DIRAC OPERATORS IN REPRESENTATION THEORY

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Lecture 1. $(\mathfrak{g}, K)$-modules

1.1. Lie groups and algebras. A Lie group G is a group which is also a smooth manifold, in such a way that the group operations are smooth. In more words, the multiplication map from $G \times G$ into G and the inverse map from G into G are required to be smooth.

The Lie algebra $\mathfrak{g}$ of G consists of left invariant vector fields on G. The left invariance condition means the following: let $l_g : G \to G$ be the left translation by $g \in G$, i.e., $l_g(h) = gh$. A vector field X on G is left invariant if $(dl_g)_h X_h = X_{gh}$ for all $g, h \in G$.

$\mathfrak{g}$ is clearly a vector space. It can be identified with the tangent space to G at the unit element e. Namely, to any left invariant vector field one can attach its value at e. Conversely, a tangent vector at e can be translated to all other points of G to obtain a left invariant vector field.

Note that we did not require our vector fields to be smooth; it is however a fact that a left invariant vector field is automatically smooth.

The operation making $\mathfrak{g}$ into a Lie algebra is the bracket of vector fields:

$$[X, Y]f = X(Yf) - Y(Xf),$$

for $X, Y \in G$ and f a smooth function on G. Here we identify vector fields with derivations of the algebra $C^\infty(G)$, i.e., think of them as first order differential operators.

The operation [,] satisfies the well known properties of a Lie algebra operation: it is bilinear and anticommutative, and it satisfies the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for any $X, Y, Z \in \mathfrak{g}$.

The main examples are various matrix groups. They fall into several classes. We are primarily interested in the semisimple connected groups, like the group $SL(n, \mathbb{R})$ of $n \times n$ matrices with determinant 1. Its Lie algebra is $\mathfrak{sl}(n, \mathbb{R})$, consisting of the $n \times n$ matrices with trace 0. There are also familiar series of compact groups: $SU(n)$, the group of unitary (complex) matrices, with Lie algebra $\mathfrak{su}(n)$ consisting of skew Hermitian matrices, and $SO(n)$, the group of orthogonal (real) matrices with Lie algebra $\mathfrak{so}(n)$ consisting of antisymmetric matrices. Further examples are the groups $Sp(n)$, $SU(p, q)$ and $SO(p, q)$; they are all defined as groups of operators preserving certain forms.

Other classes of Lie groups one needs to study are solvable groups, like the groups of upper triangular matrices; nilpotent groups like the groups of unipotent matrices and abelian groups like $\mathbb{R}^n$ or the groups of diagonal matrices. They are not of our primary interest, but they show up as subgroups of our semisimple groups and therefore have to be understood.

One also often considers reductive groups, which include semisimple groups but are allowed to have a larger center, like $GL(n,\mathbb{R})$ or $U(n)$.

The definitions are easiest to formulate for Lie algebras $\mathfrak{g}$: define $C^0\mathfrak{g} = D^0\mathfrak{g} = \mathfrak{g}$, and inductively $C^{i+1}\mathfrak{g} = [\mathfrak{g}, C^i\mathfrak{g}]$, $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$. Then $\mathfrak{g}$ is nilpotent if $C^i\mathfrak{g} = 0$ for large i, $\mathfrak{g}$ is solvable if $D^i\mathfrak{g} = 0$ for large i, $\mathfrak{g}$ is semisimple if it contains no nonzero solvable ideals, and $\mathfrak{g}$ is reductive if it contains no nonabelian solvable ideals. Abelian means that $[,] = 0$.

Instead of checking that each of the above mentioned groups is a Lie group, on can refer to a theorem of Cartan, which asserts that every closed subgroup of a Lie group is automatically a Lie subgroup in a unique way. Since each of the groups we mentioned is contained in $GL(n,\mathbb{C})$ we only need to see that $GL(n,\mathbb{C})$ is a Lie group. But $GL(n,\mathbb{C})$ is an open subset of $\mathbb{C}^{n^2}$, hence is a manifold, and the matrix multiplication and inverting are clearly smooth. Without using Cartan's theorem, one can apply implicit/inverse function theorems, as each of the above groups is given by certain equations that the matrix coefficients must satisfy.

Let us note some common features of all the mentioned examples:

• $\mathfrak{g}$ is contained in the matrix algebra $M_n(\mathbb{C})$, and $[X,Y] = XY - YX$, the commutator of matrices;

• G acts on $\mathfrak{g}$ by conjugation: $Ad(g)X = gXg^{-1}$. This is called the adjoint action. The differential of this action with respect to g gives an action of $\mathfrak{g}$ on itself, $ad(X)Y = [X,Y]$, which is also called the adjoint action.

• There is an exponential map $\exp : \mathfrak{g} \longrightarrow G$ mapping X to e^X. This is a local diffeomorphism around 0, i.e., sends a neighborhood of 0 in $\mathfrak{g}$ diffeomorphically onto a neighborhood of e in G.

1.2. Finite dimensional representations. Let V be a complex n-dimensional vector space. A representation of G on V is a continuous homomorphism

$$\pi : G \longrightarrow GL(V).$$

Any such homomorphism is automatically smooth; this is a version of the already mentioned Cartan's theorem.

Given a representation of G as above, we can differentiate it at e and obtain a homomorphism

$$d\pi = \pi : \mathfrak{g} \longrightarrow \mathfrak{g}l(V)$$

of Lie algebras.

An important special case is the case of a unitary representation. This means V has an inner product and that all operators $\pi(g)$, $g \in G$ are unitary. Then all $\pi(X)$, $X \in \mathfrak{g}$ are skew-hermitian.

The main idea of passing from G to $\mathfrak{g}$ is turning a harder, analytic problem of studying representations of G into an easier, purely algebraic (or even combinatorial in some sense)

problem of studying representations of $\mathfrak{g}$. Actually, since we are only considering complex representations, we can as well complexify $\mathfrak{g}$ and study representations of $\mathfrak{g}_{\mathbb{C}}$.

Example. The most basic example and the first one to study is the representations of $\mathfrak{g} = \mathfrak{s}l(2, \mathbb{C})$. There is an obvious basis for $\mathfrak{g}$: take

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the commutator relations are

$$[h, e] = 2e; \quad [h, f] = -2f; \quad [e, f] = h.$$

Let V be an irreducible representation of $\mathfrak{g}$ of dimension $n + 1$. Then there is a basis

$$\{v_k \big| k = -n, -n + 2, \ldots, n - 2, n\}$$

of V, such that the action of h is diagonal in this basis:

$$\pi(h)v_k = kv_k.$$

The integers k are called the weights of V. The operator $\pi(e)$ raises the weight by 2, i.e., $\pi(e)v_k$ is proportional to v_{k+2} (which is taken to be 0 if $k = n$). The operator $\pi(f)$ lowers the weight by 2. There is an obvious ordering on the set of weights (usual ordering of integers), and n is the highest weight of V; this corresponds to the fact that $\pi(e)v_n = 0$. Here is a picture of our representation V:

$$v_{-n} \underset{f}{\overset{e}{\rightleftarrows}} v_{-n+2} \underset{f}{\overset{e}{\rightleftarrows}} \cdots \underset{f}{\overset{e}{\rightleftarrows}} v_{n-2} \underset{f}{\overset{e}{\rightleftarrows}} v_n$$

Finally, the actions of e and f are also completely determined: let us normalize the choice of v_k's by taking v_n to be an arbitrary $\pi(h)$-eigenvector for eigenvalue n, and then taking $v_{n-2j} = \pi(f)^j v_n$. Then

$$(*) \qquad \pi(e)v_{n-2j} = j(n - j + 1)v_{n-2j+2}.$$

All of this is easy to prove; we give an outline here and encourage the reader to complete the details.

First, it is a general fact that the action of h diagonalizes in any finite dimensional representation, as h is a semisimple element of $\mathfrak{g}$. One can however avoid using this, and just note that the action of h will have an eigenvalue, say $\lambda \in \mathbb{C}$, with an eigenvector v_λ. By finite dimensionality, we can take λ to be the highest eigenvalue. For any $\mu \in \mathbb{C}$, let us denote by V_μ the μ-eigenspace for $\pi(h)$; of course, V_μ will be zero for all but finitely many μ.

From the commutator relations, it is immediate that $\pi(e)$ sends ¿V_μ to $V_{\mu+2}$, while $\pi(f)$ sends V_μ to $V_{\mu-2}$ for any μ. In particular, the sum of all V_μ is $\mathfrak{g}$-invariant, hence it is all of V by irreducibility. One now takes a highest weight vector v_λ, defines (as above)

$v_{\lambda-2j} = \pi(f)^j v_\lambda$, and then proves the analog of the formula (*) with λ replacing n, by induction from the commutator relations. From (*), finite dimensionality and irreducibility it now follows that ¿λ must be a non-negative integer, say n, and that V is the ¿span of the vectors $v_n, v_{n-2}, \ldots, v_{-n}$.

Thus we have described all irreducible finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-modules. Other finite dimensional representations are direct sums of irreducible ones; this is a special case of Weyl's theorem, which says this is true for any semisimple Lie algebra $\mathfrak{g}$. One way to prove this theorem is the so called unitarian trick of Weyl: one shows that there is a compact group G with the same representations as $\mathfrak{g}$. E.g., for $\mathfrak{sl}(2,\mathbb{C})$, $G = SU(2)$. Now using invariant integration one shows that every representation of a compact group is unitary. On the other hand, unitary (finite dimensional) representations are easily seen to be direct sums of irreducibles; namely, for every invariant subspace, its orthogonal is an invariant complement.

We now pass on to describe finite dimensional representations of a general semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Instead of just one element h to diagonalize, we can now have a bunch of them. They comprise a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is by definition a maximal abelian subalgebra consisting of semisimple elements. Elements of $\mathfrak{h}$ can be simultaneously diagonalized in any representation; for each joint eigenspace, the eigenvalues for various $X \in \mathfrak{h}$ are described by a functional $\lambda \in \mathfrak{h}^*$, a weight of the representation under consideration. All possible weights of finite dimensional representations form a lattice in $\mathfrak{h}^*$, called the weight lattice of $\mathfrak{g}$. The nonzero weights of the adjoint representation of $\mathfrak{g}$ on itself have a prominent role in the theory; they are called the roots of $\mathfrak{g}$, and satisfy a number of nice symmetry properties. For example, the roots of $\mathfrak{sl}(3,\mathbb{C})$ form a regular hexagon in the plane. The roots of $\mathfrak{sl}(2,\mathbb{C})$ are 2 and -2 (upon identifying $\mathfrak{h}^* = (\mathbb{C}h)^*$ with $\mathbb{C}$).

One can divide up roots into positive and negative roots, which gives an ordering on $\mathfrak{h}^*_{\mathbb{R}}$, the real span of roots, and also a notion of positive ("dominant") weights. The irreducible finite dimensional representations of $\mathfrak{g}$ are classified by their highest weights. The possible highest weights are precisely the dominant weights.

If we denote by $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) the subalgebra of $\mathfrak{g}$ spanned by all positive (respectively negative) root vectors, then we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

For example, if $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, one can take $\mathfrak{h}$ to be the diagonal matrices in $\mathfrak{g}$, $\mathfrak{n}^+$ the strictly upper triangular matrices and $\mathfrak{n}^-$ the strictly lower triangular matrices.

Let V be an irreducible finite dimensional representation with highest weight λ. The highest weight vector is unique up to scalar, and is characterized by being an eigenvector for each $X \in \mathfrak{h}$, with eigenvalue $\lambda(X)$, and by being annihilated by all elements of $\mathfrak{n}^+$.

We now address the question of "going back", i.e., getting representations of G from representations of $\mathfrak{g}$ which we have just described. This is called "integrating" or "exponentiating" representations.

It turns out there is a topological obstacle to integrating representations; this can already be seen in the simplest case of 1-dimensional (abelian) Lie groups. There are two connected 1-dimensional groups: the real line $\mathbb{R}$ and the circle group $\mathbb{T}^1$. Both have the same Lie algebra $\mathbb{R}$.

Consider the one-dimensional representations of the Lie algebra $\mathbb{R}$; each of them is given by $t \mapsto t\lambda$ for some $\lambda \in \mathbb{C}$ (we identify 1×1 complex matrices with complex numbers). All of these representations exponentiate to the group $\mathbb{R}$, and give all possible characters of $\mathbb{R}$:

$$t \longmapsto e^{t\lambda}, \quad \lambda \in \mathbb{C}.$$

However, of these characters only the periodic ones will be well defined on $\mathbb{T}^1$, and $e^{t\lambda}$ is periodic if and only if $\lambda \in 2\pi i\mathbb{Z}$.

In general, when G is connected and simply connected, then all finite dimensional representations of $\mathfrak{g}$ integrate to G. Any connected G will be covered by a simply connected $\tilde{G}$, and the representations of G are those representations of $\tilde{G}$ which are trivial on the kernel of the covering.

If G is semisimple connected with finite center, then there is a decomposition

$$G = KP$$

called the Cartan decomposition; here K is the maximal compact subgroup of G, while P is diffeomorphic to a vector space. For example, if $G = SL(n, \mathbb{R})$, then $K = SO(n)$, and P consists of positive matrices in $SL(n, \mathbb{R})$; so P is diffeomorphic to the vector space of all symmetric matrices via the exponential map. It follows that the topology of G is the same as the topology of K, and a representation of $\mathfrak{g}$ will exponentiate to G if and only if it exponentiates to K.

1.3. Infinite dimensional representations. In general, a representation of G is a continuous linear action on a topological vector space $\mathcal{H}$. Some typical choices for $\mathcal{H}$ are: a Hilbert space, a Banach space, or a Fréchet space.

The first question would be: can we differentiate a representation to get a representation of $\mathfrak{g}$? The answer is: not quite. Actually, $\mathfrak{g}$ acts, but only on the dense subspace $\mathcal{H}_\infty$ of smooth vectors (a vector $v \in \mathcal{H}$ is smooth if the map $g \mapsto \pi(g)v$ from G into $\mathcal{H}$ is smooth).

For semisimple G with maximal compact K, there is a better choice than $\mathcal{H}_\infty$: we can consider the dense subspace $\mathcal{H}_K$ of K-finite vectors in $\mathcal{H}$ (a vector $v \in \mathcal{H}$ is K-finite if the set $\pi(K)v$ spans a finite dimensional subspace of $\mathcal{H}$). One shows that K-finite vectors are all smooth, and so $\mathcal{H}_K$ becomes a Harish-Chandra module, or a $(\mathfrak{g}, K)$ module. A vector space V is a $(\mathfrak{g}, K)$-module if it has:

(1) an action of $\mathfrak{g}$;
(2) a finite action of K;
(3) the two $\mathfrak{k}$-actions obtained from (1) and (2) agree.

In case we want to allow disconnected groups, we also need

(4) the $\mathfrak{g}$-action is K-equivariant, i.e., $\pi(k)\pi(X)v = \pi(Ad(k)X)\pi(k)v$, for all $k \in K$, $X \in \mathfrak{g}$ and $v \in V$.

One usually also puts some finiteness conditions, like finite generation, or admissibility defined below.

Now a basic fact about (unitary) representations of compact groups is that they can be decomposed into Hilbert direct sums of irreducibles, which are all finite dimensional. This can be proved using the basic facts about compact operators. Thus our $\mathcal{H}$ decomposes into a Hilbert direct sum of K-irreducibles, and Harish-Chandra modules, being K-finite,

decompose into algebraic direct sums of K-irreducibles. For $\delta \in \widehat{K}$, the unitary dual of K, we denote by $V(\delta)$ the δ-isotypic component of a Harish-Chandra module, i.e., the largest K-submodule of V which is isomorphic to a sum of copies of δ. We say V is admissible if each $V(\delta)$ is finite dimensional. In other words, every $\delta \in \widehat{K}$ occurs in V with finite multiplicity. Harish-Chandra proved that for each unitary representation $\mathcal{H}$ of G, the $(\mathfrak{g}, K)$-module $\mathcal{H}_K$ is admissible.

Going back from $(\mathfrak{g}, K)$-modules to representations of G is hard. We call a representation $(\pi, \mathcal{H})$ of G a globalization of a Harish-Chandra module V, if V is isomorphic to $\mathcal{H}_K$. Every irreducible V has globalizations. In fact there are many of them; one can choose e.g. a Hilbert space globalization (not necessarily unitary), or a smooth globalization. There are also notions of minimal and maximal globalizations. A few names to mention here are Harish-Chandra, Lepowsky, Rader, Casselman, Wallach and Schmid. We will not need any globalizations, but will from now on work only with $(\mathfrak{g}, K)$-modules.

