**Eves' Equihoop, The Card Game SET, and Abstract SET**

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**Abstract**. At a Davidson College lecture in 1972, Dr. Howard Eves defined an equihoop as a binary operator $(E, \cdot)$ that satisfies the central, idempotent and medial properties with the commutative equihoop (CEH) also satisfying $ab = ba$ for all $a, b \in E$. We show the reader how to easily prove for himself that all finite CEH's $(E, \cdot)$ have $|E| = 3^k$ elements, and all CEH's $(E, \cdot)$ of order $|E| = 3^k$ are isomorphic to the direct product of $k$ copies of the basic CEH $(\{0, 1, 2, \}, \cdot)$. In 1974 Marsha Jean Falco, see [1], invented an 81 card game called SET which is mathematically identical to a CEH $(E, \cdot)$ with $|E| = 81$. Three distinct cards $\{a, b, c\}$ form a SET if $c = ab$ which is also equivalent to either $a = bc$ or $b = ac$.

Given a finite CEH $(E, \cdot)$, a common problem, which is called the SET problem, is to find a subset $S \subseteq E$ of the highest possible cardinality such that $\forall a, b \in S$, if $a \neq b$, then $ab \in E \backslash S$. Such a set $S$ is said to be maximum SET-free and we would like to compute $|S|$ for such $S$ as $|E|$ ranges over $3^k, k = 1, 2, 3, \cdots$ .

To our knowledge nobody has come even remotely close to giving a complete solution to the SET problem for the CEH, [3]. However, in this paper we precisely define and give a complete solution to the SET problem if we drop the medial property. We call such a structure Abstract SET, and it is also identical to the various collections of Steiner triples on $E$. Our solution will use the theory of round-robins, and those readers interested only in a short treatment can stop at the end of Case 1 in Section 2.4. The paper has two sections. The first section presents known results on Commutative Equihoops. The second section discusses Abstract SET and the Abstract SET Problem. Only knowledge of cyclic groups is needed for reading this paper.

# 1 Commutative Equihoops

**Definition 1** ***(Eves-Davidson College, 1972).*** *An equihoop is a non-empty set $E$ of elements $a, b, c, d, \cdots$ and a binary operation $(E, \cdot)$ that satisfies the following:*

*P1.* $a\,(ba) = b$ *for all* $a, b \in E$*. (left central property of* $x$*).*

*P2.* $aa = a$ *for all* $a \in E$*. (idempotent property of* $x$*).*

*P3.* $(ab)\,(cd) = (ac)\,(bd)$ *for all* $a, b, c, d \in E$*. (medial property of* $x$*).*

***(Eves).*** *A commutative equihoop* $(E, \cdot)$ *is an equihoop that satisfies P4.*

*P4.* $ab = ba$ *for all* $a, b \in E$*.*

**Definition 2** ***(Quasigroups).*** *A quasigroup is a binary operator* $(E, \cdot)$ *that satisfies P5.* See [6].

*P5. For all* $a, b \in E, ax = b$ *and* $ya = b$ *have unique solutions* $x$ *and* $y$ *in* $E$*.*

**Definition 3** ***(Eves).*** *A hoop is a quasigroup that also satisfies P2 and P3.*

**Definition 4** ***(medial Quasigroups).*** *A medial quasigroup is a quasigroup that also satisfies P3. D.C. Murdoch, [4], used the term Abelian quasigroup.*

***Note 1****. In 1939 D.C. Murdoch, [4], classified all medial quasigroups* $(E, \cdot)$ *that have at least one idempotent element* $0 \in E$*. That is,* $0 \cdot 0 = 0$*.*

**Definition 5** ***(Knuth).*** *Donald Knuth in 1968 defined an extremely primitive binary operator* $(E, \cdot)$ *that satisfies only P1. Note that P1 is equivalent to P1*.* $(ab)\,a = b$ *for all* $a, b \in E$*. He called* $(E, \cdot)$ *a grope because it was used to "grope" for results.*

**Theorem 1** *Suppose* $(E, 0, \cdot)$ *is a CEH where* $0 \in E$ *is arbitrary but fixed.* $\forall a, b \in E$ *define* $a + b = 0\,(ab)$*. Then* $(E, 0, +)$ *is an Abelian group with identity* $0$ *that satisfies* $3a = a + a + a = 0, \forall a \in E$*.*

*Using this* $(E, 0, +)$*, we can reverse ourselves and define* $(E, \cdot)$ *by* $\forall a, b \in E, a \cdot b = -a - b = (a + a) + (b + b)$*.*

**Proof.** We let the reader prove this easy theorem. ■

**Corollary 1** *If* $(E, \cdot)$ *is a finite CEH, then* $|E| = 3^k$*. Also if* $(E, \cdot)$ *is a CEH and* $|E| = 3^k$*, then* $(E, \cdot)$ *is isomorphic to the direct product of* $k$ *copies of the basic CEH* $(\{0, 1, 2\}, \cdot)$ *where* $0 \cdot 1 = 1 \cdot 0 = 2, 1 \cdot 2 = 2 \cdot 1 = 0, 0 \cdot 0 = 0$*, etc.*

**Proof.** Follows immediately from Abelian group theory. ■

## 2 Abstract SET and the Abstract SET Problem

In light of Marsha Jean Falco's 81 card game SET, we call the structure of the following definition 7 an Abstract SET. An Abstract SET,$(E,\cdot)$, is also identical to the various collections of Steiner triples on $\dot{E}$. This definition was also given by Kirkman.

**Definition 6** *An Abstract SET $(E,\cdot)$ is a binary operator that satisfies P1, P2 and P4.*

*P1.* $a\,(ba) = b$ *for all* $a, b \in E$.

*P2.* $aa = a$ *for all* $a \in E$.

*P4.* $ab = ba$, *for all* $a, b \in E$.

***Note 2***. *Thus, an Abstract SET $(E,\cdot)$ is a commutative, idempotent grope. However, in light of Falco's SET, we ourselves prefer to think of $(E,\cdot)$ as a CEH in which the medial property P3 has been dropped. In this paper, we abbreviate an Abstract SET $(E,\cdot)$ by ASET.*

*We now develop the most basic properties of the ASET where in some lemmas we consider $E$ to be finite.*

**Lemma 1** *Suppose $(E,\cdot)$ is an ASET. Then $\forall a, b \in E$, if $a \neq b$ then $ab \notin \{a, b\}$.*

**Proof.** Suppose $ab = a$. Then $(ab)\,a = b = a \cdot a = a$, a contradiction. ■

**Corollary 2** *$\forall a, b \in E$, the set $\{a, b, ab\}$ is either a tripleton set or a singleton set.*

**Lemma 2** *Suppose $(E,\cdot)$ is an ASET. Then $(E,\cdot)$ is a commutative quasigroup.*

**Proof.** Since $(E,\cdot)$ is commutative, we must show that $\forall a, b \in E, \exists$ a unique $x \in E$ such that $ax = b$.

First, suppose $ax = b$. Then $(ax)\,a = x = ab$. Also, $a\,(ab) = b$. ■

**Corollary 3** *$\forall a, b \in E$, if $\{a, b, ab\}$ is tripleton, then the product of any two distinct elements equals the third element. Thus, $\forall a, b, \overline{a}, \overline{b} \in E, \{a, b, ab\} \cap \left\{\overline{a}, \overline{b}, \overline{ab}\right\}$ can never be a doubleton set.*

**Observations 1.** Since the idempotent property is trivial to add, an ASET $(E,\cdot)$ can be viewed as a collection of distinct tripleton subsets of $E$ (also called Steiner triples), which we denote as $\{\Delta_1, \Delta_2, \cdots, \Delta_t\}$ or $\{l_1, l_2, \cdots, l_t\}$, that has the following properties:

1. Each $l_i = \Delta_i = \{a, b, c\}$ where $a, b, c$ are three distinct members of $E$.

2. $\forall i \neq j, l_i = \Delta_i \neq \Delta_j = l_j$. That is, our collection is a true set.

3. $\forall a, b \in E$, if $a \neq b$ then $\exists$ a unique $l_i = \Delta_i$ such that $\{a, b\} \subseteq l_i = \Delta_i$.

It is sometimes convenient to think of a Steiner triple as being either a triangle $\Delta_i$ or a line $l_i$. If we are given a collection of Steiner triples $\{\Delta_1, \Delta_2, \cdots, \Delta_t\}$ on $E$, the corresponding ASET $(E, \cdot)$ is defined as follows: $\forall a \in E, a \cdot a = a$. $\forall a, b \in E$ if $a \neq b$ then $ab$ is the 3rd member of $\Delta_i$ where $\Delta_i$ is the unique member of $\{\Delta_1, \Delta_2, \cdots, \Delta_t\}$ satisfying $\{a, b\} \subseteq \Delta_i$. Of course, if $\{a, b, c\}$ is any Steiner triple, then $ab = c, ac = b$ and $bc = a$.

Fig. 1. 7 points, $\frac{1}{3}\binom{7}{2} = 7$ lines.

