ROGAWSKI'S CONJECTURE ON THE JANTZEN FILTRATION FOR THE DEGENERATE AFFINE HECKE ALGEBRA OF TYPE A

TAKESHI SUZUKI

Abstract. The functors constructed by Arakawa and the author relate the representation theory of $\mathfrak{gl}_n$ and that of the degenerate affine Hecke algebra H_ℓ of GL_ℓ. They transform the Verma modules over $\mathfrak{gl}_n$ to the standard modules over H_ℓ. In this paper we prove that they transform the simple modules to the simple modules (in more general situations than in the previous paper). We also prove that they transform the Jantzen filtration on the Verma modules to that on the standard modules. We obtain the following results for the representations of H_ℓ by translating the corresponding results for $\mathfrak{gl}_n$ through the functors: (i) the (generalized) Bernstein-Gelfand-Gelfand resolution for a certain class of simple modules, (ii) the multiplicity formula for the composition series of the standard modules, and (iii) its refinement concerning the Jantzen filtration on the standard modules, which was conjectured by Rogawski.

Introduction

This paper is a continuation of the paper [AS], in which we gave functors from $\mathcal{O}(\mathfrak{gl}_n)$ to $\mathcal{R}(H_\ell)$. Here $\mathcal{O}(\mathfrak{gl}_n)$ denotes the Bernstein-Gelfand-Gelfand (in short, BGG) category of representations of the complex Lie algebra $\mathfrak{gl}_n$, and $\mathcal{R}(H_\ell)$ denotes the category of finite-dimensional representations of the degenerate affine Hecke algebra H_ℓ of GL_ℓ introduced by Drinfeld [Dr].

Let us review the results in [AS]. Let $\mathfrak{t}_n^*$ and W_n denote the space of weights and Weyl group of $\mathfrak{gl}_n$ respectively. For $\lambda \in \mathfrak{t}_n^*$, let $M(\lambda)$ denote the Verma module with highest weight λ and $L(\lambda)$ its simple quotient. Let $V_n = \mathbb{C}^n$ denote the vector representation of $\mathfrak{gl}_n$. For each $\lambda \in \mathfrak{t}_n^*$ and $X \in \mathcal{O}(\mathfrak{gl}_n)$, we define an action of H_ℓ on the finite-dimensional vector space $F_\lambda(X) = \mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X \otimes V_n^{\otimes \ell})$. Under the condition that $\lambda + \rho$ is dominant, we proved that the functor F_λ is exact and $F_\lambda(M(\mu))$ is isomorphic to $\mathcal{M}(\lambda, \mu)$ unless it is zero. Here $\mathcal{M}(\lambda, \mu) \in \mathcal{R}(H_\ell)$ denotes the standard module. With the restriction

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$\ell = n$, we proved that $F_\lambda(L(\mu))$ is isomorphic to the unique simple quotient $\mathcal{L}(\lambda,\mu)$ of $\mathcal{M}(\lambda,\mu)$ unless it is zero. Any simple H_ℓ-module is thus obtained. To prove the irreducibility of $F_\lambda(L(\mu))$, we compared the multiplicities of the simple modules in the composition series of $M(\mu)$ and those in $\mathcal{M}(\lambda,\mu)$ by using the Kazhdan-Lusztig type multiplicity formulas known for $\mathcal{O}(\mathfrak{gl}_n)$ and $\mathcal{R}(H_\ell)$. (See (b) (c) below.)

In the present paper, further properties of the functors are deduced from the key observation that the $\mathfrak{gl}_n$-contravariant bilinear form on a highest weight $\mathfrak{gl}_n$-module X induces the H_ℓ-contravariant bilinear form on $F_\lambda(X)$. The irreducibility of $F_\lambda(L(\mu))$ is deduced from the non-degeneracy of the bilinear form. As a consequence, we can determine the images of simple $\mathfrak{gl}_n$-modules (Theorem 3.2.2) without assuming $\ell = n$ or referring to the multiplicity formulas.

We also prove that F_λ transforms the Jantzen filtration on $M(\mu)$ to that on $F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda,\mu)$ (Theorem 4.3.5).

The followings are the consequences of these results.

(i) We obtain a resolution for a certain class of simple H_ℓ-modules by applying F_λ to the BGG resolution [BGG] and its generalization by Gabber-Joseph [GJ1] for $\mathfrak{gl}_n$-modules. This generalizes the results of Cherednik [Ch1] and Zelevinsky [Ze4].

(ii) To simplify the descriptions, we assume λ and μ are dominant integral weights. (More general cases are treated in §5.2.) Set $w \circ \mu = w(\mu+\rho) - \rho$ for $w \in W_n$ and let $w, y \in W_n$ be such that $\lambda - w\circ\mu$ and $\lambda - y\circ\mu$ are weights of $V_n^{\otimes \ell}$. We have a direct proof of the following formula:

$$[M(w\circ\mu) : L(y\circ\mu)] = [\mathcal{M}(\lambda, w\circ\mu) : \mathcal{L}(\lambda, y\circ\mu)]. \tag{a}$$

Let $P_{w,y}(q)$ denote the Kazhdan-Lusztig polynomial of W_n. The formula (a) implies the equivalence of the following two multiplicity formulas:

$$[\mathcal{M}(\lambda, w\circ\mu) : \mathcal{L}(\lambda, y\circ\mu)] = P_{w,y}(1), \tag{b}$$

$$[M(w\circ\mu) : L(y\circ\mu)] = P_{w,y}(1). \tag{c}$$

The formula (b) was proved by Ginzburg [Gi1] (see also [CG]) for affine Hecke algebras, and (c) was proved by Beilinson-Bernstein [BB1] and Brylinski-Kashiwara [BK] by using the geometric method and the theory of perverse sheaves. We remark that our proof of (a) is purely algebraic.

(iii) We have a refinement of the formula (a): Let λ, μ and w, y be as in (ii). (See §5.3 for more general cases.) Let

$$M(\mu) = M(\mu)_0 \supseteq M(\mu)_1 \supseteq M(\mu)_2 \supseteq \cdots ,$$
$$\mathcal{M}(\lambda,\mu) = \mathcal{M}(\lambda,\mu)_0 \supseteq \mathcal{M}(\lambda,\mu)_1 \supseteq \mathcal{M}(\lambda,\mu)_2 \supseteq \cdots$$

be the Jantzen filtrations on $M(\mu)$ and $\mathcal{M}(\lambda,\mu)$, respectively. Since F_λ preserves the Jantzen filtration, we have

$$[M(w \circ \mu)_j : L(y \circ \mu)] = [\mathcal{M}(\lambda, w \circ \mu)_j : \mathcal{L}(\lambda, y \circ \mu)]. \qquad \text{(a')}$$

The Jantzen filtration on standard modules over affine Hecke algebras of GL was introduced by Rogawski [Ro]. He conjectured a refinement of the formula (b) concerning the Jantzen filtration. Rogawski's conjecture was proved by Ginzburg[1]. (The result is announced in [Gi2] without details.) A degenerate affine Hecke analogue of Rogawski's conjecture is written as follows:

$$\sum_{i \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_i \mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q). \qquad \text{(b')}$$

The formula (a') implies the equivalence between (b') and the improved Kazhdan-Lusztig multiplicity formula

$$\sum_{i \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_i M(w \circ \mu) : L(y \circ \mu)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q), \qquad \text{(c')}$$

which was proved in [BB2].

