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ISOGENIES OF HECKE ALGEBRAS AND A LANGLANDS CORRESPONDENCE FOR RAMIFIED PRINCIPAL SERIES REPRESENTATIONS

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Abstract. This paper gives a Langlands classification of constituents of ramified principal series representations for split p-adic groups with connected center.

Introduction

Let F be a p-adic field, and let $\mathcal{G}$ be the group of F-rational points in a split reductive group $\underline{\mathcal{G}}$ over F. Let $\mathcal{B}$ be a Borel subgroup in $\mathcal{G}$. We denote by $\mathrm{Irr}(\mathcal{G},\mathcal{B})$ the set of irreducible representations of $\mathcal{G}$, up to equivalence, which are subquotients of representations induced from characters of $\mathcal{B}$. The aim of this paper is to classify the representations in $\mathrm{Irr}(\mathcal{G},\mathcal{B})$ in terms of the geometry of the dual group G of $\mathcal{G}$, according to predictions by Langlands and others. We must assume that p is not too small (as specified in section 5 below), and that $\underline{\mathcal{G}}$ has connected center. Our result generalizes, and depends on, the classification of constituents of unramified principal series, given by Kazhdan and Lusztig in [KL]. For inducing characters of arbitrary ramification, but which have trivial stabilizer in the Weyl group, a classification was previously obtained, with no restrictions on p, by Rodier [Rod1].

Let $\mathcal{W}_F$ be the Weil group of F, and let $\mathcal{I}_F \subset \mathcal{W}_F$ be the inertia subgroup. By "Langlands parameter" we mean a continuous homomorphism

$$\Phi : \mathcal{W}_F \times SL_2(\mathbf{C}) \longrightarrow G$$

which is rational on $SL_2(\mathbf{C})$ and such that $\Phi(\mathcal{W}_F)$ consists of semisimple elements in G. Choose a Borel subgroup B_2 in $SL_2(\mathbf{C})$, and let $S_\Phi = \Phi(\mathcal{W}_F \times B_2)$, a solvable subgroup of G. Let $\mathbf{B}$ denote the variety of Borel subgroups of G, and let $\mathbf{B}^\Phi$ denote the subvariety consisting of Borel subgroups of G containing S_Φ. Let G_Φ be the centralizer in G of the image of Φ. Then G_Φ acts naturally on $\mathbf{B}^\Phi$, and hence on the singular homology $H_*(\mathbf{B}^\Phi, \mathbf{C})$.

Theorem 1. *The representations $V \in \mathrm{Irr}(\mathcal{G},\mathcal{B})$ are in bijection with G-conjugacy classes of pairs (Φ,ρ), where Φ is a Langlands parameter such that $\mathbf{B}^\Phi$ is non-empty, and ρ is an irreducible representation of G_Φ which appears in the natural action of G_Φ on $H_*(\mathbf{B}^\Phi, \mathbf{C})$.*

The bijection in Theorem 1 contains the classification of consitutents of a given principal series representation as follows. Let $\underline{\mathcal{T}}, T$ be maximal split tori in $\underline{\mathcal{G}}, G$

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respectively, with $\underline{\mathcal{T}} \subset \underline{\mathcal{B}}$. Let X denote the rational co-character group of $\underline{\mathcal{T}}$, identified with the rational character group of T. Let $\mathcal{T}_0$ be the maximal compact subgroup of $\mathcal{T}$. A choice of uniformizer in F gives a splitting $\mathcal{T} = \mathcal{T}_0 \times X$, so characters of $\mathcal{T}$ have the form $\chi \otimes \tau$, where χ is a character of $\mathcal{T}_0$, and $\tau \in T$. Via abelian class field theory, χ gives rise to a homomorphism $\hat{\chi} : \mathcal{I}_F \longrightarrow T$. Let H be the centralizer in G of the image of $\hat{\chi}$. Then the constituents of the induced representation $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ correspond to H-conjugacy classes of triples (τ, u, ρ) where $\tau u \tau^{-1} = u^q$ and ρ is a representation of the mutual centralizer $H_{\tau,u}$ which appears in the homology of the variety $\mathbf{B}_H^{\tau,u}$ of Borel subgroups of H containing τ and u. Such a triple (τ, u, ρ) corresponds to a pair (Φ, ρ) as in Theorem 1 such that $\Phi|_{\mathcal{I}_F} = \hat{\chi}$, and the component groups of $H_{\tau,u}$ and G_Φ are naturally isomorphic to each other (see section 4).

Here is an outline of the proof of Theorem 1. Let $\mathcal{H}$ be the affine Hecke algebra attached to H, with constant parameter q= residue cardinality of F. The central characters of $\mathcal{H}$ correspond to semisimple conjugacy classes in H. In his thesis [Ro], Roche has shown that the constituents of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ are in bijection with the simple $\mathcal{H}$-modules of central character τ. In the case that the derived group H_{der} is simply connected, the simple $\mathcal{H}$-modules have been classified by Kazhdan and Lusztig [KL]. However, in our situation, H_{der} is not always simply-connected. The first main step in our proof is an extension of Kazhdan and Lusztig's result to the nonsimply-connected case. It is independent of p-adic groups.

Theorem 2. *Let H be a connected reductive complex algebraic group. Let $\mathcal{H}$ be the affine Hecke algebra whose root datum is that of H, with constant parameter q not a root of unity. Then the simple $\mathcal{H}$-modules are in bijection with H-conjugacy classes of triples (τ, u, ρ), where $\tau \in H$ is semisimple, $\tau u \tau^{-1} = u^q$, and ρ is an irreducible representation of $H_{\tau,u}$ appearing in the homology of $\mathbf{B}_H^{\tau,u}$.*

To prove Theorem 2, we begin with a recent result of Ram and Ramagge [RR]. They have observed that $\mathcal{H}$ is the fixed points of a group of automorphisms C acting on a larger Hecke algebra $\widetilde{\mathcal{H}}$ whose root datum is simply-connected, and this enables them to apply a kind of Clifford theory, developed by Macdonald and Green, to relate $\widetilde{\mathcal{H}}$-modules to $\mathcal{H}$-modules. Thus, for each simple $\widetilde{\mathcal{H}}$-module V, we have an inertia group $C_V \subset C$, a cocycle $\eta_V : C_V \times C_V \longrightarrow \mathbf{C}^\times$, and corresponding twisted group algebra $\mathcal{E}_V$, such that the restriction of V to $\mathcal{H}$ is of the form

$$V|_{\mathcal{H}} \simeq \bigoplus_{\psi \in \mathrm{Irr}(\mathcal{E}_V)} \psi \otimes {}^{\psi}V,$$

where $\mathrm{Irr}(\mathcal{E}_V)$ denotes the irreducible representations of $\mathcal{E}_V$. Ram and Ramagge show that each ${}^{\psi}V$ is either zero, or a simple $\mathcal{H}$-module, and that all simple $\mathcal{H}$-modules are obtained in this way. In sections 2-4 below, we calculate $\mathcal{E}_V$ in terms of Kazhdan and Lusztig's parameters of the $\widetilde{\mathcal{H}}$-module V, we show that ${}^{\psi}V$ is always nonzero, and we show that the pairs (V, ψ) correspond to triples (τ, u, ρ) as in Theorem 2. These are mostly calculations in equivariant K-theory, based on the work of Kazhdan and Lusztig [KL].

Combining Theorem 2 with Roche's Hecke algebra isomorphisms, we get, for fixed χ, a classification of the constituents of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ as τ varies over T.

Now, if w is in the Weyl group, the same representation V of $\mathcal{G}$ will appear in two induced representations $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ and $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi^w \otimes \tau^w)$. As Roche has pointed out to me, it does not seem obvious that his Hecke algebra isomorphisms lead to

G-conjugate Langlands parameters for V. In sections 5 and 6, we prove that they do, and this completes the proof of Theorem 1.

In section 7, we determine the pair (Φ, ρ) attached by Theorem 1 to the unique constituent of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ admitting a Whittaker model. (Recall that $\mathcal{G}$ has connected center.) The corresponding triple (τ, u, ρ) is given as follows. The centralizer H_τ acts on the variety $\mathfrak{q}_\tau = \{u \in H : \ \tau u \tau^{-1} = u^q\}$, and has a unique dense orbit $\mathfrak{q}_\tau^\circ \subset \mathfrak{q}_\tau$. Then (see (7.3.1)) the Whittaker constituent of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ corresponds to the triple (τ, u, triv) where $u \in \mathfrak{q}_\tau^\circ$, and triv denotes the trivial representation of $H_{\tau,u}$.

If G is a Levi subgroup in a larger group G', the Shahidi local L-functions $L(s, V_{\Phi,\mathrm{triv}}, r)$ are defined, for every constituent r of the action of G on $\mathfrak{g}'/\mathfrak{g}$, since $V_{\Phi,\mathrm{triv}}$ has a Whittaker model. Using [Sh, 3.5 3)], it is easy to check that

$$L(s, V_{\Phi,\mathrm{triv}}, r) = \det[1 - q^{-s}\Phi(\mathrm{Frob})|_{r_\chi}]^{-1},$$

where r_χ denotes the invariants of $\Phi(\mathcal{I}_F)$ in r. Hence the Shahidi L-functions of $V_{\Phi,\mathrm{triv}}$ agree with the Artin-Deligne L-functions of Φ, for these representations r of G.

A final remark: Theorem 2 also gives the classification of constituents of unramified principal series for arbitrary split connected groups $\underline{\mathcal{G}}$ over F [B1], without any restriction on p. The result of [KL] gives such a classification for $\underline{\mathcal{G}}$ with connected center. To get a classification for ramified principal series for all split groups, one must consider affine Hecke algebras attached to disconnected dual groups [Ro, §8]. I do not attempt this here.

This paper developed as follows. In October 1996 at the Langlands birthday conference, Roche asked me, in connection with his work [Ro], if a Langlands classification of nonsimply connected affine Hecke modules was available. It was not, until the spring of 1999 when Ram sent me the preprint [RR]. I was then able to prove Theorem 2, and also, I thought, Theorem 1. I asked Roche if we could consider these matters further in joint work. He declined, but instead offered valuable comments on a rough draft of this paper, and pointed out the difficulty mentioned three paragraphs above. I thank Arun Ram and Alan Roche for these contributions.

1. Hecke Algebras and Isogeny

To keep track of the behavior of affine Hecke algebras under isogeny, as well as the naturality properties of the Kazhdan-Lusztig construction of simple Hecke modules in the simply connected case, we shall view "Affine Hecke Algebra" as a functor on the category of based root data, where the morphisms are isogenies. I believe this pedantic foundation makes the later arguments clearer.

1.1. A based root datum $\Phi = (X, Y, R, \check{R}, S)$ consists of a pair X, Y of free abelian groups, a perfect pairing $\langle\ ,\ \rangle : \ X \times Y \longrightarrow \mathbb{Z}$, root systems $R \subset X$, $\check{R} \subset Y$, a bijection $\alpha \mapsto \check{\alpha} : R \longrightarrow \check{R}$, and a set of simple roots $S \subset R$.

Let $R^+ \subset R$ denote the roots which are nonnegative integral combinations of those in S, and set $R^- = -R^+$. The Weyl group W of Φ is the subgroup of $Aut(X)$ generated by the Coxeter reflections $s_\alpha(x) = x - \langle x, \check{\alpha}\rangle\alpha$, for $\alpha \in S$. We sometimes confound roots in S with the corresponding reflections in W. Denote by $\widetilde{W}$ the affine Weyl group $\widetilde{W} = W \ltimes X$. Let $\ell(w)$ be the length of an element $w \in \widetilde{W}$, and let $\widetilde{S}$ be the set of simple affine roots [L1, 1.4].

An *isogeny* [Sp] of based root data $f : \Phi' \longrightarrow \Phi$ is given by an injective group homomorphism $f : X' \longrightarrow X$ with finite cokernel, which restricts to a bijection $f : R' \longrightarrow R$, and whose adjoint, with respect to the pairings of Φ' and Φ, is a bijection $\check{R} \longrightarrow \check{R}'$. The isogeny f induces an injection $\widetilde{W}' \hookrightarrow \widetilde{W}$ which restricts to an isomorphism of finite Coxeter groups $(W, S) \simeq (W', S')$.

1.2. Let $\mathbf{A} = \mathbf{C}[\mathbf{q}, \mathbf{q}^{-1}]$, where $\mathbf{q}$ is the identity coordinate of $\mathbf{C}^\times$. Given a based root datum Φ, consider the associative $\mathbf{A}$-algebra $\mathcal{H}(\Phi, \mathbf{q})$ with free $\mathbf{A}$-basis $\{T_w : w \in \widetilde{W}\}$ and multiplication rules

$$T_w T_{w'} = T_{ww'} \quad \text{if } \ell(w) + \ell(w') = \ell(ww'), \tag{1.2a}$$

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - \mathbf{q}) = 0, \quad \text{for all } \alpha \in \widetilde{S}. \tag{1.2b}$$

Another presentation of $\mathcal{H}(\Phi, \mathbf{q})$, due to Bernstein, is given as follows [L1]. Let $\mathbf{A}[X]$ denote the group algebra of X, with $\mathbf{A}$-basis $\{e_\lambda : \ \lambda \in X\}$. Let

$$X^+ = \{\lambda \in X : \ \langle \lambda, \check{\alpha} \rangle \geq 0 \text{ for all } \alpha \in S\}.$$

Then the mapping $e_\lambda \mapsto \mathbf{q}^{-\ell(\lambda)/2} T_\lambda$, for $\lambda \in X^+$, extends to an embedding of algebras

$$\mathbf{A}[X] \hookrightarrow \mathcal{H}(\Phi, \mathbf{q}),$$

by means of which we view $\mathbf{A}[X]$ as a subalgebra of $\mathcal{H}(\Phi, \mathbf{q})$. Let $\mathcal{H}_0(\Phi, \mathbf{q})$ be the subalgebra spanned by $\{T_w : \ w \in W\}$. The Bernstein presentation of $\mathcal{H}(\Phi, \mathbf{q})$ is the isomorphism, given by multiplication

$$\mathcal{H}_0(\Phi, \mathbf{q}) \otimes_{\mathbf{A}} \mathbf{A}[X] \xrightarrow{\simeq} \mathcal{H}(\Phi, \mathbf{q}).$$

The corresponding multiplication in the tensor product is given, for $\alpha \in S$, by

$$e_\lambda (T_{s_\alpha} - \mathbf{q}) = (T_{s_\alpha} - \mathbf{q}) e_{s_\alpha \lambda} + (e_{s_\alpha \lambda} - e_\lambda) \frac{\mathbf{q} - e_\alpha}{1 - e_\alpha}. \tag{1.2c}$$

The center of $\mathcal{H}(\Phi, \mathbf{q})$ is the subalgebra $\mathbf{A}[X]^W$ of W-invariants in $\mathbf{A}[X]$.