1.4. Infinitesimal characters. Recall that a representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra morphism from $\mathfrak{g}$ into the Lie algebra $\mathrm{End}(V)$ of endomorphisms of V. Now $\mathrm{End}(V)$ is actually an associative algebra, which is turned into a Lie algebra by defining $[a, b] = ab - ba$; this can be done for any associative algebra. What we want is to construct an associative algebra $U(\mathfrak{g})$ containing $\mathfrak{g}$, so that representations of $\mathfrak{g}$ extend to morphisms $U(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$ of associative algebras. The construction goes like this: consider first the tensor algebra $T(\mathfrak{g})$ of the vector space $\mathfrak{g}$. Then define

$$U(\mathfrak{g}) = T(\mathfrak{g})/I,$$

where I is the two-sided ideal of $T(\mathfrak{g})$ generated by elements $X \otimes Y - Y \otimes X - [X, Y]$, $X, Y \in \mathfrak{g}$. It is easy to see that $U(\mathfrak{g})$ satisfies a universal property with respect to maps of $\mathfrak{g}$ into associative algebras; in particular, representations of $\mathfrak{g}$ (i.e., $\mathfrak{g}$-modules) are the same thing as $U(\mathfrak{g})$-modules. Some further properties are

- There is a filtration by degree on $U(\mathfrak{g})$, coming from $T(\mathfrak{g})$;
- The graded algebra associated to the above filtration is the symmetric algebra $S(\mathfrak{g})$;
- One can get a basis for $U(\mathfrak{g})$ by taking monomials over an ordered basis of $\mathfrak{g}$.

The last two properties are closely related and are the content of the Poincaré-Birkhoff-Witt theorem. Loosely speaking, one can think of $U(\mathfrak{g})$ as "noncommutative polynomials over $\mathfrak{g}$", with the commutation laws given by the bracket of $\mathfrak{g}$. If we think of elements of $\mathfrak{g}$ as left invariant vector fields on G, then $U(\mathfrak{g})$ consists of left invariant differential operators on G.

Here are some benefits of introducing the algebra $U(\mathfrak{g})$.

(1) One can use constructions from the associative algebra setting. For example, there is a well known notion of "extension of scalars": let $B \subset A$ be associative algebras and let V be a B-module. One can consider A as a right B-module for the right multiplication and form the vector space $A \otimes_B V$. This vector space is an A-module for the left multiplication in the first factor. So we get a functor from B-modules to A-modules. Another functor like this is obtained by considering $\mathrm{Hom}_B(A, V)$; now the Hom is taken with respect to the left multiplication action of B on A (and the given action on V), and the (left!) A-action on the space $\mathrm{Hom}_B(A, V)$ is given by right multiplication on A.

(2) Since $\mathfrak{g}$ is semisimple, it has no center. $U(\mathfrak{g})$ however has a nice center $Z(\mathfrak{g})$. It is a finitely generated polynomial algebra (e.g. for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ there are $n-1$ generators and their degrees are $2, 3, \ldots, n$). The importance of the center follows from a simple observation that is often used in linear algebra: if two operators commute, then an eigenspace for one of them is invariant for the other. This means that we can reduce representations by taking a joint eigenspace for $Z(\mathfrak{g})$.

Let us examine the center $Z(\mathfrak{g})$ and its use in representation theory in more detail. First, there is an element that can easily be written down; it is the simplest and most important element of $Z(\mathfrak{g})$ called the Casimir element. To define it we need a little more structure theory.

There is an invariant bilinear form on $\mathfrak{g}$, the Killing form B. It is defined by

$$B(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y), \quad X, Y \in \mathfrak{g}.$$

The invariance condition means that for a Lie group G with Lie algebra $\mathfrak{g}$, one has

$$B(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y) = B(X, Y), \quad g \in G,\ X, Y \in \mathfrak{g}.$$

The Lie algebra version of this identity is

$$B([Z, X], Y) + B(X, [Z, Y]) = 0, \quad X, Y, Z \in \mathfrak{g}.$$

Furthermore, B is nondegenerate on $\mathfrak{g}$; this is actually equivalent to $\mathfrak{g}$ being semisimple. In many cases like for $\mathfrak{sl}(n)$ over $\mathbb{R}$ or $\mathbb{C}$, one can instead of B use a simpler form $\operatorname{tr}(XY)$, which is equal to B up to a scalar.

There is a Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

$\mathfrak{k}$ and $\mathfrak{p}$ can be defined as the eigenspaces for the so called Cartan involution θ for the eigenvalues 1 respectively -1. There is also a Cartan decomposition $G = KP$ on the group level that we already mentioned; here K is the maximal compact subgroup of G (if G is connected with finite center), and $\mathfrak{k}$ is the Lie algebra of K.

Rather than defining the Cartan involution in general, let us note that for all the matrix examples in 1.1, $\theta(X)$ is minus the (conjugate) transpose of X. So e.g. for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{k}$ is $\mathfrak{so}(n)$ and $\mathfrak{p}$ is the space of symmetric matrices in $\mathfrak{g}$.

Some further properties of the Cartan decomposition: $\mathfrak{k}$ is a subalgebra, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The Killing form B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

To define the Casimir element, we choose orthonormal bases W_k for $\mathfrak{k}$ and Z_i for $\mathfrak{p}$, so that

$$B(W_k, W_l) = -\delta_{kl}; \quad B(Z_i, Z_j) = \delta_{ij}.$$

The Casimir element is then

$$\Omega = -\sum_k W_k^2 + \sum_i Z_i^2.$$

It is an element of $U(\mathfrak{g})$, and one shows by an easy calculation (which we leave as an exercise) that it commutes with all elements of $\mathfrak{g}$ and thus is an element of $Z(\mathfrak{g})$. Furthermore, it does not depend on the choice of bases W_k and Z_i.

Assume now X is an irreducible $(\mathfrak{g}, K)$-module. Then every element of $Z(\mathfrak{g})$ acts on X by a scalar. For finite dimensional X it is the well known and obvious Schur's lemma: let $z \in Z(\mathfrak{g})$ and take an eigenspace for z in X. This eigenspace is a submodule, hence has to be all of X. For infinite dimensional X the same argument is applied to a fixed K-type in X; see [V1], 0.3.19, or [Kn], Ch. VIII, §3.

Now all the scalars coming from the action of $Z(\mathfrak{g})$ on X put together give a homomorphism

$$\chi_X : Z(\mathfrak{g}) \to \mathbb{C}$$

of algebras, which is called the infinitesimal character of X.

By a theorem of Harish-Chandra, $Z(\mathfrak{g})$ is isomorphic (as an algebra) to $S(\mathfrak{h})^W$, the Weyl group invariants in the symmetric algebra of a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. (The Weyl group W is a finite reflection group generated by reflections with respect to roots.) This is obtained by taking the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ mentioned before and building a Poincaré-Birkhoff-Witt basis from bases in $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}^+$. We now get a linear map from $U(\mathfrak{g})$ into $U(\mathfrak{h}) = S(\mathfrak{h})$ by projecting along the span of all monomials that contain a factor which is not in $\mathfrak{h}$. This turns out to be an algebra homomorphism when restricted to $Z(\mathfrak{g})$, and this gives the required isomorphism.

Now we can identify $S(\mathfrak{h})$ with the algebra $P(\mathfrak{h}^*)$ of polynomials on $\mathfrak{h}^*$, and recall that any algebra homomorphism from $P(\mathfrak{h}^*)$ into $\mathbb{C}$ is given by evaluation at some $\lambda \in \mathfrak{h}^*$. It follows that the homomorphisms from $Z(\mathfrak{g})$ into $\mathbb{C}$ correspond to W-orbits $W\lambda$ in $\mathfrak{h}^*$. So, infinitesimal characters are parametrized by the space $\mathfrak{h}^*/W$. They are important parameters for classifying irreducible $(\mathfrak{g}, K)$-modules. It turns out that for every fixed infinitesimal character, there are only finitely many irreducible $(\mathfrak{g}, K)$-modules with this infinitesimal character. More details about Harish-Chandra isomorphism can be found e.g. in [KV].

To finish, let us describe an example where it is easy to explicitly write down all irreducible $(\mathfrak{g}, K)$-modules. This is the case $G = SL(2, \mathbb{R})$, whose representations correspond to $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-modules.

To see the action of $K = SO(2)$ better, we change basis of $\mathfrak{sl}(2, \mathbb{C})$ and instead of h, e, f used earlier we now use

$$W = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

Note that $W \in \mathfrak{k}_{\mathbb{C}}$. The elements W, X and Y satisfy the same relations as before:

$$[W, X] = 2X; \quad [W, Y] = -2Y; \quad [X, Y] = W.$$

Fix $\lambda \in \mathbb{C}$ and $\epsilon \in \{0, 1\}$. Define an $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-module $V_{\lambda,\epsilon}$ as follows:

- a basis of $V_{\lambda,\epsilon}$ is given by v_n, $n \in \mathbb{Z}$, n congruent to ϵ modulo 2;
- $\pi \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} v_n = e^{in\theta} v_n$;

- $\pi(W)v_n = nv_n$;
- $\pi(X)v_n = \frac{1}{2}(\lambda + (n+1))v_{n+2}$;
- $\pi(Y)v_n = \frac{1}{2}(\lambda - (n-1))v_{n-2}$.

The picture is similar to the one that we had for finite dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$, but now it is infinite:

$$\cdots \underset{f}{\overset{e}{\rightleftarrows}} v_{n-2} \underset{f}{\overset{e}{\rightleftarrows}} v_n \underset{f}{\overset{e}{\rightleftarrows}} v_{n+2} \underset{f}{\overset{e}{\rightleftarrows}} \cdots$$

Also, note that we changed normalization for the $v_n's$; now we do not have a natural place to start, like a highest weight vector, so it is best to make $\pi(X)$ and $\pi(Y)$ as symmetric as possible.

The following facts are not very difficult to check:

- $V_{\lambda,\epsilon}$ is irreducible unless $\lambda + 1$ is an integer congruent to ϵ modulo 2;
- The Casimir element Ω acts by the scalar $\lambda^2 - 1$ on $V_{\lambda,\epsilon}$;
- If $\lambda = k - 1$ where $k \geq 1$ is an integer congruent to ϵ modulo 2, then $V_{\lambda,\epsilon}$ contains two irreducible submodules, one with weights $k, k+2, \ldots$ and the other with weights $\ldots, -k-2, -k$. If $k > 1$, these are called discrete series representations, as they occur discretely in the decomposition of the representation $L^2(G)$. The quotient of $V_{\lambda,\epsilon}$ by the sum of these two submodules is an irreducible module of dimension $k-1$. For $k = 1$ the two submodules are called the limits of discrete series, and their sum is all of $V_{0,1}$.

All this can be found with many more details and proofs in Vogan's book [V1], Chapter 1. Actually, Chapters 0 and 1 of that book contain a lot of material from this lecture (plus more) and comprise a good introductory reading. Other books where a lot about $(\mathfrak{g}, K)$-modules can be found are [Kn] and [W].

Lecture 2. Clifford algebras, spinors and Dirac operators

2.1. Clifford algebras. From now on we will adopt the convention to denote real Lie algebras ¿with the subscript 0 and to refer to complexified Lie algebras with ¿the same letter but no subscript. For example, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ ¿will denote the Cartan decomposition of the real Lie algebra $\mathfrak{g}_0$, ¿while $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ will be the complexified Cartan decomposition.

We saw in 1.4 that $Z(\mathfrak{g})$ has an important role: it reduces representations, and defines a parameter (infinitesimal character) for the irreducible representations. We also defined the "smallest" non-constant element of $Z(\mathfrak{g})$, the Casimir element Ω.

Let us imagine for a moment that $Z(\mathfrak{g})$ contains a smaller (degree one) element T, such that $T^2 = \Omega$. Then T would (in principle) have more eigenvalues than Ω, as two opposite eigenvalues for T would square to the same eigenvalue for Ω. As a consequence, we would get a better reduction of representations, and infinitesimal character would be a finer invariant. Of course, such a T does not exist; degree one elements of $U(\mathfrak{g})$ are the elements of $\mathfrak{g}$ and $\mathfrak{g}$ has no center being semisimple.

The idea is then to twist $U(\mathfrak{g})$ by a finite dimensional algebra, the Clifford algebra $C(\mathfrak{p})$. The algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ will contain an element D (the Dirac element) whose square is close to $\Omega \otimes 1$.

One can define the Clifford algebra $C(\mathfrak{p})$ as an associative algebra with unit, generated by an orthonormal basis Z_i of $\mathfrak{p}$ (with respect to the Killing form B), subject to the relations

$$Z_i Z_j = -Z_j Z_i \quad (i \neq j); \qquad Z_i^2 = -1.$$

(There are variants obtained by replacing the -1 in the second relation by 1 or by $1/2$.) This definition involves choosing a basis, so let us give a nicer one:

$$C(\mathfrak{p}) = T(\mathfrak{p})/I,$$

where $T(\mathfrak{p})$ is the tensor algebra of the vector space $\mathfrak{p}$ and I is the two-sided ideal of $T(\mathfrak{p})$ generated by elements of the form

$$X \otimes Y + Y \otimes X + 2B(X, Y).$$

This definition resembles the definition of $U(\mathfrak{g})$; the similarity extends to an analog of the Poincaré-Birkhoff-Witt theorem. Namely, $C(\mathfrak{p})$ inherits a filtration by degree from $T(\mathfrak{p})$. The associated graded algebra is the exterior algebra $\bigwedge(\mathfrak{p})$. One can obtain a basis for $C(\mathfrak{p})$ by taking an (orthonormal) ordered basis Z_i for $\mathfrak{p}$ and forming monomials over it to obtain

$$Z_{i_1} \dots Z_{i_k}, \quad i_1 < \dots < i_k.$$

Note that no repetitions are allowed here, as Z_i^2 is of lower order. Together with the empty monomial 1, the above monomials form a basis of $C(\mathfrak{p})$ which thus has dimension equal to $2^{\dim \mathfrak{p}}$.

Finally, let us note that one can analogously construct a Clifford algebra for any vector space with a symmetric bilinear form.

2.2. Dirac operator. Using again our orthonormal basis Z_i of $\mathfrak{p}$, we define the Dirac operator

$$D = \sum_i Z_i \otimes Z_i \quad \in \quad U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

It is easy to show that D is independent of the choice of basis Z_i and K-invariant (for the adjoint action of K on both factors).

The adjoint action of $\mathfrak{k}_0$ on $\mathfrak{p}_0$ gives a map from $\mathfrak{k}_0$ into $\mathfrak{so}(\mathfrak{p}_0)$. On the other hand, there is a well known embedding of $\mathfrak{so}(\mathfrak{p}_0)$ into $C(\mathfrak{p}_0)$, given by

$$E_{ij} - E_{ji} \mapsto -\frac{1}{2} Z_i Z_j,$$

where E_{ij} denotes the matrix with all matrix entries equal to zero except the ij entry which is 1. One can check directly that this map is a Lie algebra morphism (where $C(\mathfrak{p}_0)$ is considered a Lie algebra in the usual way, by $[a,b] = ab - ba$). Combining these two maps, we get a map $\alpha : \mathfrak{k}_0 \longrightarrow C(\mathfrak{p}_0) \subset C(\mathfrak{p})$, and we use it to produce a diagonal embedding of $\mathfrak{k}_0$ into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, by

$$X \mapsto X \otimes 1 + 1 \otimes \alpha(X), \quad X \in \mathfrak{k}_0.$$

Complexifying this we get a map of $\mathfrak{k}$, hence also of $U(\mathfrak{k})$ and $Z(\mathfrak{k})$ into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. We denote the images by $\mathfrak{k}_\Delta$, $U(\mathfrak{k}_\Delta)$ and $Z(\mathfrak{k}_\Delta)$ (Δ for diagonal). In particular, we get $\Omega_{\mathfrak{k}_\Delta}$, the image of the Casimir element $\Omega_{\mathfrak{k}}$ of $Z(\mathfrak{k})$. Since $\Omega_{\mathfrak{k}} = -\sum_k W_k^2$, we see

$$\Omega_{\mathfrak{k}_\Delta} = -\sum_k (W_k \otimes 1 + 1 \otimes \alpha(W_k))^2.$$

Here is now the announced relationship between D^2 and the Casimir element $\Omega_{\mathfrak{g}}$:

Lemma (Parthasarathy).

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C,$$

where C is a constant that can be computed explicitly ($C = ||\rho_c||^2 - ||\rho||^2$, where ρ is the half sum of positive roots and ρ_c is the half sum of compact positive roots).

We just start the calculation and invite the reader to continue. Using the relations in $C(\mathfrak{p})$, we see the left hand side is

$$D^2 = \sum_{i,j} Z_i Z_j \otimes Z_i Z_j = -\sum_i Z_i^2 \otimes 1 + \sum_{i<j} [Z_i, Z_j] \otimes Z_i Z_j.$$

On the other hand, the right hand side is

$$(\sum_k W_k^2 \otimes 1 - \sum_i Z_i^2 \otimes 1) - \sum_k (W_k^2 \otimes 1 + 2W_k \otimes \alpha(W_k) + 1 \otimes \alpha(W_k)^2) + C.$$

One now shows easily that

$$\sum_{i<j} [Z_i, Z_j] \otimes Z_i Z_j = -2 \sum_k W_k \otimes \alpha(W_k),$$

and somewhat less easily that $\sum_k \alpha(W_k)^2$ is a constant (actually, exactly the one mentioned above).

2.3. Spinors. Let X be a $(\mathfrak{g}, K)$-module, so $U(\mathfrak{g})$ acts on X. We want to get the Dirac operator to act, so we have to tensor X with a $C(\mathfrak{p})$-module S; then $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ will act on $X \otimes S$, in particular D will act. Since we also want to stay as close as possible to X, we want to take minimal, i.e., simple S. It turns out there are only one or two choices for S, depending on whether $\dim \mathfrak{p}$ is even or odd. A simple $C(\mathfrak{p})$-module is called the spin module, or a space of spinors.

