DUAL PAIRS AND KOSTANT–SEKIGUCHI CORRESPONDENCE. I

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1. Introduction

Let (G, G') be an irreducible real reductive dual pair in $Sp(W)$, where W is an appropriate symplectic vector space (see [H1],[H2]). Let $\mathfrak{g}, \mathfrak{g}' \subseteq \mathfrak{sp}(W)$ be the Lie algebras of G and G'. Consider the moment maps

$$\tau : W \longrightarrow \mathfrak{g}^*, \quad \tau' : W \longrightarrow \mathfrak{g}'^* \tag{1}$$

defined by the formula

$$\tau(w)(x) = \langle x(w), w\rangle, \quad w \in W,\ x \in \mathfrak{g} \subseteq \mathrm{End}(W),$$

and similarly for τ'. It is easy to see that for a nilpotent coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ the set $T(\mathcal{O}) \subseteq \mathfrak{g}'^*$ defined by

$$T(\mathcal{O}) = \tau'(\tau^{-1}(\overline{\mathcal{O}}))$$

is a union of nilpotent coadjoint orbits in $\mathfrak{g}'^*$.

Problem 1.1. *Understand the structure of the set $T(\mathcal{O})$ for general reductive dual pairs.*

Our interest in the closures of orbits, rather than orbits themselves, is motivated by the fact that the wave front set of a representation of a real reductive group is a closed union of nilpotent orbits ([H3][R]). As conjectured by Howe, the map T of problem (1.1) is expected to be compatible with the correspondence of the wave front sets of representations of $\widetilde{G}, \widetilde{G'}$ (the appropriate double covers of G, G') under Howe's correspondence. For more details on Howe's conjecture the reader may consult [DP1] and [DP2].

As $\mathfrak{g}$ and $\mathfrak{g}'$ are both reductive, we will identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{g}'$ with $\mathfrak{g}'^*$. The choice of scalar factors in these identifications does not have any influence on our results as nilpotent orbits are invariant under multiplication by positive scalars. From now on all moment maps considered in this paper will have targets in Lie algebras, not in their duals.

As the first result of this paper we show (theorem 2.4) that for dual pairs of type II (that is for G, G' being general linear groups over a division algebra $\mathbb{D}$ over $\mathbb{R}$) the set $T(\mathcal{O})$ is always the closure of a single nilpotent orbit $\mathcal{O}'$. The same is true for complex groups, as was shown in [DKP1], and for dual pairs of type I in the stable range (see [DKP2]). On the other hand, there are simple examples of dual pairs of type I (e.g. $(O_{1,1}(\mathbb{R}), Sp_2(\mathbb{R}))$) such that $T(\mathcal{O})$ is not always the closure of a single nilpotent orbit (take $\mathcal{O} = \{0\} \subseteq \mathfrak{o}_{1,1}$

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- in this case $T(\mathcal{O})$ is equal to the whole nilpotent cone in $\mathfrak{sp}_2$)). In general, Problem 1.1 remains open.

In this paper we propose a method of studying problem (1.1) for a general dual pair by relating it with a similar problem (1.3) for nilpotent orbits in vector spaces associated with certain symmetric spaces (in the sense of Kostant and Rallis [KR] and Sekiguchi [S2]).

Let J be a positive definite, compatible complex structure on W ($J \in Sp(W)$, $J^2 = -1_W$ and the bilinear form $\langle J\cdot\,,\,\cdot\rangle$ is positive definite), such that the conjugation by J preserves G and G'. Then the conjugation by J defines a Cartan involution on $Sp(W)$, G and G'. We have Cartan decompositions $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$, with maximal compact subgroups $K \subseteq G$, $K' \subseteq G'$. Let $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}'_{\mathbb{C}}$, $\mathfrak{p}_{\mathbb{C}}$ and $\mathfrak{p}'_{\mathbb{C}}$ denote the complexifications of $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{p}$ and $\mathfrak{p}'$ respectively. Let $W_{\mathbb{C}}$ be the complexification of W. Let $W_{\mathbb{C}}^+$ be the $+i$-eigenspace of J on $W_{\mathbb{C}}$. The moment maps $\tau_c : W_{\mathbb{C}} \longrightarrow \mathfrak{g}_{\mathbb{C}}$, $\tau'_c : W_{\mathbb{C}} \longrightarrow \mathfrak{g}'_{\mathbb{C}}$ map $W_{\mathbb{C}}^+$ to $\mathfrak{p}_{\mathbb{C}}$, $\mathfrak{p}'_{\mathbb{C}}$ respectively, so we have the following pair of diagrams.

$$\mathfrak{g}' \xleftarrow{\tau'} W \xrightarrow{\tau} \mathfrak{g} \tag{2}$$

$$\mathfrak{p}'_{\mathbb{C}} \xleftarrow{\tau'_c} W_{\mathbb{C}}^+ \xrightarrow{\tau_c} \mathfrak{p}_{\mathbb{C}}\ . \tag{3}$$

Let $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$, $K'_{\mathbb{C}} \subseteq G'_{\mathbb{C}}$ be the complexifications of the maximal compact subgroups $K \subseteq G$, $K' \subseteq G'$. Then $K_{\mathbb{C}}$ acts via the adjoint action on $\mathfrak{p}_{\mathbb{C}}$, similarly $K'_{\mathbb{C}}$ acts on $\mathfrak{p}'_{\mathbb{C}}$. The Kostant – Sekiguchi correspondence (see [S2], [C-M], [Vo]) is a bijection

$$S : \mathcal{N}_{\mathfrak{g}}/G \longrightarrow \mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/K_{\mathbb{C}}$$

between the sets $N_{\mathfrak{g}}/G$ of nilpotent G-orbits in $\mathfrak{g}$ and $N_{\mathfrak{p}_{\mathbb{C}}}/K_{\mathbb{C}}$ of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ (here $N_{\mathfrak{g}}/G$ and $N_{\mathfrak{p}_{\mathbb{C}}}/K_{\mathbb{C}}$ are sets of orbits, with no additional structure). Similarly, we have a bijection

$$S' : \mathcal{N}_{\mathfrak{g}'}/G' \longrightarrow \mathcal{N}_{\mathfrak{p}'_{\mathbb{C}}}/K'_{\mathbb{C}}\ .$$

If $X = \bigcup \mathcal{O}_i$ is a union of nilpotent G-orbits in $\mathfrak{g}$, we will denote by $S(X)$ the union $S(X) = \bigcup S(\mathcal{O}) \subseteq \mathfrak{p}_{\mathbb{C}}$ of corresponding nilpotent $K_{\mathbb{C}}$-orbits. Similar notation will be used in the context of $\mathfrak{g}'$.

Conjecture 1.2. *Let $\mathcal{O} \subseteq \mathfrak{g}$ be a nilpotent G-orbit. Then*

$$S'(\tau'(\tau^{-1}(\overline{\mathcal{O}}))) = \tau'_c(\tau_c^{-1}(\overline{S(\mathcal{O})})),$$

so in this sense the correspondence of the closures of orbits via the moment maps is compatible with the Kostant – Sekiguchi correspondence.

The conjecture is motivated in part by the following observation: for the dual pair $(Sp(W), O_1)$ and the minimal (nonzero) nilpotent orbit $\mathcal{O} \subseteq \mathfrak{sp}(W)$, the inverse image $\tau^{-1}(\mathcal{O}) = W \setminus 0$ mapped to $W_{\mathbb{C}}^+$ by $p = (1 - iJ)/2$ and then to $\mathfrak{p}_{\mathbb{C}}$ via τ_c gives the orbit $S(\mathcal{O})$. If this conjecture is true, then instead of studying problem (1.1), it would be enough to study the following problem.

