SUPERCUSPIDAL L-PACKETS OF POSITIVE DEPTH AND TWISTED COXETER ELEMENTS

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1. INTRODUCTION

The local Langlands correspondence is a conjectural connection between representations of groups $G(k)$ for connected reductive groups G over a p-adic field k and certain homomorphisms (Langlands parameters) from the Galois (or Weil-Deligne group) of k into a complex Lie group ${}^LG$ which is dual, in a certain sense, to G and which encodes the splitting structure of G over k. More introductory remarks on the local Langlands correspondence can be found in [21].

When $G = GL_1$ this correspondence should reduce to local abelian class field theory. For $G = GL_n$, the Langlands correspondence is uniquely determined by local factors [24] and was shown to exist in [23] and [25]. So far this correspondence is not completely explicit, but much progress has been made in this direction; see [9], [10], for example.

For groups other than GL_n or PGL_n, the theory is much less advanced; new phenomena appear, arising on the arithmetic side from the difference between conjugacy and stable conjugacy and on the dual side from nontrivial monodromy of Langlands parameters. This means that a single Langlands parameter φ should determine not just one, but a finite set of representations $\Pi(\varphi)$; these are the "L-packets" of the title.

However, since local factors have not been defined in general, there is no precise characterization of an L-packet for general groups. One can, at present, only hope to define finite sets of representations $\Pi(\varphi)$ attached to Langlands parameters φ, and show that they have properties expected (or perhaps unexpected) of L-packets. (See [14, chap. 3] for some of these properties.) One is thereby proposing a definition of local factors for the representations in the sets $\Pi(\varphi)$ (cf. [4, chap.3]).

This paper is a sequel to [14]. The aim of both papers is to verify, in an explicit and natural way, the local Langlands correspondence for the simplest kinds of non-abelian extensions of k, and the simplest kinds of

Date: March 5, 2007.
Partially supported by NSF grant DMS-0207231.

supercuspidal representations of $G(k)$, where G is a fairly general reductive group.

In [14], we gave a construction of L-packets of supercuspidal representations of unramified p-adic groups and their pure inner forms, for certain tamely ramified Langlands parameters. (See also [27].) The present paper has two parts: The first part extends the construction of [14] to certain wildly ramified Langlands parameters and positive-depth supercuspidal representations. As in [14], the formal degrees (with respect to canonical Haar measures) are constant on these L-packets. There is also a new observation here: the internal parametrization of our packets (including those of [14]) has an equivariance property with respect to a natural Weyl group action which has not been considered previously. The second part of this paper investigates the canonical example of L-packets (including those of [14]) associated to twisted Coxeter elements, building on work of Springer [38].

One expected property of L-packets is stability. The L-packets of [14] have this property (assuming some restrictions on k). The positive-depth L-packets in this paper are constructed in an analogous way, from stable classes of data, but it does not yet seem possible to prove stability of the sum of characters in these L-packets.

Another expected property is a precise description of the generic representations in a tempered L-packet. The generic representations in our positive-depth packets are parameterized in the same way as those in [14]. DeBacker and I prove this in [15].

The construction of L-packets in this paper can be outlined as follows. We start with extensions of k which are abelian over their maximal unramified subextension. Thus, the number-theoretic side of this paper pertains to the Galois group of K^{ab}/k, where K is a maximal unramified extension of k and K^{ab} is the maximal abelian extension of K. The extension K^{ab}/k was described concretely by Lubin and Tate in [30], in a manner analogous to the Kronecker-Weber construction of abelian extensions of $\mathbb{Q}$. The Langlands correspondence for unramified tori is then reproved using the Lubin-Tate theory, because this route seems to me more explicit than the original proof in [29] and because it is an efficient way to ensure that the correspondence is natural and preserves depth.

Via the Langlands correspondence for tori, these Lubin-Tate extensions determine pairs (T, χ) where T is an elliptic unramified torus over k and χ is a character of $T(k)$. If we have a k-embedding of T into an inner form of G, under which χ is sufficiently regular, then a construction of Adler [1], building on earlier work of Howe [26], Carayol [11], Gerardin [18] and others, produces a "very cuspidal" representation $\pi(T, \chi)$ of $G(k)$. (Adler's construction was later generalized by Yu [42]. We hope that the methods in this paper will eventually extend to construct L-packets from Yu's representations.)

In brief, our L-packets consist of all possible very cuspidal representations one can make from a fixed character χ by varying the embedding of T into all pure inner forms of G. These embeddings are controlled by the monodromy group C_φ of the corresponding Langlands parameter φ. Thus, we get L-packets parameterized in the expected way.

The image of Frobenius under a Langlands parameter of the above type determines an elliptic element w in the Weyl group of the (possibly disconnected) L-group ${}^LG$. Various classes of Weyl group elements arise from supercuspidal L-packets, depending on G, but one case is common to all groups, namely when w is a (possibly twisted) Coxeter element. In the second part of this paper we describe these Coxeter L-packets in more detail. The results are cleanest if we assume G is adjoint and absolutely simple.

For $\mathbf{G} = \mathbf{PGL}_n$ all of our L-packets are of Coxeter type. Here $C_\varphi \simeq \mathbb{Z}/n\mathbb{Z}$ is the center of ${}^LG = SL_n(\mathbb{C})$. If ρ is a character of C_φ of order d, where $n = dm$, then $\pi(\varphi, \rho)$ is a representation of $PGL_m(D)$, where D is a division algebra of degree d over k. The representation $\pi(\varphi, \rho)$ is presumably the one associated to $\pi(\varphi, 1)$ in [3], but I have not checked this.

For a general unramified adjoint group $\mathbf{G}$, the monodromy C_φ for a Coxeter L-packet $\Pi(\varphi)$ is always the center of ${}^LG$. This means that Coxeter L-packets are as small as possible: just as for $\mathbf{PGL}_n$ there is exactly one representation in the L-packet for each inner form of $\mathbf{G}$. Each representation is induced from a parahoric subgroup, so our Langlands correspondence picks out a canonical parahoric subgroup of each inner form of $\mathbf{G}$. In the last three sections we determine these parahoric subgroups, along with the inducing data for each representation in a Coxeter L- packet $\Pi(\varphi)$. We also use the Coxeter case to illustrate other aspects of L-packets, such as stable classes of tori and their characters.

Clearly this paper owes much to my previous collaboration with Stephen DeBacker. The idea of extending [14] to the positive-depth case arose in conversations with Benedict Gross, in the course of our work on [21]. I am grateful to have worked with both of these mathematicians.

2. Basic notation and structure

2.1. **Fields and groups.** Let p be an odd rational prime, let k be a finite extension of $\mathbb{Q}_p$, and let $\mathfrak{o}, \mathfrak{p}, \mathfrak{f} = \mathfrak{o}/\mathfrak{p}$ denote the ring of integers, prime ideal, and residue field of k, respectively. Fix an algebraic closure $\bar{k}$ of k, and let K be the maximal unramified extension of k in $\bar{k}$. Let $\mathfrak{O}, \mathfrak{P}, \mathfrak{F} = \mathfrak{O}/\mathfrak{P}$ denote the ring of integers, prime ideal, and residue field of K. We fix $\varpi \in \mathfrak{p}$ such that $\mathfrak{p} = \varpi\mathfrak{o}$. Then $\mathfrak{P} = \varpi\mathfrak{O}$. Set $q = |\mathfrak{f}|$. Then $\mathfrak{f} \simeq \mathbb{F}_q$ and

$\mathfrak{F} \simeq \bar{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$. Let $\mathrm{val} : K^\times \to \mathbb{Z}$ be the valuation on K, normalized so that $\mathrm{val}(\varpi) = 1$. Then val restricts to the valuation of k. Let $\mathrm{Frob} \in \mathrm{Gal}(\bar{k}/k)$ be a geometric Frobenius element; for all $x \in \mathfrak{O}$, we have $\mathrm{Frob}(x)^q \equiv x \mod \mathfrak{P}$.

We use the following notational conventions for algebraic groups and their rational points. For any algebraic $\bar{k}$-group $\mathbf{H}$ which is defined over k, we let $H = \mathbf{H}(K)$. The action of Frob on H, arising from the given k-structure on $\mathbf{H}$, is given by an endomorphism F of H such that $\mathbf{H}(k) = H^F$. If $\mathbf{T}$ is an algebraic torus, then $X^*(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbf{GL}_1)$ and $X_*(\mathbf{T}) = \mathrm{Hom}(\mathbf{GL}_1, \mathbf{T})$ denote the algebraic character and co-character groups of $\mathbf{T}$, respectively.

Throughout this paper, $\mathbf{G}$ is a connected reductive $\bar{k}$-group which is defined over k and split over K. Let $\mathbf{Z}$ denote the identity component of the center of $\mathbf{G}$, and let $\mathbf{G}_{ad}$ denote the adjoint group of $\mathbf{G}$. If $\mathbf{T}$ is a torus in $\mathbf{G}$, then $\mathbf{T}_{ad}$ is the image of $\mathbf{T}$ in $\mathbf{G}_{ad}$.

The Bruhat-Tits building of $G_{ad} = \mathbf{G}_{ad}(K)$ is the "reduced" building of G; we denote it by $\mathcal{B}(G)$. The Frobenius endomorphism F of G induces an F-action on $\mathcal{B}(G)$ and $\mathcal{B}(G^F) = \mathcal{B}(G)^F$ is the Bruhat-Tits building of G_{ad}^F. To any maximal torus $\mathbf{T} \subset \mathbf{G}$ such that $\mathbf{T}$ is defined over k and K-split, there corresponds an F-stable apartment $\mathcal{A}(T) \subset \mathcal{B}(G)$, which is an affine space under a transitive action of the vector group $X_*(\mathbf{T}) \otimes \mathbb{R}$. This action factors through $X_*(\mathbf{T}_{ad}) \otimes \mathbb{R}$, which now acts simply-transitively on $\mathcal{A}(T)$.

We denote by G_x the parahoric subgroup of G at a point $x \in \mathcal{B}(G)$. If x is F-stable, then G_x^F is the parahoric subgroup of G^F at x.

The set of equivalence classes of irreducible admissible representations of G^F is denoted by $\mathrm{Irr}(G^F)$. If Γ is a finite or compact group then $\mathrm{Irr}(\Gamma)$ is the set of equivalence classes of irreducible representations of Γ.

2.2. **Affine root groups.** For more details in this section see [40]. Fix a K-split maximal k-torus $\mathbf{T}$ in $\mathbf{G}$, and let Φ and Ψ denote the roots and affine roots, respectively, of $\mathbf{G}$ with respect to $\mathbf{T}$. The elements of Ψ are affine functions on $\mathcal{A}(T)$. For later calculations of formal degrees, it is convenient to index the affine roots as follows. Choosing a hyperspecial point $o \in \mathcal{A}(T)$ allows us to identify $\mathcal{A}(T) = X_*(\mathbf{T}_{ad}) \otimes \mathbb{R}$, so that roots $\alpha \in \Phi$ become affine functions on $\mathcal{A}(T)$ vanishing at o and we can uniquely write each $\psi \in \Psi$ as $\psi = \alpha + n$, where $\alpha \in \Phi$ and $n = \psi(o)$. For each root $\alpha \in \Phi$ we fix a root group $u_\alpha : K^+ \to G$ such that $u_\alpha(\mathfrak{O}) = u_\alpha(K) \cap G_o$.

Then for each affine root $\psi = \alpha + n$, we have a bounded subgroup $U_\psi = U_{\alpha+n} := u_\alpha(\mathfrak{P}^n)$ of the root group $u_\alpha(K)$. The group U_ψ can also be defined as the subgroup of $u_\alpha(K)$ fixing a point in the hyperplane $\{x \in \mathcal{A}(T) : \psi(x) = 0\}$. In particular, U_ψ is independent of the choice of hyperspecial point o.

The action of the Frobenius F on $\mathcal{B}(G)$ preserves $\mathcal{A}(T)$ and acts on $\mathcal{A}(T)$ via an affine transformation. This induces an action of F on the set of affine functions on $\mathcal{A}(T)$, which preserves the set Ψ of affine roots, and correspondingly permutes the groups $U_\psi,\ \psi \in \Psi$.

The hyperspecial point o is not necessarily fixed by F. However, if $\mathbf{G}$ is k-quasisplit and $\mathbf{T}$ is contained in a Borel subgroup of $\mathbf{G}$ defined over k, then we can choose the hyperspecial point $o \in \mathcal{A}(T)$ so that $F \cdot o = o$.

2.3. **Filtration subgroups.** The parahoric subgroups in G have various filtrations. These were defined in [6] and [33] and applied to representation theory in [32]. See also [2].

Recall that we have fixed a K-split maximal k-torus $\mathbf{T}$ in $\mathbf{G}$. For $s \geq 0$, define filtration subgroups of T by

$$\begin{aligned} T_s &:= \{t \in T :\ \mathrm{val}(\chi(t)-1) \geq s \quad \text{for all} \quad \chi \in X^*(\mathbf{T})\} \\ T_{s+} &:= \{t \in T :\ \mathrm{val}(\chi(t)-1) > s \quad \text{for all} \quad \chi \in X^*(\mathbf{T})\}. \end{aligned} \tag{1}$$

Since $\mathrm{val}(K) = \mathbb{Z}$, we have $T_{s+} = T_{s+1}$ if $s \in \mathbb{Z}$ and $T_{s+} = T_s$ otherwise. The subgroup T_0 is the maximal bounded subgroup of T.

For each point $x \in \mathcal{A}(T)$ and real number $s \geq 0$, we define the subgroup

$$G_{x,s} := \langle T_s,\ U_\psi :\ \psi(x) \geq s\rangle \tag{2}$$

We have $G_{x,0} = G_x$, and $G_{x,r} \subseteq G_{x,s}$ if $r > s$. We also define $G_{x,s+} := \bigcup_{r>s} G_{x,r}$. The groups $G_{x,s},\ G_{x,s+}$ are bounded open subgroups of G. The commutator relation $[G_{x,r}, G_{x,s}] \subseteq G_{x,r+s}$ [1, 1.4.2] implies that $G_{x,r}$ is normal in $G_{x,s}$ for $r > s$. Finally, it is shown in [42, chap. 1] that the groups $G_{x,s}$ and $G_{x,s+}$ are independent of the choice of K-split maximal k-torus $\mathbf{T}$, subject to the condition $x \in \mathcal{A}(T)$.

Note that the presentations of $G_{x,s}$ and $G_{x,s+}$ above involve infinitely many groups U_ψ, almost all of which are redundant. For later computations, it will be helpful to replace these with finite presentations, using our choice of hyperspecial point o, as follows.

We fix a point $x \in \mathcal{A}(T)$. For each (linear) root $\alpha \in \Phi$, let $n(\alpha, s)$ be the largest integer such that $n(\alpha, s) \leq \alpha(x) - s$ and let $n(\alpha, s+)$ be the largest integer such that $n(\alpha, s+) < \alpha(x) - s$. We have $n(\alpha, s+) = n(\alpha, s) - 1$ if $\alpha(x) - s \in \mathbb{Z}$ and $n(\alpha, s+) = n(\alpha, s)$ otherwise. These integers depend on x, which is fixed and suppressed in the notation.

For $\psi = \alpha - n \in \Psi$, with $\alpha \in \Phi$ and $n \in \mathbb{Z}$, we have

$$\psi(x) \geq s \quad \Leftrightarrow \quad n \leq n(\alpha, s) \quad \Leftrightarrow \quad U_\psi \subseteq U_{\alpha - n(\alpha,s)}.$$

Likewise, $\psi(x) > s \Leftrightarrow U_\psi \subseteq U_{\alpha - n(\alpha, s+)}$. Hence we have finite presentations $G_{x,s} = \langle T_s,\ U_\psi :\ \psi \in \Psi_s\rangle$ and $G_{x,s+} = \langle T_s,\ U_\psi :\ \psi \in \Psi_{s+}\rangle$, where $\Psi_s = \{\alpha - n(\alpha, s) :\ \alpha \in \Phi\}$ and $\Psi_{s+} = \{\alpha - n(\alpha, s+) :\ \alpha \in \Phi\}$.