**Lemma 3** *Suppose $(E, \cdot)$ is an ASET. If $|E|$ is finite, then $|E|$ must be odd.*

**Proof.** Single out $a \in E$. $\forall x \in E \backslash \{a\}$ let us pair $\{x, ax\}$ together where we note that $ax \in E \backslash \{a\}, x \neq ax$ and $(ax)a = x$. It follows from this that $|E \backslash \{a\}|$ is even and $|E|$ is odd. $\blacksquare$

**Lemma 4** *Suppose $(E, \cdot)$ is a finite ASET. Then $3 \mid |E| \cdot (|E| - 1)$, which is equivalent to $3 \nmid |E| - 2$.*

**Proof.** Suppose $(E, \cdot)$ is represented as the Steiner triples $\{\Delta_1, \Delta_2, \cdots, \Delta_T\}$. Now each doubleton subset $\{a, b\}$ of $E$ is a subset of a unique $\Delta_i$. Also, each $\Delta_i$ produces 3 distinct doubleton subsets of $E$. Therefore, the total number of doubleton subsets of $E$ equals $3T$. Also, the number of doubleton subsets of $E$ equals $C_2^{|E|}$. Therefore, $3T = \dfrac{|E| \cdot (|E| - 1)}{2}$. $\blacksquare$

**Corollary 4** $T = \frac{1}{3} C_2^{|E|}$.

**Lemma 5** *Suppose $(E, \cdot)$ is a finite ASET. By combining lemmas 3, 4, we know that $|E| = 6k + 1$ or $|E| = 6k + 3$.*

**Remark 1** *From the theory of Steiner triples, we know that if $E$ is a finite set, then an ASET $(E, \cdot)$ exists on $E$ if and only if $|E|$ is odd and $3 \nmid |E| - 2$. However, in order to solve the abstract SET Problem, we ourselves will be forced to prove this since we must find very specialized $(E, \cdot)$'s. In general if $|E|$ is odd and $3 \nmid |E| - 2$, to construct an ASET $(E, \cdot)$ we must produce $\frac{1}{3} C_2^{|E|}$ distinct tripleton subsets of $E$, call them $l_1, l_2, l_3, \cdots$, such that $\forall i \neq j, l_i \cap l_j$ is either empty or singleton.*

**Definition 7** $(E, \cdot)$ *is a finite ASET. Using terminology from Falco's 81 card SET game, if* $\{a, b, c\}$ *is any tripleton subset of* $E$*, we say that* $\{a, b, c\}$ *is a SET if and only if* $ab = c$ *which is also equivalent to either* $b = ac$ *or* $a = bc$*. If we view* $(E, \cdot)$ *as a collection of Steiner triples* $\{\Delta_1, \Delta_2, \cdots, \Delta_t\}$*, we call these* $\Delta_i$*'s SETS.*

**Definition 8** $(E, \cdot)$ *is a finite ASET and* $S \subseteq E$*. We say that* $S$ *is SET-free if* $\forall a, b \in S$*,* $a \neq b \Rightarrow ab \in E \backslash S$*. In other words, if* $\{a, b, c\} \subseteq S$ *is any tripleton subset of* $S$*, then* $\{a, b, c\}$ *is not a SET,*

**The Abstract SET Problem.** Suppose $E$ is a finite set such that $|E|$ is odd and $3 \nmid |E| - 2$. If $|E|$ is fixed, find the positive integer $f(|E|)$ such that (a) and (b) are true for $f(|E|)$.

1. If $(E, \cdot)$ is any ASET on $E$ and $S \subseteq E$ is any subset of $E$, then [$S$ is SET-free]$\Rightarrow$ [$|S| \leq f(|E|)$].
2. $\exists$ an asset $(E, \cdot)$ on $E$ and a subset $S \subseteq E$ such that $S$ is SET-free and $|S| = f(|E|)$.

The following Main Theorem gives the complete solution to the abstract SET problem.

**Main Theorem.** Suppose $E$ is a finite set. Then an ASET $(E, \cdot)$ exists on $E$ if and only if $|E|$ is odd and $3 \nmid |E| - 2$. If $|E|$ is odd and $3 \nmid |E| - 2$, define

1. $\overline{f}(|E|) = \dfrac{|E| + 1}{2}$ if $\dfrac{|E| + 1}{2}$ is even,
2. $\overline{f}(|E|) = \dfrac{|E| - 1}{2}$ if $\dfrac{|E| + 1}{2}$ is odd.

   Then $f(|E|) = \overline{f}(|E|)$ is the solution to the abstract SET problem.
3. $\overline{f}(|E|)$ is the even integer that is closest to $\frac{|E|}{2}$.

We prove the main theorem in the rest of this paper. Section 2.1 is the easiest part of the proof.

## 2.1 Showing that $f(|E|) \leq \overline{f}(|E|)$.

The next step is the proof that $f(|E|) \leq \overline{f}(|E|)$.

**Lemma 6** $(E, \cdot)$ *is an ASET on a finite set* $E$*. Then* $\forall S \subseteq E$*, if* $S$ *is SET-free, then* $|S| \leq \overline{f}(|E|)$.

**Proof.** We will think of $(E, \cdot)$ as a collection of distinct lines $\{l_1, l_2, \cdots l_t\}$ where each line has 3 points and each pair of distinct points $\{a, b\}$ in $E$ lies on exactly one line.

$\forall x \in E \backslash S$, let $a_x$ be the number of distinct lines through $x$ that intersect $S$ in 2 points, and let $b_x$ be the number of distinct lines through $x$ that intersect $S$ in a single point. Since $S$ is SET-free, we have (1).

1. $\sum_{x \in E \backslash S} a_x = C_2^{|S|} = \frac{|S|(|S|-1)}{2}$.

   From the definitions of $a_x, b_x$ we have (2).

2. a. $\forall x \in E \backslash S, 2a_x + b_x = |S|$ which implies

   b. $2 \sum_{x \in E \backslash S} a_x + \sum_{x \in E \backslash S} b_x = |S| \cdot (|E| - |S|)$.

   From (1) and (2) we have

3. $\sum_{x \in E \backslash S} b_x = |S| \cdot (|E| + 1 - 2|S|)$, which implies $|S| \leq \dfrac{|E|+1}{2}$.

   Now if $\frac{|E|+1}{2}$ is even, then lemma 6 is true.

Therefore, suppose $\frac{|E|+1}{2}$ is odd. Now if $|S| = \frac{|E|+1}{2}$, then $|S|$ would be odd. Also, if $|S| = \dfrac{|E|+1}{2}$, then from (3) $\sum_{x \in E \backslash S} b_x = 0$, and this implies $\forall x \in E \backslash S, b_x = 0$. But if $b_x = 0$, then from 2-a, $2a_x = |S|$ is true, and this is impossible if $|S|$ is odd. Therefore, if $\dfrac{|E|+1}{2}$ is odd, we have $|S| \leq \dfrac{|E|+1}{2}$ and $|S| \neq \dfrac{|E|+1}{2}$ which implies $|S| \leq \frac{|E|-1}{2}$. ■

We now come to the hard part of actually finding an ASET $(E, \cdot)$ and a SET-free subset of $E$ satisfying $|S| = \overline{f}(|E|)$. To do this, we will use the basic theory of round-robins.

## 2.2 The Basic Theory of Round-Robins

First, suppose we have an even number of teams which are numbered $1, 2, 3, \cdots 2n+2$. Each of these teams wishes to play every other team exactly one time. They also wish to play for $2n+1$ consecutive days with $n+1$ games being played on each day, and with each team playing in exactly one game each day. The problem is to draw up a compatible schedule

The complete solution is to first define an arbitrary commutative quasigroup operator $(\{1, 2, 3, \cdots, 2n+1\}, \cdot)$. That is, $\forall a, b \in \{1, 2, 3, \cdots, 2n+1\}$, (1) $ab = ba$ and (2) $\exists$ a unique $x \in \{1, 2, 3, \cdots, 2n+1\}$ such that $ax = b$.

The schedule for teams $\{1, 2, 3, \cdots, 2n+1\} \cup \{2n+2\}$ is defined as follows. $\forall x, y \in \{1, 2, 3, \cdots, 2n+1\}$, if $x \neq y$ then teams $x, y$ play each other on day $x \cdot y$. Also, $\forall x \in \{1, 2, 3, \cdots, 2n+1\}$, team $x$ plays team $2n+2$ on day $x \cdot x$. We will show that the function $x \cdot x : \{1, 2, 3, \cdots, 2n+1\} \to \{1, 2, 3, \cdots, 2n+1\}$ is a bijection after we first give a graphical definition of the round-robin.

Suppose you have a complete undirected graph on the vertices $1, 2, 3, \cdots 2n+1$ and each vertex has a self-loop. Also, you have assigned one of the numbers $1, 2, 3, \cdots, 2n+1$ to each edge of the graph in such a way that all of the numbers $1, 2, 3, \cdots, 2n+1$ are touching each vertex. Suppose $\forall a, b \in \{1, 2, 3, \cdots, 2n+1\}$ we define $a \cdot b$ to be the number on edge $(a, b)$ where edge $(a, a)$ is the self-loop on $a$. Then $(\{1, 2, 3, \cdots, 2n+1\}, \cdot)$ is a commutative quasigroup. Now if the function $x \cdot x$ is not a bijection on $\{1, 2, 3, \cdots, 2n+1\}$, this means that at least one of the numbers $1, 2, 3, \cdots, 2n+1$, (call it $y$) is not assigned to any self-loop. Now exactly one edge having the number $y$ must be touching each vertex. Therefore, the $2n+1$ vertices must be paired off into doubleton sets by the edges assigned the number $y$. But this is impossible since $2n+1$ is odd.

Next, suppose the number of teams is odd, and they are numbered $1, 2, 3, \cdots, 2n+1$. Each team wishes to play every other team exactly one time. They also wish to play for $2n+1$ consecutive days with $n$ games being played on each day and with a different team sitting out on each day. Using an arbitrary commutative quasigroup $(\{1, 2, 3, \cdots, 2n+1\}, \cdot)$, the schedule is computed as follows. $\forall x, y \in \{1, 2, 3, \cdots, 2n+1\}$, if $x \neq y$ then teams $x, y$ play each other on day $x \cdot y$. Also, $\forall x \in \{1, 2, 3, \cdots, 2n+1\}$, team $x$ sits out on day $x \cdot x$. Of course, this is the same as adding an imaginary team $2n+2$ and having team $x$ play an imaginary game on day $x \cdot x$.

## 2.3 The Two Cases

Suppose $E$ is a finite set with $|E|$ odd and $3 \nmid |E| - 2$.

We now consider the two cases that occur in defining $\overline{f}(|E|)$. Case 1 is very easy, but case 2 is rather difficult.

Case 1. $\frac{|E|+1}{2}$ is even. We partition $E = S \cup \overline{E}$ where $|S| = \frac{|E|+1}{2}, \left|\overline{E}\right| = \frac{|E|-1}{2}$. Of course, $|S|$ is even, $|\overline{E}|$ is odd and $|S| = \left|\overline{E}\right| + 1$. We will soon find an ASET $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$ such that $S$ is SET-free, which will solve the problem.

Case 2. $\frac{|E|+1}{2}$ is odd. We partition $E = S \cup \overline{E}$ where $|S| = \frac{|E|-1}{2}, \left|\overline{E}\right| = \frac{|E|+1}{2}$. Of course, $|S|$ is even, $\left(\overline{E}\right)$ is odd and $|S| = \left|\overline{E}\right| - 1$. We will later find an ASET $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$ such that $S$ is SET-free, which will solve the problem.

We can take care of Case 1 almost immediately.

## 2.4 Case 1

We first show that $3 \nmid \left|\overline{E}\right| - 2$. Now $2\left(\left|\overline{E}\right| - 2\right) = 2\left|\overline{E}\right| - 4 = |E| - 1 - 4 = (|E| - 2) - 3$. Therefore, $3 \nmid |E| - 2$ implies $3 \nmid \left|\overline{E}\right| - 2$.