If $q \in \mathbf{C}^\times$, then replacing $\mathbf{q}$ by q results in a $\mathbf{C}$-algebra, denoted $\mathcal{H}(\Phi, q)$, or simply $\mathcal{H}(\Phi)$, if q is understood. We assume throughout that q is not a root of unity.

1.3. Let G be a complex connected reductive algebraic group over $\mathbf{C}$, with Lie algebra $\mathfrak{g}$, and let Z be the center of G. Given a pair (T, B) consisting of a maximal torus T contained in a Borel subgroup B of G, we have a based root datum $\Phi(T, B) = (X, Y, R, \check{R}, S)$, where $X = X^*(T)$, $Y = X_*(T)$ are the rational character and co-character groups of T, R and $\check{R}$ are the roots and co-roots of T in G, R^+ is the set of roots appearing in $\mathfrak{g}/\mathfrak{b}$, and $S \subset R^+$ is the corresponding set of simple roots. Note that R^- is the set of roots in $\mathfrak{b}$.

Given two such pairs (T, B), (T', B'), there is an inner automorphism ι of G such that $(\iota(T), \iota(B)) = (T', B')$. The induced isomorphism $\Phi(\iota) : \Phi(T, B) \longrightarrow \Phi(T', B')$ is independent of the choice of ι. Let $\Phi(G) = (X, Y, R, \check{R}, S)$ denote the inverse limit of the based root data $\Phi(T, B)$ with respect to the isomorphisms $\Phi(\iota)$. Then $\Phi(G)$ is a based root datum. Let $\mathcal{H}(G, \mathbf{q}) = \mathcal{H}(\Phi(G), \mathbf{q})$, $\mathcal{H}_0(G, \mathbf{q}) = \mathcal{H}_0(\Phi(G), \mathbf{q})$, etc.

For any pair (T, B), we have canonical isomorphism $\Phi(G) \longrightarrow \Phi(T, B)$. If $\lambda \in X$ and B is a Borel subgroup of G, then we have a character $\Psi_\lambda^B : B \longrightarrow \mathbf{C}^*$, by means of the canonical map $X \longrightarrow X^*(T)$ for any maximal torus $T \subset B$. By definition, we have

$$\Psi_\lambda^{gBg^{-1}}(gbg^{-1}) = \Psi_\lambda^B(b), \qquad g \in G, \quad b \in B.$$

In particular, if $z \in Z$, then $\Psi_\lambda^B(z)$ is independent of B. We set

$$\lambda(z) = \Psi_\lambda^B(z).$$

Given $b \in B$, the map $\lambda \mapsto \Psi_\lambda^B(b)$ extends linearly to an algebra homomorphism

$$\mathbf{C}[X] \longrightarrow \mathbf{C}, \qquad \theta \mapsto \Psi_\theta^B(b).$$

If $\tau \in G$ is semisimple, and $\theta \in \mathbf{C}[X]^W$ is a W-invariant element of $\mathbf{C}[X]$, then the value of $\Psi_\theta^B(\tau)$ is the same for every B containing τ, and depends only on the G-conjugacy class of τ. In this way, the ring $\mathbf{C}[X]^W$ is identified with the coordinate ring of the affine variety of semisimple conjugacy classes in G.

1.4. Suppose $f : G \longrightarrow G'$ is an isogeny. If $T \subset B$ in G are mapped by f to $T' \subset B'$ in G', then the induced map $f^* : X^*(T') \longrightarrow X^*(T)$ is an isogeny of based root data

$$f^*_{T,B} : \Phi(T', B') \longrightarrow \Phi(T, B).$$

Hence f induces an isogeny

$$\Phi(f) : \Phi(G') \longrightarrow \Phi(G).$$

For example, if $G = G'$ and f is an inner automorphism of G, then $\Phi(f)$ is the identity map on $\Phi(G)$.

Now $\Phi(f)$ induces a homomorphism $w' \mapsto w$ between the affine Weyl groups of $\Phi(G')$ and $\Phi(G)$, along with an $\mathbf{A}$-algebra homomorphism

$$\mathcal{H}(f) : \mathcal{H}(G', \mathbf{q}) \longrightarrow \mathcal{H}(G, \mathbf{q}), \qquad T_{w'} \mapsto T_w.$$

If $\pi : \mathcal{H}(G, \mathbf{q}) \longrightarrow \mathrm{End}(V)$ is an $\mathcal{H}(G, \mathbf{q})$-module, we let $f^\sharp V$ be the $\mathcal{H}(G', \mathbf{q})$-module with underlying space V on which $T \in \mathcal{H}(G', \mathbf{q})$ acts via $\pi(\mathcal{H}(f)T)$.

We have $f^\sharp V = V$ if f is an inner automorphism. More generally, if we identify $X' \subset X$ by means of $\Phi(f)$, then $\mathcal{H}(G', \mathbf{q}) \subset \mathcal{H}(G, \mathbf{q})$, and $f^\sharp V$ is the restriction of V to $\mathcal{H}(G', \mathbf{q})$.

This restriction can be analyzed by means of certain automorphisms of $\mathcal{H}(G, \mathbf{q})$. Namely, the center Z acts on $\mathcal{H}(G, \mathbf{q})$ by $\mathbf{A}$-algebra automorphisms. Using the presentation (1.2c), this action is given by

$$(T_w \otimes e_\lambda)^z = \lambda(z) T_w \otimes e_\lambda, \qquad w \in W, \ \lambda \in X, \ z \in Z.$$

The twisted $\mathcal{H}(G, \mathbf{q})$-module (π^z, V^z) is defined by $\pi^z(T^z) = \pi(T)$, for $T \in \mathcal{H}(G, \mathbf{q})$.

The central characters of $\mathcal{H}(G, \mathbf{q})$ are in canonical bijection with semisimple conjugacy classes in $G \times \mathbf{C}^\times$. If the class of $(\tau, q) \in G \times \mathbf{C}^\times$ is the central character of the simple $\mathcal{H}(G, \mathbf{q})$-module V, then the central character of V^z is the class of $(z\tau, q)$.

1.5. Let C be a finite subgroup of Z, and let $f : G \longrightarrow G/C = G'$ be the quotient map. This is an isogeny, which induces, as in (1.4), an injection of algebras

$$\mathcal{H}(f) : \mathcal{H}(G/C, \mathbf{q}) \hookrightarrow \mathcal{H}(G, \mathbf{q})$$

whose image is the subalgebra

$$\mathcal{H}(G, \mathbf{q})^C \subset \mathcal{H}(G, \mathbf{q})$$

of fixed points under the action of C on $\mathcal{H}$ (restriction of the Z-action defined in (1.4)). We identify $\mathcal{H}(G/C, \mathbf{q}) = \mathcal{H}(G, \mathbf{q})^C$ by means of $\mathcal{H}(f)$.

Now fix $q \in \mathbf{C}^\times$ not a root of unity, and let $\mathcal{H}(G) = \mathcal{H}(G, q)$. If V is an $\mathcal{H}(G)$-module, define the *inertia group* of V to be

$$C_V = \{c \in C : \ V \simeq V^c\}.$$

Choosing $\mathcal{H}(G)$-module isomorphisms $\phi_c : V \simeq V^c$ for each $c \in C_V$ defines a cocycle

$$\eta_V : C_V \times C_V \longrightarrow \mathbf{C}^\times,$$

and a twisted group algebra $\mathbf{C}[C_V, \eta_V]$, whose isomorphism class is independent of the choice of isomorphisms ϕ_c.

If $b \in C$, then $C_V = C_{V^b}$ since C is abelian, and for each $c \in C_V$, the map ϕ_c is also an $\mathcal{H}(G)$-module isomorphism $V^b \longrightarrow V^{bc}$. Hence

$$\mathbf{C}[C_{V^b}, \eta_{V^b}] = \mathbf{C}[C_V, \eta_V].$$

Note that the action of $\mathbf{C}[C_V, \eta_V]$ on V consists of $\mathcal{H}(G)^C$-module endomorphisms.

We will use the following result of Ram and Ramagge, which is based on Clifford theoretic results developed by MacDonald and Green. (The final uniqueness assertion below is not stated in [RR], but easily follows from [Mac].)

(1.5.1) Theorem [RR, App.]**.** *If V is a simple $\mathcal{H}(G)$-module, then the restriction of V to $\mathcal{H}(G)^C$ is of the form*

$$V \simeq \bigoplus_{\psi} {}^{\psi}V \otimes \psi, \tag{1.5a}$$

where ψ runs over the simple $\mathbf{C}[C_V, \eta_V]$-modules, and each ${}^{\psi}V$ is either zero, or a simple $\mathcal{H}(G)^C$-module. All simple $\mathcal{H}(G)^C$-modules are of the form ${}^{\psi}V$, and ${}^{\psi}V \simeq {}^{\psi'}V'$ if and only if there is $c \in C$ such that $V' = V^c$, and $\psi' = \psi$.

2. Equivariant K-theory and Affine Hecke Modules

In this section we assume that G has simply-connected derived group. Let Z be the center of G. Fix $q \in \mathbf{C}^\times$, not a root of unity. In this case, Kazhdan and Lusztig [KL] have shown that the simple $\mathcal{H}(G)$-modules are in bijection with G-conjugacy classes of triples (τ, u, ρ), where $\tau \in G$ is semisimple, $u \in G$ is unipotent, such that $\tau u \tau^{-1} = u^q$, and σ is an irreducible representation of the mutual centralizer $G_{\tau,u}$ appearing in the natural action of $G_{\tau,u}$ in the homology of $\mathbf{B}^{\tau,u}$. Given such a triple (τ, u, σ), the corresponding simple $\mathcal{H}(G)$-module $V_{\tau,u,\sigma}$ is the unique simple quotient of a "standard" $\mathcal{H}(G)$-module $M_{\tau,u,\sigma}$.

We analyze here the behavior of $M_{\tau,u,\sigma}$ and $V_{\tau,u,\sigma}$ under twisting by elements $z \in Z$. The construction of $M_{\tau,u,\sigma}$ is based on equivariant K-theory, and we will express the twisting by z as a direct image g_*, where $g \in G$ satisfies $\tau^g = z\tau$, see (2.7.2). Then in (2.9.1) we evaluate g_* on the constituent of the principal series of $\mathcal{H}(G)$ containing the sign character of $\mathcal{H}_0(G)$. This is needed to determine the Whittaker constituent of ramified principal series for the p-adic group $\mathcal{G}$.

We rely on the list of properties of topological equivariant K-theory laid out in [KL, §1], supplemented by results from [CG]. By work of Thomason, and [DLP], the properties in [KL, §1] are now known to hold in our setting for algebraic K-theory [L2], which is the language used here.

2.1. For any complex algebraic group H acting rationally on a variety $\mathbf{X}$, we denote by $K^H(\mathbf{X})$ the Grothendieck group, tensored with $\mathbf{C}$, of the category of coherent H-equivariant sheaves on $\mathbf{X}$. We write $K(\mathbf{X})$ when $H = 1$.

If $\mathbf{X} = pt$ is a point, an equivariant sheaf is simply a rational representation of H, so $K^H(pt) = R_H$ is the representation ring of H. More generally, if $\mathbf{X}$ has trivial H action, then $K^H(\mathbf{X}) = K(\mathbf{X}) \otimes_{\mathbf{C}} R_H$. For any H-variety $\mathbf{X}$, the space $K^H(\mathbf{X})$ is a finitely generated R_H-module.

2.2. Let $\mathbf{B}$ denote the variety of Borel subgroups of G. The group $G \times \mathbf{C}^\times$ acts on $\mathbf{B}$, with $\mathbf{C}^\times$ acting trivially. For $\lambda \in X$, let L_λ be the $G \times \mathbf{C}^\times$ equivariant line bundle (viewed as a locally free sheaf of rank one) on $\mathbf{B}$, where $(b, q) \in B \times \mathbf{C}^\times$ acts on the fiber $L_\lambda|_B$ via multiplication by $\Psi_\lambda^B(b)$.

Take a semisimple element $\tau \in G$, and let D be the algebraic subgroup of $G \times \mathbf{C}^\times$ generated by (τ, q). Take also a unipotent element $u \in G$ such that $\tau u \tau^{-1} = u^q$. Then D preserves the fixed point variety $\mathbf{B}^u$, so we have the R_D-module $K^D(\mathbf{B}^u)$.

Let $\mathbf{C}_{\tau,q}$ be the complex line on which R_D acts by evaluation of characters at (τ, q). The main object of study is the localized K-group $K^D(\mathbf{B}^u) \otimes_{R_D} \mathbf{C}_{\tau,q}$. This finite dimensional vector space has several incarnations.

