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Whittaker models and unipotent representations of p-adic groups

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Introduction

Let $G(F)$ be the rational points of a connected reductive group over a non-archimedean local field F. An irreducible admissible representation V of $G(F)$ is said to be unipotent if there is a parahoric subgroup H in $G(F)$ with pro-unipotent radical U such that the U-invariants in V contain a cuspidal unipotent representation (in the sense of Deligne-Lusztig theory) of the finite reductive group H/U. Lusztig has recently proven his own conjecture made over a decade ago, on the parametrization of unipotent representations, assuming G to be split of adjoint type. This goes as follows. Let $\hat{G}$ be the Langlands dual of G, and let q denote the cardinality of the residue field of F. Then the unipotent representations of $G(F)$ are in bijective correspondence with $\hat{G}$ conjugacy classes of triples (τ, n, ρ), where $\tau \in \hat{G}$ is semisimple, n belongs to the q^{-1}-eigenspace $Y_{\tau,q}$ of $Ad(\tau)$ in the Lie algebra of $\hat{G}$, and ρ is the isomorphism class of an irreducible representation of the component group of the mutual centralizer in $\hat{G}$ of τ and n, such that ρ is trivial on the center of $\hat{G}$. Let $V_{\tau,n,\rho}$ be the irreducible $G(F)$-module corresponding to the indicated triple. Kazhdan and Lusztig had earlier proved [KL] that the corresponding parahoric subgroup is minimal (an Iwahori subgroup) if and only if ρ appears in the homology of the mutual fixed points of τ and $\exp(n)$ on the flag manifold of $\hat{G}$. They showed moreover that if $V_{\tau,n,\rho}$ is tempered, then n must lie in the unique dense orbit of $\hat{G}_\tau$ acting on $Y_{\tau,q}$, where $\hat{G}_\tau$ is the centralizer of τ in $\hat{G}$. In [R1], it was shown that if $V_{\tau,n,\rho}$ is Iwahori spherical, then it is generic (i.e. it has a Whittaker model, as defined below) if and only if n belongs to the dense $\hat{G}_\tau$-orbit in $Y_{\tau,q}$ and ρ is trivial. The purpose of this note is to extend this to the entire L-packet (defined as the collection of unipotent representations with fixed (τ, n)), in accordance with the following general expectations.

Pairs (τ, n) correspond to admissible homomorphisms ϕ from the Weil-Deligne group of F into $\hat{G}$, and Lusztig's theorem is a special case of Langlands' conjectural parametrization of the L-packet of ϕ by representations of the component group of the centralizer of the image of ϕ [La]. Among many expected properties of the Langlands parametrization, it is believed that the $G(F)$ representation corresponding to $\phi = 1$ for a tempered L-packet should be the unique generic member of the packet. For unipotent representations, we prove the following stronger result, conjectured in [R2].

Theorem. *The unipotent representation $V_{\tau,n,\rho}$ of $G(F)$ is generic if and only if n belongs to the unique dense $\hat{G}_\tau$-orbit in $Y_{\tau,q}$ and ρ is trivial.*

Considering known results, we must only prove that if a $G(F)$ representation is both generic and unipotent, then the corresponding parahoric subgroup must be an Iwahori subgroup. This statement makes sense at least for unramified groups. Its proof in Lemma 4 below was inspired by the proof in [CS] of Rodier's theorem [Ro] on Whittaker models of parabolically induced representations. We rely on the fact that a cuspidal unipotent representation of a finite reductive group M is generic only if M is a torus, in which case the representation is trivial. If the parahoric H is maximal special, one can use the Iwasawa decomposition to lift this nonexistence to the p-adic group, as mentioned in [R2] (although the argument given there needs Lemma 3 below to be complete). An arbitrary parahoric takes a bit more work.

Some structure of *p*-adic groups

We give a brief summary of the required structure theory of G, taken from [T]. Let F be a non-archimedean local field with ring of integers $\mathscr{O}$ and finite residue field k of cardinality q a power of the prime p. Let G be a connected semisimple algebraic group over F, with maximal F-split torus A. We assume that G is unramified (quasi-split and split over a finite unramified extension of F). Let A_0 be the subgroup of $A(F)$ on which all rational characters of A have values in the unit group $\mathscr{O}^\times$. Let $\mathscr{N}$ and Z and be the normalizer and centralizer of A in G, and let Z_0 be the analogue of A_0 for $Z(F)$. As G is unramified, we may identify the lattices $A(F)/A_0 = Z(F)/Z_0 =: \Lambda$, and put $E = \mathbb{R} \otimes \Lambda$. The spherical Weyl group is $W_0 := \mathscr{N}(F)/Z(F)$ and the affine Weyl group is $W := \mathscr{N}(F)/Z_0$. The group $\mathscr{N}(F)$ acts on E by affine motions, with Z_0 acting trivially, Λ acting by translations. We may identify W_0 with the subgroup of W fixing $0 \in E$, and thus $W = W_0\Lambda$ (semidirect product). For $\lambda \in \Lambda$, we write t_λ for the element of W which acts by translation by λ on E.