Problem 1.3. *For a given nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}_c \subseteq \mathfrak{p}_{\mathbb{C}}$ understand the set $T_c(\mathcal{O}_c) := \tau_c'(\tau_c^{-1}(\overline{\mathcal{O}_c}))$.*

In this paper we prove conjecture (1.2) for pairs of type II (corollaries (3.8) and (4.10)). Unfortunately our proof does not show what really is going on, as it is not a direct proof. In fact, we solve problems (1.1) and (1.3) (proving that each of $T(\mathcal{O})$ and $T_c(\mathcal{O}_c)$ is the closure of a single nilpotent orbit), and conjecture (1.2) follows from the combinatorics involved in these solutions. The main technical result is the classification of nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $W_{\mathbb{C}}^+$ for pairs of type II, in terms of certain ab-diagrams (theorems (3.6) and (4.5)). We believe it should be possible to give a direct, intrinsic proof of conjecture (1.2) for all dual pairs, thus reducing problem (1.1) to the apparently simpler complex problem (1.3).

The methods we use to solve problem (1.3) for groups of type II are similar to those used in our paper [DKP1] and are based on some of the results and methods of Ohta ([O1]).

In our forthcoming paper [DKP2] we give a combinatorial classification of nilpotent $G \times G'$ orbits in W and $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $W_{\mathbb{C}}^+$ for all dual pairs, and for dual pairs in the stable range we solve problems 1.1 and 1.3 and verify conjecture 1.2.

In this paper we will freely use notation and terminology of [DKP1]. In particular, recall that a nilpotent element of W is by definition an element whose images via moment maps τ, τ' are nilpotent in $\mathfrak{g}$ and $\mathfrak{g}'$. The reader should also be warned that we frequently identify linear maps with their matrices in appropriate bases.

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2. Correspondence of G-orbits

Let $\mathbb{D}$ be a finite dimensional division algebra over $\mathbb{R}$, i.e. $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let V, V' be finite dimensional (right, if $\mathbb{D} = \mathbb{H}$) vector spaces over $\mathbb{D}$ and let $\dim V = n$, $\dim V' = m$. Let $G = GL_{\mathbb{D}}(V)$, $G' = GL_{\mathbb{D}}(V')$. Let $W = \mathrm{Hom}_{\mathbb{D}}(V, V') \oplus \mathrm{Hom}_{\mathbb{D}}(V', V)$. Let $\langle\ ,\ \rangle$ be the skew–symmetric form on W defined by

$$\langle (S_1, T_1), (S_2, T_2) \rangle = \mathrm{Re}\,\mathrm{Tr}(S_2 T_1 - S_1 T_2). \tag{4}$$

Let $Sp(W)$ be the isometry group of this form. Then there exist natural embeddings $G \hookrightarrow Sp(W)$, $G' \hookrightarrow Sp(W)$, and (G, G') is an irreducible dual pair in $Sp(W)$ of type II.

Let $\mathfrak{g} = \mathfrak{gl}_{\mathbb{D}}(V)$, $\mathfrak{g}' = \mathfrak{gl}_{\mathbb{D}}(V')$ and let τ, τ' be the moment maps

$$\tau : W \longrightarrow \mathfrak{g}, \qquad \tau(S, T) = T \cdot S,$$

$$\tau' : W \longrightarrow \mathfrak{g}', \qquad \tau(S, T) = S \cdot T.$$

Recall that nilpotent adjoint orbits in $\mathfrak{gl}_{\mathbb{D}}(V)$ are parametrized by partitions of $n = \dim V$, similarly nilpotent adjoint orbits in $\mathfrak{gl}_{\mathbb{D}}(V')$ are parametrized by partitions of $m = \dim V'$ ([C-M]).

An element $X = (S, T) \in W$ is called nilpotent if $T \cdot S$ is a nilpotent element of $\mathfrak{g}$ (equivalently if $S \cdot T$ is nilpotent in $\mathfrak{g}'$). The group $G \times G'$ acts on the set $\mathcal{N}_W$ of nilpotent elements of W. The classification of $G \times G'$–orbits in $\mathcal{N}_W$, in terms of combinatorial objects called ab-diagrams, was described in [K-P]. To convince the reader that the classification is valid over an arbitrary, not necessarily commutative field F, we give here an elementary proof of that result.

Let us change notation for a moment. Let U be a right vector space over F wih a direct sum decomposition

$$U = V^0 \oplus V^1.$$

For $\epsilon \in \{0, 1\}$ let $\epsilon' = 1 - \epsilon$. Let $X : U \longrightarrow U$ be a nilpotent endomorphism of U satisfying the condition

$$X(V^\epsilon) \subseteq V^{\epsilon'}.$$

Proposition 2.1. *There exists a basis*

$$\{X^\alpha v_i \ : \ 1 \leq i \leq r_1,\ 0 \leq \alpha \leq \lambda_i - 1\} \cup \{X^\beta v'_j \ : \ 1 \leq j \leq r_2,\ 0 \leq \beta \leq \mu_j - 1\}$$

of U *(called Jordan basis) such that* $v_i \in V^0$, $v'_j \in V^1$, $X^{\lambda_i} v_i = X^{\mu_j} v'_j = 0$ *for all* i, j.

Proof: Let N be the smallest integer satisfying $X^N = 0$. For $i \in \{1, \ldots, N\}$ let

$$U_i = \mathrm{Ker}(X^i), \quad V_i^\epsilon = V^\epsilon \cap U_i.$$

Choose vectors $f^\epsilon_{N,j} \in V^\epsilon_N = V^\epsilon$, $1 \leq j \leq r^\epsilon_N$, such that their cosets form a basis in $V^\epsilon_N / V^\epsilon_{N-1}$, $\epsilon = 1, 2$. Let $f^\epsilon_{N-1,j} = X(f^{\epsilon'}_{N,j})$ for $j = 1, \ldots, r^{\epsilon'}_N$.

Lemma 2.2. *The cosets of* $f^\epsilon_{N-1,j}$, $j = 1, \ldots, r^{\epsilon'}_N$ *are linearly independent in* $V^\epsilon_{N-1}/V^\epsilon_{N-2}$.

Proof: If $\sum f^\epsilon_{N-1,j}\, a_j \in V^\epsilon_{N-2}$, then $\sum f^{\epsilon'}_{N,j}\, a_j \in V^{\epsilon'}_{N-1}$, so by assumption all $a_j = 0$.

Choose $f^\epsilon_{N-1,j}$, $j = r^{\epsilon'}_N + 1, \ldots, r^\epsilon_{N-1}$ such that the cosets of all $f^\epsilon_{N-1,j}$, $j = 1, \ldots, r^\epsilon_{N-1}$ form a basis of $V^\epsilon_{N-1}/V^\epsilon_{N-2}$. Let $f^\epsilon_{N-2,j} = X(f^{\epsilon'}_{N-1,j})$ for $j = 1, \ldots, r^{\epsilon'}_{N-1}$. As before, their cosets modulo V^ϵ_{N-3} are linearly independent and we can choose $f^\epsilon_{N-2,j} \in V^\epsilon_{N-2}$ for $j = r^{\epsilon'}_{N-1} + 1, \ldots, r^\epsilon_{N-2}$ to get a basis of $V^\epsilon_{N-2}/V^\epsilon_{N-3}$. Continuing in this way we finally get a basis $f^\epsilon_{i,j}$ of U with the property that in this basis X acts as folows:

$$f^\epsilon_{i,j} \longmapsto f^{\epsilon'}_{i-1,j} \longmapsto f^\epsilon_{i-2,j} \longmapsto \ldots \longmapsto f^{\epsilon(')}_{1,j} \longmapsto 0,$$

where $\epsilon(')$ denotes either ϵ or ϵ', depending on the parity of i. Renaming the basis vectors ends the proof of the proposition.

