Nilpotent Orbits and Complex Dual Pairs

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1 Introduction

Let $(W, \langle\ ,\ \rangle)$ be a complex symplectic vector space and let $Sp(W)$ be the symplectic group. Let $G,\ G' \subseteq Sp(W)$ be a complex reductive dual pair (see Howe [H1]), i. e. G and G' are centralizers of each other in $Sp(W)$ and both act completely reductively on W. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of G and G'. We have the (unnormalized) moment maps

$$\tau : W \longrightarrow \mathfrak{g}^*, \quad \tau' : W \longrightarrow \mathfrak{g}'^* \tag{1}$$

defined by the formula

$$\tau(w)(x) = \langle x(w), w\rangle, \quad w \in W,\ x \in \mathfrak{g} \subseteq \mathrm{End}(W),$$

and similarly for τ'.

Our main theorem describes the behaviour of closures of nilpotent orbits under the action of moment maps. It is easy to see that for a nilpotent coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ the set $\tau'(\tau^{-1}(\overline{\mathcal{O}}))$ is the union of nilpotent coadjoint orbits in $\mathfrak{g}'$. It turns out that it is a closure of a single orbit:

Theorem 1.1 *Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a nilpotent coadjoint orbit. There exists a (unique) nilpotent coadjoint orbit $\mathcal{O}' \subseteq \mathfrak{g}'^*$ such that $\tau'(\tau^{-1}(\overline{\mathcal{O}})) = \overline{\mathcal{O}'}$.*

This theorem was announced in [P2], theorem 0.4. It has a potential of sheding some light on Howe's correspondence, as we shall explain below.

Let $(W_0, \langle\ ,\ \rangle)$ be a symplectic space over $\mathbb{R}$, such that $(W, \langle\ ,\ \rangle)$ coincides with the complexification of $(W_0, \langle\ ,\ \rangle)$. Let $G_0, G_0' \subseteq Sp(W_0)$ be an irreducible dual pair, as defined in [H1]. Let $\widetilde{G}_0, \widetilde{G}_0'$ be the preimages of G_0, G_0' in the metaplectic group $\widetilde{Sp}(W_0)$. Fix an oscillator representation of $\widetilde{Sp}(W_0)$, and let Π, Π' be irreducible admissible representations of $\widetilde{G}_0, \widetilde{G}_0'$ respectively,

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in Howe's correspondence, [H3]. Let I_Π denote the annihilator of the Harish-Chandra module of Π in $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of the (complexified) Lie algebra $\mathfrak{g}$ of the group $\widetilde{G}_0$. The enveloping algebra $\mathcal{U}(\mathfrak{g})$ carries a natural filtration, and the corresponding graded algebra $gr\ \mathcal{U}(\mathfrak{g})$ is isomorphic to the algebra of polynomial functions on $\mathfrak{g}^*$. The set of common zeros of grI_Π in $\mathfrak{g}^*$ is called the associated variety of the ideal I_Π, (see [B], [M]). We shall denote it by $\mathcal{V}(\Pi)$. Similarly we have $I_{\Pi'} \subseteq \mathcal{U}(\mathfrak{g}')$ and $\mathcal{V}(\Pi') \subseteq \mathfrak{g}'^*$. The following inclusion was shown in [P1], theorem 7.1:

$$\mathcal{V}(\Pi') \subseteq \tau'(\tau^{-1}(\mathcal{V}(\Pi))). \tag{2}$$

If $G_0 \neq G_0'$ are compact then $\mathcal{V}(\Pi) = \{0\}$, $\mathcal{V}(\Pi') = \{0\}$ and the inclusion (2) is proper. On the other hand, if the pair G_0, G_0' is in the stable range with G_0 the smaller member and Π is unitary, then equality holds in (2) (see [P2], theorem 7.9). Furthermore we know from a theorem of Borho - Brylinski - Jantzen that $\mathcal{V}(\Pi')$ is the closure of a single nilpotent orbit. Thus theorem 1.1 justifies the problem of trying to understand those representations in Howe's correspondence for which the two sides of (2) are equal. In fact Howe constructs the representation Π' as the unique quotient of certain finitely generated, admissible quasisimple representation Π_1' of $\widetilde{G}_0'$, (see [H3], Theorem 1A). It seems plausible that under some mild conditions equality holds in (2) if $\mathcal{V}(\Pi')$ is replaced by $\mathcal{V}(\Pi_1')$. The case of both G_0, G_0' compact shows that this can't be true in general. However in cases when it is true, it would be interesting to understand the irreducible subquotients of Π_1' having $\tau'(\tau^{-1}(\mathcal{V}(\Pi))$ as the associated variety.

Another possible application is to the study of the singularities of the closures of nilpotent orbits. In fact, some special cases of theorem 1.1 were used by Kraft and Procesi in [K-P1], [K-P2] in the study of the normality of the closures of nilpotent orbits in classical Lie algebras.

It is easy to see that it is sufficient to prove theorem 1.1 in the case of an *irreducible* dual pair (see [H1]). From now on we assume that the pair (G, G') is irreducible. We will now recall the classification of irreducible complex reductive dual pairs, due to Howe (see [H1],[H2]). There are two types of such pairs.

Pairs of type I.
Let U be a complex vector space endowed with a nondegenerate symmetric bilinear form $(\ ,\ )$. Let $O(U)$ be its isometry group. Let V be a complex vector space endowed with a nondegenerate skew–symmetric bilinear form $(\ ,\ )'$ and let $Sp(V)$ be its isometry group. Let $W = \mathrm{Hom}_{\mathbb{C}}(U, V)$. Define a symplectic form on W by

$$\langle w, w' \rangle = tr(w \cdot w'^*),$$

where the map $\mathrm{Hom}(U, V) \ni w \longmapsto w^* \in \mathrm{Hom}(V, U)$ is defined by

$$(w(u), v)' = (u, w^*(v)).$$

The groups $O(U)$, $Sp(V)$ act on W via premultiplication by the inverse and by postmultiplication, respectively. These actions embed both groups into the symplectic group $Sp(W)$, and these subgroups form an irreducible dual pair in $Sp(W)$, called *a dual pair of type I.*

Using any invariant nondegenerate symmetric bilinear forms on the orthogonal and symplectic Lie algebras $\mathfrak{o}(U)$ and $\mathfrak{sp}(V)$ we can identify them with their duals. These identifications intertwine the adjoint and coadjoint actions of the corresponding groups. It is easy to check that one can choose these identifications so that the moment maps (1) can be written as

$$\tau \quad = \pi : W \longrightarrow \mathfrak{o}(U), \qquad \pi(w) = w^* \cdot w, \tag{3}$$
$$\tau' \quad = \rho : W \longrightarrow \mathfrak{sp}(V), \qquad \rho(w) = w \cdot w^*. \tag{4}$$

Pairs of type II
Let U and V be two complex vector spaces and let $W = \mathrm{Hom}(U, V) \oplus \mathrm{Hom}(V, U)$. Define a symplectic form on W by the formula

$$\langle (w_1, w_2), (w_1', w_2') \rangle = tr(w_1 w_2' - w_1' w_2)$$

for $w_1, w_1' \in \mathrm{Hom}(U, V)$, $w_2, w_2' \in \mathrm{Hom}(V, U)$. There is an obvious action of the groups $GL(U)$, $GL(V)$ on W embedding both groups into the symplectic group $Sp(W)$, and these subgroups form an irreducible dual pair in $Sp(W)$ called *a dual pair of type II.*

As in the case of pairs of type I, we can identify $\mathfrak{gl}(U)$ with $\mathfrak{gl}(U)^*$, and $\mathfrak{gl}(V)$ with $\mathfrak{gl}(V)^*$. Under these identifications the moment maps (1) up to scalar factors can be written as

$$\pi : W \longrightarrow \mathfrak{gl}(U), \qquad \pi(w_1, w_2) = w_2 \cdot w_1, \tag{5}$$
$$\rho : W \longrightarrow \mathfrak{gl}(V), \qquad \rho(w_1, w_2) = w_1 \cdot w_2. \tag{6}$$

Remark. Since by (for example) Malcev's theorem (see [C-M, Thm. 3.4.12]), nilpotent orbits are invariant under multiplication by nonzero scalars, theorem 1.1 does not depend on a specific choice of scalars in the identifications of Lie algebras with their duals.