Let Δ be the roots of A in G, viewed as linear functionals on E, via the formula $\langle \lambda, \alpha \rangle = -val(\alpha(\tilde{\lambda}))$, where $\alpha \in \Delta$, $\tilde{\lambda} \in A$ and $\lambda = \tilde{\lambda} + A_0$. Let Δ_{aff} be the affine roots. These are affine functions on E of the form $a = \alpha + m$, where

$\alpha \in \Delta$ (the "vector part" of a) and m runs through a certain discrete subset of $\mathbb{R}$ (depending on α). For each affine root a, the group W contains an element s_a acting on E by reflection about the affine hyperplane where a vanishes.

Let P be a minimal parabolic F-subgroup of G containing A, with unipotent radical N. Let Δ^+ be the roots of A in N, and let $\Sigma \subseteq \Delta^+$ be the corresponding base of the spherical root system Δ. Let Σ_{aff} be the unique base of the affine root system Δ_{aff} containing Σ. Let C be the open subset of E defined by the conditions $0 < a < 1$ for every $a \in \Sigma_{aff}$. The boundary of C is a disjoint union of facets, parametrized by subsets of Σ_{aff}. To $J \subset \Sigma_{aff}$ corresponds the facet C_J defined by the vanishing of the affine roots in J. The affine space underlying E is an apartment in the Bruhat-Tits building X. This building is a $G(F)$-simplicial complex whose simplices are the $G(F)$-translates of the facets C_J.

Each facet determines a parahoric subgroup as follows. We begin with minimal parahoric, otherwise known as Iwahori subgroups. If G is simply connected, Iwahori subgroups are the stabilizers in $G(F)$ of open facets (translates of C) in X. In general, the Iwahori subgroup for C may be described as follows [T,3.7]. The facet $0 \in E$ corresponds to an $\mathscr{O}$-scheme $\mathscr{G}_0$ whose generic fiber is $G(F)$ and whose group of $\mathscr{O}$-points is the stabilizer in $G(F)$ of 0. Let $r : \mathscr{G}_0(\mathscr{O}) \longrightarrow \mathscr{G}_0(k)$ be the homomorphism induced by reduction modulo $\mathscr{P}$. Now $\mathscr{G}_0(k)$ is the fixed points of a Frobenius automorphism f of a connected reductive group $\overline{\mathscr{G}}_0$ defined over k, and the spherical building of $\mathscr{G}_0(k)$ may be identified with the link of 0 in X. Thus, open simplices of X having 0 in their closure are in canonical bijection, by taking stabilizers, with f-stable Borel subgroups of $\overline{\mathscr{G}}_0$. The Iwahori subgroup B corresponding to C is the inverse image $r^{-1}(\mathscr{B}^f)$, where $\mathscr{B}$ is the f-stable Borel subgroup of $\overline{\mathscr{G}}_0$ corresponding to C.

For general parahoric subgroups, take a proper subset $J \subset \Sigma_{aff}$, and let W_J be the subgroup of W generated by the reflections s_a, for $a \in J$. The set $H = H_J = BW_JB$ is a subgroup of G and is, in this paper, the parahoric subgroup corresponding to J. Note that H stabilizes the facet C_J. It is the full stabilizer if G is simply connected, or if $J = \Sigma$. As in the previous paragraph, but now with simpler notation, we have an exact sequence via reduction mod $\mathscr{P}$

$$1 \longrightarrow U \longrightarrow H \longrightarrow M \longrightarrow 1,$$

where M is the k rational points of a connected (assured by the definition of H) reductive group defined over k, and U is pro-unipotent and characteristic in H. The relative roots of M are the vector parts of the affine roots vanishing on C_J. If α is a root in M, the root group corresponding to α is $\overline{X}_\alpha := X_a/X_{a+}$, where $a = \alpha + m \in \Delta_{aff}$ vanishes on C_J, X_a is the corresponding valuated root group, and X_{a+} is the union of all $X_{a+\epsilon}$ for $\epsilon > 0$. We can also describe X_a as $H \cap N_\alpha$, where N_α is the (spherical) root subgroup of N on whose Lie algebra A acts by positive powers of α.

