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A Geometric Construction of the Discrete Series for Semisimple Lie Groups

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Dedicated to Friedrich Hirzebruch

§ 1. Introduction

In the representation theory of a compact group K, a major role is played by the Peter-Weyl theorem, which asserts that the regular representation $L^2(K)$ decomposes as a countable direct sum of irreducibles with finite multiplicity. For compact connected Lie groups this becomes much more concrete: the irreducibles are explicitly known, their characters are given by the famous Hermann Weyl formula, and there is a uniform geometrical construction for them due to Borel and Weil.

For a general locally compact group G, the general Plancherel theorem, which goes back essentially to von Neumann and is a very sophisticated generalization of the Peter-Weyl theorem, gives a direct integral decomposition of $L^2(G)$. For real semisimple Lie groups (connected and with finite center) the work of Harish-Chandra makes the Plancherel theorem quite explicit. The first and basic step is the identification of the discrete part of $L^2(G)$: the irreducible representations which enter here are (by definition) those of the discrete series. For these Harish-Chandra has given very precise results in terms of their (generalized) characters [14, 15]. Moreover, a geometrical realization of the discrete series, analogous to the Borel-Weil theorem, was conjectured by Langlands and subsequently established in various forms (cf. [21, 23, 22, 25]).

The purpose of this paper is to give a new and, to a large extent, self-contained account of the principal results concerning the discrete series. The main novelty in our presentation is that we use (a weak form of) the geometric realization to construct the discrete series representations and to obtain information about their characters. Previously things were done in the reverse order, the existence of the discrete series, proved by Harish-Chandra, being used to get the geometric realization. Analytically our proof of the existence of the discrete series rests on the L^2-index theorem [1], which gives a suitable generalization to non-compact manifolds of the Atiyah-Singer index theorem. The way in which the L^2-index theorem is used is quite analogous to the role of the Riemann-Roch theorem in the Borel-Weil-Bott approach to the representations of compact Lie groups.

In broad outline the main results concerning the discrete series may be listed as follows:

(1) *existence* of discrete series representations $\mathbf{H}_\lambda$, indexed by a suitable lattice parameter λ,

(2) *exhaustion*—proof that the representations in (1) give all the discrete series,

(3) *Geometric realization*—identification of $\mathbf{H}_\lambda$ with the space of L^2-solutions of a certain elliptic differential equation (of Dirac type) on the symmetric space G/K,

(4) *Character behavior*—discrete series characters decay at ∞ (in an appropriate sense) and are determined by their restriction to (regular points of) a compact Cartan subgroup.

(5) *Character formula*—explicit formula for the character of $\mathbf{H}_\lambda$ on a compact Cartan subgroup.

One of the features of our approach is that (1), (2) and (3) are all treated together in a rather natural manner. The character properties (4) and (5) are needed for the proof of (2). The results given in this paper are complete for (1), (2), (4) and (5), and almost complete for (3). We refer to Section 9 for a more detailed description of the situation concerning (3).

To a great extent this paper is a synthesis of existing results and methods. Its aim is to demonstrate how the theory of the discrete series may be established on geometric-analytic foundations. Because the literature is diverse, extensive and highly technical, we have thought it worthwhile to present here a reasonably complete account, hopefully written so as not to make extensive demands on the reader's knowledge of Lie group theory. As a result many relevant results are reproved (or sketched) in the version which we need—frequently this is much simpler than in the published form.

Naturally some basic results are going to be needed. In order to clarify the situation it is perhaps best to list the sort of results which we shall assume. These belong to several different categories beginning with some generalities:

(a) algebraic structure of semisimple Lie groups,

(b) basic generalities about unitary representations, including the existence of the Harish-Chandra distributional characters,

(c) representation theory of compact Lie groups,

(d) abstract Plancherel theorem.

Next come the results which are crucial to our approach:

(e) existence of uniform discrete subgroups Γ of real semisimple Lie groups,

(f) L^2-index theorem,

(g) index theorem for compact manifolds,

(h) differential-geometric computations involving curvature.

The existence of Γ is essential for the application of (f), which via (g) leads to the curvature computations in (h), exactly as in the work of Hirzebruch [17]. Finally there are three more specific results of Harish-Chandra:

(i) $\mathbf{H}\otimes V$ has a finite composition series when $\mathbf{H}$ is an irreducible unitary representation of G and V is a finite-dimensional representation,

(j) structure of the algebra of bi-invariant differential operators on G,

(k) local integrability of the Harish-Chandra characters.

Of these (i) follows from the fact that **H** appears as a subrepresentation of an induced representation (see the simple proof by Casselman [9]). The algebraic result (j) is required as the first step in the proof of the much deeper theorem (k). An alternative proof of (k), incorporating also a simple proof of (j), will be given in [3], which should therefore be viewed as foundational for the present paper.

A number of purely algebraic arguments which we reproduce in suitable form are relegated to an Appendix. These include Parasarathy's computation of the spinor Laplacian, an estimate for the action of the Casimir operator on a unitary representation, and an algebraic characterization of certain representations in terms of their K-decomposition.

We turn now to a description of the contents of the various sections, highlighting the main features. We begin in Section 2 with a general review of the abstract Plancherel theorem. This is essentially standard material included for the benefit of the reader, but it also sets the scene for Section 3, where we apply the L^2-index theorem of [1] to Dirac operators on the symmetric space G/K, with G and K having the same rank. The final result of Section 3 is Theorem (3.16), which gives an explicit formula for the difference of the Plancherel measures of two spaces $\mathscr{H}_\mu^+$ and $\mathscr{H}_\mu^-$ of square-integrable, harmonic spinors on G/K, with values in a vector bundle $\mathscr{V}_\mu$. The space $\mathscr{H}_\mu^+$ will eventually turn out to be the representation space of a discrete series representation, but the important fact at this stage is that $\mathscr{H}_\mu^+ \neq 0$, for suitable parameters μ.

In Section 4 we show that the formal difference $\mathscr{H}_\mu^+ - \mathscr{H}_\mu^-$ is a finite linear combination of irreducibles, which therefore belong to the discrete series. This is made precise by the difference formula (4.24), which in particular contains an existence theorem for the discrete series. From the identity (4.24) we derive a formula, valid on the elliptic set, for the sum of the discrete series characters which correspond to a given infinitesimal character, each multiplied by its formal degree (Theorem (4.41)). The arguments in § 4 depend on an analysis of the K-characters of representations of G, and the one non-trivial fact we use is that only finitely many irreducible representations can have a particular infinitesimal character (a consequence of the local integrability of characters).

In Section 5 we assume the parameter μ is sufficiently positive. Using an algebraic version of Parasarathy's formula (explained in the Appendix) we deduce that $\mathscr{H}_\mu^- = 0$ and that any irreducible constituent $\mathbf{H}_j$ of $\mathscr{H}_\mu^+$ has a K-decomposition with a lowest highest weight. Combined with a purely algebraic result concerning such representations (Proposition (5.14), proof sketched in Appendix), this leads rapidly to Theorem (5.20), which asserts that $\mathscr{H}_\mu^+$ is irreducible and gives its formal degree, as well as the character formula on the elliptic set.

Sections 6 and 7 are devoted to a study of growth conditions at infinity for characters, or more generally, for invariant eigendistributions. Unlike Harish-Chandra, who uses a certain Schwartz space for this purpose, we carry out our analysis in the framework of Sobolev spaces. The Sobolev spaces are technically simple to deal with, and they are already used in [1], on which our construction of the discrete series rests. We first show (Lemma (6.3)) that a discrete series character extends from $C_0^\infty(G)$ to some Sobolev space. We then enunciate the main results (Proposition (6.10) and (6.11) and Lemma (6.15)), which give the precise form of (4) above, and in particular show that the discrete series occurs if and only if $\operatorname{rank} G = \operatorname{rank} K$. As a further consequence of these results, in combination with Theorem (4.41), we also obtain an explicit description of the infinitesimal characters of the discrete series (Corollary (6.13)). Here we use (4.41), which came from the L^2-index theorem, for singular as well as non-singular weights. Thus the L^2-index theorem, as applied to G/K, embodies both an existence theorem (leading to the existence of the discrete series) and a non-existence theorem, which enters our exhaustion proof for the discrete series.

The proofs of (6.10), (6.11) and (6.15) are given in Section 7. In the first place we verify (Corollary (7.12)) that the Sobolev spaces of G and of a Cartan subgroup B are compatible in the way one would expect. By splitting B into its toroidal part and its vector part, (6.10) is eventually reduced to an easy property of Sobolev spaces in Euclidean space. The proofs of (6.11) and (6.15) are then straightforward applications of the Harish-Chandra "matching conditions", which describe the behavior of an invariant eigendistribution as one moves between two adjacent Cartan subgroups. These matching conditions are an essential supplement to the local integrability, and they will also be established in [3].

In Section 8 we use the properties of characters just described, together with Zuckerman's tensor product technique (which we explain to the extent that it is used here), to deal with those parameters μ not covered by Theorem (5.20). Zuckerman's method enables us to "shift" μ, making it sufficiently nonsingular to apply Theorem (5.20). This makes it possible to refine Theorem (4.41), which gave a formula for certain linear combinations of discrete series characters on a compact Cartan subgroup, into the corresponding formula for individual discrete series characters. The resulting character formula constitutes one part of our main theorem (8.1), the others being the computation of the formal degree and an exhaustion statement. These last two parts are easy and purely formal consequences of the same arguments which produced the character formula. Finally, at the end of Section 8, we prove Theorem (8.5), which extends to all discrete series representations the results on the "lowest highest weight" of their K-decompositions, previously established for sufficiently nonsingular parameters. Again the proof uses the Zuckerman technique to shift the parameter.

Section 9 deals with the problem of realizing the discrete series geometrically. The enumeration of the discrete series representations and the results about their K-decompositions, as described in Section 8, make it a simple matter to identify the discrete part of the spaces of square-integrable, harmonic spinors $\mathscr{H}_\mu^+, \mathscr{H}_\mu^-$ (Lemma (9.4)). In order to eliminate the continuous part from the Plancherel decomposition of $\mathscr{H}_\mu^+$, $\mathscr{H}_\mu^-$, we must appeal to Harish-Chandra's results about the explicit form of the Plancherel formula. The crucial statement appears as Lemma (9.8); although the details of its proof go well beyond the framework of this paper, we indicate at least the main ideas. The final conclusion is Theorem (9.3), which describes the spaces $\mathscr{H}_\mu^+$, $\mathscr{H}_\mu^-$, and which in particular provides a geometric realization for every discrete series representation. We end the section with some comments about ways to avoid the use of delicate properties of Plancherel measure.

The Appendix, finally, contains proofs of four technical statements, which have appeared elsewhere, and which are collected here for the convenience of the reader.

An approach to the discrete series similar to ours has recently been developed by de George and Wallach. They start in the same way by choosing a uniform discrete subgroup Γ and then computing the index of the Dirac operator on $\Gamma\backslash G/K$. However, instead of using the L^2-index theorem to relate this to G/K, they use a descending sequence $\Gamma \supset \cdots \supset \Gamma_n \supset \cdots$ of normal subgroups of finite index, intersecting in the identity. In a sense their argument is more elementary, since they only work with compact manifolds. On the other hand it appears to yield somewhat weaker results, and it does not tie in so directly with the geometric realization. A connecting link between the de George-Wallach approach and ours is to be found in a paper by Kazdan [18], which investigates $L^2(G/K)$ via the limit of $L^2(\Gamma_n\backslash G/K)$, as $n \to \infty$.

§ 2. Review of the Plancherel Theorem

Since we shall be making essential use of the Plancherel theorem, we review here the basic facts, with particular reference to those properties which we shall be using. We restrict ourselves to connected real semisimple Lie groups G with finite center and recall that these are unimodular (left and right Haar measure coincide).

The first basic fact is that any irreducible unitary representation $\mathbf{H}$ of G, when restricted to a maximal compact subgroup K, has a direct sum decomposition

$$\text{(2.1)} \quad \mathbf{H} = \bigoplus_{i \in \hat{K}} n_i V_i,$$

where the V_i are the irreducible representations of K, and the multiplicities n_i satisfy the bound

$$n_i \leqq \dim V_i.$$

This in turn implies that for any $f \in C_0^\infty(G)$, the corresponding operator

$$\pi(f) = \int_G f(g)\, \pi(g)\, dg$$

($g \mapsto \pi(g)$ denotes the action of G on $\mathbf{H}$) is of trace class, and that $f \mapsto \operatorname{trace} \pi(f)$ is continuous, hence defines a distribution Θ, called the (Harish-Chandra) character of the representation. Moreover the fact that $\pi(f)$ is in particular compact implies that G is of type I, i.e. that every factor representation involves only a type I factor. This means that every unitary representation of G can be decomposed in an appropriate sense into irreducibles. We let $\hat{G}$ denote the set of (equivalence classes of) irreducible unitary representations of G.

The Plancherel theorem is concerned with decomposing the left (right) regular representation, i.e. the action on $L^2(G)$ induced by left (right) translation. In the first instance it asserts that we have a direct integral decomposition

(2.2) $$L^2(G) \cong \int_{\hat{G}} \mathbf{H}_j \hat{\otimes} \mathbf{H}_j^* \, dj,$$

where dj is a positive measure on $\hat{G}$, $\mathbf{H}_j$ is the irreducible representation indexed by $j \in \hat{G}$, and $\mathbf{H}_j \hat{\otimes} \mathbf{H}_j^*$ is the Hilbert space tensor product of $\mathbf{H}_j$ and its dual. The isomorphism (2.2) is compatible with both the left and right actions of G. Moreover, if $\mathscr{A}$ denotes the von Neumann algebra generated by the left translations on $L^2(G)$, we have an isomorphism

(2.3) $$\mathscr{A} \cong \int_{\hat{G}} \mathscr{L}(\mathbf{H}_j) \, dj,$$

where $\mathscr{L}(\mathbf{H}_j)$ denotes the algebra of all bounded operators on $\mathbf{H}_j$. There is a similar statement for right translations which generate the commutant $\mathscr{A}'$ of $\mathscr{A}$.

A further aspect of the Plancherel theorem (which determines the Plancherel measure dj uniquely) involves the consideration of traces. On the C^∞ group algebra ($C_0^\infty(G)$ under convolution), evaluation at the identity e of G defines a natural "trace". It turns out that this can be extended to the von Neumann algebra $\mathscr{A}$. More precisely, it extends to a map

$$\operatorname{trace}_G \colon \mathscr{A}^+ \to \mathbb{R}^+ \cup \{\infty\},$$

where $\mathscr{A}^+$ denotes the cone of positive operators in $\mathscr{A}$. The elements $A \in \mathscr{A}^+$ with $\operatorname{trace}_G(A)$ finite are the positive part $\mathscr{m}^+$ of an ideal $\mathscr{m}$ of $\mathscr{A}$, on which trace_G extends as a linear functional. The elements of $\mathscr{m}$ will be said to be of G-trace class. If $A \in \mathscr{m}$ then its "Fourier components" A_j in (2.3) are of trace class almost everywhere, and

(2.4) $$\operatorname{trace}_G A = \int_{\hat{G}} \operatorname{trace} A_j \, dj.$$

In particular, taking $A = l(f)$ to be the operator representing $f \in C_0^\infty(G)$ in the left regular representation, we obtain

(2.5) $$f(e) = \int_{\hat{G}} \Theta_j(f) \, dj$$

($\Theta_j =$ Harish-Chandra character of $\mathbf{H}_j$).

If we apply (2.5) to $h = f * \tilde{f}$, where $\tilde{f}(g) = \overline{f(g^{-1})}$, then since

$$h(e) = \int_G |f(g)|^2 \, dg = \|f\|^2$$

and $\pi_j(\tilde{f}) = \pi_j(f)^*$, the result can be written in the form

(2.6) $$\|f\|^2 = \int_{\hat{G}} \operatorname{trace}(\pi_j(f) \cdot \pi_j(f)^*) \, dj \qquad \text{(Plancherel formula).}$$

The restrictions on f can be relaxed somewhat. If $f \in L^2(G)$ and $l(f) \in \mathscr{A}$ (i.e. $l(f)$ is bounded), then $\pi(f) \cdot \pi(f)^*$ is of G-trace class, and so we can apply (2.4). Moreover

$$\operatorname{trace}_G(l(f) \cdot l(f)^*) = \|f\|^2$$

follows by continuity from the corresponding statement with $f \in C_0^\infty(G)$. The easiest way to insure that $l(f)$ is bounded is to take $f \in L^2(G) \cap L^1(G)$. Thus (2.6) holds for all $f \in L^2(G) \cap L^1(G)$.

The preceeding results are all standard, but in addition we shall need:

(2.7) $f \in C^\infty(G)$ and $l(f) \in \mathscr{A}^+ \Rightarrow l(f)$ is of G-trace class and $\operatorname{trace}_G(f) = f(e)$.

As we shall explain in the next section, a proof of (2.7) is essentially given in [1].

We shall use (2.7) in the particular case that $l(f)$ is an orthogonal projection onto a subspace W of $L^2(G)$. We shall write $\dim_G W$ for $\operatorname{trace}_G \pi(f)$ which, by (2.6), is then equal to $f(e)$. Now W has a decomposition

(2.8) $$W = \int_{\hat{G}} W_j \hat{\otimes} \mathbf{H}_j^* \, dj,$$

where $W_j \subset \mathbf{H}_j$ is the image of the projection operator $\pi_j(f)$. Since, by (2.4) and (2.7), $\pi_j(f)$ is of trace class (for almost all j), this means that $\dim W_j$ is finite (for almost all j), and (2.4) becomes

(2.9) $$f(e) = \dim_G W = \int_{\hat{G}} \dim W_j \, dj.$$

In our applications the space $L^2(G)$ will be replaced by $L^2(G/K, \mathscr{F})$ – the L^2-sections of a homogeneous vector bundle $\mathscr{F}$ over the symmetric space G/K. It is a fairly simple matter to modify the above formulae involving the Plancherel measure to cover this case, as we shall now explain.

If F is a finite-dimensional unitary K-module, then $\mathscr{F} = G \times_K F$ is a homogeneous vector bundle over G/K, and $L^2(G/K, \mathscr{F})$ may be identified with the space of right K-invariants in $L^2(G) \otimes F$. Because of (2.2) this gives

(2.10) $$L^2(G/K, \mathscr{F}) \cong \int_{\hat{G}} \mathbf{H}_j \otimes W_j \, dj,$$

where W_j is the K-invariant part of $\mathbf{H}_j^* \otimes F$, which is finite-dimensional by (2.1). From (2.3) we see that the algebra $\mathscr{B}$ of G-invariant bounded operators on $L^2(G/K, \mathscr{F})$ corresponds under (2.10) to the direct integral

(2.11) $$\int_{\hat{G}} \mathscr{L}(W_j) \, dj.$$

In the algebra $\mathscr{A}' \otimes \mathscr{L}(F)$ there is a natural trace given by the tensor product of trace_G in $\mathscr{A}'$ (the algebra generated by right translation in $L^2(G)$) and the ordinary trace in $\mathscr{L}(F)$: for brevity we still denote this by trace_G and the corresponding dimension function by $\dim_G$. Restricting to the subalgebra $\mathscr{B}$ which acts on the subspace (2.10), we see that

$$\operatorname{trace}_G A = \int_{\hat{G}} \operatorname{trace} A_j \, dj;$$

here $A_j \in \mathscr{L}(W_j)$ are the components (given by (2.11)) of an operator A of G-trace class in $\mathscr{B}$. In particular, if $U \subset L^2(G/K, \mathscr{F})$ is any closed G-invariant subspace, then

$$U=\int_{\hat{G}} \mathbf{H}_j \otimes U_j\, dj, \qquad \text{with } U_j \subset W_j,$$

and

$$(2.12)\quad \dim_G U = \int_{\hat{G}} \dim U_j\, dj.$$

Suppose moreover that orthogonal projection B onto U is given by a smooth kernel b – a section of $\mathrm{Hom}(\mathscr{F},\mathscr{F})$ over $G/K \times G/K$. The corresponding A on G given by $A=p^* B p_*$ (p: $G \to G/K$ being the projection) then also has a smooth kernal a, with $a(x,y)=b(p(x),p(y))$. Hence, by (2.7), A is of G-trace class and

$$(2.13)\quad \dim_G U = \mathrm{trace}_G A = \mathrm{tr}\, a(e,e) = \mathrm{tr}\, b(0,0),$$

where $p(e)=0$ is the base point of G/K and tr denotes the usual trace in $\mathscr{L}(F)$: note that the fibre $\mathscr{F}_0$ may be identified with F.

Remark. In the above we have implicitly assumed that Haar measure on K is normalized to have total volume equal to 1, and that G/K is then given the quotient of the two Haar measures. On G Haar measure is supposed fixed once and for all – the Plancherel measure on $\hat{G}$ depends on this choice.

Returning to the general Plancherel theorem, we recall that $\mathbf{H}_{j_0}$ is said to belong to the discrete series if $j_0 \in \hat{G}$ has positive measure d_{j_0}. Then $\mathbf{H}_{j_0}$ occurs as a direct summand of the left (or right) regular representation, the corresponding projection P_0 is of G-trace class in $\mathscr{A}'$ (or $\mathscr{A}$), and (2.9) reduces to $\dim_G \mathbf{H}_{j_0} = d_{j_0}$. For this reason, d_{j_0} is called the G-dimension, or formal degree, of $\mathbf{H}_{j_0}$.

Finally we consider the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of G. For any unitary representation $\mathbf{H}$ of G, we get an action of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ on the space of C^∞ vectors in $\mathbf{H}$. If $\mathbf{H}$ is irreducible, this space of C^∞ vectors is dense in $\mathbf{H}$, the elements of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ are represented by unbounded operators, and the center $\mathfrak{Z}$ of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ acts by scalars. Thus every element $Z \in \mathfrak{Z}$ defines a scalar function on $\hat{G}$. On $L^2(G)$, the operator $l(Z)$ and its formal adjoint $l(Z^*)$ are both defined on the dense domain $C_0^\infty(G)$ (here $*$ denotes the standard anti-automorphism of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$, given by $X^* = -\bar{X}$ for $X \in \mathfrak{g}^{\mathbb{C}}$). Hence we may take the closed operator $T=$closure of $l(Z)$, and form the bounded operator $A=T(1+T^*T)^{-1/2}$, which commutes with both right and left translation and hence belongs to the center of the von Neumann algebra $\mathscr{A}$. Thus, in the decomposition (2.2), A is represented by a diagonalized operator, i.e. the Fourier components A_j are scalars (almost everywhere), and the function $j \to A_j$ is measurable on $\hat{G}$. From this it follows easily that the function which Z defines on $\hat{G}$ is also measurable (but unbounded) on $\hat{G}$. In particular these remarks apply to the Casimir operator of $\mathfrak{g}^{\mathbb{C}}$.

§ 3. The L^2-Index Theorem

In this section we shall apply the L^2-index theorem of [1] to the symmetric space G/K of a semisimple Lie group G. Formally this is quite analogous to the way in which the index theorem for compact manifolds may be used to derive the dimension formulae for irreducible representations of compact Lie groups. Computationally it is also closely related to the manner in which Hirzebruch [17] computed the dimensions of spaces of automorphic forms.

For the convenience of the reader we shall first recall the statement of the L^2-index theorem. We suppose given a discrete group Γ acting smoothly and freely on a manifold $\tilde{X}$ with $X=\Gamma\backslash\tilde{X}$ compact, and an elliptic differential operator $\tilde{D}$ on $\tilde{X}$ which is Γ-invariant (and so is the lift of an elliptic differential operator D on X). Hilbert spaces are defined by using Γ-invariant hermitian metrics, and we put

$$\mathscr{H}^+ = \text{space of } L^2\text{-solution of } \tilde{D}u=0,$$

$$\mathscr{H}^- = \text{space of } L^2\text{-solutions of } \tilde{D}^* v=0,$$

where $\tilde{D}^*$ is the adjoint differential operator. The orthogonal projections onto $\mathscr{H}^\pm$ have C^∞ kernels $K^\pm(\tilde{x},\tilde{y})$, which are Γ-invariant, i.e.

$$K^\pm(\gamma\tilde{x},\gamma\tilde{y})=K^\pm(\tilde{x},\tilde{y}) \qquad \text{for } \gamma\in\Gamma.$$

Hence we can define a (real-valued) Γ-dimension of $\mathscr{H}^+$ by

$$\dim_\Gamma \mathscr{H}^+ = \operatorname{trace}_\Gamma K^+ = \int_{\Gamma\backslash\tilde{X}} \operatorname{tr} K^+(\tilde{x},\tilde{x})\, d\tilde{x},$$

and similarly for $\mathscr{H}^-$. Here tr denotes the pointwise trace of the matrix $K^\pm(\tilde{x},\tilde{x})$ – acting on the fibers of the vector bundle involved – and the integral is taken over a fundamental domain in $\tilde{X}$ for Γ. Finally we put

$$\operatorname{index}_\Gamma \tilde{D} = \dim_\Gamma \mathscr{H}^+ - \dim_\Gamma \mathscr{H}^-.$$

The L^2-index theorem then asserts

$$\text{(3.2)} \quad \operatorname{index}_\Gamma \tilde{D} = \operatorname{index} D.$$

Note that index D is an integer, being the difference of ordinary dimensions. By the index theorem for compact manifolds [4], there is an explicit formula for index D in cohomological terms. If index $D>0$ then (3.2) implies in particular that the space $\mathscr{H}^+$ of L^2-solutions of $\tilde{D}u=0$ is non-zero. Moreover it says that $\mathscr{H}^+$ is in a certain precise sense "bigger" than $\mathscr{H}^-$. In our application to G/K these consequences will be spelled out in greater detail.