2.4. **Filtrations on Lie algebras.** Let $\mathbf{g}$ and $\mathbf{t}$ be the Lie algebras of $\mathbf{G}$ and $\mathbf{T}$, respectively, and let $\mathfrak{g} = \mathbf{g}(K)$, $\mathfrak{t} = \mathbf{t}(K)$. Since $\mathbf{T}$ splits over K, we have $\mathfrak{g} = \mathfrak{t} + \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the α-eigenspace for T in $\mathfrak{g}$.

The Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$ have analogous filtrations, namely

$$\begin{aligned} \mathfrak{t}_s &:= \{H \in \mathfrak{t} :\ \mathrm{val}(d\chi(H)) \geq s \quad \text{for all } \ \chi \in X^*(\mathbf{T})\}, \\ \mathfrak{t}_{s+} &:= \{H \in \mathfrak{t} :\ \mathrm{val}(d\chi(H)) > s \quad \text{for all } \ \chi \in X^*(\mathbf{T})\}, \end{aligned} \tag{3}$$

and for $x \in \mathcal{A}(T)$, we have $\mathfrak{O}$-lattices

$$\begin{aligned} \mathfrak{g}_{x,s} &:= \mathfrak{t}_s + \sum_{\substack{\psi\in\Psi \\ \psi(x)\geq s}} \mathfrak{u}_\psi = \mathfrak{t}_s \oplus \bigoplus_{\psi\in\Psi_s} \mathfrak{u}_\psi, \\ \mathfrak{g}_{x,s+} &:= \mathfrak{t}_{s+} + \sum_{\substack{\psi\in\Psi \\ \psi(x)> s}} \mathfrak{u}_\psi = \mathfrak{t}_{s+} \oplus \bigoplus_{\psi\in\Psi_{s+}} \mathfrak{u}_\psi. \end{aligned} \tag{4}$$

Here Ψ_s and Ψ_s^+ are as in 2.3, and $\mathfrak{u}_\psi = du_\alpha(\mathfrak{P}^n)$, where $\psi = \alpha + n \in \Psi$.

If $2s \geq r \geq s > 0$, we have canonical group isomorphisms

$$G_{x,s}/G_{x,r} \simeq \mathfrak{g}_{x,s}/\mathfrak{g}_{x,r}, \qquad\qquad T_s/T_r \simeq \mathfrak{t}_s/\mathfrak{t}_r, \tag{5}$$

along with similar isomorphisms where r is replaced by $s+$. Note that

$$\mathfrak{g}_{x,s}/\mathfrak{g}_{x,s+} = \mathfrak{t}_s/\mathfrak{t}_{s+} \oplus \bigoplus_{\substack{\psi\in\Psi_s \\ \psi(x)=s}} \mathfrak{u}_\psi/\mathfrak{u}_{\psi+1}, \tag{6}$$

and $\dim_{\mathfrak{F}}(\mathfrak{u}_\psi/\mathfrak{u}_{\psi+1}) = 1$ for each summand on the right side of (6).

3. VERY CUSPIDAL REPRESENTATIONS

In this section we recall Adler's construction of supercuspidal representations [1], along with some refinements by Yu [42].

3.1. **Minisotropic tori.** Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$ such that $\mathbf{T}$ is defined over k and K-split. Such a torus $\mathbf{T}$ is called F**-minisotropic** if any of the following equivalent conditions holds:

(1) $X_*(\mathbf{T})^F = X_*(\mathbf{Z})^F$;
(2) T^F/Z^F is compact;
(3) The group T^F has a unique fixed-point $x \in \mathcal{B}(G)^F$.

If these hold, then $T^F \subset Z^F G_x^F$.

3.2. **The inducing subgroups.** We will apply the filtrations of section 2 to a point $x \in \mathcal{A}(T)^F$, where $\mathbf{T}$ is an F-minisotropic maximal torus in $\mathbf{G}$.

Since x is fixed from now on, we suppress it from the notation, and write

$$G_s := G_{x,s}, \qquad G_{s+} := G_{x,s+}, \qquad (s \geq 0)$$
$$\mathfrak{g}_s := \mathfrak{g}_{x,s}, \qquad \mathfrak{g}_{s+} := \mathfrak{g}_{x,s+}, \qquad (s \in \mathbb{R}).$$

In particular, we now write $G_0 := G_{x,0}$. We also set

$$\mathfrak{m}_s := \bigoplus_{\psi \in \Psi_s} \mathfrak{u}_\psi, \quad \mathfrak{m}_{s+} := \bigoplus_{\psi \in \Psi_{s+}} \mathfrak{u}_\psi. \tag{7}$$

so that $\mathfrak{g}_s/\mathfrak{g}_{s+} = \mathfrak{t}_s/\mathfrak{t}_{s+} \oplus \mathfrak{m}_s/\mathfrak{m}_{s+}$. Since $F \cdot x = x$, the sets Ψ_s and Ψ_s^+ are preserved by F, so all the groups and vector spaces above are F-stable. We set

$$V_s := \mathfrak{m}_s^F/\mathfrak{m}_{s+}^F.$$

Since $T^F \subset Z^F G_0^F$, and the latter normalizes G_s^F, it follows that we have an open subgroup $K_s := T^F G_s^F$ of G^F, and K_s is compact modulo Z^F. Our eventual supercuspidal representations of G^F will be compactly induced from K_s. We have a chain of normal subgroups of K_s: $J_{s+} \trianglelefteq J_s \trianglelefteq K_s$, where $J_s := \langle T_{2s}, U_\psi : \ \psi \in \Psi_s\rangle^F$, and $J_{s+} := \langle T_{2s}, U_\psi : \ \psi \in \Psi_{s+}\rangle^F$. From (7) we have $J_s/J_{s+} \simeq V_s$. Since $K_s = T^F J_s$, the multiplication map gives an exact sequence

$$1 \longrightarrow \Delta(T_{2s}^F) \longrightarrow T^F \ltimes J_s \longrightarrow K_s = T^F J_s \longrightarrow 1, \tag{8}$$

where $\Delta(T_{2s}^F) = \{t \ltimes t^{-1} \in T_{2s}^F \ltimes J_s : \ t \in T_{2s}^F\}$. The inducing representations of K_s will come from representations of $T_{2s}^F \ltimes J_s$ which are trivial on $\Delta(T_{2s}^F)$.

Let $\chi : T^F \to \mathbb{C}^\times$ be a character of T^F which is nontrivial on T_r^F and trivial on T_{r+1}^F, for some integer $r > 0$. In the previous constructions, we take $s = \frac{1}{2}r \in \frac{1}{2}\mathbb{Z}_{>0}$. As in [1], χ gives rise to a nontrivial homomorphism

$$\hat{\chi} : J_{s+} \longrightarrow \mathbb{F}_p \tag{9}$$

which agrees with χ on T_r^F and is trivial on $\langle U_\psi : \ \psi \in \Psi_{s+}\rangle^F$.

The commutator $[J_s, J_{s+}]$ is contained in $\ker \hat{\chi}$ [42, 4.2] and we have a nondegenerate symplectic pairing

$$V_s \times V_s \longrightarrow \mathbb{F}_p, \qquad \langle u, v\rangle := \hat{\chi}([\tilde{u}, \tilde{v}]),$$

where $\tilde{u}, \tilde{v}$ are lifts of u, v in J_s.

Let $V_s^\sharp = V \times \mathbb{F}_p$, with multiplication

$$(v, a) \cdot (u, b) = (v + u, a + b + \tfrac{1}{2}\langle v, u\rangle).$$

(Recall that $p > 2$.) As in [42, chap. 10] we can extend $\hat{\chi}$ to an isomorphism

$$\tilde{\chi} : J_s/\ker \hat{\chi} \xrightarrow{\sim} V_s^\sharp. \tag{10}$$

The adjoint action of T^F gives a homomorphism $f : T^F \to Sp(V)$ which is trivial on T^F_{0+}, and $\tilde{\chi}$ is T^F-equivariant, so we have a homomorphism

$$f \ltimes \tilde{\chi} : T^F \ltimes J_s \longrightarrow Sp(V) \ltimes V^\sharp. \tag{11}$$

Let ϕ_χ be the representation of $T^F \ltimes J_s$ obtained as the pullback, via (11), of the Weil representation of $Sp(V) \ltimes V^\sharp$ with central character $\hat{\chi}$. Since (11) maps J_s surjectively onto $V^\sharp$, the representation ϕ_χ is irreducible on J_s, hence is irreducible on $T^F \ltimes J_s$.

Inflate the original character $\chi \in \mathrm{Irr}(T^F)$ to a character of $T^F \ltimes J_s$ via the natural quotient $T^F \ltimes J_s \to T^F$. The tensor product $\kappa_\chi := \chi \otimes \phi_\chi$ is trivial on $\Delta(T^F_r)$, hence gives an irreducible representation of K_s, of dimension

$$\dim(\kappa_\chi) = \dim(\phi_\chi) = |V_s|^{1/2} = q^{m/2}, \tag{12}$$

where $m = |\{\alpha \in \Phi :\ \alpha(x) \in s + \mathbb{Z}\}|$.

The compactly-induced representation

$$\pi(T, \chi) := \mathrm{ind}_{K_s}^{G^F} \kappa_\chi \tag{13}$$

will be irreducible (hence supercuspidal) when χ satisfies a certain regularity condition. To state this condition, we must interpret characters as functionals on lattices. Fix henceforth an additive character $\Lambda : k^+ \to \mathbb{C}^\times$, whose kernel is $\mathfrak{o}$.

Suppose V is a K-vector space, defined over k, with Frobenius F. Then F acts naturally on the dual space $\check{V} = \mathrm{Hom}_K(V, K)$, and we identify $\check{V}^F = \mathrm{Hom}_k(V^F, k)$, via restriction. For any integer n, define

$$\check{\mathfrak{t}}_n := \{\lambda \in \mathrm{Hom}_K(\mathfrak{t}, K) :\ \langle \lambda, \mathfrak{t}_n\rangle \subseteq \mathfrak{O}\}.$$

Then we have a bijection

$$\check{\mathfrak{t}}^F_{r+1}/\check{\mathfrak{t}}^F_r \xrightarrow{\sim} \mathrm{Irr}(\mathfrak{t}^F_r/\mathfrak{t}^F_{r+1}), \qquad \lambda \mapsto \chi_\lambda, \tag{14}$$

where $\chi_\lambda(X + \check{\mathfrak{t}}^F_r) = \Lambda(\langle \lambda, X\rangle)$. Under the isomorphism $T^F_r/T^F_{r+1} \simeq \mathfrak{t}^F_r/\mathfrak{t}^F_{r+1}$, we have $\chi = \chi_\lambda$, for some $\lambda \in \check{\mathfrak{t}}^F_{r+1}$.

Let $N(T)$ be the normalizer of T in G, and let $W(T) = N(T)/T$ be the absolute Weyl group. Then $W(T)$ acts on on $\check{\mathfrak{t}}_{r+1}/\check{\mathfrak{t}}_r$. We say that $\chi = \chi_\lambda$ is **regular** if the stabilizer of $\lambda + \check{\mathfrak{t}}_r$ in $W(T)$ is trivial. In [1], Adler proved the following

Theorem 3.1. *If χ is regular, then $\pi(T, \chi)$ is an irreducible supercuspidal representation of G^F.*

4. Formal Degrees

In this chapter we compute the formal degree of $\pi(T,\chi)$. With appropriate normalizations of Haar measures, we will see that this formal degree depends only on the k-torus $\mathbf{T}$ and the depth of $\chi \in \mathrm{Irr}(T^F)$. In particular, we will see that the normalized formal degree is independent of the embedding of $\mathbf{T}$ in $\mathbf{G}$, as well as the fixed-point x of T^F in $\mathcal{B}(G)^F$. To make this clear, we restore x to the notation.

For any connected reductive k-group $\mathbf{H}$, there exists a unique Haar measure dh on H^F such that for any $x \in \mathcal{B}(H)^F$ we have

$$\mathrm{vol}(H_x^F, dh) = |\bar{H}_x^F| \cdot |\bar{\mathfrak{h}}_x^F|^{-1/2}, \tag{15}$$

where $\bar{H}_x = H_x/H_{x,+}, \bar{\mathfrak{h}}_x = \mathfrak{h}_x/\mathfrak{h}_{x,+}$. We call dh the **canonical Haar measure** on H^F.

We choose the canonical Haar measures dg on G^F, dt on T^F and dz on Z^F. Let $\mathrm{Deg}(\cdot)$ denote the formal degree with respect to the quotient measure dg/dz. Then the formal degree $\mathrm{Deg}(St_{G,F})$ of the Steinberg representation of G^F is the same for all inner twistings of $\mathbf{G}$ (see [14, 5.2]).

Now let χ be a regular character of T^F which is nontrivial on T_r^F and trivial on T_{r+1}^F for some integer $r > 0$, and set $s = r/2$.

Proposition 4.1. *With respect to canonical Haar measures, we have*

$$\mathrm{Deg}\left(\pi(T,\chi)\right) = \frac{q^{s|\Phi|}}{\mathrm{vol}(T^F/Z^F, dt/dz)}.$$

Proof. We start with the basic formula (see [8, A.14], for example)

$$\mathrm{Deg}\left(\pi(T,\chi)\right) = \frac{\dim(\kappa_\chi)}{\mathrm{vol}(K_s/Z^F, dg/dz)}.$$

From (12) we have $\dim(\kappa_\chi) = |\mathfrak{m}_{x,s}^F/\mathfrak{m}_{x,s+}^F|^{1/2}$ and from $K_s = T^F G_{x,s}^F$ we have $K_s/Z^F G_{x,s}^F = T^F/(T^F \cap Z^F G_{x,s}^F) = T^F/Z^F T_s^F$. Using the normalization (15), it is straightforward to check that

$$\mathrm{vol}(G_{x,s}^F, dg) = \frac{|\bar{\mathfrak{g}}_x^F|^{1/2}}{[\mathfrak{g}_x^F : \mathfrak{g}_{x,s}^F]} = \frac{|\bar{\mathfrak{g}}_x^F|^{1/2}}{[\mathfrak{t}^F : \mathfrak{t}_s^F] \cdot [\mathfrak{m}_x^F : \mathfrak{m}_{x,s}^F]}. \tag{16}$$

It follows that

$$\begin{aligned} \mathrm{vol}(K_s/Z^F, dg/dz) &= \frac{[T^F : Z^F T_s^F]}{[\mathfrak{t}^F : \mathfrak{t}_s^F] \cdot \mathrm{vol}(Z_s^F, dz)} \cdot \frac{|\bar{\mathfrak{g}}_x^F|^{1/2}}{[\mathfrak{m}_x^F : \mathfrak{m}_{x,s}^F]} \\ &= \mathrm{vol}(T^F/Z^F, dt/dz) \cdot \frac{|\bar{\mathfrak{m}}_x^F|^{1/2}}{[\mathfrak{m}_x^F : \mathfrak{m}_{x,s}^F]}, \end{aligned} \tag{17}$$

so we have

$$\mathrm{Deg}\,(\pi(T,\chi)) = \frac{q^D}{\mathrm{vol}(T^F/Z^F, dt/dz)}, \tag{18}$$

where

$$D = \tfrac{1}{2}\dim_{\mathfrak{F}}(\mathfrak{m}_{x,s}/\mathfrak{m}_{x,s+}) - \tfrac{1}{2}\dim_{\mathfrak{F}}(\bar{\mathfrak{m}}_x) + \dim_{\mathfrak{F}}(\mathfrak{m}_x/\mathfrak{m}_{x,s}).$$

Note that

$$\begin{aligned} \dim_{\mathfrak{F}}(\mathfrak{m}_{x,s}/\mathfrak{m}_{x,s+}) &= |\{\alpha \in \Phi :\ \alpha(x) \in s+\mathbb{Z}\}|, \\ \dim_{\mathfrak{F}}(\bar{\mathfrak{m}}_x) &= |\{\alpha \in \Phi :\ \alpha(x) \in \mathbb{Z}\}|, \\ \dim_{\mathfrak{F}}(\mathfrak{m}_x/\mathfrak{m}_{x,s}) &= \sum_{\alpha\in\Phi} [n(\alpha,0) - n(\alpha,s)]\,. \end{aligned} \tag{19}$$

We partition the roots as $\Phi = \Phi_1 \sqcup \Phi_2 \sqcup \Phi_3 \sqcup \Phi_4$, where

$$\begin{aligned} \Phi_1 &= \{\alpha \in \Phi :\ \alpha(x) \in \mathbb{Z}\}, &\quad \Phi_2 &= \{\alpha \in \Phi :\ \alpha(x) \in (0,\tfrac{1}{2}) + \mathbb{Z}\}, \\ \Phi_3 &= \{\alpha \in \Phi :\ \alpha(x) \in (\tfrac{1}{2},1) + \mathbb{Z}\}, &\quad \Phi_4 &= \{\alpha \in \Phi :\ \alpha(x) \in \tfrac{1}{2} + \mathbb{Z}\}. \end{aligned}$$

Note that sending α to $-\alpha$ preserves Φ_1 and Φ_4, and interchanges Φ_2 and Φ_3. Let $2A = |\Phi_1|, B = |\Phi_2| = |\Phi_3|, 2C = |\Phi_4|$. Then $\dim_{\mathfrak{F}}(\bar{\mathfrak{m}}_x) = 2A$, and

$$\dim_{\mathfrak{F}}(\mathfrak{m}_{x,s}/\mathfrak{m}_{x,s+}) = \begin{cases} 2A & \text{if } \ s \in \mathbb{Z} \\ 2C & \text{if } \ s \in \frac{1}{2} + \mathbb{Z}. \end{cases} \tag{20}$$

If $s \in \mathbb{Z}$, we have $n(\alpha,s) - n(\alpha,0) = s$, so that $\dim_{\mathfrak{F}}(\mathfrak{m}_x/\mathfrak{m}_{x,s}) = s|\Phi|$.