Since $\left|\overline{E}\right|$ is odd and $3 \nmid \left|\overline{E}\right| - 2$, from the theory of Steiner Triples (which we ourselves prove later), $\exists$ an ASET $\left(\overline{E}, \cdot\right)$ on $\overline{E}$.

Let us call $\overline{E} = \left\{1', 2', 3', \cdots, \left|\overline{E}\right|'\right\}$. Also, let us call $S = \left\{1, 2, 3, \cdots, \left|\overline{E}\right| + 1\right\}$ where we look at $S$ as being an even set of teams.

Let $\left(\overline{E}, \cdot\right)$ be any arbitrary ASET on $\overline{E}$. Also, let $(S, \cdot)$ be any arbitrary round-robin on $S$ where we interpret $(S, \cdot)$ as follows. $\forall x, y \in S$, if $x \neq y$ then teams $x, y$ play each other on day $x \cdot y$ where $x \cdot y \in \left\{1, 2, 3, \cdots, \left|\overline{E}\right|\right\}$. Also, $x \cdot x$ is not defined and is not needed.

From the definition of round-robin, we have the following equality of sets. For all $x \in S$, $\left\{x \cdot 1, x \cdot 2, \cdots, x \cdot (x-1), x \cdot (x+1), \cdots, x \cdot \left(\left|\overline{E}\right| + 1\right)\right\} = \left\{1, 2, 3, \cdots, \left|\overline{E}\right|\right\}$.

From the definition of round-robin, we know that $\forall i \in \left\{1, 2, 3, \cdots, \left|\overline{E}\right|\right\}$, the teams $1, 2, 3, \cdots, \left|\overline{E}\right| + 1$ can be paired together into $\dfrac{\left|\overline{E}\right| + 1}{2}$ pairs so that each pair of teams will play each other on day $i$.

Fig. 2. $\frac{\left|\overline{E}\right|+1}{2}$ pairs playing each other on day $i$.

We now define the ASET $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$ as follows where we define the tripleton sets making up $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$.

(1) If $\{x, y, z\} \subseteq \overline{E}$ is any tripleton subset of $\overline{E}$, then $\{x, y, z\}$ is a member of $(E, \cdot)$ if and only if $\{x, y, z\}$ is a member of $\left(\overline{E}, \cdot\right)$.

(2) If $\{x, y\} \subseteq S$ is any doubleton subset of $S$, define $x \cdot y = i$ using the operator $(S, \cdot)$. In other words, teams $x, y$ play each other on day $i$.

We define the tripleton set $\{x, y, i'\}$ to be a member of $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$. Note that (1) and (2) together define a total of $C_2^{|S|} + \frac{1}{3} C_2^{\left|\overline{E}\right|} = C_2^{\frac{|E|+1}{2}} + \frac{1}{3} C_2^{\frac{|E|-1}{2}} = \frac{1}{3} C_2^{|E|}$ tripleton sets which is all the tripleton sets that we need. If $\{a, b\}$ is a doubleton subset of $E = S \cup \overline{E}$, then the 3 cases are (1) $a, b \in S$, (2) $a \in S$, $b \in \overline{E}$, (3) $a, b \in \overline{E}$.

From the definition of the ASET $\left(\overline{E}, \cdot\right)$ and from the definition of the round-robin $(S, \cdot)$ and considering Fig. 2, it is easy to consider these 3 cases to show that each doubleton subset $\{a, b\}$ of $S \cup \overline{E} = E$ is a subset of exactly one of the tripleton sets making up $(E, \cdot)$. Also, it is obvious that $S$ is a SET-free subset of $E$ and $|S| = \overline{f}\left(|E|\right)$.

The rest of the paper deals with case 2 which is the hard part.

## 2.5 Plan to solve case 2

Recall that $|E|$ is odd, $3 \nmid |E| - 2$, and in case 2, $\frac{|E|+1}{2}$ is odd. We partition $E = S \cup \overline{E}$ where $|S| = \frac{|E|-1}{2}$, $\left|\overline{E}\right| = \frac{|E|+1}{2}$. Therefore, $|S|$ is even, $\left|\overline{E}\right|$ is odd and $|S| = \left|\overline{E}\right| - 1$. We first prove that $3 \nmid \left|\overline{E}\right|$. Now $2\left|\overline{E}\right| = |E| + 1 = |E| - 2 + 3$. Therefore, $3 \nmid |E| - 2$ implies $3 \nmid \left|\overline{E}\right|$.

Since $\left|\overline{E}\right| + 2$ is odd and $3 \nmid \left(\left|\overline{E}\right| + 2\right) - 2$, this will allow us later to find a very specialized ASET on $\overline{E} \cup \{A, B\}$. In case 2, we proceed in a way that is roughly analogous to case 1. To solve case 2, we define a hybrid round-robin on $S$ and an incomplete ASET on $\overline{E}$. Analogous to case 1, we then use these two structures to define an ASET on $S \cup \overline{E} = E$ such that $S$ is SET-free. Since $|S| = \overline{f}(|E|)$ this will solve the problem. However, in case 2 no matter how we define the hybrid round-robin on $S$ and the incomplete ASET on $\overline{E}$ there is always a complicated compatibility condition that must be satisfied between these two structures. Using the simplest structures that we can find, we will proceed in the following 4 steps.

(1) We define the hybrid round-robin on $S$,

(2) We conjecture an incomplete ASET on $\overline{E}$ that will satisfy the compatibility condition,

(3) We solve the problem,

(4) We show that our conjectured incomplete ASET on $\overline{E}$ is realizable.

Step 4 is the hardest part of the paper, and some readers might wish to just grant us (4).

## 2.6 The first 3 steps in case 2

It is convenient to call $|S| = 2n, \left|\overline{E}\right| = 2n + 1$ where $3 \nmid 2n + 1$. We also call $S = \{1, 2, 3, \cdots, 2n\}, \overline{E} = \{0', 1', 2', 3', \cdots, (2n)'\}$.

Of course, $|E| = 4n + 1$. Now $|E|$ is odd. Also, $3 \nmid |E| - 2$ since $|E| - 2 = 6n - (2n + 1)$ and $3 \nmid 2n + 1$.

Step 1. We now define the hybrid round-robin on $S = \{1, 2, 3, \cdots, 2n\}$ which we denote $(S, \cdot) = (\{1, 2, \cdots 2n\}, \cdot)$. This hybrid round-robin on $\{1, 2, \cdots, 2n\}$ was obtained by first using a cyclic group to define a regular round-robin on $\{0, 1, 2, \cdots 2n + 1\}$. We then removed the two vertices $0, 1$ and then renamed the vertex $2n + 1$ vertex 1.

Let $\{1, 2, 3, \cdots, 2n\}$ denote the team numbers of $2n$ teams. Also, the days that they play are numbered $0, 1, 2, \cdots 2n$. $(C, 0, +) = (\{0, 1, 2, \cdots 2n\}, 0, +)$ is the cyclic group defined by $x + y \equiv x + y$, (mod $2n + 1$). The hybrid round-robin on $\{1, 2, \cdots, 2n\}$ is defined as follows.

(1) $\forall x, y \in \{1, 2, \cdots, 2n\}$, if $x \neq 1, y \neq 1$ and $x \neq y$, then teams $x, y$ play each other on day $x \cdot y = x + y \equiv x + y$, (mod $2n + 1$). Note that $x \cdot y \in \{0, 1, 2, \cdots 2n\}$.

(2) $\forall x \in \{1, 2, \cdots, 2n\}$, if $x \neq 1$ then teams $1, x$ play each other on day $1 \cdot x = 2x \equiv 2x$, (mod $2n + 1$),

(3) As always, $x \cdot x$ is not defined.

Using $x+y$ and $2x = x+x$, from $(C, 0+)$, we know the following.

(a) Team 1 plays the teams $\{2,3,4,\cdots,2n\}$ on the days $\{1 \cdot x : x = 2,3,4,\cdots,2n\} = \{2x : x = 2,3,4,\cdots,2n\} = \{2x : x = 0,1,2,3,\cdots,2n\} \setminus \{0,2\} = \{0,1,2,3,\cdots,2n\} \setminus \{0,2\}$, since $2n+1$ is odd. Thus, team 1 sits out on days $0, 2$.

(b) Also, $\forall i \in \{2,3,\cdots 2n\}$, team $i$ plays the teams $\{1,2,3,\cdots i-1, i+1,\cdots,2n\}$ on the days $\{i \cdot 1\} \cup \{i \cdot x : x = 2,3,\cdots,i-1,i+1,\cdots,2n\} = \{2i\} \cup \{i+x : x = 2,3,\cdots,i-1,i+$ $\{i+x : x = 0,1,2,3,\cdots,2n\} \setminus \{i,i+1\} = \{0,1,2,3,\cdots,2n\} \setminus \{i,i+1\}$. Thus, team $i$ sits out on days $i, i+1$.

Note that each team sits out two days, and the two days that each of the teams $1,2,3,\cdots,2n$ do not play is given in Fig. 3. We observe that each team plays on day 1. However, two teams sit out on each of the other days $0,2,3,4,\cdots,2n$. Also, observe that the $2n$ teams sit out 2 days according to one big $2n$-cycle, and this is as simple as we have been able to make compatibility problem.

Fig. 3. We later call $0 = 1', 1 = 0', i = i', i = 2,3,\cdots,2n$.

Thus on day 1 the $2n$ teams are paired into $n$ pairs so that each pair plays each other on day 1.

Also, on each of the other days $i \in \{0,2,3,4,\cdots,2n\}$ the $2n$ teams are paired into $n-1$ pairs so that each pair plays each other on day $i$ and the remaining two teams sit out on day $i$.

Step 2. Next, we will conjecture an incomplete ASET on $\overline{E} = \{0', 1', 2', \cdots, (2n)'\}$ which we call $(\overline{E}, \cdot)$ that has the following properties. As always, we can define $(\overline{E}, \cdot)$ as a collection of distinct tripleton subsets of $\overline{E}$.

First, we agree that none of the $2n$ doubleton sets in $(*)$ will be a subset of any tripleton set in $(\overline{E}, \cdot)$.

$$(*)\, \{1', 2'\}, \{2', 3'\}, \{3', 4'\} \cdots \{(2n-1)', (2n)'\}, \{(2n)', 1'\}.$$

For compatibility between $(S, \cdot)$ and $(\overline{E}, \cdot)$, we note that the $2n$ doubleton sets in $(*)$ are arranged in a $2n$-cycle. Also, note that we are left with $C_2^{2n+1} - 2n = (2n+1)(n) - 2n = n(2n-1)$ doubleton subsets of $\overline{E}$ that we have to deal with.