Fundamental to our applications is the following result of Borel [5] (see also Borel and Harish-Chandra [6]): every real semisimple Lie group has a discrete torsion-free subgroup Γ with $\Gamma\backslash G$ compact. If K is a maximal compact subgroup of G then Γ meets no conjugate of K, and so Γ acts freely on the symmetric space $\tilde{X}=G/K$. The quotient $X=\Gamma/\tilde{X}=\Gamma\backslash G/K$ is then a smooth manifold. We shall pick such a Γ once and for all. In our final results the group Γ will disappear; it enters only as a conveninent tool. This is to be contrasted with the superficially similar situations arising in the study of Γ-automorphic forms where Γ is the main object of interest.

The differential operators $\tilde{D}$ on G/K to which we shall apply the L^2-index theorem are the Dirac operators with coefficients in a (homogeneous) vector bundle. We begin therefore by recalling the basic facts concerning these operators; for details we refer to [8, 22].

Recall first that an oriented Riemannian manifold M is said to be a spin manifold if the structure group of its principal tangent bundle can be lifted from $SO(n)$ to $\mathrm{Spin}(n)$. A choice of such a lifting defines a spin structure, and the basic Spin representation S of $\mathrm{Spin}(n)$ then defines an associated vector bundle $\mathscr{S}$ on M, whose sections are "spinor fields". The Dirac operator is a first order formally selfadjoint, elliptic differential operator acting on spinor fields. If $\dim M = n$ is even, then the spin representation S breaks up into two half-spin representations S^+, S^- and correspondingly $\mathscr{S} = \mathscr{S}^+ \oplus \mathscr{S}^-$. The Dirac operator switches the two factors, so that it consists of an operator

$$D^+ \colon C^\infty(M, \mathscr{S}^+) \to C^\infty(M, \mathscr{S}^-);$$

together with $D^- = (D^+)^*$. If $\mathscr{V}$ is any complex vector bundle on M with a Hermitian connection, then one can define a Dirac-type operator $D_{\mathscr{V}}$ on $\mathscr{S} \otimes \mathscr{V}$ which again, when n is even, decomposes into $D_{\mathscr{V}}^+$ and its adjoint.

We now take $M = X = G/K$. The standard Riemannian metric of G/K is certainly G-invariant, but in order for the Spin structure to be G-invariant it is necessary that the representation $K \mapsto \mathrm{Aut}(\mathfrak{g}/\mathfrak{k})$, induced by the adjoint representation of G, should lift to the Spin covering of $\mathrm{Aut}(\mathfrak{g}/\mathfrak{k}) \cong SO(n)$, so that S becomes a K-module. We shall assume for the moment that this is the case. Since the Dirac operator D is canonically associated to the metric and Spin structure, it follows that it will be G-invariant. Similarly, if $\mathscr{V}$ is any homogeneous vector bundle with connection on G/K, the operator $D_{\mathscr{V}}$ will be G-invariant. Now a homogeneous vector bundle on G/K is associated to a representation V of K, and it inherits a homogeneous connection from that of the principal bundle $G \to G/K$ (defined by the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$). In this manner every finite-dimensional representation V of K defines a G-invariant operator $D_{\mathscr{V}}$.

We now assume that $\operatorname{rank} K = \operatorname{rank} G$, i.e. that G has a compact Cartan subgroup. This implies in particular that $\dim G/K$ is even, and so the spin representation S decomposes into a direct sum $S = S^+ \oplus S^-$. Thus for every V, we have a G-invariant operator

$$D_{\mathscr{V}}^+ \colon C^\infty(G/K, \mathscr{V} \otimes \mathscr{S}^+) \to C^\infty(G/K, \mathscr{V} \otimes \mathscr{S}^-),$$

and its adjoint $D_{\mathscr{V}}^-$ in the opposite direction. We now apply the L^2-index theorem to this operator: to conform with our previous notation we shall relabel it $\tilde{D}_{\mathscr{V}}^+$. Since $\tilde{D}_{\mathscr{V}}^+$ is G-invariant, it is certainly Γ-invariant, for any $\Gamma \subset G$, and $D_{\mathscr{V}}^+$ will denote the corresponding operator on $\Gamma \backslash G/K$: it is the Dirac operator on this double coset space with coefficients in the bundle $\Gamma \backslash \mathscr{V}$.

The fact that $D_{\mathscr{V}}^+$ is actually G-invariant enables us to simplify both sides of (3.2). On the one hand the Γ-dimensions can be replaced essentially by G-dimensions and related to the Plancherel measure. On the other hand the index $D_{\mathscr{V}}^+$ on $\Gamma \backslash G/K$ can be computed in terms of invariant differential forms. We proceed to describe these two aspects in detail.

For simplicity we consider first $L^2(G)$, with the two von Neumann algebras $\mathscr{A}, \mathscr{A}'$ generated by left and right translations. The operators that commute with left translation by all elements of Γ form a von Neumann algebra $\mathscr{B}$ which contains $\mathscr{A}'$. On $\mathscr{B}$ we have the Γ-trace defined in [1]. For operators T with smooth kernel $t(x, y)$, which is compactly supported on $\Gamma\backslash(G\times G)$ we have

$$\operatorname{trace}_\Gamma T = \int_{\Gamma\backslash G} t(x,x)\,dx.$$

In particular, if $T=r(f)$ is right convolution by $f\in C_0^\infty(G)$,

$$\begin{aligned}(3.3)\quad \operatorname{trace}_\Gamma r(f) &= \int_{\Gamma\backslash G} f(e)\,dg = \operatorname{vol}(\Gamma\backslash G)\,f(e)\\ &= \operatorname{vol}(\Gamma\backslash G)\operatorname{trace}_G r(f).\end{aligned}$$

Since the $r(f)$ are dense in $\mathscr{A}'$, this shows that, up to a volume factor, trace_G on $\mathscr{A}'$ is the restriction of $\operatorname{trace}_\Gamma$ on $\mathscr{B}$. In particular, for any operator $T\in\mathscr{A}'$

T is of G-trace class $\Leftrightarrow$ T is of Γ-trace class.

Thus (2.7) follows from [1, (4.8)].

An entirely similar situation holds on replacing $L^2(G)$ by $L^2(G/K,\mathscr{F})$, where $\mathscr{F}$ is the vector bundle associated to a K-module F. We now take $F=V\otimes S^\pm$ and $T=T^\pm=$projection onto the space $\mathscr{H}^\pm$ of L^2-solutions of the appropriate Dirac equation on G/K. We get

$$\begin{aligned}(\operatorname{vol}(\Gamma\backslash G))^{-1}\dim_\Gamma\mathscr{H}^\pm &= \operatorname{tr} t^\pm(0,0) && (t^\pm=\text{kernel of } T^\pm)\\ &= \dim_G\mathscr{H}^\pm && (\text{by (2.13)})\\ &= \int_{\hat G}\dim U_j^\pm\,dj && (\text{by (2.12)}),\end{aligned}$$

where

$$(3.4)\quad \mathscr{H}^\pm = \int_{\hat G}\mathbf{H}_j\otimes U_j^\pm\,dj.$$

Hence

$$(3.5)\quad (\operatorname{vol}(\Gamma\backslash G))^{-1}\operatorname{index}_\Gamma\tilde D_{\mathscr{V}}^+ = \int_{\hat G}(\dim U_j^+ - \dim U_j^-)\,dj.$$

We turn now to the other side of the problem, namely the computation of the index of the generalized Dirac operators on the compact manifold $\Gamma\backslash G/K$. Since the work of Hirzebruch [17] this has become a standard type of computation, and so we shall review it only briefly. For further details see for example [17].

The index formula of [4] gives an explicit expression in terms of the Pontrjagin classes of $\Gamma\backslash G/K$ and the Chern classes of the bundle $\mathscr{V}$. Using the differential forms which represent these characteristic classes, the index formula takes the form

$$\operatorname{index} D_{\mathscr{V}}^+ = \int_{\Gamma\backslash G/K} f(\Theta,\Phi),$$

where Θ is the curvature of $\Gamma\backslash G/K$, Φ the curvature of $\mathscr{V}$, and f an explicitly known polynomial. Since we are dealing with homogeneous spaces and homogeneous bundles, the integrand is a constant multiple of the volume form dx:

(3.6) $f(\Theta, \Phi) = C(V)\,dx,$

and hence

(3.7) $\operatorname{index} D^+_{\mathscr{V}} = C(V)\operatorname{vol}(\Gamma\backslash G/K).$

It remains to calculate the constant $C(V)$, which depends only on the K-module V. This is a purely algebraic computation in the Lie algebra of G, and the details are notationally complicated but not difficult. There is however a proportionality principle due to Hirzebruch which enables us to by-pass the computations, or rather to reduce them to well-known results for compact groups. The basic idea is to compare the index formula (3.7) with the corresponding formula for M/K where M is a "compact dual" of G.

We assume for the moment that G is a real form of the simply-connected complex semisimple group $G^{\mathbb{C}}$; we shall show later how to drop this assumption as well as the earlier spin assumption on K. The compact dual M of G is the maximal compact subgroup of $G^{\mathbb{C}}$ containing K, whose Lie algebra $\mathfrak{m}$ is related to that of G by the orthogonal decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{m} = \mathfrak{k} \oplus i\,\mathfrak{p}.$$

The Killing form is positive definite on $\mathfrak{p}$, negative definite on $i\,\mathfrak{p}$, and so (if one chooses the appropriate sign) it induces natural invariant metrics on G/K and M/K. The corresponding curvature tensors essentially coincide up to sign: in making the comparison we work only at the identity coset (since they are G- or M-invariant), and then use the correspondence $\mathfrak{p} \leftrightarrow i\,\mathfrak{p}$. Similar remarks apply to the curvature tensors of the two bundles associated to the K-module V. Hence the index of the corresponding Dirac operator on $Y = M/K$, which we denote by $D^+_{\mathscr{V}}(Y)$, is given by

(3.8) $\operatorname{index} D^+_{\mathscr{V}}(Y) = (-1)^q\, C(V)\operatorname{vol}(Y).$

Here $C(V)$ is the same constant as in (3.7), and $2q = \dim G/K = \dim Y$.

Comparing (3.7) with (3.8), we get

(3.9) $\operatorname{index} D^+_{\mathscr{V}} = \sigma_\Gamma \cdot \operatorname{index} D^+_{\mathscr{V}}(Y),$

where σ_Γ is a constant independent of V, but depending on Γ and given by

(3.10) $$\sigma_\Gamma = (-1)^q \frac{\operatorname{vol}(\Gamma\backslash G/K)}{\operatorname{vol}(M/K)}.$$

In this identity both volumes are determined by the Riemannian metrics defined by the Killing form. Since the volume of K was normalized to be 1, the invariant measures on G and G/K are related in a definite manner. It will now be convenient to renormalize Haar measure on G, and along with it the metrics on

G/K and M/K, by requiring that the total volume of M should also be equal to 1. With this choice of Haar measure, (3.10) becomes

(3.11) $\sigma_\Gamma = (-1)^q \operatorname{vol}(\Gamma \backslash G)$.

If we now apply the L^2-index theorem to $D^+_{\mathscr{V}}$ and use (3.5), (3.9) and (3.11), we get

(3.12) $\int\limits_{\hat{G}} (\dim U_j^+ - \dim U_j^-)\, dj = (-1)^q \operatorname{index} D^+_{\mathscr{V}}(Y)$,

with $U_j^\pm$ as in (3.4).

Thus we have reduced the problem to that of computing the index of a homogeneous elliptic operator on the compact homogeneous space Y. This question has been extensively studied from many different points of view, but a good account for our purposes is that given in [8]. Clearly it is sufficient to take $\mathscr{V} = \mathscr{V}_\mu$ associated to an irreducible K-module V_μ with highest weight μ. We then have the following simple result:

(3.13) $\operatorname{index} D^+_{\mathscr{V}_\mu}(Y) = (-1)^q \dim W_{\mu - \rho_n}$,

where W_λ is the irreducible M-module with highest weight λ and ρ_n the half-sum of the positive noncompact roots. The ordering of the roots must be chosen so as to make $\mu + \rho_c$ dominant (ρ_c = half-sum of the positive compact roots), and the labelling of $\mathscr{S}^+, \mathscr{S}^-$ is pinned down by requiring that ρ_n should occur as a weight for the K-module S^+. If $\mu - \rho_n$ is not a possible highest weight (i.e. if $\mu + \rho_c$ is singular), $W_{\mu - \rho_n}$ is to be interpreted as zero.

We shall explain in outline how (3.13) arises, referring to [8] for further details. First we note that the index of the M-invariant operator $D^+_{\mathscr{V}_\mu}(Y)$ can be refined to give a (virtual) character of M: the ordinary integer index is then obtained by evaluating this character index at the identity of M. Next we note that the spaces of sections $L^2(Y, \mathscr{V}_\mu \otimes \mathscr{S}^\pm)$ have well-defined formal M-characters (i.e. formal series $\sum\limits_{i \in \hat{M}} m_i \chi_i$, with $m_i \in \mathbb{Z}$ and χ_i = character of $i \in \hat{M}$). Hence the character index of $D^+_{\mathscr{V}_\mu}(Y)$ can be computed as the difference of these two formal characters. The computations give precisely the Hermann Weyl character formula for $W_{\mu - \rho_n}$, multiplied by the sign factor $(-1)^q$. The dimension formula (3.13) then follows. In fact (3.13) is closely related to the Borel-Weil-Bott construction for the irreducible representations of M, except that the flag manifold of M is here replaced by Y.

From (3.12) and (3.13) we deduce

(3.14) $\int\limits_{\hat{G}} (\dim U_j^+ - \dim U_j^-)\, dj = \dim W_{\mu - \rho_n}$.

This is then the explicit form for the L^2-index theorem on G/K. It remains now to remove the restrictions imposed on G and K.

If the representation $K \to \operatorname{Aut}(\mathfrak{g}/\mathfrak{k})$ does not lift to Spin, then the bundles $\mathscr{S}^\pm$ on G/K are not G-homogeneous, though they are $\tilde{G}$-homogeneous for a suitable double cover of G. The bundles $\mathscr{S}^\pm \otimes \mathscr{V}_\mu$ will be G-homogeneous provided the

$\tilde{K}$-modules $S^{\pm}\otimes V_\mu$ descend to K. This is easily seen to be guaranteed by assuming that $\mu-\rho_n$ is a weight of K (cf. §4 below). Thus the operators $\tilde{D}^+_{\mathscr{V}_\mu}$ are defined, as G-invariant operators, under this assumption on μ.

Next we drop the requirement that the real form G_1 of the simply connected complex group $G^{\mathbb{C}}$ coincides with G. The maximal compact subgroup K_1 is then only locally isomorphic to K. If $\mu-\rho_n$ is also a weight of K_1, our previous argument still goes through and we get (3.13).

To extend all this when $\mu-\rho_n$ is a weight of K but not a weight of K_1, we observe first that the constant $C(V_\mu)$ in (3.6) is a polynomial in μ. This follows easily from the Weyl character formula and the relations between characters and characteristic classes. Hence (3.14) continues to hold provided we replace $\dim W_{\mu-\rho_n}$ by $d(\mu-\rho_n)$, the explicit polynomial in μ which gives the dimension formula. Note that Haar measure in G is now normalized by requiring

(3.15) $\quad \operatorname{vol} K = \operatorname{vol} M/K_1 = 1.$

Collecting all our results together, we see that we have proved the following:

(3.16) **Theorem.** Let $\mu-\rho_n$ be a weight for K, such that $(\mu+\rho_c,\alpha)\geqq 0$ for all positive roots α. Let $\mathscr{H}_\mu^+,\mathscr{H}_\mu^-$ be the L^2 null spaces of the Dirac operators $D^+_{\mathscr{V}_\mu}, D^-_{\mathscr{V}_\mu}$ on G/K, and

$$\mathscr{H}_\mu^{\pm}=\int_{\hat{G}} \mathbf{H}_j\otimes U_j^{\pm}\, dj$$

their Plancherel decompositions. Then

$$\int_{\hat{G}} (\dim U_j^+ - \dim U_j^-)\, dj = d(\mu-\rho_n),$$

where $d(\lambda)$ is the polynomial in λ giving the dimension of the irreducible finite dimensional representation with highest weight λ. In this identity, Haar measure is normalized so that

$$\operatorname{vol}(K)=\operatorname{vol}(M/K_1)=1,$$

where M/K_1 denotes the simply connected compact dual of the symmetric space G/K.

The spaces $\mathscr{H}_\mu^{\pm}$ in (3.16) are also the L^2 kernels of the spinor Laplacians D^-D^+ and D^+D^- (we omit here the various subscripts). This is so because the minimal and maximal domains of D^+, and similarly of D^-, coincide, as is proved in [1]. This re-interpretation of $\mathscr{H}_\mu^{\pm}$ has the advantage that the spinor Laplacians take a particularly simple form, namely

(3.17) $\quad D^2 = -\Omega + (\mu-\rho_n, \mu-\rho_n+2\rho),$

where Ω represents the Casimir operator, acting on $L^2(G/K, \mathscr{V}_\mu\otimes\mathscr{S}^{\pm})$. The formula (3.17) is due to Parthasarathy [22] and will be proved in the Appendix. Thus $\mathscr{H}_\mu^{\pm}$ is just the eigenspace of the Casimir operator on $L^2(G/K,\mathscr{V}_\mu\otimes\mathscr{S}^{\pm})$, corresponding to the eigenvalue

(3.18) $c_\mu = (\mu - \rho_n, \mu - \rho_n + 2\rho)$.

Hence, if we consider the Plancherel decomposition

$$L^2(G/K, \mathscr{V}_\mu \otimes \mathscr{S}^\pm) = \int_{\hat{G}} \mathbf{H}_j \otimes V_j^\pm \, dj,$$

we see that the subspaces $U_j^\pm$ describing the decomposition of $\mathscr{H}_\mu^\pm$ are given by

$$\begin{aligned} U_j^\pm &= V_j^\pm \quad \text{if } \Omega \text{ acts on } \mathbf{H}_j \text{ by } c_\mu, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

The formula of Theorem (3.16) can therefore be re-written as

(3.19) $$\int_{\hat{G}_\mu} (\dim V_j^+ - \dim V_j^-) \, dj = d(\mu - \rho_n),$$

where $\hat{G}_\mu \subset \hat{G}$ is the subspace consisting of all $j \in \hat{G}$ at which the Casimir operator takes the value c_μ.

§ 4. Existence of the Discrete Series

We now use the results of the preceding section, in particular the identity (3.19), to prove the existence of discrete series representations. The crucial step consists of showing that the integrand in (3.19) vanishes for all but finitely many classes $j \in \hat{G}$. Any finite subset of $\hat{G}$, outside of the discrete series, has zero Plancherel measure. Hence only the discrete series contributes to the integral (3.19). The integrand, for any $j \in \hat{G}$, is related to the global character of j, restricted to the maximal compact subgroup K. These observations lead to an explicit formula, on K, for certain linear combinations of discrete series characters; the formula is stated as Theorem (4.41), at the end of this section. We shall deduce concrete information about individual discrete series representations in subsequent sections.

Recall the Plancherel decomposition of the spaces of $\mathscr{V}_\mu$-valued L^2-spinors:

$$L^2(G/K, \mathscr{S}^\pm \otimes \mathscr{V}_\mu) = \int_{\hat{G}} \mathbf{H}_j \otimes V_j^\pm \, dj,$$

with $V_j^\pm = K$-invariant part of $\mathbf{H}_j^* \otimes S^\pm \otimes V_\mu$; cf. (2.10). Here V_μ stands for the irreducible K-module of highest weight μ, on which the homogeneous vector bundle $\mathscr{V}_\mu$ is modelled. The half spin modules S^+, S^- are self-dual if $q = \frac{1}{2} \dim G/K$ is even, and dual to each other if q is odd. Thus we can identify the integrand in (3.19) as

(4.1) $$\begin{aligned} &\dim V_j^+ - \dim V_j^- \\ &\quad = \dim \mathrm{Hom}_K(V_\mu^*, \mathbf{H}_j^* \otimes S^+) - \dim \mathrm{Hom}_K(V_\mu^*, \mathbf{H}_j^* \otimes S^-) \\ &\quad = (-1)^q \{\dim \mathrm{Hom}_K(V_\mu, \mathbf{H}_j \otimes S^+) - \dim \mathrm{Hom}_K(V_\mu, \mathbf{H}_j \otimes S^-)\}. \end{aligned}$$

Restricted to K, the G-module $\mathbf{H}_j$ decomposes into a direct sum of K-irreducibles,

$$\mathbf{H}_j = \bigoplus_{i \in \hat{K}} n_i V_i.$$

We denote the character of V_i by χ_i. Then, as follows from the bound $n_i \leqq \dim V_i$, the formal series

$$(4.2) \quad \tau_j = \sum_{i \in \hat{K}} n_i \chi_i$$

converges to a distribution on K, the so-called K-character of $\mathbf{H}_j$. It should be noticed that

(4.3) $\dim V_j^+ - \dim V_j^- = (-1)^q \times$ multiplicity with which χ_μ occurs in the formal series $(\sigma^+ - \sigma^-)\tau_j$,

with χ_μ = character of the irreducible K-module V_μ, and $\sigma^\pm$ = character of the K-module $S^\pm$.

According to a fundamental theorem of Harish-Chandra [13, 3] the global character Θ_j of $\mathbf{H}_j$ is (integration against) a locally L^1 function on G, which moreover is real-analytic on G', the set of regular, semisimple elements of G. For more elementary reasons [12, 3], the K-character τ_j restricts to a real-analytic function on $G' \cap K$ (not necessarily an L^1 function on K, however), and

(4.4) Θ_j and τ_j agree as functions on $G' \cap K$.

We now appeal to Lemma 4.10 of [24], whose proof we shall sketch in the Appendix:

(4.5) $(\sigma^+ - \sigma^-)\tau_j$ is a finite integral linear combination of irreducible characters of K.

From (4.3–4.5), one deduces:

(4.6) $(\sigma^+ - \sigma^-)\Theta_j|_{G' \cap K}$ is a finite linear combination of irreducible characters of K, in which χ_μ occurs with coefficient $(-1)^q(\dim V_j^+ - \dim V_j^-)$.

As we shall argue next, the coefficient of χ_μ can be non-zero for only finitely many classes $j \in \hat{G}$.

We denote the complexified Lie algebra of G by $\mathfrak{g}^{\mathbb{C}}$, and the center of its universal enveloping algebra by $\mathfrak{Z}$. Then $\mathfrak{Z}$ can be naturally identified with the algebra of all bi-invariant linear differential operators on G. By its very definition, the character Θ_j is an invariant eigendistribution, i.e. a conjugation-invariant distribution, on which the algebra of bi-invariant differential operators $\mathfrak{Z}$ acts by scalars. Hence there exists a character φ_j: $\mathfrak{Z} \to \mathbb{C}$, such that

$$(4.7) \quad Z\Theta_j = \varphi_j(Z)\Theta_j, \qquad \text{for all } Z \in \mathfrak{Z}.$$

In Harish-Chandra's terminology, φ_j is the infinitesimal character of the G-module $\mathbf{H}_j$.

Corresponding to any Cartan subalgebra $\mathfrak{b}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, Harish-Chandra has constructed a canonical isomorphism

$$(4.8)\quad \gamma: \mathfrak{Z} \xrightarrow{\sim} I(\mathfrak{b}^{\mathbb{C}}),$$

between $\mathfrak{Z}$ and the algebra $I(\mathfrak{b}^{\mathbb{C}})$ of all Weyl group invariants in the symmetric algebra $S(\mathfrak{b}^{\mathbb{C}})$ [12, 26, 3]. If $\mathfrak{b}^{\mathbb{C}}$ arises as the complexified Lie algebra of a Cartan subgroup $B \subset G$, every $X \in S(\mathfrak{b}^{\mathbb{C}})$, and therefore every $\gamma(Z)$, may be viewed as a translation invariant, linear differential operator on the group B. Going to a two-fold covering of B, if necessary, one can select a square-root $\Delta_B \in C^\infty(B)$ of

$$(4.9)\quad \Delta_B^2 = (-1)^d \det\{1 - \mathrm{Ad}|_B \colon \mathfrak{g}^{\mathbb{C}}/\mathfrak{b}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}/\mathfrak{b}^{\mathbb{C}}\},$$

with $d = \frac{1}{2}(\dim G - \mathrm{rk}\, G)$. Since Δ_B^2 vanishes precisely on $B \cap G'$, Δ_B makes sense also as a smooth function on each connected component of $B \cap G'$. The isomorphism (4.8) has the following crucial property:

$$(4.10)\quad (ZF)|_{B \cap G'} = (\Delta_B^{-1} \cdot \gamma(Z) \cdot \Delta_B)\, F|_{B \cap G'},$$

for any conjugation invariant function F [12, 3].