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1. Basic definitions

1.1. **Lie algebra $\mathfrak{gl}_n$.** Let $\mathfrak{gl}_n$ denote the Lie algebra consisting of all $n \times n$ matrices with entries in $\mathbb{C}$. Let $\mathfrak{t}_n$ be the Cartan subalgebra of $\mathfrak{gl}_n$ consisting of all diagonal matrices. An inner product is defined on $\mathfrak{gl}_n$ by

$$(x|y)_n = \mathrm{tr}(xy) \qquad (1.1.1)$$

[1] I. Grojnowsky announced similar results in a series of his lectures at Kyoto 1997. He also treated affine Hecke algebras at a root of unity.

for $x, y \in \mathfrak{gl}_n$. Let $\mathfrak{t}_n^*$ denote the dual space of $\mathfrak{t}_n$. The natural pairing is denoted by $\langle\ ,\ \rangle_n : \mathfrak{t}_n^* \times \mathfrak{t}_n \to \mathbb{C}$. Let $E_{i,j}$ $(1 \leq i, j \leq n)$ denote the matrix with only nonzero entries 1 at the (i,j)-th component. Define a basis $\{\epsilon_i\}_{i=1,\ldots,n}$ of $\mathfrak{t}_n^*$ by $\epsilon_i(E_{j,j}) = \delta_{i,j}$, and define the roots by $\alpha_{ij} = \epsilon_i - \epsilon_j$ and the simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Put

$$R_n = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}, \tag{1.1.2}$$

$$R_n^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}, \quad R_n^- = R_n \setminus R_n^+, \tag{1.1.3}$$

$$\Pi_n = \{\alpha_i \mid i = 1, \ldots n-1\}. \tag{1.1.4}$$

Then $R_n \subseteq \mathfrak{t}_n^*$ is a root system of type A_{n-1}. Since the restriction of $(\ |\ )_n$ to $\mathfrak{t}_n$ is non-degenerate, we have an isomorphism $\mathfrak{t}_n^* \xrightarrow{\sim} \mathfrak{t}_n$, whose image of $\xi \in \mathfrak{t}_n^*$ is denoted by $\xi^\vee$. In particular we have $\epsilon_i^\vee = E_{i,i}$ and $\alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$.

Putting $\mathfrak{n}_n^+ = \oplus_{i<j} \mathbb{C}E_{i,j}$, $\mathfrak{n}_n^- = \oplus_{i>j} \mathbb{C}E_{i,j}$, we have a triangular decomposition $\mathfrak{gl}_n = \mathfrak{n}_n^+ \oplus \mathfrak{t}_n \oplus \mathfrak{n}_n^-$. We put $\mathfrak{b}_n^\pm = \mathfrak{n}_n^\pm \oplus \mathfrak{t}_n$.

Let σ denote the involution on $\mathfrak{gl}_n$ given by the transposition: $\sigma(E_{i,j}) = E_{j,i}$. The inner product $(\ |\ )_n$ is invariant with respect to σ: $(\sigma(x)|\sigma(y))_n = (x|y)_n$ for all $x, y \in \mathfrak{gl}_n$.

Put $\rho = \frac{1}{2}\sum_{\alpha \in R_n^+} \alpha$ and define

$$Q_n = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i, \tag{1.1.5}$$

$$D_n = \{\lambda \in \mathfrak{t}_n^* \mid \langle \lambda + \rho, \alpha \rangle_n \notin \mathbb{Z}_{<0} \text{ for all } \alpha \in R_n^+\}, \tag{1.1.6}$$

$$D_n^\circ = \{\lambda \in \mathfrak{t}_n^* \mid \langle \lambda, \alpha \rangle_n \notin \mathbb{Z}_{<0} \text{ for all } \alpha \in R_n^+\}, \tag{1.1.7}$$

$$P_n = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i, \quad P_n^+ = P_n \cap D_n^\circ. \tag{1.1.8}$$

An element of D_n° (resp. P_n, P_n^+) is called a *dominant* (resp.*integral, dominant integral*) weight.

1.2. **Weyl group.** Let $W_n \subset \mathrm{GL}(\mathfrak{t}_n^*)$ be the Weyl group associated to the root system (R_n, Π_n), which is by definition generated by the reflections s_α $(\alpha \in R_n)$ defined by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle_n \alpha \quad (\lambda \in \mathfrak{t}_n^*). \tag{1.2.1}$$

We often write $s_i = s_{\alpha_i}$ for $\alpha_i \in \Pi_n$. Note that W_n is isomorphic to the symmetric group $\mathfrak{S}_n$.

We often use another action of W_n on $\mathfrak{t}_n^*$, which is given by

$$w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in W_n,\ \lambda \in \mathfrak{t}_n^*). \tag{1.2.2}$$

For $w, y \in W_n$, we write $w \geq y$ if and only if y can be obtained as a subexpression of a reduced expression of w. The resulting relation in W_n defines a partial order called the *Bruhat order*.

1.3. **Representations of $\mathfrak{gl}_n$.** For a $\mathfrak{t}_n$-module X and $\lambda \in \mathfrak{t}_n^*$, put

$$X_\lambda = \{v \in X \mid hv = \langle \lambda, h \rangle_n v \text{ for all } h \in \mathfrak{t}_n\}, \tag{1.3.1}$$

$$X_\lambda^{\mathrm{gen}} = \{v \in X \mid (h - \langle \lambda, h \rangle_n)^k v = 0 \text{ for all } h \in \mathfrak{t}_n, \text{ some } k \in \mathbb{Z}_{>0}\}, \tag{1.3.2}$$

$$P(X) = \{\lambda \in \mathfrak{t}_n^* \mid X_\lambda \neq 0\}. \tag{1.3.3}$$

The space X_λ (resp X_λ^{gen}) is called the *weight space* (resp. *generalized weight space*) of weight λ with respect to $\mathfrak{t}_n$, and an element of $P(X)$ is called a weight of X.

Let $U(\mathfrak{gl}_n)$ denote the universal enveloping algebra of $\mathfrak{gl}_n$. Let $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n)$ denote the category of $\mathfrak{gl}_n$-modules which are finitely generated over $U(\mathfrak{gl}_n)$, $\mathfrak{n}_n^+$-locally finite and $\mathfrak{t}_n$-semisimple (see [BGG]). The category $\mathcal{O}$ is closed under the operations such as forming subquotient modules, finite direct sums, and tensor products with finite-dimensional modules. For $\lambda \in \mathfrak{t}_n^*$, let $M(\lambda) = U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{b}_n^+)} \mathbb{C}v_\lambda$ denote the Verma module with highest weight λ, where v_λ denotes the highest weight vector. The unique simple quotient of $M(\lambda)$ is denoted by $L(\lambda)$. The modules $M(\lambda)$ and $L(\lambda)$ are objects of $\mathcal{O}$.

Let $\chi_\lambda : Z(U(\mathfrak{gl}_n)) \to \mathbb{C}$ denote the infinitesimal character of $M(\lambda)$. We introduce an equivalence relation in $\mathfrak{t}_n^*$ by

$$\lambda \sim \mu \Leftrightarrow \lambda = w \circ \mu \text{ for some } w \in W_n. \tag{1.3.4}$$

Then it follows that $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$. Let $[\lambda]$ denote the equivalence class of $\lambda \in \mathfrak{t}_n^*$. Define the full subcategory $\mathcal{O}_{[\lambda]}$ of $\mathcal{O}$ by

$$\mathit{obj}\, \mathcal{O}_{[\lambda]} = \{X \in \mathit{obj}\, \mathcal{O} \mid (\mathrm{Ker}\chi_\lambda)^k X = 0 \text{ for some } k\}.$$

Then any $X \in \mathit{obj}\, \mathcal{O}$ admits a decomposition

$$X = \bigoplus_{[\lambda] \in \mathfrak{t}_n^*/\sim} X^{[\lambda]} \tag{1.3.5}$$

such that $X^{[\lambda]} \in \mathit{obj}\, \mathcal{O}_{[\lambda]}$. The correspondence $X \mapsto X^{[\lambda]}$ gives an exact functor on $\mathcal{O}$.

Lemma 1.3.1. *Let $\lambda \in D_n$. Then the natural map $(X^{[\lambda]})_\lambda \to (X/\mathfrak{n}_n^- X)_\lambda$ is bijective.*

Remark 1.3.2. (i) There also exists a canonical bijection $\mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X) \cong (X^{[\lambda]})_\lambda$ for $\lambda \in D_n$.
(ii) A proof of Lemma 1.3.1 for integral λ is given in [AS]. The generalization to non-integral cases is similarly proved.

2. Degenerate affine Hecke algebras and their representations

2.1. **Degenerate affine Hecke algebras.** For a group G, let $\mathbb{C}[G]$ denote its group ring. Let $S(\mathfrak{t}_\ell)$ denote the symmetric algebra of $\mathfrak{t}_\ell$, which is isomorphic to the polynomial ring $\mathbb{C}[\epsilon_1^\vee, \dots, \epsilon_\ell^\vee]$.

Definition 2.1.1. The *degenerate* (*or graded*) *affine Hecke algebra* H_ℓ of GL_ℓ is the unital associative algebra over $\mathbb{C}$ defined by the following properties:

(i) As a vector space, $H_\ell \cong \mathbb{C}[W_\ell] \otimes S(\mathfrak{t}_\ell)$.

