DUAL PAIR CORRESPONDENCES FOR NON-LINEAR COVERS OF ORTHOGONAL GROUPS

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Abstract. In this paper we study compact dual pair correspondences arising from smallest representations of non-linear covers of odd orthogonal groups. We identify representations appearing in these correspondences with subquotients of cohomologically induced representations.

1. Introduction

Let p be an odd positive integer and let q be an even positive integer. Let $\mathrm{SO}^0(p,q)$ be the identity component of the Lie group $\mathrm{SO}(p,q)$ and let G be the central extension of $\mathrm{SO}^0(p,q)$ with a maximal compact subgroup

$$K^0=\begin{cases}\mathrm{Spin}(p)\times\mathrm{SO}(q) \text{ if } p<q\\ \mathrm{SO}(p)\times\mathrm{Spin}(q) \text{ if } q<p.\end{cases}$$

The group G is not a linear group. In [LS], we investigated the smallest representations of G that do not factor through the linear quotient $\mathrm{SO}^0(p,q)$. (Such representations are called genuine.) We described the corresponding Harish-Chandra modules: one such module V if $p<q$ and two modules V^+ and V^- if $p>q$. These representations are interesting for a variety of reasons. For example, if G is split then V (in the case $p+1=q$) or V^+ and V^- (in the case $p-1=q$) lift to a trivial representation (of an appropriate algebraic group) via the local Shimura correspondence [ABPTV].

Let $\mathfrak{g}$ be the complexified Lie algebra of G. (Lie algebras in this paper are complex unless specified otherwise.) Let W be the Harish-Chandra module of one of the smallest representations above. We showed in [LS] that W is a $(\mathfrak{g},K)$-module where $K\supseteq K^0$ is obtained by replacing the SO-factor of K^0 be the corresponding full orthogonal group. This extension is important for investigation of dual pair correspondences arising from W. More precisely, let $K_2=\mathrm{O}(s)$. Consider a standard embedding of K_2 into the O-factor of K. Note that, by Witt's lemma, this embedding is unique up to a conjugation. Let $\mathfrak{g}_1$ be the centralizer of K_2 in $\mathfrak{g}$. Then

$$\mathfrak{g}_1=\begin{cases}\mathfrak{so}(p,r),\ r=q-s, \text{ if } p<q\\ \mathfrak{so}(r,q),\ r=p-s, \text{ if } p>q.\end{cases}$$

1991 *Mathematics Subject Classification.* 22E46, 22E47.
[0]October 31, 2007

Let G_1 be a connected subgroup of G corresponding to the Lie algebra $\mathfrak{g}_1$ and let $K_1^0 = G_1 \cap K^0$. Then W, when restricted to $\mathfrak{g}_1 \times K_2$, decomposes discretely

$$W = \sum_\tau \Theta(\tau) \otimes \tau$$

where the sum is taken over all irreducible finite dimensional representations of K_2, and $\Theta(\tau)$ is naturally a $(\mathfrak{g}_1, K_1^0)$-module. In [LS], we obtained some partial results about $\Theta(\tau)$, such as irreducibility of $\Theta(\tau)$, which were necessary to established a correspondence of infinitesimal characters.

Our objective in this paper is to give a more thorough investigation of the correspondence. Let $m = \frac{p-1}{2}$ and $m' = \frac{q}{2}$. Consider a θ-stable maximal parabolic subgroup $\mathfrak{q}_1 = \mathfrak{l}_1 + \mathfrak{n}_1$ in $\mathfrak{g}_1$ whose Levi component corresponds to a subgroup

$$L_1 = \begin{cases} \widetilde{\mathrm{U}}(m) \times \mathrm{SO}^0(1,r) \text{ if } p < q \\ \mathrm{SO}(r,0) \times \widetilde{\mathrm{U}}(m') \text{ if } p > q \end{cases}$$

in G_1. We identify $\Theta(\tau)$ with subquotients of modules with are cohomologically induced from irreducible representations of L_1 which are trivial on the SO-factor and genuine on the U-factor. In particular this implies that these cohomologically induced subquotients are unitarizable and we have a detailed information about their K_1-types, since the types of $\Theta(\tau)$ could be computed by the usual branching rules of orthogonal groups.

One can consider representations cohomologically induced from representations of L_1 which are trivial on the SO-factor and not genuine on the U-factor. It is interesting to note that these representations (of the linear quotient of G_1) appear as double lifts from compact orthogonal groups in the Howe correspondence [Lo] and [NZ].

In Section 6 we highlight a special case. Assume that $r > q$ is an odd integer. Knapp [Kn] introduced a family π'_s of $(\mathfrak{so}(r,q), \mathrm{SO}(r) \times \mathrm{Spin}(q))$-modules $s = 0, 1, 2 \ldots$. The module π'_s is a Harish-Chandra module of a genuine representation of G_1 if and only if s is even. If s is even then $p = r + s$ is odd. We show that π'_s is isomorphic to our $\Theta(0)$ where 0 denotes the trivial representation of $\mathrm{O}(s)$. These results, therefore, complement the results of Paul and Trapa [PT]. It is shown there that π'_s for s odd appear as double lifts of trivial representations of compact groups in the Howe correspondence [Lo] [NZ].

The study of our compact dual pairs unfortunately requires use of disconnected groups for technical reasons. In order to avoid the complications of treating covers of disconnected Lie groups, we will work exclusively with Harish-Chandra modules in this paper. The main results and proofs for V and $V^\pm$ are similar but each requires slightly different set of notations. Hence we will divide the paper into two parts. The first part consists of Sections 2 to 4 where we concentrate on one family of dual pairs for the smallest representation V. The main purpose is to explain the main ideas quickly and clearly without being buried by the notations. In the Section 5, we will state but without proofs the corresponding results for $V^\pm$.

Acknowledgment. The first author would like to thank the hospitality of the mathematics department at University of Utah while the initial idea for this paper was first conceived. Likewise, the second author would like to thank the hospitality of the mathematics department at National University of Singapore during 2007 when this paper was completed. We

would also like to thank Peter Trapa for some very insightful discussions. The first author was supported by an NUS grant R-146-000-085-112. The second author was supported by an NSF grant DMS 0551846.

2. The smallest representation

In Sections 2 to 4, we will assume that $p < q$. Let V be the Harish-Chandra module of the smallest representation of G as in [LS]. The module V is unitarizable and it extends to an irreducible $(\mathfrak{g}, K)$-module for $K = \mathrm{Spin}(p) \times \mathrm{O}(q)$. We need some notation in order to describe the K-types of V.

Notation. The following convention will be used throughout the paper. Given a multiple of numbers $\lambda = (\lambda_1, \dots \lambda_r, 0, \dots, 0)$ then, by adding or removing 0's at the tail, λ can be considered an s-tuple for every $s \geq r$. Let $\mathbf{1}_k := (1, \dots, 1)$ and $\mathbf{0}_k := (0, \dots, 0)$ where there are k copies of 1's and 0's respectively. We set $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 appears at the i-th position. Given $\beta = (\beta_1, \dots, \beta_r)$ and $\gamma = (\gamma_1, \dots, \gamma_s)$, we will denote $(\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_s)$ by (β, γ) if there is no fear of confusion.