Since the maximal compact subgroup K was assumed to have the same rank as G, it contains a Cartan subgroup H of G. Via exponentiation, the dual group $\hat{H}$ of the torus H becomes isomorphic to a lattice Λ,

$$(4.11)\quad \hat{H} \cong \Lambda \subset i\mathfrak{h}^*,$$

contained in $i\mathfrak{h}^*$, the real vector space of all those linear functions on $\mathfrak{h}^{\mathbb{C}}$, which assume purely imaginary values on the Lie algebra $\mathfrak{h}$. In particular, the root system $\Phi = \Phi(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ lies inside the lattice Λ. A root $\alpha \in \Phi$ is said to be compact or noncompact, depending on whether or not it is a root of the pair $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Thus Φ decomposes into a disjoint union

$$(4.12)\quad \Phi = \Phi^c \cup \Phi^n$$

of the sets Φ^c, Φ^n of all compact and noncompact roots, respectively.

As β ranges over Φ^n, e^β exhausts the set of characters by which H acts on $\mathfrak{g}^{\mathbb{C}}/\mathfrak{k}^{\mathbb{C}}$; each of these has multiplicity one. Thus Φ^n is the set of weights of the standard representation of $\mathrm{SO}(\mathfrak{g}/\mathfrak{k})$, pulled back to K via $K \to \mathrm{SO}(\mathfrak{g}/\mathfrak{k})$. Expressed in terms of the weights $\pm\mu_1, \pm\mu_2, \ldots, \pm\mu_n$ of the standard representation of $\mathrm{SO}(2n)$, the weights of the two half spin modules are

$$\tfrac{1}{2}(\pm\mu_1 \pm \mu_2 \pm \cdots \pm \mu_n),$$

with an even number of minus signs in one case, and an odd number in the other. Hence, if one appropriately chooses a positive root system $\Psi \subset \Phi$,

$$(4.13)\quad (\sigma^+ - \sigma^-)|_H = \prod_{\beta \in \Phi^n \cap \Psi} (e^{\beta/2} - e^{-\beta/2});$$

a "wrong" choice of Ψ would introduce a minus sign. It should be observed that the labelling of the half spin modules S^+, S^- is determined by (3.13), and that a positive root system Ψ gives the correct sign if it makes $\mu+\rho_c$ dominant (ρ_c = half-sum of the positive, compact roots). Once and for all, we fix[1] such a Ψ.

The function Δ_H of (4.9), corresponding to the Cartan subgroup H, can be expressed as

$$(4.14)\quad \Delta_H = \prod_{\alpha\in\Psi} (e^{\alpha/2} - e^{-\alpha/2}),$$

at least by passing to a covering of H, if necessary. Thus Δ_H equals the product of $(\sigma^+ - \sigma^-)$ with the denominator of Weyl's character formula for K. We define

$$(4.15)\quad W = N_G(H)/H = \text{Weyl group of } \mathfrak{h}^{\mathbb{C}} \text{ in } \mathfrak{k}^{\mathbb{C}}$$

(the normalizer of H in G is actually contained in K!). Because of (4.6), Weyl's character formula leads to an identity

$$(4.16\text{a})\quad \Delta_H\, \Theta_j|_{H\cap G'} = \sum_\nu n_\nu\, e^\nu,$$

where ν ranges over a finite subset of Λ, and

$$(4.16\text{b})\quad n_{w\nu} = \varepsilon(w)\, n_\nu, \qquad \text{for } w\in W$$

($\varepsilon(w)$ = sign of w). Moreover, (4.6) allows us to identify the coefficient of $e^{\mu+\rho_c}$ as

$$(4.16\text{c})\quad n_{\mu+\rho_c} = (-1)^q (\dim V_j^+ - \dim V_j^-).$$

For the usual reasons, (4.16) may hold, strictly speaking, only on a finite covering of H.

The group W is contained in $W_{\mathbb{C}}$, the Weyl group of $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$. Via the mapping γ of (4.8), $\mathfrak{Z}$ becomes isomorphic to $I(\mathfrak{h}^{\mathbb{C}})$, which may be thought of as the algebra of all $W_{\mathbb{C}}$-invariant polynomial functions on $\mathfrak{h}^{\mathbb{C}*}$. Thus every $\nu\in\mathfrak{h}^{\mathbb{C}*}$, and especially every $\nu\in\Lambda$, defines a character

$$(4.17)\quad \varphi_\nu\colon \mathfrak{Z}\to\mathbb{C},$$

with $\varphi_\nu(Z)=\gamma(Z)(\nu)$. The $W_{\mathbb{C}}$-invariant polynomials separate any two $W_{\mathbb{C}}$-orbits in $\mathfrak{h}^{\mathbb{C}*}$; hence

$$(4.18)\quad \varphi_\nu = \varphi_\mu \Leftrightarrow \mu = w\,\nu, \qquad \text{for some } w\in W_{\mathbb{C}}.$$

If $\Theta_j|_{H\cap G'}\neq 0$, the identity (4.10) makes it possible to relate the infinitesimal character φ_j to the character formula (4.16):

$$(4.19)\quad n_\nu \neq 0 \Rightarrow \varphi_j = \varphi_\nu.$$

Because of (4.16c), this implies

$$(4.20)\quad \dim V_j^+ - \dim V_j^- = 0, \quad \text{unless} \quad \varphi_j = \varphi_{\mu+\rho_c}.$$

---

[1] This is possible: one first orders Φ^c, so that μ becomes dominant for the resulting system of positive, compact roots; ρ_c is then determined, and $\mu+\rho_c$ is not only Φ^c-dominant, but also Φ^c-nonsingular. Thus, if a system of positive roots Ψ makes $\mu+\rho_c$ dominant, it necessarily induces the original ordering on Φ^c

In particular, the classes $j \in \hat{G}$ which contribute a non-zero integrand in (3.19) all have the same infinitesimal character.

A result of Harish-Chandra [12] asserts that only finitely many classes $j \in \hat{G}$ can have a given infinitesimal character. One may see this, for example, as follows. Since the global characters of non-isomorphic representations are linearly independent, it is enough to prove that the space of invariant eigendistributions, on which $\mathfrak{Z}$ acts according to a given character, has finite dimension. As a consequence of Harish-Chandra's regularity theorem [13, 3], an invariant eigendistribution Θ is completely determined by its restriction to G' – or equivalently, by the restrictions $\Theta|_{B \cap G'}$, with B running over a set of representatives for the finitely many conjugacy classes of Cartan subgroups. The intersections $B \cap G'$ have only finitely many connected components. One can now deduce the finite-dimensionality from (4.10), provided one knows: for any character $\varphi: I(\mathfrak{b}^{\mathbb{C}}) \to \mathbb{C}$, the system of differential equations

$$X f = \varphi(X) f, \qquad X \in I(\mathfrak{b}^{\mathbb{C}}),$$

on any connected open subset of B, has a finite-dimensional solution space. This is indeed the case, since $S(\mathfrak{b}^{\mathbb{C}})$ is finitely generated as a module over $I(\mathfrak{b}^{\mathbb{C}})$ [12, 26, 3].

As we have just finished arguing, only finitely many classes $j \in \hat{G}$ make a non-zero contribution to the integral (3.19). We denote the discrete series of G by $\hat{G}_d$; in other words, $\hat{G}_d \subset \hat{G}$ is the subset consisting of all square-integrable classes. A single point $j \in \hat{G}$ has positive Plancherel mass precisely when j belongs to $\hat{G}_d$. The integral (3.19) therefore remains unchanged if we integrate only over the set $\hat{G}_\mu \cap \hat{G}_d$, on which the measure is discrete:

$$\sum_{j \in \hat{G}_\mu \cap \hat{G}_d} (\dim V_j^+ - \dim V_j^-)\, d(j) = d(\mu - \rho_n); \tag{4.21}$$

here $d(j)$ denotes the Plancherel measure of $\{j\}$, which is also called the formal degree of the class $j \in \hat{G}_d$.

Via the isomorphism γ, the Casimir operator Ω corresponds to

$$\gamma(\Omega) = \sum_i X_i^2 - (\rho, \rho) \cdot 1, \tag{4.22}$$

where $\{X_i\}$ is a basis for $\mathfrak{h}^{\mathbb{C}}$, orthonormal with respect to the Killing form. Hence

$$\varphi_{\mu+\rho_c}(\Omega) = \gamma(\Omega)(\mu+\rho_c) = (\mu+\rho_c, \mu+\rho_c) - (\rho, \rho) = c_\mu$$

(cf. (3.18); $\rho = \rho_c + \rho_n$), which means that $\hat{G}_\mu$ contains the set

$$\{j \in \hat{G} \mid \varphi_j = \varphi_{\mu+\rho_c}\}. \tag{4.23}$$

According to (4.20), the summands in (4.21) vanish outside of this set. Thus, instead of summing over $\hat{G}_\mu \cap \hat{G}_d$, we may sum over the set (4.23), intersected with $\hat{G}_d$. Weyl's dimension formula gives

$$d(\mu - \rho_n) = \prod_{\alpha \in \Psi'} \frac{(\mu + \rho_c, \alpha)}{(\rho, \alpha)}.$$

We conclude:

$$(4.24)\quad \sum_j (\dim V_j^+ - \dim V_j^-)\, d(j) = \prod_{\alpha\in\Psi} \frac{(\mu+\rho_c,\alpha)}{(\rho,\alpha)},$$

with j ranging over the finite set $\{j\in\hat{G}_d \mid \varphi_j = \varphi_{\mu+\rho_c}\}$.

For the moment, we let ρ denote the half sum of the positive roots, relative to an arbitrary ordering of the roots. Then

$$(4.25)\quad \Lambda_\rho = \Lambda + \rho$$

does not depend on the particular ordering: any two possible choices for ρ differ by a sum of roots, and hence by an element of Λ. Roughly speaking, the discrete series can be parametrized in a natural manner by the W-orbits in Λ_ρ; this is the reason for introducing Λ_ρ. The Weyl group W does indeed operate on Λ_ρ, since W preserves the lattice Λ, and since any W-translate of ρ differs from ρ be a sum of roots. The complex Weyl group $W_{\mathbb{C}}$, on the other hand, need not act on Λ_ρ, unless G is linear.

For the remainder of this section, we keep fixed a particular $\lambda\in\Lambda_\rho$, and we define

$$(4.26)\quad \tilde{\Theta}_\lambda = \sum_{j\in\hat{G}_d,\ \varphi_j=\varphi_\lambda} d(j)\, \Theta_j.$$

Thus $\tilde{\Theta}_\lambda$ is a finite linear combination of discrete series characters. We shall use the identity (4.24) corresponding to various parameters μ, to compute the restriction of $\tilde{\Theta}_\lambda$ to $H\cap G'$; (4.16c) provides the link between (4.24) and the formula for $\tilde{\Theta}_\lambda$ on H.

We let Ψ denote a positive root system, which makes λ dominant. If λ is singular, there will of course be more than one possible choice. The character formulas (4.16), for the summands Θ_j, lead to an identity

$$(4.27\text{a})\quad \tilde{\Theta}_\lambda|_{H\cap G'} = \frac{\sum_\nu a_\nu e^\nu}{\prod_{\alpha\in\Psi} (e^{\alpha/2} - e^{-\alpha/2})};$$

here ν runs over a finite set, and the a_ν are real constants. Moreover

$$(4.27\text{b})\quad a_{w\nu} = \varepsilon(w)\, a_\nu, \qquad \text{for } w\in W.$$

Because of this skew-symmetry with respect to W,

$$(4.28)\quad a_\nu = 0, \qquad \text{whenever } \nu \text{ is } \Phi^c\text{-singular}.$$

All the summands which make up $\tilde{\Theta}_\lambda$ have infinitesimal character φ_λ. Hence (4.18) and (4.19) imply

$$(4.29)\quad a_\nu = 0, \quad \text{unless } \nu \text{ is } W_{\mathbb{C}}\text{-conjugate to } \lambda.$$

Although neither the numerator nor the denominator in (4.27) may be well-defined on H, the quotient necessarily is well-defined. The denominator, multiplied by $e^{-\rho}$, equals

$$\prod_{\alpha\in\Psi}(1-e^{-\alpha}),$$

which makes sense on H. It follows that the product of the numerator with $e^{-\rho}$ must also be well-defined on H, and hence equal to a finite linear combination of characters of H. In other words,

$$(4.30)\quad a_\nu \neq 0 \Rightarrow \nu\in\Lambda_\rho.$$

We remark that the preceeding observations apply equally to any linear combination of irreducible characters, provided all of the summands have the same infinitesimal character.

We ennumerate the set of all those $W_{\mathbb{C}}$-conjugates of λ in Λ_ρ, which are both Φ^c-nonsingular and dominant with respect to $\Phi^c\cap\Psi$, as

$$(4.31)\quad \lambda_1,\lambda_2,\ldots,\lambda_N.$$

Every $\nu\in\Lambda_\rho$, if it is Φ^c-nonsingular and $W_{\mathbb{C}}$-conjugate to λ, is then W-conjugate to precisely one of the λ_i. Hence there exist constants a_i, $1\leqq i\leqq N$, such that

$$(4.32)\quad \tilde{\Theta}_\lambda|_{H\cap G'}=\sum_{i=1}^N a_i\frac{\sum_{w\in W}\varepsilon(w)\,e^{w\lambda_i}}{\prod_{\alpha\in\Psi}(e^{\alpha/2}-e^{-\alpha/2})}.$$

The set (4.31) may be empty, in which case $\tilde{\Theta}_\lambda$ vanishes on H. Otherwise, replacing λ by one of its $W_{\mathbb{C}}$-conjugates, if necessary, we can arrange

$$(4.33)\quad \lambda=\lambda_1.$$

We shall assume that this has been done.

The elements of the weight lattice Λ are integral with respect to the root system Φ^c, as is ρ, which is integral even with respect to all of Φ. Thus $\lambda\in\Lambda_\rho$ must also be Φ^c-integral. Like every λ_i, λ lies in the interior of the positive Weyl chamber for $\Phi^c\cap\Psi$. Hence

$$(4.34)\quad \mu=\lambda-\rho_c$$

is at least integral and dominant with respect to $\Phi^c\cap\Psi$. In particular, μ arises as the highest weight of an irreducible $\mathfrak{k}^{\mathbb{C}}$-module V_μ. The $\mathfrak{k}^{\mathbb{C}}$-action on V_μ lifts to a representation of K if and only if μ belongs to Λ, which need not be the case. However, Λ contains $\lambda+\rho=\mu+\rho_n$ (ρ_n=half sum of the positive, noncompact roots). Since every weight of the half spin modules S^+, S^- differs from ρ_n by a sum of roots, K acts on the tensor products $V_\mu\otimes S^+$, $V_\mu\otimes S^-$. This makes it possible to define the vector bundles $\mathscr{V}_\mu\otimes\mathscr{S}^+$, $\mathscr{V}_\mu\otimes\mathscr{S}^-$ on G/K. We conclude that the identity (4.24) applies in our present context.

The summations in (4.24) and (4.26) range over the same set. Hence, using (4.16c), we can identify the coefficient a_1 in (4.32) as

$$(4.35)\quad a_1=(-1)^q\prod_{\alpha\in\Psi}\frac{(\alpha,\lambda)}{(\alpha,\rho)}.$$

In particular, if λ is singular with respect to Φ, the coefficient a_1 vanishes. Of course, $\lambda = \lambda_1$ is not really distinguished among the λ_i. We can therefore let each λ_i play the role of λ. If λ is singular, then so are all of the λ_i; hence

(4.36) $\tilde{\Theta}_\lambda|_{H \cap G'} = 0, \quad$ if λ is singular.

We now suppose that λ, and therefore its conjugates, are nonsingular. Every λ_i is made dominant by a unique positive root system Ψ_i, namely

(4.37) $\Psi_i = \{\alpha \in \Phi \,|\, (\lambda_i, a) > 0\}$.

In analogy to (4.3), we find

(4.38) $$a_i = \varepsilon_i (-1)^q \prod_{\alpha \in \Psi_i} \frac{(\alpha, \lambda_i)}{(\alpha, \rho_i)}$$

(ρ_i = half-sum of the roots in Ψ_i). The sign factor $\varepsilon_i = \pm 1$ is determined by

(4.39) $$\prod_{\alpha \in \Psi_i} (e^{\alpha/2} - e^{-\alpha/2}) = \varepsilon_i \prod_{\alpha \in \Psi} (e^{\alpha/2} - e^{-\alpha/2});$$

its presence in (4.38) is due to the fact that the denominator in (4.32) was defined in terms of Ψ, rather than Ψ_i. Each λ_i is conjugate to λ by the unique $w \in W_{\mathbb{C}}$ which maps Ψ_i onto Ψ; hence

(4.40) $$\prod_{\alpha \in \Psi_i} \frac{(\alpha, \lambda_i)}{(\alpha, \rho_i)} = \prod_{\alpha \in \Psi} \frac{(\alpha, \lambda)}{(\alpha, \rho)}.$$

Combining (4.36–4.40), we may conclude:

(4.41) **Theorem.** If $\lambda \in \Lambda_\rho$ is nonsingular, the restriction of $\tilde{\Theta}_\lambda$ to $H \cap G'$ equals

$$(-1)^q \left(\prod_{\alpha \in \Psi} \frac{(\alpha, \lambda)}{(\alpha, \rho)} \right) \sum_{i=1}^{N} \frac{\sum_{w \in W} \varepsilon(w)\, e^{w\lambda_i}}{\prod_{\alpha \in \Psi_i} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Whenever λ is singular, $\tilde{\Theta}_\lambda$ vanishes on H.

§ 5.1. The "Sufficiently Nonsingular" Case

In the last two sections, we used the L^2-index theorem to study the formal difference $\mathscr{H}_\mu^+ - \mathscr{H}_\mu^-$, of the two spaces of harmonic, $\mathscr{V}_\mu$-valued L^2-spinors $\mathscr{H}_\mu^\pm$. This difference is in particular a finite integral linear combination of discrete series representations. To obtain information about $\mathscr{H}_\mu^+$ and $\mathscr{H}_\mu^-$ individually, we shall now combine the index theorem with a suitable "vanishing theorem". Vanishing theorems in various contexts, or rather the proofs of the vanishing theorems, tend to work only in the "generic" situation. This is also the case here: we shall have to assume that the parameter μ lies far away from all of the root hyperplanes. For such values of μ, certain algebraic arguments will show that $\mathscr{H}_\mu^-$ vanishes, whereas $\mathscr{H}_\mu^+$ is irreducible. Because of what is already known about the formal difference $\mathscr{H}_\mu^+ - \mathscr{H}_\mu^-$, $\mathscr{H}_\mu^+$ must then belong to the discrete series. The algebraic arguments also lead to a formula for the character of $\mathscr{H}_\mu^+$, restricted to the maximal compact subgroup K. As a result, we obtain explicit realizations and character formulas for "most" of the discrete series. The full description of the discrete series will have to wait until § 8, following a discussion of the growth properties of discrete series characters in § 6 and § 7.

Throughout this section, we freely use the notation of § 4. A particular system of positive roots Ψ in Φ will be kept fixed. As in the past, ρ_c and ρ_n stand for the half-sums of all positive compact and noncompact roots, respectively; $\rho = \rho_c + \rho_n$ is the half-sum of all positive roots. When we talk of the highest weight of an irreducible K-module, it will always be with respect to the positive root system $\Phi^c \cap \Psi$ in Φ^c. We recall that the vector bundles $\mathscr{V}_\mu \otimes \mathscr{S}^+$, $\mathscr{V}_\mu \otimes \mathscr{S}^-$ on G/K can be defined whenever $\mu + \rho_n$ lies in the weight lattice Λ, or equivalently, whenever

$$(5.1) \quad \mu + \rho_c \in \Lambda_\rho;$$

cf. (4.25). Once and for all, we require that

$$(5.2) \quad (\mu + \rho_c - B, \alpha) > 0, \qquad \text{for every } \alpha \in \Psi,$$

if B is any sum of distinct positive, noncompact roots. Since there exist only finitely many possibilities for B, (5.2) would certainly be implied by a condition of the form

$$(5.3) \quad (\mu, \alpha) > c, \qquad \text{for } \alpha \in \Psi,$$

with a suitably chosen constant c.

For the moment, π shall denote an arbitrary irreducible unitary representation of G, and V_ν an irreducible K-module, of highest weight ν, such that

(5.4a) V_ν occurs in the restriction of π to K,

and

$$(5.4\text{b}) \quad (\nu - \rho_n, \alpha) \geqq 0, \qquad \text{if } \alpha \in \Phi^c \cap \Psi.$$

Since π is irreducible, the Casimir operator Ω acts under π by some constant $\pi(\Omega)$. According to Lemma 4.11 of [24], the hypotheses (5.4) imply

$$(5.5) \quad \pi(\Omega) \leqq (\nu - \rho_n + \rho_c, \nu - \rho_n + \rho_c) - (\rho, \rho).$$

We shall outline a proof of this estimate, which is based on an algebraic version of Parthasarathy's formula (3.17), in the Appendix.

As was shown in § 3, the spaces of square-integrable, $\mathscr{V}_\mu$-valued, harmonic spinors have Plancherel decompositions

$$(5.6) \quad \mathscr{H}_\mu^\pm = \int_{\hat{G}_\mu} \mathbf{H}_j \otimes V_j^\pm \, dj,$$

with

$$(5.7) \quad \begin{aligned} V_j^\pm &= K\text{-invariant part of } \mathbf{H}_j^* \otimes S^\pm \otimes V_\mu \\ &\cong \mathrm{Hom}_K(\mathbf{H}_j, S^\pm \otimes V_\mu), \end{aligned}$$

and $\hat{G}_\mu$ = set of all classes $j \in \hat{G}$ on which Ω acts as multiplication by

(5.8) $c_\mu = (\mu - \rho_n, \mu - \rho_n + 2\rho)$.

In the tensor product $V_\nu \otimes W$ of an irreducible K-module V_ν, of highest weight ν, with an arbitrary finite-dimensional K-module W, every irreducible constituent has a highest weight which is the sum of ν and some weight τ of W. Moreover, $\nu + \tau$ occurs as highest weight in $V_\nu \otimes W$ at most as often as the multiplicity of the weight τ in W. According to the discussion which preceeds (4.13), every weight of $S^+ \oplus S^-$ can be expressed as $\rho_n - B$, where B stands for a sum of distinct positive, noncompact roots; the weight ρ_n has multiplicity one and occurs in S^+. Thus V_j^+ and V_j^- can be non-zero only if $\mathbf{H}_j$ contains an irreducible K-module with a highest weight of the form $\mu + \rho_n - B$, and, in the case of V_j^-, $B \neq 0$.

Because of the assumption (5.2), any such highest weight $\nu = \mu + \rho_n - B$ satisfies the hypothesis (5.4b): if a weight τ, e.g. $\tau = \mu + \rho_c - B$, is dominant and nonsingular with respect to $\Phi^c \cap \Psi$, then $\tau - \rho_c$ must at least be dominant. Thus we may apply the estimate (5.5) to any class $j \in \hat{G}_\mu$, for which $V_j^\pm$ does not vanish, to conclude

(5.9) $(\mu - \rho_n, \mu - \rho_n + 2\rho) = c_\mu \leqq (\mu + \rho_c - B, \mu + \rho_c - B) - (\rho, \rho)$,

or equivalently,

(5.10) $2(\mu + \rho_c, B) \leqq (B, B)$.

On the other hand, B is a sum of positive roots, so (5.2) insures that both $\mu + \rho_c - B$ and $\mu + \rho_c$ ($B = 0$ is not excluded in (5.2)!) have a strictly positive inner product with B, unless $B = 0$. But then

$$0 < (\mu + \rho_c - B, B) + (\mu + \rho_c, B) = 2(\mu + \rho_c, B) - (B, B)$$

which contradicts (5.10).

We have shown: among the irreducible constituents of $V_\mu \otimes S^\pm$, only the one having highest weight $\mu + \rho_n$ can appear in $\mathbf{H}_j$, for any class $j \in \hat{G}_\mu$. This irreducible constituent has multiplicity one in $V_\mu \otimes S^+$, and multiplicity zero in $V_\mu \otimes S^-$. In particular, for $j \in \hat{G}_\mu$,

(5.11) $\dim V_j^- = 0$, $\dim V_j^+ =$ multiplicity, in $\mathbf{H}_j$, of the irreducible K-module of highest weight $\mu + \rho_n$.

The estimate (5.9) also proves:

(5.12) no irreducible K-module which occurs in $\mathbf{H}_j$ can have a highest weight of the form $\mu + \rho_n - \beta$, with $\beta \in \Phi^n \cap \Psi$,

again for any $j \in \hat{G}_\mu$. As one consequence of (5.11), we obtain the vanishing theorem

(5.13) $\mathscr{H}_\mu^- = 0$

(Parthasarathy [22]).

We should remark that the arguments leading up to (5.13) are really curvature estimates, in algebraic disguise. The decomposition of the K-modules $V_\mu \otimes S^\pm$ into irreducibles determines an analogous decomposition of the bundles $\mathscr{V}_\mu \otimes \mathscr{S}^\pm$. Under the assumption (5.2), the curvature properties of the bundles and of the manifold G/K force all square-integrable, harmonic spinors to take values in a certain sub-bundle of $\mathscr{V}_\mu \otimes \mathscr{S}^+$, namely the one that corresponds to the K-submodule of highest weight $\mu+\rho_n$ in $V_\mu \otimes S^+$. As happens often with differential-geometric arguments of this nature, the resulting vanishing theorem fails to be precise: the hypothesis (5.2) is unnecessarily stringent.