(ii) The subspaces $\mathbb{C}[W_\ell] \otimes \mathbb{C}$ and $\mathbb{C} \otimes S(\mathfrak{t}_\ell)$ are subalgebras of H_ℓ in a natural fashion (their images will be identified with $\mathbb{C}[W_\ell]$ and $S(\mathfrak{t}_\ell)$ respectively).

(iii) The following relations hold in H_ℓ:

$$s_\alpha \cdot \xi - s_\alpha(\xi) \cdot s_\alpha = -\langle \alpha, \xi \rangle_\ell \quad (\alpha \in \Pi_\ell,\ \xi \in \mathfrak{t}_\ell). \tag{2.1.1}$$

It is easy to verify the following lemma.

Lemma 2.1.2. *There exists a unique anti-involution ι on H_ℓ such that*

$$\iota(w) = w^{-1}\ (w \in W_\ell), \quad \iota(\xi) = \xi\ (\xi \in \mathfrak{t}_\ell).$$

For a subset $B \subseteq \Pi_\ell$, let $\mathfrak{t}_B$ denote the subspace of $\mathfrak{t}_\ell$ spanned by all $\epsilon_i^\vee$ such that $\langle \alpha, \epsilon_i^\vee \rangle \neq 0$ for some $\alpha \in B$. Put

$$H_B = \mathbb{C}[W_B] \otimes S(\mathfrak{t}_B) \subseteq H_\ell. \tag{2.1.2}$$

Then it turns out that H_B is a subalgebra of H_ℓ.

2.2. **Induced modules.** For a pair $\Delta = [a, b]$ of complex numbers such that $b - a + 1 = \ell \in \mathbb{Z}_{\geq 0}$, there exists a unique one-dimensional representation $\mathbb{C}_\Delta = \mathbb{C}\mathbf{1}_\Delta$ of H_ℓ (we put $H_0 = \mathbb{C}$ for convenience) such that

$$w\mathbf{1}_\Delta = \mathbf{1}_\Delta \quad (w \in W_\ell), \tag{2.2.1}$$

$$\epsilon_i^\vee \mathbf{1}_\Delta = (a + i - 1)\mathbf{1}_\Delta \quad (i = 1, \dots, \ell). \tag{2.2.2}$$

Let $\lambda \in \mathfrak{t}_n^*$ and $\mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P(V_n^{\otimes \ell})$. Then putting

$$\ell_i = \langle \lambda - \mu, \epsilon_i^\vee \rangle_n \in \mathbb{Z}_{\geq 0} \quad (i = 1, \dots, n), \tag{2.2.3}$$

we have $\ell = \sum_{i=1}^n \ell_i$. Let $\Pi_{\lambda,\mu} \subseteq \Pi_\ell$ be the subset associated to the partition $(\ell_1, \dots, \ell_n)$: $\Pi_{\lambda,\mu} = \{\alpha_i \mid i \neq \sum_{k=1}^j \ell_k \text{ for any } j\}$. Put

$$W_{\lambda,\mu} = W_{\Pi_{\lambda,\mu}} \subseteq W_\ell, \quad H_{\lambda,\mu} = H_{\Pi_{\lambda,\mu}} \subseteq H_\ell. \tag{2.2.4}$$

Note that $H_{\lambda,\mu} = H_{\ell_1} \otimes \cdots \otimes H_{\ell_n} = S(\mathfrak{t}_\ell) \otimes \mathbb{C}[W_{\lambda,\mu}]$. Put

$$\Delta_i = [\langle \mu + \rho,\ \epsilon_i^\vee \rangle_n, \langle \lambda + \rho, \epsilon_i^\vee \rangle_n - 1] \in \mathbb{C}^2. \tag{2.2.5}$$

Define the parabolically induced module $\mathcal{M}(\lambda,\mu)$ associated to (λ,μ) by

$$\mathcal{M}(\lambda,\mu) = H_\ell \underset{H_{\lambda,\mu}}{\otimes} (\mathbb{C}_{\Delta_1} \otimes \cdots \otimes \mathbb{C}_{\Delta_n}). \tag{2.2.6}$$

Evidently $\mathcal{M}(\lambda,\mu)$ is a cyclic module with a cyclic weight vector

$$\mathbf{1}_{\lambda,\mu} := \mathbf{1}_{\Delta_1} \otimes \cdots \otimes \mathbf{1}_{\Delta_k}, \tag{2.2.7}$$

whose weight $\zeta_{\lambda,\mu}$ is given by

$$\langle \zeta_{\lambda,\mu}, \epsilon_j^\vee \rangle_\ell = \langle \mu + \rho, \epsilon_i^\vee \rangle_n + j - \sum_{k=1}^{i-1} \ell_k - 1 \quad \text{for} \quad \sum_{k=1}^{i-1} \ell_k < j \leq \sum_{k=1}^{i} \ell_k. \tag{2.2.8}$$

It is also obvious that $\mathcal{M}(\lambda,\mu) \cong \mathbb{C}[W_\ell/W_{\lambda,\mu}]$ as a $\mathbb{C}[W_\ell]$-module and thus its dimension is given by $\dim \mathcal{M}(\lambda,\mu) = \ell!/(\ell_1! \cdots \ell_k!)$. Recall that the simple modules of W_ℓ are parameterized by unordered partitions of ℓ (or Young diagrams of size ℓ). We let S_γ denote the simple W_ℓ-module corresponding to the partition γ. Let $[\lambda - \mu]$ denote the unordered partition of ℓ obtained from $(\ell_1, \ldots, \ell_k)$ by forgetting the order. As is well-known, it holds that

$$\mathcal{M}(\lambda,\mu) \cong S_{[\lambda-\mu]} \oplus \bigoplus_{\beta \rhd [\lambda-\mu]} S_\beta^{\oplus a_\beta}, \tag{2.2.9}$$

as a $\mathbb{C}[W_\ell]$-module . Here $\rhd$ denotes the dominance order in the set of partitions, and a_β are some non-negative integers.

Let $\mathcal{Y}_\ell(n)$ denote the set of Young diagrams of size ℓ consisting of at most n rows. We say that an H_ℓ-module Y is of *level* n if $Y = \oplus_{\gamma \in \mathcal{Y}_\ell(n)} S_\gamma^{\oplus a_\gamma}$ for some $a_\gamma \in \mathbb{Z}_{\geq 0}$. The induced module $\mathcal{M}(\lambda,\mu)$ ($\lambda, \mu \in \mathfrak{t}_n^*$) is of level n. Of course, any finite-dimensional H_ℓ-module is of level ℓ.

2.3. **Zelevinsky's classification of simple modules.** The representation theory of the degenerate affine Hecke algebra is related to that of the affine Hecke algebra by Lusztig [Lu]. Thus the statements in this subsection are deduced from [Ze1, Theorem 6.1] and [Ro, §5]. (See also [Ch2].)

Theorem 2.3.1 ([Ze1, Ro]). *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$.*

(i) *In the decomposition* (2.2.9), *$S_{[\lambda-\mu]}$ generates $\mathcal{M}(\lambda,\mu)$ over H_ℓ.*

(ii) *The H_ℓ-module $\mathcal{M}(\lambda,\mu)$ has the unique simple quotient, which is denoted by $\mathcal{L}(\lambda,\mu)$.*

(iii) *The* $\mathcal{L}(\lambda,\mu)$ *contains* $S_{[\lambda-\mu]}$ *with multiplicity one as a* $\mathbb{C}[W_\ell]$*-module.*

Remark 2.3.2. The statement (i) easily follows from (ii) and (iii).

Theorem 2.3.3 ([Ze1])**.** *Any simple* H_ℓ*-module of level* n *is isomorphic to* $\mathcal{L}(\lambda,\mu)$ *for some* $\lambda \in D_n$ *and* $\mu \in \lambda - P(V_n^{\otimes \ell})$*.*

For $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$, the H_ℓ-module $\mathcal{M}(\lambda,\mu)$ is called a *standard module*. For $\eta \in \mathfrak{t}_n^*$, let $W_n[\eta]$ denote the stabilizer of η:

$$W_n[\eta] = \{w \in W_n \mid w(\eta) = \eta\}, \tag{2.3.1}$$

which is a parabolic subgroup of W_n.