For a fixed i the basis vectors $v_i, Xv_i, X^2 v_i, \ldots X^{\lambda_i - 1} v_i$ are represented by a string $abab\cdots$ of length λ_i. Similarly, for a fixed j the vectors $v'_j, Xv'_j, \ldots, X^{\mu_j - 1}$ are represented by a string $baba\cdots$ of length μ_j. The collection of strings obtained in this way is called the ab-diagram δ_X attached to X. The total number of a's in δ_X is equal to the dimension of V^1, the total number of b's in δ_X is equal to the dimension of V^2. The last (rightmost) entry in each string corresponds to a vector in the kernel of X.

Now we return to the context of dual pairs. We can embed W in $\mathrm{End}(V \oplus V')$, so that a nilpotent element $X = (S, T) \in W$ is identified with the nilpotent endomorphism X of $U = V \oplus V'$, $X(v, v') = (T(v'), S(v))$. To such an endomorphism we associate its ab-diagram δ_X as above, with $V^1 = V$, $V^2 = V'$. Then the total number of a's in δ_X is equal to n, the total number of b's is equal to m. Two nilpotent elements $X, X' \in \mathcal{N}_W$ are conjugate under $G \times G'$ if and only if $\delta_X = \delta_{X'}$, so the correspondence $X \longmapsto \delta_X$ defines a bijection of the set of nilpotent $G \times G'$-orbits in $\mathcal{N}_W$ and the set of ab diagrams consisting of n a's and m b's.

For a given ab–diagram δ, let $\tau(\delta)$ denote the partition of n counting the a's in the strings of δ, and let $\tau'(\delta)$ be the partition of m counting the b's in the strings of δ. Then, if $X \in \mathcal{N}_W$, then the nilpotent elements $\tau(X) \in \mathfrak{g}$, $\tau'(X) \in \mathfrak{g}'$ belong to the adjoint orbits corresponding to partitions $\tau(\delta_X)$, $\tau'(\delta_X)$ respectively. For more details see [K-P][Section 4], or [DKP1][Definition 3.2, Theorem 3.3].

Recall also (see e.g. [C-M],[D]) that the closure ordering on nilpotent orbits in $\mathfrak{gl}_{\mathbb{D}}(V)$ is compatible with the dominance ordering on partitions: if $\mathfrak{gl}_\lambda$ denotes the orbit corresponding to a partition λ, then $\mathfrak{gl}_\mu \subseteq \overline{\mathfrak{gl}_\lambda}$ if and only if $\mu \leq \lambda$, i.e. $\mu_1 + \ldots + \mu_i \leq \lambda_1 + \ldots + \lambda_i$ for all i, the same holds for $\mathfrak{gl}_{\mathbb{D}}(V,)$. Here, as usual, we identify a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ with the infinite sequence $(\lambda_1, \ldots, \lambda_p, 0, 0, \ldots)$.

We can now state and prove the main theorem of this section.

Definition 2.3. *Let λ be a partition of n. Let $r_1 = m$. For $i \geq 2$ let $r_i = r_i(\lambda) = m - (\lambda_1 + 1 + \lambda_2 + 1 + \cdots + \lambda_{i-1} + 1)$. Let $i_0 = i_0(\lambda)$ be the smallest $i \geq 1$ such that $r_i \leq \lambda_i$. Define a partition λ' of m by:*
$\lambda'_i = \lambda_i + 1$ *for* $i < i_0$,
$\lambda'_{i_0} = r_{i_0}$,
$\lambda'_i = 0$ *for* $i > i_0$.

Example 1. *Let $\lambda = (4, 3, 3, 2, 1)$. If $m = 3$ then $\lambda' = (3)$, if $m = 7$ then $\lambda' = (5, 2)$, for $m = (10)$ $\lambda' = (5, 4, 1)$, if $m = 21$, $\lambda' = (5, 4, 4, 3, 2, 1, 1, 1)$.*

Theorem 2.4. *Let $\mathfrak{g}_\lambda \subseteq \mathfrak{g}$ be the nilpotent orbit corresponding to a partition λ of n. Similarly let $\mathfrak{g}'_{\lambda'} \subseteq \mathfrak{g}'$ be the nilpotent orbit corresponding to a partition λ' of m. Then*

$$\tau'(\tau^{-1}(\overline{\mathfrak{g}_\lambda})) = \overline{\mathfrak{g}'_{\lambda'}}. \tag{5}$$

Proof: In the case $\mathbb{D} = \mathbb{C}$, this is exactly Theorem 4.2 of [DKP1]. The proof in the general case is almost the same with one change: instead of using an argument from invariant theory we will deduce a necessary combinatorial fact from the case $\mathbb{D} = \mathbb{C}$.

Let us recall the steps of the proof of theorem 4.2 of [DKP1]. First, from the definition of λ' (see Remark on p. 524 in [DKP1]) it immediately follows that

$$\mathfrak{g}'_{\lambda'} \subseteq \tau'(\tau^{-1}(\overline{\mathfrak{g}_\lambda})). \tag{6}$$

Then, using the fact that τ' is a quotient map in the sense of geometric invariant theory, we deduce that the right hand side of (6) is closed in $\mathfrak{g}'$, and that

$$\overline{\mathfrak{g}'_{\lambda'}} \subseteq \tau'(\tau^{-1}(\overline{\mathfrak{g}_\lambda})). \tag{7}$$

The proof of the opposite inclusion is purely combinatorial – the inclusion is equivalent to the following lemma, proved in [DKP1, p.525].

Lemma 2.5. *Let μ be a partition of n such that $\mu \leq \lambda$, and let δ be an ab-diagram such that $\tau(\delta) = \mu$. Then $\tau'(\delta) \leq \lambda'$.*

The only step of the above proof that does not carry over to the general case is the proof that the set $\tau'(\tau^{-1}(\overline{\mathfrak{g}_\lambda}))$ is closed in $\mathfrak{g}'$. But for an orbit $\mathfrak{g}'_\nu \subseteq \mathfrak{g}'$ we have:

$\mathfrak{g}'_\nu \subseteq \tau'(\tau^{-1}(\overline{\mathfrak{g}_\lambda}))$ iff there exists an ab–diagram δ with the properties $\tau'(\delta) = \nu$, $\tau(\delta) \leq \lambda$. It follows that the set $\tau'(\tau^{-1}(\overline{\mathfrak{g}_\lambda}))$ is closed if and only if the following property holds: for partitions ν, σ of m such that $\sigma \leq \nu$, if there exists an ab–diagram δ with $\tau'(\delta) = \nu$, $\tau(\delta) \leq \lambda$, then there exists an ab–diagram δ' with $\tau'(\delta') = \sigma$, $\tau(\delta') \leq \lambda$.