We shall now appeal to some algebraic results about representations of G, which can be found, for example, in [24]. The arguments of [24] work with much weaker hypotheses than (5.2), and can be simplified considerably in our more special situation. For this reason, we shall present a proof of the relevant result in the Appendix.

(5.14) **Proposition.** Suppose that μ satisfies the inequalities $(\mu+\rho-B,\alpha) \geqq 0$, for every $\alpha \in \Phi^c \cap \Psi$, and every sum B of distinct positive, noncompact roots. Up to isomorphism, there exists at most one irreducible unitary representation π of G, such that

a) $\pi|_K$ has an irreducible constituent of highest weight $\mu+\rho_n$, and

b) no irreducible constituent of $\pi|_K$ has a highest weight of the form $\mu+\rho_n-\beta$, $\beta \in \Phi^n \cap \Psi$.

In such a representation π, the irreducible K-module of highest weight $\mu+\rho_n$ occurs exactly once. The highest weight of any irreducible constituent of $\pi|_K$ can be expressed as $\mu+\rho_n+\sum n_i \beta_i$, with $\beta_i \in \Phi^n \cap \Psi$, $n_i \geqq 0$.

We first observe that (5.2) implies the hypothesis of the proposition. Indeed, as the highest weight of an irreducible K-module (which occurs in S^+), ρ_n has a non-negative inner product with every $\alpha \in \Phi^c \cap \Psi$; hence

$$(\mu+\rho-B,\alpha) \geqq (\mu+\rho_c-B,\alpha),$$

which is positive because of (5.2). According to (5.11–5.12), any class $j \in \hat{G}_\mu$ which contributes to the Plancherel decomposition of $\mathscr{H}_\mu^\pm$ has the two properties a) and b). The Plancherel measure of the totality of these classes $j \in \hat{G}_\mu$ is non-zero, as follows from (3.19). The proposition now guarantees that there can be only one such $j \in \hat{G}_\mu$, which necessarily belongs to the discrete series; also

(5.15) $\quad V_j^- = 0, \qquad \dim V_j^+ = 1.$

Since j alone enters the Plancherel decomposition of $\mathscr{H}_\mu^+$, one can identify $\mathscr{H}_\mu^+$ with $\mathbf{H}_j \otimes V_j^+ \cong \mathbf{H}_j$. We recall that the Plancherel measure of a class in the discrete series is also called the formal degree. The difference formula (3.19) gives the formal degree of j as

$$d(\mu-\rho_n) = \prod_{\alpha \in \Psi} \frac{(\alpha, \mu+\rho_c)}{(\alpha, \rho)}.$$

We summarize: in the situation (5.2), $\mathscr{H}_\mu^-$ vanishes, and $\mathscr{H}_\mu^+$ is a non-zero, irreducible, unitary G-module, belonging to the discrete series, whose formal degree is $d(\mu-\rho_n)$; moreover, $\mathscr{H}_\mu^+$ has the two properties a) and b) in the statement of Proposition (5.14).

We denote the global character of $\mathscr{H}_\mu^+ = \mathbf{H}_j$ by Θ. According to (4.16), the restriction of Θ to $H \cap G'$ can be expressed as

(5.16a) $\quad \Delta_H \Theta|_{H\cap G'} = \sum_{\nu\in\Lambda} n_\nu e^\nu$

(finite sum), with

(5.16b) $\quad n_{w(\mu+\rho_c)} = (-1)^q \varepsilon(w), \qquad$ for $w \in W$

(cf. (5.15)). The significance of the coefficients n_ν is that

(5.17) $$\tau(\sigma^+ - \sigma^-)|_{H\cap G'} = \frac{\sum_\nu n_\nu e^\nu}{\prod_{\alpha\in\Phi^c\cap\Psi} (e^{\alpha/2} - e^{-\alpha/2})};$$

here τ stands for the K-character of $\mathbf{H}_j$. In a formal sense, $\tau(\sigma^+ - \sigma^-)$ is the character of the virtual K-module $\mathbf{H}_j \otimes S^+ - \mathbf{H}_j \otimes S^-$. We set C_+ = closed cone spanned by the positive roots. As follows from (5.14), the highest weight of any K-invariant, K-irreducible component of $\mathbf{H}_j$ lies in $\mu+\rho_n+C_+$. Every weight of $S^+ \oplus S^-$ can be expressed as $-\rho_n+\beta_1+\dots+\beta_k$, with $\beta_i \in \Phi^n \cap \Psi$. If the character of an irreducible K-module appears in $\tau(\sigma^+ - \sigma^-)$, its highest weight must be the sum of a highest weight occuring in $\mathbf{H}_j$ and some weight of $S^+ \oplus S^-$, and hence lies in $\mu + C_+$. Combining this information with Weyl's character formula, we find

(5.18) $\quad n_\nu \neq 0 \Rightarrow w\nu \in \mu+\rho_c+C_+, \qquad$ for some $w \in W$.

Any two weights ν which occur in (5.16) with non-zero coefficient n_ν are $W_{\mathbb{C}}$-conjugate; this follows from (4.18–4.19). In particular,

(5.19) $\quad n_\nu \neq 0 \Rightarrow (\nu,\nu) = (\mu+\rho_c, \mu+\rho_c)$.

Since $\mu+\rho_c$ was assumed dominant with respect to Ψ, it has a strictly smaller length than all other elements of the cone $\mu+\rho_c+C_+$. Comparing (5.18) and (5.19), we may conclude that $n_\nu = 0$, unless ν is W-conjugate to $\mu+\rho_c$. This proves:

(5.20) **Theorem.** Subject to the condition (5.2), $\mathscr{H}_\mu^-$ vanishes, whereas $\mathscr{H}_\mu^+$ is a non-zero Hilbert space, on which G acts unitarily and irreducibly. The resulting representation belongs to the discrete series and has formal degree $d(\mu-\rho_n)$. Its character Θ satisfies

$$\Theta|_{H\cap G'} = (-1)^q \frac{\sum_{w\in W} \varepsilon(w)\, e^{w(\mu+\rho_c)}}{\prod_{\alpha\in\Psi} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Every irreducible K-module which occurs in $\mathscr{H}_\mu^+$ has a highest weight of the form $\mu+\rho_n+\sum n_i\beta_i$, with $\beta_i \in \Phi^n \cap \Psi$, $n_i \geq 0$. The irreducible K-module with highest weight $\mu+\rho_c$ occurs exactly once in $\mathscr{H}_\mu^+$.

§6. Characters and Sobelev Spaces

The characters of discrete series representations, unlike those of a general irreducible, unitary representation, extend continuously from $C_0^\infty(G)$ to much larger function spaces. Harish-Chandra has shown that they are in particular tempered, i.e. their domain of definition includes a suitably defined Schwartz space of rapidly decreasing functions. He used this fact to describe the discrete series characters: within the class of tempered invariant eigendistributions, a discrete series character is completely determined by its restriction to a compact Cartan subgroup.

In this section, we present Harish-Chandra's results in somewhat modified form. The vehicle for our arguments will be certain global Sobolev spaces, rather than the Schwartz space. This is not only consistent with our emphasis on L^2-methods elsewhere in our construction, but allows us also to prove the completeness of the parametrization of the discrete series within the framework of the existence proof. Sobolev spaces usually serve as a tool for studying local regularity properties of functions and distributions. Not so in our context, where they are used to measure global growth properties.

By infinitesimal right translation, the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of G acts on $C^\infty(G)$ as the Lie algebra of left-invariant complex vector fields. When this action is extended to the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$, one obtains an isomorphism

$$(6.1)\qquad r\colon\ \mathfrak{U}(\mathfrak{g}^{\mathbb{C}}) \xrightarrow{\sim} \mathscr{D}_l,$$

between $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ and the algebra $\mathscr{D}_l$ of all left-invariant linear differential operators. As a quotient of the tensor algebra of $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ has a natural filtration. We shall say that $X \in \mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ has degree at most n if it lies in the image of $\bigoplus_{k=0}^{n} (\bigotimes^{k} \mathfrak{g}^{\mathbb{C}})$. For each positive integer n, we define the n-th (left) Sobolev space $H_n(G)$ as

$$(6.2)\qquad H_n(G) = \{f \in L^2(G) \mid r(X) f \in L^2(G), \text{ for every } X \in \mathfrak{U}(\mathfrak{g}^{\mathbb{C}}) \text{ of degree at most } n\};$$

here $r(X) f$ is to be interpreted in the sense of distributions.

One can turn $H_n(G)$ into a Banach space, in an essentially natural manner: although the Banach norm is not intrinsic, the resulting topology is. The group G acts continuously on $H_n(G)$, by left translation, but $H_n(G)$ is not right invariant. When one topologizes $C_0^\infty(G)$ in the usual fashion, the inclusion of $C_0^\infty(G)$ into $H_n(G)$ becomes continuous. We remark that $C_0^\infty(G)$ lies densely in $H_n(G)$; this fact is proven, in effect, in [1], but it will not be needed here.

The next result is implicit in the proof of Harish-Chandra's lemma 76 [15].

(6.3) **Lemma.** Let Θ_π be the character of an irreducible, unitary representation π, which belongs to the discrete series. Then Θ_π extends continuously from $C_0^\infty(G)$ to $H_n(G)$, for every sufficiently large integer n.

At first glance, the statement of the lemma appears to be asymmetric, since it prefers the left Sobolev spaces over their right counterparts. However, the two possible versions of the lemma, corresponding to choices of either left or right Sobolev spaces, are quite immediately equivalent: the distribution Θ_π remains invariant under conjugation; hence, if one lets it act on $r(X)f$, $f \in C_0^\infty(G)$, the differentiation $r(X)$ can be shifted to the left, without affecting the result.

Proof of (6.3). The assertion of the lemma amounts to saying that the linear functional

$$f \mapsto \Theta_\pi(f) = \operatorname{tr} \pi(f), \qquad f \in C_0^\infty(G),$$

is continuous, relative to the topology which $H_n(G)$ induces on its subspace $C_0^\infty(G)$. It therefore suffices to prove the following two statements:

(6.4) $\pi(f)$ is a Hilbert-Schmidt operator, and $\|\pi(f)\|_{\text{H.S.}} \leqq c\|f\|_2$, with $c = c(\pi)$,

and

(6.5) there exists a Hilbert-Schmidt operator T, and some $Z \in \mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$, such that $\pi(f) = \pi(r(Z)f) \cdot T$

for every $f \in C_0^\infty(G)$. Indeed, (6.4) and (6.5) give the estimate

$$\begin{aligned} |\Theta_\pi(f)| &= |\operatorname{trace} \pi(f)| = |\operatorname{trace}(\pi(r(Z)f) \cdot T)| \\ &\leqq \|\pi(r(Z)f)\|_{\text{H.S.}} \|T\|_{\text{H.S.}} \leqq c\, \|T\|_{\text{H.S.}} \|r(Z)f\|_2; \end{aligned}$$

this bounds Θ_π in terms of the seminorm $f \mapsto \|r(Z)f\|_2$, which is continuous with respect to the topology induced by $H_n(G)$ on $C_0^\infty(G)$, provided $n \geqq \deg Z$.

The Plancherel theorem asserts that for any given $f \in L^2(G) \cap L^1(G)$, the operators $\pi_j(f)$, $j \in \hat{G}$, are Hilbert-Schmidt operators, except possibly on a set of Plancherel measure zero, and

$$\|f\|_2^2 = \int_{\hat{G}} \|\pi_j(f)\|_{\text{H.S.}}^2 \, dj.$$

Since π belongs to the discrete series, its class in $\hat{G}$ has positive Plancherel measure $d(\pi)$. Hence

$$\|f\|_2^2 \geqq d(\pi)\, \|\pi(f)\|_{\text{H.S.}}^2,$$

which implies (6.4), with $c = d(\pi)^{-1/2}$.

We now turn to (6.5) — which, incidentally, holds for any irreducible unitary representation. The various K-invariant, K-irreducible subspaces of the representation space $\mathbf{H}$ span a dense linear subspace $\mathbf{H}_\infty \subset \mathbf{H}$, which consists entirely of analytic vectors. In particular, the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of G acts on $\mathbf{H}_\infty$. This infinitesimal representation turns $\mathbf{H}_\infty$ into a module over the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$; we refer to the action of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ on $\mathbf{H}_\infty$ also by the symbol π. For $f \in C_0^\infty(G)$, $v \in \mathbf{H}_\infty$, and $Z \in \mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$, the identity

$$\pi(r(Z)f)\, v = \pi(f)\, \pi(Z)\, v$$

amounts to a tautology. Since Ω_K, the Casimir operator of K, is positive semidefinite, $\pi(1+\Omega_K)$ has a bounded inverse. Hence, setting $Z = (1+\Omega_K)^n$, one finds

(6.6) $\pi(f)=\pi(r(Z)f)\cdot\pi(1+\Omega_K)^{-n}$,

for any $f\in C_0^\infty(G)$, $n\in\mathbb{N}$. The set $\hat{K}$ of isomorphism classes of irreducible K-modules has a natural parametrization in terms of highest weights, which range over a lattice, intersected with a cone. On the irreducible K-module of highest weight μ, Ω_K acts by a constant approximately equal to $\|\mu\|^2$. Each class $i\in\hat{K}$ occurs in $\mathbf{H}$ at most as often as its degree, and this degree can be bounded by a polynomial in the length of the highest weight. Conclusion: for every sufficiently large $n\in\mathbb{N}$, $\pi(1+\Omega_K)^{-n}$ is a Hilbert-Schmidt operator. Because of (6.6), the assertion (6.5), and hence the lemma, follow.

According to Harish-Chandra's fundamental regularity theorem [13, 3], every invariant eigendistribution – in particular, every character – can be represented as integration against a locally L^1 function; this locally L^1 function is actually real-analytic on G', the set of regular, semisimple elements. We shall not distinguish between the invariant eigendistribution and the real-analytic function on G' that represents it.

We recall the definition of the rank of G: it is the minimal possible multiplicity of the eigenvalue one for the automorphisms $\operatorname{Ad} g$ of $\mathfrak{g}^{\mathbb{C}}$, as g ranges over G. Any particular $g\in G$ realizes this minimal multiplicity precisely when g is both regular and semisimple. Thus, writing

(6.7) $$\det(\lambda+1-\operatorname{Ad} g)=\sum_{k\geqq 0} D_k(g)\,\lambda^{r+k}$$

($r=$rank of G), one finds

(6.8) $$G'=\{g\in G \mid D_0(g)\neq 0\}.$$

Incidentally, D_0 assumes only real values, since $\operatorname{Ad} G$ preserves the real form $\mathfrak{g}\subset\mathfrak{g}^{\mathbb{C}}$. After passage to some finite covering of G, if necessary, the function D_0, restricted to any Cartan subgroup, admits a smooth square-root; $D_0^{1/2}$ appears as a universal denominator in character formulas. We shall amplify on these statements later. To motivate the definitions which follow, we merely remark that the singularities of a general invariant eigendistribution near the complement of G' are comparable to those of $D_0^{-1/2}$. In particular, multiplication by $|D_0|^{1/2}$ renders any invariant eigendistribution Θ locally bounded on G. For this reason, the growth properties of such a distribution Θ tend to be reflected by the behavior at infinity, along the various Cartan subgroups[1], of the function $|D_0|^{1/2}\Theta$, rather than by the behavior of the function Θ itself.

For lack of better terminology, we shall call an invariant eigendistribution Θ "bounded at infinity" if, for any Cartan subgroup B,

(6.9) $$\sup_{b\in B\cap G'} |D_0(b)|^{1/2}\,|\Theta(b)|<\infty.$$

Similarly, Θ will be said to "decay at infinity" if the restriction of $|D_0|^{1/2}\Theta$ to any Cartan subgroup B tends to zero outside of compact subsets. According to a

[1] Since $|D_0|^{1/2}\Theta$ is invariant under conjugation, its behavior at infinity must be measured transversely to the conjugacy classes, i.e. along the various Cartan subgroups

criterion of Harish-Chandra, the property of being bounded at infinity is essentially equivalent to his notion of temperedness. When Θ arises as the character of a representation, the equivalence becomes precise, as follows from a result of Fomin-Shapovalov [11]. However, we shall not use the notion of a tempered distribution.

Until the present section, G was assumed to contain a compact Cartan subgroup. This hypothesis did not play a role in Lemma (6.3), nor will it enter the next two propositions. The first of these amounts to a modified version of one direction of Harish-Chandra's temperedness criterion; the second is the analogue, in our context, of the uniqueness statement in [14, Theorem 3].

(6.10) **Proposition.** An invariant eigendistribution, which extends continuously from $C_0^\infty(G)$ to $H_n(G)$, for some $n \in \mathbb{N}$, decays at infinity.

(6.11) **Proposition.** A non-zero invariant eigendistribution which decays at infinity has a non-trivial restriction on some compact Cartan subgroup. In particular, no such invariant eigendistributions exist on G, unless G contains a compact Cartan subgroup.

The proofs of the two propositions will be given in § 7. We conclude this section with some fairly immediate corollaries.

According to the results of § 4, if G has a compact Cartan subgroup, then its discrete series is not empty. The preceeding two propositions, in conjunction with Lemma (6.3), also imply the converse:

(6.12) **Corollary** (Harish-Chandra). For the existence of a non-empty discrete series it is necessary, as well as sufficient, that G contain a compact Cartan subgroup.

Let us assume then that G does contain a compact Cartan subgroup, which we may choose to lie in K. We denote this group by H. As in § 4, Λ shall refer to the weight lattice of the torus H, and Λ_ρ to the weight lattice translated by the half-sum of the positive roots; cf. (4.25). We also recall the definition (4.17) of the characters φ_ν of $\mathfrak{Z}$ (= center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$).

(6.13) **Corollary.** The infinitesimal character of any given discrete series representation is equal to φ_λ, for some nonsingular $\lambda \in \Lambda_\rho$. Conversely, every such φ_λ arises as the infinitesimal character of some discrete series representation.

Proof. We consider a particular discrete series character Θ_j, and we express its restriction to H as in (4.16a). The argument which proves (4.30) also applies in the present situation; thus

(6.14) $n_\nu \neq 0 \Rightarrow \nu \in \Lambda_\rho$.

Because of (6.3) and (6.10–6.11), not all of the coefficients n_ν can vanish (any two compact Cartan subgroups are conjugate!). Combining this knowledge with (4.19) and (6.14), we find that Θ_j has infinitesimal character φ_λ, for some $\lambda \in \Lambda_\rho$. As a linear combination of discrete series characters, the invariant eigendistribution $\tilde{\Theta}_\lambda$ of (4.26) must decay at infinity, and hence

$$\tilde{\Theta}_\lambda = 0 \Leftrightarrow \tilde{\Theta}_\lambda|_H = 0.$$

We now appeal to Theorem (4.41):

$\tilde{\Theta}_\lambda = 0 \Leftrightarrow \lambda$ is singular.

On the other hand, $\tilde{\Theta}_\lambda$ is a linear combination, with non-zero coefficients, of all discrete series characters which correspond to the infinitesimal character φ_λ. Characters of non-isomorphic irreducible unitary representations are linearly independent. Consequently $\tilde{\Theta}_\lambda$ vanishes if and only if no discrete series representation has infinitesimal character φ_λ. This concludes the proof of the corollary.

The two Propositions (6.10–6.11) make it possible to describe the discrete series characters, uniquely within the class of invariant eigendistributions which decay at infinity, in terms of their restriction to a compact Cartan subgroup. More generally, one can give such a description within the larger class of invariant eigendistributions which are merely bounded at infinity. For this purpose, we state a lemma, due to Harish-Chandra [14], whose proof is similar to that of (6.11). It will be proved in the next section, along with the two propositions.

(6.15) **Lemma.** For every nonsingular $\lambda \in \Lambda_\rho$, there exists at most one invariant eigendistribution Θ_λ, such that Θ_λ is bounded at infinity, and

$$\Theta_\lambda|_{H \cap G'} = (-1)^q \frac{\sum_{w \in W} \varepsilon(w)\, e^{w\lambda}}{\prod_{\substack{\alpha \in \Phi, \\ (\alpha, \lambda) > 0}} (e^{\alpha/2} - e^{-\alpha/2})}.$$

§ 7. Proofs of the Preceeding Statements

We begin with the proof of Proposition (6.10). The crucial step will be to relate growth properties of invariant eigendistributions on G to their behavior on the various Cartan subgroups.

Until further notice, Θ shall denote an arbitrary invariant eigendistribution, and B a Cartan subgroup of G. We did not require G to have a faithful finite-dimensional representation, and hence B need not be abelian. Nevertheless, the identity component B^0 lies in the center of B, so that one can define a mapping

(7.1) $\xi: G/B^0 \times (B \cap G') \to (B \cap G')^G$, with $\xi(gB^0, b) = g b g^{-1}$;

as usual, $(B \cap G')^G$ stands for the open subset

$$\{g b g^{-1} | g \in G,\ b \in B \cap G'\}$$

of G. Then ξ is a covering mapping, with fibre $N_G(B)/B^0$, which is finite. At any coset gB^0, the complexified tangent space of G/B^0 may be identified, via left translation by g, with

$$\mathfrak{b}^{\mathbb{C}\perp} = \{X \in \mathfrak{g}^{\mathbb{C}} | B(X, \mathfrak{b}^{\mathbb{C}}) = 0\}$$

($B(\ ,\ )$ = Killing form of $\mathfrak{g}^{\mathbb{C}}$). Similarly, for $b \in B$, left translation by b identifies the complexified tangent space of B at b with $\mathfrak{b}^{\mathbb{C}}$. In terms of these conventions, the differential ξ_* of the mapping ξ at a point (gB^0, b) of $G/B^0 \times (B \cap G')$ is given by the formula

$$(7.2) \quad \xi_*(l_{g*}X, l_{b*}Y) = l_{h*} \operatorname{Ad} g\{(\operatorname{Ad} b^{-1} - 1)(X) + Y\},$$

for $X \in \mathfrak{b}^{\mathbb{C}\perp}$, $Y \in \mathfrak{b}^{\mathbb{C}}$, $h = gbg^{-1} = \xi(gB^0, b)$. To verify the identity, one may as well suppose that the tangent vectors X and Y are real, in which case $\xi_*(l_{g*}X, l_{b*}Y)$ becomes the tangent vector of the curve

$$\begin{aligned} t &\mapsto \xi(g\exp(tX)B^0, b\exp(tY)) \\ &= g\exp(tX)\,b\exp(tY)\exp(-tX)\,g^{-1} \\ &= gbg^{-1}\exp(t\operatorname{Ad} g \cdot \operatorname{Ad} b^{-1} X)\exp(t\operatorname{Ad} g\, Y)\exp(-t\operatorname{Ad} g\, X) \\ &= h\exp(t\operatorname{Ad} g\{(\operatorname{Ad} b^{-1} - 1)(X) + Y\} + O(t^2)), \end{aligned}$$

at $t=0$.

The identity (7.2) implies, in particular, the following standard integration formula: if the invariant measures dg on G, dg^* on G/B^0, and db on B are suitably normalized,

$$(7.3) \qquad \int_{(B\cap G')^G} f\,dg = \int_B |D_0(b)| \int_{G/B^0} f(gbg^{-1})\,dg^*\,db,$$

for every continuous function f with compact support in $(B \cap G')^G$; cf. (6.7). Indeed, $\operatorname{Ad} g$ operates as the identity on the top exterior power of $\mathfrak{g}^{\mathbb{C}}$, whereas

$$\operatorname{Ad} b^{-1} - 1 \colon \mathfrak{b}^{\mathbb{C}\perp} \to \mathfrak{b}^{\mathbb{C}\perp}$$

has determinant $\pm D_0(b)$. The top exterior power of ξ_* is therefore represented by the function $\pm D_0$, relative to translation-invariant sections of the top exterior powers of the various tangent bundles; this proves the integration formula. As one consequence of the formula,

$$(7.4) \quad \Theta(f) = \int_{B\cap G'} |D_0(b)|\,\Theta(b) \int_{G/B^0} f(gbg^{-1})\,dg^*\,db,$$

whenever $f \in C_0^\infty(G)$ has support in $(B \cap G')^G$,

for every invariant eigendistribution Θ.

Corresponding to any $\varphi \in C_0^\infty(G/B^0)$, we define a mapping

$$(7.5) \quad T_\varphi \colon C_0^\infty(B \cap G') \to C_0^\infty((B \cap G')^G)$$

as follows: for $f \in C_0^\infty(B \cap G')$ and $g \in (B \cap G')^G$, $T_\varphi f(g)$ is to be the average, extended over the fibre $\xi^{-1}(g)$, of the values of the function

$$(gB^0, b) \mapsto \varphi(gB^0)\,f(b)\,|D_0(b)|^{-1/2}.$$

An invariant eigendistribution Θ, when restricted to $B \cap G'$, remains invariant under the conjugation action of $N_G(B)$. Hence

$$\begin{aligned} (7.6) \quad \Theta(T_\varphi f) &= \int_{B\cap G'} |D_0(b)|\,\Theta(b) \int_{G/B^0} \varphi(gB^0)\,f(b)\,|D_0(b)|^{-1/2}\,dg^*\,db \\ &= \int_{G/B^0} \varphi\,dg^* \int_{B\cap G'} f\,\Theta\,|D_0|^{1/2}\,db. \end{aligned}$$

Every root α in Φ_B, the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, lifts to a character e^α of B. Since the rank of G coincides with the dimension of its Cartan subgroups, one finds

(7.7) $D_0(b) = \prod_{\alpha \in \Phi_B} (1 - e^\alpha(b)) \quad (b \in B).$

In particular, $B \cap G' = \{b \in B \mid e^\alpha(b) \neq 1, \text{ for } \alpha \in \Phi_B\}$. For $\varepsilon > 0$, we define

(7.8) $B_\varepsilon = \{b \in B \mid |e^\alpha(b) - 1| > \varepsilon, \text{ for } \alpha \in \Phi_B\}.$

The sets B_ε are open in B, and they exhaust $B \cap G'$. As a Cartan subgroup, B centralizes its own Lie algebra. Consequently every right-invariant vector field on B is automatically left-invariant, and vice versa. More generally, the two notions of invariance agree for any linear differential operator. Just as for G, we introduce global Sobolev spaces $H_n(B)$, $n \in \mathbb{N}$:

(7.9) $f \in H_n(B) \Leftrightarrow Xf \in L^2(B)$, whenever X is a translation-invariant differential operator, of order at most n.

