = \mathrm{RHom}_{\mathcal{D}_Z}^{\mathrm{top}}(\mathfrak{M} \otimes i_* i^! \mathbb{C}_Z, \mathcal{O}_{Z^{an}}).$$

We shall show, under appropriate hypotheses, that this object has the property MG. In particular, the space of hyperfunction solutions of the restricted system on $Z_{\mathbb{R}}$ will then have a natural Fréchet topology and continuous $G_{\mathbb{R}}$ -action, and the resulting representation will be admissible, of finite length.

Recall the definition of the homomorphism $\gamma : \mathcal{U}(\mathfrak{g}) \rightarrow \mathrm{End}_{\mathcal{D}_Z}(\mathfrak{M})$ in §4, and let $\mu_Z : T^*Z \rightarrow \mathfrak{g}^*$ denote the moment map. A calculation with a good filtration of $\mathfrak{M}$ by G -equivariant coherent $\mathcal{O}_Z$ -modules shows:

$$(7.2) \quad \mathrm{Ch}(\mathfrak{M}) \subset \mu_Z^{-1}(\mathrm{Ch}(\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\gamma}(\mathfrak{M}))),$$

⁶the reasons for the appearance of 2ρ in the present discussion and for its absence in the introduction were explained in §5.

where $\text{Ann}_\gamma(\mathfrak{M})$ denotes the annihilator of $\mathfrak{M}$ in $\mathcal{U}(\mathfrak{g})$ with respect to the action γ . If $\mathfrak{M}$ is not only quasi- G -equivariant as $\mathcal{D}_Z$ -module, but G -equivariant, then the entire augmentation ideal annihilates $\mathfrak{M}$, so

$$(7.3) \quad \text{Ch}(\mathfrak{M}) \subset \mu_Z^{-1}(0) \quad \text{if } \mathfrak{M} \in \text{Mod}_{G\text{-eq}}^{\text{coh}}(\mathcal{D}_Z).$$

The nilpotent cone $\mathcal{N}^* \subset \mathfrak{g}^*$ – i.e., the image of the nilpotent cone $\mathcal{N} \in \mathfrak{g}$ when $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ by means of an Ad -invariant, nondegenerate symmetric bilinear form – is the variety defined by the augmentation ideal in $S(\mathfrak{g})^G$, hence

$$(7.4) \quad \text{Ch}(\mathfrak{M}) \subset \mu_Z^{-1}(\mathcal{N}^*) \quad \text{if } \mathfrak{M} \text{ is } \mathcal{Z}(\mathfrak{g})\text{-finite};$$

here $\mathcal{Z}(\mathfrak{g})$ -finiteness means that some ideal of finite codimension $\mathcal{I} \subset \mathcal{Z}(\mathfrak{g})$ (=center of $\mathcal{U}(\mathfrak{g})$) annihilates $\mathfrak{M}$.

Borel subalgebras are solvable. Hence, by arguments in either [KMF] or [G, appendix],

$$(7.5) \quad \text{Ch}(\mathfrak{M}) \cap \mu_Z^{-1}(\mathfrak{b}^\perp) \subset T^*Z \text{ is an involutive subvariety,}$$

for any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. We shall call $\mathfrak{M}$ *admissible* if this involutive subvariety is Lagrangian, for every $\mathfrak{b}$ – or equivalently, for some $\mathfrak{b}$, since the action of G preserves $\text{Ch}(\mathfrak{M})$.

7.6 Theorem. *Let $\mathfrak{M} \in \text{Mod}_G^{\text{coh}}(\mathcal{D}_Z)$ be admissible and $\mathcal{Z}(\mathfrak{g})$ -finite, and $\mathcal{S}$ an object in the bounded equivariant derived category $D_{G_\mathbb{R}, \mathbb{R}-c}^b(\mathbb{C}_Z)$. Then $\text{RHom}_{\mathcal{D}_Z}^{\text{top}}(\mathfrak{M} \otimes \mathcal{S}, \mathcal{O}_{Z^{an}})$, as object in $D^b(\mathcal{F}_{G_\mathbb{R}})$, has the property MG.*

This statement neither involves, nor depends on, the existence of a $G_\mathbb{R}$ -invariant real form $Z_\mathbb{R}$. However, when such a real form does exist, the theorem, with $i_*i^!\mathbb{C}_Z$ in place of $\mathcal{S}$, provides the criterion alluded to at the beginning of this section.