Proposition 2.3.4 ([Ze1])**.** *Suppose that* $\lambda, \mu \in D_n$ *and* $w, y \in W_n$ *satisfy* $\lambda - w\circ\mu \in P(V_n^{\otimes \ell})$ *and* $\lambda - y\circ\mu \in P(V_n^{\otimes \ell})$*. Then the following conditions are equivalent*:

(i) $w \in W_n[\lambda+\rho]yW_n[\mu+\rho]$.

(ii) $\mathcal{M}(\lambda, w\circ\mu) \cong \mathcal{M}(\lambda, y\circ\mu)$.

(iii) $\mathcal{L}(\lambda, w\circ\mu) \cong \mathcal{L}(\lambda, y\circ\mu)$.

Remark 2.3.5. Let $\lambda, \mu \in D_n$ and $w \in W_n$ such that $\lambda - w\circ\mu \in P(V_n^{\otimes \ell})$. We often use the following fact from Proposition 2.3.4:

$$\mathcal{M}(\lambda, w\circ\mu) \cong \mathcal{M}(\lambda, w^\lambda\circ\mu) \cong \mathcal{M}(\lambda, w^\lambda_\mu\circ\mu), \tag{2.3.2}$$

$$\mathcal{L}(\lambda, w\circ\mu) \cong \mathcal{L}(\lambda, w^\lambda\circ\mu) \cong \mathcal{L}(\lambda, w^\lambda_\mu\circ\mu). \tag{2.3.3}$$

Here w^λ (resp. w^λ_μ) denotes the unique longest element in $W_n[\lambda+\rho]w$ (resp. $W_n[\lambda+\rho]wW_n[\mu+\rho]$).

3. Functors F_λ

3.1. **Construction.** Let us recall the definition of the functor

$$F_\lambda : \mathcal{O}(\mathfrak{gl}_n) \to \mathcal{R}(H_\ell)$$

introduced in [AS]. Here $\mathcal{R}(H_\ell)$ denotes the category of finite-dimensional representations of H_ℓ. Let $V_n = \mathbb{C}^n$ denote the vector representation of $\mathfrak{gl}_n$.

Lemma 3.1.1 ([AS])**.** *For any* $X \in \mathcal{O}(\mathfrak{gl}_n)$*, there exists a unique homomorphism*

$$\theta : H_\ell \to \mathrm{End}_{\,\mathfrak{gl}_n}(X \otimes V_n^{\otimes \ell}) \tag{3.1.1}$$

such that

$$s_i \mapsto \Omega_{i\,i+1} \qquad (i = 1, \ldots, \ell - 1), \tag{3.1.2}$$

$$\epsilon_i^\vee \mapsto \sum_{0 \le j < i} \Omega_{ji} + \frac{n-1}{2} \qquad (i = 1, \ldots, \ell), \tag{3.1.3}$$

where

$$\Omega_{ji} = \sum_{1 \le k,m \le n} 1^{\otimes j} \otimes E_{k,m} \otimes 1^{\otimes i-j-1} \otimes E_{m,k} \otimes 1^{\otimes \ell - i} \in \mathrm{End}\,(X \otimes V_n^{\otimes \ell}).$$

Let $\lambda \in D_n$ and $X \in obj\,\mathcal{O}(\mathfrak{gl}_n)$. We define

$$F_\lambda(X) = (X \otimes V_n^{\otimes \ell})_\lambda^{[\lambda]} \tag{3.1.4}$$

with an induced H_ℓ-module structure through the homomorphism θ. We also introduce an H_ℓ-module structure on $\Big((X \otimes V_n^{\otimes \ell})/\mathfrak{n}_n^-(X \otimes V_n^{\otimes \ell})\Big)_\lambda$. Then the bijection given in Lemma 1.3.1 gives an H_ℓ-isomorphism

$$F_\lambda(X) \cong \Big((X \otimes V_n^{\otimes \ell})/\mathfrak{n}_n^-(X \otimes V_n^{\otimes \ell})\Big)_\lambda. \tag{3.1.5}$$

Obviously F_λ defines an exact functor from $\mathcal{O}(\mathfrak{gl}_n)$ to $\mathcal{R}(H_\ell)$.

3.2. **Image of functors.** We extend the definition of $\mathcal{M}(\lambda, \mu)$ for any $\lambda, \mu \in \mathfrak{t}_n^*$ by

$$\mathcal{M}(\lambda, \mu) = 0 \text{ for } \lambda, \mu \in \mathfrak{t}_n^* \text{ such that } \lambda - \mu \notin P(V_n^{\otimes \ell}). \tag{3.2.1}$$

Let $\{u_i\}_{i=1,\ldots,n}$ denote the standard basis of $V_n = \mathbb{C}^n$. For $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$, we define an element $u_{\lambda,\mu} \in \Big((M(\mu) \otimes V_n^{\otimes \ell})\,/\,\mathfrak{n}_n^-(M(\mu) \otimes V_n^{\otimes \ell})\Big)_\lambda$ as the image of $v_\mu \otimes u_1^{\otimes \ell_1} \otimes \cdots \otimes u_n^{\otimes \ell_n} \in M(\mu) \otimes V_n^{\otimes \ell}$, where $\ell_i = \langle \lambda - \mu, \epsilon_i^\vee \rangle_n$. It was shown in [AS] that there exists an H_ℓ-homomorphism

$$\mathcal{M}(\lambda, \mu) \to \Big(M(\mu) \otimes V_n^{\otimes \ell}\,/\,\mathfrak{n}_n^-(M(\mu) \otimes V_n^{\otimes \ell})\Big)_\lambda, \tag{3.2.2}$$

which sends $\mathbf{1}_{\lambda,\mu}$ to $u_{\lambda,\mu}$, and that this is bijective. Combining (3.1.5), we have

Theorem 3.2.1 ([AS])**.** *For each $\lambda \in D_n$ and $\mu \in \mathfrak{t}_n^*$, there is an isomorphism of H_ℓ-modules*

$$F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda, \mu).$$

In particular, the H_ℓ-module $F_\lambda(M(\mu))$ has the unique simple quotient.

A proof of the following statement is given in §4.2.

Theorem 3.2.2. *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$.*

(i) *If μ satisfies the condition*

$$\langle \mu + \rho, \alpha^\vee \rangle_n \leq 0 \quad \textit{for any } \alpha \in R_n^+ \textit{ such that } \langle \lambda + \rho, \alpha^\vee \rangle_n = 0, \tag{3.2.3}$$

then we have

$$F_\lambda(L(\mu)) \cong \mathcal{L}(\lambda, \mu), \tag{3.2.4}$$

where $\mathcal{L}(\lambda, \mu)$ is the unique simple quotient of $\mathcal{M}(\lambda, \mu)$.

(ii) *If μ does not satisfy the condition* (3.2.3)*, then we have*

$$F_\lambda(L(\mu)) = 0. \tag{3.2.5}$$

Remark 3.2.3. (i) In the case $\ell = n$, Theorem 3.2.2 was proved in [AS] using the Kazhdan-Lusztig type multiplicity formula for $\mathcal{O}(\mathfrak{gl}_n)$ and that for $\mathcal{R}(H_\ell)$ (see §5.2).

(ii) Recall that $W_n[\eta] \subseteq W_n$ denotes the stabilizer of $\eta \in \mathfrak{t}_n^*$ (see (2.3.1)). Let W_n^η denote the integral Weyl group of η:

$$W_n^\eta = \{ w \in W_n \mid w \circ \eta - \eta \in Q_n \}. \tag{3.2.6}$$

(Recall that $Q_n = \oplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$.) We can express μ in Theorem 3.2.2 as

$$\mu = w \circ \tilde{\mu}$$

with $\tilde{\mu} \in D_n$ and $w \in W_n^{\tilde{\mu}}$. Then the condition (3.2.3) is equivalent to

$$\mu = w^\lambda \circ \tilde{\mu} \quad \text{or equivalently} \quad \mu = w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}.$$

Here w^λ (resp. $w_{\tilde{\mu}}^\lambda$) denotes the unique longest element in the coset $W_n[\lambda + \rho]w$ (resp. $W_n[\lambda + \rho]wW_n[\tilde{\mu} + \rho]$). (Note that $w^\lambda \circ \tilde{\mu} = w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}$.)