Let us first consider the case when $\dim \mathfrak{p}$ is even, say $2r$. The construction of S involves taking a maximal isotropic subspace $\mathfrak{u}$ of $\mathfrak{p}$ with respect to B. We can for example construct one such subspace starting from our orthonormal basis Z_i of $\mathfrak{p}_0$ and dividing it into two groups, $Z_1, \ldots, Z_r$ and $Z_{r+1}, \ldots, Z_{2r}$. Now we can take for $\mathfrak{u}$ the span of all $Z_s + iZ_{r+s}$, $s = 1, \ldots, r$. Clearly, $\bar{\mathfrak{u}}$ will be a complementary isotropic subspace (the conjugation is with respect to $\mathfrak{p}_0$), and we have

$$\mathfrak{p} = \mathfrak{u} \oplus \bar{\mathfrak{u}}.$$

Since the restriction of B to both $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ is zero, it follows that B identifies $\bar{\mathfrak{u}}$ with the dual space $\mathfrak{u}^*$ of $\mathfrak{u}$. Furthermore, the Clifford algebras over $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are equal to the exterior algebras, since $B = 0$. We take a basis u_i of $\mathfrak{u}$ and the dual basis $\bar{u}_i$ of $\bar{\mathfrak{u}}$. Let $\bar{u}_* = \bar{u}_1 \ldots \bar{u}_r$ be the corresponding basis element of $\bigwedge^{top} \bar{\mathfrak{u}}$. We define S to be the left ideal in $C(\mathfrak{p})$ generated by $\bar{u}_*$.

More explicitly, one can identify

$$S \cong \bigwedge(\mathfrak{u})\bar{u}_* \cong \bigwedge(\mathfrak{u}),$$

(since $\bar{\mathfrak{u}}$ annihilates $\bar{u}_*$). The action of $\mathfrak{p}$ is now given by

$$u \cdot (u_{i_1} \wedge \cdots \wedge u_{i_s}) = u \wedge u_{i_1} \wedge \cdots \wedge u_{i_s}$$

$$\bar{u} \cdot (u_{i_1} \wedge \cdots \wedge u_{i_s}) = -2 \sum_k B(\bar{u}, u_{i_k}) u_{i_1} \wedge \ldots \widehat{u}_{i_k} \cdots \wedge u_{i_s}$$

for $u \in \mathfrak{u}$ and $\bar{u} \in \bar{\mathfrak{u}}$. Namely, u just Clifford multiplies, or equivalently wedge multiplies, from the left, while $\bar{u}$ has to commute through, and then eventually gets killed upon meeting $\bar{u}_*$.

It is quite easy to show that S is a simple $C(\mathfrak{p})$-module, and not too difficult that it is the only one up to isomorphism. See e.g. [W], 9.2.1.

Let us now briefly examine the case when $\dim \mathfrak{p}$ is odd, say $2r+1$. We can still consider maximal isotropic $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ as before, but now there is an additional element $Z \in \mathfrak{p}$, orthogonal to both $\mathfrak{u}$ and $\bar{\mathfrak{u}}$, and we take it to be of norm 1. We again take S to be $\bigwedge \mathfrak{u}$, and we want to define an action of Z on S. From the relations it follows that Z must be scalar on every ¿homogeneous component of $\bigwedge \mathfrak{u}$, and that these scalars must alternate as the degree changes. To see this, start by observing that all elements of $\mathfrak{u}$ kill the top wedge $\bigwedge^r \mathfrak{u}$, and that there are no other elements of $\bigwedge \mathfrak{u}$ killed by all elements of $\mathfrak{u}$. Since $\bigwedge^r \mathfrak{u}$ is one dimensional, Z must act on it by a scalar. Now use anticommuting of Z and u_i^*'s to see that Z acts on $\bigwedge^{r-1} \mathfrak{u}$ by the opposite scalar, etc. From $Z^2 = -1$, the scalars by which Z can act are i and $-i$. We ¿thus have two choices: Z can be i on $\bigwedge^{even} \mathfrak{u}$ and $-i$ on $\bigwedge^{odd} \mathfrak{u}$, or vice versa. This gives us two nonisomorphic simple modules for $C(\mathfrak{p})$ in this case. One shows by a similar argument as above that these are the only simple $C(\mathfrak{p})$-modules up to isomorphism.

2.4. Spin group. Let us again first assume that $\dim \mathfrak{p}$ is even. Let $v \in \mathfrak{p}_0$ be of length 1, i.e., $B(v,v) = 1$. Then v is clearly invertible in $C(\mathfrak{p}_0)$, as $v^{-1} = -v$. Consider now the action of v on $\mathfrak{p}_0$ by conjugation in $C(\mathfrak{p}_0)$:

$$r_v(X) = vXv^{-1} = -vXv, \quad X \in \mathfrak{p}_0.$$

If X is orthogonal to v, then v and X anticommute, and hence $r_v(X) = -X$. If X is proportional to v, then $r_v(X) = X$. So we see that r_v is minus the reflection with respect to the hyperplane orthogonal to v.

All v as above generate a subgroup of the invertible elements of $C(\mathfrak{p}_0)$ which we denote by $Pin(\mathfrak{p}_0)$. The above discussion shows that we have a map

$$Pin(\mathfrak{p}_0) \to O(\mathfrak{p}_0),$$

which is surjective and has kernel $\{1, -1\}$. The connected component of $Pin(\mathfrak{p}_0)$ is the spin group $Spin(\mathfrak{p}_0)$; it coincides with the products of an even number of vectors v as above. It is a compact, semisimple group which is a double cover of the group $SO(\mathfrak{p}_0)$.

In case $\dim \mathfrak{p}$ is odd, one can do a similar construction. Instead of doing this, let us give a uniform description of $Spin(\mathfrak{p}_0)$ valid regardless of the parity of $\dim \mathfrak{p}$ (and also for forms which are not necessarily positive definite); see [Ko1], p. 282. Let α be the antiautomorphism of $C(\mathfrak{p})$ given by the identity on $\mathfrak{p}$. Then $Spin(\mathfrak{p}_0)$ is the group of all even elements g of $C(\mathfrak{p}_0)$ such that $g\alpha(g) = 1$ and $gx\alpha(g) \in \mathfrak{p}_0$ for all $x \in \mathfrak{p}_0$. For $g \in Spin(\mathfrak{p}_0)$, define $T(g) \in GL(\mathfrak{p}_0)$ by $T(g)x = gx\alpha(g)$. Then $T : Spin(\mathfrak{p}_0) \to SO(\mathfrak{p}_0)$ is a double covering. In particular, $Spin(\mathfrak{p}_0)$ is compact (since B is positive definite on $\mathfrak{p}_0$).

Since $Spin(\mathfrak{p}_0)$ is contained in $C(\mathfrak{p})$, it acts on any $C(\mathfrak{p})$-module. For $\dim \mathfrak{p}$ even, there is only one such simple module, S. Since $Spin(\mathfrak{p}_0)$ consists of even elements of $C(\mathfrak{p})$, it preserves the subspaces

$$S^+ = \bigwedge\nolimits^{even} \mathfrak{u}; \qquad S^- = \bigwedge\nolimits^{odd} \mathfrak{u}$$

of S. These are actually irreducible, and they are called spin representations.

For $\dim \mathfrak{p}$ odd, the two spaces of spinors S_1 and S_2 are irreducible for $Spin(\mathfrak{p}_0)$ and equivalent to each other. This representation is also called the spin representation.

2.5. Dirac cohomology. We consider the spin double cover $\tilde{K}$ of K, constructed from the following pullback diagram:

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & Spin(\mathfrak{p}_0) \\ \downarrow & & \downarrow \\ K & \longrightarrow & SO(\mathfrak{p}_0) \end{array}$$

Now if X is a (g,K)-module, then $\tilde{K}$ acts on $X \otimes S$ by acting on both factors: on X through K and on S through $Spin(\mathfrak{p}_0)$. Moreover, it is easy to show that $X \otimes S$ is a $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), \tilde{K})$-module. Such modules are defined analogously as $(\mathfrak{g}, K)$-modules; here $\tilde{K}$ acts on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ through the adjoint action of K on both factors, and the Lie algebra of $\tilde{K}$ embeds into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ as the diagonal $\mathfrak{k}_\Delta$ described earlier.

Dirac operator D acts on $X \otimes S$ and we define the Dirac cohomology of X to be the $\tilde{K}$ module

$$H_D(X) = \operatorname{Ker} D / \operatorname{Ker} D \cap \operatorname{Im} D.$$

If we suppose X is a unitary $(\mathfrak{g}, K)$-module, then we can define a positive definite hermitian form such that D is symmetric with respect to this form. For this we use the usual form on S for which all elements of $\mathfrak{p}_0$ are skew-hermitian (see [W], 9.2.3, or [Ch]). It now follows that if X is unitary, then $\operatorname{Ker} D \cap \operatorname{Im} D = 0$, and the Dirac cohomology of X is just $\operatorname{Ker} D$. More or less all of the above can be found in any of the following references: [Ch], [Ko1], [W] or [Kn] (even case).

Lecture 3. Dirac operators and group representations

3.1. Paul Dirac and his operator. Paul Dirac (1902-1984) was one of the greatest physicists of the 20th century. His research interests were mainly quantum mechanics and elementary particles. A free particle T in $\mathbb{R}^3$ is described by a state function $\psi(t,x)$ with $t\in\mathbb{R}$ and $x\in\mathbb{R}^3$. To understand this function, one needs to understand the square root of the wave operator

$$\Box=\frac{\partial^2}{\partial x_0^2}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_3^2}.$$

In 1928, Dirac found the square root D of $\Box$. It is a matrix valued first order differential operator, and it is now called the Dirac operator.

3.2. Square root of Laplace operator. We first describe the square root of the Laplace operator in $\mathbb{R}^n$:

$$\Delta=-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\cdots-\frac{\partial^2}{\partial x_n^2}.$$

For $n=1$, the Laplacian operator has the form $\Delta=-\frac{\partial^2}{\partial x^2}$ and the Dirac operator $D=i\frac{\partial}{\partial x}$ acts on the space of smooth functions $f\colon\mathbb{R}\to\mathbb{C}$.

For $n=2$, the Laplacian operator has the form $\Delta=-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}$. We define

$$D=\begin{pmatrix}0&i\\ i&0\end{pmatrix}\frac{\partial}{\partial x}+\begin{pmatrix}0&1\\ -1&0\end{pmatrix}\frac{\partial}{\partial+y}=\gamma_x\frac{\partial}{\partial x}+\gamma_y\frac{\partial}{\partial y},$$

which acts on the space of smooth functions $f\colon\mathbb{R}^2\to\mathbb{C}^2$. It follows from

$$\gamma_x^2=\gamma_y^2=-I,\ \gamma_x\gamma_y+\gamma_y\gamma_x=0$$

that

$$D^2=\Delta\begin{pmatrix}1&0\\ 0&1\end{pmatrix}=\Delta I.$$

For $n=3$, the Laplacian operator has the form $\Delta=-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_3^2}$. We define

$$\gamma_1=\begin{pmatrix}i&0\\ 0&-i\end{pmatrix},\gamma_2=\begin{pmatrix}0&-1\\ 1&0\end{pmatrix},\gamma_3=\begin{pmatrix}0&i\\ i&0\end{pmatrix}.$$

Then one has

$$\gamma_i^2=-I,\gamma_i\gamma_j+\gamma_j\gamma_i=0(i\neq j),i,j\in\{1,2,3\}.$$

Set

$$D=\gamma_1\frac{\partial}{\partial x_1}+\gamma_2\frac{\partial}{\partial x_2}\gamma_3\frac{\partial}{\partial+x_3},$$

which acts on the space of smooth functions $f\colon\mathbb{R}^3\to\mathbb{C}^2$. It follows that

$$D^2=\Delta\begin{pmatrix}1&0\\ 0&1\end{pmatrix}=\Delta I.$$

3.3. The original Dirac operator and its generalizations. Dirac defined the square root of the wave operator $\Box$ in terms of Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then one has

$$\sigma_i^2 = I, \sigma_j\sigma_k = -\sigma_k\sigma_j = -i\sigma_l, \text{ if } j < k \text{ and } \{j,k,l\} = \{1,2,3\}.$$

Define 4×4 matrices

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \ j = 1,2,3$$

and

$$D = \gamma_0 \frac{\partial}{\partial x_0} + \gamma_1 \frac{\partial}{\partial x_1} \gamma_2 \frac{\partial}{\partial x_2} + \gamma_3 \frac{\partial}{\partial x_3}.$$

It follows from

$$\gamma_0^2 = I, \ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -I, \ \ \gamma_j\gamma_k = -\gamma_k\gamma_j \ (j \neq k)$$

that

$$D^2 = \Box I.$$

Brauer and Weyl afterwards generalized the definition of Dirac operator to arbitrary finite-dimensional (quadratic) space of arbitrary signature. The Dirac operator defined in 2.2 is a much more recent analogue for semisimple Lie algebras; it was introduced by Parthasarathy in 1972.

3.4. Group representations. Representations of finite groups were studied by Dedekind, Frobenius, Hurwitz and Schur at the beginning of the 20th century. In 1920's, the focus of investigations was representation theory of compact Lie groups and its relations to invariant theory. Cartan and Weyl obtained the well-known classification of equivalence classes of irreducible unitary representations of compact Lie groups in terms of highest weights. In 1930's, Dirac and Wigner started the investigation of infinite-dimensional representations of noncompact Lie groups.

Harish-Chandra (1923-1983) was a Ph.D. student at the University of Cambridge under supervision of Dirac during the years 1945-1947. It was Harish-Chandra who began a systematical investigation of infinite-dimensional representations of semisimple Lie groups after his Ph.D. thesis. He lay down the foundation for further development of the theory for the last half century. In 1964 and 1965, Harish-Chandra published two papers which gave a complete parametrization of discrete series representations. Later he used this classification to prove the Plancherel formula. This classification of discrete series is also crucial to Langlands classification of admissible representations. However, Harish-Chandra did not give explicit construction of discrete series. His work was parallel to that of Cartan-Weyl for irreducible unitary representations of compact Lie groups.

In 1955, Borel and Weil gave explicit realization of irreducible unitary representations of compact Lie groups. In 1957, Bott generalized Borel-Weil theorem by considering Dolbeault cohomology of line bundles over G/T, where T is a maximal torus. In 1967, Schmid proved in his thesis the conjecture of Kostant and Langlands that discrete series representations are equivalent to certain Dolbeault cohomology of line bundles over G/T for noncompact G.

After Hotta's work on construction of holomorphic discrete series and Parthasarathy's construction of most discrete series representations by Dirac operators, Atiyah and Schmid in 1977 proved that all discrete series can be constructed as kernels of the Dirac operator over twisted spinor bundles.

The definition of the Dirac operator for a spin bundle was a major accomplishment of Atiyah-Singer, who obtained the celebrated index theorem. The Dirac cohomology is a far reaching generalization of the idea of index theory to representation theory. In the rest of the lectures we see how this simplifies proofs of many classical results such as Bott-Borel-Weil theorem and Atiyah-Schmid theorem and how it sharpens a result of Langlands and Hotta-Parthasarthy on multiplicities of automorphic forms. An important tool in our approach is the theory of $A_{\mathfrak{q}}(\lambda)$-modules which we review in the next lecture. One should bear in mind that this theory was not available at the time when the above mentioned classical results were proved.

Lecture 4. Introduction to $A_{\mathfrak{q}}(\lambda)$-modules

Mackey's construction of induction is based on real analysis, and Zuckerman's construction of induction which is needed for $A_{\mathfrak{q}}(\lambda)$-modules is based on complex analysis. It amounts to the geometric construction of representations by using Dolbeault cohomology sections of vector bundles over a noncompact complex homogeneous spaces then passing to the Taylor coefficients.

4.1. θ-stable parabolic subalgerbas. Let $H = TA$ be a fundamental Cartan in G. Let $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ be the corresponding θ-stable Cartan subalgebra. Then $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. As usual we drop the subscript 0 for the complexified Lie algebras. Let $X \in i\mathfrak{t}_0$ be such that $\mathrm{ad}(X)$ is semisimple with real eigenvalues. We define

(i) $\mathfrak{l}$ to be the zero eigenspace of $\mathrm{ad}(X)$,

(ii) $\mathfrak{u}$ to be the sum of positive eigenspaces of $\mathrm{ad}(X)$,

(iii) $\mathfrak{q}$ to be the sum of non-negative eigenspaces of $\mathrm{ad}(X)$.

Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a Levi decomposition. Furthermore, $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{q} \cap \mathfrak{g}_0$. We write L for the connected subgroup of G with Lie algebra $\mathfrak{l}_0$. Since $\theta(X) = X$, $\mathfrak{l}$,$\mathfrak{u}$ and $\mathfrak{q}$ are all invariant under θ, so

$$\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}.$$

In particular, $\mathfrak{q} \cap \mathfrak{k}$ is a parabolic subalgebra of $\mathfrak{k}$ with Levi decomposition

$$\mathfrak{q} \cap \mathfrak{k} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{u} \cap \mathfrak{k}.$$

We call such a $\mathfrak{q}$ a θ-stable parabolic subalgebra.