Let $\Lambda(n)$ denote the set of highest weights $\lambda = (\lambda_1, \dots, \lambda_{[n/2]})$ of $\mathfrak{so}(n)$. For $e = 0, \frac{1}{2}$, let $\Lambda(n, e)$ denote the subset of $\Lambda(n)$ consisting of $\lambda = (\lambda_1, \dots, \lambda_{[n/2]})$ where $\lambda_i \in \mathbb{Z} + e$. Hence $\Lambda(n) = \Lambda(n, 0) \cup \Lambda(n, \frac{1}{2})$. Let τ_n^λ denote the finite dimensional irreducible representations of $\mathfrak{so}(n)$ with the highest weight λ. If λ is in $\Lambda(n, 0)$ then τ_n^λ is an irreducible representation of the compact group $\mathrm{SO}(n)$. Otherwise it is an irreducible representation of $\mathrm{Spin}(n)$ which does not descend to $\mathrm{SO}(n)$. The trivial representation may be denoted by $\mathbb{C}_{\mathrm{SO}(n)}$. Let $\rho_n = (\frac{n-2}{2}, \frac{n-4}{2}, \dots) \in \Lambda(n)$ denote the half sum of positive roots of $\mathfrak{so}(n)$.

Next we discuss irreducible representations of $\mathrm{O}(n)$. Let $\Lambda(\mathrm{O}(n))$ denote the subset of elements in $\mathbb{Z}^n$ such of the form

$$(\lambda_1, \dots, \lambda_k, \mathbf{0}_{n-k}) \text{ or } (\lambda_1, \dots, \lambda_k, \mathbf{1}_{n-2k}, \mathbf{0}_k) \tag{1}$$

where λ_i are positive integers, and $k \leq \frac{n}{2}$. Irreducible representations of $\mathrm{O}(n)$ are parameterized by $\Lambda(\mathrm{O}(n))$ (see [GoW] and [Ho]). We will call an element λ of $\Lambda(\mathrm{O}(n))$ a *highest weight* of $\mathrm{O}(n)$. Let $\tau_{\mathrm{O}(n)}^\lambda$ denote the corresponding irreducible finite dimensional representation of $\mathrm{O}(n)$. The trivial representation of $\mathrm{O}(n)$ is sometimes denoted by $\mathbb{C}_{\mathrm{O}(n)}$.

Finally we recall a branching rule: Suppose $n > s$, then $\tau_{\mathrm{O}(n)}^\lambda$ contains $\tau_{\mathrm{O}(s)}^{\lambda'}$ if and only if $\lambda_i \geq \lambda_i' \geq \lambda_{i+n-s}$ for all $1 \leq i \leq s$.

With this notation in hand, we can now describe the K-types of V. Recall that $m = \frac{p-1}{2}$. The restriction of V to $K = \mathrm{Spin}(p) \times \mathrm{O}(q)$ is

$$V = \sum_{\lambda \in \Lambda(p,0)} \tau_p^{\lambda + \frac{q-p}{2}\mathbf{1}_m} \otimes \tau_{\mathrm{O}(q)}^\lambda.$$

Here λ in $\tau_{\mathrm{O}(q)}^\lambda$ is considered as an element of $\Lambda(\mathrm{O}(n))$ by adding 0's at the tail. In particular, the minimal K-type of V is $\tau_p^{\frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{O}(q)}$. The infinitesimal character of V is

$$\mu_{p,q} = (1, 2, \dots, \frac{p-1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, \frac{q-1}{2}).$$

We now consider the restriction of V to $\mathfrak{g}_1 \times K_2$, where $\mathfrak{g}_1 = \mathfrak{so}(p,r)$ and $K_2 = \mathrm{O}(s)$ for some integers r and s such that $r+s=q$. We obtain a direct sum

$$V = \sum_{\lambda' \in \Lambda(\mathrm{O}(s))} \Theta(\lambda') \otimes \tau^{\lambda'}_{\mathrm{O}(s)} \tag{2}$$

Note that every $\Theta(\lambda')$ is a $(\mathfrak{g}_1, K_1)$-module, where $K_1 = \mathrm{Spin}(p) \times \mathrm{O}(r)$. Since V is admissible with respect to $\mathrm{Spin}(p) \subseteq K_1$, it follows that each $\Theta(\lambda')$ is an admissible $(\mathfrak{g}_1, K_1)$-module.

The K_1-types of $\Theta(\tau)$. Let λ' be in $\Lambda(\mathrm{O}(s))$. Write $\lambda' = (\lambda'_1, \dots, \lambda'_t, 0, \dots, 0)$. We will now describe the K_1-types of

$$\Theta(\lambda') = \Theta(\tau^{\lambda'}_{\mathrm{O}(s)}).$$

Let δ_1 be a K_1-type of $\Theta(\lambda')$. Obviously, δ_1 must be isomorphic to $\tau_p^{\lambda + \frac{q-p}{2}\mathbf{1}_m} \otimes \tau^{\mu}_{\mathrm{O}(r)}$ for some $\lambda = (\lambda_1, \dots, \lambda_m)$ in $\Lambda(p,0)$, and it has to lie in the K-type $\delta = \tau_p^{\lambda + \frac{q-p}{2}\mathbf{1}_m} \otimes \tau^{\lambda}_{\mathrm{O}(q)}$ of V. Furthermore, the multiplicity of δ in $\Theta(\lambda')$ is given by

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_1 \times K_2} \left(\delta_1 \otimes \tau^{\lambda'}_{K_2}, \delta \right) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{O}(r) \times \mathrm{O}(s)} \left(\tau^{\mu}_{\mathrm{O}(r)} \otimes \tau^{\lambda'}_{\mathrm{O}(s)}, \tau^{\lambda}_{\mathrm{O}(q)} \right). \tag{3}$$

By the branching rule stated after (1), the right hand side is nonzero only if $\lambda_i \geq \lambda'_i$ for all $i \leq m$, and $\lambda'_i = 0$ for all $i > m$. In particular $\Theta(\lambda')$ is nonzero if and only if the number of nonzero integers in λ' is not greater than $(p-1)/2$, that is, $t \leq m$. (If that is the case then λ' can be viewed as a highest weight for $\mathfrak{so}(p)$.) Moreover, the branching rule implies that

$$W(\lambda') = \tau_p^{\lambda' + \frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{O}(r)}$$

appears in $\Theta(\lambda')$ with multiplicity one and it is the (unique) minimal K_1-type of $\Theta(\lambda')$.

Let $K_1^0 = \mathrm{Spin}(p) \times \mathrm{SO}(r)$ be the identity component of K_1. We can view $\Theta(\lambda')$ as a $(\mathfrak{g}_1, K_1^0)$-module. The minimal K_1-type restricts irreducibly to K_1^0, and it is not hard to see that it becomes the unique minimal K_1^0-type of $\Theta(\lambda')$.

We will now state Theorem 9.1 in [LS]. The use of disconnected K_2 is crucial here. (Note that we have just proved the second part.)