The proof of theorem 1.1 will be done separately for each type of irreducible dual pairs. It is based on the classical results on the classification of nilpotent orbits in classical Lie algebras, and on the classification of "nilpotent" $G \cdot G'$–orbits in W which, although known (see [C],[K-P1],[K-P2]), is probably less classical. The following two sections contain all the results we need. The proof of theorem 1.1 for pairs of type II is in section 4, pairs of type I are discussed in section 5.

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2 Nilpotent orbits in classical Lie algebras

The basic reference for this section is [C-M]. We begin with some combinatorial notions.

Let m be a fixed nonnegative integer. A *partition* of m is a (finite or infinite) weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0 \geq \ldots)$ such that $\lambda_1 + \lambda_2 + \ldots = m$. The numbers λ_i are called the *parts* of the partition λ. We will identify partitions that differ only in the number of parts equal to 0. The length $l(\lambda)$ of a partition λ is defined as the largest i with $\lambda_i \neq 0$.

A partition λ is *orthogonal* if each even part of λ occurs an even number of times. A partition is *symplectic*, if each odd part occurs an even number of times.

Example $(7,7,4,3,3,3,3,2,2)$ is symplectic, $(7,4,4,3,3,3,3,2,2)$ is orthogonal.

In the sequel we will use the following obvious lemma:

Lemma 2.1 *If μ is an orthogonal partition of n, then $n \equiv l(\mu) \pmod 2$*

Definition 2.2 *Let λ, μ be two partitions. We say that λ* dominates *μ, and write $\lambda \geq \mu$, if for each $i \geq 1$ we have $\lambda_1 + \ldots \lambda_i \geq \mu_1 + \ldots \mu_i$.*

Example $(3,1) \geq (2,2)$

Let V be a complex vector space of dimension m. To any endomorphism $x \in \mathrm{End}(V)$ we can associate a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of m, where $\lambda_1, \lambda_2, \ldots$ are the sizes of blocks in the Jordan normal form of x.

Theorem 2.3 *Let G be one of the groups $GL(V)$, $O(V)$, $Sp(V)$ and let $\mathfrak{g} \subseteq End(V)$ be the Lie algebra of G. Two nilpotent elements of $\mathfrak{g}$ are conjugate under the adjoint action of the group G if and only if they define the same partition. By $\mathfrak{g}_\lambda$ (resp. $\mathfrak{gl}_\lambda$, $\mathfrak{o}_\lambda$ and $\mathfrak{sp}_\lambda$) we denote the conjugacy class of nilpotent elements of $\mathfrak{g}$ (resp. $\mathfrak{gl}(V)$, $\mathfrak{o}(V)$ and $\mathfrak{sp}(V)$) corresponding to a partition λ. The nilpotent orbit $\mathfrak{sp}_\lambda$ is nonempty if and only if the partition λ is symplectic. Similarly the class $\mathfrak{o}_\lambda$ is nonempty if and only if λ is orthogonal. Moreover, if $\mathfrak{g}_\lambda$, $\mathfrak{g}_\mu$ are nonempty, then*

$$\mathfrak{g}_\mu \subseteq \overline{\mathfrak{g}_\lambda} \quad \text{if and only if} \quad \mu \leq \lambda.$$

3 Nilpotent orbits in W

In this section we will recall from the papers of Kraft, Procesi [K-P1],[K-P2], and Capparelli [C] all the informations on the orbits of $G \cdot G'$ in the space W that we will use in the proof of theorem 1.1. In fact we need only to consider the set of *nilpotent* orbits in W.

Definition 3.1 *Let W be a symplectic space and let (G, G') be an irreducible dual pair in $Sp(W)$. An element $w \in W$ is* nilpotent*, if $\tau(w) \in \mathfrak{g}$ is nilpotent (or equivalently $\tau'(w) \in \mathfrak{g}'$ is nilpotent).*

Let $\mathcal{N}_W$ denote the set of nilpotent elements of W. We will describe the orbits of $G \cdot G'$ in $\mathcal{N}_W$. We begin with pairs of type II.

3.1 Nilpotent orbits in W for pairs of type II

In this case $W = \mathrm{Hom}(U,V) \oplus \mathrm{Hom}(V,U)$, $G = GL(U)$, $G' = GL(V)$. An element $(w_1, w_2) \in W$ is nilpotent if and only if $w_1 w_2$ is a nilpotent endomorphism of V (equivalently $w_2 w_1$ is nilpotent as an endomorphism of U). Let a and b be two distinct symbols.

Definition 3.2 *An ab*-string *is a finite ordered sequence of the form aba... or bab..., i. e. a finite sequence alternating between a and b. An ab*-diagram *is a finite sequence of ab-strings. We will identify two ab-diagrams that differ only in the ordering of strings. For an ab-diagram* δ *by* $\tau(\delta)$ *we denote the partition counting the a's in the strings of* δ*, i. e. if* $\delta = (\delta_1, \delta_2, \ldots)$ *and if* α_i *denotes the number of a's in* δ_i*, then* $\tau(\delta)$ *is the partition obtained by ordering the sequence* $\alpha_1, \alpha_2, \ldots$*. Similarly,* $\tau'(\delta)$ *denotes the partition counting b's in the strings of* δ*.*

Theorem 3.3 *There is a one–to–one correspondence between the set of* $G \cdot G'$*–orbits in* $\mathcal{N}_W$ *and the set of all ab-diagrams containing* $\dim U$ *a's and* $\dim V$ *b's. The orbit corresponding to an ab-diagram* δ *will be denoted* $\mathcal{O}_\delta$*. The action of the moment maps on the orbit* $\mathcal{O}_\delta$ *in* $\mathcal{N}_W$ *is described by the formulas*

$$\tau(\mathcal{O}_\delta) = \mathfrak{gl}_{\tau(\delta)}, \qquad \tau'(\mathcal{O}_\delta) = \mathfrak{gl}_{\tau'(\delta)}.$$

We will not go into details about the precise description of the elements of the orbit $\mathcal{O}_\delta$, as we do not need any such information. The interested reader can find more details in [K-P1, section 4], [K-P2, section 6]

Lemma 3.4 *If* ν *and* λ *are two partitions for which there exists an ab-diagram* δ *such that* $\tau(\delta) = \nu$*,* $\tau'(\delta) = \lambda$*, then* $\lambda_i - 1 \leq \nu_i \leq \lambda_i + 1$ *for each i.*

Proof: By definition, $\tau(\delta)$ and $\tau'(\delta)$ are the partitions obtained by reordering the sequences $(\alpha_1, \alpha_2, \ldots)$ and $(\beta_1, \beta_2, \ldots)$, where $\alpha_i =$ the number of a's in the string δ_i, and $\beta_i =$ the number of b's in the string δ_i. If δ_i is longer than δ_{i+1}, then $\alpha_i \geq \alpha_{i+1}$ and $\beta_i \geq \beta_{i+1}$. Hence we may assume that all the strings δ_i are of the same length. But in this case the lemma is obvious.