Let $\mathfrak{f} \subset \mathfrak{q}$ be any subspace stable under $\mathrm{ad}(\mathfrak{t})$. Then there is a subset $\{\alpha_1, \dots, \alpha_r\}$ of $\mathfrak{t}^*$ and subspaces $\mathfrak{f}_{\alpha_i}$ of $\mathfrak{f}$ such that if $y \in \mathfrak{t}$ and $v \in \mathfrak{f}_{\alpha_i}$, then

$$\mathrm{ad}(y)v = \alpha_i(y)v.$$

We write

$$\Delta(\mathfrak{f}, \mathfrak{t}) = \Delta(\mathfrak{f}) = \{\alpha_1, \dots, \alpha_r\},$$

the weights or roots of $\mathfrak{t}$ in $\mathfrak{f}$. Here $\Delta(\mathfrak{f})$ is a set with multiplicities, with α_i having multiplicity $\dim \mathfrak{f}_{\alpha_i}$. Then if

$$\rho(\mathfrak{f}) = \rho(\Delta(\mathfrak{f})) = \frac{1}{2} \sum_{\alpha_i \in \Delta(\mathfrak{f}_{\alpha_i})} \alpha_i \in \mathfrak{t}^*,$$

we have

$$\rho(\mathfrak{f})(y) = \frac{1}{2} \operatorname{tr}(\mathrm{ad}(y)|_{\mathfrak{t}}) \quad (y \in \mathfrak{t}).$$

Fix a system $\Delta^+(\mathfrak{l} \cap \mathfrak{k})$ of positive roots in the root system $\Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$. (Note that we extend the meaning of root system to include the zero weights.) Then

$$\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{l} \cap \mathfrak{k}) \cup \Delta(\mathfrak{u} \cap \mathfrak{k})$$

is a positive root system for $\mathfrak{t}$ in $\mathfrak{k}$.

If Z is an $(\mathfrak{l}, L \cap K)$-module, we write $Z^{\#}$ for $Z \otimes \wedge^{\mathrm{top}} \mathfrak{u}$. We set

$$\mathrm{pro}(Z^{\#}) = \mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), Z^{\#})_{L \cap K - finite} \text{ and } \mathrm{ind}(Z^{\#}) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z^{\#}.$$

Then both $pro(Z^{\#})$ and $ind(Z^{\#})$ are $(\mathfrak{g}, L \cap K)$-modules.

4.2. Zuckerman and Bernstein functors Γ. Let V be a $(\mathfrak{g}, L\cap K)$-module. The Zuckerman functor $\Gamma = \Gamma^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}$ can be defined as follows (if K is connected): Let $\Gamma(V)$ be the sum of all finite-dimensional $\mathfrak{k}_0$-invariant subspaces of V such that the $\mathfrak{k}_0$-action can be lifted to K. For a morphism $\varphi : V \longrightarrow W$ of $(\mathfrak{g}, L\cap K)$-modules, let $\Gamma(\varphi)$ be the restriction of φ to $\Gamma(V)$.

The functor Γ is not exact, but only left exact. Therefore one also needs to study the right derived functors Γ^j. For the study of unitarity, it is useful to consider also a "left analog" of Γ, the Bernstein functor $\Pi = \Pi^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}$ and its left derived functors Π_j. Π is not so easily defined directly; however, there is a uniform way to describe Zuckerman and Bernstein functors in terms of Hecke algebras.

Let $R(K)$ be the space of distributions on K. Let $R(\mathfrak{g}, K)$ be the space of left and right K-finite distributions on G with support in K. Then

$$R(\mathfrak{g}, K) \cong R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} R(K).$$

Then one has that the Zuckerman functor on V is

$$\Gamma(V) = \mathrm{Hom}_{R(\mathfrak{g},L\cap K)}(R(\mathfrak{g}, K), V)_{K-finite},$$

and the Bernstein functor is

$$\Pi(V) = R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g},L\cap K)} V.$$

We set

$$\mathcal{R}^j(Z) = \Gamma^j(\mathrm{pro}\, Z^{\#}) \text{ and } \mathcal{L}_j(Z) = \Pi_j(\mathrm{ind}\, Z^{\#}).$$

4.3. Irreducibility and unitarity of cohomologically induced modules. The hermitian inner product is not obvious for cohomological parabolic induction. This makes studying unitarity a very difficult problem for cohomologically induced modules. Nevertheless, Vogan proved the following powerful theorem.

Theorem ([V2]). *Suppose $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ and Z is an $(\mathfrak{l}, L\cap K)$-module with infinitesimal character λ. If Z is weakly good (i.e. $\mathrm{Re}\langle\lambda + \rho(\mathfrak{u}), \alpha\rangle \geq 0$ for any $\alpha \in \Delta(\mathfrak{u})$), then*

(i) *$\mathcal{L}_j(Z) = \mathcal{R}^j(Z) = 0$ for $j \neq s$ ($s = \dim \mathfrak{u}\cap\mathfrak{k}$).*

(ii) *$\mathcal{L}_s(Z) \cong \mathcal{R}^s(Z)$.*

(iii) *If Z is irreducible, then $\mathcal{L}_s(Z)$ is irreducible or zero.*

(iv) *If Z is irreducible and in addition is good (i.e. $\mathrm{Re}\langle\lambda+\rho(\mathfrak{u}), \alpha\rangle > 0$ for any $\alpha \in \Delta(\mathfrak{u})$), then $\mathcal{L}_s(Z)$ is irreducible and nonzero.*

(v) *If Z is unitary, then $\mathcal{L}_s(Z)$ is unitary.*

Remark.. There are two dualities:

$$(\Pi^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})_j \cong (\Gamma^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})^{2s-j} \text{ and } \Pi_j(W)^h = \Gamma^j(W^h)$$

where W^h is the hermitian dual of W.

4.4. $A_q(\lambda)$-modules. Now we consider Z to be a one-dimensional representation. $\lambda: \mathfrak{l} \to \mathbb{C}$ is called admissible if it satisfies the following conditions:

(i) λ is the differential of a unitary character of L

(ii) if $\alpha \in \Delta(\mathfrak{u})$, then $\langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geq 0$.

Given $\mathfrak{q}$ and an admissible λ, define

$$\mu(\mathfrak{q}, \lambda) = \text{representation of } K \text{ of highest weight } \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p}). \tag{4.1}$$

The following theorem is due to Vogan and Zuckerman.

Theorem ([VZ], [V2]). *Suppose $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ and $\lambda: \mathfrak{l} \to \mathbb{C}$ is admissible as defined above. Then there is a unique unitary $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ with the following properties:*

(i) *The restriction of $A_{\mathfrak{q}}(\lambda)$ to $\mathfrak{k}$ contains $\mu(\mathfrak{q}, \lambda)$ as defined in (4.1);*

(ii) *$A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho$;*

(iii) *If the representation of $\mathfrak{k}$ of the highest weight δ occurs in $A_{\mathfrak{q}}(\lambda)$, then*

$$\delta = \mu(\mathfrak{q}, \lambda) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta$$

with $\mathfrak{n}_\beta$ non-negative integers. In particular, $\mu(\mathfrak{q}, \lambda)$ is the lowest K-type of $A_{\mathfrak{q}}(\lambda)$.

We note that the unitarity of $A_{\mathfrak{q}}(\lambda)$ in the above theorem was proved in [V2]. In the context of definition of θ-stable parabolic subalgebras, if we take X to be a regular element, then we obtain a minimal θ-stable subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. We call such a subalgebra $\mathfrak{b}$ a θ-stable Borel subalgebra. The corresponding representation $A_{\mathfrak{b}}(\lambda)$ is called a fundamental series representation. It is the $(\mathfrak{g}, K)$-module of a tempered representation of G. If G has a compact Cartan subgroup, then $A_{\mathfrak{b}}(\lambda)$ is the $(\mathfrak{g}, K)$-module of a discrete series representation of G. Moreover, all $(\mathfrak{g}, K)$-modules of discrete series representations of G are of this form; this will be important in Lecture 6. For the proof of the main result of [HP] (Lecture 5) it is only needed that $A_{\mathfrak{b}}(\lambda)$ has infinitesimal character $\lambda + \rho$ and the lowest K-type $\mu(\mathfrak{b}, \lambda) = \lambda + 2\rho_n$, where $\rho_n = \rho(\mathfrak{n} \cap \mathfrak{p})$. These facts are contained in Theorem 4.4.

4.5. Salamanca-Riba's classification of the unitary dual with strongly regular infinitesimal characters. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Given any weight $\Lambda \in \mathfrak{h}^*$, fix a choice of positive roots $\Delta^+(\Lambda, \mathfrak{h})$ for Λ so that

$$\Delta^+(\Lambda, \mathfrak{h}) \subset \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \mathrm{Re}\langle \Lambda, \alpha \rangle \geq 0\}.$$

Set

$$\rho(\Lambda) = \frac{1}{2} \sum_{\alpha \in \Delta^+(\Lambda, \mathfrak{h})} \alpha.$$

Definition. A weight $\Lambda \in \mathfrak{h}^*$ is said to be real if

$$\Lambda \in i\mathfrak{t}_0^* + \mathfrak{a}_0^*,$$

and to be strongly regular if it is real and

$$\langle \Lambda - \rho(\Lambda), \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta^+(\Lambda, \mathfrak{h}).$$

Salamanca-Riba [SR] proved that if an irreducible unitary $(\mathfrak{g}, K)$-module X has strongly regular infinitesimal character then $X \cong A_{\mathfrak{q}}(\lambda)$ for some θ-stable parabolic subalgebra $\mathfrak{q}$ and admissible character λ of L. Moreover, she proved the following stronger theorem which was conjectured by Vogan.

Theorem(Salamanca-Riba). *Suppose that X is an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character $\Lambda \in \mathfrak{h}^*$ satisfying*

$$\mathrm{Re}\langle \Lambda - \rho(\Lambda), \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta^+(\Lambda, \mathfrak{h}).$$

Then there exist a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and an admissible character λ of L such that X is isomorphic to $A_{\mathfrak{q}}(\lambda)$.

Lecture 5. Vogan's conjecture and its proof

In this lecture we explain Vogan's conjecture on Dirac cohomology and a proof of this conjecture. The presentation mostly follows [HP].

5.1. Vogan's conjecture. Let T be a maximal torus in K, with Lie algebra $\mathfrak{t}_0$. Let $\mathfrak{h}$ be the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$; it is a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Since $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{p}^{\mathfrak{t}}$, we get an embedding of $\mathfrak{t}^*$ into $\mathfrak{h}^*$. Therefore any element of $\mathfrak{t}^*$ determines a character of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Here we are using the standard identification $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$ via the Harish-Chandra homomorphism (W is the Weyl group), by which the characters of $Z(\mathfrak{g})$ correspond to the W-orbits in $\mathfrak{h}^*$.

We fix a positive root system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ for $\mathfrak{t}$ in $\mathfrak{k}$; let $\rho_c = \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}))$ be the corresponding half sum of the positive roots. For any finite dimensional irreducible representation (γ, E_γ) of $\mathfrak{k}$, we denote its highest weight in $\mathfrak{t}^*$ by γ again.

Vogan [V3] made a conjecture which was proved as the following theorem:

Theorem [HP]**.** *Let X be an irreducible $(\mathfrak{g}, K)$-module, such that the Dirac cohomology of X is non-zero. Let γ be a $\tilde{K}$-type contained in the Dirac cohomology. Then the infinitesimal character of X is given by $\gamma + \rho_c$.*

In view of the remarks in 2.5, in case X is unitarizable, we get the following consequence:

Corollary. *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module, such that* $\operatorname{Ker} D \neq 0$. *Let γ be a $\tilde{K}$-type contained in* $\operatorname{Ker} D$. *Then the infinitesimal character of X is given by $\gamma + \rho_c$.*

In fact, Vogan first conjectured the above corollary and then he saw that the above theorem should be the right generalization to non-unitary representations.

5.2. An algebraic reduction of the conjecture. Vogan further reduced the claim of his conjecture to an entirely algebraic statement in the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Let us first recall that in 2.2 we described a diagonal copy $\mathfrak{k}_\Delta$ of $\mathfrak{k}$ inside $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. $U(\mathfrak{k}_\Delta)$ and $Z(\mathfrak{k}_\Delta)$ denote the corresponding universal enveloping algebra and its center. It is easy to see that they are also embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$; namely, if $u \in U(\mathfrak{k})$ is a PBW monomial, then its image in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is the sum of $u \otimes 1$ and terms of the form $w \otimes a$, with w having smaller degree than u.

We can now state Vogan's algebraic conjecture that implies the theorem in 5.1.

Theorem. *Let $z \in Z(\mathfrak{g})$. Then there is a unique $\zeta(z)$ in the center $Z(\mathfrak{k}_\Delta)$ of $U(\mathfrak{k}_\Delta)$, and there are K-invariant elements $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$, such that*

$$z \otimes 1 = \zeta(z) + Da + bD.$$

To see that this theorem implies the theorem in 5.1, let $\tilde{x} \in (X \otimes S)(\gamma)$ be non-zero, such that $D\tilde{x} = 0$ and $\tilde{x} \notin \operatorname{Im} D$. Note that both $z \otimes 1$ and $\zeta(z)$ act as scalars on $\tilde{x}$. The first of these scalars is the infinitesimal character Λ of X applied to z, and the second is the $\mathfrak{k}$-infinitesimal character of γ applied to $\zeta(z)$, that is, $(\gamma + \rho_c)(\zeta(z))$.

On the other hand, since $(z \otimes 1 - \zeta(z))\tilde{x} = Da\tilde{x}$, and $\tilde{x} \notin \operatorname{Im} D$, it follows that $(z \otimes 1 - \zeta(z))\tilde{x} = 0$. Thus the above two scalars are the same, i.e., $\Lambda(z) = (\gamma + \rho_c)(\zeta(z))$.

In 5.6 we will show that under identifications $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W \cong P(\mathfrak{h}^*)^W$ and $Z(\mathfrak{k}_\Delta) \cong Z(\mathfrak{k}) \cong S(\mathfrak{t})^{W_K} \cong P(\mathfrak{t}^*)^{W_K}$ the homomorphism ζ corresponds to the restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$. Here the already mentioned inclusion of $\mathfrak{t}^*$ into $\mathfrak{h}^*$ is given by extending functionals from $\mathfrak{t}$ to $\mathfrak{h}$, letting them act by 0 on $\mathfrak{a} = \mathfrak{p}^{\mathfrak{t}}$. It follows that $\Lambda = \gamma + \rho_c$, as claimed.

5.3. A differential complex induced by Dirac operator. Let us first note that the Clifford algebra $C(\mathfrak{p})$ has a natural $\mathbb{Z}_2$-gradation into even and odd parts:

$$C(\mathfrak{p}) = C^0(\mathfrak{p}) \oplus C^1(\mathfrak{p}).$$

This gradation induces a $\mathbb{Z}_2$-gradation on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ in an obvious way.

We define a map d from $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ into itself, as $d = d^0 \oplus d^1$, where

$$d^0 : U(\mathfrak{g}) \otimes C^0(\mathfrak{p}) \to U(\mathfrak{g}) \otimes C^1(\mathfrak{p})$$

is given by

$$d^0(a) = Da - aD, \tag{5.3a}$$

and

$$d^1 : U(\mathfrak{g}) \otimes C^1(\mathfrak{p}) \to U(\mathfrak{g}) \otimes C^0(\mathfrak{p})$$

is given by

$$d^1(a) = Da + aD. \tag{5.3b}$$

In other words, if ϵ_a denotes the sign of a, that is, 1 for even a and -1 for odd a, then $d(a) = Da - \epsilon_a aD$ (for homogeneous a, i.e., those a which have sign).

We will use the formula for D^2 from Lemma 2.2, namely

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C$$

to prove that our d induces a differential on the K-invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

Proposition. *Let d be the map defined in (5.3a) and (5.3b). Then*
(i) *d is K-equivariant, hence induces a map from $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ into itself.*
(ii) *$d^2 = 0$ on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$.*

Proof. (i) is trivial, since D is K-invariant.

Let $a \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ be even or odd. Then

$$d^2(a) = d(Da - \epsilon_a aD) = D^2a - \epsilon_{Da} DaD - \epsilon_a(DaD - \epsilon_{aD} aD^2) = D^2a - aD^2,$$

since obviously $\epsilon_{aD} = \epsilon_{Da} = -\epsilon_a$. From the formula for D^2 (Lemma 2.2), we see that a will commute ¿with D^2 if and only if it commutes with $\Omega_{\mathfrak{k}_\Delta}$. If a is K-invariant, then this clearly holds, as a then commutes with all of $U(\mathfrak{k}_\Delta)$. $\square$

Thus we see that d is a differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, of degree 1 with respect to the above defined $\mathbb{Z}_2$-gradation. Note that we do not have a $\mathbb{Z}$-gradation on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ so that d is of degree 1, i.e., this is not a complex in the usual sense.

5.4. Determination of cohomology of the complex. We want to calculate the cohomology of d. Before we state the result, let us note the following:

Proposition. *$Z(\mathfrak{k}_\Delta)$ is in the kernel of d.*

Proof. Since D is K-invariant, it commutes with $\mathfrak{k}_\Delta$, and thus with $U(\mathfrak{k}_\Delta)$ and in particular with $Z(\mathfrak{k}_\Delta)$. Since $Z(\mathfrak{k}_\Delta) \subset (U(\mathfrak{g}) \otimes C^0(\mathfrak{p}))^K$, the claim follows. $\square$

We now state the following theorem which implies Theorem 5.2.