Theorem 2.1. *Recall that $\mathfrak{g}_1 = \mathfrak{so}(p,r)$, $K_2 = \mathrm{O}(s)$ and $K_1^0 = \mathrm{Spin}(p) \times \mathrm{SO}(r)$. Let $\Theta(\tau)$ be a the lift of an irreducible representation τ of K_2. Then*

(i) The $(\mathfrak{g}_1, K_1^0)$-module $\Theta(\tau)$ is either zero or irreducible.
(ii) Suppose τ and τ' are non-isomorphic irreducible representations of K_2, and suppose $\Theta(\tau)$ and $\Theta(\tau')$ are nonzero. Then the minimal K_1^0-types of $\Theta(\tau)$ and $\Theta(\tau')$ are non-isomorphic. In particular, $\Theta(\tau)$ and $\Theta(\tau')$ are non-isomorphic $(\mathfrak{g}_1, K_1^0)$-modules. □

3. Cohomological induction

The purpose of this section is to introduce cohomological induction and realize V in terms of the cohomological induction.

Notation. We recall some basic definitions and notation from [KV] and [Wa1]. We use a subscript 0 to denote a real Lie algebra. Those without are complex Lie algebras. Consider a reductive pair $(\mathfrak{g}_0, K^0)$ where K^0 is a connected compact Lie group. Let $\mathfrak{g}_0$ be the Lie algebra of K^0. Let θ be the Cartan involution of $\mathfrak{g}_0$ fixing $\mathfrak{k}_0$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let $\overline{\mathfrak{q}}$ denote its opposite parabolic subalgebra. If Z is an irreducible representation of $\mathfrak{l}$, then we put $Z^\sharp = Z \otimes \wedge^{\mathrm{top}}\mathfrak{n}$ and

$$\mathrm{ind}_{\overline{\mathfrak{q}}}^{\mathfrak{g}} Z = \mathcal{U}(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} Z.$$

We will write ind$Z$ if it is clear what $\mathfrak{g}$ and $\mathfrak{q}$ are. If Z has infinitesimal character λ_Z, then ind$Z^\sharp$ has infinitesimal character $\lambda_Z + \rho(\mathfrak{n})$. Let

$$\mathcal{L}_i(Z) = \Pi_i(\mathrm{ind} Z^\sharp)$$

where $\Pi_i = (\Pi^{\mathfrak{g},K^0}_{\mathfrak{g},L\cap K^0})_i$ is the i-th derived functor of the Bernstein functor. Given a $(\mathfrak{g}, K^0)$-module W, we set W^{h} to be the subspace of K^0-finite vectors in the conjugate linear dual vector space of W. Let $s_0 := \dim(\mathfrak{n} \cap \mathfrak{k})$. By Eq. (6.25) in [KV] and Theorem 6.3.5 in [Wa1],

$$\mathcal{L}_{s_0}(Z) = \Pi_{s_0}(\mathrm{ind} Z^\sharp) = (\Gamma^{s_0}((\mathrm{ind} Z^\sharp)^{\mathrm{h}}))^{\mathrm{h}} = \Gamma^{s_0}(\mathrm{ind} Z^\sharp)$$

where Γ^{s_0} is the s_0-th derived functor of the Zuckerman functor of taking K^0-finite vectors. If $Z = \mathbb{C}_\lambda$ is the one-dimensional character of $\mathfrak{l}$, then we denote $A_{\mathfrak{q}}(\lambda) = \Gamma^{s_0}(\mathrm{ind}\mathbb{C}^\sharp_\lambda)$ and it has infinitesimal character $\lambda + \rho(\mathfrak{g})$.

A positive root system. We now specialize to $\mathfrak{g} = \mathfrak{so}(p, q)$ and $K^0 = \mathrm{Spin}(p) \times \mathrm{SO}(q)$. Recall that $m = \frac{p-1}{2}$ and $m' = \frac{q}{2}$. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the real Lie algebras of G and K^0, respectively. Choose a compact Cartan subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{k}_0$ of $\mathfrak{g}_0$ and positive root system Φ^+ with respect to $\mathfrak{h}_0$ such that the simple roots $\varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le m-1$ belong to $\mathfrak{so}(p)$, and $\varepsilon_i - \varepsilon_{i+1}$ for $m+1 \le i \le m+m'-1$ belong to $\mathfrak{so}(q)$. The non-compact simple roots are $\varepsilon_m - \varepsilon_{m+1}$ and $\varepsilon_{m+m'}$.

Let $\lambda_0 = (\mathbf{1}_m, \mathbf{0}_{m'}) \in \sqrt{-1}\mathfrak{h}_0^*$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ be the maximal parabolic subalgebra in $\mathfrak{g}$ where $\mathfrak{l}$ is spanned by roots perpendicular to λ_0. Then $\mathfrak{q}$ is θ-stable. The Levi factor $\mathfrak{l}$ corresponds to the subgroup

$$L = \widetilde{\mathrm{U}}(m) \times \mathrm{SO}^0(1, q)$$

in G. Here $\widetilde{\mathrm{U}}(m) \subseteq \mathrm{Spin}(p)$ is a two fold cover of $\mathrm{U}(m) \subseteq \mathrm{SO}(p)$. Under the adjoint action of L, the radical $\mathfrak{n}$ decomposes as

$$\mathfrak{n} = \mathbb{C}^m \otimes \mathbb{C}^{1+q} \oplus \wedge^2(\mathbb{C}^m)$$

where $\mathbb{C}^m$ is the standard representation of $\mathrm{U}(m)$ and $\mathbb{C}^{1+q}$ the standard representation of $\mathrm{SO}^0(1, q)$. The summand $\wedge^2(\mathbb{C}^m)$ is spanned by long roots $\varepsilon_i + \varepsilon_j$ for $1 \le i < j \le m$. These long roots and short roots ε_i for $1 \le i \le m$ are precisely all compact roots contained in $\mathfrak{n}$. It follows that

$$s_0 = \dim(\mathfrak{n} \cap \mathfrak{k}) = \frac{m(m+1)}{2} = \frac{p^2 - 1}{8},$$

and this number is independent of q.

A maximal compact subgroup of L is $L \cap K^0 = \widetilde{\mathrm{U}}(m) \times \mathrm{SO}(q)$. However, since our considerations involve a disconnected group, we also need to consider a slightly larger group

$\widetilde{\mathrm{U}}(m) \times \mathrm{O}(q)$. We view $\mathbb{C}^{1+q}$, in the decomposition of $\mathfrak{n}$ above, as a natural $(\mathfrak{so}(1,q), \mathrm{O}(q))$-module. Then, as $(\mathfrak{l}, \mathrm{U}(m) \times \mathrm{O}(q))$-modules,

$$\wedge^{\mathrm{top}}\mathfrak{n} \cong \det{}_{\mathfrak{u}(m)}^{q+m} \otimes \det{}_{\mathrm{O}(q)}^{m}.$$

The action of $\mathfrak{l}$ is, of course, trivial. Recall that if Z is an $(\mathfrak{l}, \widetilde{\mathrm{U}}(m) \times \mathrm{O}(q))$-module then, using the cohomological induction, Z gives rise to a $(\mathfrak{g}, K^0)$-module

$$\mathcal{L}_{s_0}(Z) = \Gamma^{s_0}(\mathrm{ind} Z^\sharp).$$

There are two important observations to be made here: First, since $\mathrm{ind} Z^\sharp$ is already $\mathrm{SO}(q)$-finite, the functor Γ^{s_0} is simply the s_0-th derived functor of the Zuckerman functor of taking $\mathrm{Spin}(p)$-finite vectors. Using the definition and the treatment of Γ^{s_0} in Chapter 6 in [Wa1], $\Gamma^{s_0}(\mathrm{ind} Z^\sharp)$ can be computed by considering $\mathrm{ind} Z^\sharp$ as an $(\mathfrak{so}(p), \widetilde{\mathrm{U}}(m))$-module. Furthermore since $\mathrm{ind} Z^\sharp$ is an $\mathrm{O}(q)$-module, and the action of $\mathfrak{so}(p)$ commutes with the action of $\mathrm{O}(q)$, $\Gamma^{s_0}(\mathrm{ind} Z^\sharp)$ is naturally an $\mathrm{O}(q)$-module . In other words, $\mathcal{L}_{s_0}(Z)$ extends to a $(\mathfrak{g}, K)$-module.