As R_D-modules, we have, by the localization theorem [KL, 1.3(k)]

$$K^D(\mathbf{B}^u) \otimes_{R_D} \mathbf{C}_{\tau,q} \simeq K^D(\mathbf{B}^{\tau,u}) \otimes_{R_D} \mathbf{C}_{\tau,q}. \tag{2.2a}$$

The fixed points $\mathbf{B}^\tau$ are a disjoint union

$$\mathbf{B}^\tau = \mathbf{B}_1 \cup \cdots \cup \mathbf{B}_m,$$

where each $\mathbf{B}_i$ is a G_τ-orbit in $\mathbf{B}$. If we choose $B \in \mathbf{B}_1$, then B_τ is a Borel subgroup of G_τ, and B_τ has a unique fixed point $B_i \in \mathbf{B}_i$, for each i. The map

$$G_\tau / B_\tau \longrightarrow \mathbf{B}_i, \qquad hB_\tau \mapsto hB_ih^{-1}$$

is an isomorphism of G_τ-varieties. Therefore, we have the decomposition

$$K^D(\mathbf{B}^{\tau,u}) = \bigoplus_{i=1}^m K^D(\mathbf{B}_i^u), \tag{2.2b}$$

where $\mathbf{B}_i^u = \mathbf{B}_i \cap \mathbf{B}^u$. Since D acts trivially on each $\mathbf{B}_i^u$, we have

$$K^D(\mathbf{B}_i^u) \simeq K(\mathbf{B}_i^u) \otimes_{\mathbf{C}} R_D. \tag{2.2c}$$

Unlike the $\mathbf{B}_i$'s, the varieties $\mathbf{B}_i^u$, for varying i, are not generally isomorphic to one another. Some of them may be empty. However, there is always at least one Borel subgroup containing τ, u, so not all of the $\mathbf{B}_i^u$'s are empty.

If $\mathbf{B}_i^u$ is nonempty, then under the isomorphism (2.2c), the class of the line bundle L_λ in $K^D(\mathbf{B}_i^u)$ corresponds to the class of $L_\lambda \otimes (\Psi_\lambda^{B_i}|_D)$ in $K(\mathbf{B}_i^u) \otimes_{\mathbf{C}} R_D$, where we now forget the D-action on L_λ.

From (2.2a-c) it follows that

$$K^D(\mathbf{B}^u)\otimes_{R_D}\mathbf{C}_{\tau,q}\simeq\bigoplus_{i=1}^m K(\mathbf{B}_i^u)_{\tau,q}, \tag{2.2d}$$

where we have abbreviated

$$K(\mathbf{B}_i^u)_{\tau,q}:=K(\mathbf{B}_i^u)\otimes_{\mathbf{C}}\mathbf{C}_{\tau,q}.$$

2.3. We have an action of $\mathbf{A}[X]$ by R_D-linear operators on $K^D(\mathbf{B}^u)$, namely

$$e_\lambda\circ\xi=L_\lambda\otimes\xi,\qquad \lambda\in X,\quad \xi\in K^D(\mathbf{B}^u),$$

and $\mathbf{q}$ acts as multiplication by the element

$$D\hookrightarrow G\times\mathbf{C}^\times\xrightarrow{1\times\mathbf{q}}\mathbf{C}^\times$$

of R_D.

After localizing at (τ,q) and applying the isomorphism (2.2d), $e_\lambda\circ$ becomes the operator

$$e_\lambda\circ\xi=\Psi_\lambda^{B_i}(\tau)L_\lambda\otimes\xi,\qquad \xi\in K(\mathbf{B}_i^u)_{\tau,q} \tag{2.2e}$$

On $K(\mathbf{B}_i^u)$ the operator $L_\lambda\otimes$ is a unipotent operator [KL, 1.3(m3)], so (2.2e) gives the Jordan decomposition of $e_\lambda\circ$ on $K^D(\mathbf{B}^u)\otimes_{R_D}\mathbf{C}_{\tau,q}$. The action of $\mathbf{q}$ becomes multiplication by q.

2.4. Now take an element $g\in G$ such that $\tau^g=z\tau$, where $z\in Z$. Although g does not centralize D, the map $g:\mathbf{B}^u\longrightarrow\mathbf{B}^{gug^{-1}}$ commutes with the D action, since Z acts trivially on $\mathbf{B}$. By [KL, 1.3(b)] the direct image map g_* is an R_D-linear isomorphism

$$g_*:K^D(\mathbf{B}^u)\longrightarrow K^D(\mathbf{B}^{gug^{-1}}).$$

On the other hand, g permutes the components $\mathbf{B}_i$ of $\mathbf{B}^\tau$. Thus, if $g\mathbf{B}_i=\mathbf{B}_j$, we have an R_D-linear map

$$g_*:K(\mathbf{B}_i^u)_{\tau,q}\longrightarrow K(\mathbf{B}_j^{gug^{-1}})_{\tau,q}.$$

(2.4.1) Lemma. *Suppose $\tau^g=z\tau$ for some $z\in Z$. Then*

$$e_\lambda\circ g_*\xi=\lambda(z)g_*(e_\lambda\circ\xi),\qquad \lambda\in X,\quad \xi\in K(\mathbf{B}_i^u)_{\tau,q}.$$

Thus, if $\lambda(z)=1$, the operators g_* and $e_\lambda\circ$ commute.

Proof. Suppose $g\mathbf{B}_i=\mathbf{B}_j$. Since L_λ is the restriction of a G-equivariant line bundle on $\mathbf{B}$, we have $g_*(L_\lambda|_{\mathbf{B}_i^u})=L_\lambda|_{\mathbf{B}_j^{gug^{-1}}}$ in $K(\mathbf{B}_j^{gug^{-1}})$. Using [KL, 1.3f], we compute

$$\begin{aligned}
g_*^{-1}e_\lambda\circ g_*\xi&=g_*^{-1}\Psi_\lambda^{B_j}(\tau)L_\lambda|_{\mathbf{B}_j^{gug^{-1}}}\otimes g_*\xi\\
&=\Psi_\lambda^{B_j}(\tau)L_\lambda|_{\mathbf{B}_i^u}\otimes\xi\\
&=\Psi_\lambda^{B_i}(\tau^g)L_\lambda|_{\mathbf{B}_i^u}\otimes\xi\\
&=\Psi_\lambda^{B_i}(z\tau)L_\lambda|_{\mathbf{B}_i^u}\otimes\xi\\
&=\lambda(z)\Psi_\lambda^{B_i}(\tau)L_\lambda|_{\mathbf{B}_i^u}\otimes\xi\\
&=\lambda(z)e_\lambda\circ\xi.
\end{aligned}$$

□

2.5. The action of $\mathbf{A}[X]$ on $K^D(\mathbf{B}^u)$ described above extends to an action of $\mathcal{H}(G,\mathbf{q})$, as follows. It is enough to describe the action of $\mathbf{q}-T_s$, for each $s\in S$.

Let $\mathbf{P}_s$ be the conjugacy-class of rank-one parabolic subgroups corresponding to s. Let $\pi_s:\mathbf{B}\longrightarrow\mathbf{P}_s$ be the natural projection. Set $\hat{\mathbf{B}}^u=\pi_s^{-1}(\mathbf{P}_s^u)$. Thus we have a diagram

$$\begin{tikzcd}
\mathbf{B}^u \arrow[r, "i"] \arrow[d, "\pi_s^u"'] & \hat{\mathbf{B}}^u \arrow[r, "j"] \arrow[d, "\hat\pi_s^u"] & \mathbf{B} \arrow[d, "\pi_s"] \\
\mathbf{P}_s^u \arrow[r, equal] & \mathbf{P}_s^u \arrow[r, "j"] & \mathbf{P}_s,
\end{tikzcd}$$

where the vertical maps are the restrictions of π_s, and the maps i,j are the inclusions. Note that π_s, and hence $\hat\pi_s^u$, are $\mathbb{P}_1$-bundles.

Kazhdan and Lusztig define an operator t_s^u on $K_D(\mathbf{B}^u)$ as the composition

$$t_s^u: K_D(\mathbf{B}^u)\xrightarrow{(\pi_s^u)_*}K^D(\mathbf{P}_s^u)\xrightarrow{(\pi_s^u)^*}K^D(\hat{\mathbf{B}}^u)\xrightarrow{i^*}K^D(\mathbf{B}^u)\xrightarrow{(1-\mathbf{q}L_{-\alpha})\otimes}K^D(\mathbf{B}^u).$$

(2.5.1) Theorem [KL, 5.11]**.** *The assignments $T_s\mapsto\mathbf{q}-t_s^u$, $e_\lambda\mapsto L_\lambda\otimes$, preserve the Bernstein relations (1.2c). Thus we have an algebra homomorphism*

$$\mathcal{H}(G,\mathbf{q})\longrightarrow\operatorname{End}_{R_D}(K^D(\mathbf{B}^u)).$$

From now on, the spaces $K^D(\mathbf{B}^u)$ are understood to have the $\mathcal{H}(G,\mathbf{q})$-module structure described in (2.5.1).

2.6. The image of $\mathcal{H}(G,\mathbf{q})$ in $\operatorname{End}_{R_D}(K^D(\mathbf{B}^u))$ commutes with the natural action of $G_{\tau,u}$ on $K^D(\mathbf{B}^u)$. More generally, we have

(2.6.1) Proposition. *Suppose $\tau^g=z\tau$ for some $z\in Z$. Then the direct image map $g_*:K^D(\mathbf{B}^u)\longrightarrow K^D(\mathbf{B}^{gug^{-1}})$ induces an $\mathcal{H}(G,\mathbf{q})$-module isomorphism*

$$[K^D(\mathbf{B}^u)\otimes_{R_D}\mathbf{C}_{\tau,q}]^z\simeq K^D(\mathbf{B}^{gug^{-1}})\otimes_{R_D}\mathbf{C}_{\tau,q}.$$

Here the twisting by z is that of 1.4.

Proof. In view of (2.4.1), it suffices to show that $g_*:K^D(\mathbf{B}^u)\longrightarrow K^D(\mathbf{B}^{gug^{-1}})$ commutes with the action of $\mathcal{H}_0(G,\mathbf{q})$.

It is clear that $g\circ\pi_s^u=\pi_s^{gug^{-1}}\circ g$, and likewise for $\hat\pi_s^u$. We then have $g_*(\pi_s^u)_*=(\pi_s^{gug^{-1}})_*g_*$, by [KL, 1.3b]. Since the maps $\hat\pi_s^u$, $\hat\pi_s^{gug^{-1}}$ are $\mathbb{P}^1$-bundles, hence smooth, we also have $g_*(\hat\pi_s^u)^*=(\hat\pi_s^{gug^{-1}})^*g_*$, by [KL, 1.3d]. Finally, g_* commutes with $(1-qL_{-\alpha})\otimes$ by (2.4.1). It follows that g_* commutes with t_s^u, hence with the action of T_s. $\square$

2.7. We come now to the standard modules $M_{\tau,u,\sigma}$ and their simple quotients. Fix $q\in\mathbf{C}^\times$, not a root of unity. As always, $\tau\in G$ is semisimple, $u\in G$ is unipotent, and $\tau u\tau^{-1}=u^q$. As a special case of (2.6.1), the action of $\mathcal{H}(G)$ on $K^D(\mathbf{B}^u)\otimes_{R_D}\mathbf{C}_{\tau,q}$ commutes with the natural action (via direct image) of the mutual centralizer $G_{\tau,u}$. The latter action factors through a finite quotient of $G_{\tau,u}$, since the subgroup $ZG^\circ_{\tau,u}$ acts trivially on $K^D(\mathbf{B}^u)\otimes_{R_D}\mathbf{C}_{\tau,q}$. Hence for any irreducible representation σ of $G_{\tau,u}$, we have an $\mathcal{H}(G)$-module

$$M_{\tau,u,\sigma}:=\operatorname{Hom}_{G_{\tau,u}}(\sigma,K^D(\mathbf{B}^u)\otimes_{R_D}\mathbf{C}_{\tau,q}).$$

The main result in [KL] may be stated as follows.

(2.7.1) Theorem [KL, 7.12]. *The space $M_{\tau,u,\sigma}$ is nonzero if and only if σ appears in the natural action of $G_{\tau,u}$ in the homology $H_*(\mathbf{B}^{\tau,u})$. In this case, the $\mathcal{H}(G)$-module $M_{\tau,u,\sigma}$ has a unique simple quotient $V_{\tau,u,\sigma}$. Every simple $\mathcal{H}(G)$-module can be obtained this way, and two such modules are isomorphic if and only if the corresponding triples (τ,u,σ) are conjugate in G.*

If $\tau^g = z\tau$ for some $z \in Z$, and σ is a representation of $G_{\tau,u}$, let $g\sigma = \sigma \circ g^{-1}\_\, g$ be the corresponding representation of $G_{\tau,gug^{-1}}$. By (2.6.1), we have

(2.7.2) Proposition. *The direct image map g_* induces an $\mathcal{H}(G)$-module isomorphisms*

$$M^z_{\tau,u,\sigma} \xrightarrow{\simeq} M_{\tau,gug^{-1},g\sigma}, \qquad V^z_{\tau,u,\sigma} \xrightarrow{\simeq} V_{\tau,gug^{-1},g\sigma}.$$

This can be generalized as follows; the proof is entirely similar to that of (2.7.2), so we omit it. Let $f : G \longrightarrow G'$ be an isomorphism. Take a pair (τ,u) as above, and let $(\tau',u') = (f(\tau), f(u))$, $D' = f(D)$, $\sigma' = f\sigma$. Let $f^\sharp$ be the induced map on Hecke modules, as in 1.4. Then we have

(2.7.3) Proposition. *The direct image f_* induces $\mathcal{H}(G')$-module isomorphisms*

$$f^\sharp M_{\tau,u,\sigma} \xrightarrow{\simeq} M_{\tau',u',\sigma'}, \qquad f^\sharp V_{\tau,u,\sigma} \xrightarrow{\simeq} V_{\tau',u',\sigma'}.$$

2.8. As a special case of (2.7.2), we consider $u = 1$ (and omit it from the notation). The action of $\mathcal{H}(G,\mathbf{q})$ and g_* on $K^D(\mathbf{B})$ and the standard module $M_\tau = K^D(\mathbf{B}) \otimes_{R_D} \mathbf{C}_{\tau,q}$ can then be made more explicit. Our aim is to determine the effect of g_* on the $\mathcal{H}_0(G,\mathbf{q})$-eigenspaces in M_τ affording the characters

$$1_\mathbf{q} : T_s \mapsto \mathbf{q}, \qquad \text{and} \quad \epsilon_\mathbf{q} : T_s \mapsto -1.$$

Specializing $\mathbf{q} \to q$, we write 1_q, ϵ_q for the respective characters of $\mathcal{H}_0(G)$.