From Theorem 2.3.3 and Proposition 2.3.4, we have

Corollary 3.2.4. *Any finite-dimensional simple H_ℓ-module of level n is isomorphic to $F_\lambda(L(\mu))$ for some $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$ satisfying* (3.2.3)*.*

4. Contravariant forms and the Jantzen filtration

We remark on contravariant bilinear forms on $\mathfrak{gl}_n$-modules and those on H_ℓ-modules. We relate them via the functor F_λ. As a consequence, we have a proof of Theorem 3.2.2 (a similar argument can be seen in the theory of Jantzen's translation functors [Ja]). We also prove that the Jantzen filtration on the Verma modules are transformed to the Jantzen filtration on the standard modules.

4.1. **Contravariant forms.** Let $X \in obj\,\mathcal{O}(\mathfrak{gl}_n)$. A bilinear form $(\ |\ )_X : X \times X \to \mathbb{C}$ is called a $\mathfrak{gl}_n$-*contravariant form* if

$$(xv|u)_X = (v|\sigma(x)u)_X \quad \text{for all } u, v \in X,\ x \in \mathfrak{gl}_n, \tag{4.1.1}$$

where σ is the transposition (§1.1). For $Y \in obj\,\mathcal{R}(H_\ell)$, a bilinear form $(\ |\ )_Y : Y \times Y \to \mathbb{C}$ is called an H_ℓ-*contravariant form* if

$$(xv|u)_Y = (v|\iota(x)u)_Y \quad \text{for all } u, v \in Y,\ x \in H_\ell, \tag{4.1.2}$$

where ι is given in Lemma 2.1.2.

Let us recall some fundamental facts on contravariant bilinear forms. The following lemma is easily shown.

Lemma 4.1.1. (i) *Let $X \in obj\,\mathcal{O}(\mathfrak{gl}_n)$ be equipped with a $\mathfrak{gl}_n$-contravariant bilinear form $(\ |\ )_X$. Then we have*

$$X^{[\lambda]} \perp X^{[\mu]} \quad \text{unless} \quad \lambda \in W_n \circ \mu, \tag{4.1.3}$$

$$X_\lambda \perp X_\mu \quad \text{unless} \quad \lambda = \mu. \tag{4.1.4}$$

(ii) *Let $Y \in obj\,\mathcal{R}(H_\ell)$ be equipped with an H_ℓ-contravariant bilinear form $(\ |\ )_Y$. Then we have*

$$Y^{\mathrm{gen}}_\zeta \perp Y^{\mathrm{gen}}_\eta \quad \text{unless} \quad \zeta = \eta. \tag{4.1.5}$$

Lemma 4.1.2. (i) *Let $\mu \in \mathfrak{t}_n^*$. A $\mathfrak{gl}_n$-contravariant form on $M(\mu)$ is unique up to constant multiples.*

(ii) *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. An H_ℓ-contravariant form on $\mathcal{M}(\lambda, \mu)$ is unique up to constant multiples.*

Proof. (i) is well-known. We will prove (ii). Recall the decomposition (2.2.9):

$$\mathcal{M}(\lambda, \mu) \cong S_{[\lambda-\mu]} \oplus \bigoplus_{\beta \rhd [\lambda-\mu]} S_\beta^{\oplus a_\beta}$$

as a $\mathbb{C}[W_\ell]$-module. Because an H_ℓ-contravariant form is W_ℓ-invariant, its restriction to $S_{[\lambda-\mu]}$ is unique up to constant, and we have

$$S_{[\lambda-\mu]} \perp \bigoplus_{\beta \rhd [\lambda-\mu]} S_\beta^{\oplus a_\beta}. \tag{4.1.6}$$

From Theorem 2.3.1-(i), $S_{[\lambda-\mu]}$ generates $\mathcal{M}(\lambda, \mu)$ over H_ℓ. Thus the statement follows. $\square$

It is easy to construct a non-zero $\mathfrak{gl}_n$-contravariant form on $M(\mu)$. It is also known that there exists a non-zero contravariant form on $\mathcal{M}(\lambda,\mu)$ (see [Ro, CG] and also Remark 4.2.2). In the rest of this paper, we fix a canonical $\mathfrak{gl}_n$-contravariant form $(\ |\ )_{M(\mu)}$ on $M(\mu)$ by $(v_\mu|v_\mu)_{M(\mu)} = 1$. The following lemma is easily shown.

Lemma 4.1.3. (i) *Let $\mu \in \mathfrak{t}_n^*$ and let N be a unique maximal submodule of $M(\mu)$. Then*

$$N = \mathrm{rad}(\ |\ )_{M(\mu)}, \tag{4.1.7}$$

where $\mathrm{rad}(\ |\ )_{M(\mu)}$ *denotes the radical of* $(\ |\ )_{M(\mu)}$.

(ii) *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. Let $(\ |\ )_{\mathcal{M}(\lambda,\mu)}$ be a non-zero H_ℓ-contravariant form on $\mathcal{M}(\lambda,\mu)$ and let $\mathcal{N}$ be a unique maximal submodule of $\mathcal{M}(\lambda,\mu)$. Then we have*

$$\mathcal{N} = \mathrm{rad}(\ |\ )_{\mathcal{M}(\lambda,\mu)}.$$

Proof. (i) is well-known. Let us prove (ii). It is obvious that $\mathrm{rad}(\ |\ ) \subseteq \mathcal{N}$. Theorem 2.3.1 implies that $\mathcal{N} \subseteq \oplus_{\beta \rhd [\lambda-\mu]} \mathcal{S}_\beta^{\oplus\, a_\beta}$ with some $a_\beta \in \mathbb{Z}_{\geq 0}$. Thus we have $S_{[\lambda-\mu]} \perp \mathcal{N}$ by (4.1.6). Hence Theorem 2.3.1-(i) implies that $\mathcal{N} \subseteq \mathrm{rad}(\ |\ )_{\mathcal{M}(\lambda,\mu)}$. □

Let $X, Y \in obj\, \mathcal{O}(\mathfrak{gl}_n)$ with $\mathfrak{gl}_n$-contravariant forms $(\ |\ )_X$, $(\ |\ )_Y$. Then the tensor product $X \otimes Y$ is equipped with a natural $\mathfrak{gl}_n$-contravariant bilinear form such that $(u \otimes v \mid u' \otimes v')_{X \otimes Y} = (u \mid u')_X\, (v \mid v')_Y$ for $u, u' \in X$ and $v, v' \in Y$. The following simple lemma will play a key role.

Lemma 4.1.4. *Let $\lambda \in D_n$. Let X be a highest weight module (i.e. a quotient of a Verma module) of $\mathfrak{gl}_n$.*

(i) *The $\mathfrak{gl}_n$-contravariant form on $X \otimes V_n^{\otimes \ell}$ is also H_ℓ-contravariant, and thus it induces an H_ℓ-contravariant form on $(X \otimes V_n^{\otimes \ell})_\lambda^{[\lambda]} = F_\lambda(X)$.*

(ii) *If the $\mathfrak{gl}_n$-contravariant form on X is non-degenerate, then the induced contravariant form on $F_\lambda(X)$ is non-degenerate.*

Proof. (i) can be easily checked. (ii) follows from Lemma 4.1.1. □

As a consequence of Lemma 4.1.4-(i), the canonical $\mathfrak{gl}_n$-contravariant form on $M(\mu)$ induces an H_ℓ-contravariant form on $\mathcal{M}(\lambda,\mu) = F_\lambda(M(\mu))$, which we call the canonical contravariant form on $\mathcal{M}(\lambda,\mu)$. By Lemma 4.1.3-(i), the $\mathfrak{gl}_n$-contravariant form on $L(\mu)$ is non-degenerate, and it induces a non-degenerate H_ℓ-contravariant form on $F_\lambda(L(\mu))$ by Lemma 4.1.4-(ii). By Lemma 4.1.3-(ii), we have

Corollary 4.1.5. *Suppose that $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. Then the H_ℓ-module $F_\lambda(L(\mu))$ is simple unless it is zero.*

4.2. **Proof of Theorem 3.2.2.** By $F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda, \mu)$ and Corollary 4.1.5, it follows that $F_\lambda(L(\mu))$ is isomorphic to $\mathcal{L}(\lambda, \mu)$ or zero. Hence the proof of Theorem 3.2.2 is reduced to the following lemma:

Lemma 4.2.1. *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. Then $F_\lambda(L(\mu)) \neq 0$ if and only if μ satisfies the condition* (3.2.3).