Let Z_0 be a one-dimensional $(\mathfrak{l}, \widetilde{\mathrm{U}}(m) \times \mathrm{O}(q))$-module such that

$$Z_0 = \det{}_{\mathfrak{u}(m)}^{-\frac{p+q}{2}} \otimes \det{}_{\mathrm{O}(q)}^{m}. \tag{4}$$

We can apply the functor $\mathcal{L}_{s_0}$ to Z_0. One easily checks that the infinitesimal character of $\mathcal{L}_{s_0}(Z_0)$ is $\mu_{p,q}$, the infinitesimal character of V.

Lemma 3.1. *The $(\mathfrak{g}, K)$-module $\mathcal{L}_{s_0}(Z_0)$ contains the K-type $W_0 = \tau_p^{\frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{O}(q)}$ with multiplicity one. It is also the minimal K^0-type of $\mathcal{L}_{s_0}(Z_0)$. In particular, it is nonzero.*

We will derive this lemma as a corollary of the proof of Theorem 4.2 in the next section. Alternatively the lemma also follows from the Blattner formula (see Thm 5.64 in [KV]).

Since the K-type W_0 appears in $\mathcal{L}_{s_0}(Z_0)$ with multiplicity one, we define $\overline{\mathcal{L}}_{s_0}(Z_0)$ to be the unique irreducible $(\mathfrak{g}, K)$-subquotient of $\mathcal{L}_{s_0}(Z_0)$ containing W_0.

Proposition 3.2. *The irreducible $(\mathfrak{g}, K)$-modules V and $\overline{\mathcal{L}}_{s_0}(Z_0)$ are isomorphic.*

Proof. Both representations have the same infinitesimal character $\mu_{p,q}$ and the minimal K^0-type W_0. We showed in [LS] that V is the unique irreducible $(\mathfrak{g}, K^0)$-module with infinitesimal character $\mu_{p,q}$ and minimal K^0-type $\tau_p^{\frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{SO}(q)}$. Hence the two modules are isomorphic $(\mathfrak{g}, K^0)$-modules. There are two ways to extend V from a $(\mathfrak{g}, K^0)$-module to a $(\mathfrak{g}, K)$-module. One differs from the other by the determinant character of $\mathrm{O}(q)$. Hence V and $\overline{\mathcal{L}}_{s_0}(Z_0)$ are the same because they have the same minimal K-type W_0. □

4. Identifying $\Theta(\lambda')$

Let r and s be two integers such that $r + s = q$. Choose a standard embedding of $\mathrm{O}(s)$ into $\mathrm{O}(q)$, the second factor of K. Let $\mathfrak{g}_1 \cong \mathfrak{so}(p,r)$ be the centralizer of $\mathrm{O}(s)$ in $\mathfrak{g}$. Note that $\mathfrak{g}_1$ is θ-invariant. In this section we consider the restriction of V to $(\mathfrak{g}_1, K_1^0) \times K_2$ where $K_1^0 = \mathrm{Spin}(p) \times \mathrm{SO}(r)$ and $K_2 = \mathrm{O}(s)$.

Suppose $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ is in $\Lambda(\mathrm{O}(s))$ such that $\Theta(\lambda')$ in (2) is nonzero. Then by (3), $\lambda'_i = 0$ if $i > m = \frac{p-1}{2}$. In particular, λ' can be considered in element in $\Lambda(p,0)$ by adding or

removing some 0's at the tail. The irreducible $(\mathfrak{g}_1, K_1^0)$-module $\Theta(\lambda')$ has a unique minimal K_1^0-type

$$W(\lambda') = \tau_p^{\lambda' + \frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{SO}(r)}.$$

Using the θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ in $\mathfrak{g}$, we define $\mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{g}_1$. Write $\mathfrak{q}_1 = \mathfrak{l}_1 + \mathfrak{n}_1$. Then $\mathfrak{l}_1$ corresponds to a subgroup

$$L_1 = \widetilde{\mathrm{U}}(m) \times \mathrm{SO}^0(1, r)$$

in G_1. For every λ' such that $\Theta(\lambda') \neq 0$ (or equivalently $\lambda'_i = 0$ for $i > m$) let $Z(\lambda')$ be an irreducible L_1-module such that the action of $\mathrm{SO}^0(1, r)$ is trivial and

$$Z(\lambda') \cong \tau_{\mathfrak{u}(m)}^{\lambda' + \frac{q-p-2r}{2}\mathbf{1}_m}$$

as $\widetilde{\mathrm{U}}(m)$-modules. We can apply the functor $\mathcal{L}_{s_0}$ for $s_0 = \dim(\mathfrak{k}_1 \cap \mathfrak{n}_1) = \frac{p^2-1}{8}$ to $Z(\lambda')$, as specified in the previous section. Note that $\mathcal{L}_{s_0}$ is in this case also computed by Γ^{s_0}, the s_0-th derived functor of the Zuckerman functor of taking $\mathrm{Spin}(p)$-finite vectors.

Lemma 4.1. *Any* $\mathrm{Spin}(p)$*-type of the* $(\mathfrak{g}_1, K_1^0)$*-module* $\mathcal{L}_{s_0}(Z(\lambda'))$ *is isomorphic to*

$$\tau_p^{\lambda' + \frac{q-p}{2}\mathbf{1}_m + \kappa}$$

where κ *is an* m*-tuple of non-negative integers. The module* $\mathcal{L}_{s_0}(Z(\lambda'))$ *contains the* K_1^0*-type* $W(\lambda')$ *with multiplicity one.*

We will prove Lemma 4.1 together with Theorem 4.2 below. One could also verify this lemma directly using the Blattner's formula.

Let $\overline{\mathcal{L}}_{s_0}(Z(\lambda'))$ denote the unique irreducible subquotient of $\mathcal{L}_{s_0}(Z(\lambda'))$ containing the minimal K_1^0-type $W(\lambda')$. We can now state the main result of this section.