(7.10) **Lemma.** For any fixed $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$T_\varphi \colon C_0^\infty(B_\varepsilon) \to C_0^\infty((B \cap G')^G)$$

is continuous with respect to the topologies induced by $H_n(B)$ and $H_n(G)$.

Proof. We consider a vector field $r(Z)$ on G, with $Z \in \mathfrak{g}^{\mathbb{C}}$; cf. (6.1). The mapping ξ pulls back $r(Z)$ to a vector field $\xi^* Z$ on $G/B^0 \times (B \cap G')$. We denote the projections of $\mathfrak{g}^{\mathbb{C}}$ onto $\mathfrak{b}^{\mathbb{C}\perp}$ and $\mathfrak{b}^{\mathbb{C}}$ by p and q, respectively. According to (7.2), at points (gB^0, b) of $G/B^0 \times (B \cap G')$, $\xi^* Z$ takes the value $(l_{g*} X, l_{b*} Y)$, with

$$X = (\operatorname{ad} b^{-1} - 1)^{-1} \cdot p \cdot \operatorname{Ad} g^{-1}(Z), \qquad Y = q \cdot \operatorname{Ad} g^{-1}(Z).$$

The automorphism $(\operatorname{ad} b^{-1} - 1)^{-1}$ of $\mathfrak{b}^{\mathbb{C}\perp}$ is semisimple with eigenvalues $m_\alpha(b) = (e^{-\alpha}(b) - 1)^{-1}$, indexed by $\alpha \in \Phi_B$. As functions on B_ε, the m_α and all their derivatives with respect to translation-invariant differential operators are uniformly bounded. For the purposes of this proof, the space of all such functions will be referred to as $U^\infty(B_\varepsilon)$. As follows from our observations, there exist smooth vector fields X_i on G/B^0, translation-invariant vector fields Y_j on B, and functions $u_i \in U^\infty(B_\varepsilon)$, $v_j \in C^\infty(G/B^0)$, such that

$$\xi^* Z = \sum_i u_i X_i + \sum_j v_j Y_j.$$

To understand the effect of $r(Z)$ on $T_\varphi f$, for $f \in C^\infty(B_\varepsilon)$, we average the product

$$\varphi f \in C_0^\infty(G/B^0 \times B_\varepsilon)$$

with respect to the finite group $N_G(B)/B^0$, which operates on G/B^0 by right translation, and on B by conjugation; it should be observed that the action preserves $B_\varepsilon \subset B$. The averaged product can be expressed as $\sum_l \tilde{\varphi}_l \tilde{f}_l$, with $\tilde{\varphi}_l \in C_0^\infty(G/B^0)$, $\tilde{f}_l \in C_0^\infty(B_\varepsilon)$. Moreover,

$$(T_\varphi f)(g b g^{-1}) = \sum_l \tilde{\varphi}_l(gB^0)\,|D_0(b)|^{-1/2}\tilde{f}_l(b),$$

since D_0 already is $N_G(B)/B^0$-symmetric. In particular,

$$r(Z)(T_\varphi f)(g b g^{-1}) = \sum_{i,l}(X_i\tilde{\varphi}_l)(gB^0)\,u_i(b)\,|D_0(b)|^{-1/2}\tilde{f}_l(b) + \sum_{j,l} v_j(gB^0)\,\tilde{\varphi}_l(gB^0)\,Y_j(|D_0|^{-1/2}\tilde{f}_l)(b).$$

Because of (7.7), the functions $|D_0|^{1/2}(Y_j|D_0|^{-1/2})$ lie in $U^\infty(B_\varepsilon)$. Hence one obtains an identity

$$r(Z)(T_\varphi f)(g b g^{-1}) = \sum_{j,l}\varphi_{jl}(gB^0)\,|D_0(b)|^{-1/2}(Y_j\tilde{f}_l)(b) + \sum_{k,l}\psi_{kl}(gB^0)\,h_k(b)\,|D_0(b)|^{-1/2}\tilde{f}_l(b),$$

with suitably chosen $\varphi_{j,l}, \psi_{k,l} \in C_0^\infty(G/B^0)$, $h_k \in U^\infty(B_\varepsilon)$, which depend on Z and φ. Inductively this procedure leads to a formula of the following type: if $Z \in \mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ has degree n,

$$r(Z)(T_\varphi f)(g b g^{-1}) = \sum_{i,j,l}\varphi_{ijl}(gB^0)\,h_i(b)\,|D_0(b)|^{-1/2}(Y_j\tilde{f})(b).$$

Here Y_j runs over a basis of translation-invariant differential operations on B, of order up to n, $\{\varphi_{ijl}\}$ is a collection of functions in $C_0^\infty(G/B^0)$, and the $h_i \in C^\infty(B \cap G')$ are uniformly bounded on B_ε. Except for a constant factor, the mapping $f \to \tilde{f}_l$ is translation by some element of $N_G(B)/B^0$, hence continuous in the topology of $H_n(B)$. The lemma now follows from an application of the integration formula (7.3).

We select an ordering $>$ on Φ_B, and we let ρ_B denote one-half of the sum of the positive roots. Then

$$\begin{aligned} D_0(b) &= \prod_{\substack{\alpha\in\Phi_B,\\ \alpha>0}} \{(1-e^\alpha(b))(1-e^{-\alpha}(b))\} \\ &= (-1)^d\, e^{2\rho_B}(b) \prod_{\substack{\alpha\in\Phi_B,\\ \alpha>0}} (1-e^{-\alpha}(b))^2, \end{aligned}$$

with d equal to half of the cardinality of Φ_B. Passing to a finite covering of G, if necessary, one can arrange that ρ_B lifts to a character e^{ρ_B} of B. In this situation,

$$(7.11)\quad \begin{aligned} \Delta_B &= e^{\rho_B} \prod_{\substack{\alpha\in\Phi_B,\\ \alpha>0}} (1-e^{-\alpha}) \\ &= \prod_{\substack{\alpha\in\Phi_B,\\ \alpha>0}} (e^{\alpha/2}-e^{-\alpha/2}) \end{aligned}$$

becomes a well-defined function on B, such that

$$D_0(b) = (-1)^d\,\Delta_B(b)^2 \qquad (b\in B).$$

On any connected component of $B \cap G'$, $|D_0|^{1/2}$ coincides with a constant multiple of Δ_B (D_0 is real-valued!). Hence (7.6) and (7.10) imply:

(7.12) **Corollary.** If the invariant eigendistribution Θ extends continuously from $C_0^\infty(G)$ to $H_n(G)$, then the linear functional

$$f \mapsto \int_B f \Delta_B \Theta \, db$$

on $C_0^\infty(B_\varepsilon)$ is continuous with respect to the topology induced by $H_n(B)$, for any $\varepsilon > 0$.

At this point, we have to recall certain facts about the local structure of an invariant eigendistribution. As before, B will denote a Cartan subgroup, with Lie algebra $\mathfrak{b}$, and Φ_B the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$. Every root α of the sub-root system

(7.13) $\Phi_{B,\mathbb{R}} = \{\alpha \in \Phi_B \mid \alpha \text{ is real-valued on } \mathfrak{b}\}$

lifts to a character e^α of B, which assumes only real values[1]. If Θ is an invariant eigendistribution, the function $\Delta_B \Theta$ on $B \cap G'$ has a real-analytic extension to the larger subset

(7.14) $B'' = \{b \in B \mid e^\alpha(b) \neq 1, \text{ for } \alpha \in \Phi_{B,\mathbb{R}}\}$

of B; this is part of Harish-Chandra's "matching condition" [13, 3]. The group B can be expressed as a direct product

(7.15a) $B = B_+ B_-$,

such that B_+ is a compact group, and B_- a vector group. Via the exponential map, B_- becomes isomorphic to its own Lie algebra $\mathfrak{b}_-$, i.e.

(7.15b) $\exp: \mathfrak{b}_- \xrightarrow{\sim} B_-$.

The identity component B_+^0 of B_+ is a torus, with Lie algebra $\mathfrak{b}_+$, so that

(7.15c) $B_+^0 = \exp(\mathfrak{b}_+)$.

We observe that

(7.16) $e^\alpha(B_+) \subset \{\pm 1\}, \quad \text{if } \alpha \in \Phi_{B,\mathbb{R}}$,

since e^α takes only real values.

We enumerate the connected components of B'' as $B_1'', \ldots, B_N''$. For each j,

(7.17) $\Phi_{B,\mathbb{R},j} = \{\alpha \in \Phi_{B,\mathbb{R}} \mid e^\alpha > 0 \text{ on } B_j''\}$

is then a sub-root system of $\Phi_{B,\mathbb{R}}$, and

(7.18) $\Phi_{B,\mathbb{R},j}^+ = \{\alpha \in \Phi_{B,\mathbb{R},j} \mid e^\alpha > 1 \text{ on } B_j''\}$

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[1] This is not totally obvious, since B may have several connected components. One should observe that the statement really concerns the adjoint group, which is an algebraic group over $\mathbb{R}$, and which contains the image of B as an algebraic subgroup. The character e^α is also defined over $\mathbb{R}$, and hence must assume real values on all of B

a system of positive roots in $\Phi_{B,\mathbb{R},j}$, which corresponds to the Weyl chamber

(7.19) $C_j = \{X \in \mathfrak{b}_- \mid \langle \alpha, X \rangle > 0, \text{ for } \alpha \in \Phi^+_{B,\mathbb{R},j}\}$

in $\mathfrak{b}_-$. We claim that B''_j can be decomposed into a product

(7.20) $B''_j = b_j B^0_+ \exp C_j, \quad$ for some fixed $b_j \in B_+$.

Indeed, since B_+ meets every connected component of B, one can choose a $b_j \in B_+$, such that B''_j lies in the connected component $b_j B^0_+ \exp \mathfrak{b}_-$. But then B''_j must be a connected component of the open subset

(7.21) $\{b \in b_j B^0_+ \exp \mathfrak{b}_- \mid e^\alpha(b) \neq 1, \text{ for } a \in \Phi_{B,\mathbb{R}}\}$

of B. According to (7.16), for any $\alpha \in \Phi_{B,\mathbb{R}}$, e^α is identically equal to $+1$ or -1 on $b_j B^0_+$, depending on whether or not α belongs to $\Phi_{B,\mathbb{R},j}$. The set (7.21) therefore coincides with

$$\{b_j b_0 \exp X \mid b_0 \in B^0_+, X \in \mathfrak{b}_-, \langle \alpha, X \rangle \neq 0 \text{ for } \alpha \in \Phi_{B,\mathbb{R},j}\},$$

i.e. the disjoint union of the connected, open subsets $b_j B^0_+ \exp C$, where C runs over the collection of Weyl chambers, in $\mathfrak{b}_-$, of the root system $\Phi_{B,\mathbb{R},j}$. One of these is B''_j; the description (7.18) of $\Phi^+_{B,\mathbb{R},j}$ shows that it can only be $b_j B^0_+ \exp C_j$.

We now focus our attention on a particular invariant eigendistribution Θ, and we keep fixed a connected component B''_j of B'', as in (7.20). The Weyl group $W_{B,\mathbb{C}}$ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ operates on $\mathfrak{b}^{\mathbb{C}}$ in the usual manner, and by duality also on the dual space $\mathfrak{b}^{\mathbb{C}*}$. As a preliminary step in the proof of Harish-Chandra's regularity theorem [13, 3], one obtains the following result: there exist polynomial functions $p_{j,w}$ on $\mathfrak{b}_-$, indexed by $w \in W_{B,\mathbb{C}}$, and a linear function μ on $\mathfrak{b}^{\mathbb{C}}$, such that

(7.22) $(\Delta_B \Theta)(b_j \exp(X+Y)) = \sum_{w \in W_{B,\mathbb{C}}} p_{j,w}(X)\, e^{\langle w\mu, X+Y \rangle}$,
whenever $X \in C_j$, $Y \in \mathfrak{b}_+$.

If μ is nonsingular, i.e. $w\mu \neq \mu$ for $w \neq 1$, the $p_{j,w}$ are actually constants, and they are uniquely determined. In general, to make the $p_{j,w}$ unique, one should sum not over $W_{B,\mathbb{C}}$, but over the quotient of $W_{B,\mathbb{C}}$ by the stabilizer of μ.

According to (7.12), if Θ extends continuously to $H_n(G)$, the linear functional

(7.23) $f \mapsto \int_B f \Delta_B \Theta \, db, \quad f \in C_0^\infty(B''_j \cap B_\varepsilon)$,

becomes bounded in the topology which $H_n(B)$ induces on $C_0^\infty(B''_j \cap B_\varepsilon)$; $\varepsilon > 0$ is arbitrary. To complete the proof of Proposition (6.10), we must deduce:

(7.24) $p_{j,w} \neq 0 \Rightarrow \operatorname{Re}\langle w_\mu, X \rangle < 0, \quad$ for any non-zero X in the closure of C_j.

By separating out the toroidal variable, we shall reduce the problem to one about functions on Euclidean spaces. For this purpose, we re-interpret the identity (7.22): there exist distinct characters $\eta_1, \ldots, \eta_N$ of the torus B_+^0, and functions $h_1, \ldots, h_N \in C^\infty(B_-)$, such that

$$(7.25) \quad (\Delta_B \Theta)(b_j\, b \exp X) = \sum_i \eta_i(b)\, h_i(\exp X),$$

if $b \in B_+^0$, $X \in C_j$. Moreover,

$$(7.26) \quad h_i(\exp X) = \sum_{j=1}^{M_i} p_{ij}(X)\, e^{\langle \nu_{ij}, X\rangle}, \qquad X \in \mathfrak{b}_-,$$

where the p_{ij} are polynomial functions on $\mathfrak{b}_-$, and $\nu_{ij} \in \mathfrak{b}_-^{\mathbb{C}*}$. We let C_j' denote the translate of C_j by some fixed $X_0 \in C_j$; thus C_j contains the closure of C_j'. As will be argued shortly, one can select a non-empty open subset $U \subset B_+^0$, and some $\varepsilon > 0$, which have the property that

$$(7.27) \quad b_j\, U \exp C_j' \subset B_j'' \cap B_\varepsilon.$$

No non-trivial linear combination of the η_i is perpendicular to $C_0^\infty(U)$ in $L^2(B_+^0)$. Hence, for suitably chosen functions $f_i \in C_0^\infty(U)$,

$$\int_{B_+^0} f_i\, \eta_j\, db = \delta_{ij}.$$

Testing the linear functional (7.23) against products $f_i f$, with $f \in C_0^\infty(\exp C_j')$, one finds that

$$(7.28) \quad f \mapsto \int_{B_-} f\, h_i\, db, \qquad f \in C_0^\infty(\exp C_j'),$$

is bounded, relative to the topology of $H_n(B_-)$,

for $1 \leqq i \leqq N$; the Sobolev space $H_n(B_-)$ for the vector group $B_- \cong \mathfrak{b}_-$ is defined in the usual fashion.

We still must produce U and ε, as in (7.27). For any real root α, e^α is uniformly bounded away from one, on the entire set $b_j\, B_+^0 \exp C_j'$. Every $\alpha \in \Phi_B$, whether or not it lies in $\Phi_{B,\mathbb{R}}$, assumes only real values on $\mathfrak{b}_-$, so that $e^\alpha > 0$ on B_-. On the other hand, $|e^\alpha| \equiv 1$ on B_+. Hence, if δ is sufficiently small,

$$|e^\alpha(b_j\, b) - 1| > \delta \Rightarrow |e^\alpha(b_j\, b \exp X) - 1| > \tfrac{1}{2}\delta,$$

whenever $b \in B_+^0$, $X \in \mathfrak{b}_-$. Thus (7.27) will be satisfied for any open, relatively compact subset U of

$$(7.29) \quad \{b \in B_+^0 \mid e^\alpha(b_j\, b) \neq 1 \text{ for } \alpha \in \Phi_B,\ \alpha \notin \Phi_{B,\mathbb{R}}\},$$

if only $\varepsilon > 0$ is small enough. No character e^α, with $\alpha \in \Phi_B$, $\alpha \notin \Phi_{B,\mathbb{R}}$, remains constant on B_+^0. The set (7.29) is therefore non-empty, and we can indeed choose U and ε.

Because of (7.28), the verification of the assertion (7.24) amounts to a problem about functions on Euclidean space. In order to state the relevant result, we let Q denote the positive quadrant in $\mathbb{R}^d$,

$$Q=\{(x_1,\dots,x_d)\in\mathbb{R}^d \mid x_i>0,\ 1\leqq i\leqq d\}.$$

Via the inclusion $\mathbb{R}^d \hookrightarrow \mathbb{C}^d$ and the natural pairing $\mathbb{C}^d\times\mathbb{C}^d\mapsto\mathbb{C}$, every $\xi\in\mathbb{C}^d$ defines a complex-valued linear function $x\mapsto\langle\xi,x\rangle$ on $\mathbb{R}^d$.

(7.30) **Lemma.** Let $\xi_1,\dots,\xi_N$ be distinct elements of $\mathbb{C}^d$, and $p_1,\dots,p_N$ non-zero polynomial functions on $\mathbb{R}^d$. The distribution

$$f\mapsto\int_{\mathbb{R}^d} f(x)\sum_i p_i(x)\,e^{\langle\xi_i,x\rangle}\,dx,\qquad f\in C_0^\infty(Q),$$

does not extend continuously to the closure of $C_0^\infty(Q)$ in any Sobolev space $H_n(\mathbb{R}^d)$, unless the real part of each ξ_i lies in $-Q$.

In terms of suitable linear coordinates $x_1,\dots,x_d$ on $\mathfrak{b}_-$, the Weyl chamber C_j can be described as

$$\{(x_1,\dots,x_d)\in\mathbb{R}^d \mid x_i>0,\ 1\leqq i\leqq k\},$$

for some $k\leqq d$. Except for a finite number of hyperplanes, this set decomposes into 2^{d-k} copies of Q. We also note that translation by some $x_0\in\mathbb{R}^d$ transforms the distribution described in the lemma into another one, of the same type, with the same exponents ξ_i. In particular, the conclusion remains unchanged if the domain $C_0^\infty(Q)$ of the distribution in question is replaced by $C_0^\infty(Q+x_0)$. For these reasons, the lemma implies the statement (7.24).

Proof of (7.30). If a distribution is continuous in the topology of the n-th Sobolev space, then its image under a constant coefficient differential operator, of order k, is continuous in the topology of the $(n+k)$-th Sobolev space. For any polynomial q of d variables,

$$q\left(\frac{\partial}{\partial x_i}\right)\Big(\sum_j p_j(x)\,e^{\langle\xi_j,x\rangle}\Big)=\sum_j\sum_I\frac{1}{I!}\left(\frac{\partial}{\partial x_I}p_j\right)(x)\left(\frac{\partial}{\partial x_I}q\right)(\xi_j)\,e^{\langle\xi_j,x\rangle},$$

with I running over all d-tuples of indices $I=(i_1,\dots,i_d)$ and $I!=\prod_l i_l!$. Hence, by an appropriate choice of the constant coefficient operator $q\left(\frac{\partial}{\partial x_i}\right)$, we can arrange that

$$q\left(\frac{\partial}{\partial x_i}\right)\Big(\sum_j p_j(x)\,e^{\langle\xi_j,x\rangle}\Big)=e^{\langle\xi_l,x\rangle}.$$

In this fashion, we reduce the problem to the case of a single exponential term $e^{\langle\xi,x\rangle}$, without a polynomial coefficient. A further reduction is possible: by testing the distribution $e^{\langle\xi,x\rangle}$ against products

$$f(x_1,\dots,x_d)=f_1(x_1)\cdot f_2(x_2)\cdot\dots\cdot f_d(x_d),\qquad f_i\in C_0^\infty(\mathbb{R}^+),$$

one can deal with one variable at a time. In other words, we may and shall assume that $d=1$. We now evaluate the distribution $e^{\xi x}$ on a sequence of functions $f_k(x)=f(x-k)$, for a fixed $f\in C_0^\infty(\mathbb{R}^+)$. The boundedness of the distribution in some Sobolev topology gives, at least, the estimate $\operatorname{Re}\xi\leqq 0$; we must exclude the possibility that $\operatorname{Re}\xi=0$. If ξ were purely imaginary, translating the distribution would change it only by a multiplicative constant, of modulus one. Every $f\in C_0^\infty(\mathbb{R})$ has a translate whose support lies in $\mathbb{R}^+$. Consequently, the distribution $e^{\xi x}$ would be bounded, in the topology of some Sobolev space, not only on $C_0^\infty(\mathbb{R}^+)$, but actually on all of $C_0^\infty(\mathbb{R})$. That is absurd: evaluation of the Fourier transform $\hat{f}$ at some point $y\in\mathbb{R}$, for $f\in C_0^\infty(\mathbb{R})$, fails to be continuous with respect to any Sobolev topology.

The proofs of Proposition (6.11) and of Lemma (6.15) are straightforward applications of Harish-Chandra's "matching conditions". We shall briefly recall what is involved. Until further notice, we keep fixed the following data: an invariant eigendistribution Θ, a Cartan subgroup B of G, a connected component B_j'' of B'', as in (7.20), and a root $\gamma\in\Phi_{B,\mathbb{R},j}$ which is simple with respect to the positive root system $\Phi^+_{B,\mathbb{R},j}$. To any real root, and in particular to γ, one can associate a so-called Cayley transform, an inner automorphism of $\mathfrak{g}^{\mathbb{C}}$, which maps $\mathfrak{b}^{\mathbb{C}}$ onto the complexification of a Cartan subalgebra $\mathfrak{b}_\gamma\subset\mathfrak{g}$; the corresponding Cartan subgroup B_γ meets B along

$$B\cap B_\gamma=\{b\in B\mid e^\gamma(b)=1\},$$

and the dimension of its compact part $B_{\gamma,+}$ exceeds that of B_+ by one. The open subsets $(B\cap G')^G$ and $(B_\gamma\cap G')^G$ of G have a hypersurface S_γ as common boundary, which contains $B\cap B_\gamma$. As an invariant eigendistribution, Θ satisfies certain differential equations, which are used, in particular, to obtain expressions of the type (7.22). An investigation of the same differential equations, near the hypersurface S_γ, leads to relations between the restrictions of Θ to B and B_γ, respectively. These are the matching conditions [13, 3].

Since γ is simple with respect to $\Phi^+_{B,\mathbb{R},j}$, the intersection $B\cap B_\gamma$ – equivalently, the kernel of the character e^γ on B – contains a whole "wall" of B_j'', namely

(7.31) $\{b_j\, b\exp X\mid b\in B^0_+,\ X\in\text{closure of } C_j,\ \langle\gamma, X\rangle=0\}$.

According to (7.22), the restriction of $\Delta_B\,\Theta$ to B_j'' extends as an analytic function across the wall (7.31); for simplicity, the extension shall be referred to as φ. We now choose a non-zero, translation-invariant vector field X_γ on B, which is normal to the codimension one subgroup $B\cap B_\gamma$. The matching conditions assert that, for each odd integer n, $X_\gamma^n\,\varphi$ coincides, on the subset (7.31) of $B\cap B_\gamma$, with a similar expression derived from the restriction of Θ to B_γ. In particular, if $\Theta\equiv 0$ on B_γ, all of the odd derivatives $X_\gamma^n\,\varphi$ vanish on the wall (7.31). Equivalently, $\Delta_B\,\Theta\,|_{B_j''}$ must then be symmetric with respect to the reflection about the root γ. The reflections about all simple roots in $\Phi^+_{B,\mathbb{R},j}$ generate the full Weyl group of the root system $\Phi_{B,\mathbb{R},j}$. Hence, if Θ vanishes on B_γ, for every simple root $\gamma\in\Phi^+_{B,\mathbb{R},j}$, then $\Delta_B\,\Theta\,|_{B_j''}$ will be symmetric with respect to this entire Weyl group. The compact factor of every such B_γ has dimension one greater than the compact factor of B. We conclude:

(7.32) $\Delta_B\,\Theta\,|_{B_j''}$ is symmetric with respect to the Weyl group of $\Phi_{B,\mathbb{R},j}$, provided $\dim B_+$ is maximal, among all Cartan subgroups on which Θ does not vanish identically.

Our arguments depend only on this one consequence of the matching conditions.