Let $\overline{J}$ be the vector parts of the roots in J. Then the subgroup U_1 of M generated by $\overline{X}_\alpha$ for $\alpha \in \overline{J}$ is a Sylow p-subgroup of M. Morover, $\overline{J}$ is the base of a sub-root system $\Delta_J \subseteq \Delta$, whose Weyl group $W_{0,J} \subseteq W_0$ is generated by the reflections about the kernels of the roots in $\overline{J}$. Let Δ_J^+ be the unique positive system of Δ_J containing $\overline{J}$. Note that Δ_J^+ is not generally contained in Δ^+. Let $w \mapsto \overline{w}$ be the natural map from W to W_0. If $a = \alpha + m$ is an affine root vanishing on C_J, then $\overline{s_a} = s_\alpha$, so $W_{0,J} = \{\overline{w} :\ w \in W_J\}$. Now let

$$W_0^J = \{w \in W_0 :\ w^{-1}\overline{J} \subset \Delta^+\}.$$

It is a standard fact about sub-root systems that W_0^J meets every coset $W_{0,J}x$ for $x \in W_0$.

Lemma 1. *The set $W_0^J \Lambda \subset W$ meets all cosets $W_J x$ for $x \in W$.*

Proof. Say $x = wt_\lambda$, with $w \in W_0$, $\lambda \in \Lambda$. Write $w = yz$, with $y \in W_{0,J}$, $z \in W_0^J$. Then $y = \overline{u}$ for some $u \in W_J$, so $y = ut_\nu$ for some $\nu \in \Lambda$. Hence $x = ut_\nu z t_\lambda = uzt_{z^{-1}\nu+\lambda}$, so $zt_{z^{-1}\nu+\lambda} = u^{-1}x \in W_0^J \Lambda \cap W_J x$. □

Lemma 2. *For any $x \in \mathcal{N}(F)$, the image of $H \cap {}^xN$ in M is a Sylow p-subgroup of M.*

Proof. The pre-image of W_J in $\mathcal{N}(F)$ is contained in H, so we can suppose that, modulo Z_0, $x = wt_\lambda$, with $w \in W_0^J$, by Lemma 1. Thus $H \cap {}^xN = H \cap {}^wN$ (with no ambiguity caused by the abuse of notation). Let $\alpha \in \overline{J}$, and consider the root group $\overline{X}_\alpha = X_a / X_{a+}$ as above. Our choice of w implies that $w^{-1}X_a w \subset N$, and it follows that $\overline{X}_\alpha$ is in the reduction modulo $\mathscr{P}$ of $H \cap {}^wN$, so the image of $H \cap {}^wN$ in M contains the Sylow p-subgroup U_1. Being a finite subquotient of a pro-unipotent group, the image is itself a p-group, hence cannot exceed U_1. □

Generic representations

A complex valued character of U_1 is called "generic" if it is nontrivial on $\overline{X}_\alpha$ for every $\alpha \in \overline{J}$, and trivial on $\overline{X}_\alpha$ for $\alpha \in \Delta_J^+ - \overline{J}$. The last condition is superfluous if $q > 3$ [DM, p.129], as the subgroup $U_1^* \subset U_1$ generated by the nonsimple root groups is then also the commutator subgroup of U_1. An irreducible complex representation of M (hence of H) is "generic" if its restriction to U_1 contains a generic character of the latter, and "nongeneric" otherwise.

Lemma 3. *If M is not a torus, a cuspidal nongeneric representation of M contains no character of U_1 which is trivial on U_1^*.*

Proof. Let θ be the character afforded by a U_1-invariant line in a cuspidal nongeneric representation of M. Being nongeneric and trivial on U_1^*, θ must be trivial on $\overline{X}_\alpha$ for some $\alpha \in \overline{J}$. But then θ is trivial on the unipotent radical of the maximal parabolic subgroup of M whose Levi subgroup has simple roots $\overline{J} - \{\alpha\}$. This contradicts cuspidality. □

We turn now to generic representations of $G(F)$. Let N^* be the product of those spherical root groups N_α with $\alpha \in \Delta^+ - \Sigma$. A character of $N(F)$ is "generic" if it is nontrivial on each simple root group and trivial on $N^*(F)$. This last condition may be superfluous, and certainly is for split groups by [H, Lemma

7], since the p-adic field F is infinite. An irreducible admissible representation of $G(F)$ is "generic" if it may be realized as a submodule of $\mathrm{Ind}_{N(F)}^{G(F)}\psi$ for some generic character ψ of $N(F)$. Here Ind denotes smooth induction on which $G(F)$ acts by right translations, and later ind will mean compact induction. We have now arrived at the main point.