If $s \in \frac{1}{2} + \mathbb{Z}$, we have

$$n(\alpha,s) - n(\alpha,0) = \begin{cases} s + \frac{1}{2} & \text{if } \ \alpha \in \Phi_1 \cup \Phi_2 \\ s - \frac{1}{2} & \text{if } \ \alpha \in \Phi_3 \cup \Phi_4, \end{cases} \tag{21}$$

which implies that $\dim_{\mathfrak{F}}(\mathfrak{m}_x/\mathfrak{m}_{x,s}) = s|\Phi| + A - C$. It follows that $D = s|\Phi|$ in both cases. From (18) we get the formal degree claimed in 4.1. □

The following alternate viewpoint is suggestive. The inducing group K_s is contained in $Z^F G_x^F$, so we can also view $\pi(T,\chi)$ as induced from the finite-dimensional irreducible representation

$$R(T,\chi) := \mathrm{Ind}_{K_s}^{Z^F G_x^F}\, \kappa_\chi \tag{22}$$

on $Z^F G_x^F$. Set $\bar{G}_x = G_x/G_{x+},\ \bar{T} = T_0/T_1$. Using the equations used to compute D above, one finds that

$$\dim R(T,\chi) = q^{s|\Phi|} \cdot [\bar{G}_x^F : \bar{T}^F]_{p'}, \tag{23}$$

where $[\cdots]_{p'}$ is the largest factor of the index not divisible by p. In the depth-zero case [14, chap.5], the inducing representation arises from a completely different, cohomological construction [16] and has dimension $[\bar{G}_x^F : \bar{T}^F]_{p'}$. Hence equation (23) reduces the proof of the constancy of formal degrees to the depth-zero case, which was proved in [14, chap.5], and suggests that $R(T,\chi)$ should be a positive-depth analogue of a Deligne-Lusztig representation (cf. [31]).

5. Lubin-Tate extensions and tori

The Langlands correspondence for tori is well known [29]. However, we need two properties of it which do not seem to be in the literature: We require our correspondence to preserve depth, and to be natural with respect to automorphisms. These requirements are easily seen to hold if we reformulate the correspondence (for unramified tori only) in terms of Lubin and Tate's explicit form of abelian class field theory.

The method is essentially that used in [14, 4.3]. There, depth-zero characters were parametrized using the Weil group of the maximal tame extension k_t/k of k. Note that k_t is abelian over K. Here, for arbitrary depth, the relevant Weil group is that of K^{ab}/k, where K^{ab} is the maximal abelian extension of K.

5.1. Lubin-Tate extensions.

In this section we review some results in [30]. Recall that K is the maximal unramified extension of k contained in a fixed algebraic closure $\bar{k}$ of k. For $d \geq 1$ an integer, let $k_d \subset K$ be the unramified extension of k of degree d. We let $\mathfrak{o}_d$ be the ring of integers of k_d and let $\mathfrak{p}_d$ be the prime ideal of $\mathfrak{o}_d$.

Lubin and Tate construct the maximal abelian extension k_d^{ab} of k_d in the form of a tower

$$k_d \subset K \subset K_d^{(1)} \subset K_d^{(2)} \subset \cdots \bigcup_{n\geq 1} K_d^{(n)} = k_d^{ab},$$

as follows. Fix a prime element $\varpi \in k$, and consider the polynomial $f_d = \varpi X + X^{q^d} \in \mathfrak{o}[X]$. Let $\Lambda_d^{(n)} \subset \bar{k}$ be the set of zeros of the n-fold iteration $f_d^{(n)} := f_d \circ \cdots \circ f_d$. Then $K_d^{(n)} = K(\Lambda_d^{(n)})$ is the field generated over K by $\Lambda_d^{(n)}$.

It is easy to see that $f_d^{(n)}(X) = X h_1(X) \cdots h_n(X)$ where each $h_i(X)$ is an Eisenstein polynomial in $\mathfrak{o}_d[X]$. This implies that the degree of $K_d^{(n)}/K$ is given by

$$[K_d^{(n)} : K] = q^{d(n-1)}(q^d - 1). \tag{24}$$

According to Lubin-Tate, the Galois groups $\mathrm{Gal}(K_d^{(n)}/K)$ can be described in a manner analogous to those of cyclotomic extensions of $\mathbb{Q}$, with the group $\bar{\mathbb{Q}}^\times$ replaced by the unique (one-dimensional commutative) formal group $G_d(X,Y) \in \mathfrak{o}_d[[X,Y]]$ admitting f_d as an endomorphism. The key fact is that for each $\alpha \in \mathfrak{o}_d$, there is a unique power series $[\alpha]_d$ of the form

$$[\alpha]_d = \alpha T + \text{(higher order terms)} \in T\mathfrak{o}_d[[T]]$$

commuting with f_d under composition. Then $[\alpha]_d \in \mathrm{End}(G_d)$ and the map $\alpha \mapsto [\alpha]_d$ is a ring homomorphism $[\ \ ]_d : \mathfrak{o}_d \to \mathrm{End}(G_d)$, such that $[\varpi^n]_d = f_d^{(n)}$ for all $n \geq 1$.

Let $\bar{\mathfrak{p}}$ be the subring of $\bar{k}$ whose elements have (extended) norm < 1. Since the series $G_d(x,y)$ converges for $x, y \in \bar{\mathfrak{p}}$, we can put a new abelian group structure on $\bar{\mathfrak{p}}$ via the addition rule $x \dot{+} y = G_d(x,y)$. Let $G_d(\bar{\mathfrak{p}})$ denote the group $(\bar{\mathfrak{p}}, \dot{+})$. It is an $\mathfrak{o}_d$-module, via the endomorphisms $[\alpha]_d$. Since $f_d^{(n)} \equiv X^{q^{nd}} \mod \mathfrak{p}$, it follows that $\Lambda_d^{(n)} \subset \bar{\mathfrak{p}}$. Since $\Lambda_d^{(n)} = \ker f_d^{(n)} = \ker[\varpi]_d^n$, the set $\Lambda_d^{(n)}$ is an $\mathfrak{o}_d$ - submodule of $G_d(\bar{\mathfrak{p}})$. By construction, the annihilator of $\Lambda_d^{(n)}$ is $\mathfrak{p}_d^n$, so we have $\Lambda_d^{(n)} \simeq \mathfrak{o}_d/\mathfrak{p}_d^n$, as $\mathfrak{o}_d$-modules.

The action of $\mathrm{Gal}(K_d^{(n)}/K)$ on $\Lambda_d^{(n)}$ commutes with the $\mathfrak{o}_d$-action, so we have an injection $\mathrm{Gal}(K_d^{(n)}/K) \hookrightarrow \mathrm{Aut}_{\mathfrak{o}_d}(\Lambda_d^{(n)}) = \mathfrak{o}_d^\times/(1+\mathfrak{p}_d^n)$, and (24) shows that this injection is also surjective. In this way, we get the reciprocity isomorphism

$$\mathsf{r}_d^{(n)} : \mathrm{Gal}(K_d^{(n)}/K) \xrightarrow{\sim} \mathfrak{o}_d^\times/(1+\mathfrak{p}_d^n), \tag{25}$$

characterized by the property that $[\mathsf{r}_d^{(n)}(\gamma)]_d = \gamma^{-1} \in \mathrm{Aut}_{\mathfrak{o}_d}(\Lambda_d^{(n)})$ for any $\gamma \in \mathrm{Gal}(K_d^{(n)}/K)$.

Finally, Lubin and Tate show that field $K_d^{(n)}$ and the homomorphism $\mathsf{r}_d^{(n)}$ are independent of the choice of prime element ϖ used to define f_d and that $\bigcup_{n\geq 1} K_d^{(n)}$ is indeed the maximal abelian extension k_d^{ab} of k_d. The maps $\mathsf{r}_d^{(n)}$ piece together to give an isomorphism

$$\mathsf{r}_d : \mathrm{Gal}(k_d^{ab}/K) \xrightarrow{\sim} \varprojlim_n \mathfrak{o}_d^\times/(1+\mathfrak{p}_d^n) \ = \ \mathfrak{o}_d^\times.$$

In terms of inertia groups, the above reciprocity isomorphisms read as follows. Let $\mathcal{W}(k_d)$ be the absolute Weil group of k_d. Note that $\mathcal{I} = \mathrm{Gal}(\bar{k}/K)$ is the inertia subgroup of $\mathcal{W}(k_d)$ for every d. Let $\mathcal{I}_d := \mathrm{Gal}(\bar{k}/k_d^{ab})$, so that $\mathcal{I}/\mathcal{I}_d = \mathrm{Gal}(k_d^{ab}/K)$. Pulling back via the quotient $\mathcal{I} \to \mathcal{I}/\mathcal{I}_d$, the reciprocity map r_d may be viewed a surjective homomorphism

$$\mathsf{r}_d : \mathcal{I} \to \mathcal{I}/\mathcal{I}_d \xrightarrow{\sim} \mathfrak{o}_d^\times \tag{26}$$

whose kernel is $\mathcal{I}_d$. If we set $\mathcal{I}_d^{(n)} := \mathsf{r}_d^{-1}(1+\mathfrak{p}_d^n)$, for $n \geq 1$, then the original isomorphism (25) now reads as

$$\mathsf{r}_d^{(n)} : \mathcal{I}/\mathcal{I}_d^{(n)} \xrightarrow{\sim} \mathfrak{o}_d^\times/(1+\mathfrak{p}_d^n). \tag{27}$$

5.2. **The maximal abelian extension of K.** We have so far considered d as fixed; now we study the effect of varying d. If $c \mid d$, we have [39] $\mathsf{r}_c = N_{c|d} \circ \mathsf{r}_d$, where $N_{c|d} : k_d^\times \to k_c^\times$ is the norm homomorphism. Since k_d/k is unramified, we have [36] $N_{c|d}(1+\mathfrak{p}_d^n) = 1+\mathfrak{p}_c^n$. Since $\mathcal{I}_d^{(n)} = \mathsf{r}_d^{-1}(1+\mathfrak{p}_d^n)$, we have $\mathcal{I}_d^{(n)} \subset \mathcal{I}_c^{(n)}$ and $K_c^{(n)} \subset K_d^{(n)}$. The natural quotient map $j_{c|d} : \mathcal{I}/\mathcal{I}_d^{(n)} \to \mathcal{I}/\mathcal{I}_c^{(n)}$ fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{I}/\mathcal{I}_d^{(n)} \arrow[r, "\mathsf{r}_d^{(n)}"] \arrow[d, "j_{c|d}"'] & \mathfrak{o}_d^\times/(1+\mathfrak{p}_d^n) \arrow[d, "N_{c|d}"] \\
\mathcal{I}/\mathcal{I}_c^{(n)} \arrow[r, "\mathsf{r}_c^{(n)}"] & \mathfrak{o}_c^\times/(1+\mathfrak{p}_c^n).
\end{tikzcd}$$

Thus, if we set $\mathcal{I}^{(n)} = \bigcap_{d\geq 1} \mathcal{I}_d^{(n)}$, the reciprocity maps $\mathsf{r}_d^{(n)}$ fit together to make an isomorphism

$$\mathsf{r}^{(n)} : \mathcal{I}/\mathcal{I}^{(n)} \xrightarrow{\sim} \varprojlim_d \mathfrak{o}_d^\times/(1+\mathfrak{p}_d^n), \tag{28}$$

where the transition functions in the projective limit are induced by the norm maps $N_{c|d}$. The isomorphism $\mathsf{r}^{(n)}$ intertwines the automorphism $\mathrm{Ad}(\mathrm{Frob})$ on $\mathcal{I}/\mathcal{I}^{(n)}$ (induced by conjugation by Frob on $\mathcal{I}$) with the automorphism on the projective limit induced by the Galois action of Frob on each group $\mathfrak{o}_d^\times$.

For each $d \geq 1$, the canonical projection $\mathcal{I}/\mathcal{I}^{(n)} \to \mathfrak{o}_d^\times/(1+\mathfrak{p}_d^n)$ induces an isomorphism

$$[\mathcal{I}/\mathcal{I}^{(n)}]_{\mathrm{Ad}(\mathrm{Frob}^d)} \xrightarrow{\sim} \mathfrak{o}_d^\times/(1+\mathfrak{p}^n) \tag{29}$$

where $[\mathcal{I}/\mathcal{I}^{(n)}]_{\mathrm{Ad}(\mathrm{Frob}^d)}$ denotes the co-invariants of $\mathrm{Ad}(\mathrm{Frob}^d)$ in $\mathcal{I}/\mathcal{I}^{(n)}$.

In terms of Galois groups, we have $\mathcal{I}/\mathcal{I}^{(n)} = \mathrm{Gal}(K^{(n)}/K)$, where $K^{(n)} = \bigcup_{d\geq 1} K_d^{(n)}$. The field $K^{(1)} = k_t$ is the maximal tame extension of k and is also the maximal tame abelian extension of K. The union $\bigcup_{n\geq 1} K^{(n)}$ is the maximal abelian extension K^{ab} of K. The intermediate fields $k \subset L \subset K^{ab}$ are exactly those extensions L of k in $\bar{k}$ which are abelian over their unramified part, that is, those for which $L/L\cap K$ is abelian.

5.3. **Langlands correspondence for unramified tori.** Let $\mathbf{T}$ be a k-torus splitting over K, let F be the Frobenius endomorphism of $T=\mathbf{T}(K)$, and abbreviate $X:=\mathrm{Hom}(\mathbf{GL}_1,\mathbf{T}), Y:=\mathrm{Hom}(\mathbf{T},\mathbf{GL}_1)$. Then F acts on X via an automorphism σ of finite order, say d. Evaluation at ϖ gives an embedding $X\hookrightarrow T$, which allows us to identify $T=X\otimes K^\times$ and $F=\sigma\otimes\mathrm{Frob}$. Note that $\mathbf{T}$ splits over k_d, so that $\mathbf{T}(k_d)=T^{F^d}=X\otimes k_d^\times$. Let

$$N_\sigma: T^{F^d}\longrightarrow T^F,\qquad N_\sigma(t)=tF(t)\cdots F^{d-1}(t) \tag{30}$$

be the norm mapping.