Now, this property is purely combinatorial and it follows from the fact that the set $\tau'(\tau^{-1}(\overline{\mathfrak{g}_\lambda}))$ is closed in the case $\mathbb{D} = \mathbb{C}$. This ends the proof of Theorem 2.4.

3. Pairs of type II over $\mathbb{R}$ – correspondence of $K_{\mathbb{C}}$-orbits

Let G, G' and W be as in the previous section, with $\mathbb{D} = \mathbb{R}$. We fix bases of V and V' and we identify elements of W with pairs of matrices. Let $J : W \longrightarrow W$ be the compatible complex structure defined by

$$J(S,T) = (-T^t, S^t). \tag{8}$$

Let $V_{\mathbb{C}}$, $V'_{\mathbb{C}}$ denote the complexifications of V, V'. Define $W_{\mathbb{C}}^+$ as in the introduction. Let $p : W \longrightarrow W_{\mathbb{C}}^+$ be defined by $p = (1/2)(1 - iJ)$, so

$$p(S,T) = \frac{1}{2} \cdot (Z, -iZ^t) \tag{9}$$

where $Z = S + iT^t$. In this way the space $W_{\mathbb{C}}^+$ can be identified with the space $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, V'_{\mathbb{C}})$, via

$$\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, V'_{\mathbb{C}}) \ni Z \longmapsto \frac{1}{2} \cdot (Z, -iZ^t) \in W_{\mathbb{C}}^+, \tag{10}$$

or, equivalently, with the subspace $L = L(V_{\mathbb{C}}, V'_{\mathbb{C}}) \subseteq \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}} \oplus V'_{\mathbb{C}})$ consisting of the homomorphisms whose matrices have the form

$$\begin{bmatrix} 0 & -Z^t \\ Z & 0 \end{bmatrix}, \quad Z \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, V'_{\mathbb{C}}). \tag{11}$$

The factor i in (10) is dropped in (11), as it changes the moment maps (12) only by a scalar factor, which does not change their behaviour with respect to nilpotent orbits. The same remark applies to $1/2$, and to other scalar factors that may appear in similar contexts.

Let $G_{\mathbb{C}} = GL(V_{\mathbb{C}})$, $G'_{\mathbb{C}} = GL(V'_{\mathbb{C}})$ denote the complexifications of G and G', let $K_{\mathbb{C}} = O(V_{\mathbb{C}}) \subseteq G_{\mathbb{C}}$, $K'_{\mathbb{C}} = O(V'_{\mathbb{C}}) \subseteq G'_{\mathbb{C}}$ be the complexifications of maximal compact subgroups

$K = O(V) \subseteq G$, $K' = O(V') \subseteq G'$, here all orthogonal groups are the isometry groups of the symmetric forms defined by the identity matrices (recall that the Cartan involutions on G and G' are equal to the conjugation by J). Then the identification (10) preserves natural actions of $K_{\mathbb{C}} \times K'_{\mathbb{C}}$.

Now we are ready to describe the solution to problem 1.3 in the case of the dual pair $(GL(V), GL(V'))$. Under the identifications (10) the diagram

$$\mathfrak{p}'_{\mathbb{C}} \xleftarrow{\tau'_c} W^+_{\mathbb{C}} \xrightarrow{\tau_c} \mathfrak{p}_{\mathbb{C}}$$

is equal to

$$\mathfrak{p}'_{\mathbb{C}} \xleftarrow{\tau'_c} \mathrm{Hom}(V_{\mathbb{C}}, V'_{\mathbb{C}}) \xrightarrow{\tau_c} \mathfrak{p}_{\mathbb{C}},$$

with $\mathrm{Hom}(V_{\mathbb{C}}, V'_{\mathbb{C}})$ identified with the space $M_{m,n}(\mathbb{C})$ of $m \times n$ matrices, $\mathfrak{p}'_{\mathbb{C}}$ identified with the space of symmetric $m \times m$ matrices, $\mathfrak{p}_{\mathbb{C}}$ identified with the space of symmetric $n \times n$ matrices ($m = \dim V'$, $n = \dim V$), and (up to nonessential scalar factors) the maps τ_c, τ'_c given by

$$\tau_c(Z) = Z^t \cdot Z, \qquad {\tau_c}'(Z) = Z \cdot Z^t. \tag{12}$$

It is known (see [O1],[S1]) that every nilpotent $GL(V_{\mathbb{C}})$-orbit in $\mathfrak{gl}(V_{\mathbb{C}})$ intersects the subspace $\mathfrak{p}_{\mathbb{C}}$ along a single $O(V_{\mathbb{C}})$-orbit, so in a natural way nilpotent $O(V_{\mathbb{C}})$-orbits in $\mathfrak{p}_{\mathbb{C}}$ are parametrized by partitions λ of n. Let $\mathfrak{p}_\lambda$ be the orbit corresponding to λ. Similarly, by $\mathfrak{p}'_\nu$ we will denote the $O(V'_{\mathbb{C}})$-orbit in $\mathfrak{p}'_{\mathbb{C}}$ corresponding to a partition ν of m.

Recall the following theorem of Ohta, [O1, Thm 1, p 447].

Theorem 3.1. *Let λ, μ be two partitions of n, then $\mathfrak{p}_\mu \subseteq \overline{\mathfrak{p}_\lambda}$ if and only if $\mu \leq \lambda$ (i.e. $\mu_1 + \ldots + \mu_i \leq \lambda_1 + \ldots + \lambda_i$ for all i).*

The main result of this section describes the orbit correspondence in this case.

Theorem 3.2. *Let λ be a partition of n, let λ' be the partition of m defined as in* 2.3. *Then*

$$\tau'_c({\tau_c}^{-1}(\overline{\mathfrak{p}_\lambda})) = \overline{\mathfrak{p}'_{\lambda'}}.$$

For the proof of this theorem we will need some information on the classification of nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $W^+_{\mathbb{C}}$.

Consider the space $L' = L'(V_{\mathbb{C}}, V'_{\mathbb{C}}) \subseteq \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}} \oplus V'_{\mathbb{C}})$ consisting of the homomorphisms whose matrices have the block form

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}, \quad A \in \mathrm{Hom}(V_{\mathbb{C}}, V'_{\mathbb{C}}), \;\; B \in \mathrm{Hom}(V'_{\mathbb{C}}, V_{\mathbb{C}}). \tag{13}$$

This space contains the subspace L defined above. We note the following obvious lemma.

Lemma 3.3. *The matrix (*13*) is nilpotent if and only if $A \cdot B$ is a nilpotent endomorphism of $V'_{\mathbb{C}}$, if and only if $B \cdot A$ is a nilpotent endomorphism of $V_{\mathbb{C}}$.*

It follows that in order to classify all nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $W^+_{\mathbb{C}}$ it will be sufficient to classify nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in L.

We have the following proposition, due to Ohta [O1, Prop. 4, p. 459], see also [MWZ, Prop. 2.1].

Proposition 3.4. *Two nilpotent matrices in L are conjugate under the group $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ if and only if they are conjugate under the group $G_{\mathbb{C}} \times G'_{\mathbb{C}}$, thus if a $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit in L' intersects L, then it intersects L along a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit.*

It follows that nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in L are in one-to-one correspondence with those nilpotent $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbits in L' which intersect L.