3.2 Nilpotent orbits in W for pairs of type I

In this case $W = \mathrm{Hom}(U, V)$, and $G = O(U)$, $G' = Sp(V)$. The moment maps are denoted by π and ρ.

Definition 3.5 *An ab-diagram* δ *is* orthosymplectic *if it consists of the following types of strings or pairs of strings:*

abab...ba *with an odd number of a's,*

baba...ab *with an odd number of a's,*

aba...ba
aba...ba *with an even number of a's in each string,*

bab...ab
bab...ab *with an even number of a's in each string,*

ab...ab
ab...ab .

For an orthosymplectic ab-diagram δ arising in the context of a dual pair of type I we will write $\pi(\delta)$ and $\rho(\delta)$ for the partitions counting a's and b's in the strings of δ. Note that for an orthosymplectic ab-diagram δ the partition $\pi(\delta)$ is orthogonal and the partition $\rho(\delta)$ is symplectic.

Theorem 3.6 *There is a one–to–one correspondence between the set of $G \cdot G'$–orbits in $\mathcal{N}_W$ and the set of all orthosymplectic ab-diagrams containing* $\dim U$ *a's and* $\dim V$ *b's. The orbit corresponding to an ab-diagram δ will be denoted $\mathcal{O}_\delta$. The action of the moment maps on the orbit $\mathcal{O}_\delta$ in $\mathcal{N}_W$ is described by the formulas*

$$\pi(\mathcal{O}_\delta) = \mathfrak{o}_{\pi(\delta)}, \qquad \rho(\mathcal{O}_\delta) = \mathfrak{sp}_{\rho(\delta)}.$$

Remark: Lemma 3.4 is valid in the orthosymplectic context as well.

3.3 Moment maps

Let us recall from [K-P1],[K-P2, section 1] one more fact that we will need in the sequel. The First Fundamental Theorem of the Classical Invariant Theory (see [W], or [G-W]), says that the moment maps $\tau : W \longrightarrow \mathfrak{g}$ defined in the introduction are quotient maps for the action of the corresponding group G'. In particular we have the following lemma:

Lemma 3.7 *The image $\tau(X) \subseteq \mathfrak{g}$ of a closed, G'-invariant subset X of W is closed in $\mathfrak{g}$.*

4 Pairs of type II

In this section we prove theorem 1.1 for an irreducible dual pair of type II. In this case $W = \mathrm{Hom}(U,V) \oplus \mathrm{Hom}(V,U)$, $G = GL(U)$, $G' = GL(V)$, $\dim U = n$, $\dim V = m$. We begin with a sketch of the main idea. Let λ be a partition of n. We define a partition λ' of m by setting $\lambda'_1 = \lambda_1 + 1$, $\lambda'_2 = \lambda_2 + 1$, $\ldots$, untill we reach $\lambda'_1 + \lambda'_2 + \ldots = m$. More precisely

Definition 4.1 *Let λ be a partition of n. Let $r_1 = m$. For $i \geq 2$ let $r_i = r_i(\lambda) = m - (\lambda_1 + 1 + \lambda_2 + 1 + \ldots + \lambda_{i-1} + 1)$ for $i \geq 1$. Let $i_0 = i_0(\lambda)$ be the smallest $i \geq 1$ such that $r_i \leq \lambda_i$. Define a partition λ' of m by:*
$\lambda'_i = \lambda_i + 1$ *for* $i < i_0$,
$\lambda'_{i_0} = r_{i_0}$,
$\lambda'_j = 0$ *for* $j > i_0$.

Remark: It is clear that there exists an ab-diagram δ such that $\tau(\delta) \leq \lambda$, $\tau'(\delta) = \lambda'$. Indeed, since $r_{i_0} \leq \lambda_{i_0}$, we may find δ so that $\tau(\delta) = (\lambda_1, \lambda_2, \ldots, \lambda_{i_0 - 1}, r_{i_0}, 1, \ldots, 1)$, 1's occur if $r_{i_0} < \lambda_{i_0} + \lambda_{i_0+1} + \ldots$. Lemma 4.3 implies that λ' is the "largest" partition with these properties.

Theorem 4.2 *Let λ be a partition of n. Then*

$$\tau'(\tau^{-1}(\overline{\mathfrak{gl}_\lambda})) = \overline{\mathfrak{gl}_{\lambda'}}. \tag{7}$$

Proof: It follows from the remark following definition 4.1 that

$$\mathfrak{gl}_{\lambda'} \subseteq \tau'(\tau^{-1}(\overline{\mathfrak{gl}_\lambda})).$$

As $\tau^{-1}(\overline{\mathfrak{gl}_\lambda})$ is G-invariant, by lemma 3.7 the set $\tau'(\tau^{-1}(\overline{\mathfrak{gl}_\lambda}))$ is closed and it follows that the relation $\supseteq$ holds in (7). It remains to prove the following claim:

Claim: Let μ be a partition of n such that $\mu \leq \lambda$, and let δ be an ab-diagram such that $\tau(\delta) = \mu$. Then $\tau'(\delta) \leq \lambda'$.

The claim follows immediately from the following two lemmas:

Lemma 4.3 *Let $\mu = \tau(\delta)$. Then $\tau'(\delta) \leq \mu'$, where μ' is defined in terms of μ according to definition 4.1.*

Lemma 4.4 *If $\mu \leq \lambda$ are two partitions of n then $\mu' \leq \lambda'$.*

Proof of lemma 4.3: Let $\tau'(\delta) = \nu$ and $i_0 = i_0(\mu)$. For $i < i_0$ we have $\mu'_i = \mu_i + 1 \geq \nu_i$ (lemma 3.4), so $\nu_1 + \ldots + \nu_i \leq \mu'_1 + \ldots + \mu'_i$ for all $i < i_0$. Also, $\nu_1 + \ldots + \nu_{i_0} \leq m = \mu'_1 + \ldots + \mu'_{i_0}$, so $\nu \leq \mu'$.

Proof of lemma 4.4: Let $\mu \leq \lambda$ be two partitions of n. In order to prove that $\mu' \leq \lambda'$ it suffices to consider the special case, when μ and λ are adjacent in the order $\leq$ (using the terminology of Kraft-Procesi, when $\mu < \lambda$ is a "minimal degeneration").

There are two possible types of pairs of adjacent partitions (see [K-P1, p. 229]):

$$\begin{array}{ll}
\text{(A)} & \lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l', \lambda_{k+2}, \ldots) \\
 & \mu = (\lambda_1, \ldots, \lambda_{k-1}, l-1, l'+1, \lambda_{k+2}, \ldots) \\
 & (l \geq l'+2) \\
\text{(B)} & \lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, \underbrace{l-1, \ldots, l-1}_{j-k-1}, l-2, \lambda_{j+1}, \ldots) \\
 & \mu = (\lambda_1, \ldots, \lambda_{k-1}, \underbrace{l-1, \ldots, l-1}_{j-k+1}, \lambda_{j+1}, \ldots)
\end{array}$$

Let $\mu < \lambda$ be one of the adjacent pairs listed above and let $i_0 = i_0(\lambda)$. Let k be as in (A) or (B) above. We will consider several cases depending on the relative position of k and i_0.

$i_0 < k$:
In this case, as $(\mu_1, \ldots, \mu_{k-1}) = (\lambda_1, \ldots, \lambda_{k-1})$, we have $i_0(\lambda) = i_0(\mu)$ and $\mu' = \lambda'$.

$i_0 \geq k$:
In this case $\lambda'_i = \lambda_i + 1 = \mu_i + 1 = \mu'_i$ for all $i < k$. It follows that we can assume $k = 1$.