Remark 4.2.2. Lemma 4.2.1 implies that the canonical $\mathfrak{gl}_n$-contravariant form on $M(\mu)$ induces a non-zero H_ℓ-contravariant form on $F_\lambda(M(\mu))$ if and only if the condition (3.2.3) is satisfied. By Remark 2.3.5 and Remark 3.2.3, it follows that any standard module admits a non-zero H_ℓ-contravariant form.

Proof of Lemma 4.2.1. First we show the "only if" part. Suppose that μ does not satisfy (3.2.3). Then there exists $\alpha \in R_n^+$ such that $\langle \mu+\rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$ and $\langle \lambda+\rho, \alpha^\vee \rangle = 0$. The first inequality implies $M(s_\alpha \circ \mu) \subset M(\mu)$, and the second equality implies $\mathcal{M}(\lambda, \mu) \cong \mathcal{M}(\lambda, s_\alpha \circ \mu)$ (Proposition 2.3.4). Hence we have $F_\lambda(L(\mu)) = 0$, because it is a quotient of $F_\lambda(M(\mu))/F_\lambda(M(s_\alpha \circ \mu)) = 0$.

Let us prove the "if" part. We can write μ as

$$\mu = w \circ \tilde{\mu},$$

where $\tilde{\mu} \in D_n$ and w is an element of the integral Weyl group $W_n^{\tilde{\mu}}$ (see (3.2.6)).

Then the condition (3.2.3) implies $\mu = w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}$, where $w_{\tilde{\mu}}^\lambda$ is the longest element in $W_n[\lambda + \rho] w_{\tilde{\mu}} W_n[\tilde{\mu} + \rho]$ (see Remark 3.2.3). In the Grothendieck group of $\mathcal{O}(\mathfrak{gl}_n)$, we write

$$M(w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}) = L(w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}) + \sum_{y_{\tilde{\mu}}} a_{y_{\tilde{\mu}}} L(y_{\tilde{\mu}} \circ \tilde{\mu}). \tag{4.2.1}$$

Here the sum runs over those elements $y_{\tilde{\mu}} \in W_n$ such that $y_{\tilde{\mu}}$ is longest in $y_{\tilde{\mu}} W_n[\mu + \rho]$ and

$$y_{\tilde{\mu}} > w_{\tilde{\mu}}^\lambda. \tag{4.2.2}$$

Applying F_λ to (4.2.1) we have

$$\mathcal{M}(\lambda, w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}) = F_\lambda(L(w_{\tilde{\mu}}^\lambda \circ \tilde{\mu})) + \sum_{y_{\tilde{\mu}}} a_{y_{\tilde{\mu}}} F_\lambda(L(y_{\tilde{\mu}} \circ \tilde{\mu})) \tag{4.2.3}$$

in the Grothendieck group of $\mathcal{R}(H_\ell)$. Assuming that $F_\lambda(L(w_{\tilde{\mu}}^\lambda \circ \tilde{\mu})) = 0$, we will deduce a contradiction. Since the multiplicity of $\mathcal{L}(\lambda, w_{\tilde{\mu}}^\lambda \circ \tilde{\mu})$ in $\mathcal{M}(\lambda, w_{\tilde{\mu}}^\lambda \circ \tilde{\mu})$ is non-zero, Corollary 4.1.5 implies

$$\mathcal{L}(\lambda, w_{\tilde{\mu}}^\lambda \circ \tilde{\mu}) = F_\lambda(L(y_{\tilde{\mu}} \circ \tilde{\mu})) = \mathcal{L}(\lambda, y_{\tilde{\mu}} \circ \tilde{\mu})$$

for some $y_{\tilde{\mu}}$. But this implies $y_{\tilde{\mu}} \in W_n[\lambda+\rho]w^{\lambda}_{\tilde{\mu}}W_n[\tilde{\mu}+\rho]$ by Proposition 2.3.4, and thus we have $l(y_{\tilde{\mu}}) \leq l(w^{\lambda}_{\tilde{\mu}})$. This contradicts (4.2.2). □

4.3. **The Jantzen filtrations.** Throughout this subsection, we fix a weight $\delta \in \mathfrak{t}_n^*$. Let $A = \mathbb{C}[\,t\,]_{(t)}$ denote the localization of $\mathbb{C}[\,t\,]$ at the prime ideal (t). We use the notation: $\eta^t = \eta + \delta t \in \mathfrak{t}_n^* \otimes A$ for $\eta \in \mathfrak{t}_n^*$.

For $\mu \in \mathfrak{t}_n^*$, let $M(\mu^t)$ be the Verma module of $\mathfrak{gl}_n \otimes A$ with highest weight μ^t:

$$M(\mu^t) = (U(\mathfrak{gl}_n) \otimes A) \underset{U(\mathfrak{b}_n^+)\otimes A}{\otimes} (Av_{\mu^t}).$$

The canonical $\mathfrak{gl}_n$-contravariant bilinear form on $M(\mu)$ can be naturally extended to a $\mathfrak{gl}_n \otimes A$-contravariant form $(\ |\ )_{M(\mu^t)}$ on $M(\mu^t)$ (with respect to the anti-involution $\sigma \otimes \mathrm{id}_A$) with values in A.

Define

$$M(\mu^t)_j = \{v \in M(\mu^t) \mid (v \mid u)_{M(\mu^t)} \in t^j A \text{ for all } u \in M(\mu^t)\}. \tag{4.3.1}$$

Putting $M(\mu)_j = M(\mu^t)_j \,/ (tM(\mu^t) \cap M(\mu^t)_j)$ we have a filtration

$$M(\mu) = M(\mu)_0 \supseteq M(\mu)_1 \supseteq M(\mu)_2 \supseteq \cdots \tag{4.3.2}$$

by $\mathfrak{gl}_n$-modules called the *Jantzen filtration* [Ja].

Our next aim is to define the Jantzen filtration on the standard module, which was introduced in [Ro]. Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$. Analogously to §2.2, we define an $H_\ell \otimes A$-module $\mathcal{M}(\lambda^t, \mu^t)$ by

$$\mathcal{M}(\lambda^t, \mu^t) = (H_\ell \otimes A) \underset{H_{\lambda,\mu}\otimes A}{\otimes} (A\mathbf{1}_{\lambda^t,\mu^t}).$$

Put $X = M(\mu^t) \otimes V_n^{\otimes \ell}$, which is equipped with a $\mathfrak{gl}_n \otimes A$-contravariant form $(\ |\ )_X$. Then $\mathfrak{t}_n^* \otimes A$ acts semisimply on X and it follows that

$$X = \bigoplus_{\eta^t \in \mu^t + P_n} X_{\eta^t}, \tag{4.3.3}$$

$$X_{\eta^t} \perp X_{\nu^t} \text{ unless } \mu = \nu. \tag{4.3.4}$$

Let $\chi_{\eta^t} : Z(U(\mathfrak{gl}_n) \otimes A) \to A$ be the infinitesimal character of $M(\eta^t)$. Following [GJ2], we define for $\eta \in \mathfrak{t}_n^*$ an ideal J_{η^t} of $Z(U(\mathfrak{gl}_n) \otimes A)$ by

$$J_{\eta^t} = \cap_{w \in W_n} \mathrm{Ker}\chi_{(w\circ\eta)^t},$$

and define

$$X^{[\eta^t]} = \{v \in X \mid J^k_{\eta^t} v = 0 \ \text{ for some } k\}. \tag{4.3.5}$$

Obviously $X^{[\eta^t]}$ depends only on the equivalence class $[\eta]$ of η with respect to the equivalence relation (1.3.4).

Lemma 4.3.1 ([GJ2]). *We have*

$$X = \bigoplus_{[\eta] \in \mathfrak{t}_n^* / \sim} X^{[\eta^t]}, \tag{4.3.6}$$

$$X^{[\eta^t]} \perp X^{[\nu^t]} \quad \text{unless } [\eta] = [\nu]. \tag{4.3.7}$$

On the $\mathfrak{gl}_n \otimes A$-module $X = M(\mu^t) \otimes V_n^{\otimes \ell}$, we can define an action of $H_\ell \otimes A$ commuting with $\mathfrak{gl}_n \otimes A$ as in Lemma 3.1.1. We define an induced $H_\ell \otimes A$-module structure on the following spaces:

$$(X/\mathfrak{n}_n^- X)_{\lambda^t}, \quad (X^{[\lambda^t]})_{\lambda^t}. \tag{4.3.8}$$

With respect to this action, the natural map

$$(X^{[\lambda^t]})_{\lambda^t} \to (X/\mathfrak{n}_n^- X)_{\lambda^t} \tag{4.3.9}$$

is an $H_\ell \otimes A$-homomorphism.