The filtration groups T_r are F-stable. For $r>s>0$ and any $d\geq 0$, we have

$$T_r^{F^d}/T_s^{F^d}=(T_r/T_s)^{F^d},\quad \text{and}\quad (T_0/T_r)^{F^d}=X\otimes(\mathfrak{o}_d^\times/1+\mathfrak{p}_d^r).$$

Lemma 5.1. *For every $r\geq 0$ we have an exact sequence*

$$1\longrightarrow T_r^F\longrightarrow T_r^{F^d}\xrightarrow{1-F} T_r^{F^d}\xrightarrow{N_\sigma} T_r^F\longrightarrow 1.$$

Proof. Exactness at the first two terms (reading from the left) is clear. Exactness at the third term follows from the profinite version of Lang's theorem, which allows us to write any $t\in T_r^{F^d}$ in the form $s^{-1}F(s)$ for some $s\in T_r$. One checks that, if $N_\sigma(t)=1$, then $s\in T^{F^d}$.

It remains to show that N_σ is surjective. Replacing T_r by T_r/T_{r+1}, we get a sequence

$$1\to(T_r/T_{r+1})^F\to(T_r/T_{r+1})^{F^d}\xrightarrow{1-F}(T_r/T_{r+1})^{F^d}\xrightarrow{\bar N_\sigma}(T_r/T_{r+1})^F\to 1.$$

Taking the Euler characteristic, we see that the image of $|\bar N_\sigma|$ has cardinality that of $(T_r/T_{r+1})^F$, so $\bar N_\sigma$ is surjective. It follows [37] that N_σ is surjective. $\square$

Let $\alpha\mapsto\hat\alpha:\mathrm{Aut}(X)\to\mathrm{Aut}(Y)$ be the anti-automorphism given by duality. Then $\hat\sigma=\hat\sigma\otimes\mathrm{Id}$ acts on the dual torus $\hat T:=Y\otimes\mathbb{C}^\times$, and we can form the semidirect product ${}^LT:=\langle\hat\sigma\rangle\ltimes\hat T$.

Now we consider the group of characters of T_0^F which are trivial on T_{r+1}^F. Given automorphisms α,β of abelian groups A,B, respectively, let $\mathrm{Hom}_{\alpha,\beta}(A,B)$ denote the set of homomorphisms $f:A\to B$ such that

$f \circ \alpha = \beta \circ f$. We have

$$\begin{aligned}\mathrm{Hom}(T_0^F/T_{r+1}^F, \mathbb{C}^\times) &\overset{5.1}{=} \mathrm{Hom}_{F,\mathrm{Id}}\left(T_0^{F^d}/T_{r+1}^{F^d}\,,\ \mathbb{C}^\times\right)\\ &= \mathrm{Hom}_{F,\mathrm{Id}}\ \left(X \otimes (\mathfrak{o}_d^\times/1+\mathfrak{p}_d^{r+1})\,,\ \mathbb{C}^\times\right)\\ &= \mathrm{Hom}_{\mathrm{Frob},\hat{\sigma}}\left(\mathfrak{o}_d^\times/1+\mathfrak{p}_d^{r+1}\,,\ \hat{T}\right)\\ &\simeq \mathrm{Hom}_{\mathrm{Ad\,Frob},\hat{\sigma}}\left(\mathcal{I}/\mathcal{I}_d^{(r+1)}\,,\ \hat{T}\right),\end{aligned} \tag{31}$$

the last isomorphism coming from abelian reciprocity (27). Since $\hat{\sigma}$ has order d, the latter group consists exactly of the restrictions to $\mathcal{I}/\mathcal{I}^{(r+1)}$ of continuous homomorphisms

$$\varphi : \mathcal{W}(k)/\mathcal{I}^{(r+1)} \longrightarrow \langle\hat{\sigma}\rangle \ltimes \hat{T} \tag{32}$$

for which $\varphi(\mathrm{Frob}) \in \hat{\sigma} \ltimes \hat{T}$.

The $\hat{T}$-conjugacy class of those φ with a given restriction to $\mathcal{I}/\mathcal{I}^{(r+1)}$ is determined by the $\hat{\sigma}$-twisted $\hat{T}$-conjugacy class of the element $\tau \in \hat{T}$, where $\varphi(\mathrm{Frob}) = \hat{\sigma} \ltimes \tau$. In turn, the $\hat{\sigma}$-twisted conjugacy class of τ is nothing but a character of X^σ. Since T^F is a direct product $T^F = X^\sigma \times T_0^F$, we have shown that the characters of T^F which are trivial on T_{r+1}^F are in bijection with $\hat{T}$-conjugacy classes of Langlands parameters φ, as in (32).

To summarize the bijection: the character $\chi_\varphi : T^F/T_{r+1}^F \longrightarrow \mathbb{C}^\times$ corresponding to the parameter φ in (32) is determined by the two equations:

$$\chi_\varphi \circ N_\sigma\left(\lambda \otimes \mathsf{r}_d(x)\right) = \lambda\left(\varphi(x)\right) \quad \text{and} \quad \chi_\varphi(\mu) = \mu(\tau) \tag{33}$$

for all $\lambda \in X,\ \mu \in X^\sigma,\ x \in \mathcal{I}$, where r_d is the reciprocity map (26), N_σ is the norm mapping (30), and $\tau \in \hat{T}$ is given by $\varphi(\mathrm{Frob}) = \hat{\sigma} \ltimes \tau$.

This correspondence $\varphi \mapsto \chi_\varphi$ has the following naturality property. Let α be a k-automorphism of $\mathbf{T}$. Then $\alpha \in \mathrm{Aut}(X)$ commutes with σ and $\hat{\alpha} \in \mathrm{Aut}(Y)$ commutes with $\hat{\sigma}$. We can therefore extend $\hat{\alpha}$ to an automorphism of ${}^LT$. A computation identical to that of [14, 4.3.1] shows that

$$\chi_\varphi \circ \alpha = \chi_{\hat{\alpha}\circ\varphi}. \tag{34}$$

6. L-PACKETS

In this section we construct our L-packets. The elements of these packets are certain equivalence classes that generalize the notion of representation. We briefly explain this first, before embarking on the construction.

6.1. **Galois cohomology and representations.** For more details in this section, see [14, chaps. 2,3]. Let $\mathbf{G}$ be a connected reductive k-group with Frobenius automorphism F on $G := \mathbf{G}(K)$. Each element

$$u \in Z^1(F,G) = \{u \in G :\ u \cdot F(u) \cdots F^{n-1}(u) = 1,\ \text{for some } n \geq 1\}.$$

arises from a k-structure on $\mathbf{G}$ with Frobenius $\mathrm{Ad}(u) \circ F$ on G. Denoting $\mathbf{G}$ with this new k-structure by $\mathbf{G}_u$, we have

$$\mathbf{G}_u(k) = G^{\mathrm{Ad}(u)\circ F}.$$

The group G acts on $Z^1(F,G)$ by $g * u = guF(g)^{-1}$; the set of G-orbits in $Z^1(F,G)$ is denoted $H^1(F,G)$. Evaluating cocycles at Frob gives a bijection $H^1(k,\mathbf{G}) \xrightarrow{\sim} H^1(F,G)$, where $H^1(k,\mathbf{G})$ denotes the first Galois cohomology set of $\mathbf{G}$.

For each $u \in Z^1(F,G)$, the map $\mathrm{Ad}(g)$ intertwines $\mathrm{Ad}(u)\circ F$ and $\mathrm{Ad}(g*u) \circ F$. It follows that $\mathrm{Ad}(g)$ sends $\mathbf{G}_u(k)$ to $\mathbf{G}_{g*u}(k)$ and hence induces a bijection

$$\mathrm{Irr}\left(\mathbf{G}_u(k)\right) \longrightarrow \mathrm{Irr}\left(\mathbf{G}_{g*u}(k)\right), \quad \text{by} \quad \pi \mapsto {}^g\pi = \pi \circ \mathrm{Ad}(g)^{-1}.$$

Hence G acts on the set of pairs

$$\mathcal{R}(F,G) := \{(u,\pi) :\ u \in Z^1(F,G),\ \pi \in \mathrm{Irr}\left(\mathbf{G}_u(k)\right)\},$$

by the rule $g \cdot (u,\pi) := (g*u, {}^g\pi)$. We let $[u,\pi] \in \mathcal{R}(F,G)/G$ denote the G-orbit of (u,π). Projecting onto $Z^1(F,G)$ gives a partition

$$\mathcal{R}(F,G)/G = \coprod_{\omega \in H^1(F,G)} \mathcal{R}(F,G,\omega)/G,$$

where for each class $\omega \in H^1(F,G)$, the set $\mathcal{R}(F,G,\omega)$ consists of those pairs $(u,\pi) \in \mathcal{R}(F,G)$ for which $u \in \omega$.

6.2. **Unramified groups.** We now adopt our previous set-up. That is, we assume that $\mathbf{G}$ is a connected reductive k-group which is K-split and k-quasisplit. We write F for the corresponding Frobenius endomorphism of $G = \mathbf{G}(K)$. (The change from F to F signifies that F arises from a quasisplit k-structure on $\mathbf{G}$.) Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ defined over k, and let $\mathbf{T}$ be a maximal torus of $\mathbf{B}$. Then $\mathbf{T}$ is defined over k and split over K. (Note: This torus is different from the minisotropic torus used in chapter 3. We will eventually apply the construction of chapter 3 to minisotropic twists of the present $\mathbf{T}$.) Let $\mathbf{N}$ be the normalizer of $\mathbf{T}$ in $\mathbf{G}$, and write $X = \mathrm{Hom}(\mathbf{GL}_1,\mathbf{T}), Y = \mathrm{Hom}(\mathbf{T},\mathbf{GL}_1)$ as before. Let $W = N/T_0$ be the affine Weyl group of G. For $\lambda \in X$, let t_λ be the image of $\lambda(\varpi)$ in W. Thus we view X as a subgroup of W.

The Frobenius F acts on X and W via an automorphism ϑ of finite order. Moreover, ϑ preserves a hyperspecial vertex $o \in \mathcal{A}(T)$, since $\mathbf{G}$ is

k-quasisplit. The affine Weyl group decomposes as $W = X \rtimes W_o$. By duality we have $\hat\vartheta \in \mathrm{Aut}(Y)$ and a $\hat\vartheta$-stable subgroup $\hat W_o \subset \mathrm{Aut}(Y)$. The action of the group $\langle\hat\vartheta\rangle \ltimes \hat W_o$ on Y extends to the dual torus $\hat T = Y \otimes \mathbb{C}^\times$, acting trivially on $\mathbb{C}^\times$. We identify $X = \mathrm{Hom}(\hat T, \mathbb{C}^\times)$.

Let $\hat G$ be the dual group of $\mathbf{G}$, so that $\hat T$ is a maximal torus of $\hat G$. We identify $\hat W_o$ with the Weyl group of $\hat T$ in $\hat G$. Fix a pinning in $\hat G$ containing $\hat T$. There is a unique extension of $\hat\vartheta$ to an automorphism of $\hat G$ preserving the pinning. Set ${}^LG := \langle\hat\vartheta\rangle \ltimes \hat G$, and let $\hat Z^{\hat\vartheta}$ denote the fixed-points of $\hat\vartheta$ in the center $\hat Z$ of $\hat G$. In fact, $\hat Z^{\hat\vartheta}$ is the center of ${}^LG$.

6.3. **Langlands parameters.** Let $\mathcal{W} = \mathcal{W}(k)$ be the Weil group of k. We consider Langlands parameters $\varphi : \mathcal{W} \to {}^LG$ satisfying the following three conditions:

(1) The map φ is trivial on $\mathcal{I}^{(r+1)}$ and nontrivial on $\mathcal{I}^{(r)}$, for some integer $r > 0$.
(2) The centralizer of $\varphi(\mathcal{I}^{(r)})$ in $\hat G$ is a maximal torus of $\hat G$.
(3) $\varphi(\mathrm{Frob}) \in \hat\vartheta \ltimes \hat G$, and the centralizer of $\varphi(\mathcal{W})$ in $\hat G$ is finite, modulo $\hat Z^{\hat\vartheta}$.

These are the conditions of [14] except that here φ is not required to be trivial on the wild inertia group $\mathcal{I}^{(1)}$. Condition 1 implies that $\varphi(\mathcal{I})$ is abelian. Condition 2 is the regularity condition and Condition 3 is the ellipticity condition. We may and shall always choose φ in its $\hat G$-conjugacy class so that the torus of Condition 2 is $\hat T$. That implies in particular that $\varphi(\mathcal{I}) \subset \hat T$. Since Frob normalizes $\mathcal{I}$, Condition 3 implies that $\varphi(\mathrm{Frob}) = \hat\vartheta \ltimes \hat n$, for some $\hat n \in N_{\hat G}(\hat T)$ which projects to an element $\hat w \in \hat W_o$. We say that the dual element $w \in W_o$ is **associated to** φ.

The $\hat G$-centralizer of φ is given by $C_{\hat G}(\varphi) = \hat T^{\varphi(\mathrm{Frob})} = \hat T^{\hat\vartheta \ltimes \hat w} = \hat T^{\widehat{w\vartheta}}$, hence, if C_φ denotes the component group of $C_{\hat G}(\varphi)$, we have an isomorphism

$$[X/(1-w\vartheta)X]_{\mathrm{tor}} \xrightarrow{\sim} \mathrm{Irr}(C_\varphi),$$

given by restriction, where $[\cdots]_{\mathrm{tor}}$ denotes torsion subgroup. Let X_φ be the set of elements of X whose coset in $X/(1-w\vartheta)X$ belongs to $[X/(1-w\vartheta)]_{\mathrm{tor}}$. (Note that $X_\varphi = X$ if G^{F} has compact center, or equivalently, if $\hat Z^{\hat\vartheta}$ is finite.) For $\lambda \in X_\varphi$, we let $\rho_\lambda \in \mathrm{Irr}(C_\varphi)$ be the restriction of λ to C_φ.

6.4. **Vertices and pure inner forms.** Let φ be a Langlands parameter satisfying the conditions of section 6.3, with associated $w \in W_o$. Let $\lambda \in X_\varphi$. To this data we associate, as in [14], a point $x_\lambda \in \mathcal{A}(T)$ which will play the role of x in the earlier chapters, along with a cocycle $u_\lambda \in Z^1(\mathrm{F}, G)$. This goes as follows. By Condition 3, the element $t_\lambda w\vartheta \in W\vartheta$ has a unique fixed-point $x_\lambda \in \mathcal{A}(T)$, given by $x_\lambda = (1 - w\vartheta)^{-1}t_\lambda \cdot o$.

Choose an alcove C_λ in $\mathcal{A}(T)$ containing x_λ in its closure. We can uniquely write

$$t_\lambda w\vartheta = w_\lambda y_\lambda \vartheta, \tag{35}$$

where w_λ belongs to the subgroup W_{x_λ} of W generated by reflections about the affine root hyperplanes in $\mathcal{A}(T)$ containing x_λ, and $y_\lambda \in W$ is such that $y_\lambda\vartheta \cdot C_\lambda = C_\lambda$.

From [14, 2.6] the element y_λ has a lift $u_\lambda \in N \cap Z^1(\mathrm{F}, N)$. As in section 6.1, this gives a twisted k-group $\mathbf{G}_\lambda = \mathbf{G}_{u_\lambda}$ (no longer k-quasisplit, in general) with Frobenius $\mathrm{F}_\lambda := \mathrm{Ad}(u_\lambda) \circ \mathrm{F}$. We have $\mathbf{G}_\lambda(K) = \mathbf{G}(K) = G$ and $\mathbf{G}_\lambda(k) = G^{\mathrm{F}_\lambda}$.