Theorem. *Let d be the differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ constructed above. Then* $\operatorname{Ker} d = Z(\mathfrak{k}_\Delta) \oplus \operatorname{Im} d$. *In particular, the cohomology of d is isomorphic to $Z(\mathfrak{k}_\Delta)$.*

The proof uses the standard method of filtering the algebra (the filtration comes from the usual filtration on $U(\mathfrak{g})$), and then passing to the graded algebra. This graded algebra is of course $S(\mathfrak{g}) \otimes C(\mathfrak{p})$. The analogue of our theorem in the graded setting is easy; the complex we get is closely related to the standard Koszul complex associated to the vector space $\mathfrak{p}$. Namely, the operator $\bar{d}$ induced by d on $S(\mathfrak{g}) \otimes C(\mathfrak{p})$ is given by supercommuting with

$$\bar{D} = \sum_i Z_i \otimes Z_i,$$

and one easily calculates that

$$\bar{d}(U \otimes Z_{i_1} \dots Z_{i_k}) = -2 \sum_{j=1}^{k} u Z_{i_j} \otimes Z_{i_1} \dots \hat{Z_{i_j}} \dots Z_{i_k}.$$

Upon identifying $C(\mathfrak{p})$ and $\bigwedge(\mathfrak{p})$ as vector spaces and writing

$$S(\mathfrak{g}) \otimes C(\mathfrak{p}) = S(\mathfrak{k}) \otimes (S(\mathfrak{p}) \otimes \bigwedge(\mathfrak{p})),$$

we see that

$$\bar{d} = (-2) id \otimes d_{\mathfrak{p}},$$

where d_p is the Koszul differential for the vector space $\mathfrak{p}$. In particular, $\bar{d}$ is a differential with cohomolgy $S(\mathfrak{k}) \otimes \mathbb{C}$, which embeds into $S(\mathfrak{g}) \otimes C(\mathfrak{p})$ by embedding $\mathbb{C}$ into $S(\mathfrak{p}) \otimes \bigwedge(\mathfrak{p})$ as the constants. Passing to K-invariants, we see that

$$\operatorname{Ker} \bar{d} = S(\mathfrak{k})^K \otimes 1 \oplus \operatorname{Im} \bar{d}$$

on $(S(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. This is the graded version of our theorem.

One can now go back to the original setting by an easy induction on the degree of the filtration. The main point is that one can reconstruct an element of $Z(\mathfrak{k}_\Delta)$ from its top term.

We refer the reader to our paper [HP] for the details of the above proof. Let us note a consequence, which immediately proves Vogan's conjecture, just put $b = a$.

Corollary. *Let $z \in Z(\mathfrak{g})$. Then there is a unique $\zeta(z) \in Z(\mathfrak{k}_\Delta)$, and there is an $a \in (U(\mathfrak{g}) \otimes C^1(\mathfrak{p}))^K$, such that*

$$z \otimes 1 = \zeta(z) + Da + aD.$$

Proof. This follows at once from Theorem 5.4, if we just notice that $z \otimes 1$ commutes with D (indeed, it is in the center of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$), and being even, it is thus in $\operatorname{Ker} d$. Hence, it is of the form $\zeta(z) + d(a) = \zeta(z) + Da + aD$.

5.5. Dirac inequality and unitary representations with nonzero Dirac cohomology. We first indicate how to check if a unitarizable X has non-zero Dirac cohomology.

Proposition. *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Assume that $X \otimes S$ contains a $\tilde{K}$-type γ, i.e., $(X \otimes S)(\gamma) \neq 0$. Assume further that $||\Lambda|| = ||\gamma + \rho_c||$. Then the Dirac cohomology of X, $\operatorname{Ker} D$, contains $(X \otimes S)(\gamma)$.*

Proof. This again uses the formula for D^2 from Lemma 2.2. The formula implies that D^2 acts on $(X \otimes S)(\gamma)$ by the scalar

$$-(||\Lambda||^2 - ||\rho||^2) + (||\gamma + \rho_c||^2 - ||\rho_c||^2) + (||\rho_c||^2 - ||\rho||^2) = 0.$$

It follows from self-adjointness of D that $D = 0$ on $(X \otimes S)(\gamma)$. $\square$

Note that Corollary 5.1 implies the converse of the above proposition: if X is irreducible unitarizable, with Dirac cohomology containing $(X \otimes S)(\gamma)$, then the infinitesimal character of X is $\Lambda = \gamma + \rho_c$. Hence $||\Lambda|| = ||\gamma + \rho_c||$. We note that all irreducible unitary representations with nonzero Dirac cohomology and strongly regular infinitesimal characters were described in [HP]. They are all $A_{\mathfrak{q}}(\lambda)$-modules. See Proposition 5.6 where this is explained in a special case.

Finally, combining the above proposition with Corollary 5.1, we sharpen the Parthasarathy's Dirac inequality:

Theorem (Extended Dirac Inequality). *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character Λ. If $(X \otimes S)(\gamma) \neq 0$, then*

$$||\Lambda|| \leq ||\gamma + \rho_c||.$$

The equality holds if and only if some W conjugate of Λ is equal to $\gamma + \rho_c$.

5.6. Fundamental series and determination of ζ.

Proposition. *Let X be an $A_{\mathfrak{b}}(\lambda)$-module (as in Theorem 4.4) with $\mathfrak{b}$ a θ-stable Borel subalgebra, i.e., X is a fundamental series representation. Assume that $\lambda|_{\mathfrak{a}} = 0$. Then the Dirac cohomology of X contains a $\tilde{K}$-type E_γ of highest weight $\gamma = \lambda + \rho_n$.*

Proof. The lowest K-type $\mu(\mathfrak{b}, \lambda)$ has highest weight $\lambda + 2\rho_n$. Since $-\rho_n$ is a weight of S, E_γ occurs in $\mu(\mathfrak{b}, \lambda) \otimes S$, hence in $X \otimes S$. Since the infinitesimal character of X is $\lambda + \rho = \gamma + \rho_c$, it follows that E_γ is in the kernel of the Dirac operator D, i.e., in the Dirac cohomology of X. $\square$

Now we can describe the homomorphism ζ explicitly.

Theorem. *The homomorphism ζ satisfies the following commutative diagram:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\ \zeta\ } & Z(\mathfrak{k}) \\ \downarrow & & \downarrow \\ S(\mathfrak{h})^W & \xrightarrow{\ \mathrm{Res}\ } & S(\mathfrak{t})^{W_K} \end{array}$$

Here the vertical arrows are the Harish-Chandra homomorphisms, and the map Res *corresponds to the restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$ under the identifications $S(\mathfrak{h})^W = P(\mathfrak{h}^*)^W$ and $S(\mathfrak{t})^{W_K} = P(\mathfrak{t}^*)^{W_K}$. As before, we can view $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$ by extending functionals from $\mathfrak{t}$ to $\mathfrak{h}$, letting them act by 0 on $\mathfrak{a}$.*

Proof. Let $\bar{\zeta} : P(\mathfrak{h}^*)^W \to P(\mathfrak{t}^*)^{W_K}$ be the homomorphism induced by ζ under the identifications via Harish-Chandra homomorphisms. Furthermore, let $\tilde{\zeta} : \mathfrak{t}^*/W_K \to \mathfrak{h}^*/W$ be the morphism of algebraic varieties inducing the homomorphism $\bar{\zeta}$. We have to show that $\bar{\zeta}$ is the restriction map, or alternatively that $\tilde{\zeta}$ is given by the inclusion map.

We know from the above proposition that the fundamental series representation $A_{\mathfrak{b}}(\lambda)$ has the lowest K-type

$$\mu(\mathfrak{b}, \lambda) = \lambda + 2\rho_n,$$

and infinitesimal character

$$\Lambda = \lambda + \rho.$$

On the other hand, it follows from Proposition 5.5 that if $\lambda|_{\mathfrak{a}} = 0$, then the Dirac cohomology of $A_{\mathfrak{b}}(\lambda)$ contains the $\tilde{K}$-type of highest weight $\gamma = \lambda + \rho_n$.

When proving that Theorem 5.2 implies Theorem 5.1, we saw that $\Lambda(z) = (\gamma + \rho_c)(\zeta(z))$, for all $z \in Z(\mathfrak{g})$. In our present situation we however have

$$\Lambda = \lambda + \rho = (\lambda + \rho_n) + \rho_c = \gamma + \rho_c,$$

so it follows that $\Lambda(\zeta(z)) = \Lambda(z)$ for all $z \in Z(\mathfrak{g})$. This means that $\tilde{\zeta}(\Lambda) = \Lambda$, for all infinitesimal characters Λ of the above fundamental series representations.

It is clear that when λ ranges over all admissible weights in $\mathfrak{h}^*$ such that $\lambda|_{\mathfrak{a}} = 0$, then $\Lambda = \lambda + \rho$ form an algebraically dense subset of $\mathfrak{t}^*$. To see this, it is enough to note that such λ span a lattice in $\mathfrak{t}^*$. Hence $\tilde{\zeta}$ is indeed the inclusion map. $\square$

5.7. Remark on finite-dimensional representations. Both Vogan and Kostant pointed out to us that Theorem 5.6 can also be proved by considering finite-dimensional representations with non-zero Dirac cohomology. These are the representations V_λ of highest weight $\lambda \in \mathfrak{t}^* \subset \mathfrak{h}^*$. See [HP, Remark 5.6.]

Lecture 6. Generalized Bott-Borel-Weil Theorem and construction of discrete series

6.1. Kostant cubic Dirac operator. Let G be a compact semisimple Lie group and R be a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{r}$ be the complexifications of the corresponding Lie algebras. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}$ be the orthogonal decomposition with respect to the Killing form. Choose an orthonormal basis $Z_1, \dots, Z_n$ of $\mathfrak{p}$ with respect to the Killing form $\langle\ ,\ \rangle$. Kostant [K2] defines his cubic Dirac operator to be the element

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i + 1 \otimes v \in U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

where $v \in C(\mathfrak{p})$ is the image of the fundamental 3-form $\omega \in \bigwedge^3(\mathfrak{p}^*)$,

$$\omega(X, Y, Z) = -\frac{1}{2}\langle X, [Y, Z]\rangle$$

under the Chevalley identification $\bigwedge(\mathfrak{p}^*) \to C(\mathfrak{p})$. Kostant's cubic Dirac operator reduces to the ordinary Dirac operator when $(\mathfrak{g}, \mathfrak{r})$ is a symmetric pair, since $\omega = 0$ for the symmetric pair. Kostant ([K2, Theorem 2.16]) shows that

$$D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{r}_\Delta} + C, \tag{6.1}$$

where C is the constant $||\rho||^2 - ||\rho_{\mathfrak{r}}||^2$. This is the generalization of Lemma 2.2. The sign change comes from the fact that Kostant uses a slightly different definition of $C(\mathfrak{p})$, requiring Z_i^2 to be 1 and not -1. Over $\mathbb{C}$, there is no substantial difference between the two conventions.

Now we can define the cohomology of the complex $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^R$ using Kostant's cubic Dirac operator exactly as in Lecture 5, i.e., by $d(a) = Da - \epsilon_a aD$. As before, $d^2 = 0$ on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^R$. Since the degree of the cubic term is zero in the filtration of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ used in Lecture 5, the proof Theorem 5.4 goes through without change and we get

Theorem. *Let d be the differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^R$ defined by Kostant's cubic Dirac operator as above. Then* $\operatorname{Ker} d = \operatorname{Im} d \oplus Z(\mathfrak{r}_\Delta)$. *In particular, the cohomology of d is isomorphic to $Z(\mathfrak{r}_\Delta)$.*

As a consequence we get an analogous homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{r})$ for a reductive subalgebra $\mathfrak{r}$ in a semisimple Lie algebra $\mathfrak{g}$ and a more general version of Vogan's conjecture.

6.2. Kostant's theorem on cohomology of homogeneous spaces. If we fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{r}$ and extend $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then ζ is induced by the Harish-Chandra homomorphism exactly as in Theorem 5.6. This was proved in [K3], by constructing a sufficiently large family of highest weight modules with known infinitesimal characters and nonzero Dirac cohomology.

Moreover, the homomorphism ζ induces the structure of a $Z(\mathfrak{g})$-module on $Z(\mathfrak{r})$, which has topological significance. Namely, Kostant has shown that from a well-known theorem of H. Cartan [C], which is by far the most comprehensive result on the real (or complex) cohomology of a homogeneous space, one has

Theorem [K3]. *There exists an isomorphism*

$$H^*(G/R, \mathbb{C}) \cong \mathrm{Tor}_*^{Z(\mathfrak{g})}(\mathbb{C}, Z(\mathfrak{r})).$$

6.3. A generalized Weyl character formula. We assume that $\mathrm{rank}\, R = \mathrm{rank}\, G$. Let $W^1 \subset W_{\mathfrak{g}}$ be the subset of Weyl group elements that map the positive Weyl chamber for $\mathfrak{g}$ into the positive Weyl chamber for $\mathfrak{r}$. If λ is a dominant weight for $\mathfrak{g}$, then $\lambda + \rho(\mathfrak{g})$ lies in the interior of the Weyl chamber for $\mathfrak{g}$. It follows that $w(\lambda + \rho(\mathfrak{g}))$ lies in the interior of the Weyl chamber for $\mathfrak{r}$ for any $w \in W^1$. Thus, $w \cdot \lambda = w(\lambda + \rho(\mathfrak{g})) - \rho(\mathfrak{r})$ is a dominant weight for $\mathfrak{r}$.

Theorem [GKRS].

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{w \in W^1} (-1)^{l(w)} U_{w \cdot \lambda}.$$

It follows that

$$ch(V_\lambda) = \frac{\sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \cdot \lambda})}{\sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \cdot 0})}.$$

Note that the above formula reduces to the Weyl character formula when R is a maximal torus T.

6.4. A generalized Bott-Borel-Weil Theorem. We assume that U_μ is an irreducible representation of R (or $\tilde{R}$) so that $S \otimes U_\mu$ is a representation of R. The Dirac operator acts on the smooth and L^2-sections on the twisted spinor bundles over G/R, if we let $Z_i \in \mathfrak{g}$ act by differentiating from the right. So we have

$$D \colon L^2(G) \otimes_R (S \otimes U_\mu) \to L^2(G) \otimes_R (S \otimes U_\mu).$$

We write this action in another form:

$$D \colon \mathrm{Hom}_{\tilde{R}}(U_\mu^*, L^2(G) \otimes S) \to \mathrm{Hom}_{\tilde{R}}(U_\mu^*, L^2(G) \otimes S).$$

Then D is formally self-adjoint. By Peter-Weyl theorem, one has $L^2(G) \cong \oplus_{\lambda \in \widehat{G}} V_\lambda \otimes V_\lambda^*$. It follows that

$$\mathrm{Ker}\, D = \bigoplus_{\lambda \in \widehat{G}} V_\lambda \otimes \mathrm{Ker}\{D \colon \mathrm{Hom}_{\tilde{R}}(U_\mu^*, V_\lambda^* \otimes S) \circlearrowleft\}.$$

The proved Vogan's conjecture implies $\mathrm{Ker}\, D \neq 0$ if and only if $\lambda + \rho(\mathfrak{g})$ is conjugate to $\mu + \rho(\mathfrak{r})$ by the Weyl group. Further consideration of the multiplicity results in

Theorem. *One has* $\mathrm{Ker}\, D = V_{w(\mu+\rho(\mathfrak{r}))-\rho(\mathfrak{g})}$ *if there exists a* $w \in W_{\mathfrak{g}}$ *so that* $w(\mu+\rho(\mathfrak{r}))-\rho(\mathfrak{g})$ *is dominant, and* $\mathrm{Ker}\, D$ *is zero if no such* w *exists.*

In the case $R = T$ a maximal torus, this is a version of Borel-Weil theorem.

Corollary. *Consider*

$$D^+: L^2(G) \otimes_R (S^+ \otimes U_\mu) \to L^2(G) \otimes_R (S^- \otimes U_\mu)$$

and the adjoint

$$D^-: L^2(G) \otimes_R (S^- \otimes U_\mu) \to L^2(G) \otimes_R (S^+ \otimes U_\mu).$$

One has $\operatorname{Index} D = \dim \ker D^+ - \dim \operatorname{Ker} D^- = (-1)^{l(w)} \dim V_{w(\mu+\rho(\mathfrak{r}))-\rho(\mathfrak{g})}$ *if there exists a* $w \in W_\mathfrak{g}$ *so that* $w(\mu + \rho(\mathfrak{r})) - \rho(\mathfrak{g})$ *is dominant and it is zero if no such* w *exists.*

6.5. Geometric construction of discrete series. Let G be a linear semisimple noncompact Lie group. Let K be a maximal compact subgroup of G. Assume that $\operatorname{rank} G = \operatorname{rank} K$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of the Lie algebra of G. Then $\mathfrak{u}_0 = \mathfrak{k}_0 + i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = \mathfrak{g}_0 \otimes_\mathbb{R} \mathbb{C}$. Let U be the compact analytic subgroup in the complexification $G_\mathbb{C}$ of G with Lie algebra $\mathfrak{u}_0$.