Definition 3.5. *An ab-diagram is of type BDI, if it consists only of strings of the form*

$$\begin{array}{ll} (i) & baba\dots ab, \\ (ii) & abab\dots ba, \end{array}$$

or pairs of strings:

$$(iii) \quad \begin{array}{c} ba\dots ba \\ ab\dots ab, \end{array} \quad \textit{both strings of the same length.}$$

Theorem 3.6. *Let $X \in L'$ be a nilpotent element. Then the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit of X intersects L if and only if its ab-diagram δ_X is of type BDI. Thus the nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $W^+_{\mathbb{C}}$ are in one-to-one correspondence with ab-diagrams of type AI containing n a's and m b's.*

Remark 3.7. *This theorem was proved by Ohta in* [O2, Prop. 2] *as the classification of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ for symmetric pairs of type BDI. One reason that we give the proof is that we want to keep this paper self-contained, and the second reason is that we believe our proof is more transparent.*

Proof: Assume that $X \in L$. The proof that δ_X is of type BDI will proceed along the lines of the proof of the clasification theorem of nilpotent orbits in complex orthogonal Lie algebras, with a refinement coming from Vinberg's theory of nilpotent elements in graded Lie algebras [Vi].

We must prove that in δ_X strings of even length come in pairs, as in (iii) in the definition 3.5. Fix bases of $V_{\mathbb{C}}$ and $V'_{\mathbb{C}}$ and let $(\ , \ )$ be the symmetric bilinear form on $V_{\mathbb{C}} \oplus V'_{\mathbb{C}}$ such that the union of these bases is an orthonormal basis of $V_{\mathbb{C}} \oplus V'_{\mathbb{C}}$. Let $\mathfrak{o} = \mathfrak{o}(V_{\mathbb{C}} \oplus V'_{\mathbb{C}})$ be the Lie algebra of the isometry group $O(V_{\mathbb{C}} \oplus V'_{\mathbb{C}})$. Let $\mathfrak{o}_0 = \mathfrak{o}(V_{\mathbb{C}}) \times \mathfrak{o}(V'_{\mathbb{C}}) \subseteq \mathfrak{o}$ be the subspace consisting of matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and let $\mathfrak{o}_1 = L$ be the subspace of matrices of the form (11). Then $\mathfrak{o} = \mathfrak{o}_0 \oplus \mathfrak{o}_1$ is a $\mathbb{Z}_2$–grading of $\mathfrak{o}$ and our nilpotent matrix X belongs to $\mathfrak{o}_1$. By the Vinberg's refinement of the Jacobson–Morozov theorem ([Vi, Thm. 1(1)], see also Kostant and Rallis [KR]) there exists a Jacobson–Morozov triple (X, Y, H) with $Y \in \mathfrak{o}_1$ and $H \in \mathfrak{o}_0$. Now we proceed as in [C-M, Section 5.1]. Let $\mathfrak{a} = \mathrm{span}(X, Y, H) \subseteq \mathfrak{o}$, $\mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C})$, and let

$$V_{\mathbb{C}} \oplus V'_{\mathbb{C}} = \bigoplus_{r \geq 0} M(r)$$

be the isotypic decomposition of $V_{\mathbb{C}} \oplus V'_{\mathbb{C}}$ as an $\mathfrak{a}$-module, where $M(r)$ is the sum of irreducible representations of $\mathfrak{a}$ of highest weight r (and dimension $r+1$). Let $H(r) \subseteq M(r)$ be the highest weight space, i.e. the kernel $\operatorname{Ker}(X : M(r) \longrightarrow M(r))$. As $X(V_{\mathbb{C}}) \subseteq V'_{\mathbb{C}}$ and $X(V'_{\mathbb{C}}) \subseteq V_{\mathbb{C}}$, we have a decomposition

$$H(r) = (H(r) \cap V_{\mathbb{C}}) \oplus (H(r) \cap V'_{\mathbb{C}}). \tag{14}$$

Assume that r is odd (so $r+1$ is even). Then the dimension of $H(r) \cap V_{\mathbb{C}}$ is equal to the number of strings of δ_X of the form $ba \dots ba$ of length $r+1$, and similarly, the dimension of $H(r) \cap V'_{\mathbb{C}}$ is equal to the number of strings of δ_X of the same length, but of the form $ab \dots ab$. We will show that

$$\dim H(r) \cap V_{\mathbb{C}} = \dim H(r) \cap V'_{\mathbb{C}}.$$

This will show that δ_X is of type BDI. Let $(\ , \ )_r$ be the bilinear form on $H(r)$ defined by $(w_1, w_2)_r = (w_1, Y^r w_2)$. By Lemmas 5.1.11 and 5.1.14 in [C-M] this form is skew–symmetric and nondegenerate. Moreover, for $w_1, w_2 \in V_{\mathbb{C}}$ we have $Y^r w_2 \in V'_{\mathbb{C}}$ (r is odd!), so $(w_1, Y^r w_2) = 0$, hence $H(r) \cap V_{\mathbb{C}}$ is an isotropic subspace of $H(r)$ with respect to $(\ , \ )_r$. The same is true about $H(r) \cap V'_{\mathbb{C}}$, hence by (14) $H(r) \cap V_{\mathbb{C}}$ and $H(r) \cap V'_{\mathbb{C}}$ are complementary maximal isotropic subspaces of $H(r)$, in particular their dimensions must be equal. This proves the claim.

It remains to show that if $X \in L'$ has ab-diagram of type BDI, then the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit of X intersects L. We will show that if $X \in L'$ is a nilpotent element whose ab-diagram δ_X is one of (i), (ii), (iii) in Definition (3.5), then the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit through X contains a skew-symmetric matrix. From this, the proposition follows easily.

Assume first that $\delta_X = bab \cdots ab$, with k a's, $k+1$ b's, is of the form (i). Then in an appropriate basis

$$X = \begin{bmatrix} 0_{k\times k} & B \\ A & 0_{(k+1)\times(k+1)} \end{bmatrix},$$

$$\text{where } A = \begin{bmatrix} 0_{1\times k} \\ I_k \end{bmatrix}, \qquad B = \begin{bmatrix} I_k & 0_{k\times 1} \end{bmatrix}.$$

Let $(g, g') \in GL_k(\mathbb{C}) \times GL_{k+1}(\mathbb{C})$. The condition

$$(g, g') \cdot X \cdot (g, g')^{-1} \text{ is skew-symmetric}$$

can be written at the matrix equation

$$g \cdot B \cdot g'^{-1} = -(g^t)^{-1} \cdot A^t \cdot g'^t,$$

or equivalently, as

$$(g^t \cdot g) \cdot B = -A^t \cdot (g'^{\,t} \cdot g'). \tag{15}$$

Let

$$F_k = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 1 & \dots & 0 & 0 \end{bmatrix} \in GL_k(\mathbb{C}) \tag{16}$$

and similarly $F_{k+1} \in GL_{k+1}(\mathbb{C})$ be the "antidiagonal identity" matrices. As F_k and $-F_{k+1}$ are symmetric, there exist $g \in GL_k(\mathbb{C})$ and $g' \in GL_{k+1}(\mathbb{C})$ such that $g^t \cdot g = F_k$ and $g'^t \cdot g' = -F_{k+1}$. Then (15) holds and the proposition is true in this case.

In the same way we can prove the existence of a skew-symmetric matrix in the $GL_k(\mathbb{C}) \times GL_{k+1}(\mathbb{C})$-orbit of a nilpotent matrix $X \in L'$ whose ab-diagram is of the form (ii).