By construction, we have $\mathrm{F}_\lambda \cdot x_\lambda = x_\lambda$, and in fact x_λ is a vertex in $\mathcal{B}(G)^{\mathrm{F}_\lambda}$ (though x_λ is not always a vertex in $\mathcal{B}(G)$). Let G_{x_λ} be the parahoric subgroup of G at x_λ. There is an element $p_\lambda \in G_{x_\lambda}$ such that $p_\lambda^{-1}\,\mathrm{F}_\lambda(p_\lambda)$ belongs to N and is a lift of w_λ. Let $\mathbf{T}_\lambda := p_\lambda \mathbf{T} p_\lambda^{-1}$. Then $\mathrm{Ad}(p_\lambda) : \mathbf{T} \to \mathbf{T}_\lambda$ is a k-isomorphism which intertwines $w\,\mathrm{F}$ on T with F_λ on T_λ. The torus $\mathbf{T}_\lambda$ is an F_λ-minisotropic maximal torus in $\mathbf{G}_\lambda$, and x_λ is the unique fixed-point of $T_\lambda^{\mathrm{F}_\lambda}$ in $\mathcal{B}(G)^{\mathrm{F}_\lambda}$.

6.5. **Invoking the abelian Langlands correspondence.** Let φ be a Langlands parameter satisfying the conditions of section 6.3, with associated $w \in W_o$ and set $\sigma = w\vartheta$. We will construct from φ a $\hat{T}$-conjugacy class of Langlands parameters $\varphi_T : \mathcal{W} \to {}^L\hat{T}$, such that $\varphi_T = \varphi$ on $\mathcal{I}$, and such that $\varphi_T(\mathrm{Frob})$ and $\varphi(\mathrm{Frob})$ have the same action on $\hat{T}$. Since ${}^LT$ is not a subgroup of ${}^LG$, this is not quite immediate. We will have $\varphi_T(\mathrm{Frob}) = \hat{\sigma} \ltimes \tau$ for some $\tau \in \hat{T}$, which is only defined up to $\hat{\sigma}$-twisted conjugacy. The coset of τ in $\hat{T}/(1-\hat{\sigma})\hat{T}$ is defined as in [14], as follows. Let $\hat{G}'$ be the derived group of $\hat{G}$, let $\hat{T}' = \hat{T} \cap \hat{G}'$ and let $\hat{G}_{ab} = \hat{G}/\hat{G}'$. Condition 2 implies that the map $\tau \mapsto \tau\hat{\sigma}(\tau)^{-1}$ has finite kernel on $\hat{T}'$, which means that $(1-\hat{\sigma})\hat{T}' = \hat{T}'$. Hence the inclusion $\hat{T} \hookrightarrow \hat{G}$ induces a bijection $\hat{T}/(1-\hat{\sigma})\hat{T}' \xrightarrow{\sim} \hat{G}_{ab}$. It follows that $\hat{T} \hookrightarrow \hat{G}$ induces a bijection

$$\hat{T}/(1-\hat{\sigma})\hat{T} \xrightarrow{\ \sim\ } \hat{G}_{ab}/(1-\hat{\vartheta})\hat{G}_{ab}. \tag{36}$$

Now, if $\varphi(\mathrm{Frob}) = \hat{\vartheta} \ltimes \hat{n}$, we take any $\tau \in \hat{T}$ whose class in $\hat{T}/(1-\hat{\sigma})\hat{T}$ corresponds under (36) to the image of $\hat{n}$ in $\hat{G}_{ab}/(1-\hat{\vartheta})\hat{G}_{ab}$. This procedure gives the desired Langlands parameter φ_T.

Let r be the largest integer such that φ is nontrivial on $\mathcal{I}^{(r)}$. By the Langlands correspondence for tori, as given in section 5.3, the parameter

φ_T gives a character $\chi_\varphi \in \operatorname{Irr}(T^{w\,\mathrm{F}})$ which is nontrivial on $T_r^{w\,\mathrm{F}}$ and trivial on $T_{r+1}^{w\,\mathrm{F}}$; we say χ_φ has **depth** r.

Conjugating by $\operatorname{Ad}(p_\lambda)$, we get a character

$$\chi_\lambda := \chi_\varphi \circ \operatorname{Ad}(p_\lambda)^{-1} \in \operatorname{Irr}(T^{\mathrm{F}_\lambda}). \tag{37}$$

Since $p_\lambda \in G$, it follows that χ_λ also has depth r. By the naturality property (34), the regularity Condition 2 on φ implies that χ_λ satisfies the regularity condition of Theorem 3.1.

6.6. **Supercuspidal L-packets.** Let φ be a Langlands parameter satisfying the conditions of section 6.3. We can now apply the construction of chapter 3 to the group $\mathbf{G}_\lambda$ with Frobenius F_λ, the F_λ-minisotropic torus $\mathbf{T}_\lambda$ with unique fixed-point x_λ of $T_\lambda^{\mathrm{F}_\lambda}$ in $\mathcal{B}(G)^{\mathrm{F}_\lambda}$, and the character χ_λ of $T_\lambda^{\mathrm{F}_\lambda}$; this gives an irreducible supercuspidal representation $\pi_\lambda := \pi(T_\lambda, \chi_\lambda)$ of G^{F_λ}.

Lemma 4.4.2 of [14], which does not depend on the depth of representations, shows that, for fixed φ and λ, the isomorphism class of π_λ is independent of the choices made in the construction.

We thus have infinitely many groups G^{F_λ} and representations π_λ. However, these form only finitely many equivalence classes, in the sense of section 6.1.

Lemma 6.1. *Let φ be a Langlands parameter satisfying the conditions of section 6.3 and let $\lambda,\ \mu \in X_\varphi$. Then we have*

$$[u_\lambda, \pi_\lambda] = [u_\mu, \pi_\mu] \quad \Leftrightarrow \quad \rho_\lambda = \rho_\mu.$$

Proof. If $\rho_\lambda = \rho_\mu$, then the first half of the proof of [14, 4.5.2], which does not depend on depth, shows that there is $g \in G$ such that $g * u_\lambda = u_\mu,\ g \cdot x_\lambda = x_\mu$ and

$$\operatorname{Ad}(g) \cdot (T_\lambda, \chi_\lambda) = (T_\mu, \chi_\mu).$$

This implies that $\operatorname{Ad}(g)_* \cdot \kappa_\lambda = \kappa_\mu$, so that $g \cdot (u_\lambda, \pi_\lambda) = (u_\mu, \pi_\mu)$. Hence we have $[u_\lambda, \pi_\lambda] = [u_\mu, \pi_\mu]$, as claimed.

Conversely, suppose there is $g \in G$ such that $g \cdot (u_\lambda, \pi_\lambda) = (u_\mu, \pi_\mu)$. Then ${}^g\pi_\lambda$ and π_μ are equivalent representations of G^{F_μ}.

Thus, we have two pairs $({}^gT_\lambda, {}^g\chi_\lambda)$ and (T_μ, χ_μ) in G^{F_μ} giving rise, via Adler's construction, to equivalent representations of G^{F_μ}. The two pairs must then be conjugate in G^{F_μ}, as follows from a character calculation [13] or more general results on distinguished representations [22, Cor. 6.9]. Hence, modifying g by an element of G^{F_μ}, we may assume that

$$({}^gT_\lambda, {}^g\chi_\lambda) = (T_\mu, \chi_\mu).$$

Recall that ${T_\lambda}^{\mathrm{F}_\lambda}$ and $T_\mu^{\mathrm{F}_\mu}$ have unique fixed-points x_λ and x_μ in $\mathcal{B}(G)^{\mathrm{F}_\lambda}$ and $\mathcal{B}(G)^{\mathrm{F}_\mu}$. Since $g * u_\lambda = u_\mu$, it follows that $g \cdot x_\lambda = x_\mu$. The last paragraph of the proof of [14, 4.5.2], repeated verbatim, now shows that $\rho_\lambda = \rho_\mu$. $\square$

Lemma 6.1 allows us to define

$$\pi(\varphi,\rho) := [u_\lambda, \pi_\lambda],$$

for any $\lambda \in X_\varphi$ such that $\rho_\lambda = \rho$. Our supercuspidal L-packet is then

$$\Pi(\varphi) := \{\pi(\varphi,\rho) :\ \rho \in \mathrm{Irr}(C_\varphi)\}.$$

From 4.1, it follows that all representations in $\Pi(\varphi)$ have the same formal degree, with respect to canonical Haar measures.

6.7. **A simple case.** The L-packets $\Pi(\varphi)$ simplify greatly $\mathbf{G}$ is k-split and simply-connected. In this case, $\vartheta = 1$ and X is the co-root lattice of $\mathbf{T}$ in $\mathbf{G}$. For any $\lambda \in X$, we have $w_\lambda = t_\lambda w$ and $y_\lambda = 1$ so we may take $u_\lambda = 1$ and $\mathrm{F}_\lambda = \mathrm{F}$. It follows that for each $\rho \in X/(1-w)X$, we may identify the class $\pi(\varphi,\rho)$ with the G^{F}-isomorphism class of representations π_λ, for $\lambda \in \rho$. Thus, the L-packet $\Pi(\varphi)$ consists of isomorphism classes of representations of the single group G^{F}.

6.8. **A useful complement.** The construction of π_λ involves several choices, among which is a choice of alcove C_λ whose closure contains x_λ. The variability of C_λ can be inconvenient when working out particular cases of our L-packets. One might hope to fix an alcove C, and that for each $\rho \in \mathrm{Irr}(C_\varphi)$ one can find $\lambda \in X_\varphi$ such that $\rho_\lambda = \rho$ and $C_\lambda = C$. Unfortunately, this is not always possible. Recall, however, that the pair (φ,ρ) is only taken up to conjugacy by $\hat{G}$. This extra freedom allows us to fix C.

Lemma 6.2. *Let C be an alcove in $\mathcal{A}(T)$. Then any pair (φ,ρ), where φ satisfies the conditions of 6.3 and $\rho \in \mathrm{Irr}(C_\varphi)$, may be chosen in its $\hat{G}$-conjugacy class so that $\varphi(\mathcal{I}) \subset \hat{T}$, and so that there exists $\mu \in X_\varphi$ with $\rho_\mu = \rho$ and $C_\mu = C$.*

Proof. We already know we can arrange that $\varphi(\mathcal{I}) \subset \hat{T}$. Choose any $\lambda \in X_\varphi$ such that $\rho_\lambda = \rho$, and choose any alcove C_λ containing x_λ in its closure $\bar{C}_\lambda$. Let $\dot{w} \in N$ be a representative of w.

Now choose $n \in N$ such that $n \cdot C_\lambda = C$. Let $v \in W$ be the image of n, and let v_o be the projection of v to W_o. Then $n\dot{w}\,\mathrm{F}(n)^{-1}$ projects to $w' := vw\vartheta(v)^{-1} \in W_o$. The action of v_o on X gives an isomorphism

$$[X/(1-w\vartheta)X]_{\mathrm{tor}} \xrightarrow{v_*} [X/(1-w'\vartheta)X]_{\mathrm{tor}} \tag{38}$$

such that $v_*\rho_\lambda = \rho'_{v\lambda}$ is the image of $v_o\lambda$ in the right side of (38).

Conjugating both sides of the equation $t_\lambda w\vartheta = w_\lambda y_\lambda \vartheta$ by v, we get an analogous equation

$$t_{v\lambda} w'\vartheta = w'_{v\lambda} y'_{v\lambda}\vartheta, \tag{39}$$

where the unique fixed-point in $\mathcal{A}(T)$ of both sides of (39) is $x'_{v\lambda} := v \cdot x_\lambda \in \bar{C}$. Moreover, we have $y'_{v\lambda}\vartheta \cdot C = C$.

Let $\hat{v}_o \in \hat{W}_o$ correspond to v_o under duality, and let $\hat{n} \in N_{\hat{G}}(\hat{T})$ be a lift of $\hat{v}_o$. Conjugating (φ, ρ) by $\hat{n}$ gives a new pair (φ', ρ') such that $\rho' = \rho'_{v\lambda}$ and $C'_{v\lambda} = C$. Replacing (φ, ρ) by (φ', ρ') and taking $\mu = v\lambda$ satisfies the conclusion of the lemma. □

A warning: If one fixes the alcove C, and uses Lemma 6.2 to construct an L-packet with all inducing data on points in $\bar{C}$, then the element w will vary for each representation. However, w will only vary within its ϑ-conjugacy class in W_o.

6.9. **Stable classes of tori and their characters.** The results in [14] on stable classes of tori and their characters do not depend on the depth of the characters. In this section we recall these results and show how they apply to our positive-depth L-packets $\Pi(\varphi)$.

Let F be a Frobenius endomorphism of G arising from a given K-split k-structure on $\mathbf{G}$. We denote the set of F-stable K-split maximal tori in G by $\mathfrak{T}(G, F)$ and we say that two tori $S_1,\ S_2 \in \mathfrak{T}(G, F)$ are (G, F)**-stably conjugate** if there is $g \in G$ such that ${}^g(S_1^F) = S_2^F$. This is an equivalence relation on $\mathfrak{T}(G, F)$ whose classes we call (G, F)**-stable classes**. We write $[\mathfrak{T}(G, F)]_{\mathrm{st}}$ for the set of (G, F)-stable classes in $\mathfrak{T}(G, F)$.

Any $S \in \mathfrak{T}(G, F)$ is of the form $S = {}^gT$ for some $g \in G$, and the element $n = g^{-1}F(g)$ belongs to N. By [12], two such tori S_1 and S_2, corresponding to n_1 and n_2, are (G, F)-stably conjugate if and only if n_1T and n_2T belong to the same F-twisted conjugacy-class in N/T . This gives an injective mapping

$$[\mathfrak{T}(G, F)]_{\mathrm{st}} \hookrightarrow H^1(F, N/T). \tag{40}$$

Suppose $F = \mathrm{F}_u$, where F is the Frobenius for a quasisplit k-structure on $\mathbf{G}$, and $u \in Z^1(F, N)$. The map $z \mapsto zu$ induces a bijection

$$H^1(\mathrm{F}_u, N/T) \xrightarrow{\sim} H^1(\mathrm{F}, N/T). \tag{41}$$

Since F is a quasisplit Frobenius, there is an F-stable hyperspecial vertex $o \in \mathcal{A}(T)$, and we may identify $N/T = W_o$ as F-groups. Let ϑ be the automorphism of W induced by F. Then ϑ preserves W_o, and the map $w \mapsto w\vartheta$ identifies the cohomology set $H^1(\mathrm{F}, W_o)$ with the set of W_o-orbits, via ordinary conjugation, on $W_o\vartheta$. An element $w\vartheta \in W_o\vartheta$ is called **elliptic** if it has no fixed-points in the root lattice of $\mathbf{T}$ in $\mathbf{G}$.

Combining (40) and (41), we get an injective mapping

$$\Psi_u : [\mathfrak{T}(G, \mathrm{F}_u)]_{\mathrm{st}} \hookrightarrow W_o\vartheta/W_o \tag{42}$$

which sends each F_u-minisotropic class in $\mathfrak{T}(G, \mathrm{F}_u)$ to an elliptic class in $W_o\vartheta$. If $u \neq 1$, the map Ψ_u is not necessarily surjective, but we have the following immediate consequence of [14, 9.6.1].

Lemma 6.3. *If $w\vartheta \in W_o\vartheta$ is elliptic, and $u \in Z^1(\mathrm{F}, N)$, then there is a G-stable class $\mathcal{T}_{w,u} \subset \mathfrak{T}(G, \mathrm{F}_u)$ such that $\Psi_u(\mathcal{T}_{w,u})$ is the W_o-orbit of $w\vartheta$.*

We now construct a "covering" of $\mathfrak{T}(G, F)$ by adding an extra piece of data. Let

$$\hat{\mathfrak{T}}(G, F) := \{(S, \theta) : \ S \in \mathfrak{T}(G, F) \quad \text{and} \quad \theta \in \mathrm{Irr}(S^F)\}.$$

We say two pairs (S_1, θ_1), (S_2, θ_2) in $\hat{\mathfrak{T}}(G, F)$ are (G, F)-stably conjugate if there is $g \in G$ such that ${}^g(S_1^F) = S_2^F$ and ${}^g\theta_1 = \theta_2$.