Let $r = r_{i_0}(\lambda)$. If $r = 0$, then $\lambda' = (\lambda_1 + 1, \ldots, \lambda_{i_0-1} + 1) \geq (\mu_1 + 1, \ldots, \mu_{i_0-1} + 1, \ldots) \geq \mu'$.

In the following we assume $r > 0$. Let t denote the greatest index with the property $\mu_t \neq \lambda_t$ (i. e. $t = k+1 = 2$ in the case (A) and $t = j$ in the case (B)). If $i_0 > t$, then $i_0(\lambda) = i_0(\mu)$, $\lambda'_i = \lambda_i + 1$, $\mu'_i = \mu_i + 1$ for $i < i_0$, $\lambda'_i = \mu'_i$ for $i \geq i_0$ and we have $\mu' < \lambda'$ (it is an adjacent pair of the same type as $\mu < \lambda$). It remains to consider the case $1 \leq i_o \leq t$. Here is the detailed case by case analysis:

$$\begin{array}{llllll}
(A) & i_0 = 1 & \lambda' = (r), & \mu' = (r), & \mu' = \lambda', \\
 & i_0 = 2 & \lambda' = (l+1, r), & \mu' = (l, r+1), & \mu' < \lambda', \\
(B) & i_0 = 1 & \lambda' = (r), & \mu' = (r), & \mu' = \lambda', \\
 & 2 \leq i_0 \leq j & \lambda' = (l+1, l, \ldots, l, r), & \mu' = (l, \ldots, l, r+1), & \mu' < \lambda'.
\end{array}$$

This ends the proof of lemma 4.4 and theorem 4.2.

5 Pairs of type I

In this section we prove theorem 1.1 for pairs of type I. We use the following notation:
$W = \mathrm{Hom}(U, V)$, $\dim U = n$, $\dim V = m$.

5.1 The case $G = Sp(V)$, $G' = O(U)$

The main idea of the construction is completely analogous to the case of a dual pair of type II. Given a symplectic partition λ, we construct an orthogonal partition λ^o putting $\lambda^o_i = \lambda_i + 1$ as long as possible, but the final part of the construction is more delicate, as we want to get an orthogonal partition of n.

Definition 5.1 *Let λ be a symplectic partition of m. Define an orthogonal partition λ^o of n in the following way.*
Let $r_1 = n$. For $i \geq 2$ define $r_i = r_i(\lambda) = n - (\lambda_1 + 1 + \lambda_2 + 1 + \ldots + \lambda_{i-1} + 1)$ for $i \geq 1$. Let $i_0 = i_0(\lambda)$ be the smallest $i \geq 1$ such that $(\lambda_1, \ldots, \lambda_{i-1})$ is a symplectic partition and either $r_i \leq \lambda_i$ or λ_i is odd and $r_i \leq 2\lambda_i + 1$.
For $1 \leq i < i_0$ define $\lambda^o_i = \lambda_i + 1$.
Moreover:
(a) if $r_{i_0} = 0$ define $\lambda^o_{i_0} = 0$, and $\lambda^o_j = 0$ for $j > i_0$,
(b) if $0 < r_{i_0} \leq \lambda_{i_0}$ then:
(b1) if r_{i_0} is odd, define $\lambda^o_{i_0} = r_{i_0}$, and $\lambda^o_j = 0$ for $j > i_0$,
(b2) if r_{i_0} is even, define $\lambda^o_{i_0} = r_{i_0} - 1$, $\lambda^o_{i_0+1} = 1$, and $\lambda^o_j = 0$ for $j > i_0 + 1$,
(c) if $\lambda_{i_0} < r_{i_0} \leq 2\lambda_{i_0} + 1$ (so $\lambda_{i_o} = \lambda_{i_o+1}$ is odd), then:
(c1) if r_{i_0} is even, define $\lambda^o_{i_0} = \lambda_{i_0}$, $\lambda^o_{i_0+1} = r_{i_0} - \lambda_{i_0}$, and $\lambda^o_j = 0$ for $j \geq i_0 + 2$,
(c2) if r_{i_0} is odd, define $\lambda^o_{i_0} = \lambda_{i_0}$, $\lambda^o_{i_0+1} = r_{i_0} - \lambda_{i_0} - 1$, $\lambda^o_{i_0+2} = 1$, and $\lambda^o_j = 0$ for $j \geq i_0 + 3$.

Remark: It follows from the definition that λ^o is an orthogonal partition and that there exists an orthosymplectic ab-diagram δ such that $\rho(\delta) \leq \lambda$, $\pi(\delta) = \lambda^o$. We will prove below (lemma 5.3) that λ^o is the "largest" partition with these properties.

Theorem 5.2 *Let λ be a symplectic partition of $m = \dim V$. Then*

$$\pi(\rho^{-1}(\overline{\mathfrak{sp}_\lambda})) = \overline{\mathfrak{o}_{\lambda^o}}. \tag{8}$$

Proof: From the remark following definition 5.1 it follows that

$$\mathfrak{o}_{\lambda^o} \subseteq \pi(\rho^{-1}(\overline{\mathfrak{sp}_\lambda})).$$

As $\rho^{-1}(\overline{\mathfrak{sp}_\lambda})$ is $Sp(V)$-invariant, by lemma 3.7 the set $\pi(\rho^{-1}(\overline{\mathfrak{sp}_\lambda}))$ is closed and it follows that the relation $\supseteq$ holds in (8). It remains to prove the following claim:
Claim: Let μ be a symplectic partition of m such that $\mu \leq \lambda$, and let δ be an orthosymplectic ab-diagram such that $\rho(\delta) = \mu$. Then $\pi(\delta) \leq \lambda^o$.

The claim follows immediately from the following two lemmas:

Lemma 5.3 *In the situation of the claim, $\pi(\delta) \leq \mu^o$.*

Lemma 5.4 *If $\mu \leq \lambda$ are two symplectic partitions of m, then $\mu^o \leq \lambda^o$.*

Proof of lemma 5.3: Let $\pi(\delta) = \mu'$, $i_0 = i_0(\mu)$. For $i < i_0$ we have $\mu_i^o = \mu_i + 1 \geq \mu_i'$ (lemma 3.4), so $\mu_1' + \ldots + \mu_i' \leq \mu_1^o + \ldots + \mu_i^o$ for all $i < i_0$. If the length $l(\mu^o) \leq i_0$ (cases (a), (b1) of definition 5.1), then also $\mu_1' + \ldots + \mu_{i_0}' \leq \mu_1^o + \ldots + \mu_{i_0}^o = n$, so $\mu' \leq \mu^o$.

Consider now case (b2) of definition 5.1 (so $\mu_{i_0}^o = r_{i_0} - 1$, $\mu_{i_0+1}^o = 1$). In this case the only possibility for $\mu' \not\leq \mu^o$ is the length $l(\mu') = i_0$, but by lemma 2.1 this cannot happen (otherwise $i_0 \equiv i_0 + 1 \pmod 2$).

In case (c) we have $\mu_{i_0}^o = \mu_{i_0}$. As before $\mu_i^o = \mu_i + 1$ for $i < i_0$. If $\mu' \not\leq \mu^o$, then there are two possibilities: either

(α) $\qquad \mu_1' + \ldots + \mu_{i_0}' > \mu_1^o + \ldots + \mu_{i_0}^o,$

or

(β) $\qquad \mu_1' + \ldots + \mu_{i_0}' \leq \mu_1^o + \ldots + \mu_{i_0}^o,$ and
$\qquad \mu_1' + \ldots + \mu_{i_0+1}' > \mu_1^o + \ldots + \mu_{i_0+1}^o.$

As $\mu_{i_0}' \leq \mu_{i_0} + 1 = \mu_{i_0}^o + 1$, and $\mu_i' \leq \mu_i^o$ for $i < i_0$, (α) implies $\mu_{i_0}' = \mu_{i_0}^o + 1$ and $\mu_i' = \mu_i^o$ for $i < i_0$. The partition $(\mu_1^o, \ldots, \mu_{i_0-1}^o)$ is orthogonal, so $(\mu_1', \ldots, \mu_{i_0-1}')$ is orthogonal. This, together with the facts that μ' is orthogonal and that $\mu_{i_0}' = \mu_{i_0} + 1$ is even implies that $\mu_{i_0}' = \mu_{i_0+1}' = \mu_{i_0} + 1$, but this is not possible as $\mu_{i_0}' + \mu_{i_0+1}' \leq n - (\mu_1' + \ldots + \mu_{i_0-1}') = r_{i_0} < 2\mu_{i_0} + 1$.