Let us suppose now that Θ decays at infinity. The exponents $w\mu$ which appear with non-zero coefficient $p_{j,w}$ in (7.22) then satisfy $\langle w\mu, X\rangle < 0$, for every $X \neq 0$ in the closure of C_j. If $\Delta_B \Theta|_{B_j''}$ were symmetric with respect to the Weyl group of $\Phi_{B,\mathbb{R},j}$, each $w\mu$ would have to assume strictly negative values also on the various translates of the Weyl chamber C_j, i.e. on all of $\mathfrak{b}_-$. This is impossible unless $\Theta|_{B_j''}=0$, or $\mathfrak{b}_-=0$, in which case B is compact. The assertion of Proposition (6.11) follows.

We argue similarly to prove Lemma (6.15). Let us suppose that Θ_1, Θ_2 are two distinct invariant eigendistributions, which both have the properties mentioned in the lemma. As was described in § 4, the explicit formula for $\Theta_i|_H$ makes it possible to identify the infinitesimal character of Θ_i, namely

$$(7.33)\quad Z \mapsto \gamma_H(Z)(\lambda), \qquad Z \in \mathfrak{Z};$$

$\gamma_H\colon \mathfrak{Z} \xrightarrow{\sim} I(\mathfrak{h}^{\mathbb{C}})$ is the isomorphism (4.8) corresponding to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$, and $\gamma_H(Z)$ is viewed as a polynomial function on $\mathfrak{h}^{*\mathbb{C}}$. Since Θ_1 and Θ_2 have the same infinitesimal character, their difference $\Theta=\Theta_1-\Theta_2$ is again an invariant eigendistribution. We now choose a Cartan subgroup B and a connected component B_j'' of B'', such that Θ does not vanish identically on B_j'', but does vanish on any Cartan subgroup whose compact factor has a larger dimension than B_+. Then B cannot be compact: any two compact Cartan subgroups are conjugate, and $\Theta|_H \equiv 0$. Because of our assumptions, Θ remains bounded at infinity. Hence, when the restriction of $\Delta_B \Theta$ to B_j'' is expressed as in (7.22), every exponent $w\mu$ which occurs with a non-trivial coefficient $p_{j,w}$ satisfies $\operatorname{Re} w\mu \leq 0$, on C_j. The assertion (7.32) allows us to conclude

$$(7.34)\quad p_{j,w} \neq 0 \Rightarrow \operatorname{Re} w\mu \equiv 0 \quad \text{on } \mathfrak{b}_-.$$

To complete the proof of (6.15), we must derive a contradiction.

The reasoning which led to the description (7.33) of the infinitesimal character of Θ also gives the alternate description

$$Z \mapsto \gamma_B(Z)(\mu), \qquad Z \in \mathfrak{Z}$$

in terms of the isomorphism $\gamma_B\colon \mathfrak{Z} \xrightarrow{\sim} I(\mathfrak{b}^{\mathbb{C}})$. Because of the canonical nature of the Harish-Chandra isomorphism (4.8), γ_B and γ_H are related: $\gamma_H = c \cdot \gamma_B$, whenever $c\colon \mathfrak{b}^{\mathbb{C}} \xrightarrow{\sim} \mathfrak{h}^{\mathbb{C}}$ is induced by an inner automorphism of $\mathfrak{g}^{\mathbb{C}}$. Hence $c^* \lambda$ and μ lie in the same Weyl group orbit, i.e.

$$(7.35)\quad \mu = u\, c^* \lambda, \qquad \text{for some } u \in W_{B,\mathbb{C}}.$$

The elements of the weight lattice Λ of H can be expressed as $\mathbb{Q}$-linear combinations of roots $\alpha \in \Phi$. Since every root $\alpha \in \Phi_B$ assumes real values on the split part $\mathfrak{b}_-$ of $\mathfrak{b}$, and since $c^* \Phi = \Phi_B$, the Weyl group translates of μ must be real on $\mathfrak{b}_-$. Thus (7.34) can be sharpened:

$$(7.36)\quad p_{j,w} \neq 0 \Rightarrow w\mu \equiv 0 \quad \text{on } \mathfrak{b}_-.$$

The real roots $\alpha \in \Phi_{B,\mathbb{R}}$ vanish on $\mathfrak{b}_+$, and the decomposition $\mathfrak{b} = \mathfrak{b}_+ \oplus \mathfrak{b}_-$ is orthogonal with respect to the Killing form. Thus (7.36) implies

$$(w\mu, \alpha) = 0, \qquad \text{for } \alpha \in \Phi_{B,\mathbb{R}},$$

provided $p_{j,w} \neq 0$. This is the case with at least one $w \in W_{B,\mathbb{C}}$, because $\Theta|_{B_j'} \neq 0$. A Cartan subgroup has an empty system of real roots precisely when it is fundamental, i.e. when its compact part is of maximal possible dimension. In our situation, G contains a compact Cartan subgroup, so that B cannot be fundamental. We conclude that μ is singular. In view of (7.35), this contradicts our assumption on λ.

§ 8. Complete Description of the Discrete Series

As was shown in § 6, G has a non-empty discrete series if and only if it contains a compact Cartan subgroup. We assume that this is the case, and we select a particular compact Cartan subgroup $H \subset G$. We shall follow the notation of § 4. In particular, $\Lambda_\rho \subset i\mathfrak{h}^*$ is the lattice formed by the differentials of characters of H, translated by the half-sum of the positive roots; cf. (4.25). We recall the notion of an invariant eigendistribution which is bounded at infinity, as defined in § 6. With these ingredients, we can now state and prove Harish-Chandra's fundamental result on the discrete series [15]:

(8.1) **Theorem.** Corresponding to any nonsingular $\lambda \in \Lambda_\rho$, there exists exactly one invariant eigendistribution Θ_λ, which is bounded at infinity, and such that

$$\Theta_\lambda|_{H \cap G'} = (-1)^q \frac{\sum\limits_{w \in W} \varepsilon(w)\, e^{w\lambda}}{\prod\limits_{\substack{\alpha \in \Phi,\\ (\alpha,\lambda) > 0}} (e^{\alpha/2} - e^{-\alpha/2})}$$

($q = \frac{1}{2} \dim G/K$). Every such Θ_λ arises as the character of a discrete series representation, whose formal degree equals

$$d(\lambda - \rho) = \prod_{\substack{\alpha \in \Phi,\\ (\alpha,\lambda) > 0}} \frac{(\alpha, \lambda)}{(\alpha, \rho)}, \qquad \text{with } \rho = \tfrac{1}{2} \sum_{(\alpha,\lambda) > 0} \alpha.$$

Conversely every discrete series character occurs among the Θ_λ.

Because of the uniqueness which is asserted in the theorem, two of the Θ_λ coincide precisely when they have the same restriction to H, and this is the case whenever their parameters are related by the action of W, the Weyl group of H in G. In particular, then, the theorem provides a one-to-one parametrization of the discrete series, in terms of the quotient

(8.2) $\{\lambda \in \Lambda_\rho \mid \lambda \text{ is nonsingular}\}/W$.

▲

Our proof of the theorem also leads to information about the decomposition of discrete series representations under the maximal compact subgroup K. We consider a particular discrete series representation π_λ, with character Θ_λ. The parameter λ determines a system of positive roots Ψ in Φ, namely

(8.3) $\Psi = \{\alpha \in \Phi \mid (\lambda, \alpha) > 0\}$.

As before, we set

(8.4) $\rho_c = \frac{1}{2} \sum_{\Phi^c \cap \Psi} \alpha, \quad \rho_n = \frac{1}{2} \sum_{\Phi^n \cap \Psi} \alpha.$

The next statement is a weaker, but useful version of Blattner's conjecture.

(8.5) **Theorem.** In the restriction of π_λ to K, the irreducible K-module of highest weight $\lambda + \rho_n - \rho_c$ occurs exactly once. Any irreducible constituent of $\pi_\lambda|_K$ has a highest weight of the form $\lambda + \rho_n - \rho_c + A$, where A stands for a sum of roots in Ψ.

Turning to the proof of the two theorems, we note that the uniqueness of the Θ_λ was established in § 6. We now fix a nonsingular $\lambda \in \Lambda_\rho$, and we define Ψ, ρ_c, ρ_n in terms of λ, as above. As was argued below (4.34),

(8.6) $\mu = \lambda - \rho_c$

is the highest weight of an irreducible $\mathfrak{k}^{\mathbb{C}}$-module V_μ, and the action of $\mathfrak{k}^{\mathbb{C}}$ on the tensor products $V_\mu \otimes S^\pm$ lifts to K. Thus we can apply Theorem (5.20): if

(8.7) $\min_{\alpha \in \Phi} |(\alpha, \lambda)| > c,$

for some appropriately chosen constant c, there does exist a discrete series representation π_λ, which has the properties described in (8.5), and whose character Θ_λ satisfies the conditions of (8.1). In the case of a general λ, Theorem (4.41) provides at least a partial answer. We shall use a method of Zuckerman [27] to deduce the assertions of (8.1) and (8.5). Since our arguments depend only on a specialized version of Zuckerman's technique, we shall develop his ideas to the extent that they are needed here.

Although we are concerned with unitary representations, it is necessary at this point to work in the wider context of representations on Banach spaces. We briefly recall the important properties of such representations; details can be found in [26], for example. A representation π of G on a Banach space is said to be admissible if each class $i \in \hat{K}$ occurs with finite multiplicity n_i in $\pi|_K$. Every irreducible unitary representation has this property; it is unknown whether irreducibility implies admissibility in general. Each admissible representation π gives rise to an infinitesimal representation of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$, on the space of K-finite vectors. The sub-representations of π correspond precisely to the invariant subspaces for the infinitesimal representations. If the infinitesimal representations attached to global representations are isomorphic, one calls the global representations infinitesimally equivalent. Informally, one may think of infinitesimally equivalent representations as being identical, except for a modification of the topology on the underlying vector spaces. Among unitary representations, the notions of infinitesimal equivalence and unitary equivalence coincide. Whenever π is both admissible and irreducible, the K-multiplicities n_i satisfy the bound $n_i \leq$ degree of i, just as in the unitary case; moreover, the center $\mathfrak{Z}$ of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ then acts by scalars. Conversely, if π is admissible, and if $\mathfrak{Z}$ operates on the space of K-finite vectors according to a character, then π has at least a finite composition series.

Admissible, irreducible representations have global characters, for essentially the same reasons as in the unitary case. More generally, the definition of the character of an admissible representation π makes sense, provided only π has a finite composition series. One can describe the character explicitly as the sum, in the sense of distributions, of the diagonal matrix coefficients, relative to a "basis" consisting of vectors in the various K-invariant, K-irreducible subspaces. Characters cannot distinguish between infinitesimally equivalent representations: two admissible, irreducible representations have the same character precisely when they are infinitesimally equivalent. In fact, the characters corresponding to any finite set of infinitesimally distinct, irreducible representations are linearly independent.

We now consider a particular admissible, irreducible representation π, and a finite-dimensional representation τ, whose characters we denote by Θ_π and χ_τ. As is not hard to check, the tensor product will then be admissible, too. Furthermore, it is known that

(8.8) $\pi \otimes \tau$ has a finite composition series.

One can see this, for example, as follows. Like any admissible, irreducible representation, π is infinitesimally equivalent to a sub-representation of some principal series representation[1], which need not be unitary, of course. Hence, in verifying (8.8), we may as well assume that π itself is a principal series representation, instead of being irreducible. Thus π is induced, from a minimal parabolic subgroup $P \subset G$, by a finite-dimensional, irreducible representation σ of P, i.e. $\pi = \operatorname{ind}_P^G \sigma$. For essentially formal reasons,

$$\pi \otimes \tau \cong \operatorname{ind}_P^G(\sigma \otimes \tau).$$

Any composition series of the finite-dimensional representation $\sigma \otimes \tau$ of P determines a filtration of $\operatorname{ind}_P^G(\sigma \otimes \tau)$, of finite length, whose quotients are again principal series representations. A principal series representation, finally, does have a finite composition series, as follows from the fact that it has an infinitesimal character, and that it is admissible.

As the character of the tensor product $\pi \otimes \tau$, $\chi_\tau \Theta_\pi$ equals the sum of the characters of the composition factors. For each character φ of the algebra $\mathfrak{Z}$, we let $(\chi_\tau \Theta_\pi)_\varphi$ denote the sum of the characters of those composition factors on which $\mathfrak{Z}$ acts according to φ. Then

$$(8.9) \quad \chi_\tau \Theta_\pi = \sum_\varphi (\chi_\tau \Theta_\pi)_\varphi,$$

with φ ranging over a finite set of characters of $\mathfrak{Z}$.

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[1] Casselman [9] has given a simple proof of this fact

For the purpose of stating the next lemma, we fix a system of positive roots $\Psi \subset \Phi$, and some $\lambda \in \Lambda_\rho$, such that λ is dominant with respect to Ψ, but not necessarily nonsingular. Although G itself need not be linear, it is at least a finite covering of a linear group. Hence arbitrarily large positive multiples of λ occur as highest weights of irreducible, finite-dimensional representations of G. We suppose that τ is such an irreducible, finite-dimensional representation, of highest weight $(m-1)\lambda$, $m \geqq 1$; χ_τ denotes the character of τ, and χ_τ^* the character of the representation dual to τ. We shall use the symbols $\mathscr{C}_\lambda$ and $\mathscr{C}_{m\lambda}$ to refer to the sets of characters of irreducible, admissible representations, with infinitesimal character φ_λ and $\varphi_{m\lambda}$, respectively; cf. (4.17).

(8.10) **Lemma** (Zuckerman [27]): The mapping

$$S:\ \Theta \mapsto (\chi_\tau \Theta)_{\varphi_{m\lambda}}$$

establishes a bijection between the sets $\mathscr{C}_\lambda$ and $\mathscr{C}_{m\lambda}$, whose inverse is given by

$$T:\ \Theta \mapsto (\chi_\tau^* \Theta)_{\varphi_\lambda}.$$

If $\Theta \in \mathscr{C}_\lambda$ decays at infinity, then so does $S\Theta$, and conversely.

Proof. In the obvious manner, S and T extend to mappings between the linear spans of $\mathscr{C}_\lambda$ and $\mathscr{C}_{m\lambda}$, in the appropriate spaces of invariant eigendistributions. By virtue of their definition, both S and T associate to each irreducible character an integral linear combination of irreducible characters, with non-negative coefficients. We shall show that

$$(8.11)\quad T\Theta \neq 0, \qquad \text{if } \Theta \in \mathscr{C}_{m\lambda}$$

and

$$(8.12)\quad T \cdot S = \text{identity}, \qquad \text{on } \mathscr{C}_\lambda,$$

as well as the corresponding statements with reversed roles for S and T. Thus, if $S\Theta$ were a sum of more than one irreducible character, $\Theta = T \cdot S\Theta$ would also have to be a sum of several irreducible characters, contradicting the irreducibility of Θ. This will prove the first half of the lemma. The second assertion will follow from an explicit description of S.

In order to understand S and T, we consider a particular $\Theta \in \mathscr{C}_\lambda$, restricted to a Cartan subgroup B. On any connected component B_j'' of B'', as in (7.20), Θ can be expressed by a formula like (7.22):

$$(8.13)\quad (\Delta_B \Theta)(b_j \exp(X+Y)) = \sum_{w \in W_{B,\mathbb{C}}} p_{j,w}(X)\, e^{\langle w\mu, X+Y\rangle}, \qquad \text{for } X \in C_j,\ Y \in \mathfrak{b}_+.$$

Since Θ is a character, the coefficients $p_{j,w}$ are known to be constants, not polynomials [11], but this fact turns out to be irrelevant for our purposes. The Cartan subalgebras $\mathfrak{b}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ are related by an inner automorphism of $\mathfrak{g}^{\mathbb{C}}$, say

$$(8.14)\quad c:\ \mathfrak{b}^{\mathbb{C}} \xrightarrow{\sim} \mathfrak{h}^{\mathbb{C}}.$$

The arguments which precede (7.35) also apply in the present situation. Thus, modifying c by an element of $W_{\mathbb{C}}$, if necessary, one can arrange

(8.15) $\mu = c^* \lambda$.

Every weight ν of τ, relative to the Cartan subalgebra $\mathfrak{b}^{\mathbb{C}}$, lifts to a character e^ν of B. We set n_ν = multiplicity of the weight ν; then

$$\chi_\tau|_B = \sum_\nu n_\nu e^\nu.$$

The Weyl group $W_{B,\mathbb{C}}$ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ leaves χ_τ invariant, so that

$$(\Delta_B \chi_\tau \Theta)(b_j \exp(X+Y))$$
$$= \sum_\nu \sum_{w \in W_{B,\mathbb{C}}} n_\nu e^{w\nu}(b_j) p_{j,w}(X) e^{\langle w(\mu+\nu), X+Y \rangle},$$

if $X \in C_j$, $Y \in \mathfrak{b}_+$. In this formula, the contribution of $S\Theta$ consists of those terms which belong to the infinitesimal character $\varphi_{m\lambda}$, i.e. the terms for which $\mu+\nu$ is $W_{B,\mathbb{C}}$-conjugate to $c^*(m\lambda) = m\mu$.

We claim: $\mu+\nu$ lies in the $W_{B,\mathbb{C}}$-orbit of $m\mu$ only if $\nu = (m-1)\mu$, in which case $n_\mu = 1$. Indeed, $(m-1)\mu$ occurs as an extreme weight of the finite-dimensional, irreducible representation τ, and hence has multiplicity one. The weights of τ all lie in the convex body spanned by the extreme weights, i.e. by the $W_{B,\mathbb{C}}$-translates of $(m-1)\mu$. When this convex body is shifted by μ, among the new vertices, $m\mu$ lies farthest from the origin. The claim follows, and we deduce:

(8.16) $(\Delta_B(S\Theta))(b_j \exp(X+Y))$
$$= \sum_{w \in W_{B,\mathbb{C}}} e^{(m-1)w\mu}(b_j) p_{j,w}(X) e^{m\langle w\mu, X+Y \rangle},$$

whenever $X \in C_j$, $Y \in \mathfrak{b}_+$. Similarly, if the restriction of some $\Theta \in \mathscr{C}_{m\lambda}$ to B_j'' is given by

$$(\Delta_B \Theta)(b_j \exp(X+Y)) = \sum_{w \in W_{B,\mathbb{C}}} q_{j,w}(X) e^{m\langle w\mu, X+Y \rangle},$$

then

(8.17) $(\Delta_B(T\Theta))(b_j \exp(X+Y))$
$$= \sum_{w \in W_{B,\mathbb{C}}} e^{-(m-1)w\mu}(b_j) q_{j,w}(X) e^{\langle w\mu, X+Y \rangle},$$

again for all $X \in C_j$, $Y \in \mathfrak{b}_+$; the verification is entirely analogous to that of (8.16).

Every invariant eigendistribution is completely determined by its restrictions to the various Cartan subgroups. Hence the explicit formulas (8.16–8.17) imply the two assertions (8.11–8.12). We recall that an invariant eigendistribution Θ decays at infinity if, and only if, in terms of the notation of (8.13),

(8.18) $p_{j,w} \neq 0 \Rightarrow w\mu$ assumes strictly negative values on the closure of C_j,

except of course at the origin, for all choices of B and B_j''. The condition (8.18) remains unchanged by an application of S or T, which has the effect of multiplying the exponents by a positive constant. This completes the proof of the lemma.

When the arguments leading up to (8.17) are carried out for the compact Cartan subgroup H, one finds:

(8.19) **Corollary.** If $\Theta\in\mathscr{C}_{m\lambda}$ satisfies

$$\Delta_H \Theta|_H = \sum_{w\in W_{\mathbb{C}}} a_w\, e^{mw\lambda},$$

then

$$\Delta_H (T\Theta)|_H = \sum_{w\in W_{\mathbb{C}}} a_w\, e^{w\lambda}.$$

We return to the proof of the main theorem. Thus a nonsingular $\lambda\in\Lambda_\rho$ is given, and Ψ is the system of positive roots which makes λ dominant. We enumerate as

(8.20) $\lambda_1=\lambda, \lambda_2, \ldots, \lambda_N$

those $W_{\mathbb{C}}$-conjugates of λ which lie in Λ_ρ and are dominant with respect to $\Phi^c\cap\Psi$; this is consistent with the notation of Theorem (4.41). The correspondences S and T of (8.10) depend on the choice of the integer m. We make m so large that the multiples $m\lambda_i$ of the various λ_i become sufficiently nonsingular, in the sense of (8.7). By assumption, $(m-1)\lambda$ occurs as weight of a finite-dimensional representation of G, hence lies in Λ, as do its $W_{\mathbb{C}}$-conjugates. Thus Λ_ρ contains not only the various λ_i, but also their multiples $m\lambda_i$. Theorem (8.1) has already been established for every nonsingular parameters. In particular, there exist discrete series characters $\Theta_{m\lambda_i}$, such that

$$\Theta_{m\lambda_i}|_{H\cap G'} = (-1)^q \frac{\sum_{w\in W} \varepsilon(w)\, e^{mw\lambda_i}}{\prod_{\substack{\alpha\in\Phi,\\ (\alpha,\lambda_i)>0}} (e^{\alpha/2}-e^{-\alpha/2})}.$$

We define

(8.21) $\Theta_{\lambda_i} = T\Theta_{m\lambda_i}$.

Then, because of (8.10) and (8.19), Θ_{λ_i} is the character of an irreducible representation,

$$(8.22)\quad \Theta_{\lambda_i}|_{H\cap G'} = (-1)^q \frac{\sum_{w\in W} \varepsilon(w)\, e^{w\lambda_i}}{\prod_{\substack{\alpha\in\Phi,\\ (\alpha,\lambda_i)>0}} (e^{\alpha/2}-e^{-\alpha/2})},$$

and Θ_{λ_i} decays at infinity.

The invariant eigendistribuition $\tilde{\Theta}_\lambda$ of (4.26) was defined as the sum of the discrete series characters which correspond to the infinitesimal character φ_λ, each multiplied by its formal degree. As a linear combination of discrete series characters, $\tilde{\Theta}_\lambda$ decays at infinity. If two invariant eigendistribution agree on a compact Cartan subgroup, and if they both decay at infinity, than they coincide; this follows from (6.11) (any two compact Cartan subgroups are conjugate!). Thus, comparing (4.41) and (8.22), we find

$$(8.23)\quad \tilde{\Theta}_\lambda = \left(\prod_{\alpha\in\Psi} \frac{(\alpha,\lambda)}{(\alpha,\rho)}\right) \sum_{i=1}^{N} \Theta_{\lambda_i}.$$

An invariant eigendistribution can be expressed as a linear combination of irreducible characters in only one way, if at all. Also, the coefficient appearing in (8.23) is non-zero. We conclude: the Θ_{λ_i}, $1 \leqq i \leqq N$, are precisely the discrete series characters corresponding to the infinitesimal character φ_λ; each has formal degree

$$d(\lambda-\rho) = \prod_{\alpha\in\Psi} \frac{(\alpha,\lambda)}{(\alpha,\rho)}.$$

The infinitesimal character of any discrete series representation equals φ_λ, for some nonsingular $\varphi\in\Lambda_\rho$, as was shown in §6. The proof of Theorem (8.1) is therefore complete.

For very nonsingular parameters $\lambda\in\Lambda_\rho$, the assertion of Theorem (8.5) is part of Theorem (5.20). If $\lambda\in\Lambda_\rho$ is nonsingular, but otherwise arbitrary, we choose the integer m as in the previous argument, so that

$$\Theta_\lambda = T\Theta_{m\lambda} = (\chi_\tau^* \Theta_{m\lambda})_{\varphi_\lambda}.$$

Thus, if π_λ and $\pi_{m\lambda}$ are discrete series representations with global character Θ_λ and $\Theta_{m\lambda}$, respectively, then π_λ can be realized as a composition factor of $\pi_{m\lambda}\otimes\tau^*$, up to infinitesimal equivalence. Here τ^* stands for the representation dual to τ, i.e. the finite-dimensional, irreducible representation which has $-(m-1)\lambda$ as lowest weight. In particular, all irreducible constituents of $\pi_\lambda|_K$ are among those of $(\pi_{m\lambda}\otimes\tau^*)|_K$. The highest weight of any irreducible constituent of the tensor product can be expressed as the sum of the highest weight of a constituent of $\pi_{m\lambda}|_K$ with some weight of τ^*. We already know that every highest weight in $\pi_{m\lambda}|_K$ is of the form $m\lambda+\rho_n-\rho_c+A$, with A equal to a sum of positive roots. Any weight of τ^* differs from the lowest weight by a sum of positive roots. All this implies the second assertion of (8.5).

Since $\pi_{m\lambda}|_K$ contains the irreducible K-module of highest weight $m\lambda+\rho_n-\rho_c$ only once, and since the lowest weight $-(m-1)\lambda$ of τ^* has multiplicity one, the irreducible K-module of highest weight $\lambda+\rho_n-\rho_c$ cannot occur more than once in $\pi_\lambda|_K$. We must check that it does occur. The results in the beginning of §4, especially (4.16c), coupled with the explicit formula for $\Theta_\lambda|_{H\cap G'}$, show that

(8.24) $\lambda-\rho_c$ is the highest weight of an irreducible constituent of $(\pi_\lambda|_K)\otimes(s^+\oplus s^-)$

(s^+, s^- refer to the action of K on the half spin modules S^+, S^-). Every weight of $S^+\oplus S^-$ can be written as $-\rho_n+\beta_1+\dots+\beta_s$, with $\beta_i\in\Phi^n\cap\Psi$, and every highest weight in $\pi_\lambda|_K$ as $\lambda+\rho_n-\rho_c+A$, A being a sum of positive roots. Hence (8.24) is possible only if $\pi_\lambda|_K$ contains the irreducible K-module of highest weight $\lambda+\rho_n-\rho_c$. This we have proved Theorem (8.6).