Borel showed that there exists a torsion free discrete subgroup Γ of G so that $\Gamma\backslash G$ and $X = \Gamma\backslash G/K$ are compact smooth manifolds. For any $\mu \in \widehat{K}$, the Dirac operator D acts on the smooth sections of the twisted spin bundle in a similar way described in 6.4.

$$D: C^\infty(G/K, S \otimes E_\mu) \to C^\infty(G/K, S \otimes E_\mu).$$

Note that the above action of D commutes with the left action of G. So we can consider the elliptic operator

$$D^+_\mu(X): C^\infty(\Gamma\backslash G/K, S^+ \otimes E_\mu) \to C^\infty(\Gamma\backslash G/K, S^- \otimes E_\mu).$$

The index of $D^+_\mu(X)$ can be computed by Atiyah-Singer Index Theorem

$$\operatorname{Index} D^+_\mu(X) = \int_X f(\Theta, \Phi),$$

where Θ is the curvature of X and Φ is the curvature of the twisted spinor bundle over G/K. By the homogeneity, $f(\Theta, \Phi)$ is a multiple of the volume form depending only on μ, i.e., $f(\Theta, \Phi) = c(\mu)dx$. Thus

$$\operatorname{Index} D^+_\mu(X) = c(\mu) vol(\Gamma\backslash G/K).$$

Let $Y = U/K$ be the compact homogeneous space. By Hirzebruch proportionality principle, the index of

$$D^+_\mu(Y): C^\infty(U/K, S^+ \otimes E_\mu) \to C^\infty(U/K, S^- \otimes E_\mu)$$

can be computed in the same way and

$$\operatorname{Index} D^+_\mu(Y) = (-1)^q c(\mu) vol(U/K),$$

where $q = \dim G/K = \dim U/K$. It follows that

$$\text{Index}\, D^+_\mu(X) = (-1)^q \frac{vol(\Gamma\backslash G/K)}{vol(U/K)} \text{Index}\, D^+_\mu(Y).$$

If we normalize the Haar measure so that $vol(U) = 1$, then

$$\text{Index}\, D^+_\mu(X) = (-1)^q vol(\Gamma\backslash G) \text{Index}\, D^+_\mu(Y).$$

Let $L^2(G) \cong \int_{\widehat{G}} H_j \otimes H_j^* d\mu(j)$ be the abstract Plancherel decomposition. Then the L^2 sections of the twisted spinor bundles are decomposed as

$$L^2(G/K, S \otimes E_\mu) \cong \int_{\widehat{G}} H_j \otimes \text{Hom}_K(H_j, S \otimes E_\mu) d\mu(j).$$

It follows that the L^2-sections of $\text{Ker}\, D$ are decomposed as

$$\text{Ker}\, D \cong \int_{\widehat{G}} H_j \otimes \text{Ker}\{D\colon \text{Hom}_{\tilde{K}}(E^*_\mu, , H_j^* \otimes S) \circlearrowleft\} d\mu(j).$$

By the proved conjecture of Vogan, if H_j occurs in the decomposition of $\text{Ker}\, D$ then it has infinitesimal character $\mu + \rho_c$. There are at most finitely many representations with a fixed infinitesimal character. Thus, if $\text{Ker}\, D$ is nonzero, the occurred H_j must be in the discrete spectrum, i.e., a discrete series representation. It follows from Corollary 6.4 and the above discussion of the indices that $w(\mu + \rho_c) - \rho$ is dominant. In other words, the infinitesimal character is strongly regular and therefore H_j is an $A_{\mathfrak{q}}(\lambda)$-module. Write λ for $w(\mu + \rho_c)$. In case λ is regular, this discrete series is isomorphic to $A_{\mathfrak{b}}(\lambda)$ with the lowest K-type $w(\mu + \rho_c) + \rho_c - \rho_n$. Therefore, we proved the following theorem due to Atiyah and Schmid.

Theorem (Atiyah-Schmid [AS]**).** *Let G be a linear group. Then the kernel of the Dirac operator D acting on the L^2-sections of the twisted spinor bundle corresponding to E_μ is a discrete series representation if there exists a $w \in W_{\mathfrak{g}}$ so that $w(\mu + \rho_c) - \rho$ is dominant, and* $\text{Ker}\, D$ *is zero otherwise.*

We note that Atiyah and Schmid also extended this geometric construction of discrete series to nonlinear groups.

Lecture 7. Lie algebra cohomology

7.1. Definition. Let $\mathfrak{g}$ be a complex Lie algebra and V a $\mathfrak{g}$-module. The space of invariants of V with respect to $\mathfrak{g}$ is the vector space

$$V^{\mathfrak{g}} = \{v \in V \big| Xv = 0, \forall X \in \mathfrak{g}\}.$$

Note that this definition is consistent with the definition of invariant (fixed) vectors under a group action. Upon differentiation, the condition of being fixed becomes the condition of being annihilated.

One also considers invariants not with respect to whole $\mathfrak{g}$ but with respect to some subalgebra. Let us describe an important example. Let $\mathfrak{g}$ be semisimple, $\mathfrak{h}$ a Cartan subalgebra, Δ the set of roots for $\mathfrak{g}$ with respect to $\mathfrak{h}$, Δ^+ a set of positive roots, and $\mathfrak{n} = \mathfrak{n}^+$ and $\mathfrak{n}^-$ the corresponding subalgebras like in Section 1.2.

Then one is interested in the space $V^{\mathfrak{n}}$ of $\mathfrak{n}$-invariants in V, which is now not only a vector space but an $\mathfrak{h}$-module. We already know why this space is interesting; if V is finite dimensional irreducible, then $V^{\mathfrak{n}}$ is one dimensional (the highest weight subspace), and the $\mathfrak{h}$-action on $V^{\mathfrak{n}}$, i.e., the highest weight of V, determines V. If V is still finite dimensional but reducible, $V = \oplus_i V_i$, then each V_i contributes a weight vector to $V^{\mathfrak{n}}$ and $V^{\mathfrak{n}}$ thus encodes the information about this decomposition.

We consider the functor $V \mapsto V^{\mathfrak{n}}$ from the category $\mathcal{M}(\mathfrak{g})$ of $\mathfrak{g}$-modules into the category $Vect_{\mathbb{C}}$ of complex vector spaces. (Analogously, $V \mapsto V^{\mathfrak{n}}$ would be a functor from $\mathcal{M}(\mathfrak{g})$ into $\mathcal{M}(\mathfrak{h})$.) This functor is in general left exact, but not exact. This means that if

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is a short exact sequence of $\mathfrak{g}$-modules (which is a slightly more precise way to write $W = V/U$), then

$$0 \longrightarrow U^{\mathfrak{g}} \longrightarrow V^{\mathfrak{g}} \longrightarrow W^{\mathfrak{g}}$$

is exact, but $V^{\mathfrak{g}} \longrightarrow W^{\mathfrak{g}}$ is not surjective in general. Thus we define the $\mathfrak{g}$-cohomology functors to be the right derived functors of the functor $V \mapsto V^{\mathfrak{g}}$.

To motivate the use of derived functors, let us consider the following simple example. Knowing how one can benefit from the usual duality operation for vector spaces, $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, one would like to have something similar for modules, say over $\mathbb{Z}$ (for simplicity). If one tries to consider $V^* = \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$, then one notices that for free modules of finite rank it works fine (including double dual giving back the same module), but for say $\mathbb{Z}_2$, one has

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) = 0.$$

Namely, $\mathbb{Z}_2$ is not free, and its generator 1 can not be mapped anywhere but to 0. We can however resolve $\mathbb{Z}_2$ by free modules:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Thus the complex $0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow 0$, concentrated in degrees -1 and 0, should replace the module $\mathbb{Z}_2$. If we take this resolution and plug it in the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ instead of the module $\mathbb{Z}_2$ it resolves, we get the complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow 0$$

which is concentrated in degrees 0 and 1 (by contravariance, the arrows changed direction). Its zeroth cohomology is 0, corresponding to the observed fact $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) = 0$. However, its first cohomology is $\mathbb{Z}_2$ (this is $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z})$, the first derived functor of Hom). If we now apply duality again, we get back to our resolution of $\mathbb{Z}_2$, that is $\mathbb{Z}_2$ in the zeroth cohomology.

To make the above completely precise, one would need to pass to derived category, which is a way of making modules equal (isomorphic) to their resolutions. However, the point we wanted to make is visible: the functor itself maybe sometimes gives nothing, but considering also the derived functors may give more.

We now get back to our covariant, left exact functor $V \mapsto V^{\mathfrak{g}}$. In principle, the right derived functors are defined using injective resolutions $0 \longrightarrow V \longrightarrow I^{\cdot}$; so the i-th $\mathfrak{g}$-cohomology space of V would be

$$H^i(\mathfrak{g}; V) = H^i((I^{\cdot})^{\mathfrak{g}}).$$

Namely, as above, we replace V by its resolution, plug the resolution in the functor, get a complex with induced differential, and then take cohomology. However, this is not very explicit, as injective resolutions are rather complicated to write down. Fortunately we have a better possibility: note that

$$V^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V),$$

where $\mathbb{C}$ is the trivial $\mathfrak{g}$-module; simply identify every map $\phi : \mathbb{C} \longrightarrow V$ with $\phi(1) \in V^{\mathfrak{g}}$. This means that our $\mathfrak{g}$-cohomology $H^i(\mathfrak{g}; V)$ is equal to $\mathrm{Ext}^i_{\mathfrak{g}}(\mathbb{C}, V)$, and to define it (calculate it) we can resolve $\mathbb{C}$ by free modules (or projectives) instead of resolving V by injectives. This is done by the so called standard, or Koszul complex

$$U(\mathfrak{g}) \otimes \bigwedge^{\cdot} \mathfrak{g} \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0,$$

where the $\mathfrak{g}$-action is given by left multiplication in the $U(\mathfrak{g})$-factor, the differential is the deRham differential

$$d(u \otimes X_1 \wedge \cdots \wedge X_k) = \sum_i (-1)^{i-1} u X_i \otimes X_1 \wedge \ldots \widehat{X}_i \cdots \wedge X_k + \sum_{i<j} (-1)^{i+j} u \otimes [X_i, X_j] \wedge X_1 \wedge \ldots \widehat{X}_i \ldots \widehat{X}_j \cdots \wedge X_k,$$

and ϵ is the augmentation map, given by $1 \mapsto 1$ and $\mathfrak{g}U(\mathfrak{g}) \mapsto 0$.

To see that this is indeed a resolution, one considers the graded version, $S(\mathfrak{g}) \otimes \bigwedge \mathfrak{g}$ and is thus lead to the analogous question of resolving the trivial module $\mathbb{C}$ over a polynomial algebra. Now for polynomials in one variable, it is obvious how to do this:

$$0 \longrightarrow \mathbb{C}[X] \xrightarrow{X\cdot} \mathbb{C}[X] \longrightarrow \mathbb{C} \longrightarrow 0$$

is clearly the required resolution. To increase the number of variables, this is tensored with itself several times; the introduction of signs which leads to the exterior algebra is forced by the requirement $d^2 = 0$. The reader is invited to try to construct a resolution for two variables from scratch and see how the exterior algebra appears quite naturally.

Now $H^i(\mathfrak{g}; V)$ is the i-th cohomology of the complex

$$\mathrm{Hom}^{\cdot}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes \bigwedge^{\cdot} \mathfrak{g}, V) = \mathrm{Hom}^{\cdot}_{\mathbb{C}}(\bigwedge^{\cdot} \mathfrak{g}, V)$$

with the induced differential

$$d\alpha(X_1 \wedge \cdots \wedge X_k) = \sum_i X_i \alpha(X_1 \wedge \ldots \widehat{X}_i \cdots \wedge X_k) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j] \wedge \ldots),$$

for $\alpha \in \mathrm{Hom}^{k-1}_{\mathbb{C}}(\bigwedge^{\cdot} \mathfrak{g}, V)$. Of course, $H^0(\mathfrak{g}; V) = V^{\mathfrak{g}}$.

7.2. Properties and basic applications. Here are some properties of these functors:

(1) Any short exact sequence

$$0 \to U \to V \to W \to 0$$

of $\mathfrak{g}$-modules gives rise to a long exact sequence of cohomology

$$0 \to U^{\mathfrak{g}} \to V^{\mathfrak{g}} \to W^{\mathfrak{g}} \to H^1(\mathfrak{g}; U) \to H^1(\mathfrak{g}; V) \to H^1(\mathfrak{g}; W) \to H^2(\mathfrak{g}; U) \to \ldots$$

This is a completely general property of derived functors. It can be used to calculate cohomology in some case, or for various proofs, like the one below.

(2) If $\mathfrak{g}$ is semisimple and if V has nontrivial infinitesimal character, then $H^i(\mathfrak{g}; V) = 0$ for all i. This can be proved by noting that since any $Z \in Z(\mathfrak{g})$ with no constant term vanishes on the trivial module $\mathbb{C}$, then by a standard homological argument it follows that the action of Z on the standard complex which is a resolution of $\mathbb{C}$ must be homotopic to 0.

From this statement one can obtain Weyl's theorem mentioned in 1.2: any finite dimensional module over a semisimple Lie algebra is a direct sum of irreducibles. This statement is equivalent to the statement that any short exact sequence

$$0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$$

of finite dimensional $\mathfrak{g}$ modules is split, i.e., there is an $s : W \to V$ such that $p \circ s$ is the identity on W. Indeed, such a splitting exhibits V as a direct sum of U and W and we can then decompose modules completely by induction on the dimension.

Now to get a splitting, it suffices to know that the sequence

$$\mathrm{Hom}_{\mathfrak{g}}(W, V) \xrightarrow{p\circ} \mathrm{Hom}_{\mathfrak{g}}(W, W) \to 0$$

is exact; then we get our splitting s as a preimage of id_W. But the functor $\mathrm{Hom}_{\mathfrak{g}}(W, -)$ is a composition of two exact functors: $\mathrm{Hom}_{\mathbb{C}}(W, -)$, which is clearly exact, and $(-)^{\mathfrak{g}}$ which is also exact on finite dimensional modules by the long exact sequence and the fact that $H^1(\mathfrak{g}; -)$ vanishes on all finite dimensional modules. The vanishing of H^1 has to be checked separately for the trivial module $\mathbb{C}$; this is an easy calculation.

(3) Similarly, Levi's theorem that any Lie algebra is a semidirect product of the radical and a semisimple subalgebra can be obtained from the fact that the second

cohomology of finite dimensional modules over a semisimple Lie algebra vanishes. Namely, by definition the radical $\mathfrak{r}$ of $\mathfrak{g}$ is the biggest solvable ideal, so there is an exact sequence

$$0 \longrightarrow \mathfrak{r} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{l} \longrightarrow 0$$

of Lie algebras, with $\mathfrak{l}$ semisimple. Now the nontrivial extensions of $\mathfrak{l}$ by $\mathfrak{r}$ can be seen to correspond to $H^2(\mathfrak{l};\mathfrak{r})$. As the last space is 0, every extension is trivial and hence the above sequence splits.

7.3. Theorems of Casselman-Osborne and Kostant. We next consider a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, with a Levi decomposition

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u};$$

this is a slightly different Levi decomposition from the one mentioned above, with $\mathfrak{u}$ the nilradical (the largest nilpotent ideal) and $\mathfrak{l}$ a reductive subalgebra. A special case arises in the situation already studied; for a Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{n}$ coming from a choice of positive roots, define

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}.$$

This is a maximal solvable subalgebra of $\mathfrak{g}$ and such are called Borel subalgebras. A parabolic subalgebra of $\mathfrak{g}$ can be defined as any subalgebra containing a Borel subalgebra. One should think of Borel subalgebras as algebras of upper triangular matrices, and of parabolic subalgebras as algebras of block upper triangular matrices with some fixed shape of blocks. Then the Levi factor $\mathfrak{l}$ is the algebra of corresponding block diagonal matrices, and the nilradical $\mathfrak{u}$ is the algebra of all upper triangular matrices that are zero on blocks. This is for example exactly true (in some complex basis) if $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$.

We consider the $\mathfrak{u}$-cohomology spaces $H^i(\mathfrak{u};V)$ for a $\mathfrak{g}$-module V. These spaces are actually $\mathfrak{l}$-modules in a natural way; namely, $\mathfrak{l}$ acts on the complex defining $H^i(\mathfrak{u};V)$ by acting both on $\bigwedge \mathfrak{u}$ and on V. This action commutes with the differential, hence descends to cohomology.

Now $Z(\mathfrak{g})$, the center of the enveloping algebra of $\mathfrak{g}$, acts on the complex $\mathrm{Hom}^{\cdot}(\bigwedge \mathfrak{u}, V)$ and its cohomology $H^i(\mathfrak{u};V)$ in two ways. One action is through the Harish-Chandra homomorphism $Z(\mathfrak{g}) \longrightarrow Z(\mathfrak{l})$ and the $\mathfrak{l}$ (i.e., $U(\mathfrak{l})$) action described above, and the other is acting on V only. This second action is well defined only for the center and not for the rest of $U(\mathfrak{g})$.

Casselman-Osborne theorem. *The above two actions of $Z(\mathfrak{g})$ agree on cohomology $H^i(\mathfrak{u};V)$.*

This is a similar statement to Property 2) of 7.2, about the action of the center on the standard complex which was used to get vanishing of $\mathfrak{g}$-cohomology. A proof can be found in [V], Theorem 3.1.5, or [KV], Theorem 4.149. We will also see how to get this in a different way, using Clifford algebra actions and Dirac type operators.