Assume thus that δ_X is of the type (iii) with the length of each string equal to $2k$. In an appropriate basis

$$X = \begin{bmatrix} 0_{2k\times 2k} & B \\ A & 0_{2k\times 2k} \end{bmatrix},$$

where

$$A = \begin{bmatrix} C & 0_{k\times k} \\ 0_{k\times k} & I_k \end{bmatrix}, \quad B = \begin{bmatrix} I_k & 0_{k\times k} \\ 0_{k\times k} & C \end{bmatrix}.$$

with

$$C = \begin{bmatrix} 0_{1\times(k-1)} & 0 \\ I_{k-1} & 0_{(k-1)\times 1} \end{bmatrix}.$$

Then $F_{2k} \cdot B \cdot (-F_{2k}) = -A^t$, so as before, for $g, g' \in GL_{2k}(\mathbb{C})$ satisfying $g^t \cdot g = F_{2k}$, $g'^t \cdot g' = -F_{2k}$ we get a skew-symmetric matrix $(g, g') \cdot X \cdot (g, g')^{-1}$ in the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit of X, which proves the proposition.

Proof of theorem 3.2. The proof is identical as the proof of theorem 4.2 in [DKP1], once we observe the following: for each partition λ of n there exists an ab-diagram δ of type BDI such that (notation as in section 2) $\tau(\delta) \leq \lambda$, $\tau'(\delta) = \lambda'$ (note that to construct such δ we need only strings of odd length, so of the form (i) and (ii) in 3.5). This allows us to deduce that

$$\mathfrak{p}'_{\lambda'} \subseteq \tau'_c(\tau_c^{-1}(\overline{\mathfrak{p}_\lambda})). \tag{17}$$

Now, the map τ' is a quotient map ([O1][Theorem 3, p. 453]), so the set $\tau'_c(\tau_c^{-1}(\overline{\mathfrak{p}_\lambda}))$ is closed, and the rest of the proof carries over with no change (lemma 2.5 applies here).

Corollary 3.8. *Let $\mathcal{O} \subseteq \mathfrak{gl}(V)$ be a nilpotent orbit in the real Lie algebra $\mathfrak{gl}(V)$, then*

$$S'(\tau'(\tau^{-1}(\overline{\mathcal{O}}))) = \tau'_c(\tau_c^{-1}(\overline{S(\mathcal{O})})).$$

Proof: Immediate from theorems 2.4 and 3.2, if we use the fact that on the level of partitions the Kostant–Sekiguchi correspondence for $GL(V)$ is just the identity (it follows from the fact that both $\mathcal{O}$ and $S(\mathcal{O})$ are contained in the same $GL(V_{\mathbb{C}})$-orbit in $\mathfrak{gl}(V_{\mathbb{C}})$ and all three orbits correspond to the same partition).

4. Pairs of type II over over $\mathbb{H}$ – correspondence of $K_{\mathbb{C}}$-orbits

Let V, V' be right vector spaces over $\mathbb{H}$, of dimensions $n = \dim_{\mathbb{H}} V$, $m = \dim_{\mathbb{H}} V'$. Fix a basis $v_1, \ldots, v_n$ of V over $\mathbb{H}$, then $v_1, \ldots, v_n, v_1 j, \ldots, v_n j$ is a basis of V over $\mathbb{C}$, where $j \in \mathbb{H}$ is one of the standard generating quaternions i, j, k. In this way the group $G = GL_{\mathbb{H}}(V)$ can be identified with the subgroup of $GL_{2n}(\mathbb{C})$ consisting of matrices of the form

$$\begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix},$$

and $G_{\mathbb{C}} = GL_{\mathbb{C}}(V)$.

Similarly, $G' = GL_{\mathbb{H}}(V')$ can be identified with a subgroup of $G'_{\mathbb{C}} = GL_{2m}(\mathbb{C})$, and the Lie algebras $\mathfrak{g} = \mathfrak{gl}_{\mathbb{H}}(V)$, $\mathfrak{g}' = \mathfrak{gl}_{\mathbb{H}}(V')$ with the appropriate subalgebras of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_{\mathbb{C}}(V)$ and $\mathfrak{g}'_{\mathbb{C}} = \mathfrak{gl}_{\mathbb{C}}(V')$.

Let

$$W = \mathrm{Hom}_{\mathbb{H}}(V, V') \oplus \mathrm{Hom}_{\mathbb{H}}(V', V).$$

As above, W can be identified with the appropriate subspace of

$$W_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(V, V') \oplus \mathrm{Hom}_{\mathbb{C}}(V', V).$$

Let

$$J_V = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad J'_V = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}.$$

For $S \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ let $S^* = -J_V \cdot S^t \cdot J_{V'} \in \mathrm{Hom}_{\mathbb{C}}(V', V)$, and for $T \in \mathrm{Hom}_{\mathbb{C}}(V', V)$ let $T^* = -J_{V'} \cdot T^t \cdot J_V \in \mathrm{Hom}_{\mathbb{C}}(V, V')$.

Let $J : W \longrightarrow W$ be a complex structure defined by

$$J(S, T) = (-T^*, S^*)$$

(as usual, we identify linear maps with matrices in appropriate bases). Let $\langle\ ,\ \rangle$ be the skew–symmetric form on W defined in the introduction (formula (4)). Then J is a compatible positive definite complex structure. Conjugation by J is a Cartan involution on G and also on G'. Let $K \subseteq G$ and $K' \subseteq G'$ be the corresponding maximal compact subgroups and let $K_{\mathbb{C}} = Sp_{\mathbb{C}}(V) \subseteq G_{\mathbb{C}}$, and $K'_{\mathbb{C}} = Sp_{\mathbb{C}}(V') \subseteq G'_{\mathbb{C}}$ be their complexifications, equal to the isometry groups of the skew–symmetric forms J_V on $V_{\mathbb{C}}$ and $J_{V'}$ on $V'_{\mathbb{C}}$.

Let $p : W \longrightarrow W_{\mathbb{C}}$ be defined by

$$p = \frac{1}{2}(1 - iJ).$$

Then

$$p(S, T) = \frac{1}{2}(Z, -iZ^*), \quad \text{for } Z = S + iT^*.$$

It follows that $W^+_{\mathbb{C}} = \mathrm{Im}(p)$ can be identified with $\mathrm{Hom}_{\mathbb{C}}(V, V')$ by

$$\mathrm{Hom}_{\mathbb{C}}(V, V') \ni Z \longmapsto \frac{1}{2}(Z, -iZ^*) \in W^+_{\mathbb{C}}. \tag{18}$$

As in Section 3, $W^+_{\mathbb{C}}$ can also be identified with certain subspace $L = L(V, V')$ of $\mathrm{End}_{\mathbb{C}}(V \oplus V')$. This time L consists of matrices of the form

$$\begin{bmatrix} 0 & -Z^* \\ Z & 0 \end{bmatrix}, \tag{19}$$

where $Z \in W^+_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(V, V')$ will be identified with the matrix (19). All these identifications preserve natural actions of $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$.