We have a well-known isomorphism (cf. [KL, 2.15])

$$\mathbf{A}[X] \simeq K^{G\times\mathbf{C}^\times}(\mathbf{B}) \tag{2.8a}$$

where $\mathbf{q}$ maps to the bundle $\mathbf{B} \times \mathbf{C}$ with the canonical $\mathbf{C}^\times$ action in the fibers, and λ maps to the line bundle L_λ (see 2.2). This isomorphism commutes with the respective actions of $\mathbf{A}[X]^W \simeq R_{G\times\mathbf{C}^\times}$.

The operators t_s described in 2.4 are defined just as well on $K^{G\times\mathbf{C}^\times}(\mathbf{B})$, and the assignments $T_s \longrightarrow \mathbf{q} - t_s$, $e_\lambda \mapsto L_\lambda \otimes$ define an $\mathcal{H}(G,\mathbf{q})$ module structure on $K^{G\times\mathbf{C}^\times}(\mathbf{B})$. The corresponding action on $\mathbf{A}[X]$, via (2.8a), is [KL, 3.10]

$$\begin{aligned} T_s \circ e_\lambda &= \frac{e_{s\lambda} - e_{\lambda+\alpha}}{e_\alpha - 1} + \mathbf{q}\frac{e_{\lambda+\alpha} - e_{s\lambda-\alpha}}{e_\alpha - 1}, \\ e_\mu \circ e_\lambda &= e_{\mu+\lambda}. \end{aligned} \tag{2.8b}$$

Let

$$\rho = \frac{1}{2}\sum_{\beta\in R^+} \beta.$$

Using (2.8b) it is easy to check that

$$T_s \circ e_{-\rho} = \mathbf{q} e_{-\rho}. \tag{2.8c}$$

Since e_ρ is a unit in $\mathbf{A}[X]$, this means the $\mathcal{H}(G,\mathbf{q})$-module $\mathbf{A}[X]$ is generated by a $1_\mathbf{q}$-eigenvector for the subalgebra $\mathcal{H}_0(G,\mathbf{q})$.

On the other hand, consider the elements

$$\phi_\pm = \prod_{\alpha\in R^+} (1 - \mathbf{q} e_{\pm\alpha}) \in \mathbf{A}[X].$$

Since the product $\phi_+\phi_-$ is invariant under W and hence belongs to the center of $\mathcal{H}(G, \mathbf{q})$, a simple computation using [KL, 3.10(c)] shows that

$$T_s \circ \phi_- = -\phi_-. \tag{2.8d}$$

This time, the $\epsilon_{\mathbf{q}}$-eigenvector ϕ_- is not a unit in $\mathbf{A}[X]$, hence it is not a generator.

Let us now localize the isomorphism (2.8a) at the conjugacy class of (τ, q) in $G \times \mathbf{C}^\times$. We have

$$\begin{aligned}
\mathbf{A}[X] \otimes_{\mathbf{A}[X]^W} \mathbf{C}_{\tau,q} &\simeq K^{G\times\mathbf{C}^\times}(\mathbf{B}) \otimes_{R_{G\times\mathbf{C}^\times}} \mathbf{C}_{\tau,q} \\
&\simeq K^{G\times\mathbf{C}^\times}(\mathbf{B}) \otimes_{R_{G\times\mathbf{C}^\times}} R_D \otimes_{R_D} \mathbf{C}_{\tau,q} \\
&\simeq K^D(\mathbf{B}) \otimes_{R_D} \mathbf{C}_{\tau,q} \qquad \text{by [CG, 6.2.3(6)]} \\
&= M_\tau.
\end{aligned}$$

By (2.8c) M_τ is generated by a 1_q-eigenvector for $\mathcal{H}_0(G)$, and it also has central character τ. The matrix entries of T_s are polynomial functions of (τ, q) by (2.8b). Evaluating these entries at $(1, 1)$, we get the natural representation of W on $\mathbf{C}[X] \otimes_{\mathbf{C}[X]^W} \mathbf{C}$, which is well-known to be the regular representation of W. Since $\mathcal{H}_0(G)$ is semisimple, it follows that for any (τ, q), the restriction of M_τ to $\mathcal{H}_0(G)$ is the regular representation of $\mathcal{H}_0(G)$.

(2.8.1) Lemma. *There exists a Borel subgroup $B \in \mathbf{B}^\tau$ which contains all unipotent elements u satisfying $\tau u^q \tau^{-1} = u$.*

Proof. Choose any maximal torus T containing τ. There is a choice of positive roots for T in G which all have modulus ≤ 1 on τ. This defines a Borel subgroup $B \in \mathbf{B}^\tau$ whose Lie algebra contains the q^{-1}-eigenspace of $Ad(\tau)$, and hence, by exponentiation, B contains all the required u's. □

Choose B as in (2.8.1) and let $\tau_B : \mathbf{C}[X] \longrightarrow \mathbf{C}$ be the homomorphism defined by

$$\tau_B(e_\lambda) = \Psi^B_\lambda(\tau).$$

Set

$$M(\tau_B) = \mathcal{H}(G) \otimes_{\mathbf{C}[X]} \mathbf{C}_{\tau_B}.$$

This is an $\mathcal{H}(G)$-module, via left multiplication.

(2.8.2) Proposition. *Let M be an $\mathcal{H}(G)$-module which restricts to the regular representation of $\mathcal{H}_0(G)$, is generated by a 1_q-eigenvector for $\mathcal{H}_0(G)$, and has central character τ. Then $M \simeq M(\tau_B)$, as $\mathcal{H}(G)$-modules, and this isomorphism is unique up to scalar. In particular, we have $M_\tau \simeq M(\tau_B)$.*

Proof. The choice of B in (2.8.1) ensures that if $\tau_B(e_\alpha) = q^{-1}$, then Ψ^B_α is a root of T in B, so $\alpha \in R^-$. By [R1, 6.6], the $\mathcal{H}(G)$-module $M(\tau_B)$ is generated by its 1_q-eigenspace for $\mathcal{H}_0(G)$. This implies that $M(\tau_B)$ has a unique simple quotient $V(\tau_B)$. Hence, $V(\tau_B)$ is the unique simple $\mathcal{H}(G)$-module containing a 1_q-eigenvector for $\mathcal{H}_0(G)$, and having central character τ.

Let $\mathfrak{m}(\tau_B) \subset \mathbf{C}[X]$ be the kernel of τ_B. It follows from [R2, 4.2] that the subspace of $M(\tau_B)$ consisting of vectors killed by some power of $\mathfrak{m}(\tau_B)$ maps isomorphically to the corresponding subspace in $V(\tau_B)$. In particular, $V(\tau_B)$ is also the unique simple $\mathcal{H}(G)$-module containing the eigencharacter τ_B for $\mathbf{C}[X]$.

Our given module M is generated by a 1_q-eigenvector for $\mathcal{H}_0(G)$, so we have a surjection $M \longrightarrow V(\tau_B)$. Therefore M contains a τ_B-eigenvector for $\mathbf{C}[X]$, which induces a nonzero map $f : M(\tau_B) \longrightarrow M$. The image of f contains a 1_q-eigenvector for $\mathcal{H}_0(G)$, which must be the given generator for M, since the regular represention of $\mathcal{H}_0(G)$ has a one-dimensional 1_q-eigenspace. Therefore f is surjective. It is injective since both sides have dimension $|W|$.

Finally, any isomorphism $M \simeq \mathcal{H}(G) \otimes_{\mathbf{C}[X]} \mathbf{C}_{\tau_B}$ is determined by its effect on the 1_q-eigenspace, hence is unique up to scalar. □

2.9. We consider next the submodule of M_τ containing the ϵ_q-eigenspace for $\mathcal{H}_0(G)$. For the eventual purpose of identifying the Langlands parameters of the Whittaker-constituent of a ramified principal series, we want to know the effect of g_* on this ϵ_q-eigenspace. This will involve $u \neq 1$, but we still abbreviate $M_\tau = M_{\tau,1,\mathrm{triv}}$ as in 2.8.

Let

$$\mathfrak{q}_\tau = \{u \in G : \ \tau u \tau^{-1} = u^q\}.$$

For any $u \in \mathfrak{q}_\tau$ we have an $\mathcal{H}(G, \mathbf{q})$-module homomorphism

$$j_*^u : K^D(\mathbf{B}^u) \otimes_{R_D} \mathbf{C}_{\tau,q} \longrightarrow K^D(\mathbf{B}) \otimes_{R_D} \mathbf{C}_{\tau,q}$$

induced by the inclusion $j^u : \mathbf{B}^u \hookrightarrow \mathbf{B}$. Now $G_{\tau,u} \subseteq G_\tau$, and the latter is connected since G_{der} is simply-connected. It follows that j_*^u vanishes on $M_{\tau,u,\sigma}$ unless σ is the trivial representation. On the other hand, the map j_*^u is not identically zero, since the fundamental class of $\mathbf{B}^u$ gives a nonzero homology class in $\mathbf{B}$. Therefore we have a nonzero $\mathcal{H}(G)$-module map

$$j_*^u : M_{\tau,u,\mathrm{triv}} \longrightarrow M_\tau \simeq M(\tau_B)$$

(with B as in (2.8.1)).

Now the group G_τ acts on $\mathfrak{q}_\tau$ with finitely many orbits, and there is a unique dense orbit $\mathfrak{q}_\tau^\circ$. If $u \in \mathfrak{q}_\tau^\circ$, then $M_{\tau,u,\mathrm{triv}}$ is irreducible by [KL, 5.15a]; so we have an injection

$$j_*^u : M_{\tau,u,\mathrm{triv}} \hookrightarrow M_\tau \simeq M(\tau_B), \qquad u \in \mathfrak{q}_\tau^\circ. \tag{2.9a}$$

It is known [R2, 10.1] that the image of (2.9a) is the unique simple $\mathcal{H}(G)$-submodule of $M(\tau_B)$, and that it contains the ϵ_q-eigenspace of $\mathcal{H}_0(G)$.

(2.9.1) Proposition. *If $\tau^g = z\tau$ for some $z \in Z$, and moreover g centralizes $u \in \mathfrak{q}_\tau^\circ$, then g_* acts as the identity on the ϵ_q-eigenspace for $\mathcal{H}_0(G)$ in $M_{\tau,u,triv}$.*

Proof. Since g_* clearly commutes with j_*^u, it is enough to prove that g_* acts trivially on the ϵ_q-eigenspace for $\mathcal{H}_0(G)$ in M_τ. Under the isomorphism

$$\mathbf{A}[X] \otimes_{\mathbf{A}[X]^W} \mathbf{C}_{\tau,q} \simeq K^D(\mathbf{B}) \otimes_{R_D} \mathbf{C}_{\tau,q},$$

the operator g_* corresponds by (2.4.1) to the operator on $\mathbf{A}[X] \otimes_{\mathbf{A}[X]^W} \mathbf{C}_{\tau,q}$ given by $e_\lambda \otimes 1 \mapsto \lambda(z) e_\lambda \otimes 1$.

For $\theta \in \mathbf{A}[X]$, let $\theta^{\tau,q} = \theta \otimes 1 \in \mathbf{A}[X] \otimes_{\mathbf{A}[X]^W} \mathbf{C}_{[\tau,q]}$. Let B be a Borel subgroup as in (2.8.1), and let B^- be another Borel subgroup such that $B \cap B^-$

is a torus containing τ. Then the Lie algebra of B^- contains $\mathfrak{q}_\tau$. It follows that ϕ_- acts invertibly in the summand $K(\mathbf{B}_i)_{\tau,q}$, where $B^- \in \mathbf{B}_i$, and in particular, $\phi_-^{\tau,q}$ is not identically zero in $\mathbf{A}[X] \otimes_{\mathbf{A}[X]^W} \mathbf{C}_{\tau,q}$. Hence, $\phi_-^{\tau,q}$ is an ϵ_q-eigenvector in $\mathbf{A}[X] \otimes_{\mathbf{A}[X]^W} \mathbf{C}_{\tau,q}$. By (2.4.1) we have $g_*\phi_-^{\tau,q} = \phi_-^{\tau,q}$, since all roots are trivial on z. □

3. Inertia Groups

Recall that G_{der} is simply-connected, and Z is the center of G. Take a finite subgroup $C \subseteq Z$, and put $G' = G/C$. The natural projection $f : G \longrightarrow G'$ is an isogeny. We can now calculate the inertia groups in C of simple $\mathcal{H}(G)$-modules, and thereby classify the simple $\mathcal{H}(G')$-modules as in Theorem 2 of the introduction.

3.1. The group G acts by conjugation on G' via f. Let τ be a semisimple element of G, and set $\tau' = f(\tau)$. Let G_τ, $G'_{\tau'}$ denote the centralizers of τ, τ' in G, G', respectively, and let G_τ^+ denote the centralizer of τ' in G. That is,

$$G_\tau^+ = \{g \in G : \ \tau^g \in \tau C\}.$$

Since G_{der} is simply-connected, the centralizer G_τ is the identity component of G_τ^+, and we put

$$R_\tau := G_\tau^+/G_\tau.$$

We have an isomorphism

$$R_\tau \xrightarrow{\simeq} C_\tau \subseteq C \qquad g \mapsto c_g := \tau^{-1}\tau^g \tag{3.1a}$$

from R_τ onto a subgroup $C_\tau \subseteq C$. In particular, R_τ is abelian.