Suppose $F = \mathrm{F}_u$, let φ be a Langlands parameter satisfying the conditions of section 6.3 with associated $w \in W_o$, and let $\chi = \chi_\varphi \in \mathrm{Irr}(T^{w\,\mathrm{F}})$. We define

$$\hat{\mathcal{T}}_{w,u,\chi} := \{(S, \theta) \in \hat{\mathfrak{T}}(G, \mathrm{F}_u) : \ \exists g \in G \ \text{ such that } \ S^{\mathrm{F}_u} = {}^g(T^{w\,\mathrm{F}}) \text{ and } \theta = {}^g\chi\}.$$

Then G^{F_u} acts by conjugation on $\hat{\mathcal{T}}_{w,u,\chi}$, with a finite number of orbits. These orbits can be parametrized as follows. First, Kottwitz' isomorphism $\mathrm{Irr}(C_\varphi) \to H^1(\mathrm{F}, G)$ (see [28]) factors as

$$\mathrm{Irr}(C_\varphi) = [X/(1 - w\vartheta)X]_{\mathrm{tor}} \xrightarrow{r_w} [\Omega/(1 - \vartheta)\Omega]_{\mathrm{tor}} = H^1(\mathrm{F}, G), \tag{43}$$

where $\Omega = \mathrm{Irr}(\hat{Z})$ and r_w is induced by the restriction from $\hat{T}$ to $\hat{Z}$.

Now, in [14, 9.6.1] it is shown (via a proof that does not depend on the depth of characters) that the map $\lambda \mapsto (T_\lambda, \chi_\lambda)$ induces a bijection

$$r_w^{-1}[u] \xrightarrow{\sim} \hat{\mathcal{T}}_{w,u,\chi}/G^{\mathrm{F}_u}, \tag{44}$$

where $[u] \in H^1(\mathrm{F}, G)$ is the class of the cocycle u.

It follows that the classes in our L-packet $\Pi(\varphi)$ which contain representations on a given pure inner form G^{F_u} are constructed from a complete set of representatives of G^{F_u}-orbits in the stable class $\hat{\mathcal{T}}_{w,u,\chi_\varphi}$ corresponding to the fiber over $[u]$ of the natural map $\mathrm{Irr}(C_\varphi) \longrightarrow H^1(\mathrm{F}, G)$.

6.10. **An equivariance property.** Let φ be a Langlands parameter satisfying the conditions of section 6.3, with associated $w \in W_o$.

The centralizer $C(w\vartheta)$ of $w\vartheta$ in W_o acts naturally on the parameter space $[X/(1 - w\vartheta)]_{\mathrm{tor}}$ of the L-packet $\Pi(\varphi)$. Moreover, for any $\lambda \in X_\varphi$, the group $C(w\vartheta)$ may be identified with the F_λ-rational points in the Weyl group $W(T_\lambda)$ of T_λ in G. In this picture, the subgroup $C(w\vartheta, \lambda)$ stabilizing the class of λ in $X/(1 - w\vartheta)X$ consists of those elements of $W(T_\lambda)$ which can be represented by elements in G^{F_λ}. These facts are proved in [14, 2.11.2].

It follows that $C(w\vartheta)$ acts on the characters of $T_\lambda^{\mathrm{F}_\lambda}$, so for any $h \in C(w\vartheta)$ we can compare the representation $\pi(T_\lambda, \chi)$ with its "twist" $\pi(T_\lambda, \chi^h)$. By the remarks above, these representations will be equivalent if $h \in C(w\vartheta, \lambda)$. Thus we expect a relation between the twisting action of $C(w\vartheta)$ on representations and the natural action of $C(w\vartheta)$ on $[X/(1-w\vartheta)X]_{\mathrm{tor}}$.

This relation can be expressed as an equivariance property for the pairing $(\varphi, \rho) \mapsto \pi(\varphi, \rho)$, where φ is a Langlands parameter as considered above, and $\rho \in \mathrm{Irr}(C_\varphi)$. Indeed, if we view W_o as the Weyl group of the dual torus $\hat{T}$, we have a natural action of $C(w\vartheta)$ on the set of Langlands parameters φ satisfying conditions of section 6.3. Namely, given $h \in C(w\vartheta)$, we can form the twisted parameter φ^h, defined by

$$\varphi^h(\mathrm{Frob}) = \varphi(\mathrm{Frob}), \qquad \varphi^h(\gamma) = \varphi(\gamma)^h, \quad \text{for} \quad \gamma \in \mathcal{I}.$$

The action of $C(w\vartheta)$ on $\hat{T}$ also preserves $C_\varphi = \hat{T}^{\widehat{w\vartheta}}$, hence $C(w\vartheta)$ acts on $\mathrm{Irr}(C_\varphi)$; we denote this action by $\rho \mapsto {}^h\rho = \rho \circ h^{-1}$. The equivariance property can then be stated as follows:

Proposition 6.4. *Let φ be a Langlands parameter satisfying the conditions of section 6.3, with associated $w \in W_o$. Then for $\rho \in \mathrm{Irr}(C_\varphi)$ and $h \in C(w\vartheta)$ we have $\pi(\varphi^h, \rho) = \pi(\varphi, {}^h\rho)$.*

Proof. We are asserting an equality of G-orbits of pairs (u, π). We calculate this G-action as follows.

Let $n \in N$ be a lift of $h \in C(w\vartheta)$ and let $\lambda \in X_\varphi$. As in equation (35), we have two expressions for the elements

$$t_\lambda w\vartheta = w_\lambda y_\lambda \vartheta \quad \text{and} \quad t_{h\lambda} w\vartheta = w_{h\lambda} y_{h\lambda} \vartheta \tag{45}$$

in $W\vartheta$. Let $u_\lambda \in Z^1(\mathrm{F}, N)$ be a lift of y_λ, as in section 6.4. I first claim that the element $n * u_\lambda = n u_\lambda \, \mathrm{F}(n)^{-1} \in N$ is a lift of $y_{h\lambda}$. Since $h \in C(w\vartheta)$ we have

$$t_{h\lambda} w\vartheta = h t_\lambda w\vartheta h^{-1}. \tag{46}$$

The left side of (46) has unique fixed-point $x_{h\lambda}$ in $\mathcal{A}(T)$, while the right side has unique fixed-point $h \cdot x_\lambda$, so we have $x_{h\lambda} = h \cdot x_\lambda$. Using the first equation in (45), we get

$$h t_\lambda w\vartheta h^{-1} = h \cdot w_\lambda y_\lambda \vartheta \cdot h^{-1} = {}^h w_\lambda \cdot h y_\lambda ({}^\vartheta h^{-1}) \cdot \vartheta.$$

This must be the corresponding factorization of $t_{h\lambda} w\vartheta$, by equation (46). Therefore, we have $w_{h\lambda} = {}^h w_\lambda$ and $y_{h\lambda} = h y_\lambda ({}^\vartheta h^{-1})$. Since $h y_\lambda ({}^\vartheta h^{-1})$ is the image of $n * u_\lambda = n u_\lambda \, \mathrm{F}(n)^{-1}$ in W, the claim is proved. Therefore we can take $u_{h\lambda} := n * u_\lambda$ and define $\mathrm{F}_{h\lambda} := \mathrm{Ad}(u_{h\lambda}) \circ \mathrm{F}$.

Let $p_\lambda \in G_{x_\lambda}$ be as in section 6.4, so that $p_\lambda^{-1} \, \mathrm{F}_\lambda(p_\lambda) \in N \cap G_{x_\lambda}$ is a lift of w_λ. It is straightforward to check that the element ${}^n p_\lambda{}^{-1} \cdot \mathrm{F}_{h\lambda}({}^n p_\lambda)$

is a lift of ${}^h w_\lambda = w_{h\lambda}$, so we may take $p_{h\lambda} = np_\lambda n^{-1} \in N \cap G_{x_{h\lambda}}$. By definition, $T_{h\lambda} = \mathrm{Ad}(p_{h\lambda})T$, so we have $\mathrm{Ad}(n)T_\lambda = T_{h\lambda}$.

Let $\chi = \chi_\varphi \in \mathrm{Irr}(T^{w\,\mathrm{F}})$. The naturality property (34) implies that $\chi^h = \chi_{\varphi^h}$. By definition, we have

$$(\chi^h)_\lambda = (\chi^h)\circ\mathrm{Ad}(p_\lambda)^{-1} = \chi\circ\mathrm{Ad}(np_\lambda^{-1}) = \chi\circ\mathrm{Ad}(p_{h\lambda}^{-1}n) = (\chi_{h\lambda})\circ\mathrm{Ad}(n).$$

We have shown that $\mathrm{Ad}(n)\cdot(T_\lambda, (\chi^h)_\lambda) = (T_{h\lambda}, \chi_{h\lambda})$. Putting everything together, we have

$$\begin{aligned} \pi(\varphi^h, \rho) &= \left[u_\lambda, \pi(T_\lambda, (\chi^h)_\lambda)\right] = \left[n * u_\lambda, {}^n\pi(T_\lambda, (\chi^h)_\lambda)\right] \\ &= \left[u_{h\lambda}, \pi(T_{h\lambda}, \chi_{h\lambda})\right] = \pi(\varphi, \rho_{h\lambda}) = \pi(\varphi, {}^h\rho), \end{aligned} \tag{47}$$

as claimed. □

6.11. **An example in E_8.** Take $\mathbf{G}$ of type E_8. Up to conjugacy, the Weyl group W_o contains a unique elliptic element w of order three. We consider L-packets $\Pi(\varphi)$ where $\varphi(\mathrm{Frob}) \in N(\hat{T})$ is a lift of $\hat{w}$. The lattice X is the E_8-root lattice, on which we normalize the W_o-invariant Euclidean metric $\langle\ ,\ \rangle$ such that $\langle\alpha, \alpha\rangle = 2$ for each root α. It can be shown that the finite group $X/(1-w)X$ is a four-dimensional vector space over the field of three elements; we set $V_w := X/(1-w)X$. The pairing $(x, y) \mapsto \langle(1-w)x, y\rangle$ induces a nondegenerate symplectic form on V_w, which is preserved by the centralizer $C(w)$. The resulting map $C(w) \longrightarrow Sp(V_w)$ is surjective, with kernel of order three, generated by w. These facts are proved in [34]. It follows that $C(w)$ is transitive on non-zero vectors in V_w.

The class $\pi(\varphi, 1)$ is supported on hyperspecial vertices. By Proposition 6.4, the remaining 80 classes in $\Pi(\varphi)$ all contain representations of G^{F} arising from twists of the character χ_φ on a single minisotropic torus in G^{F} stabilizing a non-hyperspecial vertex x. Since x must have the property that W_x contains elliptic elements $t_\lambda w$ of order three, we see that x has type $A_2 + E_6$.

7. Twisted Coxeter elements

This section is preparation for studying a canonical example of supercuspidal L-packets, where $w\vartheta$ is a ϑ-Coxeter element (see section 7.1 below for definitions). We will describe the L-packets, the classes of tori, and corresponding inducing data which arise in this case. As part of this calculation, we must determine the factorizations

$$t_\lambda w\vartheta = w_\lambda y_\lambda \vartheta \tag{48}$$

and the vertices x_λ from section 6.4. We will show that the element in (48) is in fact a $y_\lambda\vartheta$-Coxeter element in $W_{x_\lambda} y_\lambda\vartheta$, and that this fact determines x_λ.

The passage from a ϑ-Coxeter element to a $y_\lambda\vartheta$-Coxeter element is a property of Coxeter elements that might be of independent interest; it can be explained purely in the context of affine Weyl groups, so this chapter is independent of what has gone before. We begin with some background, following Springer [38], on twisted Coxeter elements. Springer only treats the case of irreducible root systems, whereas we must allow our root systems to have finitely many components, which are permuted transitively by the twisting automorphism. Springer's proofs can be adapted with only minor modifications, which we leave to the reader.

7.1. **Definition and basic properties of twisted Coxeter elements.** Let W be a finite Weyl group with root system Φ, let $V = \mathrm{Hom}(\mathbb{Z}\Phi, \mathbb{C})$ be the complexified reflection representation of W and set $n = \dim V$. We view W as a subgroup of $GL(V)$.

Let $\sigma \in GL(V)$ be a linear transformation of finite order which preserves some base Π of Φ. Hence σ preserves Φ itself and normalizes W, so W acts by conjugation on the coset $W\sigma$. The W-orbits in $W\sigma$ are called **σ-conjugacy classes**.

Let $\Pi_1, \dots \Pi_{n_\sigma}$ be the orbits of $\langle\sigma\rangle$ in Π. For each i, choose $\alpha_i \in \Pi_i$ arbitrarily, and let $r_i \in W$ denote the corresponding reflection. Let w be the product of $r_1, \dots, r_{n_\sigma}$ in any order. The element $w\sigma \in W\sigma$ thus obtained is called a **σ-Coxeter element**. If $\sigma = 1$ we omit the prefix "$\sigma-$".

It follows from the simple transitivity of W on bases that two σ-stable bases are conjugate by the group W^σ of σ-fixed-points in W. Using also [38, 7.5], we see that the W-orbit of $w\sigma$ in $W\sigma$ is independent of the choices of the base Π, the representatives α_i, or their ordering. Hence the σ-Coxeter elements form a single σ-conjugacy class in $W\sigma$.

This definition of σ-Coxeter elements is a bit unsatisfactory, since it depends on a particular base Π. One can give a more intrinsic characterization of σ-Coxeter elements, as follows. We first need two definitions. Let

$$V^{\mathsf{reg}} := V - \bigcup_{\alpha\in\Phi} \ker\alpha$$

be the complement of the root hyperplanes in V. An element $w\sigma \in W\sigma$ is **regular** if $w\sigma$ has an eigenvector in V^{reg}. We call the corresponding eigenvalue "regular" as well. Next, we say that $w\sigma \in W\sigma$ is **elliptic** if $V^{w\sigma} = 0$.

We assume from now on that the group $\langle\sigma\rangle$ generated by σ acts transitively on the irreducible components $\Phi_1, \dots, \Phi_k$ of Φ. We have $W = W_1 \times \cdots \times W_k$, accordingly. Let h_σ be the maximal order of an eigenvalue of an element of $W_i\sigma$ (it is the same for any i), and recall that n_σ is the number of orbits of $\langle\sigma\rangle$ in the given σ-stable base Π of Φ. The basic properties of σ-Coxeter elements are collected in the following proposition, whose proof is an easy reduction to the irreducible case treated in [38, chap. 7] and will be omitted here.

Proposition 7.1. *Let $w\sigma \in W\sigma$ be a σ-Coxeter element. Then the following hold.*

(1) *$w\sigma$ is elliptic and regular, and has order h_σ.*
(2) *$w\sigma$ has a regular eigenvalue of order h_σ, with multiplicity one.*
(3) *Each orbit of $\langle w\sigma\rangle$ in Φ has cardinality h_σ, and $|\Phi| = n_\sigma h_\sigma$.*
(4) *There is an ordering $\Phi = \Phi^+ \sqcup \Phi^-$ such that each $\langle w\sigma\rangle$-orbit in Φ contains exactly one root $\alpha \in \Phi^+$ for which $w\sigma\alpha \in \Phi^-$.*
(5) *The centralizer of $w\sigma$ in W is cyclic, generated by $(w\sigma)^s$, where s is the order of σ.*
(6) *The σ-Coxeter elements in $W\sigma$ are precisely those elliptic regular elements of $W\sigma$ having a regular eigenvalue of order h_σ.*

The last item is of particular importance, as it allows us to recognize σ-Coxeter elements by intrinsic properties.