We begin the sketch of the proof of (7.6) with some general remarks. The forgetful functor

$$(7.7a) \quad \text{Mod}_{G\text{-eq}}(\mathcal{D}_Z) \longrightarrow \text{Mod}_G(\mathcal{D}_Z)$$

has a left adjoint,

$$(7.7b) \quad EQ : \text{Mod}_G(\mathcal{D}_Z) \longrightarrow \text{Mod}_{G\text{-eq}}(\mathcal{D}_Z),$$

given by $EQ(\mathfrak{M}) = \mathfrak{M}/\gamma(\mathfrak{g})\mathfrak{M}$; here $\gamma(\mathfrak{g})\mathfrak{M}$ denotes the image in $\mathfrak{M}$ of the quasi- G -equivariant $\mathcal{D}_Z$ -module $\mathfrak{g} \otimes \mathfrak{M}$. The functor EQ is visibly right exact.

Coherent, $\mathcal{Z}(\mathfrak{g})$ -finite, quasi- G -equivariant $\mathcal{D}_Z$ -modules admit finite filtrations such that the successive quotients are modules with an infinitesimal character, i.e., modules on which $\mathcal{Z}(\mathfrak{g})$ acts by a character. Also, if any two objects in a distinguished triangle in $D^b(\mathcal{F}_{G_\mathbb{R}})$ have the property MG, then so does the third. This allows us to assume, without loss of generality, that $\mathfrak{M}$ itself has an infinitesimal character. To be consistent with our notational choices in §§5,6, we index characters of $\mathcal{Z}(\mathfrak{g})$ by linear functionals λ on the

universal Cartan without the customary shift by ρ (= half sum of the positive roots); in other words, $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ denotes the character by which $\mathcal{Z}(\mathfrak{g})$ acts on the Verma module with highest weight λ . Then $\chi_\lambda = \chi_\mu$ if and only if $\lambda + \rho$ is conjugate to $\mu + \rho$ under the action of the Weyl group W . We let $\text{Mod}_G^\lambda(\mathcal{D}_Z)$ denote the full subcategory of $\text{Mod}_G(\mathcal{D}_Z)$ consisting of modules with infinitesimal character χ_λ , and $\text{Mod}_G^{\text{coh},\lambda}(\mathcal{D}_Z)$ the full subcategory of coherent modules in $\text{Mod}_G^\lambda(\mathcal{D}_Z)$. Because of our earlier assumption, $\mathfrak{M}$ belongs to one of these subcategories:

$$(7.8) \quad \mathfrak{M} \in \text{Mod}_G^{\text{coh},\lambda}(\mathcal{D}_Z).$$

Replacing $\lambda + \rho$ by an appropriate W -translate, we can arrange

$$(7.9) \quad \langle \check{\alpha}, \lambda + \rho \rangle \notin \mathbb{Z}_{<0}, \quad \text{for every positive coroot } \check{\alpha};$$

we shall refer to this condition by saying that $\lambda + \rho$ is *integrally dominant*.

For the moment, we do not assume (7.9). We shall consider modules over the ring of twisted differential operators $\mathcal{D}_{X \times Z, \lambda}$ on the product of the flag variety X with Z ; the twisting is confined to the factor X , and is indexed by the parameter λ . We write $p : X \times Z \rightarrow X$, $q : X \times Z \rightarrow Z$ for the two projections. With these ingredients, we define functors

$$(7.10) \quad \begin{aligned} \Delta : \text{Mod}_G^\lambda(\mathcal{D}_Z) &\longrightarrow \text{Mod}_{G-eq}(\mathcal{D}_{X \times Z, \lambda}), & \Delta(\mathfrak{N}) &= EQ(\mathcal{D}_{X, \lambda} \boxtimes \mathfrak{N}), \\ \Psi : \text{Mod}_{G-eq}(\mathcal{D}_{X \times Z, \lambda}) &\longrightarrow \text{Mod}_G^\lambda(\mathcal{D}_Z), & \Psi(\mathfrak{L}) &= q_*(\mathfrak{L}). \end{aligned}$$

The fact that Ψ is well defined requires verification: the G -equivariant projection q has projective fibers, so the sheaf direct image q_* exists as a left exact functor between the categories of quasi- G -equivariant, quasi-coherent $\mathcal{D}$ -modules; a small calculation shows that the $\mathcal{D}_{X \times Z, \lambda}$ -module structure of $\mathfrak{L}$ imposes the infinitesimal character χ_λ on $q_*(\mathfrak{L})$.

By construction, Δ is right exact, Ψ left exact. We note also that Ψ is the right adjoint of Δ . The next statement, we shall see, formally contains the Beilinson-Bernstein equivalence of categories [BB1].

7.11 Theorem. *If $\lambda + \rho$ is integrally dominant,*

- a) $R^n q_*(\mathfrak{L}) = 0$, for every $\mathfrak{L} \in \text{Mod}(\mathcal{D}_{X \times Z, \lambda})$ and every $n > 0$;
- b) $\Psi \Delta(\mathfrak{N}) \simeq \mathfrak{N}$, for every $\mathfrak{N} \in \text{Mod}_G^\lambda(\mathcal{D}_Z)$.