Since $3 \nmid 2n+1$ we know that $3 | (2n)(2n-1)$ which implies $3 | n(2n-1)$. Therefore, in step 4, we will be able to specify $\dfrac{n(2n-1)}{3}$ tripleton sets that make up the incomplete ASET $\left(\overline{E}, \cdot\right)$ so that (1) and (2) are true.

(1) None of the $2n$ doubleton sets in $(*)$ is a subset of any tripleton set in $\left(\overline{E}, \cdot\right)$.

(2) Each of the $n(2n-1)$ remaining doubleton subsets of $\overline{E}$ is a subset of exactly one tripleton set in $\left(\overline{E}, \cdot\right)$

Note that (1), (2) implies that each doubleton subset $\{0', b'\}$ of $\overline{E}$ is a subset of exactly one tripleton set in $\left(\overline{E}, \cdot\right)$ since $0'$ does not appear in $(*)$.

Step 3. Before we specify the tripleton sets that make up $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$, we must do the following. In the hybrid round-robin $(S, \cdot) = (\{1, 2, 3, \cdots, 2n\}, \cdot)$ defined in step 1, the teams are numbered $1, 2, \cdots, 2n$ and they play on days $0, 1, 2, 3, \cdots, 2n$. However, for compatibility between $(S, \cdot)$ and $\left(\overline{E}, \cdot\right)$, let us now rename the days that they play as follows.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $\cdots$ | $2n$ |
| $1'$ | $0'$ | $2'$ | $3'$ | $4'$ | $5'$ | $6'$ | $\cdots$ | $(2n)'$ |

From Fig. 3 this means that on day $0'$ all of the teams play, but on each of the other days $1', 2', \cdots, (2n)'$, two teams sit out, and this pattern defines one big $2n$-cycle.

The tripleton sets of $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$ are defined as follows.

(1′) If $\{x, y, z\} \subseteq \overline{E}$ is any tripleton subset of $\overline{E}$, then $\{x, y, z\}$ is a member of $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$ if and only if $\{x, y, z\}$ is a member of $\left(\overline{E}, \cdot\right)$. Of course, this gives a total of $\dfrac{n(2n-1)}{3}$ tripleton sets.

(2′) If $\{x, y\} \subseteq S$ is any doubleton subset of $S$, define $x \cdot y = i'$ using the operator $(S, \cdot)$. In other words, teams $x, y$ play each other on day $i'$ where we call $0 = 1', 1 = 0', i = i', i = 2, 3, \cdots, 2n$. We define $\{x, y, i'\}$ to be a member of $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$. This gives a total of $C_2^{|S|} = n(2n-1)$ tripleton sets.

(3′) From the definition of $\left(\overline{E}, \cdot\right)$, we know that none of the $2n$ doubleton sets $(*)$ $\{1', 2'\}$ $\{2', 3'\}, \{3', 4'\}, \cdots$ is a member of any tripleton set $\{x, y, z\}$ that makes up $\left(\overline{E}, \cdot\right)$. Also, from Fig. 3 (after calling $0 = 1', 1 = 0', i = i', i = 2, 3, \cdots, 2n$) we know that none of the following doubleton sets is a subset of any tripleton set that we have defined in step (1) and step (2): $\{1, 1'\}, \{1, 2'\}$ and $\{2, 2'\}, \{2, 3'\}$ and $\{3, 3'\}, \{3, 4'\}$ and $\{4, 4'\}, \{4, 5'\}$ and $\cdots$, and $\{2n, (2n)'\}, \{2n, 1'\}$. This gives a total of $2n + 2 \cdot 2n = 6n$ doubleton subsets of $S \cup \overline{E}$ that we have not dealt with. We now put the following tripleton sets in $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot) : \{1', 2', 1\}, \{2', 3', 2\}, \{3', 4', 3\}, \{4', 5', 4\}, \cdots \{(2n)', 1', 2n\}$. This gives a total of $2n$ tripleton sets, and this takes care of the remaining $6n$ doubleton sets.

The total number of tripleton sets from $(1'), (2')\,(3')$ equals $\frac{n(2n-1)}{3} + n(2n-1) + 2n = \frac{2n(4n+1)}{3}$. Since $|E| = |S| + \left|\overline{E}\right| = 4n+1$, the number of tripleton sets making up $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$ must equal $\frac{1}{3}C_2^{4n+1} = \frac{2n(4n+1)}{3}$.

If $\{a, b\}$ is any doubleton subset of $E = S \cup \overline{E}$, then the 3 cases are (1) $a, b \in S$, (2) $a \in S, b \in \overline{E}$, (3) $a, b \in \overline{E}$. From the definitions of $(S, \cdot), \left(\overline{E}, \cdot\right)$ and $\left(S \cup \overline{E}, \cdot\right) = (E, \cdot)$ that we have used, it is easy to consider these 3 cases to show that each doubleton subset $\{a, b\}$ of $S \cup \overline{E} = E$ is a subset of exactly one of the tripleton sets making up $(E, \cdot)$. Also, it is obvious that $S$ is SET-free and $|S| = \overline{f}(|E|)$.

The rest of this paper deals with step 4 in which we show that the conjectured incomplete ASET $\left(\overline{E}, \cdot\right)$ that we specified in step 2 actually exists. Showing this is the hardest part of the paper.

### 2.7 Finishing the solution (step 4)

We finish the solution by solving two problems.

In problem 1 we show that an ASET $(E, \cdot)$ exists on $E$ when $|E| = 2n+3$ and $3 \nmid 2n+1$. We represent $E$ by $E = \{0, 1, 2, \cdots, 2n\} \cup \{A, B\}$.

In problem 2 we use the solution to problem 1 to construct a very specialized ASET $(E, \cdot)$ on $E$ when $|E| = 2n+3$ $3 \nmid 2n+1$. Immediately after stating problem 2, we show how the solution, $(E, \cdot)$, to problem 2 can easily be used to show that the conjectured incomplete ASET $\left(\overline{E}, \cdot\right)$ of step 2, section 2.6 actually exists.

**Problem 1**

Suppose $E$ is a finite set, $|E|$ is odd and $3 \nmid |E| - 2$. Show that an ASET $(E, \cdot)$ exists on $E$.

**Solution**

Since $|E| \in \{1, 3\}$ is trivial, let us assume that $|E| \geq 7$. Therefore, since $|E| \geq 7$, we can write $|E| = (2n+1) + 2$ where $3 \nmid 2n+1$ and $2n+1 \geq 5$.

Let us represent $E$ as $E = \{0, 1, 2, \cdots, 2n\} \cup \{A, B\}$. What we now wish to do is define $\frac{1}{3}C_2^{2n+3}$ distinct tripleton subsets $\{x, y, z\}$ of $E$ such that $(**)$ is true. $(**)$. Each distinct doubleton subset $\{a, b\}$ of $E$ is a subset of exactly one $\{x, y, z\}$. However, if we define a collection of tripleton subsets $\{x, y, z\}$ of $E$ that has this property $(**)$, we do not need to prove that the number of these will be $\dfrac{1}{3}C_2^{2n+3}$. This will be automatic.

As always, let $(C, 0, +) = (\{0, 1, 2, \cdots 2n\}, 0, +)$ be the cyclic group defined by $\forall a, b \in \{0, 1, 2, \cdots, 2n\}, a + b \equiv a + b (\text{mod } 2n+1)$. We now define an incomplete binary operator $(C, \odot)$ as follows.

(1) $\forall x \in C, x \odot x = x$.

(2) $\forall x, y \in C$, if $x \neq y$ and $\{x, y, -x-y\}$ is a tripleton set, then we define $x \odot y = y \odot x = -x - y$.

(3) $\forall x, y \in C$, if $x \neq y$ and $\{x, y, -x-y\}$ is not a tripleton set, then $x \odot y$ and $y \odot x$ are left undefined.

Of course, $(C, \odot)$ is idempotent everywhere, and it is commutative when it is defined. Also, $\forall x, y \in C$, if $x \neq y$ and $x \odot y$ is defined, then the $\odot$ product of any two members of the tripleton set $\{x, y, x \odot y = -x-y\}$ is defined and equals the third member of the set.

For example, $x \odot (x \odot y) = -x - (x \odot y) = -x - (-x-y) = y$.

$\forall x, y \in C$ we observe that $\{x, y, -x-y\}$ is a tripleton set if and only if (1) $x \neq y$, (2) $y \neq -2x$ and (3) $x \neq -2y$. Of course, by symmetry the three conditions (1), (2) and (3) are equivalent to the three conditions $x \neq -x-y, -x-y \neq -2x$ and $x \neq -2(-x-y)$.

Also, by symmetry (1), (2) and (3) are equivalent to the three conditions $y \neq -x-y, -x-y \neq -2y$ and $y \neq -2(-x-y)$.

We also observe that if $x = 0$ and $y \neq 0$ then $\{x, y, -x-y\} = \{0, y, -y\}$ is always a tripleton set since $y \neq 0, -y \neq 0$ and since $y = -y$ implies $2y \equiv 0 (\text{mod } 2n+1)$ which is impossible. Therefore, we know that $\forall x \in C, x \odot x$ and $0 \odot x$ are always defined. Also, $\forall x, y \in C$, if $x \neq 0, y \neq 0$ then $x \odot y$ is not defined if and only if (1) $x \neq y$ and $y = -2x$ or (2) $x \neq y$ and $x = -2y$.

Perhaps a better way of looking at what we have done is as follows. First, we define all distinct tripleton subsets $\{x, y, z\}$ of $C$ that satisfy $x + y + z \equiv 0 (\text{mod } 2n+1)$. Thus if $\{x, y, z\}, \{\overline{x}, \overline{y}, \overline{z}\}$ are any two such distinct tripleton subsets of $C$, then $\{x, y, z\} \cap \{\overline{x}, \overline{y}, \overline{z}\}$ is either empty or singleton. Also, if $\{a, b\}$ is any doubleton subset of $C$ satisfying $a \neq -2b$ and $b \neq -2a$, then $\{a, b\}$ is a subset of exactly one $\{x, y, z\}$ namely $\{x, y, z\} = \{a, b, a \odot b = -a-b\}$.