7.2. **The long root example.** Several examples of twisted Coxeter groups W occur naturally as long root subgroups in larger Weyl groups $\tilde{W}$. In this section we show that such twisted Coxeter elements in W are actually ordinary Coxeter elements in $\tilde{W}$. This hereditary property of Coxeter elements will also appear in our study of L-packets. One could check this property case-by-case, but we can give a uniform treatment, illustrating the use of 7.1. (Note, however, that the proof of 7.1 in [38] relies on some checking of cases.)

Let $\tilde{W}$ be a Weyl group of type $B_n,\ C_n,\ G_2,\ F_4$. The root system $\tilde{\Phi}$ for $\tilde{W}$ is irreducible, with two root lengths. Let Φ be the set of long roots in $\tilde{\Phi}$. Let $\tilde{\Pi}$ be a base of $\tilde{\Phi}$, and write $\tilde{\Pi} = \Pi_l \sqcup \Pi_s$, where Π_l and Π_s are the sets of long and short roots in $\tilde{\Pi}$, respectively. Let W_s be the subgroup of $\tilde{W}$ generated by the reflections from Π_s, and let W be the subgroup of $\tilde{W}$ generated by the reflections from Φ. Then W is normal in $\tilde{W}$, and the latter is a semidirect product

$$\tilde{W} = W \rtimes W_s. \tag{49}$$

Moreover, the group W_s, being simply-laced, irreducible and without branch node, is of type A_m, where $m = |\Pi_s|$, see [19, chap. 5]. In this section, we show that the decomposition (49) also produces natural examples of σ-Coxeter elements.

First, we need another fact about the decomposition in (49). The choice of $\tilde{\Pi}$ determines a base Π of Φ. Namely, if we let $\tilde{\Phi}^+$ be the positive system in $\tilde{\Phi}$ containing $\tilde{\Pi}$, then $\Phi^+ := \Phi \cap \tilde{\Phi}^+$ is a positive system in Φ, and Π is the unique base contained in Φ^+. If $\sigma \in W_s$, then σ is a product of short reflections, so $\sigma\Phi^+ = \Phi^+$, hence $\sigma\Pi = \Pi$.

Lemma 7.2. *If σ is a Coxeter element in W_s, then Π_l is a set of representatives for the σ-orbits on Π.*

Proof. First note that $\Pi_l \subset \Pi$. For otherwise, some $\alpha \in \Pi_l$ could be written $\alpha = \sum_i c_i\alpha_i$, with $\alpha_i \in \Pi$, all $c_i \in \mathbb{Z}_{\geq 0}$, and $\sum c_i > 1$. But since $\Pi \subset \tilde{\Phi}^+$ and $\Pi_l \subset \tilde{\Pi}$, this means $\alpha \notin \Pi_l$, a contradiction.

Now let σ be a Coxeter element in $W_s \simeq S_{m+1}$. Let $\beta \in \Pi_l$ be the unique root not orthogonal to Π_s, and let $\alpha \in \Pi_s$ be the unique root not orthogonal to Π_l. The functional $\langle \cdot, \check{\beta}\rangle$ is a dominant weight for Π_s. Hence the stabilizer of β in W_s is generated by the reflections from $\Pi_s \setminus \{\alpha\}$, so is isomorphic to S_m. This subgroup contains no nontrivial power of an $m+1$-cycle, so the stabilizer of β in $\langle\sigma\rangle$ is trivial. Hence the σ-orbit of β has exactly $m+1$ elements.

Let $\Pi_l' = \Pi_l \setminus \{\beta\}$. We must show that

$$\Pi = \Pi_l' \sqcup \{\beta, \sigma\beta, \ldots, \sigma^m\beta\}. \tag{50}$$

The two sets on the right are disjoint, since σ fixes each root in Π_l'. It suffices, then, to show that $|\Pi| = |\Pi_l'| + m + 1$. But $|\Pi| = |\tilde{\Pi}| = |\Pi_l| + |\Pi_s| = 1 + |\Pi_l'| + m$. The lemma is proved. □

Let w be the product of the reflections from Π_l, and let σ be the product of the reflections in Π_s, both products taken in any order. Then $w\sigma$ is a Coxeter element of $\tilde{W}$. By Lemma 7.2, $w\sigma$ is also a σ-Coxeter element of $W\sigma$. Since the Coxeter graph of Π_l is a tree, it follows from equation (50) that Φ is σ-irreducible, so the σ-Coxeter number h_σ of $W\sigma$ is defined, and in fact h_σ is also the Coxeter number of $\tilde{W}$.

This element $w\sigma$ is a carefully chosen Coxeter element in $\tilde{W}$. The next result shows that this is immaterial.

Lemma 7.3. *Let $\tilde{w}$ be any Coxeter element of $\tilde{W}$. Write $\tilde{w} = w'\sigma'$ as in (49), with $w' \in W$ and $\sigma' \in W_s$. Then σ' is a Coxeter element of W_s and $w'\sigma'$ is a σ'-Coxeter element of $W\sigma'$.*

Proof. There is $\tilde{x} \in \tilde{W}$ such that $\tilde{x}\tilde{w}\tilde{x}^{-1} = w\sigma$, where $w\sigma$ is the carefully chosen Coxeter element defined above. Projecting to W_s, we see that σ' is W_s-conjugate to σ. This implies that σ' is a Coxeter element in W_s, that Φ is σ'-irreducible, and that the maximal orders of a regular eigenvalue of $W\sigma$ and $W\sigma'$ are the same. Hence $h_\sigma = h_{\sigma'}$ is also the σ'-Coxeter number of $W\sigma'$.

Being Coxeter in $\tilde{W}$, the element $\tilde{w}$ is elliptic, and is regular with respect to $\tilde{\Phi}$. Hence $\tilde{w}$ is also regular with respect to Φ. Since $\tilde{w}$ has a regular eigenvalue of order $h_\sigma = h_{\sigma'}$, Proposition 7.1 implies that $\tilde{w}$ is a σ'-Coxeter element of $W\sigma'$. $\square$

7.3. **Affine Weyl groups.** In this section we describe another hereditary property of twisted Coxeter elements, arising from finite reflection subgroups of affine Weyl groups.

We now denote the (finite) Weyl group and (spherical) root system considered in the two previous sections by W_o and Ψ_o, respectively. We set $X := \mathrm{Hom}(\mathbb{Z}\Psi_o, \mathbb{Z})$, $\mathcal{A} := \mathrm{Hom}(\mathbb{Z}\Psi_o, \mathbb{R})$, $V_o := \mathrm{Hom}(\mathbb{Z}\Psi_o, \mathbb{C})$, and now write ϑ for the automorphism $\sigma \in GL(V_o)$ considered above. Here, o is the zero element of $\mathcal{A}$. Assume that Ψ_o is irreducible (not just ϑ-irreducible). Then X is a lattice in $\mathcal{A}$, and W_o and ϑ preserve X. The affine Weyl group is the semi-direct product $W := X \rtimes W_o$. and is contained in the larger group $W \rtimes \langle\vartheta\rangle$. The latter group acts on $\mathcal{A}$ by affine transformations: an element $\lambda \in X$ acts via the translation $t_\lambda : x \mapsto \lambda + x$ on $\mathcal{A}$.

Let Ψ be the set of affine functions $\alpha - m$, for $\alpha \in \Psi_o$ and $m \in \mathbb{Z}$. Let $H_{\alpha,m}$ be the hyperplane in $\mathcal{A}$ defined by the vanishing of $\alpha - m$. These hyperplanes partition $\mathcal{A}$ into a disjoint union of **facets** (see [5, V.1]). A **vertex** is a facet consisting of a single point. An **alcove** is a facet which is open in $\mathcal{A}$. The alcoves are also the connected components of the complement in $\mathcal{A}$ of all affine root hyperplanes $H_{\alpha,m}$. The orthogonal reflection about $H_{\alpha,m}$ is the element $s_{\alpha,m} = t_{m\check{\alpha}}s_\alpha \in W$, where $\check{\alpha} \in X$ is the co-root corresponding to α. These reflections generate a subgroup $W^\circ \subset W$ which acts simply transitively on the set of alcoves.

For each $x \in \mathcal{A}$, let Ψ_x be the set of affine roots in Ψ which vanish at x, and let $\mathfrak{m}_x$ be the ideal of polynomial functions on V which vanish at x. We identify Ψ_x with its image in the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Thus, the affine roots Ψ_x are linear functionals on the tangent space $V_x := (\mathfrak{m}_x/\mathfrak{m}_x^2)\check{}$.

For $f \in \mathfrak{m}_x$, $v \in V_o$ and t a variable, let $\langle v, d_x(f)\rangle$ denote the coefficient of t in $f(x + tv)$. Then d_x induces the local differential mapping $d_x : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \check{V}_o$ on the dual spaces. If $\alpha - m \in \Psi_x$, then $d_x(\alpha - m) = \alpha$.

Let $W_x^\star$ be the stabilizer of x in the group $W\langle\vartheta\rangle$, and let W_x be the subgroup of $W_x^\star$ generated by the reflections $s_{\alpha,m}$ for $\alpha - m \in \Psi_x$. Then $W_x^\star$ acts on V_x and the normal subgroup W_x of $W_x^\star$ is a reflection group on V_x with root system $\Psi_x \subset \check{V}_x$.

Let V_x^{reg} denote the set of vectors in V_x on which no root in Ψ_x vanishes. Since $d_x(\Psi_x) \subset \Psi_o$, it follows that the adjoint $\delta_x : V_o \longrightarrow V_x$ of d_x satisfies

$$\delta_x(V_o^{\mathsf{reg}}) \subset V_x^{\mathsf{reg}}. \tag{51}$$

The set of connected components of V_x^{reg} is in bijection with the set of alcoves in $\mathcal{A}$ having x in their closure (each of the former contains a unique one of the latter). Hence W_x acts simply transitively on this set of alcoves.

It follows that if we fix an alcove C_x with $x \in \bar{C}_x$, then we can express $W_x^\star$ as a semidirect product

$$W_x^\star = W_x \rtimes \Sigma_x, \tag{52}$$

where $\Sigma_x := \{\sigma \in W_x^\star : \ \sigma \cdot C_x = C_x\}$. The set $\Psi_x^+ := \{\alpha - m \in \Psi_x : \alpha(y) > m \ \text{ for all } y \in C_x\}$ is a positive system of Ψ_x, containing a unique base Π_x. Both Ψ_x and Π_x are preserved by Σ_x.

Now suppose we have $\lambda \in X$ and $w \in W_o$ such that $t_\lambda w\vartheta$ fixes a point $x_\lambda \in \mathcal{A}$. It is easy to check that

$$t_\lambda w\vartheta \circ \delta_{x_\lambda} = \delta_{x_\lambda} \circ w\vartheta. \tag{53}$$

Choose an alcove $C_{x_\lambda} \subset \mathcal{A}$ containing x_λ in its closure. According to (52), we have a unique factorization

$$t_\lambda w\vartheta = w_\lambda \sigma_\lambda, \tag{54}$$

with $w_\lambda \in W_{x_\lambda}$, and $\sigma_\lambda \in \Sigma_{x_\lambda}$. Note that both w_λ and σ_λ fix x_λ. In particular, σ_λ acts on Ψ_{x_λ} as well as on the tangent space V_{x_λ}.

From property (51) and equation (53) it follows that if $w\vartheta$ is regular on V_o then $w_\lambda\sigma_\lambda$ is regular on V_x. Moreover, the eigenvalues of $w\vartheta$ on V_o are the same as the eigenvalues of $w_\lambda\sigma_\lambda$ on V_x. Using part 6 of Proposition 7.1, this proves:

Proposition 7.4. *Let $w\vartheta$ be a ϑ-Coxeter element in $W_o\vartheta$. For $\lambda \in X$, write $t_\lambda w\vartheta = w_\lambda\sigma_\lambda$ as in* (54)*, and let $x_\lambda \in \mathcal{A}$ be the unique fixed-point of $t_\lambda w\vartheta$. Assume that Ψ_{x_λ} is σ_λ-irreducible. Then $w_\lambda\sigma_\lambda$ is a σ_λ-Coxeter element in $W_{x_\lambda}\sigma_\lambda$.*

We will see in the next section that the irreducibility assumption in 7.4 always holds, although I do not have a uniform argument for this.

7.4. **Coxeter facets.** In this section we determine the points $x_\lambda \in \mathcal{A}$ arising as the fixed-points of lifted ϑ-Coxeter elements, as in Proposition 7.4.

Let C be a fixed ϑ-stable alcove containing o in its closure $\bar{C}$, and let $\Omega = \{y \in W : \ y \cdot C = C\}$, so that $W = W^\circ \rtimes \Omega$.

Let $\sigma \in \Omega\vartheta$. The fixed-point set $\mathcal{A}^\sigma$ inherits a simplicial structure from $\mathcal{A}$, whose facets are of the form J^σ, where J is a σ-stable facet in $\mathcal{A}$. The alcove C is σ-stable and C^σ is an alcove in $\mathcal{A}^\sigma$. A point $x \in \mathcal{A}^\sigma$ is vertex exactly when $\{x\} = J^\sigma$, for some σ-stable facet J in $\mathcal{A}$.

A **σ-Coxeter facet** is a σ-stable facet $J \subset \bar{C}$ for which there exists a σ-Coxeter element of $W_J\sigma$ projecting to a ϑ-Coxeter element in $W_o\vartheta$, under the natural projection $W\vartheta \to W_o\vartheta$. From [14, 4.4.1] it follows that if J is a σ-Coxeter facet then J^σ is a vertex in $\mathcal{A}^\sigma$.

For $x \in \mathcal{A}$, the objects W_x, Ψ_x, Π_x depend only on the facet J in $\mathcal{A}$ containing x. We now write W_J, Ψ_J, Π_J, respectively, where Π_J is the base of Ψ_J determined by C. We also say that a σ-stable facet $J \subset \bar{C}$ is σ**-irreducible** if $\sigma \cdot J = J$ and the root system Ψ_J is σ-irreducible.

If $J \subset \bar{C}$ is a σ-irreducible facet, the σ-Coxeter number $h_\sigma(J)$ is defined for $W_J\sigma$. Recall that h_ϑ is the ϑ-Coxeter number for $W_o\vartheta$.

Proposition 7.5. *Let $\sigma \in \Omega\vartheta$. Then the following hold.*

(1) *σ-Coxeter facets $J \subset \bar{C}$ exist, and form a single orbit under Ω^ϑ-conjugacy.*
(2) *Each σ-Coxeter facet J is σ-irreducible.*
(3) *The vertex $x = J^\sigma$ is special in $\mathcal{A}^\sigma$ (*[5, V.3.10]*).*
(4) *If $\sigma = \vartheta$, then J is a hyperspecial vertex in $\mathcal{A}$.*

Proof. First, note that $\Omega^\vartheta = \Omega^\sigma$, since Ω is abelian, so the uniqueness assertion in item 1 makes sense.

Since $\ker[W \to W_o] = X$ is torsion free, an element of $W\sigma$ of finite order projects to an element of the same order in $W_o\vartheta$. By Proposition 7.4, the proofs of item 2 and the existence part of item 1 amount to finding a minimal σ-stable, σ-irreducible facet J for which $h_\sigma(J) = h_\sigma$. We have

$$h_\sigma(J) = \frac{|\Psi_J|}{n_\sigma(J)}, \qquad h_\vartheta = \frac{|\Psi_o|}{n_\vartheta},$$

where $n_\sigma(J)$ is the number of σ-orbits on Π_J.