Under the identifications (18),(19) the moment maps

$$\tau_{\mathbb{C}} : W^+_{\mathbb{C}} \longrightarrow \mathfrak{p}_{\mathbb{C}} \subseteq \mathfrak{gl}_{\mathbb{C}}(V),$$
$$\tau'_{\mathbb{C}} : W^+_{\mathbb{C}} \longrightarrow \mathfrak{p}'_{\mathbb{C}} \subseteq \mathfrak{gl}_{\mathbb{C}}(V'),$$

will be identified with

$$\tau_{\mathbb{C}} : \mathrm{Hom}_{\mathbb{C}}(V, V') \longrightarrow \mathfrak{p}_{\mathbb{C}}, \quad \tau_{\mathbb{C}}(Z) = Z^* \cdot Z,$$
$$\tau'_{\mathbb{C}} : \mathrm{Hom}_{\mathbb{C}}(V', V) \longrightarrow \mathfrak{p}'_{\mathbb{C}}, \quad \tau'_{\mathbb{C}}(Z) = Z \cdot Z^*.$$

Here $\mathfrak{p}_{\mathbb{C}} = \{X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid XJ_V = -(XJ_V)^t\}$, $\mathfrak{p}'_{\mathbb{C}} = \{X \in \mathfrak{gl}_{2m}(\mathbb{C}) \mid XJ_{V'} = -(XJ_{V'})^t\}$.

Recall from [S1], [O1] the basic facts about the nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$.

For a partition $\mu = (\mu_1, \mu_2, \dots) \in \mathcal{P}(n)$ let $\mu^2 = (\mu_1, \mu_1, \mu_2, \mu_2, \dots) \in \mathcal{P}(2n)$ denote its *double*. Let $\mathcal{P}(n)^2 = \{\mu^2 \mid \mu \in \mathcal{P}(n)\}$.

Proposition 4.1. *Nilpotent $K_{\mathbb{C}}(V)$-orbits in $\mathfrak{p}_{\mathbb{C}}$ are parametrized by partitions $\lambda \in \mathcal{P}(n)^2$. Let $\mathfrak{p}_\lambda$ denote the orbit corresponding to a partition λ. Then for $\lambda, \mu \in \mathcal{P}(n)^2$*

$$\mathfrak{p}_\mu \subseteq \overline{\mathfrak{p}_\lambda} \quad \textit{iff} \quad \mu \leq \lambda.$$

Now we will describe the classification of $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $W^+_{\mathbb{C}}$. The idea is similar as in theorem 3.2. Let $W_{\mathbb{C}}$ be identified with the space L' of complex matrices

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}. \tag{20}$$

Lemma 4.2. *The matrix (20) is nilpotent if and only if $A \cdot B$ is a nilpotent endomorphism of V', if and only if $B \cdot A$ is a nilpotent endomorphism of V.*

A variant (also due to Ohta [O1, Prop. 4, p. 459]) of proposition 3.4 holds also in this case.

Proposition 4.3. *Two nilpotent matrices in L are conjugate under the group $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ if and only if they are conjugate under the group $G_{\mathbb{C}} \times G'_{\mathbb{C}}$, thus if a $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit in L' intersects L, then it intersects L along a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit.*

It follows that nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in L are in one-to-one correspondence with the set of nilpotent $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbits in L' which intersect L. As before, $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbits in L' are classified by ab-diagrams containing $\dim_{\mathbb{C}} V = 2n$ a's and $\dim_{\mathbb{C}} V' = 2m$ b's.

Definition 4.4. *An ab-diagram is of type CII, if it consists only of pairs of strings of the form*

(i)	$\begin{matrix} ba\dots ab \\ ba\dots ab \end{matrix}$	*both strings of the same odd length,*
(ii)	$\begin{matrix} ab\dots ba \\ ab\dots ba \end{matrix}$	*both strings of the same odd length,*
(iii)	$\begin{matrix} ba\dots ba \\ ab\dots ab \end{matrix}$	*both strings of the same even length.*

Theorem 4.5. *Let $X \in L'$ be a nilpotent element. Then the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$–orbit of X intersects L if and only if its ab-diagram δ_X is of type CII. Thus the nilpotent $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbits in $W^+_{\mathbb{C}}$ are in one-to-one correspondence with ab-diagrams of type CII containing $2n$ a's and $2m$ b's.*

Remark 4.6. *As in the case of theorem* 3.6 *this result was proved by Ohta* [O2, Prop. 2] *as the classification of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ for symmetric pairs of type CII.*

Proof: The proof is similar to the proof of theorem 3.6. Assume that $X \in L$. Fix bases of V and V' such that in these bases

$$J_V = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \qquad J_{V'} = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$

Let $(\ ,\ )$ be the skew–symmetric bilinear form on $V \oplus V'$ defined by the matrix

$$\begin{bmatrix} J_V & 0 \\ 0 & J_{V'} \end{bmatrix}.$$

Let $\mathfrak{sp} = \mathfrak{sp}(V \oplus V')$ be the Lie algebra of the isotropy group $Sp(V \oplus V')$. Let $\mathfrak{sp}_0 = \mathfrak{sp}(V) \oplus \mathfrak{sp}(V') \subseteq \mathfrak{sp}$ be the subspace consisting of matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and let $\mathfrak{sp}_1 \subseteq \mathfrak{sp}$ be the subspace of matrices of the form (19). Then $\mathfrak{sp} = \mathfrak{sp}_0 \oplus \mathfrak{sp}_1$ is a $\mathbb{Z}_2$–grading of $\mathfrak{sp}$ and our nilpotent element $X \in \mathfrak{sp}_1$. As in the proof of theorem 3.6 we consider a Jacobson–Morozov triple (X, Y, H) with $Y \in \mathfrak{sp}_1$ and $H \in \mathfrak{sp}_0$, and study the highest weight space $H(r)$ in the sum $M(r)$ of irreducible $\mathfrak{sl}_2$-submodules of $V \oplus V'$ of dimension $r + 1$.

Let $(\ ,\ )_r$ be the bilinear form on $H(r)$ defined by $(w_1, w_2)_r = (w_1, Y^r w_2)$. As before this form is nondegenerate.

Assume that r is odd. In this case the form $(\ ,\ )_r$ on $H(r)$ is symmetric and as in the proof of theorem 3.6 we show that

$$H(r) = (H(r) \cap V) \oplus (H(r) \cap V') \tag{21}$$

is a decomposition of $H(r)$ into the sum of complementary isotropic subspaces, so the dimension of $H(r) \cap V$ (equal to the number of strings of δ_X of the form $ba \ldots ba$ of length $r+1$) is equal to the dimension of $H(r) \cap V'$ (equal to the number of strings of δ_X of the form $ab \ldots ab$ of the same length $r + 1$). This shows that strings of even length can be paired as in (iii) of definition 4.4.

If r is even, then the form $(\ ,\ )_r$ is skew–symmetric, so the complex dimension of the space $H(r)$ is even. This dimension is equal to the number of strings of δ_X of length $r+1$, so the number of strings of δ_X of each odd length is even (this is of course well known and follows from the classification of nilpotent orbits in complex symplectic Lie algebras). We must show that the number of strings $ab \ldots ba$ of length $r + 1$ is even and that the number of strings $ba \ldots ab$ of the same length is also even. This follows easily from the fact that partitions corresponding to nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$, $\mathfrak{p}'_{\mathbb{C}}$ belong to $\mathcal{P}(n)^2$, $\mathcal{P}(m)^2$ respectively (see proposition 4.1), so partitions counting either a's or b's in the strings of δ_X must have this property.

This proves that the strings of odd length come in pairs as in (i) and (ii) of definition 4.4 and shows that δ_X is of type CII.

It remains to show that if a nilpotent $X \in L'$ has ab-diagram of type CII, then the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$–orbit of X intersects L. We will show this fact in the case when X is a nilpotent element whose ab-diagram is one of (i), (ii), (iii) in definition 4.4, the general case follows easily from this special case.