Since $E = C \cup \{A, B\}$, we observe that up to now we have not defined any tripleton sets $\{x, y, z\}$ that contain either $A$ or $B$. Therefore, all doubleton subsets of $E$ containing $A$ and/or $B$ must be worked in. Also, if $\{a, b\}$ is any doubleton subset of $C$ we have not defined a tripleton set $\{x, y, z\}$ having $\{a, b\}$ as a subset when either $a = -2b$ or $b = -2a$. With this in mind, we now plan to extend the incomplete binary operator $(C, \odot)$ to define the remaining tripleton subsets of $E = C \cup \{A, B\}$. This will give us the ASET $(E, \cdot) = (C \cup \{A, B\}, \cdot)$ that we wish to find.

Let us now define the function $f : C \setminus \{0\} \to C \setminus \{0\}$ as follows.

$\forall x \in C \setminus \{0\}, f(x) = -x - x \equiv -2x, (\text{mod } 2n+1)$. Since $2n+1$ is odd, we see that $f$ is a bijection on $C \setminus \{0\} = \{1, 2, 3, \cdots, 2n\}$. This means that $f$ can be partitioned into the union of pairwise disjoint cycles $C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_t$ where each $C_i = \left\{x_{i1}, x_{i2}, x_{i3}, \cdots, x_{i,k(i)}\right\}$ with $-2x_{i\theta} = x_{i,\theta+1}$ when $\theta = 1, 2, \cdots, k(i) - 1$ and $-2x_{i,k(i)} = x_{i1}$. For convenience, we are going to call $k(i) = k$ since this omission of $i$ will cause no confusion. Of course, $-C_i = \{-x_{i1}, -x_{i2}, -x_{i3}, \cdots - x_{ik}\}$ is also a cycle since $(-2)(-x_{i\theta}) = -x_{i,\theta+1}$ is true if and only if $-2x_{i\theta} = x_{i,\theta+1}$. This implies $\forall i \in \{1, 2, \cdots, t\}, C_i = -C_i$ or $C_i \cap (-C_i) = \phi$. Since

$\forall x \in C\backslash\{0\}, -x \neq x$ and $-(-x) = x$, we see that if $C_i = -C_i$, then $|C_i|$ must be even. Therefore, if $|C_i|$ is odd then $C_i \cap (-C_i) = \phi$.

We now show that $\forall x \in C\backslash\{0\}$, all four elements of the set $\left\{x, -2x, (-2)^2 x, (-2)^3 x\right\} = \{x, -2x, 4x, -8x\}$ are distinct.

This is true if and only if $x \neq -2x,\ x \neq 4x,\ x \neq -8x,\ -2x \neq 4x,\ -2x = -8x$ and $4x \neq -8x$ which is equivalent to the four conditions $3x \not\equiv 0 (\text{mod } 2n+1),\ 9x \not\equiv 0 (\text{mod } 2n+1),\ 6x \not\equiv 0 (\text{mod } 2n+1)$ and $12x \not\equiv 0 (\text{mod } 2n+1)$.

Since $2 \nmid 2n+1$ and $3 \nmid 2n+1$, these 4 conditions are obviously true since $x \in C\backslash\{0\}$. This implies $\forall i \in \{1, 2, \cdots, t\}, |C_i| \geq 4$. In Fig. 4-a we have drawn a cycle $C_i$ to illustrate the case where $|C_i|$ is even. In Fig. 4-b, we have drawn two cycles $C_j, -C_j$ to illustrate the case where $|C_j|$ is odd which implies $C_j \cap (-C_j) = \phi$. Given such a pair $\{C_j, -C_j\}$ we have arbitrarily chosen one of $\{C_j, -C_j\}$, namely $C_j$, to be the top cycle and the other, namely $-C_j$, to be the bottom cycle in the drawing.

In drawing (b) we know that $|C_j| = |-C_j| \geq 5$ since $|C_j|$ is odd and $|C_j| \geq 4$.

In Fig. 4 we have also drawn the three elements $0, A, B$. Technically we should use $k = k(i)$ in (a), which gives $|C_i| = k(i)$, and use $k = k(j)$ in (b), which gives $|C_i| = |-C_j| = k(j)$. But again this technical omission should cause no confusion.

Fig. 4. $|C_i|$ even, $|C_j| = |-C_j|$ odd, $x_{i1} \in C_i, x_{j1} \in C_j$ fixed.

Of course, in Fig. 4, $x_{i1} \in C_i, x_{j1} \in C_j$ can be arbitrarily chosen. However, we assume that we have chosen a fixed $x_{i1} \in C_i$ for each $C_i$ such that $|C_i|$ is even. Also, for each pair $\{C_j, -C_j\}$ where $|C_j| = |-C_j|$ is odd and $C_j$ is chosen to be the top cycle and $-C_j$ is chosen to be the bottom cycle, we assume that we have chosen a fixed $x_{j1} \in C_j$. In Fig. 4, it is convenient to define $x_{i,k+1} = x_{i1}$ and $x_{j,k+1} = x_{j1}$. $\forall C_i$ and $\forall C_j \cup (-C_j)$ illustrated in Fig. 4, we note that none of the doubleton sets $\{x_{i\theta}, x_{i,\theta+1}\}, \theta = 1, 2, 3, \cdots$ and none of the doubleton sets $\{x_{j\theta}, x_{j,\theta+1}\}, \{-x_{j\theta}, -x_{j,\theta+1}\}, \theta = 1, 2, 3, \cdots$ are subsets of any of the tripleton sets that we have defined up to now. This is because $-2x_{i\theta} = x_{i,\theta+1}, -2x_{j\theta} = x_{j,\theta+1}$ and $-2(-x_{j\theta}) = x_{j,\theta+1}$.

We now see the incomplete binary operator $(C, \odot) = (\{0, 1, 2, \cdots, 2n\}, \odot)$ to define the ASET $(E, \cdot) = (C \cup \{A, B\}, \cdot)$ in the steps 1, 2, 3,4, 5, 6 that follow.

(1) $\forall x, y \in C = \{0, 1, 2, \cdots 2n\}, x \cdot y = x \odot y$ if $x \odot y$ is defined in $(C, \odot)$ with the following exceptions. Suppose $\{C_j, -C_j\}$ is any pair satisfying $|C_j| = |-C_j|$ is odd with $C_j$ the top cycle and $-C_j$ the bottom cycle of such a pair as illustrated in Fig. 4-b.

As always, we wrote $C_j = \{x_{j1}, x_{j2}, \cdots, x_{jk}\}$ where $x_{j1}$ has been chosen and where technically we should call $k = k(j)$. Also, $-C_j = \{-x_{j1}, -x_{j2}, \cdots, -x_{jk}\}$. Of course, $x_{j,\theta+1} = -2x_{j\theta}$ when $\theta = 1, 2, \cdots, k$.

For all such pairs $\{C_j, -C_j\}$ with $x_{j1} \in C_j$ chosen for each pair, we now agree that none of the following have been defined in $(E, \cdot) : 0 \cdot x_{j1}, 0 \cdot (-x_{j1})$ and $x_{j1} \cdot (-x_{j1})$. In other words, we assume that the tripleton set $\{0, x_{j1}, -x_{j1}\}$ is not used in $(E, \cdot)$.

Of course, without this agreement, we would have $0 \cdot x_{j1} = -x_{j1},\ 0 \cdot (-x_{j1}) = x_{j1}$ and $x_{j1} \cdot (-x_{j1}) = 0$ in $(E, \cdot)$.

By symmetry we also agree that none of the following have been defined in $(E, \cdot)$ : $0 \cdot x_{jk}, 0 \cdot (-x_{jk})$ and $x_{jk} \cdot (-x_{jk})$. In other words, we assume that the tripleton set $\{0, x_{jk}, -x_{jk}\}$ is not used in $(E, \cdot)$.

Steps 2-6 will come after the following discussion.

Let $C_i$ be any cycle such that $|C_i|$ is even. Of course, $|C_i| \geq 4$. In Fig. 4-a, we have drawn the directed edges of $C_i$ which are $x_{i1} \to x_{i2} \to x_{i3} \to \cdots\ x_{ik} \to x_{i1}$ where $k \geq 4$ and $k$ is even. Since $k$ is even, we are now able to alternate coloring these directed edges dark and light as we have illustrated in Fig. 4-a. Of course, this can be done in two different ways, and we choose one of these 2 ways for each cycle that satisfies $|C_i|$ is even.

Let $C_j$ be the top cycle and $-C_j$ be the bottom cycle in Fig. 4-b of any pair $\{C_j, -C_j\}$ such that $|C_j| = |-C_j|$ is odd. Also, $x_{j1} \in C_j$ has been chosen.

In Fig. 4-b, we have drawn the following directed edges where some are backwards in $(***)$ which is of no concern:

$(***)$ $x_{j1} \to x_{j2} \to x_{j3} \to \cdots \to x_{j,k-1} \to x_{jk} \to -x_{jk} \leftarrow -x_{j,k-1} \leftarrow -x_{j,k-2} \leftarrow \cdots \leftarrow -x_{j3} \leftarrow -x_{j2} \leftarrow -x_{j1} \to x_{j1}$.

Note that the two directed edges $x_{j1} \leftarrow x_{jk}$ and $-x_{jk} \to -x_{j1}$ have been $\times$'ed out in Fig. 4-b and are not used.

The number of directed edges in $(***)$ is even so we are now able to alternate coloring these directed edges dark and light as we have illustrated in Fig. 4-b. This also can be done in two different ways, and we choose one of these 2 ways for each $\{C_j, -C_j\}$ that satisfies $|C_j| = |-C_j|$ is odd. Again we emphasize that the edges $x_{j1} \leftarrow x_{jk}$ and $-x_{jk} \to -x_{j1}$ have not been colored. Step 1 started our definition of $(E, \cdot)$. In steps 2, 3, ..., 6, we complete the definition of $(E, \cdot)$ by defining the additional tripleton sets that we need. We use Fig. 4 to do this where in Fig. 4 $C_i$ is any arbitrary cycle satisfying $|C_i|$ is even and $\{C_j, -C_j\}$ is any arbitrary pair of cycles satisfying $|C_j| = |-C_j|$ is odd and where one of $\{C_j, -C_j\}$ is chosen to be the top cycle. Also, $x_{i1} \in C_i, x_{j1} \in C_j$ are fixed.

(2) We use $(0, A, B)$ in $(E, \cdot)$. Thus, $0 \cdot A = B, 0 \cdot B = A, A \cdot B = 0$.