Similarly to (3.2.2), we can construct an $H_\ell \otimes A$-homomorphism

$$\mathcal{M}(\lambda^t, \mu^t) \to (X/\mathfrak{n}_n^- X)_{\lambda^t}. \tag{4.3.10}$$

The following lemma is elementary.

Lemma 4.3.2. *Let M and N be free A-modules of finite rank, and let $f : M \to N$ be an A-homomorphism. If the specialization*

$$\bar{f} : M/tM \to N/tN$$

at $t = 0$ is a $\mathbb{C}$-isomorphism, then f is an A-isomorphism.

Using Lemma 4.3.2, we get

Proposition 4.3.3. *The $H_\ell \otimes A$-homomorphisms* (4.3.9) *and* (4.3.10) *are bijective*:

$$(X^{[\lambda^t]})_{\lambda^t} \cong (X/\mathfrak{n}_n^- X)_{\lambda^t} \cong \mathcal{M}(\lambda^t, \mu^t). \tag{4.3.11}$$

Proof. The specialization of (4.3.9) (resp. (4.3.10)) at $t = 0$ gives the isomorphism in Lemma 1.3.1 (resp. (3.2.2)). Therefore by Lemma 4.3.2, it is enough to show that $(X^{[\lambda^t]})_{\lambda^t}$, $(X/\mathfrak{n}_n^- X)_{\lambda^t}$ and $\mathcal{M}(\lambda^t, \mu^t)$ are all free A-modules of finite rank. Obviously they are finitely generated over A. It is also clear that $\mathcal{M}(\lambda^t, \mu^t)$ is free. Since A is a principal ideal domain and X is a free A-module, its subspace $(X^{[\lambda^t]})_{\lambda^t}$ is a free A-module. Finally, let us show that $(X/\mathfrak{n}_n^- X)_{\lambda^t}$ is a free A-module. By the isomorphism

$$X = M(\mu^t) \otimes V_n^{\otimes \ell} \cong (U(\mathfrak{gl}_n) \otimes A) \underset{U(\mathfrak{b}_n^+) \otimes A}{\otimes} (A v_{\mu^t} \otimes V_n^{\otimes \ell}) \tag{4.3.12}$$

as $U(\mathfrak{gl}_n) \otimes A$-modules, it follows that

$$(X/\mathfrak{n}_n^- X)_{\lambda^t} \cong (V_n^{\otimes \ell})_{\lambda - \mu} \otimes A \tag{4.3.13}$$

as A-modules. This is a free A-module. $\square$

It follows that the $\mathfrak{gl}_n \otimes A$-contravariant form on $X = M(\mu^t) \otimes V_n^{\otimes \ell}$ is also $H_\ell \otimes A$-contravariant. Through the isomorphism

$$\mathcal{M}(\lambda^t, \mu^t) \cong (X^{[\lambda^t]})_{\lambda^t} \subset X, \tag{4.3.14}$$

we introduce an A-valued $H_\ell \otimes A$-contravariant form on $\mathcal{M}(\lambda^t, \mu^t)$.

Assume that μ satisfies the condition (3.2.3) in Theorem 3.2.2. Then the induced contravariant form is non-zero (since its specialization at $t = 0$ is non-zero). Therefore we have a filtration

$$\mathcal{M}(\lambda, \mu) = \mathcal{M}(\lambda, \mu)_0 \supseteq \mathcal{M}(\lambda, \mu)_1 \supseteq \mathcal{M}(\lambda, \mu)_2 \supseteq \cdots \tag{4.3.15}$$

by H_ℓ-modules, which we call the Jantzen filtration. Recall that any standard module is isomorphic to $\mathcal{M}(\lambda, \mu)$ for some $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$ satisfying (3.2.3) (Remark 2.3.5).

Remark 4.3.4. In [Ro], the deformation direction δ is restricted by a certain condition. The construction above gives the definition of the Jantzen filtration for an arbitrary direction δ.

Theorem 4.3.5. *Suppose that $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$ satisfy the condition* (3.2.3)*. Then $F_\lambda(M(\mu)_j) = \mathcal{M}(\lambda, \mu)_j$.*

Proof. It is easy to check that $F_\lambda(M(\mu)_j) \subseteq \mathcal{M}(\lambda, \mu)_j$. To prove the opposite inclusion, let

$$p : M(\mu^t) \otimes V_n^{\otimes \ell} \to (M(\mu^t) \otimes V_n^{\otimes \ell})_{\lambda^t}^{[\lambda^t]} = \mathcal{M}(\lambda^t, \mu^t)$$

denote the natural projection. Note that $(M(\mu^t) \otimes V_n^{\otimes \ell}))_{\lambda^t}^{[\lambda^t]} \perp \mathrm{Ker} p$ by (4.3.4) and (4.3.7). Fix any orthonormal basis $\{b_i\}_{i=1}^{n^\ell}$ of $V_n^{\otimes \ell}$ with respect to the $\mathfrak{gl}_n$-contravariant form $(\ |\ )_{V_n^{\otimes \ell}}$.

Take any $u \in \mathcal{M}(\lambda^t, \mu^t)_j \subseteq (M(\mu^t) \otimes V_n^{\otimes \ell}))_{\lambda^t}^{[\lambda^t]}$ and write as $u = \sum_i a_i \otimes b_i$ with $a_i \in M(\mu^t)$. Then for any $v \in M(\mu^t)$ and k, we have

$$\begin{aligned}(a_k \mid v)_{M(\mu^t)} &= (u \mid v \otimes b_k)_{M(\mu^t) \otimes V_n^{\otimes \ell}} = (u \mid p(v \otimes b_k))_{M(\mu^t) \otimes V_n^{\otimes \ell}} \\ &= (u \mid p(v \otimes b_k))_{(M(\mu^t) \otimes V_n^{\otimes \ell})_{\lambda^t}^{[\lambda^t]}} \in t^j A.\end{aligned}$$

This implies $a_k \in M(\mu^t)_j$ and thus $u \in (M(\mu^t)_j \otimes V_n^{\otimes \ell})_{\lambda^t}^{[\lambda^t]}$. Therefore we have $F_\lambda(M(\mu)_j) \supseteq \mathcal{M}(\lambda, \mu)_j$. $\square$

5. Consequences

5.1. **BGG resolution.** Recall the generalization of the BGG resolution for certain simple $\mathfrak{gl}_n$-modules given by Gabber-Joseph [GJ1].

We fix $\mu \in \mathfrak{t}_n^*$ such that $-(\mu+\rho)$ is dominant and regular, i.e. $\langle -(\mu+\rho), \alpha^\vee\rangle_n \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in R_n^+$. Set $R_n^\mu = \{\alpha \in R_n \mid \langle \mu, \alpha^\vee\rangle_n \in \mathbb{Z}\}$. It is known that R_n^μ is a root system and its Weyl group coincides with the integral Weyl group

$$W_n^\mu = \{w \in W_n \mid w \circ \mu - \mu \in Q_n\}. \tag{5.1.1}$$

Set $R_n^{\mu+} = R_n^\mu \cap R_n^+$ and let Π_n^μ be the set of simple roots of $R_n^{\mu+}$.

Fix $B \subseteq \Pi_n^\mu$. The length function l_B and the Bruhat order of W_B are defined with respect to the set of simple roots B. Let w_B be a unique longest element of W_B with respect to l_B. Put $\mu_B = w_B \circ \mu$. Gabber-Joseph constructed the exact sequence

$$0 \leftarrow L(\mu_B) \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \tag{5.1.2}$$

of $\mathfrak{gl}_n$-modules, where

$$C_i = \bigoplus_{y \in W_B,\ l_B(y)=i} M(y \circ \mu_B).$$

We apply F_λ to the sequence (5.1.2). Then Theorem 3.2.1 and Theorem 3.2.2 imply the following:

Theorem 5.1.1. *Let μ and B as above. Suppose that $\lambda \in D_n \cap (\mu_B + P(V_n^{\otimes \ell}))$ satisfies $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for any $\alpha \in B$. Then there exists an exact sequence*

$$0 \leftarrow \mathcal{L}(\lambda, \mu_B) \leftarrow \mathcal{C}_0 \leftarrow \mathcal{C}_1 \leftarrow \cdots \tag{5.1.3}$$

of H_ℓ-modules, where

$$\mathcal{C}_i = \bigoplus_{y \in W_B,\ l_B(y)=i} \mathcal{M}(\lambda, y \circ \mu_B).$$

Remark 5.1.2. In the case $\mu_B \in P_n^+$ and $B = \Pi_\ell$ (the original BGG case [BGG]), the corresponding sequence has been obtained by Cherednik [Ch1] by a different method (see also [Ze4, AST]).