Kostant's theorem. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra corresponding to a choice of positive roots. Let V_λ be the irreducible finite dimensional $\mathfrak{g}$-module with highest weight λ. Then*

$$H^i(\mathfrak{n};V_\lambda) = \bigoplus_{w\in W, l(w)=i} \mathbb{C}_{w(\lambda+\rho)-\rho}$$

as $\mathfrak{h}$-modules. Here $\mathbb{C}_\mu$ is the one dimensional $\mathfrak{h}$-module with weight μ, W is the Weyl group, and $l(w)$, the length of $w \in W$, is the smallest number of simple root reflections needed to write w as their product.

A proof of this statement and also its generalization to the case of $\mathfrak{u}$-cohomology can be found e.g. in [KV], theorems 4.135 and 4.139. The original proof is greatly simplified by using the (more recent) Casselman-Osborne theorem to conclude that only the weights $w(\lambda+\rho)-\rho$ can appear.

Kostant interpreted this theorem as an algebraic version of the Borel-Weil-Bott theorem, which realizes finite dimensional $\mathfrak{g}$-modules as global sections or cohomology of line bundles on the flag variety $\mathcal{B}$ of $\mathfrak{g}$. (The flag variety consists of all Borel, i.e., maximal solvable subalgebras of $\mathfrak{g}$). He further used it to obtain an algebraic proof of the Weyl character formula, which explicitly calculates the (global) character of irreducible representations of compact groups.

7.4. $\mathfrak{n}$-homology and Casselman subrepresentation theorem. Lie algebra homology is defined in a similar way as cohomology. One starts by defining coinvariants

$$V_{\mathfrak{g}} = V/\mathfrak{g}V = V \otimes_{U(\mathfrak{g})} \mathbb{C}$$

of a $\mathfrak{g}$-module V; this is the largest quotient of V on which $\mathfrak{g}$ acts trivially. This is a right exact functor, and Lie algebra homology functors are the left derived functors

$$H_i(\mathfrak{g};V) = \mathrm{Tor}_i^{\mathfrak{g}}(\mathbb{C},V) = H_i(U(\mathfrak{g}) \otimes \bigwedge{}^{\cdot}\mathfrak{g} \otimes_{U(\mathfrak{g})} V) = H_i(\bigwedge{}^{\cdot}\mathfrak{g} \otimes V).$$

The differential is again induced by the deRham differential of the standard complex, which is used to resolve the variable $\mathbb{C}$ and thus define the derived functors.

To illustrate the importance of ($\mathfrak{n}$-)homology, we mention a very standard construction of representations, namely the real parabolic induction. Start with the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

of a real semisimple Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{a}_0 \subset \mathfrak{p}_0$ be a maximal abelian subalgebra (subspace). Then one can consider the $(\mathfrak{g}_0, \mathfrak{a}_0)$-roots, by the same principle as for $\mathfrak{h}$ in $\mathfrak{g}$ (the adjoint action of $\mathfrak{a}_0$ on $\mathfrak{g}_0$ diagonalizes etc.) Let $\mathfrak{n}_0$ be the sum of positive root spaces (for a choice of positive roots). Then one shows that

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0;$$

this is called the Iwasawa decomposition of $\mathfrak{g}_0$. There is also a group version of this fact: multiplication defines a diffeomorphism

$$K \times A \times N \to KAN = G$$

where A and N are the connected subgroups of G corresponding to $\mathfrak{a}_0$ and $\mathfrak{n}_0$. We denote by L the centralizer of A in G; then $L = MA$, where $M = L \cap K$ is the centralizer of A in K. The subgroup

$$P = MAN$$

is called a minimal (real) parabolic subgroup of G. Note a slight problem with notation: the Lie algebra of P is not $\mathfrak{p}$ from Cartan decomposition (that one is not a Lie algebra at all). Nevertheless, this is common notation.

Here is what is meant by induction from P to G. Let V be a (finite dimensional) representation of P. Consider the principal bundle $G \to G/P$ and construct the associated G-equivariant vector bundle

$$\begin{array}{c} G \times_P V \\ \big\downarrow \\ G/P \end{array}$$

The space $G \times_P V$ consists of classes of the equivalence relation on $G \times V$ defined by setting

$$(gp, v) \sim (g, p \cdot v), \quad g \in G, p \in P, v \in V.$$

The continuous sections of this bundle form a representation $\mathrm{Ind}_c(V)$ of G called the continuously induced representation. Here G acts on the sections by left translation. (One can also consider smooth sections, or L^2 sections of this bundle.) The sections can also be interpreted as functions from G into V with the appropriate transformation property for P; in this way one can avoid the language of bundles. We will denote by $\mathrm{Ind}(V)$ the $(\mathfrak{g}, K)$-module of K-finite vectors in $\mathrm{Ind}_c(V)$.

Let us consider the case

$$V = \sigma \otimes \lambda \otimes 1,$$

where σ is an irreducible finite dimensional representation of M, λ is a character of A (which is the same as a character of $\mathfrak{a}$), and 1 denotes the trivial representation of N. This is basically the only case one needs to study. In this case, $\mathrm{Ind}(V)$ is called the principal series representation.

Casselman proved the following version of Frobenius reciprocity: for any $(\mathfrak{g}, K)$-module W,

$$\mathrm{Hom}_{(\mathfrak{g},K)}(W, \mathrm{Ind}(V)) \xrightarrow{\cong} \mathrm{Hom}_{(\mathfrak{p},M)}(W, V).$$

Here $\mathfrak{p}$ is the Lie algebra of P; note that M is a maximal compact subgroup of P. Since the action of $\mathfrak{n}$ on V is trivial, the second Hom-space is further equal to

$$\mathrm{Hom}_{(\mathfrak{p},M)}(W/\mathfrak{n}W, V).$$

If $W/\mathfrak{n}W$ is not equal to zero, then one can choose V so that the last space is nonzero. It follows that W maps nontrivially into $\mathrm{Ind}(V)$, and if W is irreducible this map has to be an embedding. So we get Casselman's subrepresentation theorem. Namely, $W/\mathfrak{n}W$ is not equal to zero; in fact, this space contains leading exponents of asymptotic expansions of matrix coefficients of W. See [CM].

Let us finish this lecture by mentioning briefly two more facts. First, there is a version of Poincaré duality for Lie algebra cohomology. Second, there is a strong relationship between $\mathfrak{n}$-cohomology and Beilinson-Bernstein localization theory. In this theory, irreducible $(\mathfrak{g}, K)$-modules are realized as global sections (or cohomology) of certain sheaves on the flag variety $\mathcal{B}$ of $\mathfrak{g}$. The relationship is the following: the geometric fiber of the sheaf corresponding to a $(\mathfrak{g}, K)$-module V at a point $\mathfrak{b} \in X$ is exactly the $\mathfrak{n}$-cohomology of V, where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$.

Lecture 8. $(\mathfrak{g}, K)$-cohomology

8.1. Definition. One can study relative Lie algebra cohomology for the pair $(\mathfrak{g}, \mathfrak{k})$. It has however become more usual to study pairs $(\mathfrak{g}, K)$. The main case we are interested in is the one suggested by our choice of notation: $\mathfrak{g}$ is a semisimple complex Lie algebra and K is a maximal compact subgroup of a Lie group G with complexified Lie algebra $\mathfrak{g}$. This setting can however be generalized; K could be another compact subgroup of G, or a complex reductive subgroup of a complex group with Lie algebra $\mathfrak{g}$. In principle K does not even have to be reductive, but then it is not as easy as below to write down resolutions. One can also consider similar pairs $(\mathcal{A}, K)$ where $\mathcal{A}$ is an associative algebra. $\mathcal{A}$ could be $U(\mathfrak{g})$, or a quotient U_θ corresponding to an infinitesimal character (θ is a Weyl group orbit of an element of $\mathfrak{h}^*$).

For the purpose of this lecture, let us concentrate on the usual (and the most interesting) case of $(\mathfrak{g}, K)$, the first one mentioned above.

Formally, $(\mathfrak{g}, K)$-cohomology is analogous to $\mathfrak{g}$-cohomology. Namely, one can consider the functor

$$V \mapsto V^{\mathfrak{g},K} = \{v \in V \big| Xv = 0, kv = v, \text{ for all } X \in \mathfrak{g}, k \in K\}$$

of taking $(\mathfrak{g}, K)$-invariants. It is a functor from the category $\mathcal{M}(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$-modules into the category of complex vector spaces, which is left exact. The $(\mathfrak{g}, K)$-cohomology functors $V \mapsto H^i(\mathfrak{g}, K; V)$ are the right derived functors of $V \mapsto V^{\mathfrak{g},K}$.

As before, one can write

$$V^{\mathfrak{g},K} = \mathrm{Hom}_{(\mathfrak{g},K)}(\mathbb{C}, V),$$

and thus

$$H^i(\mathfrak{g}, K; V) = \mathrm{Ext}^i_{(\mathfrak{g},K)}(\mathbb{C}, V).$$

As before, rather than resolving V by injectives, we use a projective resolution of the trivial module $\mathbb{C}$. This is the relative standard complex

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bigwedge^{\cdot}(\mathfrak{g}/\mathfrak{k}) \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0;$$

Since we are considering compact K, we can replace $\mathfrak{g}/\mathfrak{k}$ by the K-invariant direct complement $\mathfrak{p}$. The differential d of the above complex and the map ϵ are similar as before:

$$d(u \otimes X_1 \wedge \cdots \wedge X_k) = \sum_i (-1)^{i-1} u X_i \otimes X_1 \wedge \ldots \widehat{X}_i \cdots \wedge X_k + \sum_{i<j} (-1)^{i+j} u \otimes [X_i, X_j]_{\mathfrak{p}} \wedge X_1 \wedge \ldots \widehat{X}_i \ldots \widehat{X}_j \cdots \wedge X_k,$$

for any compact K; here $[X_i, X_j]_{\mathfrak{p}}$ denotes the projection of $[X_i, X_j]$ to $\mathfrak{p}$ along $\mathfrak{k}$. If K is a symmetric subgroup, like in our main case when K is the maximal compact subgroup, then this projection is always zero, so the second sum actually vanishes. ϵ is as before the augmentation map, given by $1 \otimes 1 \mapsto 1$ and $\mathfrak{g}U(\mathfrak{g}) \otimes 1 \mapsto 0$.

The relative standard complex is obtained from the standard complex for $\mathfrak{g}$, by taking coinvariants with respect to the $\mathfrak{k}$-action given by right multiplication on $U(\mathfrak{g})$ and adjoint

action on $\bigwedge(\mathfrak{g})$, and also with respect to the action of $\bigwedge(\mathfrak{k})$ on $\bigwedge(\mathfrak{g})$ by (exterior) multiplication. Exactness is proved using this and exactness of the standard complex for $\mathfrak{g}$. The fact that this is a projective resolution follows from general homological principles: any finite K-module is projective, and the functor $W \mapsto U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} W$ from finite K-modules to $(\mathfrak{g}, K)$-modules preserves projectives, as it is left adjoint to the exact functor of forgetting the $\mathfrak{g}$-action.

Using the above resolution, we can now identify $H^i(\mathfrak{g}, K; V)$ with the i^{th} cohomology of the complex

$$\mathrm{Hom}^{\cdot}_{(\mathfrak{g},K)}(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bigwedge^{\cdot}(\mathfrak{p}), V) = \mathrm{Hom}^{\cdot}_{K}(\bigwedge^{\cdot}(\mathfrak{p}), V),$$

with differential

$$df(X_1 \wedge \cdots \wedge X_k) = \sum_i (-1)^{i-1} X_i \cdot f(X_1 \wedge \ldots \widehat{X}_i \cdots \wedge X_k)$$

if K is symmetric, and for other K there is another sum, over $i < j$.

There is also a less often mentioned theory of $(\mathfrak{g}, K)$-homology. It is constructed by deriving the functor of $(\mathfrak{g}, K)$-coinvariants; so

$$H_i(\mathfrak{g}, K; V) = \mathrm{Tor}_i^{(\mathfrak{g},K)}(\mathbb{C}, V).$$

which is calculated using the same resolution of $\mathbb{C}$ as above.

8.2. Applications. A good reference for learning about various applications of $(\mathfrak{g}, K)$-cohomology to the theory of automorphic forms is a recent survey article [LS]. Let us mention just one very classical application, the Matsushima formula. Let $\Gamma \subset G$ be a cocompact lattice and let E be a finite dimensional representation of G. Then the group cohomology of Γ with coefficients in E, which is also equal to the cohomology of the space $\Gamma \backslash G/K$ with coefficients in E, can be expressed as

$$H^*(\Gamma, E) \cong \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; H_\pi \otimes E),$$

where $m(\pi, \Gamma)$ is the multiplicity of the unitary representation (π, H_π) of G in $L^2(\Gamma \backslash G)$.

Another application is a construction of derived Zuckerman functors. Namely, let $(\mathfrak{g}, K)$ be a pair as above, with K complex algebraic (e.g., the complexification of a maximal compact subgroup). Let $T \subset K$ be a closed reductive subgroup. Let $R(K)$ be the algebra of regular functions on K. Then one can express the derived Zuckerman modules of a $(\mathfrak{g}, T)$-module V as

$$\Gamma^i_{K,T}(V) = H^i(\mathfrak{k}, T; R(K) \otimes V).$$

Here the $(\mathfrak{k}, T)$ cohomology is taken with respect to the tensor product of the regular action with the given action on V. K acts by right translation on $R(K)$, and the $\mathfrak{g}$-action is obtained by twisting the given action π_V on V: if we regard an element of $R(K) \otimes V$ as a regular function $F : K \to V$, then for $X \in \mathfrak{g}$ the function $\pi(X)F$ is given by

$$(\pi(X)F)(k) = \pi_V(\mathrm{Ad}(k)X)(F(k)).$$

There are several versions of this construction, due to Wallach ([W], Chapter 6), Duflo-Vergne, and Miličić-Pandžić.

8.3. The Vogan-Zuckerman classification. The central result about $(\mathfrak{g}, K)$-cohomology (for K the maximal compact subgroup of G) is the classification of irreducible unitary $(\mathfrak{g}, K)$-modules with nonzero $(\mathfrak{g}, K)$-cohomology obtained by Vogan and Zuckerman in [VZ]. This result can be stated as follows.

Let V be irreducible unitary, of the same infinitesimal character as a finite dimensional representation F. Note that there is only one possible F for any given V. Then $V \otimes F^*$ has nonzero $(\mathfrak{g}, K)$-cohomology if and only if V is an $A_{\mathfrak{q}}(\lambda)$ module, as described in previous lectures. (We also saw that in this case the lowest K-type of V gives rise to Dirac cohomology.)

Note that if V and F do not have the same infinitesimal character then $H^*(\mathfrak{g}, K; V \otimes F^*) = 0$. In case there is cohomology, it is equal to

$$\mathrm{Hom}_{L\cap K}(\textstyle\bigwedge^{i-\dim(\mathfrak{u}\cap\mathfrak{p})}(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C}),$$

where L is the Levi subgroup involved in the definition of $A_{\mathfrak{q}}(\lambda)$, $\mathfrak{l}$ is the (complexified) Lie algebra of L and $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$.

For more details, see [LS] or [VZ].

Lecture 9. Relationship of Dirac cohomology to other kinds of cohomology

In this lecture we briefly sketch some results from [HPR], which grew out of the ideas of [V4].

9.1. Kostant's cubic Dirac operator. In [K2], Kostant has constructed a cubic Dirac operator D corresponding to any decomposition

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$$

of a semisimple Lie algebra $\mathfrak{g}$ with $\mathfrak{r}$ a reductive subalgebra such that the Killing form is non-degenerate on $\mathfrak{r}$. The definition is as follows: let Z_i be an orthonormal basis for $\mathfrak{s}$ and let $v \in \bigwedge^3 \mathfrak{s}$ correspond to the alternating trilinear form $(X, Y, Z) \mapsto B([X, Y], Z)$ under the isomorphism $(\bigwedge^3 \mathfrak{s})^* \cong \bigwedge^3 \mathfrak{s}$ induced by the Killing form. Then

$$D = \sum_i Z_i \otimes Z_i + 1 \otimes v \ \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

It is easy to see that D is independent of the choice of basis Z_i and $\mathfrak{r}$-invariant for the adjoint action. Following Kostant, we now use defining relations $Z_i^2 = 1$ instead of $Z_i^2 = -1$ for $C(\mathfrak{s})$.

Kostant has shown that the formula we had for the Dirac operator corresponding to the Cartan decomposition, i.e.,

$$D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{r}_\Delta} + (||\rho||^2 - ||\rho_{\mathfrak{r}}||^2)$$

still holds in this more general situation. The ingredients of this formula are defined analogously as before.

Applications to $\mathfrak{u}$-cohomology. Let us consider a special case when $\mathfrak{r} = \mathfrak{l}$ and $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$ correspond to a θ-stable parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}.$$

Note that $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are both isotropic for B, that B identifies $\bar{\mathfrak{u}}$ with $\mathfrak{u}^*$, and that $\bar{\mathfrak{u}}$ is complex conjugate to $\mathfrak{u}$ with respect to the real form $\mathfrak{g}_0$ implicit in the above definitions.