3.2. Fix $q \in \mathbf{C}^\times$, not a root of unity, and recall the variety

$$\mathfrak{q}_\tau = \{u \in G : \tau u \tau^{-1} = u^q\}.$$

The isogeny $f : G \longrightarrow G'$ restricts to an isomorphism from $\mathfrak{q}_\tau$ to the analogous variety $\mathfrak{q}_{\tau'} \subset G'$, and we will not distinguish between $\mathfrak{q}_\tau$ and $\mathfrak{q}_{\tau'}$.

The finitely many G_τ-orbits in $\mathfrak{q}_\tau$ are permuted by R_τ. For $u \in \mathfrak{q}_\tau$, set $G_{\tau,u}^+ := G_\tau^+ \cap G_u$ and let $R_{\tau,u}$ be the image of $G_{\tau,u}^+$ in R_τ. It is easy to see that $R_{\tau,u}$ is the stabilizer in R_τ of the G_τ-orbit through u.

3.3. Fix τ, u with $u \in \mathfrak{q}_\tau$, and set

$$A_{\tau,u} := \pi_0(G_{\tau,u}), \qquad A_{\tau,u}^+ := \pi_0(G_{\tau,u}^+).$$

Note that $A_{\tau,u}^+$ is isomorphic, via the isogeny $f : G \longrightarrow G'$, to the component group of $G'_{\tau',u'}$. Moreover, we have an exact sequence

$$1 \longrightarrow A_{\tau,u} \to A_{\tau,u}^+ \longrightarrow R_{\tau,u} \longrightarrow 1.$$

In particular, the group $R_{\tau,u}$ permutes the irreducible representations σ of $A_{\tau,u}$. Let $R_{\tau,u,\sigma}$ be the stabilizer in $R_{\tau,u}$ of the isomorphism class of $\sigma \in \mathrm{Irr}(A_{\tau,u})$. For each $r \in R_{\tau,u,\sigma}$ we choose an $A_{\tau,u}$-isomorphism $f_r : \sigma \longrightarrow \sigma^r$. These choices define a 2-cocycle $\mu_{\tau,u,\sigma}$ on $R_{\tau,u,\sigma}$ with values in $\mathbf{C}^\times$, such that $f_r f_s = \mu_{\tau,u,\sigma}(r,s) f_{rs}$. Let $\mathcal{E}_{\tau,u,\sigma}$ denote the corresponding twisted group algebra. Then by Mackey's theorem, we may identify

$$\mathcal{E}_{\tau,u,\sigma} = \mathrm{End}_{A_{\tau,u}^+}(\mathrm{Ind}_{A_{\tau,u}}^{A_{\tau,u}^+} \sigma).$$

Hence we have a decomposition as $\mathcal{E}_{\tau,u,\sigma}\otimes \mathbf{C}[A^+_{\tau,u}]$ modules

$$\operatorname{Ind}_{A_{\tau,u}}^{A^+_{\tau,u}}\sigma \simeq \bigoplus_{\psi\in \operatorname{Irr}(\mathcal{E}_{\tau,u,\sigma})} \psi\otimes\rho_\sigma^\psi,$$

where $\operatorname{Irr}(\mathcal{E}_{\tau,u,\sigma})$ is the set of simple $\mathcal{E}_{\tau,u,\sigma}$-modules up to isomorphism, and each ρ_σ^ψ is an irreducible representation of $A^+_{\tau,u}$. The map $\psi\mapsto\rho_\sigma^\psi$ is a bijection between $\operatorname{Irr}(\mathcal{E}_{\tau,u,\sigma})$ and the set of irreducible representations of $A^+_{\tau,u}$ which contain σ upon restriction to $A_{\tau,u}$.

3.4. Let σ be an irreducible representation of $A_{\tau,u}$ occurring in $H_*(\mathbf{B}^{\tau,u})$. We can now calculate the inertia group $C_{\tau,u,\sigma}$ of the simple $\mathcal{H}(G)$-module $V_{\tau,u,\sigma}$.

Let $c\in C$. By our remarks in 1.4, the central character of the twisted module $V^c_{\tau,u,\sigma}$ is $c\tau$. (Recall that q has been fixed.) Hence if $V^c_{\tau,u,\sigma}\simeq V_{\tau,u,\sigma}$, there is $g\in G$ such that $c\tau=\tau^g$. We have $g\in G^+_\tau$, and $c=c_g$. It follows that $C_{\tau,u,\sigma}\subset C_\tau$.

By (2.7.2), we have an $\mathcal{H}(G)$-module isomorphism

$$V^{c_g}_{\tau,u,\sigma}\simeq V_{\tau,gug^{-1},g\sigma}. \tag{3.4a}$$

Hence c_g belongs to the inertia group of $V_{\tau,u,\sigma}$ if and only if there is $h\in G_\tau$ such that $(gug^{-1},g\sigma)=(huh^{-1},h\sigma)$. This proves

(3.4.1) Proposition. *The inertia group $C_{\tau,u,\sigma}$ of the simple $\mathcal{H}(G)$-module $V_{\tau,u,\sigma}$ is the image of $R_{\tau,u,\sigma}$ under the embedding $R_\tau\hookrightarrow C$ from 3.1a.*

3.5. Fix τ,u, with $u\in\mathfrak{q}_\tau$. We now determine the twisted group algebra $\mathbf{C}[C_V,\eta_V]$, defined in 1.5, for $V=V_{\tau,u,\sigma}$. We set

$$M_{\tau,u}:=K^D(\mathbf{B}^u)\otimes_{R_D}\mathbf{C}_{\tau,q}.$$

From (2.4.1) and (2.6.1), the pair $\mathcal{H}(G)$ and $A^+_{\tau,u}$ act on $M_{\tau,u}$, but do not commute, while the pairs $(\mathcal{H}(G),A_{\tau,u})$ and $(\mathcal{H}(G'),A^+_{\tau,u})$ do commute. Let $N_{\tau,u}$ be the intersection of the maximal proper $\mathcal{H}(G)$-submodules of $M_{\tau,u}$, let $V_{\tau,u}=M_{\tau,u}/N_{\tau,u}$, and let $p:M_{\tau,u}\longrightarrow V_{\tau,u}$ be the quotient map. From (2.4.1), it follows that $A^+_{\tau,u}$ preserves each $\mathcal{H}(G)$-submodule of $M_{\tau,u}$, so $A^+_{\tau,u}$ acts on $V_{\tau,u}$.

Now

$$V_{\tau,u,\sigma}=\operatorname{Hom}_{A_{\tau,u}}(\sigma,V_{\tau,u})\simeq\operatorname{Hom}_{A^+_{\tau,u}}(\operatorname{Ind}_{A_{\tau,u}}^{A^+_{\tau,u}}\sigma,V_{\tau,u}),$$

and $\mathcal{E}_{\tau,u,\sigma}$ acts on the latter space by composition, commuting with the natural $\mathcal{H}(G')$ action. Thus we have an algebra homomorphism

$$\mathcal{E}_{\tau,u,\sigma}\longrightarrow\operatorname{End}_{\mathcal{H}(G')}(V_{\tau,u,\sigma}). \tag{3.5a}$$

(3.5.1) Lemma. *Every irreducible representation of $\mathcal{E}_{\tau,u,\sigma}$ appears in this action on $V_{\tau,u,\sigma}$. In particular, the homomorphism (3.5a) is injective.*

Proof. Recall from 2.2 the decomposition $\mathbf{B}^{\tau,u}=\mathbf{B}^u_1\cup\cdots\cup\mathbf{B}^u_m$. Each subvariety $\mathbf{B}^u_i$ is stable under $G_{\tau,u}$, hence the subspace $K(\mathbf{B}^u_i)_{\tau,q}\subset M_{\tau,u}$ (see (2.2d)) is stable under $A_{\tau,u}$. If we set $V_{\tau,u}(i):=p(K(\mathbf{B}^u_i)_{\tau,q})\subset V_{\tau,u}$, then $V_{\tau,u}=\bigoplus_i V_{\tau,u}(i)$, and each summand is stable under $A_{\tau,u}$. Therefore σ appears in $V_{\tau,u}(i)$ for some i.

Now the group G^+_τ permutes the components $\mathbf{B}_i$. I claim that the stabilizer of any component is exactly G_τ. If we admit this, and recall that $G^+_{\tau,u}=G^+_\tau\cap G_u$, then the $A^+_{\tau,u}$-submodule of $V_{\tau,u}$ generated by $V_{\tau,u}(i)$ is a direct sum

$$\bigoplus_{g\in R_{\tau,u}} gV_{\tau,u}(i).$$

It follows that

$$\mathrm{Ind}_{A_{\tau,u}}^{A^+_{\tau,u}}\sigma \subset \mathrm{Ind}_{A_{\tau,u}}^{A^+_{\tau,u}} V_{\tau,u}(i) \simeq \mathbf{C}[A^+_{\tau,u}]\cdot V_{\tau,u}(i);$$

so $\mathrm{Ind}_{A_{\tau,u}}^{A^+_{\tau,u}}\sigma$ is isomorphic to an $A^+_{\tau,u}$-submodule of $V_{\tau,u}$, and this implies (3.5.1).

It remains to verify the claim. Suppose $g \in G^+_\tau$ preserves $\mathbf{B}_i$. Let $B \in \mathbf{B}_i$. Then B_τ is a Borel subgroup in G_τ, and B is the unique fixed point of B_τ in $\mathbf{B}_i$. Choose a maximal torus $T \subseteq B_\tau$. Modifying g by an element of G_τ, we may assume that g normalizes T and B_τ. But then gBg^{-1}, being another fixed point of B_τ in $\mathbf{B}_i$, must equal B, so $g \in B \cap N_T = T$, hence $g \in G_\tau$. This proves the claim and the lemma. ∎

Let $C_{\tau,u,\sigma}$ be as in 3.4.1, and let $\eta_{\tau,u,\sigma}$ be the cocycle on $C_{\tau,u,\sigma}$ defined in 1.5, for $V = V_{\tau,u,\sigma}$. Combining (3.5.1) with (1.5.1), we have

$$|R_{\tau,u,\sigma}| = \dim \mathcal{E}_{\tau,u,\sigma} \le \dim \mathrm{End}_{\mathcal{H}(G')}(V_{\tau,u,\sigma}) \le \dim \mathbf{C}[C_{\tau,u,\sigma},\eta_{\tau,u,\sigma}] = |R_{\tau,u,\sigma}|.$$

We have proved:

(3.5.2) Lemma. *The map (3.5a) is an isomorphism*

$$\mathcal{E}_{\tau,u,\sigma} \simeq \mathrm{End}_{\mathcal{H}(G')}(V_{\tau,u,\sigma}).$$

Hence, the restriction of $V_{\tau,u,\sigma}$ to $\mathcal{H}(G')$ is given by

$$V_{\tau,u,\sigma}|_{\mathcal{H}(G')} \simeq \bigoplus_{\psi\in \mathrm{Irr}(\mathcal{E}_{\tau,u,\sigma})} {}^{\psi}V_{\tau,u,\sigma}\otimes\psi.$$

Each ${}^{\psi}V_{\tau,u,\sigma}$ is nonzero, and is a simple $\mathcal{H}(G')$-module. Every simple $\mathcal{H}(G')$-module arises in this way, and ${}^{\psi}V_{\tau,u,\sigma} \simeq {}^{\psi'}V_{\tau,u,\sigma'}$ if and only if there is $g \in G^+_{\tau,u}$ such that

$$\sigma' \simeq \sigma^g, \qquad \psi' = \psi. \tag{3.5b}$$

From (3.5.2) we see that the simple $\mathcal{H}(G')$ modules are in bijection with quadruples (τ,u,σ,ψ) as in (3.5.2) modulo the equivalence

$$(\tau_1,u_1,\sigma_1,\psi_1) \sim (\tau_2,u_2,\sigma_2,\psi_2)$$

iff there is $g \in G'$ such that

$$(\tau_2,u_2,\sigma_2,\psi_2) = (f(\tau_1)^g, f(u_1)^g, \sigma_1^g, \psi_1^g), \tag{3.5c}$$

where $f : G \longrightarrow G'$ is the isogeny 3.1, and when writing σ_1^g, we are identifying A_{τ_1,u_1} with its image in $\pi_0(G'_{f(\tau_1),f(u_1)})$ under f; see 3.3.

(3.5.3) Lemma. *There is a bijection between G-conjugacy classes of quadruples (τ,u,σ,ψ) as in (3.5c), and G' conjugacy classes of triples (τ',u',ρ'), where $\tau' \in G'$ is semisimple, $u' \in \mathfrak{q}_{\tau'}$ and ρ' is a representation of $G'_{\tau',u'}$ appearing in the homology of $\mathbf{B}^{\tau',u'}$.*

Proof. Given (τ,u,σ,ψ), set

$$\tau' = f(\tau), \quad u' = f(u), \quad \rho' = \rho_\psi.$$

On the other hand, given (τ',u',ρ'), choose τ,u to be lifts of τ',u' in G, let σ be an irreducible representation of $A_{\tau,u}$ appearing in the restriction of ρ' to $A_{\tau,u}$, and let $\psi = \mathrm{Hom}_{A_{\tau,u}}(\sigma,\rho)$. We let the reader check that these assignments respect the appropriate equivalence relations.

Note that $\mathbf{B}^{\tau',u'} = \mathbf{B}^{\tau,u}$. The proof of (3.5.1) shows that ρ' appears in $H_*(\mathbf{B}^{\tau,u})$ if and only if $\rho' = \rho^\psi_\sigma$ for some pair (σ,ψ) where σ appears in $H_*(\mathbf{B}^{\tau,u})$. ∎

We have proved Theorem 2 of the introduction: (We now replace G' by G.)