Theorem 4.2. *The irreducible* $(\mathfrak{g}_1, K_1^0)$*-modules* $\Theta(\lambda')$ *and* $\overline{\mathcal{L}}_{s_0}(Z(\lambda'))$ *are isomorphic. In particular* $\overline{\mathcal{L}}_{s_0}(Z(\lambda'))$ *is unitarizable and it has* K_1^0*-types given by the branching* (3)*.*

Remarks. It is interesting to note that $\mathcal{L}_{s_0}(Z(\lambda'))$ is not always in the good or weakly good range (see Def. 0.49 in [KV]). Hence it may be reducible. It is of separate interest that the image of the bottom layer map induces an unitarizable subquotient. The infinitesimal character of $\overline{\mathcal{L}}(Z(\lambda'))$ is

$$(\lambda' + \frac{q-p-2r}{2}\mathbf{1}_m, \mathbf{0}_{[\frac{r+1}{2}]}) + \rho_{p+r}.$$

Hence Theorem 4.2 gives an alternative proof of the correspondence of infinitesimal characters of $\mathfrak{so}(p, r)$ and $\mathfrak{so}(s)$, Theorem 1.2 in [LS].

The rest of this section contains the proofs of Lemma 3.1, Lemma 4.1 and Theorem 4.2. It is inspired by the work of [GW] and [Wa2].

Recall that $\mathfrak{n}_1 \subseteq \mathfrak{n}$. We have a decomposition $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ such that $\mathfrak{n}_2 = \mathbb{C}^m \otimes \mathbb{C}^s$ is a tensor product of standard representations of $\mathrm{U}(m)$ and $\mathrm{O}(s)$, while the group $\mathrm{SO}^0(1, r)$ acts trivially on it. We extend $\mathfrak{n}_2$ to a representation of $\mathrm{U}(m) \times \mathrm{U}(s)$. It is well known that (see [GoW] and [Ho])

$$\mathrm{Sym}^n \mathfrak{n}_2 = \sum_{\mu} \tau^{\mu}_{\mathrm{U}(m)} \otimes \tau^{\mu}_{\mathrm{U}(s)} \tag{5}$$

where the sum is taken over all partitions μ of n of length not longer than $\min(m,s)$. (So every such partition can be viewed as a highest weight for both $\mathrm{U}(m)$ and $\mathrm{U}(s)$.) We further restrict the summand $\tau^{\mu}_{\mathrm{U}(s)}$ to $\mathrm{O}(s)$

$$\tau^{\mu}_{\mathrm{U}(s)} = \sum_{\lambda' \uparrow \mu} \tau^{\lambda'}_{\mathrm{O}(s)}. \tag{6}$$

The notation $\lambda' \uparrow \mu$ simply means that $\tau^{\lambda'}_{\mathrm{O}(s)}$ is a subrepresentation of $\tau^{\mu}_{\mathrm{U}(s)}$, and the sum is taken with multiplicities. Note that $\tau_{\mathrm{O}(s)}$ appears in the restriction from $\tau_{\mathrm{U}(s)}$ with multiplicity one. Using this notation, we get

$$\mathrm{Sym}^n \mathfrak{n}_2 = \sum_{\mu} \sum_{\lambda' \uparrow \mu} \tau^{\mu}_{\mathrm{U}(m)} \otimes \tau^{\lambda'}_{\mathrm{O}(s)} \tag{7}$$

as a sum of irreducible representations of $\mathrm{U}(m) \times \mathrm{O}(s)$.

We now recall the definition of Z_0 from (4). One easily sees that the restriction of $Z_0^\sharp$ to $L_1 \times \mathrm{O}(s)$ is given by

$$Z_0^\sharp = \det{}^{\frac{q-1}{2}}_{\mathfrak{u}(m)} \otimes \mathbb{C}_{\mathfrak{so}(1,r)} \otimes \mathbb{C}_{\mathrm{O}(s)}.$$

By the Poincare-Birkhoff-Witt theorem,

$$\mathrm{ind} Z_0^\sharp = \mathcal{U}(\mathfrak{n}_1) \otimes \mathcal{U}(\mathfrak{n}_2) \otimes Z_0^\sharp$$

as $L_1 \times \mathrm{O}(s)$-modules. Let $\mathcal{U}_n(\mathfrak{n}_2)$ denote the natural filtration of $\mathcal{U}(\mathfrak{n}_2)$ by degrees. We define $\mathcal{F}_n$ to be the $(\mathfrak{g}_1, L_1 \cap K_1^0)$-submodule of $\mathrm{ind} Z_0^\sharp$ generated by $1 \otimes \mathcal{U}_n(\mathfrak{n}_2) \otimes Z_0^\sharp$. Hence $\{\mathcal{F}_n : n = 0, 1, 2, \ldots\}$ forms an exhaustive increasing filtration of $\mathfrak{g}_1 \times \mathrm{O}(s)$-submodules of $\mathrm{ind} Z_0^\sharp$. We will now state a special case of a known fact which is used in proof of the Blattner formula in [KV].

Lemma 4.3. *For every positive integer n, we have an isomorphism of $\mathfrak{g}_1 \times \mathrm{O}(s)$-modules*

$$\mathcal{F}_n / \mathcal{F}_{n-1} = \sum_{\mu} \sum_{\lambda' \uparrow \mu} \mathrm{ind}_{\mathfrak{q}_1}^{\mathfrak{g}_1} (\tau_{\mathfrak{u}(m)}^{\mu + (\frac{q-1}{2})\mathbf{1}_m} \otimes \mathbb{C}_{\mathfrak{so}(1,r)}) \otimes \tau^{\lambda'}_{\mathrm{O}(s)}$$

where μ is any partition of n of length not more than $\min(m,s)$ and $\tau^{\lambda'}_{\mathrm{O}(s)}$ is counted with multiplicity with which it appears in the restriction of $\tau^{\mu}_{\mathrm{U}(s)}$. □

We shall use the filtration $\mathcal{F}_n$ to compute $\Gamma^{s_0}(\mathrm{ind} Z_0^\sharp)$.

Case 1. We first consider the filtration $\mathcal{F}_n$ in the case $r = 0$ and $s = q$. In particular, $\mathfrak{g}_1 = \mathfrak{so}(p)$. Put

$$V(\mu) = \mathrm{ind}_{\mathfrak{q}_1}^{\mathfrak{g}_1} (\tau_{\mathfrak{u}(m)}^{\mu + (\frac{q-1}{2})\mathbf{1}_m}).$$

The infinitesimal character of $V(\mu)$ is the same as the infinitesimal of $\tau_p^{\mu + \frac{q-p}{2}\mathbf{1}_m}$. In particular, these infinitesimal characters are pairwise different for different partitions μ. It follows that the filtration $\mathcal{F}_n$ splits:

$$\mathrm{ind} Z_0^\sharp = \sum_{\mu} \sum_{\lambda' \uparrow \mu} V(\mu) \otimes \tau^{\lambda'}_{\mathrm{O}(q)}. \tag{8}$$

Here the first sum is taken over all partitions μ of length no more than $m = \frac{p-1}{2}$, and $\tau^{\lambda'}_{\mathrm{O}(q)}$ is counted with multiplicity with which it appears in $\tau^{\mu}_{\mathrm{O}(q)}$.