It remains to consider case (β) which can happen only if μ^o is as in (c2), i. e. r_{i_0} is odd, $\mu_{i_0}^o = \mu_{i_0}$, $\mu_{i_0+1}^o = r_{i_0} - \mu_{i_0} - 1$, $\mu_{i_0+2}^o = 1$. It follows that the second inequality of (β) is possible only if the length $l(\mu') = i_0 + 1$, but by lemma 2.1 this cannot happen (otherwise $i_0 + 1 \equiv i_0 + 2 \pmod 2$).

Proof of lemma 5.4: Assume that $\mu \leq \lambda$ are two symplectic partitions. To prove that $\mu^o \leq \lambda^o$ it suffices to consider the special case, when μ and λ are adjacent in the order $\leq$ (using the terminology of Kraft-Procesi, when $\mu < \lambda$ is a "minimal degeneration").

The list of all possible adjacent pairs of symplectic partitions is as follows:

(A) $\qquad \lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l-2, \lambda_{k+2}, \ldots)$
$\qquad \mu = (\lambda_1, \ldots, \lambda_{k-1}, l-1, l-1, \lambda_{k+2}, \ldots)$

(l is even)

(B) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l', \lambda_{k+2}, \ldots)$

$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-2, l'+2, \lambda_{k+2}, \ldots)$

(both l, l' are even, $l \geq l'+4$)

(C) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l', l', \lambda_{k+3}, \ldots)$

$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-2, l'+1, l'+1, \lambda_{k+3}, \ldots)$

(l is even, $l \geq l'+3$)

(D) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l, l', \lambda_{k+3}, \ldots)$

$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-1, l-1, l'+2, \lambda_{k+3}, \ldots)$

(l' is even, $l \geq l'+3$)

(E) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l, l', l', \lambda_{k+3}, \ldots)$

$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-1, l-1, l'+1, l'+1, \lambda_{k+3}, \ldots)$

($l \geq l'+2$)

(F) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l, \underbrace{l-1, \ldots, l-1}_{j-k-2}, l-2, l-2, \lambda_{j+2}, \ldots)$

$\mu = (\lambda_1, \ldots, \lambda_{k-1}, \underbrace{l-1, \ldots, l-1}_{j-k+2}, \lambda_{j+2}, \ldots)$

(l is odd; even l, although possible, does not give a minimal degeneration)

(G) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, \underbrace{l-1, \ldots, l-1}_{j-k-1}, l-2, \lambda_{j+1}, \ldots)$

$\mu = (\lambda_1, \ldots, \lambda_{k-1}, \underbrace{l-1, \ldots, l-1}_{j-k+1}, \lambda_{j+1}, \ldots)$

(l is even, $j-k-1$ is even, and $j > k+1$)

The above list differs slightly from the one given in (Kraft-Procesi), as our case (F) contains their two cases (f) and (h).

Let $\mu < \lambda$ be one of the minimal degenerations listed above, and let $i_0 = i_0(\lambda)$. Let k be as in the classification of minimal degenerations, so it is the smallest index with the property $\lambda_k \neq \mu_k$. We will consider several cases depending on the relative position of k and i_0.

$\boldsymbol{i_0 < k}$**:**
In this case, as $(\mu_1, \ldots, \mu_{k-1}) = (\lambda_1, \ldots, \lambda_{k-1})$, we have $i_0(\lambda) = i_0(\mu)$ and $\mu^o = \lambda^o$.

$\boldsymbol{i_0 \geq k}$**:**
Notice that in all cases of minimal degenerations the partition $(\lambda_1, \ldots, \lambda_{k-1})$ is symplectic, so both $(\lambda_k, \lambda_{k+1}, \ldots)$ and $(\mu_k, \mu_{k+1}, \ldots)$ are symplectic as well. Also, $\lambda_i^o = \lambda_i + 1 = \mu_i + 1 = \mu_i^o$ for all $i < k$. It follows that from now on we can assume that $k = 1$.

Let $r = r_{i_0}(\lambda)$. If $r = 0$, then $\lambda^o = (\lambda_1 + 1, \ldots, \lambda_{i_0-1} + 1)$ and as $\mu < \lambda$, obviously $\mu^o \leq \lambda^o$ (as $\mu_i^o \leq \mu_i + 1$ for all i).

In the following we always assume $r > 0$.

If r is odd and $r \leq \lambda_{i_o}$, then $\lambda^o = (\lambda_1 + 1, \ldots, \lambda_{i_o-1} + 1, r)$, and obviously $\mu^o \leq \lambda^o$.

Let t denote the greatest index with the property $\mu_t \neq \lambda_t$ (i. e. $t = k+1 = 2$ in the cases (A) and (B), $t = k+2 = 3$ in the cases (C) and (D), $t = 4$ in the case (E), $t = j$ in the case (G), and $t = j+1$ in the case (F)). If $i_0 > t$, then $i_0(\lambda) = i_0(\mu)$, $\lambda_i^o = \lambda_i + 1$, $\mu_i^o = \mu_i + 1$ for $i < i_0$, $\lambda_i^o = \mu_i^o$ for $i \geq i_0$ and we have $\mu^o < \lambda^o$.

(*) It follows from the above that it remains to consider the case $1 \leq i_0 \leq t$, and either $r = r_{i_0}(\lambda) > 0$ even, or r odd and $\lambda_{i_0} < r \leq 2\lambda i_0 + 1$ (in the last case $\lambda_{i_o} = \lambda_{i_o+1}$ is odd).

$i_0 = 1$, r even:
In this case either $\lambda^o = (r-1, 1)$ (if $r \leq \lambda_{i_0}$), or $\lambda^o = (\lambda_{i_0}, r - \lambda_{i_0})$ (if $\lambda_{i_0} < r$).
In the first case the only possibility for $\mu^o \not\leq \lambda^o$ is $\mu^o = (r)$, but this is not possible, as r is even, and (r) is not orthogonal.
In the second case the only possibility for $\mu^o \not\leq \lambda^o$ is $\mu_1^o > \lambda_1$, but $\mu_1^o \leq \mu_1 + 1 < \lambda_1 + 1$, so $\mu_1^o \leq \lambda_1$.

We see that in both cases $\mu^o \leq \lambda^o$.

$i_0 = 1$, r odd:
In this case $\lambda_{i_0} < r \leq 2\lambda_{i_0} + 1$, so $\lambda^o = (\lambda_1, r - \lambda_1 - 1, 1)$. As $\mu_1 \leq \lambda_1$, the only possibility for $\mu^o \not\leq \lambda^o$ is $\mu^o = (\mu_1^o, \mu_2^o)$. But $\mu_1^o + \mu_2^o = r$, and as r is odd, μ^o cannot be orthogonal.