If $\lambda + \rho$ is both integrally dominant and regular, the functors Δ, Ψ define equivalences of categories, and are quasi-inverses to each other.

The proof of this theorem amounts to a reduction to the analogous statements in [BB1]. The vanishing of the higher derived images $R^n q_*(\mathfrak{L})$, for example, is local with respect to Z , so one may as well suppose that Z is affine. But then $R^n q_*(\mathfrak{L})$ is determined by its space of global sections, and that space coincides with the n -th cohomology of the quasi-coherent – for affine Z – $\mathcal{D}_{X, \lambda}$ -module $p_*(\mathfrak{L})$, so the Beilinson-Bernstein vanishing theorem applies. The other assertions can be verified by similar arguments.

To make the formal connection with [BB1], let us look at an algebraic subgroup $H \subset G$. The theorem, with $Z = G/H$ and $\lambda + \rho$ integrally dominant regular, asserts an equivalence of categories

$$(7.12a) \quad \text{Mod}_G^\lambda(\mathcal{D}_{G/H}) \simeq \text{Mod}_{G-eq}(\mathcal{D}_{X \times G/H, \lambda}).$$

But $\text{Mod}_{G-eq}(\mathcal{D}_{X \times G/H, \lambda}) \simeq \text{Mod}_{H-eq}(\mathcal{D}_{X, \lambda})$ for formal reasons – “induction from H to G ”. On the other hand, the equivalence of categories (4.4) identifies $\text{Mod}_{G-eq}(\mathcal{D}_{X \times G/H, \lambda})$ with $\text{Mod}^\lambda(\mathfrak{g}, H)$, the category of algebraic $(\mathfrak{g}, H)$ -modules with infinitesimal character χ_λ . Thus (7.12a) is tantamount to the equivalence of categories

$$(7.12b) \quad \text{Mod}_{H-eq}(\mathcal{D}_{X, \lambda}) \simeq \text{Mod}^\lambda(\mathfrak{g}, H).$$

This, of course, follows from the Beilinson-Bernstein equivalence and, in fact, reduces to it precisely when $H = \{e\}$.

Our next statement can be verified directly, by keeping track of the effect of the functors Δ, Ψ on characteristic varieties; both (7.3) and (7.5) are crucial ingredients of the argument.

7.13 Lemma. *The functor Δ assigns a holonomic module in $\text{Mod}_{G-eq}(\mathcal{D}_{X \times Z, \lambda})$ to any admissible module $\mathfrak{M} \in \text{Mod}_G^\lambda(\mathcal{D}_Z)$. Conversely, Ψ assigns admissible modules to holonomic modules.*

For the proof of (7.6), we may suppose that the admissible, quasi- G -equivariant $\mathcal{D}_Z$ -module $\mathfrak{M}$ satisfies the additional hypotheses (7.8-9). Because of (7.11) and (7.13), there exists a holonomic, hence coherent module $\mathfrak{L} \in \text{Mod}_{G-eq}(\mathcal{D}_{X \times Z, \lambda})$, such that

$$q_* \mathfrak{L} = \mathfrak{M}, \quad R^n q_* \mathfrak{L} = 0 \text{ if } n \neq 0.$$

Thus, as a consequence of the definition of the direct image functor (4.12), we obtain the isomorphism

$$(7.14) \quad \mathfrak{M} \simeq \int_q (Lp^*(\mathcal{D}_{X, -\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1}) \otimes_{\mathcal{O}_{X \times Z}} \mathfrak{L})$$

in the derived category $D_{G, coh}^b(\mathcal{D}_Z)$; here d , it should be recalled, denotes the dimension of X . We now argue as we did in §5: for $\mathcal{S} \in D_{G_{\mathbb{R}}, \mathbb{R}-c}^b(\mathbb{C}_Z)$, and with $\mathcal{L} = DR_{X \times Z}(\mathfrak{L})$,

$$(7.15) \quad \begin{aligned} & \text{R Hom}_{\mathcal{D}_Z}^{top} \left(\int_q (Lp^*(\mathcal{D}_{X, -\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1}) \otimes_{\mathcal{O}_{X \times Z}} \mathfrak{L}) \otimes \mathcal{S}, \mathcal{O}_{Z^{an}} \right) [2 \dim Z] \simeq \\ & \simeq \text{R Hom}_{\mathcal{D}_{X, -\lambda}}^{top} (\mathcal{D}_{X, -\lambda} \otimes_{\mathcal{O}_X} (\Omega_X^d)^{-1} \otimes Rp_*(\mathcal{L} \otimes q^{-1} \mathcal{S}), \mathcal{O}_{X^{an}}(-\lambda)) [d], \end{aligned}$$

as objects in $D^b(\mathcal{F}_{G_{\mathbb{R}}})$. Except for the concrete choices of $\mathcal{S}$ and λ , the object on the right in (7.15) coincides with the object (6.14), and thus has the property MG.

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