First suppose that $\sigma = \vartheta$ (the "quasi-split" case). Then we may take J to be a ϑ-stable hyperspecial vertex in $\mathcal{A}$. Let us prove uniqueness in this case. Let J be any ϑ-irreducible facet in $\bar{C}$ such that J^ϑ is a vertex in $\mathcal{A}^\vartheta$. Then $n_\vartheta(J) = n_\vartheta$ and $|\Psi_J| \le |\Psi_o|$. Hence $h_\vartheta(J) = h_\vartheta$ if and only if $|\Psi_J| = |\Psi_o|$. The latter condition implies that J^ϑ is special. If $\vartheta = 1$, then Ω is transitive on special vertices, proving uniqueness. There are four cases where $\vartheta \ne 1$, namely where (W, ϑ) has type ${}^2A_n,\ {}^2D_n,\ {}^3D_4,\ {}^2E_6$. One checks in each case that if J^ϑ is special and $|\Psi_J| = |\Psi_o|$, then J is a ϑ-stable hyperspecial vertex in $\mathcal{A}$. These vertices are permuted transitively by Ω^ϑ, completing the uniqueness proof for $\sigma = \vartheta$.

For $\sigma \ne \vartheta$, we argue case-by-case, as follows. It is easy to see that if J is a σ-Coxeter facet, then there is $I \subseteq \bar{J}$ such that Ψ_I is a σ-irreducible factor of Ψ_J and $h_I(\sigma) = h_\vartheta$. We compute h_ϑ, and $h_\sigma(I)$ for each σ-irreducible facet $I \subset \bar{C}$. We find in each case a unique such facet J, up to Ω^ϑ conjugacy, such that $h_\sigma(J) = h_\vartheta$. Moreover, this J is in each case a minimal σ-stable facet, as claimed.

The results are given in the table below. In the first column, we indicate the type of W and σ using the "name" of [40]. Since $\sigma \ne \vartheta$, we list only those names which are those of non-quasisplit groups. The second column shows a subdiagram of the affine Dynkin diagram, namely the one whose vertices are the simple affine roots vanishing on J, and for which $h_\sigma(J) =$

h_ϑ. If J is the product of k copies of an irreducible type J_1, permuted transitively by σ, we write $J_1^{(k)}$.

Name	J	$h_\sigma(J) = h_\vartheta$
${}^dA_{dm-1}$	$A_d^{(m)}$	dm
${}^2A''_{2m-1}$	${}^2A_{2m-2}$	$4m-2$
${}^2B_n$	${}^2D_n$	$2n$
${}^2C_{2m}$	$C_m^{(2)}$	$4m$
${}^2C_{2m+1}$	${}^2A_{2m}$	$4m+2$
${}^2D'_n$	${}^2D_{n-1}$	$2n-2$
${}^2D''_{2m}$	${}^2A_{2m-1}$	$4m-2$
${}^2D''_{2m+1}$	${}^2A_{2m}$	$4m+2$
${}^4D_{2m}$	${}^2D_m^{(2)}$	$4m$
${}^4D_{2m+1}$	${}^2D_m^{(2)}$	$4m$
${}^3E_6$	${}^3D_4$	12
${}^2E_7$	${}^2E_6$	18

This completes the proof of Proposition 7.5. □

8. Coxeter tori

We return now to p-adic groups, and consider first the stable class of tori (see section 6.9) corresponding to a ϑ-Coxeter element in $W_o\vartheta$. We now assume that **G** is simple, of adjoint type. The latter condition means that $X = X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{Z}\Phi, \mathbb{Z})$.

Let $u \in Z^1(\mathrm{F}, G)$ be a cocycle, giving the twisted Frobenius $\mathrm{F}_u = \mathrm{Ad}(u) \circ \mathrm{F}$. We define an F_u**-Coxeter torus** in G to be a torus in $\mathfrak{T}(G, \mathrm{F}_u)$ whose (G, F_u)-stable class corresponds, via the map Ψ_u in (42), to the class of ϑ-Coxeter elements in $W_o\vartheta$. Since ϑ-Coxeter elements are elliptic, such tori exist by Lemma 6.3. Let $\mathcal{T}_{\mathsf{cox}} \subset \mathfrak{T}(G, \mathrm{F}_u)$ be the (G, F_u)-stable class of F_u-Coxeter tori in G.

Proposition 8.1. *For $u \in Z^1(\mathrm{F}, N)$, the following hold.*

(1) *The F_u-Coxeter tori in G form a single conjugacy class under G^{F_u}.*
(2) *If S is an F_u-Coxeter torus in G, then the natural map*

$$H^1(\mathrm{F}_u, S) \to H^1(\mathrm{F}_u, G)$$

is a bijection; both groups are isomorphic to $\Omega/(1-\vartheta)\Omega$.
(3) *If S is an F_u-Coxeter torus in G, with normalizer $N_G(S)$, then the natural map*

$$N_G(S)^{\mathrm{F}_u}/S^{\mathrm{F}_u} \to (N_G(S)/S)^{\mathrm{F}_u}$$

is a bijection; both groups are cyclic of order h_ϑ/t, where t is the order of ϑ.

Proof. Part 3 follows from [14, 10.2] and part 5 of 7.1.

Let $\omega \in \Omega/(1-\vartheta)\Omega$ correspond to the class of u under Kottwitz' isomorphism

$$\Omega/(1-\vartheta)\Omega \simeq H^1(\mathrm{F}, G); \tag{55}$$

see [28] and [14, chap.2]. Let $C(w\vartheta)$ denote the centralizer of $w\vartheta$ in W_o. From (44), the G^{F_u}-orbits in $\mathcal{T}_{\mathsf{cox}}$ are in bijection with the $C(w\vartheta)$-orbits in the fiber $r_w^{-1}(\omega)$ of the map $X/(1-w\vartheta)X \xrightarrow{r_w} \Omega/(1-\vartheta)\Omega$ in (43). Under Kottwitz' isomorphism (55), the map r_w may be identified with the map in part 2; see [14, 2.6.1]. Thus, parts 1 and 2 of Proposition 8.1 both amount to the claim that r_w is bijective.

It is clear that r_w is surjective. If $\vartheta = 1$, injectivity is equivalent to the fact, due to Steinberg (see exercise 22 in [5, chap. 6]), that $(1-w)X$ is the co-root lattice of $\mathbf{T}$ in $\mathbf{G}$.

For $\vartheta \neq 1$ we compute $|\det(1-w\vartheta)|$, in the following table. In the top row, the upper-left superscript is the order of ϑ.

(W_o, ϑ) :	${}^2A_{2m}$	${}^2A_{2m-1}$	${}^2D_{2m}$	${}^2D_{2m+1}$	${}^3D_4$	${}^2E_6$
Ω :	μ_{2m+1}	μ_{2m}	$\mu_2\times\mu_2$	μ_4	$\mu_2\times\mu_2$	μ_3
$\lvert\det(1-w\vartheta)\rvert$:	1	2	2	2	1	1

In this table, when Ω is cyclic, the action of ϑ is inversion. For ${}^2D_{2m}$ the action of ϑ switches the factors in Ω, and for ${}^3D_4$, the action of ϑ cyclically permutes the nontrivial elements of Ω. It follows that in each case, we have $|\det(1-w\vartheta)| = |\Omega/(1-\vartheta)\Omega|$. □

8.1. **Remarks on $H^1(\mathrm{F}, G)$.** We have seen that if $w\vartheta$ is a ϑ-Coxeter element in $W_o\vartheta$, then

$$H^1(\mathrm{F}, G) \simeq \Omega/(1-\vartheta)\Omega \simeq X/(1-w\vartheta)X.$$

Let us take a closer look at the group $\Omega/(1-\vartheta)\Omega$.

The map $\omega \mapsto \omega\cdot o$ is a bijection from Ω to the set of hyperspecial vertices in the closure of C. This bijection is given explicitly as follows. Let $\mu_2, \dots, \mu_f \in X$ be the minuscule co-weights [5, p.240], and let W_i be the stabilizer of μ_i in W_o. The points

$$o, \qquad x_i := t_{\mu_i}\cdot o, \qquad 2 \le i \le f$$

are the hyperspecial vertices in $\bar{C}$. Set $\omega_i := t_{\mu_i} w_o w_i$, where w_o and w_i are the longest elements (with respect to Π) of W_o and W_i, respectively. Then [5, p. 189] we have $\Omega = \{1, \omega_2, \dots, \omega_f\}$, and it is clear that $\omega_i \cdot o = x_i$. It follows that ϑ acts on Ω according to the way ϑ permutes the minuscule coweights. The latter is easily determined from the action of ϑ on the Dynkin diagram of $\mathbf{G}$.

For $\vartheta \neq 1$ one can choose representatives for $\Omega/(1-\vartheta)\Omega$ as follows. Note that $\Omega/(1-\vartheta)\Omega$ is nontrivial only in types ${}^2A_{2m-1}$ and ${}^2D_n$, where the dual group of $\mathbf{G}$ is $SL_{2m}(\mathbb{C})$ and $\mathrm{Spin}_{2n}(\mathbb{C})$, respectively. The nontrivial element of $\Omega/(1-\vartheta)\Omega$ is represented by the highest weight μ of the standard representation of $SL_{2m}(\mathbb{C})$ and either spin representation of $\mathrm{Spin}_{2n}(\mathbb{C})$, respectively.

9. Coxeter L-packets

We continue to assume, as in the previous section, that $\mathbf{G}$ is simple, of adjoint type. As above, let F be the Frobenius endomorphism of $G = \mathbf{G}(K)$ arising from a quasi-split k-structure on $\mathbf{G}$, fixing the hyperspecial vertex $o \in \mathcal{A}(\mathbf{T})$, and let ϑ be the automorphism of X induced by F. Recall the construction of L-packets from section 6: Given a Langlands parameter $\varphi : \mathcal{W} \to {}^LG$ satisfying the conditions in section 6.3, the image of Frobenius $\varphi(\mathrm{Frob})$ determines an elliptic element $w\vartheta \in W_o\vartheta$. For each $\lambda \in X$ we have a twisted Frobenius endomorphism $\mathrm{F}_\lambda = \mathrm{Ad}(u_\lambda)\circ \mathrm{F}$ and an irreducible supercuspidal representation $\pi_\lambda = \pi(T_\lambda, \chi_\lambda) \in \mathrm{Irr}(G^{\mathrm{F}_\lambda})$. The pair (u_λ, π_λ) determines a G-equivalence class $\pi(\varphi, \rho) = [u_\lambda, \pi_\lambda]$ which depends only on the image ρ of λ in $X/(1-w\vartheta)X$. These classes form the L-packet

$$\Pi(\varphi) = \{[u_\lambda, \pi_\lambda] :\ \lambda \in X/(1-w\vartheta)X\}.$$

In this chapter we explicate these L-packets when $w\vartheta$ is a ϑ-Coxeter element in $W_o\vartheta$. Since $\mathbf{G}$ is adjoint, the classes in $H^1(\mathrm{F}, G)$ parametrize the inner forms of $\mathbf{G}$. Using Proposition 8.1, it follows that $|\Pi(\varphi)| = |H^1(\mathrm{F}, G)|$, and for each class $\omega \in H^1(\mathrm{F}, G)$, there is exactly one class $[u_\lambda, \pi_\lambda]$ in $\Pi(\varphi)$ with $u_\lambda \in \omega$.

From section 8.1 we see that if $\mathbf{G}$ is split ($\vartheta = 1$), then

$$\Pi(\varphi) = \{[1, \pi_0],\ [u_2, \pi_{\mu_2}], \dots, [u_f, \pi_{\mu_f}]\},$$

where $u_i = u_{\mu_i}$, and the μ_i are the minuscule weights as in Section 8.1. If $\mathbf{G}$ is not split then $\Pi(\varphi) = \{[1, \pi_0]\}$ is a singleton, except when $\mathbf{G}$ has type ${}^2A_{2m-1}$ or ${}^2D_n$, in which case $\Pi(\varphi) = \{[1, \pi_0], [u_\mu, \pi_\mu]\}$.

The inducing data for the representations π_λ is given as follows. By Lemma 6.2 we may choose our ϑ-Coxeter element $w\vartheta \in W_o\vartheta$ and $\lambda \in X$ that $x_\lambda \in \bar{C}$.

Proposition 9.1. *With the set-up as just described, the following hold.*

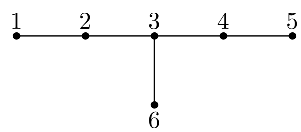
(1) *The element $t_\lambda w\vartheta = w_\lambda\sigma_\lambda$ is a σ_λ-Coxeter element in $W_\lambda\sigma_\lambda$.*

(2) *The facet* J_λ *containing* x_λ *is a* σ_λ*-Coxeter facet in* $\mathcal{A}$*, and* $x_\lambda = J_\lambda^{\sigma_\lambda}$.
(3) *The torus* $\mathbf{T}_\lambda$ *is an* F_λ*-Coxeter torus in* $\mathbf{G}$.

Proof. Item 1 is immediate from Proposition 7.4. Item 2, along with the classification of the various facets J_λ, follows from section 7.4. The construction of T_λ shows that the (G, F_λ)-stable class of T_λ corresponds, under the map Ψ_{u_λ} of Section 8, to the W_o-orbit of $w\vartheta$ in $W_o\vartheta$. This proves item 3. $\square$

9.1. **An example in E_6.** To illustrate, we consider the split adjoint group $\mathbf{G}$ of type E_6. Then ${}^LG = \hat{G}$ is the simply-connected form of $E_6(\mathbb{C})$. Suppose $w \in W_o = W(E_6)$ is a Coxeter element. Then $C_\varphi = Z(\hat{G}) \simeq \mathbb{Z}/3\mathbb{Z}$ and the two nontrivial characters of C_φ are the restrictions of the two minuscule weights $\mu, \mu' \in X$ of $\hat{T}$. The groups G^{F_μ} and $G^{\mathrm{F}_{\mu'}}$ are isomorphic, of type ${}^3E_6$.

Fix an alcove $C \subset \mathcal{A}(T)$ and number the simple roots Π corresponding to C as follows.



Let r_i be the corresponding simple reflections, and let r_0 be the reflection about the highest root. Then $\mu,\ \mu'$ are the fundamental weights corresponding to $\alpha_1,\ \alpha_5$ respectively. As Coxeter element for which $x_\mu,\ x_{\mu'} \in \bar{C}$ (see Lemma 6.2), we take $w = (r_0r_6r_3r_2)(r_1r_3r_5r_2r_4r_6)(r_2r_3r_6r_0)$. Here we have written w in non-reduced form to show that it is indeed a Coxeter element in E_6. One checks that $w\mu = -\mu + \mu', \quad w\mu' = -\mu$. It follows that $x_\mu = x_{\mu'} = \frac{1}{3}(o + \mu + \mu')$. This point is the barycenter of the triangle $J \subset \bar{C}$ whose vertices are the three hyperspecial vertices $o, \mu, \mu' \in \bar{C}$. Hence $W_{x_\mu} = W_{x_{\mu'}} = W_J$ is the pointwise stablizer of J in W, and has type D_4.

The L-packet $\Pi(\varphi)$ has the form $\Pi(\varphi) = \{\pi_0,\ \pi_\mu,\ \pi_{\mu'}\}$, (suppressing the cocycles) where π_0 is induced from the hyperspecial parahoric G_o^{F} in the split form of G and π_μ is induced from the special parahoric $G_{x_\mu}^{\mathrm{F}_\mu}$ in the non-split inner form of G, and $\pi_{\mu'}$ is the "same" representation on the isomorphic group $G^{\mathrm{F}_{\mu'}}$.

The decomposition $t_\mu w = w_\mu y_\mu$ of (35) is obtained as follows. The element y_μ must be a nontrivial rotation of C. Since w_μ fixes J pointwise, it follows that y_μ is the rotation of J sending $o \mapsto \mu$, and $y_{\mu'}$ is the opposite rotation. This means y_μ and $y_{\mu'}$ act on $W_J = W(D_4)$ by triality automorphisms, so the reductive quotient of $\bar{G}_J$ has rational type ${}^3D_4$ over $\mathfrak{f}$. From Proposition 9.1 it follows that w_μ is a twisted Coxeter element in W_Jy_μ, and likewise for $w_{\mu'}$. One can also show this directly, but the computation is tedious.

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