5.2. **Kazhdan-Lusztig formulas.** For a module M and simple module L, let $[M : L]$ denote the multiplicity of L in the composition series of M.

Recall that W_n^μ denotes the integral Weyl group of $\mu \in \mathfrak{t}_n^*$ (see (3.2.6)). The following formula is a direct consequence of Theorem 3.2.1 and Theorem 3.2.2:

Theorem 5.2.1. *Let $\lambda, \mu \in D_n$ and let $w, y \in W_n^\mu$ such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P(V_n^{\otimes \ell})$. Then we have*

$$[\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = [M(w \circ \mu) : L(y^\lambda \circ \mu)], \tag{5.2.1}$$

where y^λ denotes the longest element in $W_n[\lambda + \rho]y$.

Let $\lambda, \mu \in D_n$ and $w, y \in W_n^\mu$ be as in Theorem 5.2.1. The equality (5.2.1) has been known through the following two multiplicity formulas:

$$[M(w \circ \mu) : L(y \circ \mu)] \quad = P_{w,y_\mu}(1), \tag{5.2.2}$$

$$[\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = P_{w,y_\mu^\lambda}(1). \tag{5.2.3}$$

Here $P_{w,y}(q) \in \mathbb{Z}[q, q^{-1}]$ denotes the Kazhdan-Lusztig polynomial [KL1] of the Hecke algebra associated to W_n^μ (we put $P_{w,y}(q) = 0$ for $w \not\leq y$ for convenience), and y_μ (resp. y_μ^λ) denotes the longest element in $yW_n[\mu + \rho]$ (resp. $W_n[\lambda + \rho]yW_n[\mu + \rho]$).

Remark 5.2.2. It follows from (5.2.2) and (5.2.3) that $P_{w,y_\mu}(1) = P_{w_\mu,y_\mu}(1)$ and $P_{w,y_\mu^\lambda}(1) = P_{w_\mu,y_\mu^\lambda}(1) = P_{w_\mu^\lambda,y_\mu^\lambda}(1)$. The latter is expressed in terms of the intersection cohomology concerning nilpotent orbits on the quiver variety [Ze3].

The formula (5.2.2) was conjectured by Kazhdan-Lusztig [KL1] and proved by Beilinson-Bernstein [BB1] and Brylinski-Kashiwara [BK]. The formula (5.2.3) was conjectured by Zelevinsky [Ze2] (see also [Ze3]) and proved by Ginzburg [Gi1] (see also [CG]). The theory of perverse sheaves plays an essential role in these proofs.

Theorem 5.2.1 (proved in a purely algebraic way) says that the Kazhdan-Lusztig formula (5.2.2) is equivalent to its degenerate affine Hecke analogue (or its p-adic analogue) (5.2.3). The implication (5.2.2) $\Rightarrow$ (5.2.3) is obvious. The implication (5.2.3) $\Rightarrow$ (5.2.2) is proved as follows. Take any $\mu \in D_n$ and $w, y \in W_n^\mu$. Then we can find $\ell \in \mathbb{Z}_{\geq 2}$ and $\lambda \in D_n^\circ$ such that

$$\lambda - z \circ \mu \in P(V_n^{\otimes \ell}) \text{ for all } z \in W_n^\mu.$$

In this case $F_\lambda(L(z \circ \mu))$ never vanishes and thus it is isomorphic to $\mathcal{L}(\lambda, z \circ \mu)$. Now (5.2.3) implies (5.2.2).

5.3. **Rogawski's conjecture.** Let $\{M(\mu)_j\}_j$ and $\{\mathcal{M}(\lambda, \mu)_j\}_j$ be the Jantzen filtrations defined in §4.3. As a direct consequence of Theorem 3.2.2 and Theorem 4.3.5, we have

Theorem 5.3.1. *Let $\lambda, \mu \in D_n$ and $w, y \in W_n^\mu$ (see (3.2.6)) be such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P(V_n^{\otimes \ell})$. Then we have*

$$[\mathcal{M}(\lambda, w \circ \mu)_j : \mathcal{L}(\lambda, y \circ \mu)] = [M(w^\lambda \circ \mu)_j : L(y^\lambda \circ \mu)], \tag{5.3.1}$$

where w^λ and y^λ denote the longest element in $W_n[\lambda+\rho]w$ and $W_n[\lambda+\rho]y$ respectively.

A priori the Jantzen filtrations depend on the choice of the deformation direction $\delta \in \mathfrak{t}_n^*$. It has been known that the Jantzen filtration on $M(\mu)$ does not depend on the choice of δ for which $(\ |\ )_{M(\mu^t)}$ is non-degenerate [Ba]. Now Theorem 4.3.5 implies

Proposition 5.3.2. *Let $\lambda \in D_n$ and $\mu \in \lambda - P(V_n^{\otimes \ell})$ satisfy (3.2.3). Then the Jantzen filtration on $\mathcal{M}(\lambda, \mu)$ does not depend on the choice of δ such that*

$$\langle \delta, \alpha^\vee \rangle_n \neq 0 \ \text{ for any } \alpha \in R_n^+ \text{ such that } \langle \mu + \rho, \alpha^\vee \rangle_n \in \mathbb{Z}_{>0}. \tag{5.3.2}$$

Remark 5.3.3. For λ and μ as in Proposition 5.3.2, the condition (5.3.2) is equivalent to the condition that the $H_\ell \otimes A$-contravariant form $(\ |\ )_{\mathcal{M}(\lambda^t, \mu^t)}$ is non-degenerate.

We say that the Jantzen filtration $\{M(\mu)_j\}_j$ (or $\{\mathcal{M}(\lambda, \mu)_j\}_j$) is *regular* if the deformation direction δ satisfies (5.3.2). The following formula was conjectured in [GJ2, GM], and proved in [BB2].

Theorem 5.3.4 ([BB2])**.** *Let $\mu \in D_n$ and $w, y \in W_n^\mu$. Suppose that w and y are the longest elements in $wW_n[\mu+\rho]$ and $yW_n[\mu+\rho]$, respectively. For the regular Jantzen filtration $\{M(w \circ \mu)_j\}_j$, we have*

$$\sum_{j \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_j M(w \circ \mu) : L(y \circ \mu)] q^{(l_\mu(y) - l_\mu(w) - j)/2} = P_{w,y}(q), \tag{5.3.3}$$

where $P_{w,y}(q)$ denotes the Kazhdan-Lusztig polynomial of W_n^μ, and l_μ denotes the length function on W_n^μ.

Combining with Theorem 5.3.1, the improved Kazhdan-Lusztig formula (5.3.3) implies its degenerate affine Hecke analogue, which was conjectured in [Ro].

Theorem 5.3.5. (c.f. [Gi2, Theorem 2.6.1]) *Let $\lambda, \mu \in D_n$ and $w, y \in W_n^\mu$ be such that $\lambda - w \circ \mu$, $\lambda - y \circ \mu \in P(V_n^{\otimes \ell})$. Suppose that w and y are the longest elements in $W_n[\lambda+\rho]wW_n[\mu+\rho]$ and $W_n[\lambda+\rho]yW_n[\mu+\rho]$, respectively. For the regular Jantzen filtration $\{\mathcal{M}(\lambda, w \circ \mu)_j\}_j$, we have*

$$\sum_{j \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_j \mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] q^{(l_\mu(y) - l_\mu(w) - j)/2} = P_{w,y}(q), \tag{5.3.4}$$

where $P_{w,y}(q)$ denotes the Kazhdan-Lusztig polynomial of W_n^μ, and l_μ denotes the length function on W_n^μ.

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Research Institute for Mathematical Sciences, Kyoto University, Japan
E-mail address: `takeshi@kurims.kyoto-u.ac.jp`