(3.5.4) Theorem. *Let G be a connected reductive complex algebraic group, with affine Hecke algebra $\mathcal{H}(G)$, where the parameter q is not a root of unity. Then the simple $\mathcal{H}(G)$-modules are in bijection with the set $\Pi(G)$ of G-conjugacy classes of triples (τ, u, ρ), where $\tau \in G$ is semisimple, $u \in \mathfrak{q}_\tau$, and ρ is an irreducible representation of $G_{\tau,u}$ appearing in $H_*(\mathbf{B}^{\tau,u})$. The bijection is given as follows. Choose an isogeny $\widetilde{G} \longrightarrow G$ with $\widetilde{G}_{der}$ simply connected. Given (τ, u, ρ), let $\tilde{\tau}, \tilde{u}$ be lifts in $\widetilde{G}$, and let $\sigma \in \mathrm{Irr}(A_{\tilde{\tau},\tilde{u}})$, $\psi \in \mathrm{Irr}(\mathcal{E}_{\tilde{\tau},\tilde{u},\sigma})$ be such that $\rho = \rho_\sigma^\psi$. Then the corresponding simple $\mathcal{H}(G)$-module is ${}^\psi V_{\tilde{\tau},\tilde{u},\sigma}$.*

4. Langlands parameters

4.1. Let G be a complex Lie group with simply-connected derived group. Suppose we are given a prime number p, and a chain of subgroups

$$\Delta_1 \subseteq \Delta \subseteq \Gamma \subseteq G,$$

such that Γ consists of semisimple elements, Δ is finite and normal in Γ, Δ_1 is a normal p-subgroup of Γ, Δ/Δ_1 is cyclic of order prime to p, and Γ/Δ is cyclic, generated by the coset of a given element $s \in \Gamma$.

Such a chain of subgroups arises from a continuous homomorphism $\mathcal{W}_F \longrightarrow G$, where $\mathcal{W}_F$ is the Weil group of a nonarchimedean local field F. Then Γ is the image of $\mathcal{W}_F$, s is the image of a given Frobenius element in $\mathcal{W}_F$, and Δ, Δ_1 are respectively the images of the inertia and wild inertia subgroups of $\mathcal{W}_F$.

Let $\mathbf{B}$ be the variety of Borel subgroups of G, and let $\mathbf{B}^\Gamma$ denote the set of Γ-fixed points in $\mathbf{B}$.

(4.1.1) Lemma. *The variety $\mathbf{B}^\Gamma$ is nonempty if and only if Γ is contained in a maximal torus of G.*

Proof. See [B2, 10.6(5)]. □

(4.1.2) Proposition. *Recall that G_{der} is simply-connected. Assume that p is not a torsion prime for G. Let G_Δ denote the centralizer of Δ in G. Then the following are equivalent:*

(1) Γ *is contained in G_Δ,*
(2) Γ *is contained in a maximal torus of G,*
(3) Γ *is abelian,*
(4) $\mathbf{B}^\Gamma$ *is nonempty.*

Proof. Implications $2 \Rightarrow 3 \Rightarrow 1$ are obvious, and $1 \Rightarrow 3$ follows from the cyclicity of Γ/Δ. $2 \Leftrightarrow 4$ is (4.1.1). Finally, $3 \Rightarrow 2$ is a special case of Steinberg's result [St, 2.25], and requires our assumptions on G and p. □

4.2. Let $\mathcal{W}_F$ be the Weil group of F, and let $\mathcal{I}_F$ be the inertia subgroup of $\mathcal{W}_F$. Choose a Frobenius element $\mathrm{Frob} \in \mathcal{W}_F$. We assume that the residue characteristic p of F is not a torsion prime for G.

Given a Langlands parameter

$$\Phi : \mathcal{W}_F \times SL_2(\mathbf{C}) \longrightarrow G,$$

we let $\mathbf{B}^\Phi$ denote the variety of Borel subgroups of G fixed by $\Phi(\mathcal{W}_F \times B_2)$, where B_2 is the upper triangular Borel subgroup of $SL_2(\mathbf{C})$. By (4.1.2), and Borel's fixed point theorem, the variety $\mathbf{B}^\Phi$ is nonempty if and only if Φ factors through the topological abelianization $\mathcal{W}_F^a$ of $\mathcal{W}_F$.

Let $\mathcal{I}_F^a$ be the image of $\mathcal{I}_F$ in $\mathcal{W}_F^a$. Fix a continuous homomorphism $\hat{\chi} : \mathcal{I}_F^a \longrightarrow G$, and let $\Phi : \mathcal{W}_F^a \times SL_2(\mathbf{C}) \longrightarrow G$ be a Langlands parameter whose restriction to $\mathcal{I}_F^a$ is $\hat{\chi}$. We want to classify such Φ's in terms of conjugacy classes in the group $H := G_{\hat{\chi}}$.

Since $\Phi(\mathcal{W}_F^a)$ is abelian, we have $\Phi : \mathcal{W}_F^a \times SL_2(\mathbf{C}) \longrightarrow H$. Set

$$\tau = \Phi\big(\mathrm{Frob} \times \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\big), \qquad u = \Phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $\tau \in H$ is semisimple, $u \in H$ is unipotent, and $\tau u \tau^{-1} = u^q$.

On the other hand, given such a pair (τ, u) in H, we may choose a homomorphism $\varphi_u : SL_2(\mathbf{C}) \longrightarrow H$ such that $\varphi_u \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u$, and such that the element $\tau_u = \varphi_u \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$, commutes with τ (cf. [KL, 2.4g]). Then the element $s := \tau_u^{-1}\tau$ centralizes u.

The choice of Frob determines a splitting

$$\mathcal{W}_F^a = \mathcal{I}_F^a \times \langle \mathrm{Frob} \rangle;$$

so we can extend $\hat{\chi}$ to a homomorphism $\hat{\chi}_s : \mathcal{W}_F^a \longrightarrow G$ by setting $\hat{\chi}_s(\mathrm{Frob}) = s$. By (4.3b) below, the image of $\hat{\chi}_s$ commutes with the image of φ_u. We can therefore define

$$\Phi : \mathcal{W}_F^a \times SL_2(\mathbf{C}) \longrightarrow G$$

by

$$\Phi(w, x) = \hat{\chi}_s(w)\varphi_u(x), \qquad w \in \mathcal{W}_F^a, \quad x \in SL_2(\mathbf{C}).$$

It is straightforward to check that the processes $\Phi \leftrightarrow (\tau, u)$ are inverse to one another, and that Φ^g corresponds to $(\tau^g, u^g) \in H^g = G_{\hat{\chi}^g}$, for $g \in G$.

4.3. We compare the component groups of centralizers of Φ and (τ, u) in 4.2. Given (τ, u) in H as above, set $M = H_{\varphi_u}$, and let N be the unipotent subgroup of H whose Lie algebra is the span of the eigenspaces of $Ad_H(\tau_u)$ with eigenvalues > 1. Then N_u is the unipotent radical of H_u, and we have a semidirect product

$$H_u = MN_u. \tag{4.3a}$$

The decomposition (4.3a) is preserved under conjugation by τ_u, which has no non-trivial fixed points in N_u, and acts trivially on M. It follows that

$$M = H_{\tau_u, u}. \tag{4.3b}$$

A similar argument with τ shows that

$$H_{\tau,u} = M_s N_{\tau,u}. \tag{4.3c}$$

Finally, $N_{\tau,u}$ is connected, since N_u is unipotent, and this implies

$$M_s^\circ = M_s \cap H_{\tau,u}^\circ. \tag{4.3d}$$

Now, if $\Phi : \mathcal{W}_F^a \times SL_2(\mathbf{C}) \longrightarrow G$ corresponds to (τ, u) as in 4.2, then

$$G_\Phi = M_s. \tag{4.3e}$$

Set

$$A_\Phi = \pi_0(G_\Phi), \qquad A_{\tau,u} = \pi_0(H_{\tau,u}).$$

Combining (4.3c-e) proves

(4.3.1) Lemma. *The inclusion $G_\Phi \hookrightarrow H_{\tau,u}$ induces an isomorphism*

$$A_\Phi \xrightarrow{\simeq} A_{\tau,u}.$$

4.4. If a group A acts on a variety $\mathbf{X}$, let $\mathcal{R}(A,\mathbf{X})$ denote the set of irreducible representations of A appearing in the homology $H_*(\mathbf{X})$.

(4.4.1) Lemma. *The isomorphism of (4.3.1) induces a bijection*

$$\mathcal{R}(A_\Phi, \mathbf{B}^\Phi) \longrightarrow \mathcal{R}(A_{\tau,u}, \mathbf{B}_H^{\tau,u}).$$

Proof. From (4.3e) we have

$$\mathcal{R}(A_\Phi, \mathbf{B}^\Phi) = \mathcal{R}(M_s, \mathbf{B}^\Phi). \tag{4.4a}$$

Since u, τ_u generate a dense subgroup of $\Phi(B_2)$, we have

$$\mathbf{B}^\Phi = \mathbf{B}^{\Delta,\tau,u,\tau_u}.$$

Under the H-action, $\mathbf{B}^\Delta$ is a disjoint union of copies of $\mathbf{B}_H$, so $\mathbf{B}^\Phi$ is a disjoint union of copies of $\mathbf{B}_H^{\tau,u,\tau_u}$. Therefore

$$\mathcal{R}(M_s, \mathbf{B}^\Phi) = \mathcal{R}(M_s, \mathbf{B}_H^{\tau,u,\tau_u}). \tag{4.4b}$$

Let D be the torus generated by τ_u. Note that D and M commute. In particular, the D-action on $\mathbf{B}_H^{\tau,u}$ commutes with the M_s-action. Since $\mathbf{B}_H^{\tau,u}$ has no odd homology [DLP], we have

$$\mathcal{R}(M_s, \mathbf{B}_H^{\tau,u,\tau_u}) = \mathcal{R}(M_s, \mathbf{B}_H^{\tau,u,D}) = \mathcal{R}(M_s, \mathbf{B}_H^{\tau,u}), \tag{4.4c}$$

the last equality following from [CG, 2.5.1]. Finally,

$$\mathcal{R}(M_s, \mathbf{B}_H^{\tau,u}) = \mathcal{R}(A_{\tau,u}, \mathbf{B}_H^{\tau,u}), \tag{4.4d}$$

by (4.3c). Combining (4.4a-d) proves the result. □

4.5. We summarize what has been proved in sections 1-4 of this paper. Recall that we assume G_{der} is simply connected, that p is a nontorsion prime for G, and we have chosen a Frobenius element Frob $\in \mathcal{W}_F$. Let $\hat{\chi} : \mathcal{I}_F^a \longrightarrow G$ be a continuous homomorphism, and let H be the centralizer of the image of $\hat{\chi}$. Let $\mathcal{H}$ be the affine Hecke algebra whose root datum is that of H, and whose parameter q (a power of p) is the residue cardinality of F. Let $\Pi_{\hat{\chi}}(G,\mathbf{B})$ be the set of G-conjugacy classes of pairs (Φ,ρ), where $\Phi : \mathcal{W}_F \times SL_2(\mathbf{C}) \longrightarrow G$ is a Langlands parameter such that $\mathbf{B}^\Phi$ is nonempty, $\Phi|_{\mathcal{I}_F}$ is G-conjugate to $\hat{\chi}$, and $\rho \in \mathcal{R}(A_\Phi, \mathbf{B}^\Phi)$. Let $\Pi(H)$ be the set of H-conjugacy classes of triples (τ,u,ρ), where $\tau \in H$ is semisimple, $u \in H$ is unipotent, $\tau u \tau^{-1} = u^q$, and $\rho \in \mathcal{R}(A_{\tau,u}, \mathbf{B}_H^{\tau,u})$. We have established the following.

(4.5.1) Theorem. *The simple $\mathcal{H}$-modules are in bijection with $\Pi_{\hat{\chi}}(G,\mathbf{B})$, and also with $\Pi(H)$. These bijections are given by combining (3.5.4), 4.2 and (4.4.1).*

5. p-ADIC GROUPS

We now combine the previous results with those of Roche, in [Ro]. As in that work, we impose restrictions on the residue characteristics p for each type of group G, as follows:

$$\begin{aligned}&A_n: \ p > n+1,\\ &B_n, C_n, D_n: \ p \neq 2,\\ &G_2, E_6: \ p \neq 2,3,5,\\ &E_7, E_8: \ p \neq 2,3,5,7.\end{aligned}$$

In particular, the torsion primes of G are excluded.

5.1. Recall we have chosen a Frobenius element $\mathrm{Frob} \in \mathcal{W}_F$. Let $r_F : \mathcal{W}_F^a \longrightarrow F^\times$ be the reciprocity isomorphism of abelian class-field theory, and set $\varpi = r_F(\mathrm{Frob})$, a prime element in F.

Let $\mathcal{G}$ be the group of rational points in a connected, split, reductive F-group $\underline{\mathcal{G}}$ having connected center. Let $\underline{\mathcal{T}} \subset \underline{\mathcal{B}}$ be a maximal split torus and Borel subgroup in $\underline{\mathcal{G}}$, and let $\underline{\mathcal{U}}$ be the unipotent radical of $\underline{\mathcal{B}}$. These choices determine a based root datum $\Phi(\underline{\mathcal{T}}, \underline{\mathcal{B}})$ for $\mathcal{G}$.

There exists a triple (G, B, T) consisting of a complex connected reductive group G, a Borel subgroup B in G, and a maximal torus $T \subset B$ such that the based root datum $\Phi(T, B) = (X, Y, R, \check{R}, S)$ is dual to $\Phi(\underline{\mathcal{T}}, \underline{\mathcal{B}})$. The group G has simply-connected derived group.