Let $S = \bigwedge^{\cdot} \mathfrak{u}$ be a space of spinors for $C(\mathfrak{s})$. Since $\mathfrak{s}$ is even-dimensional, S is unique up to isomorphism. There are two $\mathfrak{l}$-actions on S; one is given by the adjoint action of $\mathfrak{l}$ on $\mathfrak{u}$, and the other is the spin action, coming from $\mathfrak{l} \longrightarrow \mathfrak{s}o(\mathfrak{s}) \longrightarrow C(\mathfrak{s})$. These two actions are related by a twist by $\rho_{\bar{\mathfrak{u}}}$; see [K4].

Let X be a $(\mathfrak{g}, K)$-module. Then the space

$$X \otimes S = \bigwedge^{\cdot} \mathfrak{u} \otimes X$$

has an action of D and of the $\mathfrak{u}$-homology operator ∂. Moreover, since we can identify

$$\bigwedge^{\cdot} \mathfrak{u} \otimes X \cong \mathrm{Hom}(\bigwedge^{\cdot} \bar{\mathfrak{u}}, X)$$

using $(\bigwedge^{\cdot} \mathfrak{u})^* \cong \bigwedge^{\cdot} \mathfrak{u}^* \cong \bigwedge^{\cdot} \bar{\mathfrak{u}}$, we also get an action of the $\bar{\mathfrak{u}}$-cohomology operator d on $X \otimes S$.

One checks that

$$D = d + 2\partial$$

on $X \otimes S$. To see this, take a basis u_i for $\mathfrak{u}$ and a dual basis u_i^* for $\bar{\mathfrak{u}}$; so $B(u_i, u_j^*) = \delta_{ij}$. Now write

$$D = \sum_i u_i^* \otimes u_i + \sum_i u_i \otimes u_i^* - \frac{1}{4} \sum_{i,j} 1 \otimes [u_i^*, u_j^*] u_i u_j - \frac{1}{4} \sum_{i,j} 1 \otimes [u_i, u_j] u_i^* u_j^*.$$

Denote by C (respectively C^-) the element of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ composed of the first and the third term (respectively the second and the fourth term) in the above expression for D. Then show that the action of C on $X \otimes S$ induces the differential d, while the action of C^- induces 2∂.

The "half Diracs" C and C^- are independent of the choice of basis u_i and $\mathfrak{l}$-invariant. They however do depend on the choice of $\mathfrak{u}$ inside $\mathfrak{s}$, while D does not. Furthermore, both C and C^- square to zero, and their supercommutator $CC^- + C^-C$ is equal to D^2.

It is convenient to introduce another element,

$$E = -\frac{1}{2} \sum_i 1 \otimes u_i^* u_i.$$

This operator acts as a degree operator on $X \otimes S$. It satisfies the following commuting relations:

$$[E, C] = C; \quad [E, C^-] = -C^-; \quad [E, D^2] = 0.$$

It follows that E, C, C^- and D^2 span a four dimensional superalgebra inside $(U(\mathfrak{g}) \otimes C(\mathfrak{s}))^{\mathfrak{l}}$. This algebra is actually $\mathbb{Z}$-graded, if we set C^- to be of degree -1, D^2 and E of degree 0 and C of degree 1. This grading is compatible with the obvious grading of $X \otimes S$ coming from the standard grading of $\bigwedge^{\cdot} \mathfrak{u}$.

This superalgebra was used by physicists under the name supersymmetric algebra. It is denoted by $l(1,1)$ in Kac's classification [Kac]. It is completely solvable and its representation theory is easy but not trivial.

As an application of the above facts, let us mention that one can easily prove that the main result of [HP] holds for C and C^- and in this way get the Casselman-Osborne theorem (see 7.3) as a corollary. Thus the formal analogy between the two results becomes more concrete.

There are two questions that arise from the above considerations. The first one, asked by Vogan in [V4], is to relate the Dirac cohomology, $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology of a $(\mathfrak{g}, K)$-module X. This could be useful for example to pass between different choices for $\mathfrak{u}$ within the same $\mathfrak{s}$. The second question was implicitly posed by Kostant in [K4]. His remark was that $X \otimes S$ can be formed not just in the "Levi factor case", i.e., for $\mathfrak{r} = \mathfrak{l}$, but also for a wider class of $\mathfrak{r}$ described above. On $X \otimes S$ there is always a cubic Dirac operator, and its square is the Laplacian. So it looks natural to try and study this more general setting and see how to make use of that.

At this moment, we have some results regarding the first question. Namely, in certain special cases all three (co)homology modules are equal up to appropriate twists. The twist is coming from the already mentioned identification of spin and adjoint actions on $\bigwedge(\mathfrak{u})$, and it is given by $\rho_{\bar{\mathfrak{u}}}$.

9.3. Hodge decomposition. Let us take X to be unitary, so it carries an invariant positive definite hermitian form. We need to put a similar form on S. There is a natural, $\mathfrak{l}$-invariant one, given by the Killing form B on $\bigwedge \mathfrak{u}$. This form is however indefinite if $\mathfrak{u}$ intersects both $\mathfrak{k}$ and $\mathfrak{p}$ as will usually be the case. The operators d and 2∂ can be shown to be adjoint with respect to this form, but one can not in general obtain a "Hodge decomposition".

A good case where one can get the "Hodge theory approach" to work is when $\mathfrak{l} = \mathfrak{k}$ so that $\mathfrak{u} \subset \mathfrak{p}$ and B is positive definite on $\mathfrak{u}$. This is possible only in the Hermitian symmetric case (i.e., when $\mathfrak{k}$ has a center). In this case the Dirac operator D is "ordinary" (i.e., has no cubic term) - it is the same D we studied in Lecture 5. We know that in this situation D^2 is a scalar on each $\tilde{K}$-type, so in particular, D^2 acts (locally) finitely on $X \otimes S$. Note also that D^2 is negative semidefinite in the present situation. Namely, the relations in the Clifford algebra are now $Z_i^2 = 1$, not -1, hence D is skew-symmetric and not symmetric as before. So all eigenvalues of D^2 are non-positive.

Because of these facts, one can use the easy variant of Hodge theory for finite dimensional spaces (see [W], Scholium 9.4.4), and conclude that

$$\operatorname{Ker} D = \operatorname{Ker} D^2 = \operatorname{Ker} d \cap \operatorname{Ker} \partial;$$

$$\operatorname{Ker} d = \operatorname{Ker} D \oplus \operatorname{Im} d; \quad \operatorname{Ker} \partial = \operatorname{Ker} D \oplus \operatorname{Im} \partial.$$

In particular, the cohomology of both d and ∂ is equal to the Dirac cohomology of D, $\operatorname{Ker} D$, as vector spaces. To compare them as $\mathfrak{l} = \mathfrak{k}$-modules involves the above mentioned twist.

The same argument proves an analogous result for X finite dimensional; here one uses the "admissible form" on X, i.e., the one invariant for the compact form of $\mathfrak{g}$. This case was known to Vogan and it is also implicit in [K4].

9.4. A counterexample. Here is a simple example which shows that the equality of the three cohomology modules does not hold for general $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-modules. Consider the module V which is a nontrivial extension of the discrete series representation of highest weight -2 by the trivial module $\mathbb{C}$:

$$0 \to \mathbb{C} \to V \to W \to 0.$$

V is a submodule of the module $V_{-1,0}$ from the end of Lecture 1. The weights of V (for the basis element $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ of $\mathfrak{k}$) are $\cdots -4, -2, 0$. We are considering the case $\mathfrak{l} = \mathfrak{k}$, $\mathfrak{u}$ is spanned by $u = X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $\bar{\mathfrak{u}}$ is spanned by $u^* = Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$. Now

$$V \otimes S = V \otimes 1 \oplus V \otimes u,$$

with $d : V \otimes 1 \to V \otimes u$ given by $d(v \otimes 1) = u^* \cdot v \otimes u$, and $\partial : V \otimes u \to V \otimes 1$ given by $\partial(v \otimes u) = u \cdot v \otimes 1$.

By an easy direct calculation one sees that the $\mathfrak{u}$-homology of V is given by

$$H_0(\partial) = 0; \quad H_1(\partial) = \mathbb{C}v_0 \otimes u,$$

the $\bar{\mathfrak{u}}$-cohomology of V is given by

$$H^0(d) = \mathbb{C}v_0 \otimes 1; \quad H^1(d) = \mathbb{C}v_0 \otimes u \oplus \mathbb{C}v_{-2} \otimes u,$$

and the Dirac cohomology of V is given by

$$H_D(V) = \operatorname{Ker} D = \mathbb{C}v_0 \otimes u.$$

So we see

$$H_D(V) = H_\cdot(\partial) \neq H^\cdot(d).$$

9.5. Dirac cohomology and $(\mathfrak{g}, K)$-cohomology. In the rest of this lecture we make a few comments on the relationship of Dirac cohomology and $(\mathfrak{g}, K)$-cohomology. It was proved in [HP] that if X is unitary and has $(\mathfrak{g}, K)$-cohomology, i.e.,

$$H^*(\mathfrak{g}, K; X \otimes F^*) = H^*(\operatorname{Hom}^\cdot_K(\textstyle\bigwedge^\cdot \mathfrak{p}, X \otimes F^*)) \neq 0$$

for a finite dimensional F (which then necessarily has the same infinitesimal character as X), then X also has Dirac cohomology.

In the following we assume that $\dim \mathfrak{p}$ is even. Then we can write $\mathfrak{p}$ as a direct sum of isotropic vector spaces $\mathfrak{u}$ and $\bar{\mathfrak{u}} \cong \mathfrak{u}^*$. One considers the spinor spaces $S = \bigwedge^\cdot \mathfrak{u}$ and $S^* = \bigwedge^\cdot \bar{\mathfrak{u}}$; then

$$S \otimes S^* \cong \textstyle\bigwedge^\cdot(\mathfrak{u} \oplus \bar{\mathfrak{u}}) = \bigwedge^\cdot \mathfrak{p}.$$

It follows that we can identify the $(\mathfrak{g}, K)$-cohomology of $X \otimes F^*$ with

$$H^*(\operatorname{Hom}^\cdot_{\tilde{K}}(F \otimes S, X \otimes S)).$$

There are several possible actions of the Dirac operator D on the above complex; similarly as before, they can be related to the coboundary operator d and the boundary operator ∂ for $(\mathfrak{g}, K)$-homology, which also acts on the same complex after appropriate identifications.

Now if X is unitary, Wallach has proved that $d = 0$ (see [W], Proposition 9.4.3, or [BW]). Using similar arguments one can analize the above mentioned Dirac actions and the actions of the corresponding "half-Diracs". In particular, it follows that

$$H^*(\mathfrak{g}, K; X \otimes F^*) = \operatorname{Hom}^\cdot_{\tilde{K}}(H_D(F), H_D(X)).$$

This can be concluded from the fact that the eigenvalues of D^2 are of opposite signs on $F \otimes S$ and $X \otimes S$; see [W], 9.4.6. One may hope to generalize some of these facts, either with respect to X, or with respect to F.

We finish by a remark that for nice and natural proofs it would be useful to formalize some constructions in the category of modules over the Clifford algebra, although this may not seem necessary as the category is extremely simple. For example, one can construct a coproduct of $C(\mathfrak{p})$ by defining

$$c(Z) = \frac{1}{\sqrt{2}}(Z \otimes 1 + 1 \otimes Z)$$

for $Z \in \mathfrak{p}$. This coproduct is an algebra morphism in the graded sense. It is not coassociative, but it is cocommutative. There is no counit, but there is an antiautomorphism, the already mentioned α (see 2.4) which can be used instead of an antipode. It seems worthwhile to try to use this structure to clarify notions like tensor products and dual modules.

Lecture 10. Multiplicities of automorphic forms

In this lecture we prove a formula for multiplicities of automorphic forms which sharpens the result of Langlands and Hotta-Parthasarathy. Let G be a linear semisimple noncompact Lie group. Let K be a maximal compact subgroup of G. Assume that $\operatorname{rank} G = \operatorname{rank} K$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of the Lie algebra of G. Then $\mathfrak{u}_0 = \mathfrak{k}_0 + i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Let U be the compact analytic subgroup in the complexification $G_{\mathbb{C}}$ of G with Lie algebra $\mathfrak{u}_0$.

10.1. Hirzebruch proportionality principle. Let Γ be a torsion free discrete subgroup of G so that $\Gamma\backslash G$ and $X = \Gamma\backslash G/K$ are compact smooth manifolds. Borel showed that such a Γ always exists. Then the regular representation on $\Gamma\backslash G$ is decomposed discretely with finite multiplicities:

$$L^2(\Gamma\backslash G) \cong \bigoplus_{\pi\in\widehat{G}} m(\Gamma,\pi)\mathcal{H}_\pi.$$

Let X_π be the Harish-Chandra module of $\mathcal{H}_\pi$. For any $\mu \in \widehat{K}$, we define the Dirac operators D, $D^+_\mu(X)$ and $D^+_\mu(Y)$ as in 6.5. As we saw in 6.5, if we normalize the Haar measure so that $vol(U) = 1$, then

$$\operatorname{Index} D^+_\mu(X) = (-1)^q vol(\Gamma\backslash G) \operatorname{Index} D^+_\mu(Y).$$

On the other hand,

$$\operatorname{Index} D^+_\mu(X) = \sum_{\pi\in\widehat{G}} m(\Gamma,\pi) \operatorname{Index} D^+_\mu(X_\pi),$$

where $D^+_\mu(X_\pi) : \operatorname{Hom}_{\tilde{K}}(E^*_\mu, X_\pi \otimes S^+) \to \operatorname{Hom}_{\tilde{K}}(E^*_\mu, X_\pi \otimes S^-)$ is the linear map defined by $\phi \mapsto D \circ \phi$ for any $\phi \in \operatorname{Hom}_{\tilde{K}}(E^*_\mu, X_\pi \otimes S^+)$.

10.2. Dimension of automorphic forms. If $\operatorname{Index} D^+_\mu(X_\pi) \neq 0$, then the Dirac cohomology $H_D(X_\pi)$ contains E_{μ^*}. It follows from the proved Vogan's conjecture that the infinitesimal character of X_π is given by $\mu^* + \rho_c$. If we assume that $\lambda = w(\mu^* + \rho_c) - \rho$ is dominant for some $w \in W$, then X_π is isomorphic to $A_{\mathfrak{q}}(\lambda)$ for some θ-stable parabolic subalgebra $\mathfrak{q}$. If in addition we assume that λ is regular with respect to the noncompact roots $\Delta^+(\mathfrak{p})$, then X_π is uniquely determined as a discrete series $A_{\mathfrak{b}}(\lambda)$. Since $\operatorname{Index} D^+_\mu(A_{\mathfrak{b}}(\lambda)) = \dim D^+_\mu(A_{\mathfrak{b}}(\lambda)) - \operatorname{codim} D^+_\mu(A_{\mathfrak{b}}(\lambda)) = (-1)^q$ and $\operatorname{Index} D^+_\mu(Y)$ has been calculated in Corollary 6.4., we obtain the following theorem.

Theorem. *Let $\pi = A_{\mathfrak{b}}(\lambda)$ be a discrete series representation with λ regular with respect to all noncompact roots. Assume that λ is dominant and can be written as $\lambda = \mu - \rho_n$ for some highest weight $\mu \in \widehat{K}$. Then*

$$m(\Gamma,\pi) = vol(\Gamma\backslash G) d_\pi,$$

where d_π is the formal degree of π:

$$d_\pi = \frac{\Pi_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\lambda+\rho,\alpha)}{\Pi_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\rho,\alpha)}.$$

This sharpens the result of Langlands [L] and Hotta-Parthasarathy [HoP], who proved the above formula for discrete series representations whose matrix coefficients are in $L^1(G)$. Trombi-Varadarajan proved that if the matrix coefficients of the discrete series $A_{\mathfrak{b}}(\lambda)$ are in $L^1(G)$, then

$$\langle \lambda + \rho, \alpha \rangle > \text{Max}\,\{\, |\langle w\rho, \alpha\rangle|, \forall w \in W_{\mathfrak{g}} \text{ and } \forall \alpha \in \Delta^+(\mathfrak{p})\,\}.$$

Hecht-Schmid proved this is also a sufficient condition. Our assumption on the regularity of λ with respect to the noncompact roots amounts to the condition

$$\langle \lambda + \rho, \alpha \rangle > \text{Max}\,\{\, |\langle w\rho, \alpha\rangle|, \forall \alpha \in \Delta^+(\mathfrak{p}) \text{ with } w = 1.\,\}.$$

Therefore, our condition is weaker than that assumed by Langlands and Hotta-Parthasarathy.

10.3. A final remark. Borel and Wallach proved that for any finite-dimensional representation of G,

$$H^*(\Gamma, F) = \bigoplus_{\pi \in \widehat{G}} m(\Gamma, \pi) H^*(\mathfrak{g}, K, X_\pi \otimes F).$$

We still assume that $\text{rank}\, G = \text{rank}\, K$. If the highest weight of F is regular, then it follows from a similar argument as in 10.2. that X_π is uniquely determined as a discrete series $A_{\mathfrak{b}}(\lambda)$, and therefore,

$$\dim H^*(\Gamma, F) = vol(\Gamma\backslash G) d_\pi \dim H^*(\mathfrak{g}, K, X_\pi \otimes F).$$

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