$i_0 = 2$: Let $r = r_2(\lambda)$.
(A)
Assume that the degeneration $\mu < \lambda$ is of type (A). As $\lambda_2 = l - 2$ is even, the only remaining case is that of even r. In that case $\lambda^o = (l+1, r-1, 1)$. As $\mu_1^o \leq \lambda_1 = l$, the only possibility for $\mu^o \not\leq \lambda^o$ is $l(\mu^o) = 2$, but then $\mu_1^o + \mu_2^o = l + r + 1$ is odd, so μ^o cannot be orthogonal.
(B):
In the case of a degeneration of type (B), as $\lambda_2 = l'$ is even, the only case to consider is that of even r. In this case $\lambda^o = (l+1, r-1, 1)$ and as $\mu_1^o \leq \mu_1 + 1 = l - 1$, we conclude that $\mu^o \leq \lambda^o$.
(C):
The same argument as above works in case (C) if $\lambda_2 = l'$ is even, so it remains to consider the case of odd l'. We have three possibilities:

1. $r \leq \lambda_2$ and r is even.
   In this case $\lambda^o = (l + 1.r - 1, 1)$ and the same argument as in the case (B) works here,

2. $\lambda_2 < r \leq 2\lambda_2 + 1$ and r is even,
   In this case $\lambda^o = (l+1, l', r-l')$. As $\mu_1^o \leq l-1$ and $\mu_2^o \leq \mu_2 + 1 = l' + 2$, we have $\mu^o < \lambda^o$.

3. $\lambda_2 < r \leq 2\lambda_2 + 1$ and r is odd.
   In this case $\lambda^o = (l+1, l', r - l' - 1, 1)$. As $\mu^o = (l-1, \mu_2^o, \mu_3^o, \ldots)$ ($\mu_1^o = l - 1$ since $\mu_1 = l - 2$ is even, and $r_1(\mu) \geq \mu_1 + 1$), the only possibility for $\mu^o \not\leq \lambda^o$ is $l(\mu^o) \leq 3$. But then $\mu_2^o + \mu_3^o = r + 2$, and (as $r + 2$ is odd) μ_2^o must be even and at the same time equal to $l - 1$. This is not possible, since l is even. Hence $\mu^o < \lambda^o$ in this case.

(D), (E):
In the cases (D) and (E) l must be even (if l is odd, $i_0(\lambda)$ cannot be equal to 2), so the only

remaining case is that of even r. But then $\lambda^o = (l+1, r-1, 1)$ and as in the case (A) (with $i_0 = 2$) we get $\mu^o \leq \lambda^o$.

(F):

This case cannot occur, as $\lambda_1 = \lambda_2$ is odd, so $i_0(\lambda) \neq 2$.

(G):

In this case $\lambda_2 = \lambda_3 = l - 1$ is odd, so we have the following possibilities:

1. $r \leq \lambda_2$ and r is even.
   In this case $\lambda^o = (l+1, r-1, 1)$, and as $\mu_1^o \leq l$, we can reason as in the case (A).

2. $\lambda_2 < r \leq 2\lambda_2 + 1$ and r is even.
   In this case $\lambda^o = (l+1, l-1, r-l+1)$, and as $\mu_1^o \leq l$, and $\mu_2^o \leq l$, we have $\mu^o < \lambda^o$.

3. $\lambda_2 < r \leq 2\lambda_2 + 1$ and r is odd.
   In this case $\lambda^o = (l+1, l-1, r-l, 1)$ and the same reasoning as in the case 3. of case (C) gives the result.

This ends the case $i_0(\lambda) = 2$.

$i_0(\lambda) \geq 3$:

Here the cases to consider are (C), (D), (E), (F) and (G).

$i_0 = 3$, (C), (D):

Let $r = r_3(\lambda)$.

In the case (C) $i_0(\lambda) = 3$ can occur only if $\lambda_3 = l'$ is even. By definition the same is true in the case (D), so we consider both cases together. By **(*)** the only remaining case is that of even r. Hence $\lambda^o = (\lambda_1 + 1, \lambda_2 + 1, r-1, 1)$, and as $\mu_1^o < \lambda_1^o$ and $\mu_1^o + \mu_2^o \leq \lambda_1^o + \lambda_2^o$, the only possibility for $\mu^o \not\leq \lambda^o$ is $l(\mu^o) = 3$.

In the case (C) we have $\mu^o = (l-1, l'+2, \mu_3^o)$, and as $\mu_1^o + \mu_2^o + \mu_3^o = \lambda_1^o + \lambda_2^o + \lambda_3^o + \lambda_4^o = l + 1 + l' + 1 + r$, we have $\mu_3^o = r + 1$, so the only even part of μ^o is equal to $l' + 2$, but this is not possible as μ^o is orthogonal.

In the case (D) we have $\mu^o = (l, l, \mu_3^o)$, and as $\mu_1^o + \mu_2^o + \mu_3^o = l + 1 + l + 1 + r$ we get $\mu_3^o = r + 2$ and this is not possible as r is even.

$i_0 = 3$, (E):

We have to consider the cases of l' even or odd separately. Let, as before, $r = r_3(\lambda)$.

l' even:

By **(*)** the only case to consider is that of even r. In this case $\lambda^o = (l+1, l+1, r-1, 1)$ and the only possibility for $\mu^o \not\leq \lambda^o$ is $\mu^o = (l, l, r+2)$, but this is not possible as $r + 2$ is even.

l' odd:

We have three subcases:

1. $r \leq \lambda_3 = l'$ and r is even.
   Here $\lambda^o = (l+1, l+1, r-1, 1)$ and we continue as in the case of even l'.

2. $\lambda_3 < r \leq 2\lambda_3 + 1$ and r is even.
   In this case $\lambda^o = (l+1, l+1, l', r-l')$. Since $\mu_1^o = l$, $\mu_2^o = l$ and $\mu_3^o \leq l' + 2$, we have $\mu^o < \lambda^o$.

3. $\lambda_3 < r \leq 2\lambda_3 + 1$ and r is odd.
   Here $\lambda^o = (l+1, l+1, l', r-l'-1, 1)$. Now, as $\mu_1^o = l$, $\mu_2^o = l$ and $\mu_3^o \leq l'+2$, it could happen that $\mu^o \not\leq \lambda^o$ only if $\mu^o = (l, l, \mu_3^o, \mu_4^o)$, but in that case $\mu_3^o + \mu_4^o = r+2$ and this is not possible as $r+2$ is odd and μ^o is orthogonal.

$\boldsymbol{i_0 \geq 3}$**:**
We are left with the cases (F) and (G) with $i_0(\lambda) \geq 3$.

(F), $\boldsymbol{3 \leq i_0(\lambda) < j}$**:**
As $\lambda_{i_0} = l-1$ is even, the only interesting case is that of even r. In that case $\lambda^o = (l+1, l+1, l, \ldots, l, r-1, 1)$, and as $\mu_p = l$ for $1 \leq p \leq i_0 - 1$, we could have $\mu^o \not\leq \lambda^o$ only if $\mu^o = (l, l, \ldots, l, r+2)$, but this is not possible since $r+2$ is even.
(F), $\boldsymbol{i_0(\lambda) = j}$**:**
As $\lambda_j = l-2$ is odd, we have three cases to consider.

1. $r = r_j(\lambda) \leq \lambda_j$ and r is even.
   Here $\lambda^o = (l+1, l+1, l, \ldots, l, r-1, 1)$ and, as before, $\mu^o \not\leq \lambda^o$ could happen only if $\mu^o = (l, l, \ldots, l, r+2)$, but this is not possible since $r+2$ is even.

2. $\lambda_j < r \leq 2\lambda_j + 1$ and r is even.
   Here $\lambda^o = (l+1, l+1, l, \ldots, l, l-2, r-l+2)$ and we have $\mu^o \leq \lambda^o$ as $\mu_p^o \leq l$ for all p.