We have $T = Y \otimes \mathbf{C}^\times$, and X may be identified with both the character group of T and co-character group of $\underline{\mathcal{T}}$. For $\lambda \in X = X_*(\underline{\mathcal{T}})$, we set $t_\lambda = \lambda(\varpi)$. This is an embedding of X in $\mathcal{T}$, and gives a splitting

$$\mathcal{T} = \mathcal{T}_0 \times X \tag{5.1a}$$

where $\mathcal{T}_0$ is the maximal compact subgroup of $\mathcal{T}$.

Take a continuous homomorphism

$$\chi : \mathcal{T}_0 \longrightarrow \mathbf{C}^\times.$$

According to the splitting (5.1a), the extensions of χ to $\mathcal{T}$ are uniquely of the form $\chi \otimes \tau$, where $\tau \in T$.

Let $\hat{\chi} : \mathcal{I}_F^a \longrightarrow T$ be the unique homomorphism satisfying

$$\lambda \circ \hat{\chi} = \chi \circ \lambda \circ r_F \tag{5.1b}$$

for all $\lambda \in X$, where λ is viewed as a character of T on the left side of (5.1b) and as a co-character of $\mathcal{T}$ on the right side. Let $G_{\hat{\chi}}$ denote the centralizer of the image of $\hat{\chi}$ in G. It is reductive, and connected, since G_{der} is simply connected, and p is not a torsion prime for G [St].

The choice of (T, B) determines a based root datum $\Phi(T, B_{\hat{\chi}})$ for $G_{\hat{\chi}}$, hence an affine Hecke algebra $\mathcal{H}(\Phi(T, B_{\hat{\chi}}))$, with $\mathbf{q} = q$. We identify $\mathcal{H}(\Phi(T, B_{\hat{\chi}})) = \mathcal{H}(G_{\hat{\chi}})$ via the canonical isomorphism $\Phi(T, B_{\hat{\chi}}) \simeq \Phi(G_{\hat{\chi}})$.

5.2. Roche has constructed a compact open subgroup $\mathcal{J}_\chi$ containing $\mathcal{T}_0$, and an extension ρ_χ of χ to $\mathcal{J}_\chi$, such that $\mathcal{H}(G_{\hat{\chi}})$ is isomorphic to the Hecke algebra

$$\mathcal{H}(\mathcal{G}, \rho_\chi) = \mathrm{End}_{\mathcal{G}}(\mathrm{ind}_{\mathcal{J}_\chi}^{\mathcal{G}} \rho_\chi),$$

where ind denotes induction with compact supports. The isomorphism is not unique; we determine it as follows. First normalize Haar measures on $\mathcal{G}$ and $\mathcal{T}$ to give volume one to $\mathcal{J}_\chi$ and $\mathcal{T}_0$, respectively.

Let $\mathcal{H}(\mathcal{T},\chi) = \mathrm{End}_{\mathcal{T}}(\mathrm{ind}_{\mathcal{T}_0}^{\mathcal{T}}\chi)$. This algebra has a linear basis $\{\epsilon_\lambda * :\ \lambda \in X\}$, where $\epsilon_\lambda *$ acts on $\mathrm{ind}_{\mathcal{T}_0}^{\mathcal{T}}\chi$ as convolution by the unique function $\epsilon_\lambda : \mathcal{T} \longrightarrow \mathbf{C}$ supported on $\mathcal{T}_0 t_\lambda$, such that $\epsilon_\lambda(t_\lambda) = 1$. We have an isomorphism

$$\psi_\chi : \mathbf{C}[X] \longrightarrow \mathcal{H}(\mathcal{T},\chi), \qquad \psi_\chi(e_\lambda) = \epsilon_\lambda * .$$

Let $\phi_\lambda^\chi : \mathcal{J}_\chi t_\lambda \mathcal{J}_\chi \longrightarrow \mathbf{C}$ be the unique function supported on $\mathcal{J}_\chi t_\lambda \mathcal{J}_\chi$ such that

$$\phi_\lambda^\chi(jt_\lambda j') = \frac{\rho_\chi(jj')}{vol(\mathcal{J}_\chi t_\lambda \mathcal{J}_\chi)^{1/2}}.$$

Then the convolution operator $\phi_\lambda^\chi *$ belongs to $\mathcal{H}(\mathcal{G},\chi)$. There is a unique embedding of algebras [Ro, eq. (5.2)]

$$t_u : \mathcal{H}(\mathcal{T},\chi) \hookrightarrow \mathcal{H}(\mathcal{G},\chi)$$

such that for $\lambda \in X^+$, we have

$$t_u(\epsilon_\lambda *) = \phi_\lambda^\chi * .$$

It now follows from [Ro, 6.3] that there is a unique support preserving isomorphism of Hecke algebras

$$\Psi_\chi : \mathcal{H}(G_{\hat\chi}) \longrightarrow \mathcal{H}(\mathcal{G},\chi) \tag{5.2a}$$

such that

$$\Psi_\chi(e_\lambda) = \phi_\lambda^\chi * .$$

5.3. Let $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ denote the space of locally constant functions $f : \mathcal{G} \longrightarrow \mathbf{C}$ such that $f(bg) = \delta^{1/2}(b)\tau(b)\chi(b)f(g)$ for all $b \in \mathcal{B},\ g \in \mathcal{G}$, where δ is the modular function of $\mathcal{B}$. Then $\mathcal{G}$ acts on $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ by right translations.

Let $\mathrm{Irr}_\chi(\mathcal{G},\mathcal{B})$ denote the set of irreducible representations of $\mathcal{G}$, up to equivalence, which appear in $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$, for some $\tau \in T$. Let $\mathrm{Irr}\,\mathcal{H}(\mathcal{G},\chi)$ denote the set of simple $\mathcal{H}(\mathcal{G},\chi)$-modules, up to equivalence. If V is a smooth representation of $\mathcal{G}$, then the space

$$V^\chi := \mathrm{Hom}_{\mathcal{J}_\chi}(\rho_\chi, V)$$

is an $\mathcal{H}(\mathcal{G},\chi)$ module, in a natural way. According to [Ro, Cor. 7.9], the assignment $V \mapsto V^\chi$ is a bijection

$$\mathrm{Irr}_\chi(\mathcal{G},\mathcal{B}) \longrightarrow \mathrm{Irr}\,\mathcal{H}(\mathcal{G},\chi), \tag{5.3a}$$

Let

$$\Psi_\chi^\sharp : \mathrm{Irr}\,\mathcal{H}(\mathcal{G},\chi) \longrightarrow \mathrm{Irr}\,\mathcal{H}(G_{\hat\chi}) \tag{5.3b}$$

be the bijection induced by the isomorphism (5.2a). Combining (5.3a,b) we have a bijection

$$\mathrm{Irr}_\chi(\mathcal{G},\mathcal{B}) \longrightarrow \mathrm{Irr}\,\mathcal{H}(G_{\hat\chi}) \tag{5.3c}$$

with the property [Ro, Thm. 9.4] that a constituent of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ corresponds to an $\mathcal{H}(G_{\hat\chi})$-module with central character $\tau \in T/W_\chi$.

Combining (5.3c) with (4.5) we therefore have a bijection

$$i_\chi : \mathrm{Irr}_\chi(\mathcal{G},\mathcal{B}) \longrightarrow \Pi_{\hat\chi}(G,\mathbf{B}). \tag{5.3d}$$

5.4. In the next section we will show that bijection (5.3d) is independent of χ, in the following sense. Suppose V is an irreducible representation of $\mathcal{G}$ appearing in two induced representations $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ and $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi' \otimes \tau')$. Then χ and χ' are conjugate by the common Weyl group W of $\mathcal{G}$ and G, as are $\hat{\chi}$ and $\hat{\chi}'$. Hence

$$\mathrm{Irr}_{\chi}(\mathcal{G}, \mathcal{B}) = \mathrm{Irr}_{\chi'}(\mathcal{G}, \mathcal{B}), \quad \text{and} \quad \Pi_{\hat{\chi}}(G, \mathbf{B}) = \Pi_{\hat{\chi}'}(G, \mathbf{B}).$$

In the next section, we will prove that $i_{\chi}(V) = i_{\chi'}(V)$. It will then follow that the bijections i_{χ} piece together to give a bijection

$$\mathrm{Irr}(\mathcal{G}, \mathcal{B}) \xrightarrow{\simeq} \Pi(G, \mathbf{B}) \tag{5.4a}$$

between the set $\mathrm{Irr}(\mathcal{G}, \mathcal{B})$ of irreducible constituents of all principal series representations of $\mathcal{G}$, and the set $\Pi(G, \mathbf{B})$ of G-conjugacy classes of all pairs (Φ, ρ), where $\Phi(\mathcal{W}_F)$ is contained in a Borel subgroup of G, and ρ is an irreducible representation of the centralizer of the image of Φ appearing in the variety of Borel subgroups of G containing $\Phi(\mathcal{W}_F \times B_2)$. Thus Theorem 1 of the introduction will be proved.

6. Independence of χ

6.1. Let V, χ, χ' be as in 5.4. We want to show that

$$i_{\chi}(V) = i_{\chi'}(V). \tag{6.1a}$$

There is $w \in W$ such that

$$(\chi \otimes \tau)^w = \chi' \otimes \tau'.$$

To prove (6.1a) we may assume that $w = s$ is a simple reflection, and that $\chi^s \neq \chi$. We let $\dot{s}$ denote a representative of s, taken in $\mathcal{G}$ or G, where appropriate. Conjugation by $\dot{s}$ in G gives an isomorphism $f_{\dot{s}} : G_{\hat{\chi}} \longrightarrow G_{\hat{\chi}^s}$. By (1.4) this induces an isomorphism of Hecke algebras

$$\mathcal{H}(f_{\dot{s}}) : \mathcal{H}(G_{\hat{\chi}}) \longrightarrow \mathcal{H}(G_{\hat{\chi}^s}),$$

sending $T_w \mapsto T_{sws}$ for $w \in \widetilde{W}_{\chi}$.

We will find an algebra isomorphism

$$\mathcal{H}(\mathcal{G}, \chi) \xrightarrow{\Theta_s} \mathcal{H}(\mathcal{G}, \chi^s) \tag{6.1b}$$

such that

$$\Theta_s^* V^{\chi^s} \simeq V^{\chi}, \tag{6.1c}$$

and for which the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}(G_{\hat{\chi}}) & \xrightarrow{\mathcal{H}(f_{\dot{s}})} & \mathcal{H}(G_{\hat{\chi}^s}) \\ {\scriptstyle \Psi_{\chi}}\big\downarrow & & {\scriptstyle \Psi_{\chi^s}}\big\downarrow \\ \mathcal{H}(\mathcal{G}, \chi) & \xrightarrow{\Theta_s} & \mathcal{H}(\mathcal{G}, \chi^s), \end{array} \tag{6.1d}$$

where $\Psi_{\chi},\ \Psi_{\chi^s}$ are as in (5.2a).

Suppose we have found such a map Θ_s. Let (τ, u, ρ) and (τ', u', ρ') be triples corresponding to $\Psi_{\chi}^* V^{\chi}$, and $\Psi_{\chi^s}^* V^{\chi^s}$, respectively, under the bijection in 4.5. Then

$$V_{\tau,u,\rho} \simeq \Psi_{\chi}^* V^{\chi} \simeq \Psi_{\chi}^* \Theta_s^* V^{\chi^s} \simeq f_{\dot{s}}^{\sharp} \Psi_{\chi^s} V^{\chi^s} \simeq f_{\dot{s}}^{\sharp} V_{\tau',u',\rho'} \simeq V_{s\tau' s^{-1}, \dot{s}u'\dot{s}^{-1}, \dot{s}\rho'},$$

this last by (2.7.3). It follows that (τ, u, ρ) and (τ', u', ρ') are G-conjugate, so $i_{\chi}(V) = i_{\chi'}(V)$, and (6.1a) will be proved.

The existence of an isomorphism (6.1b) with properties (6.1c,d) will be proved first for the tamely ramified case, then for the essentially tame case, and finally we will reduce the general case to the essentially tame case, using the Bushnell-Kutzko theory of covers. This pattern of argument was used already in [Ro].

6.2. Assume in this section that $\hat\chi$ is tamely ramified. Then $\mathcal{J}_\chi = \mathcal{J}_{\chi^s}$ is the Iwahori subgroup of $\mathcal{G}$ corresponding to $\mathcal{B}$. We denote it by $\mathcal{J}$. The group $\mathcal{T}_0$ is a quotient of $\mathcal{J}$, and χ, χ' inflate to characters of $\mathcal{J}$. Thus

$$\mathcal{H}(\mathcal{G},\chi) = \mathrm{End}_{\mathcal{G}}(\mathrm{ind}_{\mathcal{J}}^{\mathcal{G}} \chi).$$

The algebra $\mathcal{H}(\mathcal{G},\chi)$ has a linear basis $\{T_w^\chi :\ w \in \widetilde{W}_\chi\}$ defined as follows. Let $\mathcal{N}$ be the normalizer of $\mathcal{T}_0$ in $\mathcal{G}$, and let $\mathcal{N}_\chi$ be the stabilizer of χ in $\mathcal{N}$. Choose an extension $\tilde\chi$ of χ to $\mathcal{N}_\chi$, as in [Ro, p. 385]. Choose representatives $\{\dot w :\ w \in \widetilde{W}\}$ for $\widetilde{W}$ in $\mathcal{N}$.

For $w \in \widetilde{W}$, we have an intertwining map

$$\theta_{\dot w}^\chi : \mathrm{ind}_{\mathcal{J}}^{\mathcal{G}} \chi \longrightarrow \mathrm{ind}_{\mathcal{J}}^{\mathcal{G}} \chi^w$$

defined by

$$\theta_{\dot w}^\chi f(x) = \frac{1}{vol(\mathcal{J}_1)} \int_{\mathcal{J}_1} f(\dot w^{-1} u x)\ du,$$

where $\mathcal{J}_1$ is the pro-unipotent radical of $\mathcal{J}$.