Lemma 4.4. *Let μ be a partition of length not more than m. Then $V(\mu)$ is an irreducible $\mathfrak{so}(p)$-module.*

Proof. Since $V(\mu)$ is $\mathfrak{u}(m)$-finite, any submodule of $V(\mu)$ must be a quotient of $V(\mu')$ where $\mu' + \frac{q-1}{2}\mathbf{1}_m$ is a weight of $V(\mu)$. It follows, from the definition of $V(\mu)$, that any weight of $V(\mu)$ is of the form $\mu + (\frac{q-1}{2})\mathbf{1}_m + \sum \alpha_i$ where the α_i's are the roots of $\mathfrak{n}_1$. Since the roots of $\mathfrak{n}_1$ are of the form ε_i or $\varepsilon_i + \varepsilon_j$, we conclude that $\mu' = \mu + \kappa$ where κ is an m-tuple of positive integers. However, if $\kappa \neq \mathbf{0}_m$ then the infinitesimal characters of $V(\mu)$ and $V(\mu')$ are different (since these are the infinitesimal characters of two different finite-dimensional irreducible representations of $\mathrm{Spin}(p)$). This shows that $V(\mu')$ cannot map to $V(\mu)$. The lemma is proved. □

Recall that the Zuckerman functor Γ^j is computed in the category of $(\mathfrak{so}(p), \widetilde{\mathrm{U}}(m))$-modules. If we apply Γ^j to both sides of (8) then

$$\Gamma^j(Z_0^\sharp) = \sum_{\mu} \sum_{\lambda' \uparrow \mu} \Gamma^j(V(\mu)) \otimes \tau^{\lambda'}_{\mathrm{O}(q)}. \tag{9}$$

Since $s_0 = \dim(\mathfrak{n}_1 \cap \mathfrak{k}_1) = \frac{m(m+1)}{2} = \frac{p^2-1}{8}$, by the Borel-Weil-Bott-Kostant theorem, $\Gamma^j(V(\mu)) = 0$ if $j \neq s_0$ and $\Gamma^{s_0}(V(\mu)) = \tau_p^{\mu + \frac{q-p}{2}\mathbf{1}_m}$. The reader may recognize that we have essentially followed the proof of the Blattner formula in [KV] to compute K-types of $\mathcal{L}_{s_0}(Z_0)$. Now we have the following conclusions:

(1) A K-type of $\mathcal{L}_{s_0}(Z_0)$ is of the form $\tau_p^{\mu + \frac{q-p}{2}\mathbf{1}_m} \otimes \tau^{\lambda'}_{\mathrm{O}(q)}$, where $\tau^{\lambda'}_{\mathrm{O}(q)}$ appears in the restriction from $\tau^{\mu}_{\mathrm{U}(q)}$. In particular, the minimal K-type is $W_0 = \tau_p^{\frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{O}(q)}$ which occurs with multiplicity one. It is also the image of the bottom layer map. With this, we have proven Lemma 3.1.
(2) The module $\mathcal{L}_{s_0}(Z_0)$ is admissible with respect to $\mathrm{Spin}(p)$. This also follows from a very general criterion in [Ko].

Case 2. Now we return to the general r for $\mathfrak{g}_1 = \mathfrak{so}(p, r)$. Consider the filtration $\mathcal{F}_n$ in this situation. We abbreviate

$$L(\mu) = \mathrm{ind}_{\bar{\mathfrak{q}}_1}^{\mathfrak{g}_1}(\tau_{\mathfrak{u}(m)}^{\mu + (\frac{q-1}{2})\mathbf{1}_m} \otimes \mathbb{C}_{\mathfrak{so}(1,r)}).$$

Then, $\mathcal{F}_n/\mathcal{F}_{n-1}$ is a direct sum of $L(\mu)$ where μ is a partition of n of length not more than $\min(m, s)$. By (8), $L(\mu)$ is a direct sum of various $V(\mu')$. In particular, in order to understand $\mathrm{Spin}(p)$-types of $\Gamma^{s_0}(L(\mu))$, we must describe μ' such that $V(\mu') \subseteq L(\mu)$. Let $\mathfrak{h}_m$ denote the maximal Cartan subalgebra of $\mathfrak{u}(m)$. Arguing as in the proof of Lemma 4.4, any $\mathfrak{h}_m$-weight of $L(\mu)$ is of the form $\mu + (\frac{q-1}{2})\mathbf{1}_m + \sum \alpha_i$ where the α_i's are the roots of $\mathfrak{n}_1$. Since the roots of $\mathfrak{n}_1$ are of the form ε_i, $\varepsilon_i + \varepsilon_j$ or $\varepsilon_i - \varepsilon_j$ with $j > m$, any $\mathfrak{h}_m$-weight of $L(\mu)$ is of the form

$$\mu + (\frac{q-1}{2})\mathbf{1}_m + \kappa \tag{10}$$

where κ is an m-tuple of positive integers. The lowest $\mathfrak{u}(m)$-type of $V(\mu')$ has the highest $\mathfrak{h}_m$-weight $\mu' + (\frac{q-1}{2})\mathbf{1}_m$. Combining with the equation (10), we see that $\mu' = \mu + \kappa$ if $V(\mu') \subseteq L(\mu)$. In addition, $L(\mu)$ contains a unique copy of $V(\mu)$ and $\mathrm{SO}(r)$ acts trivially on it. It follows that $\mathrm{Spin}(p)$-types of $\Gamma^{s_0}(L(\mu))$ are $\tau_p^{\mu+\frac{q-p}{2}\mathbf{1}_m+\kappa}$, here κ is an m-tuple of positive integers, and the K_1^0-type $W(\mu) = \tau_p^{\mu+\frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{SO}(r)}$ occurs with multiplicity one. This, in particular, proves Lemma 4.1.

Lemma 4.5. *Let $\mathcal{F}_n$ be the filtration as in Lemma 4.3. Then $\Gamma^j(\mathcal{F}_n) = 0$ if $j \neq s_0$. Furthermore we have an exact sequence*

$$0 \to \Gamma^{s_0}(\mathcal{F}_{n-1}) \to \Gamma^{s_0}(\mathcal{F}_n) \to \Gamma^{s_0}(\mathcal{F}_n/\mathcal{F}_{n-1}) \to 0.$$

Proof. Let $\bar{\mathcal{F}}_n := \mathcal{F}_n/\mathcal{F}_{n-1}$. By the naturality of the Zuckerman functor, we may compute $\Gamma^j\bar{\mathcal{F}}_n$ in the category of $(\mathfrak{so}(p), \widetilde{\mathrm{U}}(m))$-modules. Then $\mathcal{F}_n$ and $\bar{\mathcal{F}}_n$ are direct sums of $V(\mu)$'s in (8). Hence $\Gamma^j(\mathcal{F}_n)$ and $\Gamma^j(\bar{\mathcal{F}}_n)$ are direct sums of $\Gamma^j(V(\mu))$ and we have shown that these are zeros if $j \neq s_0$. Finally we apply the functor Γ to the exact sequence

$$0 \to \mathcal{F}_{n-1} \to \mathcal{F}_n \to \bar{\mathcal{F}}_n \to 0$$

to get the long exact sequence. The exact sequence in the lemma follows immediately. □

Lemma 4.6. *In the category of $(\mathfrak{g}_1, K_1^0)$-modules, $\Gamma^{s_0}(\mathcal{F}_n)$ is an exhaustive increasing filtration of $\mathcal{L}_{s_0}(Z_0)$ and*

$$\Gamma^{s_0}(\mathcal{F}_n)/\Gamma^{s_0}(\mathcal{F}_{n-1}) = \Gamma^{s_0}(\mathcal{F}_n/\mathcal{F}_{n-1}) = \sum_{\mu} \sum_{\lambda' \uparrow \mu} \Gamma^{s_0}(L(\mu)) \otimes \tau^{\lambda'}_{\mathrm{O}(s)}.$$

Here the first sum is taken over all partitions of length no more than $\min(m, s)$ and $\tau^{\lambda'}_{\mathrm{O}(s)}$ is counted with multiplicity with which it appears in $\tau^{\mu}_{\mathrm{O}(s)}$.