(i) Assume that

$$\delta_X = \begin{matrix} ba\dots ab \\ ba\dots ab \end{matrix},$$

both strings of length $2k+1$. In appropriate bases of V and V'

$$X = \begin{bmatrix} 0_{2k\times 2k} & B \\ A & 0_{(2k+1)\times(2k+1)} \end{bmatrix},$$

where $A = \begin{bmatrix} 0_{1\times k} & 0_{1\times k} \\ I_k & 0_{k\times k} \\ 0_{1\times k} & 0_{1\times k} \\ 0_{k\times k} & I_k \end{bmatrix}$, and $B = \begin{bmatrix} I_k & 0_{k\times 1} & 0_{k\times k} & 0_{k\times 1} \\ 0_{k\times k} & 0_{k\times 1} & I_k & 0_{k\times 1} \end{bmatrix}$.
We must find $(g, g') \in GL_{\mathbb{C}}(V) \times GL_{\mathbb{C}}(V')$ such that

$$(g, g') \cdot X \cdot (g, g')^{-1} = \begin{bmatrix} 0 & gBg'^{-1} \\ g'Ag^{-1} & 0 \end{bmatrix} \in L,$$

i.e. $g'Ag^{-1} = -(gBg'^{-1})^* = J_{V'}(g'^t)^{-1}B^t g^t J_V$, so $A = (g'^{-1}J_{V'}(g'^{-1})^t) \cdot B^t \cdot (g^t J_V g)$, which is equivalent to finding two skew-symmetric matrices $C_1 \in GL_{2k+2}(\mathbb{C})$, $C_2 \in GL_{2k}(\mathbb{C})$ such that

$$A = C_1 B^t C_2. \tag{22}$$

Let $C_1 = \begin{bmatrix} 0 & F_{k+1} \\ -F_{k+1} & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & -F_k \\ F_k & 0 \end{bmatrix}$, where F_k is as in (16). Then (22) holds, proving that the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit through X intersects L.

(ii) If $\delta_X = \begin{matrix} ab\dots ba \\ ab\dots ba \end{matrix}$ then the proof is almost identical as above (transpose all the matrices).

(iii) Let $\delta_X = \begin{matrix} ba\dots ba \\ ab\dots ab \end{matrix}$, with both strings of length $2k$. Then in appropriate bases

$$X = \begin{bmatrix} 0_{2k\times 2k} & B \\ A & 0_{2k\times 2k} \end{bmatrix},$$

where $A = \begin{bmatrix} C & 0_{k\times k} \\ 0_{k\times k} & I_k \end{bmatrix}$, $B = \begin{bmatrix} I_k & 0_{k\times k} \\ 0_{k\times k} & C \end{bmatrix}$ with $C = \begin{bmatrix} 0_{1\times(k-1)} & 0 \\ I_{k-1} & 0_{(k-1)\times 1} \end{bmatrix}$.

Let $C_1 = \begin{bmatrix} 0 & F_k \\ -F_k & 0 \end{bmatrix}$ and $C_2 = -C_1$, then $A = C_1 \cdot B^t \cdot C_2$, so as before the $G_{\mathbb{C}} \times G'_{\mathbb{C}}$-orbit through X intersects L. This ends the proof of theorem 4.5.

Definition 4.7. *Let $\lambda \in \mathcal{P}(n)^2$. Let $r_1 = m$. For $i \geq 2$ let $r_i = m - (\lambda_1 + 1 + \lambda_3 + 1 + \ldots + \lambda_{2i-3} + 1)$. Let $i_0 = i_0(\lambda)$ be the smallest $i \geq 1$ such that $r_i \leq \lambda_{2i}$. Define a partition $\lambda^h \in \mathcal{P}(m)^2$ by*

$$\begin{aligned} \lambda_i^h &= \lambda_i + 1 \quad \text{for } i < 2i_0 - 1, \\ \lambda_{2i_0-1}^h &= \lambda_{2i_0}^h = r_{i_0}, \\ \lambda_i^h &= 0 \quad \text{for } i > 2i_0. \end{aligned}$$

In other words, if $\lambda = \nu^2$, then $\lambda^h = \nu'^2$, where ν' is the partition of m defined as in definition (2.3).

Theorem 4.8. *Let $\lambda \in \mathcal{P}(n)^2$. Then*

$$\tau_c'(\tau_c^{-1}(\overline{\mathfrak{p}_\lambda})) = \overline{\mathfrak{p}'_{\lambda^h}}.$$

Proof: The inclusion

$$\mathfrak{p}'_{\lambda^h} \subseteq \tau_c'(\tau_c^{-1}(\overline{\mathfrak{p}_\lambda})) \tag{23}$$

follows from the definition of λ^h. As the map τ_c' is a quotient map ([O1, Thm. 3, p. 453]), the right hand side of (23) is closed, so $\overline{\mathfrak{p}'_{\lambda^h}} \subseteq \tau_c'(\tau_c^{-1}(\overline{\mathfrak{p}_\lambda}))$. The opposite inclusion follows from the following lemma.

Lemma 4.9. *Let $\lambda, \mu \in \mathcal{P}(n)^2$ be two partitions such that $\mu \leq \lambda$ and let δ be an ab-diagram of type CII such that $\tau(\delta) = \mu$. Then $\tau'(\delta) \leq \lambda^h$.*

Proof: For a partition $\lambda \in \mathcal{P}(n)^2$, let $\lambda^{1/2} \in \mathcal{P}(n)$ denote the partition such that $(\lambda^{1/2})^2 = \lambda$. For an ab-diagram δ of type CII let $\delta^{1/2}$ denote the ab-diagram constructed as follows: from each pair of strings $ab \ldots ba$ of δ of the same length choose one string, from each pair of strings $ba \ldots ab$ of δ of the same length choose one string, from each pair of strings $ab \ldots ab$, $ba \ldots ba$ of δ of the same length choose one string $ab \ldots ab$. The lemma follows from Lemma 2.5 and from the following observations:
1. for $\lambda, \mu \in \mathcal{P}(n)^2$ we have $\mu \leq \lambda \iff \mu^{1/2} \leq \lambda^{1/2}$,
2. for an ab-diagram δ of type CII we have $\tau(\delta) = \mu \iff \tau(\delta^{1/2}) = \mu^{1/2}$, similarly for τ'.

Corollary 4.10. *Let $\mathcal{O} \subseteq \mathfrak{gl}_{\mathbb{H}}(V)$ be a nilpotent orbit, then*

$$S'(\tau'(\tau^{-1}(\overline{\mathcal{O}}))) = \tau_c'(\tau_c^{-1}(\overline{S(\mathcal{O})})).$$

Proof: Immediate from theorems 2.4 and 4.8, if we use the fact that on the level of partitions the Kostant–Sekiguchi correspondence for $GL_{\mathbb{H}}(V)$ is just the map $\mathcal{P}(n) \ni \lambda \longmapsto \lambda^2 \in \mathcal{P}(n)^2$ (it follows from the fact that both $\mathcal{O}$ and $S(\mathcal{O})$ are contained in the same $GL_{\mathbb{C}}(V)$-orbit and all three orbits correspond to the same partition, where this time nilpotent elements of $\mathfrak{gl}_{\mathbb{H}}(V)$ are considered as complex matrices of the double size - the effect of this identification on partitions is precisely the identification $\lambda \longmapsto \lambda^2$).

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