3. $\lambda_j < r \leq 2\lambda_j + 1$ and r is odd.
   Here $\lambda^o = (l+1, l+1, l, \ldots, l, l-2, r-l+1, 1)$, and as $\mu_1^o = \ldots = \mu_{j-1}^o = l$, we could get $\mu^o \not\leq \lambda^o$ only if $\mu^o = (l, \ldots, l, \mu_j^o, \mu_{j+1}^o)$, but then $\mu_j^o + \mu_{j+1}^o = r+2$ and only one part of μ^o would be even, which would contradict the orthogonality of μ^o.

(F), $\boldsymbol{i_o(\lambda) = j+1}$**:**
This case cannot occur as $\lambda_j = \lambda_{j+1}$ are odd.

(G), $\boldsymbol{3 \leq i_0(\lambda) < j}$**:**
The same argument as for (G), $i_0 = 2$, works in this case.
(G), $\boldsymbol{i_0 = j}$**:**
The only case to consider is that of even $r = r_j(\lambda)$. Then $\lambda^o = (l+1, l, \ldots, l, r-1, 1)$. We could get $\mu^o \not\leq \lambda^o$ only if $\mu^o = (l, l, \ldots, l, r+1)$ but this cannot happen as μ^o would have an odd number of even l's, which would contradict the orthogonality of μ^o.

This was the last case to consider and the lemma 5.4 is proved. This also ends the proof of the theorem 5.2.

5.2 The case $G = O(U)$, $G' = Sp(V)$

The main idea of the construction is completely analogous to the previously considered cases. The technical problems are fortunately a little bit simpler than in the proof of theorem 5.2.

Definition 5.5 *Let λ be an orthogonal partition of n. Define a symplectic partition λ^s of m in the following way.*

Let $r_1 = m$. For $i \geq 2$ let $r_i = r_i(\lambda) = m - (\lambda_1 + 1 + \lambda_2 + 1 + \ldots + \lambda_{i-1} + 1)$. Let $i_0 = i_0(\lambda)$ be the smallest $i \geq 1$ such that $(\lambda_1, \ldots, \lambda_{i-1})$ is an orthogonal partition and either $r_i \leq \lambda_i$ or λ_i is even and $\lambda_i < r_i \leq 2\lambda_i + 1$.
For $1 \leq i < i_0$ define $\lambda_i^s = \lambda_i + 1$.
Moreover:
(a) if $r_{i_0} = 0$ define $\lambda_j^s = 0$ for $j \geq i_0$,
(b) if $0 < r_{i_0} \leq \lambda_{i_0}$ then define $\lambda_{i_0}^s = r_{i_0}$, and $\lambda_j^s = 0$ for $j > i_0$,
(c) if $\lambda_{i_0} < r_{i_0} \leq 2\lambda_{i_0} + 1$ (so $\lambda_{i_o} = \lambda_{i_o+1}$ is even), then define $\lambda_{i_0}^s = \lambda_{i_0}$, $\lambda_{i_0+1}^s = r_{i_0} - \lambda_{i_0}$, and $\lambda_j^s = 0$ for $j \geq i_0 + 2$,

Remark: Immediately from the definition it follows that λ^s is a symplectic partition (note that by lemma 2.1 the number r_{i_0} is even!), and that there exists an orthosymplectic ab-diagram δ such that $\rho(\delta) = \lambda^s$, $\pi(\delta) \leq \lambda$. We will show later that λ^s is the largest partition with these properties.

Theorem 5.6 *Let λ be an orthogonal partition of $n = \dim U$. Then*

$$\rho(\pi^{-1}(\overline{\mathfrak{o}_\lambda})) = \overline{\mathfrak{sp}_{\lambda^s}}. \tag{9}$$

Proof: From the remark following definition 5.5 it follows that

$$\mathfrak{sp}_{\lambda^s} \subseteq \rho(\pi^{-1}(\overline{\mathfrak{o}_\lambda})).$$

As $\pi^{-1}(\overline{\mathfrak{o}_\lambda})$ is $O(U)$-invariant, by lemma 3.7 the set $\rho(\pi^{-1}(\overline{\mathfrak{o}_\lambda}))$ is closed and it follows that the relation $\supseteq$ holds in 9. It remains to prove the following claim:

Claim: Let μ be an orthogonal partition of n such that $\mu \leq \lambda$, and let δ be an orthosymplectic ab-diagram such that $\pi(\delta) = \mu$. Then $\rho(\delta) \leq \lambda^s$.

The claim follows immediately from the following two lemmas:

Lemma 5.7 *In the situation of the claim, $\rho(\delta) \leq \mu^s$.*

Lemma 5.8 *If $\mu \leq \lambda$ are two orthogonal partitions of n, then $\mu^s \leq \lambda^s$.*

Proof of lemma 5.7:
Let $\rho(\delta) = \mu'$. As by (3.4) $\mu'_j \leq \mu_j + 1 = \mu_j^s$ for all $j < i_0$, we have $\mu'_1 + \ldots + \mu'_i \leq \mu_1^s + \ldots + \mu_i^s$ for all $i < i_0$. If $l(\mu^s) = i_0$, also $\mu'_1 + \ldots + \mu'_{i_0} \leq m = \mu_1^s + \ldots + \mu_{i_0}^s$, so the only case that remains is case (c) of the definition of μ^s, when $l(\mu^s) = i_0 + 1$. But in that case the only possibility for $\mu' \not\leq \mu^s$ is $\mu'_1 + \ldots + \mu'_{i_0} > \mu_1^s + \ldots + \mu_{i_0}^s$. This can happen only if $\mu'_i = \mu_i + 1$ for all $i \leq i_0$, but then for μ' to be symplectic, μ'_{i_0} has to be equal to μ'_{i_0+1} (as μ_{i_0} is even) and then $\mu'_1 + \ldots + \mu'_{i_0+1} > m$ gives a contradiction.

Proof of lemma 5.8:

As before it suffices to consider pairs of adjacent orthogonal partitions. The list of all possible adjacent pairs of symplectic partitions is as follows:

(A) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l-2, \lambda_{k+2}, \ldots)$
$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-1, l-1, \lambda_{k+2}, \ldots)$
(l is odd)

(B) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l', \lambda_{k+2}, \ldots)$
$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-2, l'+2, \lambda_{k+2}, \ldots)$
(both l, l' are odd, $l \geq l'+4$)

(C) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l', l', \lambda_{k+3}, \ldots)$
$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-2, l'+1, l'+1, \lambda_{k+3}, \ldots)$
(l is odd, $l \geq l'+3$)

(D) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l, l', \lambda_{k+3}, \ldots)$
$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-1, l-1, l'+2, \lambda_{k+3}, \ldots)$
(l' is odd, $l \geq l'+3$)

(E) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l, l', l', \lambda_{k+4}, \ldots)$
$\mu = (\lambda_1, \ldots, \lambda_{k-1}, l-1, l-1, l'+1, l'+1, \lambda_{k+4}, \ldots)$
($l \geq l'+2$)

(F) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, l, \underbrace{l-1, \ldots, l-1}_{j-k-2}, l-2, l-2, \lambda_{j+2}, \ldots)$
$\mu = (\lambda_1, \ldots, \lambda_{k-1}, \underbrace{l-1, \ldots, l-1}_{j-k+2}, \lambda_{j+2}, \ldots)$
(l is even; odd l, although possible, does not give a minimal degeneration)

(G) $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, l, \underbrace{l-1, \ldots, l-1}_{j-k-1}, l-2, \lambda_{j+1}, \ldots)$
$\mu = (\lambda_1, \ldots, \lambda_{k-1}, \underbrace{l-1, \ldots, l-1}_{j-k+1}, \lambda_{j+1}, \ldots)$
(l is odd, $j-k-1$ is even, and $j > k+1$)

Let $\mu < \lambda$ be one of the adjacent pairs listed above and let $i_0 = i_0(\lambda)$.