Proof. This follows from Lemmas 4.5 and 4.3. □

We are finally ready to prove Theorem 4.2, that is, compute $\Theta(\lambda')$ where λ' is in $\Lambda(\mathrm{O}(s))$ of length not more than $\min(m, s)$. We define $S(\lambda')$ as the set of all partitions μ of length not more than $\min(m, s)$ such that $\tau^{\mu}_{\mathrm{U}(s)}$ contains $\tau^{\lambda'}_{\mathrm{O}(s)}$. Since V is an irreducible subquotient of $\mathcal{L}_{s_0}(Z_0)$, it follows that $\Theta(\lambda')$ is an irreducible subquotient of $\mathcal{L}_{s_0}(Z_0)$, considered as $(\mathfrak{g}_1, K_1^0)$-module. It follows that $\Theta(\lambda')$ is an irreducible subquotient of $\Gamma^{s_0}(L(\mu))$ for some μ in $S(\lambda')$. We now need the following lemma.

Lemma 4.7. *Let μ be in $S(\lambda')$. The K_1^0-type $W(\lambda') = \tau_p^{\lambda'+\frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}_{\mathrm{SO}(r)}$ occurs in $\Gamma^{s_0}(L(\mu))$ if and only if $\mu = \lambda'$.*

Proof. We check $\mathrm{Spin}(p)$-types. If $W(\lambda')$ is contained in $\Gamma^{s_0}(L(\mu))$ for some then, as we have just seen, $\lambda' = \mu + (\kappa_1, \dots, \kappa_m)$ where $\kappa_i \geq 0$. On the other hand, since $\tau^{\mu}_{\mathrm{U}(s)}$ contains $\tau^{\lambda'}_{\mathrm{O}(s)}$, this is possible only if $\mu = \lambda'$ as desired. □

Since $\Theta(\lambda')$ contains $W(\lambda')$ the lemma implies that $\Theta(\lambda')$ is an irreducible subquotient of $\Gamma^{s_0}(L(\lambda'))$. This proves Theorem 4.2.

5. The smallest representation V^+

In this section, we will extend Theorems 2.1 and 4.2 to representations V^+ and V^-. Since the proofs are almost identical to those in the previous sections, we will only state the main results.

Let $\mathfrak{g} = \mathfrak{so}(p,q)$ and $K = \mathrm{O}(p) \times \mathrm{Spin}(q)$. Recall that $m = \frac{p-1}{2}$ and $m' = \frac{q}{2}$. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the real Lie algebras of G and K^0, respectively. Choose a compact Cartan subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{k}_0$ of $\mathfrak{g}_0$ and positive root system Φ^+ such that the simple roots $\varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq m'-1$ belong to $\mathfrak{so}(q)$ and, $\varepsilon_i - \varepsilon_{i+1}$ for $m'+1 \leq i \leq m+m'-1$ and $\varepsilon_{m'+m}$ belong to $\mathfrak{so}(p)$. The non-compact simple root is $\varepsilon_{m'} - \varepsilon_{m'+1}$.

We refer to the notations on cohomological inductions in the beginning of Section 3. We set $\lambda_0 = (\mathbf{1}_{m'}, \mathbf{0}_m) \in \sqrt{-1}\mathfrak{h}_0^*$ and we let $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ be the corresponding the parabolic subalgebra. The algebra $\mathfrak{l}$ corresponds to subgroup

$$L = \mathrm{SO}(p) \times \widetilde{\mathrm{U}}(m')$$

in G. We have $s_0 = \frac{m'(m'-1)}{2} = \frac{q(q-2)}{8}$. Let Z_0 be a one dimensional $\mathrm{O}(p) \times \widetilde{\mathrm{U}}(m')$-module

$$Z_0 = \det{}_{\mathrm{O}(p)}^{m'} \otimes \det{}_{\mathfrak{u}(m')}^{-(\frac{p+q}{2})}.$$

We consider the $(\mathfrak{g}, K)$-module $\mathcal{L}_{s_0}(Z_0)$. It is equal to $A_{\mathfrak{q}}(\lambda)$ where $\lambda = -\frac{p+q}{2}\lambda_0$. The following is essentially a result of [Kn] and [T]. The only difference is that we consider K and not K^0. See Section 6 for more details.

Theorem 5.1. *Recall that $p > q$ and $K = \mathrm{O}(p) \times \mathrm{Spin}(q)$.*

(i) The minimal K-type of $\mathcal{L}_{s_0}(Z_0)$ is $W_0 = \mathbb{C}_{\mathrm{O}(p)} \otimes \tau_q^{\frac{p-q}{2}\mathbf{1}_{m'}}$ and it occurs in $\mathcal{L}_{s_0}(Z_0)$ with multiplicity 1.

(ii) Let $V^+ = \overline{\mathcal{L}}_{s_0}(Z_0)$ denote the irreducible subquotient of $\mathcal{L}_{s_0}(Z_0)$ generated by W_0. Then V^+ is an unitarizable $(\mathfrak{g}, K)$-module. □

The restriction of V^+ to $K = \mathrm{O}(p) \times \mathrm{Spin}(q)$ is

$$V^+ = \sum_{\lambda \in \Lambda(q+1,0)} \tau_{\mathrm{O}(p)}^{\lambda} \otimes \tau_q^{\lambda + \frac{p-q}{2}\mathbf{1}_{m'}}.$$

Its infinitesimal character is $\left(\frac{q}{2}, \frac{q-2}{2}, \ldots, 1, \frac{p-2}{2}, \frac{p-4}{2}, \ldots, \frac{1}{2}\right)$. The module V^+ remains irreducible as a $(\mathfrak{g}, K^0)$-module. In [LS] we call V^+ a smallest representation of the non-linear cover of $\mathrm{SO}(p,q)$, and there is also an outline of a construction of V^+ using Gelfand-Zetlin bases.

Remark. We note that by an outer automorphism action of the pair $(\mathfrak{so}(p,q), K)$ on V^+, we get another smallest representation V^-. All the results in this paper on V^+ would immediately give corresponding results for V^- via this outer automorphism. Therefore we will only work with V^+.

Choose a standard embedding of $K_2 = \mathrm{O}(s)$ into $\mathrm{O}(p)$, the first factor of K. Let $\mathfrak{g}_1 \cong \mathfrak{so}(r,q)$ be the centralizer of $\mathrm{O}(s)$ in $\mathfrak{g}$. Note that $\mathfrak{g}_1$ is θ-invariant. In this section we consider

the restriction of V^+ to $K_2 \times (\mathfrak{g}_1, K_1)$ where $K_1 = \mathrm{O}(r) \times \mathrm{Spin}(q)$.

$$V^+ = \sum_{\lambda' \in \Lambda(\mathrm{O}(s))} \tau^{\lambda'}_{\mathrm{O}(s)} \otimes \Theta(\lambda').$$

Since $\mathrm{O}(s)$ is compact, the right hand side is a direct sum. Furthermore V^+ is admissible with respect to $\mathrm{Spin}(q)$, so $\Theta(\lambda')$ is an admissible $(\mathfrak{g}_1, K_1)$-module.