Lemma 4. *Suppose $V \subset \mathrm{Ind}_{N(F)}^{G(F)}\psi$ is a generic representation of $G(F)$, and the parahoric H is not an Iwahori subgroup. Then the U-invariants in V contain no cuspidal nongeneric representation σ of M.*

Proof. In this proof, let us abbreviate $G = G(F)$, $N = N(F)$, $\mathscr{N} = \mathscr{N}(F)$. Suppose the U-invariants in V contain such a σ. Then

$$0 \neq \mathrm{Hom}_H(\sigma, V) = \mathrm{Hom}_G(\mathrm{ind}_H^G\sigma, V) \subseteq \mathrm{Hom}_G(\mathrm{ind}_H^G\sigma, \mathrm{Ind}_N^G\psi),$$

so there is a nonzero linear functional $T : \mathrm{ind}_H^G\sigma \longrightarrow \mathbb{C}$ satisfying $T(R_n f) = \psi(n)T(f)$, for every $f \in \mathrm{ind}_H^G\sigma$ and $n \in N$, where $R_n f(g) = f(gn)$ for $g \in G$. Let $K = \mathscr{G}_0(\mathscr{O})$ be the parahoric subgroup stabilizing $0 \in E$, as above. We have an Iwasawa decomposition $G = KAN$, hence $G = BWN$, hence $G = HWN$, since $B \subseteq H$. Let $x \in \mathscr{N}$ represent $wt_\lambda \in W$ with $w \in W_0^J$ as in Lemma 1. Let I_x be the space of functions in $\mathrm{ind}_H^G\sigma$ which are supported on HxN. As an N-module,

$$I_x \simeq \mathrm{ind}_{H^x \cap N}^N \sigma^x$$

via the map $f \mapsto f_x$, $f_x(n) = f(xn)$. Suppose T is nonzero on I_x. Taking contragredients, there is a nonzero function $f \in \check{I}_x = \mathrm{Ind}_{H^x\cap N}^N \check{\sigma}^x$ transforming under N by ψ^{-1}. The nonzero vector $v = f(1) \in \check{\sigma}$ therefore satisfies, for every $h \in H \cap {}^xN$, the relation

$$\psi^{-1}(x^{-1}hx)v = f(x^{-1}hx) = \check{\sigma}^x(x^{-1}hx)v = \check{\sigma}(h)v.$$

So the restriction of $\check{\sigma}$ to $H \cap {}^xN$ contains the character ${}^x\psi^{-1}$. In particular, ${}^x\psi^{-1}$ is trivial on $U \cap {}^x\underline{N}$, and is therefore the inflation of a character θ on U_1.

Let $\alpha \in \Delta_J^+ - \overline{J}$, that is, α is a nonsimple root in U_1. The sub-root system $w^{-1}\Delta_J$ has the base $w^{-1}\overline{J}$, and $w^{-1}\alpha$ is not simple with respect to this base. Since $w \in W_0^J$, we have $w^{-1}\overline{J} \subseteq \Delta^+$, so $w^{-1}\alpha$ is a nontrivial sum of at least two roots in Δ^+. Thus $w^{-1}\alpha \in \Delta^+ - \Sigma$. The character ${}^\lambda\psi^{-1}$ (which does not depend on the representative chosen for λ) is also generic, hence is trivial on $N_{w^{-1}\alpha}$. Therefore ${}^x\psi^{-1}$ is trivial on N_α, implying that θ is trivial on $\overline{X}_\alpha$.

We have found in $\check{\sigma}$ a character θ of U_1 which is trivial on U_1^*. The properties of being cuspidal and nongeneric are preserved by taking contragredients, so we have a contradiction by Lemma 3. $\square$

We now prove the theorem as stated in the introduction. The only unipotent generic representation of M is the Steinberg representation. If the algebraic group underlying M has connected center, this is spelled out in [Car, p.379]. In general, it follows immediately from [DM 14.49]. The Steinberg representation is cuspidal if and only if it is trivial, if and only if M is a torus, so by Lemma 4, any unipotent generic $G(F)$-module V contains a vector fixed under an Iwahori subgroup and thus is a subquotient of an unramified principal series representation $I(\tau)$ of $G(F)$. If G is adjoint, there is only one generic subquotient of $I(\tau)$. This follows from the uniqueness theorem of Rodier [Ro] and the fact that for adjoint G there is only one orbit of generic characters under $A(F)$. If G is moreover split, it is shown in [R,10.1] that this generic subquotient is none other than $V_{\tau,n,1}$, where n belongs to the dense $\hat{G}_\tau$-orbit in $Y_{\tau,q}$.

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