(3) From each $C_j \cup (-C_j)$ we use the tripleton sets $\{0, x_{j1}, x_{jk}\}$ and $\{0, -x_{j1}, -x_{jk}\}$ in $(E, \cdot)$. Thus, $0 \cdot x_{j1} = x_{jk}$, etc.

Recall in step (1) that the tripleton sets $\{0, x_{j1}, -x_{j1}\}$ and $\{0, x_{jk}, -x_{jk}\}$ were not to be used in $(E, \cdot)$.

(4) In the $C_i$ drawing of Fig. 4-a of an arbitrary $C_i$ with $|C_i|$ even, if the directed edge $x_{i\theta} \rightarrow x_{i,\theta+1}$ is colored dark, where $\theta = 1, 2, \cdots, k$, we use the tripleton set $\{x_{i\theta}, x_{i,\theta+1}, B\}$ in $(E, \cdot)$. Of course, this applies to $x_{ik} \rightarrow x_{i1}$ since we are calling $x_{i,k+1} = x_{i1}$.

If the directed edge $x_{i\theta} \rightarrow x_{i,\theta+1}$ is colored light, we use the tripleton set $\{x_{i\theta}, x_{i,\theta+1}, A\}$ in $(E, \cdot)$.

(5) In the $C_j \cup (-C_j)$ drawing in Fig. 4-b of an arbitrary $C_j \cup (-C_j)$ with $|C_j|$ odd, we use the alternating dark and light coloring for the edges in the sequence $(***)$ that was defined earlier. Suppose $x_\theta, x_\phi \in C_j \cup (-C_j)$ and $x_\theta \rightarrow x_\phi$ or $x_\theta \leftarrow x_\phi$ in the sequence $(***)$.

If this directed edge $x_\theta \rightarrow x_\phi$ (or $x_\theta \leftarrow x_\phi$) is colored dark, we use the tripleton set $\{x_\theta, x_\phi, B\}$ in $(E, \cdot)$.

If this directed edge $x_\theta \rightarrow x_\phi$ (or $x_\theta \leftarrow x_\phi$) is colored light, we use the tripleton set $\{x_\theta, x_\phi, A\}$ in $(E, \cdot)$.

(6) Finally, for completeness, $\forall x \in E$, we specify that $x \cdot x = x$.

Steps 1-6 give the complete definition of all tripleton sets that make up $(E, \cdot)$ as well as $x \cdot x = x$.

To show that this collection of tripleton sets $(E, \cdot)$ is indeed an ASET, we need to show that if $\{a, b\}$ is any doubleton subset of $E = C \cup \{A, B\}$, then $\{a, b\}$ is a subset of exactly one of the tripleton subsets $\{x, y, z\}$ that we have defined.

To see this we go through the construction that we have given for $\{a, b, a \cdot b\}$ when the doubleton subset $\{a, b\}$ of $E$ lies in each of the following 5 cases.

**Case 1.** $\{a, b\} \subseteq C \setminus \{0\}$ with $a \not\equiv -2b \pmod{2n+1}$ and $b \not\equiv -2a \pmod{2n+1}$. This uses step 1 and the part of step 5 that fills in the gap caused by the omitted tripleton sets in step 1.

**Case 2.** $\{a, b\} \subseteq C \setminus \{0\}$ with $a \equiv -2b \pmod{2n+1}$ or $b \equiv -2a \pmod{2n+1}$. This uses steps 3-5.

**Case 3.** $\{a, b\} = \{0, b\}$ with $b \in C \setminus \{0\}$. This uses step 1 and step 3 to fill in the gap caused by the omitted tripleton sets in step 1.

**Case 4.** $\{a, b\} = \{A, b\}$ with $b \in C \setminus \{0\}$ or $\{a, b\} = \{B, b\}$ with $b \in C \setminus \{0\}$. This uses steps 4, 5.

**Case 5.** $\{a,b\}=\{0,A\}$ or $\{a,b\}=\{0,B\}$ or $\{a,b\}=\{A,B\}$. This uses step 2. ■

We will soon tamper with the ASET $(E,\cdot)$ that we just constructed to create a more specialized ASET $(E,\cdot)$ that satisfies the conditions required in Problem 2.

**Lemma 7** *Suppose $(E,\cdot)$ is a finite ASET. $\forall a,b\in E$, if $a\neq b$ then we can partition $E$ into pairwise disjoint sets $E=\overline{C}_0\cup\overline{C}_1\cup\overline{C}_2\cup\cdots\cup\overline{C}_t$ such that $\overline{C}_0,\overline{C}_1,\cdots,\overline{C}_t$ have the following properties.*

*(1)* $\overline{C}_0=\{a,b,ab\}$.

*(2)* $\forall i\in\{1,2,\cdots t\}$, *we can write* $\overline{C}_i=\left\{x_{i1},x_{i2},\cdots,x_{i,2k(i)}\right\}$ *where* $2k(i)\geq 4$ *and where* $\forall\theta\in\{1,2,\cdots,2k(i)\}$, *the following is true.*

*(a)* *If $\theta$ is odd,* $x_{i\theta}\cdot x_{i,\theta+1}=b$.

*(b)* *If $\theta$ is even,* $x_{i\theta}\cdot x_{i,\theta+1}=a$ *where we define* $x_{i,2k(i)+1}=x_{i1}$.

*Fig. 5.* $\left|\overline{C}_0\right|=3$ *and* $\forall i\geq 1,\left|\overline{C}_i\right|\geq 4$ *and even.*

**Proof.** First consider $E\backslash\overline{C}_0$. Choose any $x_{1,1}\in E\backslash\overline{C}_0$. Define $x_{1,2}=x_{1,1}\cdot b,\quad x_{1,3}=x_{1,2}\cdot a,\ x_{1,4}=x_{1,3}\cdot b,\ x_{1,5}=x_{1,4}\cdot a\cdots$. Using the properties of $(E,\cdot)$ it is fairly easy to see that $\overline{C}_1$ forms a cycle, $\left|\overline{C}_1\right|$ is even and $\left|\overline{C}_1\right|\geq 4$.

Next choose any $x_{2,1}\in E\backslash\left(\overline{C}_0\cup\overline{C}_1\right)$ and do the same thing, etc. ■

**Observation 2**. $(E,\cdot)$ is a finite ASET. Choosing any $a,b\in E,a\neq b$, let us partition $E=\overline{C}_0\cup\overline{C}_1\cup\overline{C}_2\cup\cdots\cup\overline{C}_t$ as in Lemma 7. Let us now pick out a specific $\overline{C}_i=\left\{x_{i1},x_{i,2},\cdots,x_{i,2k(i)}\right\}$.

It follows from Lemma 7 that

A. $\{x_{i\theta},x_{i,\theta+1},b\},\theta=1,3,5,\cdots,2k(i)-1$ and $\{x_{i\theta},x_{i,\theta+1},a\},\theta=2,4,6,\cdots,2k(i)$ are tripleton sets used in $(E,\cdot)$ where $x_{i,2k(i)+1}=x_{i1}$.

Suppose we interchange $a$ and $b$ in $\overline{C}_i$. That is, for the above tripleton sets A we substitute in $(E,\cdot)$ the new tripleton sets given in B.

B. $\{x_{i\theta},x_{i,\theta+1},a\},\theta=1,3,5,\cdots,2k(i)-1$ and $\{x_{i\theta},x_{i,\theta+1},b\},\theta=2,4,6,\cdots,2k(i)$.

After we make this substitution and keep the other tripleton sets in $(E,\cdot)$ the same, it is fairly obvious that the new collection of tripleton sets that results is also an ASET on $E$. This follows since all of the doubleton subsets appearing in the tripleton sets in A also appear in the tripleton sets in B.

**Problem 2** Suppose $E$ is a finite set with $|E| \geq 7$. Also, $|E| = 2n+3$ and $3 \nmid 2n+1$. Therefore, $2n+1 \geq 5$.

We wish to construct an ASET $(E, \cdot)$ that has the following properties. $\exists a, b \in E, a \neq b$, such that the partition of $E$ defined in lemma 7 for $a, b$ has only two sets namely $E = \overline{C}_0 \cup \overline{C}_1$. Of course, $\left|\overline{C}_0\right| = 3, \left|\overline{C}_1\right| = 2n$.

**Step. 4**. Before solving problem 2, we show that the solution $(E, \cdot)$ of problem 2 easily solves the conjectured incomplete ASET $\left(\overline{E}, \cdot\right)$ conjectured in step 2, section 2.6. Let us define $\overline{E} = E \backslash \{a, b\} = \{ab\} \cup \overline{C}_1 = \{ab\} \cup \{x_{11}, x_{12}, x_{13}, \cdots x_{1,2n}\}$.

We define an incomplete ASET $\left(\overline{E}, \cdot\right)$ on $\overline{E}$ as follows. If $\{x, y, z\}$ is any tripleton subset of $\overline{E}$, then $\{x, y, z\}$ is a member of $\overline{E}$ if and only if $\{x, y, z\}$ is a member of $(E, \cdot)$. In other words, $\left(\overline{E}, \cdot\right)$ keeps those tripleton set $\{x, y, z\}$ of $(E, \cdot)$ that satisfy $\{x, y, z\} \cap \{a, b\} = \phi$ and throws away those tripleton sets $\{x, y, z\}$ of $(E, \cdot)$ that satisfy $\{x, y, z\} \cap \{a, b\} \neq \phi$.

Of course, from the notation of lemma 7, this means that $\{x_{11}, x_{12}, b\}, \{x_{12}, x_{13}, a\}, \{x_{13}, x_{14}, b\}, \{x_{14}, x_{1}$ have been thrown away. Also, if we now call $0' = ab, 1' = x_{11}, 2' = x_{12}, 3' = x_{13}, \cdots, (2n)' = x_{1,2n}$, then it is obvious that none of the $2n$ doubleton sets $(*)$. $\{1', 2'\}, \{2', 3'\}, \{3', 4'\}, \cdots, \{(2n-1)', (2n)$ will be a subset of any tripleton set in $\left(\overline{E}, \cdot\right)$. Also, all of the remaining doubleton subsets $\{a, b\}$ of $\overline{E}$ will be a subset of a unique tripleton set $\{x, y, z\}$ in $\left(\overline{E}, \cdot\right)$. ■

**Solution to Problem 2**. The plan is to first construct an ASET $(E, \cdot)$ such that $\exists a, b \in E, a \neq b$, such that the partition of $E$ defined by lemma 7 for $a, b$ satisfies (a) or (b).