The K_1-types of $\Theta(\lambda')$ can be computed using branching rules similar to (3). More precisely, suppose $\delta_1 = \tau^{\mu}_{\mathrm{O}(r)} \otimes \tau_q^{\lambda + \frac{q-p}{2}\mathbf{1}_{m'}}$ is a K_1-type of $\Theta(\lambda')$. Then δ_1 has to lie in the K-type $\delta = \tau^{\lambda}_{\mathrm{O}(p)} \otimes \tau_q^{\lambda + \frac{q-p}{2}\mathbf{1}_{m'}}$ of V^+. The multiplicity of δ in $\Theta(\lambda')$ is given by

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_1 \times \mathrm{O}(s)} \left(\delta_1 \otimes \tau^{\lambda'}_{\mathrm{O}(s)}, \delta \right) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{O}(r) \times \mathrm{O}(s)} \left(\tau^{\mu}_{\mathrm{O}(r)} \otimes \tau^{\lambda'}_{\mathrm{O}(s)}, \tau^{\lambda}_{\mathrm{O}(p)} \right). \tag{11}$$

By the right hand side of (11), $\Theta(\lambda')$ is nonzero if and only if nonzero entries of λ' is not greater than $\frac{q}{2}$. The minimal K_1^0-type of $\Theta(\lambda')$ is

$$W(\lambda') = \mathbb{C}_{\mathrm{SO}(r)} \otimes \tau_q^{\lambda' + \frac{p-q}{2}\mathbf{1}_{m'}}. \tag{12}$$

We compare the next theorem with Theorem 2.1.

Theorem 5.2. *Recall that $\mathfrak{g}_1 = \mathfrak{so}(r,q)$, $K_2 = \mathrm{O}(s)$ and $K_1^0 = \mathrm{SO}(r) \times \mathrm{Spin}(q)$. Let $\Theta(\tau)$ be the lift of an irreducible representation τ of K_2. Then*

(i) The $(\mathfrak{g}_1, K_1^0)$-module $\Theta(\lambda')$ is either zero or irreducible.
(ii) Suppose $\Theta(\lambda')$ and $\Theta(\eta')$ are nonzero. Then $\Theta(\lambda')$ and $\Theta(\eta')$ are isomorphic $(\mathfrak{g}_1, K_1^0)$-modules if and only if $\lambda' = \eta'$.

Part (i) follows the same argument as that of Theorem 9.1 in [LS]. We will omit the proof. Part (ii) is a consequence of (12) because if $\lambda' \neq \eta'$, then $\Theta(\lambda')$ and $\Theta(\eta')$ have distinct minimal K_1^0-types.

Cohomological induction. We would like to identify $\Theta(\lambda')$ as a subquotient of a cohomological induced module.

Suppose $\Theta(\lambda')$ is nonzero. Then the number of nonzero entries in λ' is not greater than m'. Let $\mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{g}_1$ be a theta-stable parabolic subalgebra of $\mathfrak{g}_1$. Its Levi subalgebra $\mathfrak{l}_1$ corresponds to a subgroup

$$L_1 = \mathrm{SO}(r) \times \widetilde{\mathrm{U}}(m')$$

in G_1. Let $Z(\lambda')$ be an irreducible L_1-module which is trivial on $\mathrm{SO}(r)$ and such that the restriction to $\widetilde{\mathrm{U}}(m')$ is

$$Z(\lambda') \cong \tau_{\mathfrak{u}(m')}^{\lambda' + \frac{p-q-2r}{2}\mathbf{1}_{m'}}.$$

We consider the cohomologically induced representation $\mathcal{L}_{s_0}(Z(\lambda'))$. Its minimal K_1^0-type is $W(\lambda')$ in (12) and it occurs with multiplicity one. Let $\overline{\mathcal{L}}_{s_0}(Z(\lambda'))$ denote the unique irreducible $(\mathfrak{g}_1, K_1^0)$-subquotient of $\mathcal{L}_{s_0}(Z(\lambda'))$ containing $W(\lambda')$. The next theorem is proved in the same way as Theorem 4.2.

Theorem 5.3. *The irreducible $(\mathfrak{g}_1, K_1^0)$-modules $\Theta(\lambda')$ and $\overline{\mathcal{L}}_{s_0}(Z(\lambda'))$ are isomorphic. In particular, $\overline{\mathcal{L}}_{s_0}(Z(\lambda'))$ is nonzero and unitarizable.* □

6. On results of Knapp and Trapa

The aim of this section is to relate our results to some results of Knapp and Trapa. Assume that r is an integer and $r \geq q$. For every non-negative integer s, Knapp [Kn] defined an $(\mathfrak{so}(r,q), K_1^0)$-module π'_s as a certain (naturally unitarizable) subquotient of $A_{\mathfrak{q}}(\lambda)$ where $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$, $\mathfrak{l} = \mathfrak{u}(m') + \mathfrak{so}(r)$ and

$$\lambda = \left(\frac{s-r-q}{2} \mathbf{1}_{m'}, \mathbf{0}_{[\frac{r}{2}]} \right).$$

The module π'_s contains the minimal K_1^0-type of $A_{\mathfrak{q}}(\lambda)$. Trapa showed in [T] that π'_s is irreducible. We now focus our attention to nonnegative integral values of s so that $A_{\mathfrak{q}}(\lambda)$ is a faithful representation of K_1^0. This implies that $\frac{s-r-q}{2} \in \mathbb{Z} + \frac{1}{2}$, that is, $r+s$ is odd.

Consider $W = V^+$ and the dual pair $(\mathfrak{g}_1, K_1) \times \mathrm{O}(s)$ where $\mathfrak{g}_1 = \mathfrak{so}(r,q)$, $K_1 = \mathrm{O}(r) \times \mathrm{Spin}(q)$ and $p = r+s$. Let $\Theta(0)$ denote the theta lift of the trivial representation of $\mathrm{O}(s)$. Then $\Theta(0)$ is an $(\mathfrak{so}(r,q), K_1^0)$-module. The next theorem follows from Theorem 5.3.

Theorem 6.1. *Let r and s be two positive integers such that $r \geq q$ and $p = r+s$ is odd. Then the $(\mathfrak{so}(r,q), K_1^0)$-module $\Theta(0)$ is isomorphic to π'_s.* □

We note that Knapp computed K_1^0-types of π'_s. His computation shows that K_1^0-types of π'_s coincide with K_1^0-types of $\Theta(0)$. Hence this paper gives an independent proof of the fact that π'_s is irreducible (see [T]).

An interesting way to formulate the above result for odd r is as follows: Let π'_0, $\pi'_2, \ldots$ be Knapp's family for $\mathfrak{so}(p,q)$, where $p > q$. Then $\pi'_{2a} \cong \Theta(0)$ where $\Theta(0)$ is the theta lift of the trivial representation of $\mathrm{O}(2a)$. Again, we note that Paul and Trapa studied how π'_{2a+1} appear in the Howe correspondence [PT].

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