§9. Realization of the Discrete Series

Although the spaces $\mathscr{H}_\mu^+, \mathscr{H}_\mu^-$ of square-integrable, harmonic spinors play a crucial role in our construction, we have not yet described them completely, except for very nonsingular parameters μ. It is known that each $\mathscr{H}_\mu^\pm$ either vanishes, or is an irreducible unitary G-module, which belongs to the discrete series; moreover, every discrete series representation can be realized in this manner [22, 24]. For the sake of completeness, we now recall the precise statement and discuss its proof.

As in the past, μ shall denote the highest weight of an irreducible $\mathfrak{k}^{\mathbb{C}}$-module V_μ, and Ψ a system of positive roots, such that

(9.1) $(\mu+\rho_c, \alpha) \geqq 0, \quad$ for $\alpha \in \Psi$.

We also assume

(9.2) $\mu+\rho_n \in \Lambda$,

which insures that the twisted spin bundles $\mathscr{V}_\mu \otimes \mathscr{S}^\pm$ can be defined on G/K.

(9.3) **Theorem.** Both $\mathscr{H}_\mu^+$ and $\mathscr{H}_\mu^-$ vanish whenever $\mu+\rho_c$ is singular. Otherwise, for nonsingular $\mu+\rho_c$, only $\mathscr{H}_\mu^-$ vanishes, whereas G acts irreducibly on $\mathscr{H}_\mu^+$, according to the discrete series representation with character $\Theta_{\mu+\rho_c}$.

If $\lambda \in \Lambda_\rho$ is nonsingular, and if Ψ is the positive root system which makes λ dominant, then $\mu=\lambda-\rho_c$ has the required properties. In particular, the theorem provides a concrete realization for every discrete series representation.

Turning to the proof of the theorem, we recall the Plancherel decomposition (5.6) of $\mathscr{H}_\mu^\pm$. The discrete series contributes to it discretely; any $j \in \hat{G}_d$ occurs as often as the dimension of $V_j^\pm$, if j lies in $\hat{G}_\mu$, and does not occur at all for $j \notin \hat{G}_\mu$. The explicit enumeration of the discrete series representations, combined with the information about their K-decompositions in Theorem (8.5), makes it possible to determine these multiplicities:

(9.4) **Lemma.** Suppose j is a class in the discrete series, which belongs to $\hat{G}_\mu$. Then $V_j^+ + V_j^- = 0$, except in the following situation: $\mu+\rho_c$ is nonsingular, and j has character $\Theta_{\mu+\rho_c}$, in which case $\dim V_j^+ = 1$, $V_j^- = 0$.

Proof. We consider a particular class $j \in \hat{G}_d \cap \hat{G}_\mu$, with character Θ_λ. For the time being, Ψ shall denote the system of positive roots

$$\{\alpha \in \Phi \mid (\alpha, \lambda) > 0\},$$

which may be inconsistent with (9.1). However, replacing λ by one of its W-translates, if necessary, we can arrange that

(9.5) $(\mu+\rho_c, \alpha) > 0, \quad$ for $\alpha \in \Phi^c \cap \Psi$.

As was remarked in § 5, $V_j^{\pm}$ can be non-zero only if $\mathbf{H}_j$ contains a K-invariant, K-irreducible subspace with a highest weight of the form $\mu+\rho_n-B$; here B stands for a sum of distinct positive, noncompact roots and, in the case of V_j^-, $B \neq 0$. This statement, incidentally, depends only on the property (9.5) of Ψ, except for the labelling of $V_j^{\pm}$, which is determined by the stronger condition (9.1). According to (8.5), every such highest weight can be expressed as $\lambda+\rho_n-\rho_c+A$, with A equal to a sum of positive roots. Thus $V_j^{\pm}=0$, unless

$$\mu+\rho_c=\lambda+A+B.$$

The preceding equality implies

$$(9.6)\quad (\mu+\rho_c, \mu+\rho_c) \geqq (\lambda,\lambda)+2(\lambda, A+B);$$

because of our choice of Ψ, $(\lambda, A+B)$ is strictly positive, except for $A=B=0$.

Since j lies in $\hat{G}_\mu$, and since φ_λ is its infinitesimal character, we find

$$(\mu+\rho_c, \mu+\rho_c)-(\rho,\rho)=c_\mu=\varphi_\lambda(\Omega)=(\lambda,\lambda)-(\rho,\rho);$$

cf. (4.22). Equivalently,

$$(\mu+\rho_c, \mu+\rho_c)=(\lambda,\lambda),$$

which is compatible with (9.6) only if $A=B=0$, $\lambda=\mu+\rho_c$. Thus we may as well assume that $\mu+\rho_c$ is nonsingular and equal to λ. In this situation, the positive root system Ψ does have the property (9.1), and hence gives the correct labelling of V_j^+, V_j^-. If V_j^- were non-zero, the inequality (9.6) would have to hold with $B \neq 0$, which is impossible. Finally,

$$\dim V_j^+ = \dim V_j^+ - \dim V_j^- = 1,$$

as follows from (4.16c).

In order to complete the proof of Theorem (9.3), we must show that the complement of the discrete series does not contribute to the Plancherel decomposition (5.6) — or equivalently, that the set

$$(9.7)\quad \{j \in \hat{G}_\mu \mid j \notin \hat{G}_d,\ V_j^+ \oplus V_j^- \neq 0\}$$

has zero Plancherel measure. At present, only one argument is known which proves this last statement in full generality. It depends on Harish-Chandra's work on the explicit form of Plancherel measure, as we shall now explain.

To each conjugacy class of Cartan subgroups, Harish-Chandra attaches a series of unitary representations, which are induced from a parabolic subgroup; the inducing representations, restricted to the semisimple part of a Levi component, belong to the discrete series. At one extreme, the discrete series corresponds to the conjugacy class of compact Cartan subgroups (here the inducing process becomes trivial: G is viewed as a parabolic subgroup of itself), at the other extreme lies the unitary principal series. The union of these so-called "nondegenerate series" supports the Plancherel measure. Roughly speaking, the series attached to a Cartan subgroup B is parametrized by the dual group $\hat{B}$, modulo the action of the Weyl group. In terms of this parametrization, the

Plancherel measure is completely continuous with respect to Haar measure on $\hat{B}$. The action of the Casimir operator can be described, in terms of the same parametrization, as the Killing form plus a constant, transferred to a function on $\hat{B}$ via the exponential map. The set of points where this function assumes a given value has zero Haar measure, except if $\hat{B}$ is discrete, in which case $\hat{B}$ parametrizes the discrete series. The preceding results, which go well beyond the bounds of this paper, directly imply the following statement:

(9.8) **Lemma** (Harish-Chandra [16]). The set of classes $j \in \hat{G}$, outside of the discrete series, at which the Casimir operator takes any given value, has measure zero.

In particular, the set (9.7) has zero Plancherel measure, and this completes the proof of Theorem (9.3). The proof is not quite satisfactory, of course, since it uses almost the full strength of Harish-Chandra's explicit Plancherel formula. We shall now mention two alternate approaches to the problem of realizing the discrete series geometrically, which are independent of Lemma (9.8).

Instead of working with the Dirac operator, we could have equally well carried out our construction in the framework of L^2-cohomology, as originally suggested by Langlands [20]. This has the disadvantage of making some of the arguments technically more difficult. On the other hand, a comparatively elementary result of Casselman and Osborne [10] then takes the place of Lemma (9.8). Because of it, all classes $j \in \hat{G}$ which enter the Plancherel decomposition of one of the L^2-cohomology groups have the same infinitesimal character, and consequently only the discrete series can contribute. For details the reader is referred to [25].

The second alternative applies only in the case of a linear group G, and it can deal only with nonsingular values of $\mu+\rho_c$ (which however is enough to realize all discrete series representations). In this situation, because $\mu+\rho_c$ is a dominant and nonsingular weight, $\mu-\rho_n=(\mu+\rho_c)-\rho$ is at least dominant. Thus $\mu-\rho_n$ occurs as the highest weight of an irreducible $\mathfrak{k}^{\mathbb{C}}$-module $V_{\mu-\rho_n}$. Since ρ_n is the highest weight of an irreducible constituent of S^+, there exists an injective K-homomorphism

(9.9) $V_\mu \hookrightarrow V_{\mu-\rho_n} \otimes (S^+ \oplus S^-)$.

The direct sum of the two half-spin modules may be viewed as a square-root of the exterior algebra of the standard representation. In our context, this means that

$$(S^+ \oplus S^-) \otimes (S^+ \oplus S^-) \cong \wedge \mathfrak{p}^{\mathbb{C}}$$

($\mathfrak{p}$ = orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$). Tensoring both sides of (9.9) with $(S^+ \oplus S^-)$, one finds

(9.10) $V_\mu \otimes (S^+ \oplus S^-) \hookrightarrow V_{\mu-\rho_n} \otimes \wedge \mathfrak{p}^{\mathbb{C}}$.

We recall the isomorphism

(9.11) $V_j^{\pm} \cong \operatorname{Hom}_K(\mathbf{H}_j, V_\mu \otimes S^{\pm})$.

Because of (9.10–9.11), the set (9.7) becomes a subset of

(9.12) $\{j \in \hat{G}_\mu | j \notin \hat{G}_d,\ \mathrm{Hom}_K(\mathbf{H}_j, V_{\mu-\rho_n} \otimes \wedge \mathfrak{p}^{\mathbb{C}}) \neq 0\}$.

It will be enough to prove that this set has zero Plancherel measure, or more specifically, that it is finite.

If M is a $(\mathfrak{g}^{\mathbb{C}}, K)$-module (i.e. a $\mathfrak{g}^{\mathbb{C}}$-module such that the action of $\mathfrak{k}^{\mathbb{C}}$ lifts to the group K), the relative Lie algebra cohomology groups $H^*(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}; M)$ can be computed in terms of the standard complex $C^*(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}; M)$, with

(9.13) $C^p(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}; M) = \mathrm{Hom}_K(\wedge^p \mathfrak{p}^{\mathbb{C}}, M)$.

We apply this remark to the tensor product $M = \mathbf{H}_\infty \otimes F$, where $\mathbf{H}_\infty$ is the space of K-finite vectors in an irreducible, unitary G-module $\mathbf{H}$, and F an irreducible, finite-dimensional G-module. The cochain groups (9.13) are then finite-dimensional and carry a natural inner product. With respect to this inner product, the formal Laplacian turns out to be the difference of the constants by which the Casimir operator Ω acts on $\mathbf{H}_\infty$ and F [7]. In particular,

(9.14) $H^p(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}; \mathbf{H}_\infty \otimes F) \cong \mathrm{Hom}_K(\wedge^p \mathfrak{p}^{\mathbb{C}}, \mathbf{H} \otimes F)$, provided Ω acts on $\mathbf{H}_\infty$ and F by the same constant.

We let $F_{\mu-\rho_n}$ denote the irreducible, finite-dimensional G-module of highest weight $\mu-\rho_n$, and $F^*_{\mu-\rho_n}$ its contragredient. Then $F_{\mu-\rho_n}$ contains $V_{\mu-\rho_n}$ as a K-submodule, and Ω operates on both $F_{\mu-\rho_n}$ and $F^*_{\mu-\rho_n}$ as multiplication by the constant c_μ of (3.18). Hence, for any $j \in \hat{G}_\mu$,

(9.15) $\mathrm{Hom}_K(\mathbf{H}_j, V_{\mu-\rho_n} \otimes \wedge^p \mathfrak{p}^{\mathbb{C}}) \neq 0 \Rightarrow H^p(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}; \mathbf{H}_{j,\infty} \otimes F^*_{\mu-\rho_n}) \neq 0$,

as follows from (9.14).

The category of $(\mathfrak{g}^{\mathbb{C}}, K)$-modules contains enough projectives, so that one can define the derived functors $\mathrm{Ext}^*_{\mathfrak{g}^{\mathbb{C}}}$ of the functor

$M \mapsto \mathrm{Hom}_{\mathfrak{g}^{\mathbb{C}}}(M, N)$.

Arguing purely formally, one obtains isomorphisms

(9.16) $\mathrm{Ext}^p_{\mathfrak{g}^{\mathbb{C}}}(M, N) \cong H^p(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}; \mathrm{Hom}_{\mathbb{C}}(M, N))$,

since both sides agree for $p=0$. The Ext groups classify exact sequences of $(\mathfrak{g}^{\mathbb{C}}, K)$-modules, beginning with N and ending with M, modulo a certain equivalence relation (Yoneda equivalence). As was pointed out by D. Wigner, this implies in particular: if M, N are $(\mathfrak{g}^{\mathbb{C}}, K)$-modules with infinitesimal characters χ_M and χ_N, then

(9.17) $\mathrm{Ext}^p_{\mathfrak{g}^{\mathbb{C}}}(M, N) = 0$ whenever $\chi_M \neq \chi_N$.

Combining (9.15–9.17), we see that the elements of the set (9.12) all have the same infinitesimal character as the G-module $F_{\mu-\rho_n}$. Consequently the set must be finite.

To our knowledge, the preceding argument was originally put together by Zuckerman, although others may have been aware of it independently. The details which we left out can be found in the first two sections of Borel-Wallach [7].

Appendix

In order to keep this paper reasonably self-contained, we shall sketch the proofs of some auxiliary results which were used in our construction, and which have already appeared elsewhere, namely:

a) Parthasarathy's formula (3.17) for the square of the spinor Laplacian [22],
b) the characterization (4.5) of the singularities of the K-character [24],
c) the bound (5.5) for the action of the Casimir operator [24], and
d) Proposition (5.14).

The first three of these are closely related and have fairly simple proofs. Proposition (5.14) is a special case of the main result of [24]; in the situation which we consider, its verification can be simplified quite a bit.

We begin with the proof of (3.17). The Lie algebra of the maximal compact subgroup K has a unique $\mathrm{Ad}K$-invariant complement $\mathfrak{p}$ in $\mathfrak{g}$. Since the K-modules S^+, S^- were introduced as the half-spin representations of the orthogonal group of $\mathfrak{p}$, there exist distinguished K-homomorphisms

(A.1) $\quad \mathfrak{p}^{\mathbb{C}} \otimes S^+ \to S^-, \qquad \mathfrak{p}^{\mathbb{C}} \otimes S^- \to S^+$

which we write as

$$X \otimes s \mapsto c(X)\, s;$$

$c(X) \in \mathrm{Hom}(S^{\pm}, S^{\mp})$ is "Clifford multiplication" by X. The mappings (2.1) are induced by an action of the Clifford algebra on $S^+ \oplus S^-$, which implies

(A.2) $\quad c(X)^2 = -B(X,X), \qquad \text{for } X \in \mathfrak{p}^{\mathbb{C}}$

(B = Killing form of $\mathfrak{g}^{\mathbb{C}}$). As irreducible modules for the orthogonal group of $\mathfrak{p}$ – or more precisely, for the corresponding spin group – S^+ and S^- carry essentially unique Hermitian metrics. When these metrics are suitably normalized,

(A.3) $\quad -c(\bar{X})$ is the adjoint of $c(X)$;

here X denotes the complex conjugate of $X \in \mathfrak{p}^{\mathbb{C}}$, relative to the real form $\mathfrak{p}$. Since B is positive definite on $\mathfrak{p}$, we can choose an orthonormal basis $\{X_i\}$. The action $s^{\pm}(Z)$ of any $Z \in \mathfrak{k}^{\mathbb{C}}$ on $S^{\pm}$ is then given by

(A.4) $\quad s^{\pm}(X) = -\frac{1}{4} \sum_i c([Z, X_i])\, c(X_i).$

In fact, the analogous identity holds for every Z in the Lie algebra of $SO(\mathfrak{p})$ [2].

The twisted spin bundles $\mathscr{V}_\mu \otimes \mathscr{S}^{\pm}$ are associated to the principal bundle

$$K \to G \to G/K$$

by the action of K on $V_\mu \otimes S^{\pm}$. Hence there exist natural G-isomorphisms

(A.5) $\quad C^\infty(G/K, \mathscr{V}_\mu \otimes \mathscr{S}^{\pm}) \cong (C^\infty(G) \otimes V_\mu \otimes S^{\pm})_K;$

$(\dots)_K$ refers to the subspace of K-invariants, with K acting on $C^\infty(G)$ by right translation, and on $V_\mu \otimes S^\pm$ in the obvious manner.

We again let $\{X_i\}$ be an orthonormal basis of $\mathfrak{p}$. Each X_i determines a left-invariant vector field $r(X_i)$, by infinitesimal right translation. In terms of the isomorphisms (A.5), the Dirac operators on $\mathscr{V}_\mu \otimes \mathscr{S}^\pm$ can be expressed as

$$\text{(A.6)}\quad \sum_i r(X_i) \otimes 1 \otimes c(X_i)\colon (C^\infty(G) \otimes V_\mu \otimes S^\pm)_K \to (C^\infty(G) \otimes V_\mu \otimes S^\mp)_K.$$

Indeed, one can check that these operators preserve the K-invariants in $C^\infty(G) \otimes V_\mu \otimes S^\pm$, and that they commute with the action of G. Consequently they define G-invariant first order operators between $\mathscr{V}_\mu \otimes \mathscr{S}^+$ and $\mathscr{V}_\mu \otimes \mathscr{S}^-$. When the fibres of $\mathscr{V}_\mu \otimes \mathscr{S}^\pm$ at the identity coset are identified with $V_\mu \otimes S^\pm$, and the cotangent space with $\mathfrak{p}^{\mathbb{C}*} \cong \mathfrak{p}^{\mathbb{C}}$ (this latter isomorphism comes from the Killing form), the symbols of the operators (A.6) are given by the linear maps

$$\mathfrak{p}^{\mathbb{C}} \otimes V_\mu \otimes S^\pm \to V_\mu \otimes S^\mp, \qquad X \otimes v \otimes s \mapsto v \otimes c(X)\, s,$$

which are also the symbols of the twisted Dirac operators. A translation-invariant first order operator between $\mathscr{V}_\mu \otimes \mathscr{S}^+$ and $\mathscr{V}_\mu \otimes \mathscr{S}^-$ is completely determined by its symbol: since S^+ and S^- have no weights in common [1],

$$\text{(A.7)}\quad \operatorname{Hom}_K(V_\mu \otimes S^+, V_\mu \otimes S^-) = 0,$$

which implies that there exist no non-trivial, translation-invariant bundle maps between $\mathscr{V}_\mu \otimes \mathscr{S}^+$ and $\mathscr{V}_\mu \otimes \mathscr{S}^-$.

The spinor Laplacian on $\mathscr{V}_\mu \otimes \mathscr{S}^\pm$ is the composition of the two Dirac operators on $\mathscr{V}_\mu \otimes \mathscr{S}^\pm$, which are adjoint to each other; it therefore equals

$$\begin{aligned}
&\Big(\sum_i r(X_i) \otimes 1 \otimes c(X_i)\Big)^2 \\
&\quad = \sum_{i,j} r(X_j)\, r(X_i) \otimes 1 \otimes c(X_j)\, c(X_i) \\
&\quad = \tfrac{1}{2} \sum_{i,j} \{ r(X_j)\, r(X_i) \otimes 1 \otimes c(X_j)\, c(X_i) + r(X_i)\, r(X_j) \otimes 1 \otimes c(X_i)\, c(X_j) \} \\
&\quad = \tfrac{1}{2} \sum_{i,j} r(X_j)\, r(X_i) \otimes 1 \otimes (c(X_j)\, c(X_i) + c(X_i)\, c(X_j)) \\
&\qquad + \tfrac{1}{2} \sum_{i,j} r([X_i, X_j]) \otimes 1 \otimes c(X_i)\, c(X_j) \\
&\quad = -\sum_i r(X_i)^2 \otimes 1 \otimes 1 + \tfrac{1}{2} \sum_{i,j} r([X_i, X_j]) \otimes 1 \otimes c(X_i)\, c(X_j).
\end{aligned}$$

In the last step, we have used the identity

$$c(X_i)\, c(X_j) + c(X_j)\, c(X_i) = -2\delta_{ij},$$

---

[1] According to the discussion above (4.13), if this were not the case, there would exist an odd number of noncompact roots which add up to zero. When a noncompact root, positive or negative, is expressed as an integral linear combination of simple roots, the sum of the coefficients of the noncompact simple roots is an odd integer. An odd number of noncompact roots can therefore never add up to zero

which follows from (A.2). We now choose an orthonormal basis $\{Z_l\}$ for $\mathfrak{k}^{\mathbb{C}}$. The Casimir operators Ω of $\mathfrak{g}^{\mathbb{C}}$ and Ω_K of $\mathfrak{k}^{\mathbb{C}}$ can then be expressed as

$$\Omega=\sum_i X_i^2+\sum_l Z_l^2, \qquad \Omega_K=\sum_l Z_l^2.$$

Hence

$$\text{(A.8)}\quad (\sum_i r(X_i)\otimes 1\otimes c(X_i))^2$$
$$= -r(\Omega)\otimes 1\otimes 1+r(\Omega_K)\otimes 1\otimes 1+\tfrac{1}{2}\sum_{i,j} r([X_i,X_j])\otimes 1\otimes c(X_i)\,c(X_j);$$

as in the past, r refers to the right infinitesimal action of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ on $C^\infty(G)$.

Since $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition, $[\mathfrak{p},\mathfrak{p}]$ lies in $\mathfrak{k}$, and $[\mathfrak{k},\mathfrak{p}]$ in $\mathfrak{p}$. Thus we find:

$$\tfrac{1}{2}\sum_{i,j} r([X_i,X_j])\otimes 1\otimes c(X_i)\,c(X_j)$$
$$=\tfrac{1}{2}\sum_{i,j,l} B([X_i,X_j],Z_l)\,r(Z_l)\otimes 1\otimes c(X_i)\,c(X_j)$$
$(\{Z_l\}$ is an orthonormal basis of $\mathfrak{k}^{\mathbb{C}})$
$$=-\tfrac{1}{2}\sum_{i,j,l} B([Z_l,X_j],X_i)\,r(Z_l)\otimes 1\otimes c(X_i)\,c(X_j)$$
(B is $\operatorname{Ad} G$-invariant)
$$=-\tfrac{1}{2}\sum_{j,l} r(Z_l)\otimes 1\otimes c([Z_l,X_j])\,c(X_j)$$
$(\{X_i\}$ is an orthonormal basis of $\mathfrak{p}^{\mathbb{C}})$
$$=2\sum_l r(Z_l)\otimes 1\otimes s^{\pm}(Z_l)$$
(because of (2.4)).

The various operators act on the K-invariants in $C^\infty(G)\otimes V_\mu\otimes S^{\pm}$. Hence, if τ_μ denotes the representation of $\mathfrak{k}^{\mathbb{C}}$ on V_μ,

$$\sum_l r(Z_l)\otimes 1\otimes s^{\pm}(Z_l)=-\sum_l 1\otimes\tau_\mu(Z_l)\otimes s^{\pm}(Z_l)-1\otimes 1\otimes s^{\pm}(\Omega_K).$$

Similarly,

$$r(\Omega_K)\otimes 1\otimes 1=1\otimes(\tau_\mu\otimes s^{\pm})(\Omega_K)$$
$$=1\otimes\tau_\mu(\Omega_K)\otimes 1+1\otimes 1\otimes s^{\pm}(\Omega_K)+2\sum_l 1\otimes\tau_\mu(Z_l)\otimes s^{\pm}(Z_l).$$

All this allows us to rewrite the identity (A.8) as follows:

$$\text{(A.9)}\quad (\sum_i r(X_i)\otimes 1\otimes c(X_i))^2$$
$$=-r(\Omega)\otimes 1\otimes 1+1\otimes\tau_\mu(\Omega_K)\otimes 1-1\otimes 1\otimes s^{\pm}(\Omega_K).$$

On the irreducible $\mathfrak{k}^{\mathbb{C}}$-module with highest weight μ, Ω_K acts as multiplication by

(A.10) $\tau_\mu(\Omega_K)=(\mu+\rho_c,\mu+\rho_c)-(\rho_c,\rho_c)$.

In order to complete the proof of (3.17), we must identify $s^\pm(\Omega_K)$.

We recall the character formula (4.13). Applying Weyl's denominator formula for G, one finds

$$\begin{aligned}(\text{A.11})\quad &(\operatorname{trace} s^+ - \operatorname{trace} s^-)|_H \\ &= \prod_{\beta\in\Phi^n\cap\Psi}(e^{\beta/2}-e^{-\beta/2}) \\ &=\Big(\prod_{\alpha\in\Phi^c\cap\Psi}(e^{\alpha/2}-e^{-\alpha/2})\Big)^{-1}\prod_{\alpha\in\Psi}(e^{\alpha/2}-e^{-\alpha/2}) \\ &=\Big(\prod_{\alpha\in\Phi^c\cap\Psi}(e^{\alpha/2}-e^{-\alpha/2})\Big)^{-1}\sum_{w\in W}\varepsilon(w)\,e^{w\rho}.\end{aligned}$$

The K-modules S^+,S^- have no irreducible constituent in common; this is a special case of (A.7), with $\mu=0$. Since there can be no cancellation in (A.11), every irreducible summand of $S^+\oplus S^-$ must have a highest weight of the form $w\rho-\rho_c$, for some $w\in W_{\mathbb{C}}$. The Casimir operator Ω_K therefore acts as multiplication by

$$s^\pm(\Omega_K)=(w\rho,w\rho)-(\rho_c,\rho_c)=(\rho,\rho)-(\rho_c,\rho_c).$$

Combining this with (A.9) and (A.10), we obtain Parthasarathy's formula (3.17).