1. $E = \overline{C}_0 \cup \overline{C}_1$ with $\left|\overline{C}_0\right| = 3, \left|\overline{C}_1\right| = 2n$ or
2. $E = \overline{C}_0 \cup \overline{C}_1 \cup \overline{C}_2$ with $\left|\overline{C}_0\right| = 3, \left|\overline{C}_1\right| = \left|\overline{C}_2\right| = n$.

Of course (a) solves the problem. In (b) we show that we can slightly modify the construction of $(E, \cdot)$ to obtain a final $(E, \cdot)$ such that the partition of $E$ of lemma 7 for $a, b$ is $E = \overline{C}_0 \cup \overline{C}_1^*$ where $\left|\overline{C}_0\right| = 3, \left|\overline{C}_1^*\right| = 2n$. Of course, $\overline{C}_0, \overline{C}_1^*$ solves the problem.

To construct $(E, \cdot)$ we call $E = \{0, 1, 2, \cdots, 2n\} \cup \{A, B\}$ and then use the same construction that was used to solve problem 1. In that construction, each $|C_i|$ of Fig. 4-a was even with $|C_i| \geq 4$. Also, each $|C_j| = |-C_j|$ of Fig. 4-b was odd with $|C_j| = |-C_j| \geq 5$. Also, for each pair $\{C_j, -C_j\}$ we arbitrary chose $C_j$ to be the top and $-C_j$ to be the bottom in Fig. 4-b.

In that construction, for each $C_j \cup (-C_j)$ we need to place one restriction on the $x_{j1} \in C_j$ that we choose. We agree to choose $x_{j1} \in C_j$ so that $(****)$ $\{x_{j1}, -x_{j1}, x_{jk}, -x_{jk}\} \cap \{-1, 1\} = \phi$ where $-1 = (2n+1) - 1 = 2n$ in $(C, 0, +) = (\{0, 1, 2, \cdots, 2n\}, 0, +)$. We can do this since $|C_j| \geq 5 > 3$, and we note that $|C_j| \geq 3$ would be sufficient to do this since if one of $-1, 1$ appears in $C_j$, then the other must appear in $-C_j$.

Let us partition $E = \{0, -1, 1\} \cup \{2, 3, 4, \cdots, 2n-1, A, B\}$.

From restriction $(****)$ and from step 1 in the construction of $(E,\cdot)$ in problem 1, we know that $(-1)\cdot(1) = -(-1)-1 = 0$ in $(E,\cdot)$. This implies that $\{0,-1,1\}$ is one of the tripleton sets making up $(E,\cdot)$.

Let us call $-A = B, -B = A$. Using $(C,0,+) = (\{0,1,2,\cdots,2n\},0,+)$, we now write

$$\begin{aligned}\{1,2,3,4,\cdots,2n\} &= \{1,2,3,\cdots,n\}\cup\{n+1,n+2,\cdots,2n\}\\ &= \{1,2,3,\cdots,n\}\cup\{-1,-2,-3,\cdots,-n\}\\ &= \{1,-1\}\cup\{2,-2\}\cup\{3,-3\}\cup\cdots\{n,-n\}.\end{aligned}$$

Since each $|C_j| = |-C_j| \geq 5$ in Fig. 4-b, it follows from step 1 in the construction of $(E,\cdot)$ in problem 1 that if $n$ is even then at least $\frac{n}{2}+1$ of the doubleton sets $\{i,-i\}, i = 1,2,\cdots,n$, must have the property that $i\cdot(-i) = 0$ in $(E,\cdot)$. Therefore, if $n$ is even then at least $\frac{n}{2}+1$ of the tripleton sets $\{-i,i,0\}, i = 1,2,3,\cdots,n$, are used in $(E,\cdot)$.

Recall that $\{A,B,0\}$ is a tripleton set used in $(E,\cdot)$. Since $\{A,-A\} = \{B,-B\} = \{A,B\}$, if we substitute $\{A,-A\}$ for $\{-1,1\}$ then when $n$ is even it follows that at least $\frac{n}{2}+1$ of the tripleton sets $\{-i,i,0\}, i = 2,3,4,\cdots,n,A$, are used in $(E,\cdot)$.

For the $(E,\cdot)$ that we have now specified, let us define $a = -1, b = 1$. Therefore, from lemma 7, $\overline{C}_0 = \{-1,1,0\}$ since the tripleton set $\{-1,1,0\}$ is used in $(E,\cdot)$.

In Figures 6 and 7 we have drawn two cases that we must carefully study. In Fig. 6, $n$ is even, and we call this case 1. In Fig. 7, $n$ is odd, and we call this case 2.

Fig. 6. Case 1, $n$ is even. Fig. 7. Case 2, $n$ is odd.

In both Figs. 6, 7 we temporarily consider $x,y,z,v$ to be unknown, but we soon show that $x\in\{A,B\}\, y\in\{A,B\}, z\in\{A,B\}, V\in\{A,B\}$

In both Figs. 6, 7 we observe that $E = C\cup\{A,B\}$ where $C = \{0,1,2,\cdots,2n\} = \{0,-1,1\}\cup\{2,3,\cdots,n\}\cup\{-2,-3,\cdots,-n\}$. From step 1 in the problem 1 construction of $(E,\cdot) = (C\cup\{A,B\},\cdot)$, we recall that $\forall x,y\in C$, if $x\neq y, x\neq -2y$ and $y\neq -2x$ in $(C,0,+)$, then $x\cdot y$ is computed inside the cyclic group $(C,0,+)$ by $x\cdot y = -x-y$ with one exception. In each $C_j\cup(-C_j)$ in Fig. 4-b, the two tripleton sets $\{0,x_{j1},-x_{j1}\}$ and $\{0,x_{jk},-x_{jk}\}$ were not used in $(E,\cdot)$. This exception can occur only if 0 is a member of the tripleton set which allows us to easily keep track of possible exceptions.

In both cases 1, 2, the following is true. As we know, the tripleton set $\{-1,1,0\}$ is used in $(E,\cdot)$ since $\{-1,1\}\cap\{x_{j1},-x_{j1},x_{jk},-x_{jk}\} = \phi$ is true for each $C_j\cup(-C_j)$ in Fig. 4-b.

Also, $\forall i = 2,3,4,\cdots,n-1, i\cdot 1 = -i-1$ since $1\in C, i\in C$ and since $i\neq 1, i\neq -2\cdot 1, 1\neq -2i$ in $(C,0,+)$ and since $0\notin\{i,1,-i-1\}$. Also, $\forall i = 2,3,4,\cdots,n-1, (-i)\cdot(-1) = i+1$

since -1 $\in C, -i \in C$ and since $-i \neq -1, -i \neq -2(-1), -1 \neq -2(-i)$ in $(C, 0, +)$ and since $0 \notin \{-i, -1, i+1\}$.

In Figs. 6-7, let us now study $2 \cdot (-1) = x, (-2) \cdot (1) = z, n \cdot 1 = y$ and $(-n) \cdot (-1) = v$.

In computing $2 \cdot (-1)$ we observe that $2 = -2(-1)$. In computing $(-2) \cdot (1)$ we observe that $-2 = -2(1)$.

In computing $n \cdot 1$ we observe that $1 = -2n$ since $2n+1 \equiv 0 (\text{mod } 2n+1)$. In computing $(-n) \cdot (-1)$ we observe that $-1 = -2(-n)$ since $2n+1 \equiv 0 (\text{mod } 2n+1)$.

Since $\{-1, 1\} \cap \{x_{j1}, -x_{j1}, x_{jk}, -x_{jk}\} = \phi$ for each $C_j \cup (-C_j)$ in Fig. 4-b, it follows from steps 4, 5 in the problem 1 construction of $(E, \cdot)$ that the following is true in both cases 1 and 2: $2 \cdot (-1) = x \in \{A, B\}, n \cdot 1 = y \in \{A, B\}, (-2) \cdot (1) = z \in \{A, B\}$ and $(-n) \cdot (-1) = v \in \{A, B\}$. Since $x \cdot (-1) = 2, v \cdot (-1) = -n, y \cdot 1 = n, z \cdot 1 = -2$, it is obvious that $x \neq v$ and $y \neq z$.

We now study cases 1, 2 separately.

Case 1. Since $n \cdot 1 = y \in \{A, B\}$, by the symmetry in the way that $\{A, B\}$ is used in the construction of $(E, \cdot)$, there is no loss of generality in assuming that $n \cdot 1 = y = A$. Therefore, $z = B$. This leaves 2 possibilities for $(x, v)$ namely (a) $x = B, v = A$ and (b) $x = A, v = B$.

(a) $x = B, y = A, z = B, v = A$ leads to $E = \overline{C}_0 \cup \overline{C}_1$, where $|\overline{C}_0| = 3, |\overline{C}_1| = 2n$ and this solves the problem.

(b) $x = A, y = A, z = B, v = B$ leads to $E = \overline{C}_0 \cup \overline{C}_1 \cup \overline{C}_2$ where $|\overline{C}_0| = 3, |\overline{C}_1| = |\overline{C}_2| = n$. We will soon modify (b) to solve the problem.

Case 2. Again by symmetry there is no loss of generality in assuming that $n \cdot 1 = y = A$. Since $y \neq z$ this implies $z = B$. There are two possibilities for $(x, v)$ namely (a) $x = A, v = B$ or (b) $x = B, v = A$. In either (a) or (b) it is obvious that $E = \overline{C}_0 \cup \overline{C}_1$, where $|\overline{C}_0| = 3, |\overline{C}_1| = 2n$ which solves the problem.

Case 1-b. In case 1-b, $n$ is even, $x = y = A$ and $z = v = B = -A$.

For convenience, we will call $\overline{C}_1 = \{A, 2, -3, 4, -5, 6, \cdots, n\} = \{x_1, x_2, \cdots, x_n\}$ in that order. That is, $x_1 = A, x_2 = 2, x_3 = -3, \cdots$.

Also, $\overline{C}_2 = \{B, -2, 3, -4, 5, -6, \cdots, -n\} = \{-x_1, -x_2, \cdots, x_n\}$ in that order.

Also, $x_i \cdot x_{i+1} = -1$ if $i$ is odd and $x_i \cdot x_{i+1} = 1$ if $i$ is even where $x_{n+1} = x_1$.

Also, $(-x_i) \cdot (-x_{i+1}) = 1$ if $i$ is odd and $(-x_i) \cdot (-x_{i+1}) = -1$ if $i$ is even where $-x_{n+1} = -x_1$.