If $\chi^w = \chi$, then the basis element T_w^χ is given by

$$T_w^\chi = q^{\ell(w)} \tilde\chi(\dot w) \theta_{\dot w}^\chi,$$

where ℓ is the length function on $\widetilde{W}_\chi$.

For $h \in \mathcal{H}(\mathcal{G},\chi)$, we define

$$\Theta_s(h) = (\theta_{\dot s}^{\chi^s})^{-1} h \theta_{\dot s}^{\chi^s}.$$

Straightforward calculations, using the formulas in [M, 5.10, 6.6] show that

$$\Theta_s(T_w^\chi) = T_{sws}^{\chi^s},$$

so diagram (6.1d) commutes. Using the maps

$$V^\chi = \mathrm{Hom}_{\mathcal{J}}(\chi, V) \simeq \mathrm{Hom}_{\mathcal{G}}(\mathrm{ind}_{\mathcal{J}}^{\mathcal{G}} \chi, V) \xrightarrow{\ \_\circ\theta_{\dot s}^{\chi^s}\ } \mathrm{Hom}_{\mathcal{G}}(\mathrm{ind}_{\mathcal{J}}^{\mathcal{G}} \chi^s, V) \simeq V^{\chi^s},$$

one verifies property (6.1c) by a formal calculation. Thus, (6.1a) is proved in the tame case.

6.3. We turn next to the essentially tame case. Let $\Delta_F^{(1)}$ be the wild ramification subgroup of $\mathcal{W}_F^a$, and set $\Delta_1 = \hat\chi(\Delta_F^{(1)})$. The corresponding group for $\hat\chi^s$ is Δ_1^s.

Suppose $\Delta_1 \subseteq Z$. Since $\Delta_F^{(1)}$ is a direct factor of $\mathcal{W}_F^a$, there is a character $\hat\chi_1 : \mathcal{W}_F^a \longrightarrow Z$ restricting to $\hat\chi^{-1}$ on $\Delta_F^{(1)}$. The map $\hat\chi\hat\chi_1 : \mathcal{I}_F^a \longrightarrow G$ is tamely ramified. Clearly $G_{\hat\chi\hat\chi_1} = G_{\hat\chi}$. Since conjugation in G does not affect $\hat\chi_1$, the map $\hat\chi^s\hat\chi_1 : \mathcal{I}_F^a \longrightarrow G$ is also tamely ramified.

On the p-adic side, the homomorphism $\hat\chi_1 : \mathcal{W}_F^a \longrightarrow Z$ corresponds to a character $\chi_1 : \mathcal{G} \longrightarrow \mathbf{C}^\times$. Multiplication by χ_1 induces an isomorphism

$$\mathcal{H}(\mathcal{G},\chi) \xrightarrow{\simeq} \mathcal{H}(\mathcal{G},\chi\chi_1),$$

likewise for χ^s, so the existence of a map Θ_s with properties (6.1c,d) follows from the tame case considered in (6.2).

6.4. Suppose χ, χ^s have arbitrary ramification. Let $L = G_{\Delta_1}$ be the centralizer of Δ_1 in G. Since Δ_1 is a finite abelian p-group, and p is not a torsion prime for G, the subgroup L is a Levi subgroup of G, and $L \neq G$ exactly when χ is not essentially tame. Let $\mathcal{L}$ be the Levi subgroup of $\mathcal{G}$ dual to L. The corresponding groups for χ^s are L^s and $\mathcal{L}^s$.

It follows from [Ro, 5.1] that there is a support-preserving isomorphism of Hecke algebras

$$r_\chi : \mathcal{H}(\mathcal{G}, \chi) \longrightarrow \mathcal{H}(\mathcal{L}, \chi).$$

Thus, we have a diagram of support-preserving algebra isomorphisms

$$\begin{array}{ccccccc} \mathcal{H}(G_{\hat\chi}) & = & \mathcal{H}(L_{\hat\chi}) & \xrightarrow{\mathcal{H}(f_{\dot s})} & \mathcal{H}(L_{\hat\chi^s}) & = & \mathcal{H}(G_{\hat\chi^s}) \\ \Psi_\chi \downarrow & & \Psi'_\chi \downarrow & & \Psi'_{\chi^s} \downarrow & & \Psi_{\chi^s} \downarrow \\ \mathcal{H}(\mathcal{G}, \chi) & \xrightarrow{r_\chi} & \mathcal{H}(\mathcal{L}, \chi) & \xrightarrow{?} & \mathcal{H}(\mathcal{L}^s, \chi^s) & \xleftarrow{r_{\chi^s}} & \mathcal{H}(\mathcal{G}, \chi^s). \end{array} \tag{6.4b}$$

It suffices to replace ? by a map Θ'_s making (6.4b) commute, and for which the map

$$\Theta_s := r_{\chi^s}^{-1} \circ \Theta'_s \circ r_\chi \tag{6.4c}$$

satisfies (6.1c).

If $\dot s \in \mathcal{L}$, then we get the desired map Θ'_s from the essentially tame case in (6.3).

Suppose, then, that $\dot s \notin \mathcal{L}$. We must examine more closely the subgroup $\mathcal{J}_\chi$, defined in [Ro, §3], to which we refer for notation. There the convex function f_χ has the property that

$$f_\chi(s\beta) = f_{\chi^s}(\beta),$$

as long as $\beta \neq \pm\alpha$ (here α is the root for s). This is the case if β is a root in $\mathcal{L}$. It follows that

$$\mathcal{L}^s \cap \mathcal{J}_{\chi^s} = [\mathcal{L} \cap \mathcal{J}_\chi]^{\dot s},$$

and $\rho_{\chi^s} = (\rho_\chi)^{\dot s}$ on this subgroup. Therefore, conjugation by $\dot s$ is a bijection

$$\theta'_s : \operatorname{ind}_{\mathcal{L}\cap\mathcal{J}_\chi}^{\mathcal{L}} \rho_\chi \longrightarrow \operatorname{ind}_{[\mathcal{L}\cap\mathcal{J}_\chi]^{\dot s}}^{\mathcal{L}^s} (\rho_\chi)^{\dot s} = \operatorname{ind}_{\mathcal{L}^s\cap\mathcal{J}_{\chi^s}}^{\mathcal{L}^s} \rho_{\chi^s}.$$

We let Θ'_s be conjugation by θ'_s. Verifying (6.1c) and the commutativity of (6.4b) is straightforward, and left to the reader. The proof of (6.1a) is now complete.

7. Whittaker Models

Let χ be a character of $\mathcal{T}_0$, and let $\tau \in T$. Since $\mathcal{G}$ has connected center, there is a unique irreducible constituent $U(\chi, \tau)$ of $\operatorname{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi \otimes \tau)$ which admits a Whittaker model [Rod2]. We determine here the parameter (Φ, ρ) attached to $U(\chi, \tau)$ by Theorem 1.

7.1. Let $\mathcal{L}, L$ be the Levi subgroups defined in 6.4, and let $\mathcal{B}_{\mathcal{L}}$ be a Borel subgroup of $\mathcal{L}$ containing $\mathcal{T}$. Denote by $U_{\mathcal{L}}(\chi, \tau)$ the Whittaker constituent of $\operatorname{Ind}_{\mathcal{B}_{\mathcal{L}}}^{\mathcal{L}}(\chi \otimes \tau)$. Since $\operatorname{im}\Phi \subset H \subset L$, the same pair (Φ, ρ) is attached by Theorem 1 to both $U(\chi, \tau)$ and $U_{\mathcal{L}}(\chi, \tau)$. Therefore, we may assume that χ is essentially tame. Arguing as in 6.3, we may reduce further and assume, as we shall, that χ is in fact tamely ramified.

7.2. The $\mathcal{H}(G_{\hat\chi})$-module $[\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi\otimes\tau)]^{\chi}$ restricts to the regular representation of $\mathcal{H}_0(G_{\hat\chi})$. It follows that there is a unique $\mathcal{H}(G_{\hat\chi})$-subquotient $E(\chi,\tau)$ of $[\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi\otimes\tau)]^{\chi}$ containing the character ϵ_q upon restriction to $\mathcal{H}_0(G_{\hat\chi})$. In this section, we will show that

$$U(\chi,\tau)^{\chi}=E(\chi,\tau). \tag{7.2a}$$

Let $\mathcal{K}=\mathcal{J}W_{\chi}\mathcal{J}$, a hyperspecial maximal compact subgroup containing $\mathcal{J}$. Let $\mathcal{K}_1$ be the pro-unipotent radical of $\mathcal{K}$. Then $\mathcal{K}_1\subset\mathcal{J}\subset\mathcal{K}$, and $\mathcal{K}/\mathcal{K}_1=\mathcal{G}(q)$, the reductive group of the same type as $\mathcal{G}$ with points in $\mathbb{F}_q$. Let $\dot w_0\in\mathcal{K}$ be a representative of the longest element of W.

Recall that $\mathcal{U}$ is the unipotent radical of $\mathcal{B}$, and put $\mathcal{U}_0=\mathcal{U}\cap\mathcal{J}$, $\mathcal{U}_1=\mathcal{U}\cap\mathcal{K}_1$. Since $\mathcal{G}$ has connected center, there is a single $\mathcal{T}$-orbit of generic characters Ψ of $\mathcal{U}$. We choose Ψ so that $\ker\Psi\circ x_\alpha$ is the prime ideal in the ring of integers of F, for any simple root group $x_\alpha:F\longrightarrow\mathcal{U}$. Now $\mathcal{U}(q)=\mathcal{U}_0/\mathcal{U}_1$, and we let ψ be the character of $\mathcal{U}(q)$ obtained from the restriction of Ψ to $\mathcal{U}_0$. The Whittaker functional

$$\Omega\in\mathrm{Hom}_{\mathcal{U}}(\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\chi\otimes\tau,\Psi)$$

is given by the principal-valued integral

$$\Omega(f)=\int_{\mathcal{U}}f(\dot w_0u)\Psi^{-1}(u)\ du.$$

Since $\mathcal{B}\dot w_0\mathcal{I}=\mathcal{B}\dot w_0\mathcal{U}_0$, it follows that there is a nonzero function $f\in[\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi\otimes\tau)]^{\mathcal{K}_1}$ supported on $\mathcal{B}\dot w_0\mathcal{I}$. It is easy to check that $\Omega(f)\neq0$. Thus, Ω restricts to a nonzero functional on $[\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi\otimes\tau)]^{\mathcal{K}_1}$. The latter space may be identified with $\mathrm{Ind}_{\mathcal{B}(q)}^{\mathcal{G}(q)}\chi$. Therefore Ω restricts to a nonzero functional

$$\omega\in\mathrm{Hom}_{\mathcal{U}(q)}(\mathrm{Ind}_{\mathcal{B}(q)}^{\mathcal{G}(q)}\chi,\psi).$$

There is a unique irreducible $\mathcal{G}(q)$-constituent $\sigma\subset\mathrm{Ind}_{\mathcal{B}(q)}^{\mathcal{G}(q)}\chi$ on which ω does not vanish [C, 8.1.5]. The endomorphism algebra of $\mathrm{Ind}_{\mathcal{B}(q)}^{\mathcal{G}(q)}\chi$ is isomorphic to $\mathcal{H}_0(G_{\hat\chi})$, and the χ-isotypic component σ^{χ} is an irreducible representation of $\mathcal{H}_0(G_{\hat\chi})$, whose generic degree $d(\sigma^{\chi})$ has the same p-adic valuation as $\dim\sigma$. On the other hand, by Deligne-Lusztig theory (cf. [C, 8.4.9]), the p-part of $\dim\sigma$ is $q^{\nu(\chi)}$, where $\nu(\chi)$ is the number of reflections in W_χ. By [A], this is the maximum power of q that can divide a generic degree, and it is attained only for the sign character of $\mathcal{H}_0(G_{\hat\chi})$. It follows that σ^{χ} affords the sign character of $\mathcal{H}_0(G_{\hat\chi})$.

Let V be the smallest $\mathcal{G}$-submodule of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi\otimes\tau)$ such that $\sigma\subset V^{\mathcal{K}_1}$. Then Ω does not vanish on V, so $U(\chi,\tau)=V/V\cap\ker\Omega$ by the uniqueness of $U(\chi,\tau)$. It follows that σ^{χ} appears in $U(\chi,\tau)^{\chi}$, proving (7.2a).

7.3. We can now determine the parameter of the Whittaker constituent. Recall that pairs (Φ,ρ) correspond to triples (τ,u,ρ), as in (4.4.1).

(7.3.1) Proposition. *Let (Φ,ρ) be the pair attached to $U(\chi,\tau)$ by Theorem 1. Then ρ is the trivial represenatation of G_Φ, and Φ corresponds to (τ,u), where u belongs to the dense $G_{\hat\chi,\tau}$-orbit in $\mathfrak{q}_\tau$.*

Proof. Let $H=G_{\hat\chi}$, and let $\widetilde H$ be isogenous to H, with simply-connected derived group. Let $\tilde\tau$ be a lift of τ in $\widetilde H$. By [R2, §10], the character ϵ_q of $\mathcal{H}_0(\widetilde H)$ appears in the simple $\mathcal{H}(\widetilde H)$-module $V_{\tilde\tau,u,\mathrm{triv}}$, where u belongs to the dense orbit $\mathfrak{q}_\tau^\circ$ under the centralizer of $\tilde{\tau}$ in $\widetilde{H}$. This orbit is unique, so $R_{\tau,u} = R_\tau$ (see 3.2). By (2.9.1), the character ϵ_q of $\mathcal{H}_0(H)$ appears in the simple $\mathcal{H}(H)$-module $[V_{\tilde{\tau},u,\mathrm{triv}}]^{R_\tau}$ This implies, by (7.2a), that

$$[V_{\tilde{\tau},u,\mathrm{triv}}]^{R_\tau} \simeq U(\chi,\tau)^\chi.$$

The result follows now from 4.5. □

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