$\boldsymbol{i_0 < k}$**:**
In this case, as $(\mu_1, \ldots, \mu_{k-1}) = (\lambda_1, \ldots, \lambda_{k-1})$, we have $i_0(\lambda) = i_0(\mu)$ and $\mu^s = \lambda^s$.

$\boldsymbol{i_0 \geq k}$**:**
Notice that in all cases of minimal degenerations the partition $(\lambda_1, \ldots, \lambda_{k-1})$ is orthogonal, so both $(\lambda_k, \lambda_{k+1}, \ldots)$ and $(\mu_k, \mu_{k+1}, \ldots)$ are orthogonal as well. Also, $\lambda_i^s = \lambda_i + 1 = \mu_i + 1 = \mu_i^s$ for all $i < k$. It follows that from now on we can assume that $k = 1$.

Let $r = r_{i_0}(\lambda)$. If $r = 0$ (so λ^s is as in definition 5.5(a)), then $\lambda^s = (\lambda_1 + 1, \ldots, \lambda_{i_0-1} + 1)$ and as $\mu < \lambda$, obviously $\mu^s \leq \lambda^s$ (as $\mu_i^s \leq \mu_i + 1$ for all i). In the following we always assume

$r > 0$.

If λ^s is as in definition 5.5(b), $\lambda^s = (\lambda_1 + 1, \dots, \lambda_{i_0-1} + 1, r)$, then from $\mu_i^s \leq \mu_i + 1$ and $\mu \leq \lambda$ we immediately get $\mu^s \leq \lambda^s$, so the only case that remains is λ^s as in definition 5.5(c), i.e.

$$\lambda^s = (\lambda_1 + 1, \dots, \lambda_{i_0-1} + 1, \lambda_{i_0}, r - \lambda_{i_0}),$$

with both r and λ_{i_0} even.

The same argument as above shows that $\mu_1^s + \ldots + \mu_i^s \leq \lambda_1^s + \ldots + \lambda_i^s$ for all $i \neq i_0$, so it remains to show that

$$\mu_1^s + \ldots + \mu_{i_0}^s \leq \lambda_1^s + \ldots + \lambda_{i_0}^s. \tag{10}$$

We will show it by a case by case analysis similar to those in previous proofs.

Let t be the greatest number such that $\mu_t \neq \lambda_t$, i.e. $t = k + 1 = 2$ in cases (A) and (B), $t = k + 2 = 3$ in cases (C) and (D), $t = 4$ in case (E), $t = j$ in case (G) and $t = j + 1$ in case (F).

As in previous proofs, if $i_0 > t$ then the inequality (10) is immediate, so it remains to consider cases $1 \leq i_0 \leq t$.

$i_0 = 1$:
In this case $\lambda^s = (\lambda_1, r - \lambda_1)$, so it is enough to prove that $\mu_1^s \leq \lambda_1$. But this is clear, as in all cases $\mu_1 < \lambda_1$.

$i_0 = 2$:
(A):
This case cannot occur, as $\lambda_{i_0} = l - 2$ should be even for (c), but it should be odd for (A).
(B):
This case canot occur as $\lambda_{i_0} = l'$ should be even for (c), but odd for (B).
(C):
$\lambda = (l, l', l', \ldots)$,
$\mu = (l - 2, l' + 1, l' + 1, \ldots)$,
l is odd, $l' = \lambda_{i_0}$ is even, $r > l'$ implies $r \geq l' + 2$ (both are even), so
$\lambda^s = (l + 1, l', r - l')$,
$\mu^s = (l - 1, l' + 2, r - l' - 2)$, and (10) holds in this case.
(D), (E), (F):
These cases cannot occur as on the one hand in all these cases $\lambda_{i_0} > \lambda_{i_0+1}$, on the other hand (c) requires $\lambda_{i_0} = \lambda_{i_0+1}$.
(G):
$\lambda = (l, l - 1, l - 1, \ldots)$,
$\mu = (l - 1, l - 1, l - 1, \ldots)$, so
$\lambda^s = (l + 1, l - 1, r - l + 1)$, $\mu^s = (l, l, r - l + 1)$, and (10) holds in this case.

$i_0 = 3$:
Here the only cases to consider are (C), (D), (E), (F) and (G) (otherwise $i_0 > t$).
(C):
This case cannot occur as (C) requires $\lambda = (l, l', l', \ldots)$ with l odd and l' even (see the paragraph preceeding (10)), while definition 5.5(c) requires the partition (λ_1, λ_2) to be orthogonal.

(D):
This case cannot occur as (D) requires $\lambda = (l, l, l', \ldots)$ with l' odd, and definition 5.5(c) requires $\lambda_{i_0} = l'$ to be even.
(E):
$\lambda = (l, l, l', l', \ldots)$,
$\mu = (l-1, l-1, l'+1, l'+1, \ldots)$, $l \geq l'+2$,
so
$\lambda^s = (l+1, l+1, l', r-l')$,
$\mu^s = (l, l, l'+2, r-l'-2)$ and (10) holds in this case.
(F):
This case cannot occur as (F) requires $\lambda = (l, l, l-1, \ldots)$ with even l, while for 5.5(c) $l-1$ should be even.
(G):
This case cannot occur as (G) requires $\lambda = (l, l-1, \ldots)$, while for 5.5(c) the partition $(l, l-1)$ should be orthogonal.

This ends the study of the case $i_0 = 3$. The only cases that remain are $i_0 = 4$ for a pair of type (E), and $i_0 \geq 4$ for pairs of types (F) and (G).

$\boldsymbol{i_0 = 4}$**; (E):**
This case cannot occur, as (E) requires $\lambda = (l, l, l', l', \ldots)$ with $l > l'$, and now by 5.5(c) the partition (l, l, l') should be orthogonal, so l' must be odd, and at the same time $\lambda_{i_0} = l'$ should be even.

$\boldsymbol{4 \leq i_0 < j}$**:**
(F):
This case cannot occur as (F) requires even l, while for 5.5(c) $\lambda_{i_0} = l-1$ should be even.
(G):
$\lambda = (l, l-1, \ldots, \lambda_{i_0} = l-1, l-1, \ldots, l-1, \lambda_j = l-2, \ldots)$,
$\mu = (l-1, \ldots, l-1, \mu_j = l-1, \ldots)$,
so
$\lambda^s = (l+1, l, \ldots, l, \lambda^s_{i_0} = l-1, r-l+1)$,
$\mu^s = (l, l, \ldots, l, \mu^s_{i_0} = l, r-l+1)$, and (10) holds in this case.

$\boldsymbol{i_0 = j}$**:**
(F):
$\lambda = (l, l, l-1, \ldots, l-1, \lambda_{i_0} = l-2, l-2, \ldots)$,
$\mu = (l-1, l-1, \ldots, \mu_{i_0} = l-1, \ldots)$,
so
$\lambda^s = (l+1, l+1, l, \ldots, l, \lambda^s_{i_0} = l-2, r-l+2)$,
$\mu^s = (l, l, \ldots, \mu^s_{i_0} = l, r-l+2)$,, and (10) holds in this case.
(G):
This case cannot occur as (G) requires l to be odd, while for 5.5(c) $\lambda_{i_0} = l-2$ should be even.

$i_0 = j + 1$; (F):
This case cannot occur as (F) requires even l, while for 5.5(c) the partition $(l, l, l-1, \ldots, l-1, l-2)$ should be orthogonal, so $l-2$ should be odd.

This was the last case to consider and this ends the proof of lemma 5.8 and theorem 5.6.

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