For the proof of (4.5) and (5.5), we consider an irreducible unitary representation π of G, on a Hilbert space $\mathbf{H}$. The various K-invariant, K-irreducible subspaces span a dense linear subspace $\mathbf{H}_\infty\subset\mathbf{H}$, which consists entirely of analytic vectors. Thus $\mathfrak{g}^{\mathbb{C}}$, and hence $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$, act on $\mathbf{H}_\infty$ by differentiation. We shall let π denote also this infinitesimal action. It is irreducible, and determines π up to unitary equivalence. Again we choose an orthonormal basis $\{X_i\}$ of $\mathfrak{p}^{\mathbb{C}}$. The two K-invariant linear mappings

$$\begin{aligned}(\text{A.12})\quad &d_\pm:\ \mathbf{H}_\infty\otimes S^\pm\to\mathbf{H}_\infty\otimes S^\mp,\\ &d_\pm:\ v\otimes s\mapsto\sum_i\pi(X_i)\,v\otimes c(X_i)\,s,\end{aligned}$$

become adjoint to each other, when $\mathbf{H}_\infty$ is viewed as a pre-Hilbert space. This follows from (A.3) and is formally analogous to the self-adjointness of the Dirac operators. Parthasarathy's formula has a direct analogue in the present context:

$$(\text{A.13})\quad d_-d_+=d_+d_-=(\pi\otimes s^\pm)(\Omega_K)-\pi(\Omega)\otimes 1-(\rho,\rho)\,1+(\rho_c,\rho_c)\,1;$$

its proof is virtually identical to that of (3.17).

As a function on the dual $\hat{K}$ of K, Ω_K tends to $+\infty$ outside of finite subsets. Since each irreducible K-module occurs only finitely often in $\mathbf{H}_\infty$, and hence in $\mathbf{H}_\infty\otimes S^\pm$, the identity (A.13) forces d_+ to have finite-dimensional kernel and cokernel. For purely formal reasons

$$\mathbf{H}_\infty\otimes S^+-\mathbf{H}_\infty\otimes S^-=\ker d_+-\operatorname{coker} d_+,$$

as virtual K-modules. Thus, if τ denotes the K-character of π, and $\sigma^\pm$ the character of $S^\pm$,

$$\tau(\sigma^+ - \sigma^-) = \operatorname{char}(\mathbf{H}_\infty \otimes S^+ - \mathbf{H}_\infty \otimes S^-) = \operatorname{char}(\ker d_+ - \operatorname{coker} d_+).$$

In particular, $\tau(\sigma^+ - \sigma^-)$ is a finite integral linear combination of characters of irreducible K-modules, as asserted by (4.5). We should remark that the preceding argument does not really use the unitary structure. Thus (4.5) holds more generally for irreducible, admissible representations on Banach spaces.

If V and W are irreducible K-modules, with V having highest weight ν and W lowest weight η, then $\nu+\eta$ occurs as the highest weight of an irreducible constituent of $V \otimes W$, provided

$$(\nu+\eta, \alpha) \geqq 0, \qquad \text{for } \alpha \in \Phi^c \cap \Psi;$$

this follows, for example, from Weyl's character formula. With respect to any ordering compatible with Ψ, $-\rho_n$ is lowest among weights of $S^+ \oplus S^-$, and hence is the lowest weight of an irreducible summand of $S^+ \oplus S^-$. The conditions (5.4) therefore guarantee that $\mathbf{H}_\infty \otimes (S^+ \oplus S^-)$ contains the irreducible K-module of highest weight $\nu - \rho_n$ at least once. On this K-module, Ω_K operates as multiplication by

$$(\nu - \rho_n + \rho_c, \nu - \rho_n + \rho_c) - (\rho_c, \rho_c).$$

If one applies the positive semi-definite operator (A.13) to the submodule in question, one is led to the estimate (5.5).

Let us turn to the proof of Proposition (5.14)! We shall show that the two conditions a) and b) determine the infinitesimal representation of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ on $\mathbf{H}_\infty$. The unitary structure will again be irrelevant, and hence (5.14) applies more generally to irreducible, admissible representations on Banach spaces: up to infinitesimal equivalence, there exists at most one such representation which has the two properties a) and b).

The irreducible K-module of highest weight $\mu + \rho_n$, $V_{\mu+\rho_n}$, is a left $\mathfrak{U}(\mathfrak{k}^{\mathbb{C}})$-module, and $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ is a right $\mathfrak{U}(\mathfrak{k}^{\mathbb{C}})$-module, via right multiplication. Thus one can form the tensor product

$$\text{(A.14)} \qquad M = \mathfrak{U}(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{k}^{\mathbb{C}})} V_{\mu+\rho_n},$$

on which $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ operates by left multiplication. The $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-module M has the universal mapping property

$$\text{(A.15)} \qquad \operatorname{Hom}_{\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})}(M, L) \cong \operatorname{Hom}_{\mathfrak{U}(\mathfrak{k}^{\mathbb{C}})}(V_{\mu+\rho_n}, L),$$

for any $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-module L. From $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$, M inherits a filtration

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_n \subset M_{n+1} \subset \cdots \subset M,$$

with

$$M_0 = 1 \otimes V_{\mu+\rho_n}, \qquad M_{n+1} = M_n + \mathfrak{g}^{\mathbb{C}} M_n;$$

it is $\mathfrak{k}^{\mathbb{C}}$-stable, because the adjoint action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ preserves the filtration of the latter. As a consequence of the Birkhoff-Witt theorem, there exist natural $\mathfrak{k}^{\mathbb{C}}$-isomorphisms

(A.16) $M_n/M_{n-1} \cong \mathfrak{p}^{\mathbb{C}(n)} \otimes V_{\mu+\rho_n}$

($\mathfrak{p}^{\mathbb{C}(n)} = n$-th symmetric power of $\mathfrak{p}^{\mathbb{C}}$). In particular,

(A.17) $M_1 \cong (\mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n}) \oplus V_{\mu+\rho_n}$;

this splitting is canonical: the irreducible $\mathfrak{k}^{\mathbb{C}}$-module $V_{\mu+\rho_n}$ does not embed into $\mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n}$.

Every irreducible component of $\mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n}$ has a highest weight of the form $\mu+\rho_n+\beta$, $\beta \in \Phi^n$. Lumping together these components for which β is, respectively, positive or negative, one obtains submodules U_+ and U_-, which decompose $\mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n}$:

(A.18) $\mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n} = U_+ \oplus U_-$.

The isomorphism (A.17) provides an inclusion $U_- \hookrightarrow M_1$. We set

$N = \mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-submodule of M generated by U_-,

$Q = M/N$.

The $\mathfrak{k}^{\mathbb{C}}$-invariant filtration of M induces a filtration $\{Q_n\}$ of the quotient Q.

We now suppose that the representation π has the two properties a) and b) mentioned in Proposition (5.14). The former, in conjunction with the mapping property (A.15), guarantees the existence of a non-zero $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-homomorphism $M \to \mathbf{H}_\infty$. Because of the latter, any such homomorphism must annihilate N. Thus one can produce a non-trivial $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-map $Q \to \mathbf{H}_\infty$, which is necessarily surjective, since $\mathbf{H}_\infty$ is known to be irreducible. Consequently,

(A.19) $\mathbf{H}_\infty$ is isomorphic, as $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-module, to an irreducible quotient of Q,

and this reduces the proposition to:

(A.20) **Lemma.** The $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-module Q has a unique irreducible quotient. Under the action of $\mathfrak{k}^{\mathbb{C}}$, Q breaks up into a direct sum of irreducible $\mathfrak{k}^{\mathbb{C}}$-modules, each occuring with finite multiplicity. Every irreducible constituent has a highest weight which can be expressed as $\mu+\rho_n+\beta_1+\dots+\beta_m$, with $\beta_1, \dots, \beta_m \in \Phi^n \cap \Psi$ and $m \geqq 0$; the highest weight $\mu+\rho_n$ appears at most once.

To begin with, we shall argue that the last two assertions imply the first. Indeed, since M_0 generates the $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$-module M, Q_0 must generate Q. As a quotient of $M_0 \cong V_{\mu+\rho_n}$, Q_0 either vanishes, in which case $Q=0$, or must itself be isomorphic to $V_{\mu+\rho_n}$, and hence irreducible under the action of $\mathfrak{k}^{\mathbb{C}}$. In particular, no proper submodule of Q can meet Q_0. Since the irreducible $\mathfrak{k}^{\mathbb{C}}$-module of highest weight $\mu+\rho_n$ is known to occur only once in Q, it cannot lie in any proper submodule, nor in the linear span of any number of proper submodules. Hence Q has a unique maximal proper submodule, or equivalently, a unique irreducible quotient.

The adjoint action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ lifts to the group K, which operates by conjugation. Since K operates on $V_{\mu+\rho_n}$ as well, the action of $\mathfrak{k}^{\mathbb{C}}$ on M, and hence on Q, also lifts to K. The finite-dimensional $\mathfrak{k}^{\mathbb{C}}$-submodules Q_n exhaust Q. It follows that Q breaks up, under $\mathfrak{k}^{\mathbb{C}}$, into a direct sum of irreducibles, although conceivably with infinite multiplicities. In particular,

(A.21) $Q \cong \bigoplus_{n=0}^{\infty} Q_n/Q_{n-1}$, as $\mathfrak{k}^{\mathbb{C}}$-module.

The positive roots lie in a closed cone, which is properly contained in a half-space. Each weight can therefore be expressed as a sum of positive roots in at most finitely many ways, and a non-empty sum of positive roots can never equal the zero weight. Hence the lemma becomes a consequence of the following statement,

(A.22) each irreducible constituent of the $\mathfrak{k}^{\mathbb{C}}$-module Q_n/Q_{n-1} has a highest weight of the form

$$\mu+\rho_n+\beta_1+\dots+\beta_n, \quad \text{with} \quad \beta_1, \dots, \beta_n \in \Phi^n \cap \Psi,$$

which will be verified next.

The inclusion of U_- in $\mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n}$, tensored with the identity on $\mathfrak{p}^{\mathbb{C}(n-1)}$, and followed by multiplication, determines a $\mathfrak{k}^{\mathbb{C}}$-homomorphism

(A.23) $h\colon \mathfrak{p}^{\mathbb{C}(n-1)} \otimes U_- \to \mathfrak{p}^{\mathbb{C}(n)} \otimes V_{\mu+\rho_n}$.

Under the isomorphism (A.16), the image of h will certainly go into $N \cap M_n/N \cap M_{n-1}$, i.e., into the kernel of the projection $M_n/M_{n-1} \to Q_n/Q_{n-1}$. Thus:

(A.24) Q_n/Q_{n-1} is isomorphic, as $\mathfrak{k}^{\mathbb{C}}$-module, to a quotient of the cokernel of h.

The homomorphism h is induced, in a certain sense, from a homomorphism between modules of a Borel subalgebra of $\mathfrak{k}^{\mathbb{C}}$. To describe the induction process, we use a sequence of functors, which were introduced in [24]. We shall briefly summarize their definition and main properties.

The root spaces in $\mathfrak{g}^{\mathbb{C}}$ corresponding to all negative, compact roots span a maximal nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{k}^{\mathbb{C}}$, which is normalized by $\mathfrak{h}^{\mathbb{C}}$. Hence

$$\mathfrak{b} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{n}$$

becomes a Borel subalgebra of $\mathfrak{k}^{\mathbb{C}}$. For any $\mathfrak{b}$-module E, the cohomology groups $H^p(\mathfrak{n}, E)$ have natural $\mathfrak{h}^{\mathbb{C}}$-module structures, since $\mathfrak{h}^{\mathbb{C}}$ acts on both E and $\mathfrak{n}$. The subspace of $\mathfrak{h}^{\mathbb{C}}$-invariants will be denoted by $H^p(\mathfrak{n}, E)_{\mathfrak{h}^{\mathbb{C}}}$. In the following, we only consider finite-dimensional $\mathfrak{b}$-modules E, such that the action of $\mathfrak{h}^{\mathbb{C}}$ on E lifts to the torus H. In this situation, E and its $\mathfrak{n}$-cohomology groups become completely reducible, as $\mathfrak{h}^{\mathbb{C}}$-modules. In particular, any short exact sequence of such $\mathfrak{b}$-modules

$$0 \to E' \to E \to E'' \to 0$$

gives rise to a long exact sequence

$$\to H^p(\mathfrak{n}, E')_{\mathfrak{h}^{\mathbb{C}}} \to H^p(\mathfrak{n}, E)_{\mathfrak{h}^{\mathbb{C}}} \to H^p(\mathfrak{n}, E'')_{\mathfrak{h}^{\mathbb{C}}} \to H^{p+1}(\mathfrak{n}, E')_{\mathfrak{h}^{\mathbb{C}}} \to .$$

For each $i \in \hat{K}$, we select a $\mathfrak{k}^{\mathbb{C}}$-module W_i which represents the isomorphism class i, and we define

(A.25) $\quad I^p(E) = \bigoplus_{i \in \hat{K}} W_i \otimes H^p(\mathfrak{n}, W_i^* \otimes E)_{\mathfrak{h}^{\mathbb{C}}}$;

here W_i^*, the $\mathfrak{k}^{\mathbb{C}}$-module dual to W_i, is regarded as $\mathfrak{b}$-module by restriction. With $\mathfrak{k}^{\mathbb{C}}$ acting trivially on the right factors $H^p(\mathfrak{n}, W_i^* \otimes E)_{\mathfrak{h}^{\mathbb{C}}}$, $I^p(E)$ becomes a completely reducible $\mathfrak{k}^{\mathbb{C}}$-module. The definition (A.25) is functorial in E. Hence:

(A.26) $\quad I^p$, $0 \leqq p \leqq \dim \mathfrak{n}$, is a sequence of functors from the category of finite-dimensional $\mathfrak{b}$-modules, for which the action of $\mathfrak{h}^{\mathbb{C}}$ lifts to H, to the category of completely reducible $\mathfrak{k}^{\mathbb{C}}$-modules.

Moreover,

(A.27) $\quad$ every short exact sequence $0 \to E' \to E \to E'' \to 0$ determines a long exact sequence

$$0 \to I^0(E') \to I^0(E) \to E^0(E'') \to I^1(E') \to \cdots;$$

this follows from the analogous property of the functors $E \to H^p(\mathfrak{n}, E)_{\mathfrak{h}^{\mathbb{C}}}$. Loosely speaking, $I^0(E)$ is obtained by inducing E holomorphically from the complex Lie group with Lie algebra $\mathfrak{b}$ to the complexification of K. This process is left exact, and the functors $I^1, I^2, \ldots$ measure the obstruction to its exactness on the right.

Three properties of the functors I^p will be crucial. The first of these is directly implied by Kostant's Lie algebra version of the Borel-Weil-Bott theorem [19]. For any weight $\mu \in \Lambda$, L_μ shall denote the one-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}^{\mathbb{C}}$ acts according to the linear functional μ. Then

(A.28) $\quad I^0(L_\mu)$ is irreducible, with highest weight μ, provided μ is dominant with respect to $\Phi^c \cap \Psi$; in all remaining cases, $I^0(L_\mu) = 0$; if $(\mu + \rho_c, \alpha) \geqq 0$ for every $\alpha \in \Phi^c \cap \Psi$, $I^p(L_\mu)$ vanishes for $p > 0$.

Since $\mathfrak{b}$ is solvable, every finite-dimensional $\mathfrak{b}$-module E has a composition series

(A.29a) $\quad 0 \subset E_0 \subset E_1 \subset \cdots \subset E_m = E$,

with one-dimensional quotients. In this situation, there exists an injective $\mathfrak{k}^{\mathbb{C}}$-homomorphism

(A.29b) $\quad I^p(E) \hookrightarrow \bigoplus_{l=0}^{m} I^p(E_l/E_{l-1})$,

which need not be functorial, however. Indeed, for each l,

$$I^p(E_{l-1}) \to I^p(E_l) \to I^p(E_l/E_{l-1})$$

is an exact sequence of completely reducible $\mathfrak{k}^{\mathbb{C}}$-modules, so that the assertion can be verified inductively. Finally,

(A.30) $\quad$ for each finite-dimensional K-module V, there exist $\mathfrak{k}^{\mathbb{C}}$-isomorphisms $I^p(V \otimes E) \cong V \otimes I^p(E)$, which are functorial in both V and E.

To see this, one should observe that

$$W_i^* \otimes V \otimes E \cong \bigoplus_{j \in \hat{K}} (W_i^* \otimes V \otimes W_j)_K \otimes (W_j^* \otimes E),$$

where $(\ldots)_K$ denotes the subspace of K-invariants, equipped with the trivial $\mathfrak{b}$-action. Hence

$$I^p(V \otimes E) \cong \bigoplus_{i,\, j \in \hat{K}} W_i \otimes (W_i^* \otimes V \otimes W_j)_K \otimes H^p(\mathfrak{n}, W_j^* \otimes E)_{\mathfrak{h}^{\mathbb{C}}}$$

$$\cong \bigoplus_{i \in \hat{K}} W_i \otimes (W_i^* \otimes V \otimes I^p(E))_K \cong V \otimes I^p(E),$$

which proves (A.30).

As a particular consequence of (A.28) and (A.30),

(A.31) $\quad I^0(L_{\mu+\rho_n}) \cong V_{\mu+\rho_n}, \qquad I^0(\mathfrak{p}^{\mathbb{C}} \otimes L_{\mu+\rho_n}) \cong \mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n};$

these isomorphisms will be regarded as identities. The root spaces indexed by the various negative, noncompact roots span a $\mathfrak{b}$-submodule $\mathfrak{p}_-$ of $\mathfrak{p}^{\mathbb{C}}$. In any composition series of $\mathfrak{p}_-$, precisely the $\mathfrak{b}$-modules $L_{-\beta}$, $\beta \in \Phi^n \cap \Psi$, occur as the one-dimensional quotients; similarly, the $\mathfrak{b}$-modules L_β, $\beta \in \Phi^n \cap \Psi$, decompose $\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_-$. Appealing to (A.28) and (A.29), one finds that every irreducible summand of $I^0(\mathfrak{p}_- \otimes L_{\mu+\rho_n})$ has a highest weight of the form $\mu+\rho_n-\beta$, with β positive and noncompact. None of these highest weights occur in the $\mathfrak{k}^{\mathbb{C}}$-module U_+ of (A.18), so that

$$\operatorname{Hom}_{\mathfrak{k}^{\mathbb{C}}}(I^0(\mathfrak{p}_- \otimes L_{\mu+\rho_n}), U_+) = 0.$$

For completely analogous reasons

$$\operatorname{Hom}_{\mathfrak{k}^{\mathbb{C}}}(U_-, I^0(\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_- \otimes L_{\mu+\rho_n})) = 0.$$

Since

$$0 \to I^0(\mathfrak{p}_- \otimes L_{\mu+\rho_n}) \to I^0(\mathfrak{p}^{\mathbb{C}} \otimes L_{\mu+\rho_n}) \to I^0(\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_- \otimes L_{\mu+\rho_n})$$

is exact, the subspace $U_- \subset \mathfrak{p}^{\mathbb{C}} \otimes V_{\mu+\rho_n} = I^0(\mathfrak{p}^{\mathbb{C}} \otimes L_{\mu+\rho_n})$ must coincide with the image of $I^0(\mathfrak{p}_- \otimes L_{\mu+\rho_n})$. We conclude: under the identifications

$$\mathfrak{p}^{\mathbb{C}(n-1)} \otimes U_- \cong \mathfrak{p}^{\mathbb{C}(n-1)} \otimes I^0(\mathfrak{p}_- \otimes L_{\mu+\rho_n}) \cong I^0(\mathfrak{p}^{\mathbb{C}(n-1)} \otimes \mathfrak{p}_- \otimes L_{\mu+\rho_n}),$$

$$\mathfrak{p}^{\mathbb{C}(n)} \otimes V_{\mu+\rho_n} \cong I^0(\mathfrak{p}^{\mathbb{C}(n)} \otimes L_{\mu+\rho_n}),$$

the $\mathfrak{k}^{\mathbb{C}}$-homomorphism h of (A.23) corresponds to the mapping

(A.32) $\quad I^0(\mathfrak{p}^{\mathbb{C}(n-1)} \otimes \mathfrak{p}_- \otimes L_{\mu+\rho_n}) \to I^0(\mathfrak{p}^{\mathbb{C}(n)} \otimes L_{\mu+\rho_n}),$

which arises from the inclusion $\mathfrak{p}_- \hookrightarrow \mathfrak{p}^{\mathbb{C}}$, followed by multiplication $\mathfrak{p}^{\mathbb{C}(n-1)} \otimes \mathfrak{p}^{\mathbb{C}} \to \mathfrak{p}^{\mathbb{C}(n)}$. We must identify the cokernel of this homomorphism.

The decomposition $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_- \oplus \bar{\mathfrak{p}}_-$ ($\bar{\mathfrak{p}}_-$ = complex conjugate of $\mathfrak{p}_-$) determines a complex structure on the real vector space $\mathfrak{p}$. When the polynomial version of the Dolbeaut lemma is dualized, one obtains an exact sequence

$$0 \to \mathfrak{p}^{\mathbb{C}(n-q)} \otimes \wedge^q \mathfrak{p}_- \to \mathfrak{p}^{\mathbb{C}(n-q+1)} \otimes \wedge^{q-1} \mathfrak{p}_- \to \cdots$$
$$\cdots \to \mathfrak{p}^{\mathbb{C}(n-1)} \otimes \mathfrak{p}_- \to \mathfrak{p}^{\mathbb{C}(n)} \to (\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_-)^{(n)} \to 0,$$

in which all arrows are $\mathfrak{b}$-homomorphisms. It remains exact when it is tensored with $L_{\mu+\rho_n}$:

(A.33) $$0 \to \mathfrak{p}^{\mathbb{C}(n-q)} \otimes \wedge^q \mathfrak{p}_- \otimes L_{\mu+\rho_n} \to \mathfrak{p}^{\mathbb{C}(n-q+1)} \otimes \wedge^{q-1} \mathfrak{p}_- \otimes L_{\mu+\rho_n} \to \cdots$$
$$\cdots \to \mathfrak{p}^{\mathbb{C}(n-1)} \otimes \mathfrak{p} \otimes L_{\mu+\rho_n} \to \mathfrak{p}^{\mathbb{C}(n)} \otimes L_{\mu+\rho_n} \to (\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_-)^{(n)} \otimes L_{\mu+\rho_n} \to 0.$$

The one-dimensional quotients in a composition series of the $\mathfrak{b}$-module $\wedge^s \mathfrak{p}_- \otimes L_{\mu+\rho_n}$ all belong to weights $\mu+\rho_n-B$, where B stands for a sum of s distinct positive, noncompact roots. Hence the statements (A.28–A.30), coupled with the hypothesis of Proposition (5.14), guarantee that

(A.34) $$I^p(\mathfrak{p}^{\mathbb{C}(n-s)} \otimes \wedge^s \mathfrak{p}_- \otimes L_{\mu+\rho_n}) \cong \mathfrak{p}^{\mathbb{C}(n-s)} \otimes I^p(\wedge^s \mathfrak{p}_- \otimes L_{\mu+\rho_n}) = 0,$$
$$\text{for } 0 \leqq s \leqq q,\ p \geqq 1.$$

Any exact $\mathfrak{b}$-module sequence $0 \to E_n \to E_{n-1} \to \cdots \to E_0 \to E \to 0$, which has the property that $I^p(E_s)=0$ if $0 \leqq s \leqq n$ and $p \geqq 1$, is transformed into an exact sequence by the functor I^0; this is purely formal, and can be checked by induction. In particular,

$$I^0(\mathfrak{p}^{\mathbb{C}(n-1)} \otimes \mathfrak{p}_- \otimes L_{\mu+\rho_n}) \to I^0(\mathfrak{p}^{\mathbb{C}(n)} \otimes L_{\mu+\rho_n}) \to I^0((\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_-)^{(n)} \otimes L_{\mu+\rho_n}) \to 0$$

is exact.

To complete the proof of Proposition (5.14), we must verify the statement (A.22). According to (A.24), Q_n/Q_{n-1} is isomorphic to a quotient of the cokernel of h, hence to a quotient of the cokernel of the homomorphism (A.32), hence finally to a quotient of $I^0((\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_-)^{(n)} \otimes L_{\mu+\rho_n})$. The $\mathfrak{b}$-module $(\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_-)^{(n)}$ has a composition series with quotients $L_{\beta_1+\cdots+\beta_n}$, $\beta_1, \ldots, \beta_n \in \Phi^n \cap \Psi$. Thus (A.28) and (A.29) allow us to identify the highest weights of the potential irreducible constituents of $I^0((\mathfrak{p}^{\mathbb{C}}/\mathfrak{p}_-)^{(n)} \otimes L_{\mu+\rho_n})$: they all can be expressed as $\mu+\rho_n+\beta_1+\cdots+\beta_n$, with $\beta_i \in \Phi^n \cap \Psi$. This proves (A.22), and along with it Proposition (5.14).

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