Since $n$ is even, let us define the $\dfrac{n}{2}$ doubleton sets $\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \cdots, \{x_{n-1}, x_n\}$ and the corresponding $\dfrac{n}{2}$ doubleton sets $\{-x_1, -x_2\}, \{-x_3, -x_4\}, \{-x_5, -x_6\}, \cdots, \{-x_{n-1}, -x_n\}$.

As we noted earlier, if $n$ is even and we defined $-A = B, -B = A$, then at least $\dfrac{n}{2} + 1$ of the tripleton sets $\{i, -i, 0\}, i = A, 2, 3, \cdots, n$, must appear in $(E, \cdot)$.

From this it follows that $\exists \theta \in \{1, 3, 5, \cdots, n-1\}$ such that the doubleton set $\{x_\theta, x_{\theta+1}\}$ and the corresponding $\{-x_\theta, -x_{\theta+1}\}$ have the property that both of the tripleton sets $\{x_\theta, -x_\theta, 0\}$ and $\{x_{\theta+1}, -x_{\theta+1}, 0\}$ appear in $(E, \cdot)$.

Since $\theta$ is odd we know that $x_\theta \cdot x_{\theta+1} = -1$ and $(-x_\theta) \cdot (-x_{\theta+1}) = 1$.

Since $x_1 \cdot x_2 = -1, x_2 \cdot x_3 = 1, x_3 \cdot x_4 = -1, \cdots, x_n \cdot x_1 = 1$, as in observation 2, let us now interchange 1 and $-1$ in $\overline{C}_1 = \{x_1, x_2, \cdots, x_n\}$ and then substitute the new tripleton sets that we obtain for the old in $(E, \cdot)$. That is, for $\{x_1, x_2, -1\}$ we substitute $\{x_1, x_2, 1\}$, for $\{x_2, x_3, 1\}$ we substitute $\{x_2, x_3, -1\}$, etc. This will give us a new modified ASET on $E$ which we call new $(E, \cdot)$ where $x_i \cdot x_{i+1} = 1$ if $i$ is odd and $x_i \cdot x_{i+1} = -1$ if $i$ is even. Of course, we still have $(-x_i) \cdot (-x_{i+1}) = 1$ if $i$ is odd and $(-x_i) \cdot (-x_{i+1}) = -1$ if $i$ is even.

This means that the following 4 tripleton sets appear in new $(E, \cdot) : \{x_\theta, x_{\theta+1}, 1\}, \{-x_\theta, -x_{\theta+1}, 1\}, \{x_\theta, -$

Together these 4 tripleton sets contain the following 12 doubleton sets: $\{x_\theta, x_{\theta+1}\}, \{x_\theta, 1\}, \{x_{\theta+1}, 1\}, \{-x_\theta,$

We now interchange $0, 1$ in the above 4 tripleton sets to create the following 4 new tripleton sets: $\{x_\theta, x_{\theta+1}, 0\}, \{-x_\theta, -x_{\theta+1}, 0\}, \{x_\theta, -x_\theta, 1\}, \{x_{\theta+1}, -x_{\theta+1}, 1\}$.

Together these 4 new tripleton sets contain the same 12 doubleton sets as the 4 old tripleton sets.

Therefore, if we now substitute these 4 new tripleton sets for the 4 old tripleton sets in new $(E, \cdot)$, we will have created an ASET on $E$ which we now call final $(E, \cdot)$.

Still using $a = -1, b = 1$ in lemma 7 with this final $(E, \cdot)$, if the reader draws a picture of how the two cycles $\overline{C}_1, \overline{C}_2$ have been changed in going from $(E, \cdot)$ to new $(E, \cdot)$ to final $(E, \cdot)$, it is obvious that the lemma 7 partition for final $(E, \cdot)$ now satisfies $E = \overline{C}_0 \cup \overline{C}_{1(final)}$ where $\left|\overline{C}_0\right| = 3, \left|\overline{C}_{1(final)}\right| = 2n$. ■

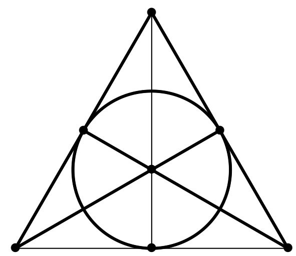


Fig 1. Seven points, seven lines.

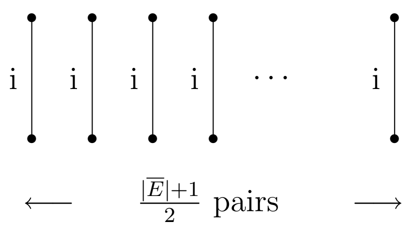



Fig. 2. $\frac{|\overline{E}|+1}{2}$ pairs playing on day $i$.

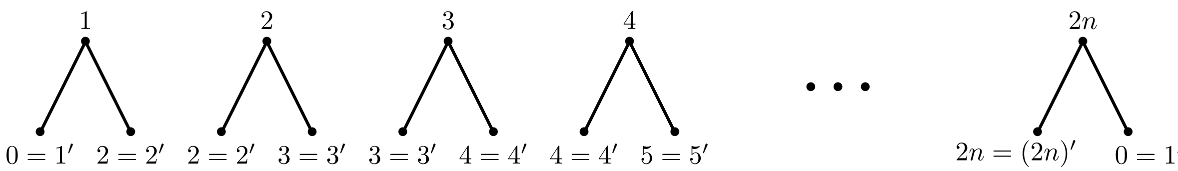



Fig. 3. We later call $0 = 1', 1 = 0', i = i', i = 2, 3, \ldots, 2n$.

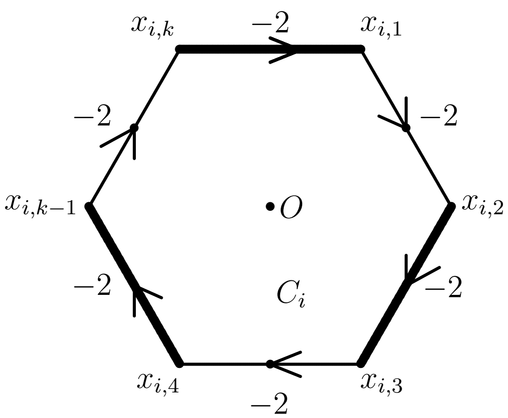



Fig. 4a. $|C_i| \geq 4$ and even

Fig. 4. $|C_i|$ even, $|C_j| = |-C_j|$ odd, $x_{i1} \in C_i, x_{j1} \in C_j$ fixed

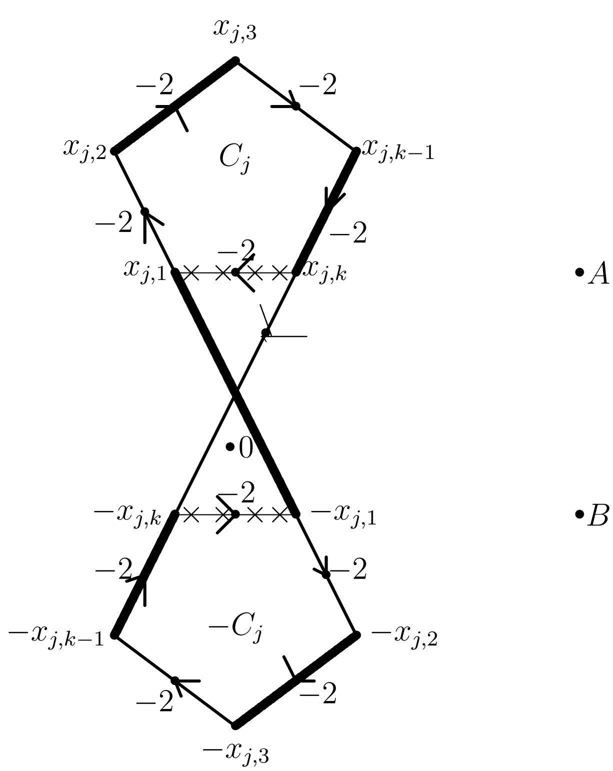
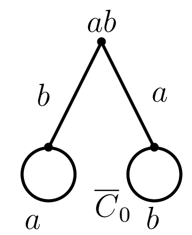



Fig. 4b. $|C_j| = |-C_j| \geq 5$ and odd



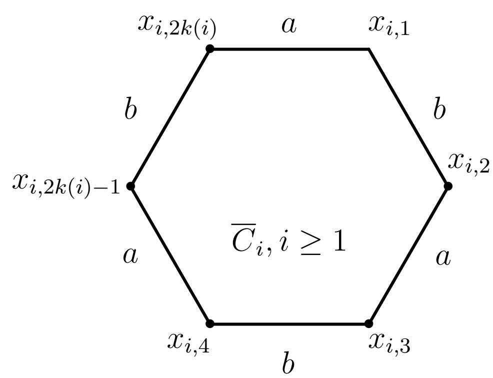




Fig.5. $|\overline{C}_0| = 3$ and $\forall i \geq 1, |\overline{C}_i| \geq 4$ and even

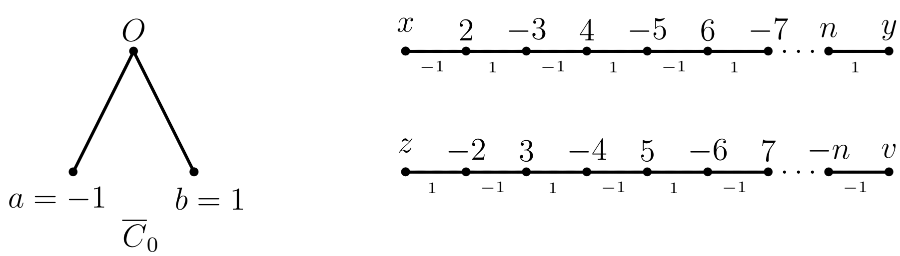



Fig. 6. Case 1, $n$ is even.

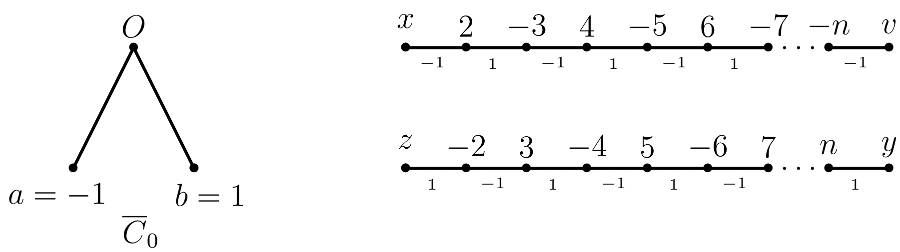



Fig. 7. Case 2, $n$ is odd.

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