HERMITIAN FORMS FOR AFFINE HECKE ALGEBRAS

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Abstract. We study star operations for Iwahori-Hecke algebras and invariant hermitian forms for finite dimensional modules over (graded) affine Hecke algebras, with a view towards a signature algorithm.

1. Introduction

In this paper, we study star operations for Iwahori-Hecke algebras and invariant hermitian forms for the (graded) affine Hecke algebras that appear in the theory of reductive p-adic groups. There are three main parts to our paper. We explain them next.

1.1. We classify the star operations (conjugate-linear involutive anti-automorphims) for the graded affine Hecke algebra $\mathbb{H}$ with unequal parameters which preserve a natural filtration of $\mathbb{H}$ (section 2). This can be viewed as an analogue of the problem of classifying the star operations for the enveloping algebra $U(\mathfrak{g})$ of a complex semisimple Lie algebra which preserve $\mathfrak{g}$. The first result, Proposition 2.4.3, says that essentially there are only two such star operations: $*$ and $\bullet$, Definition 2.3.2.

The anti-automorphism $*$ is known to correspond to the natural star operation of the Hecke algebra of a reductive p-adic group, i.e., $f^*(g) = \overline{f(g^{-1})}$, see [BM1, BM2].

On the other hand, the anti-automorphism $\bullet$ is the Hecke algebra analogue of the "compact star operation" for $(\mathfrak{g}, K)$-modules studied by Adams-van Leeuwen-Trapa-Vogan [ALTV] and Yee [Y]. The operation $\bullet$ also arises naturally in conjunction with Macdonald theory for affine Hecke algebras, and from this perspective, it was studied by Opdam [Op2].

1.2. We investigate the basic properties of the signature of $\bullet$-invariant hermitian forms for finite dimensional $\mathbb{H}$-modules (sections 3–6). We prove that every irreducible $\mathbb{H}$-module with real central character admits a nondegenerate $\bullet$-invariant hermitian form, Corollary 5.1.3, and moreover, when $\mathbb{H}$ is of geometric type, this form can be normalized canonically so that it is positive definite on every isotypic component of a lowest W-type, Corollary 5.3.3. For the first claim, we explicitly determine in Theorem 3.7.5 the $\bullet$-hermitian dual of any given simple $\mathbb{H}$-module, in terms of the Langlands datum, and we exhibit in Proposition 3.9.1 an explicit invariant hermitian form. The second claim follows by comparing the Langlands classification with the geometric classification of simple and standard $\mathbb{H}$-module [Lu2], together with an argument involving the "signature at infinity" of the form.

These results represent the Hecke algebra analogue of the similar results about c-invariant forms of $(\mathfrak{g}, K)$-modules [ALTV]. Motivated by the algorithm of [ALTV]

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(see also [Vo]), we define in section 6, based on the Jantzen filtration, hermitian Kazhdan-Lusztig polynomials (see Definition 6.3.2). We conjecture (Conjecture 6.3.2) a simple relation with the Kazhdan-Lusztig polynomials for graded Hecke algebras [Lu3]. In the remainder of section 6, we offer some evidence for this conjecture, by analyzing the case of regular central character, and the interesting examples of the subregular central character in types B_2 and G_2.

The algebra $\mathbb{H}$ has a large abelian subalgebra $\mathbb{A}$. Since $\bullet$ preserves $\mathbb{A}$ (unlike the classical $\star$), it is interesting to consider the weight spaces for $\mathbb{A}$ and study signatures of forms in this way. We prove a number of results along these lines, for example, a linear independence result for $\mathbb{A}$-characters of irreducible $\mathbb{H}$-modules, Theorem 4.4.2, as well as explicit formulas for the $\bullet$-forms when the $\mathbb{A}$-parameter is sufficiently dominant, e.g., Corollary 3.10.3. These results enter in an essential way in our proof of Conjecture 6.3.2 in the regular case.

1.3. We give an explanation of the occurence of the $\bullet$ operation from the perspective of the theory of Bernstein projective modules (section 7). We prove that in the Iwahori-spherical case for split p-adic groups, when the Iwahori-Hecke algebra $\mathcal{H}$ is viewed as the endomorphism algebra of a projective generator $\mathcal{P}$ ([Be]), then $\mathcal{H}$ acquires a natural hermitian inner product and the $\bullet$ star operation (with respect to right adjointness), see Theorem 7.8.4. We expect that a similar result holds in the generality of [He].

2. Star operations

2.1. **Graded affine Hecke algebra.** We fix an $\mathbb{R}$-root system $\Phi = (V, R, V^\vee, R^\vee)$. This means that $V, V^\vee$ are finite dimensional $\mathbb{R}$-vector spaces, with a perfect bilinear pairing $(\ ,\ ) : V \times V^\vee \to \mathbb{R}$, where $R \subset V \setminus \{0\}$, $R^\vee \subset V^\vee \setminus \{0\}$ are finite subsets in bijection

$$R \longleftrightarrow R^\vee,\ \alpha \longleftrightarrow \alpha^\vee,\ \text{ satisfying } (\alpha, \alpha^\vee) = 2. \tag{2.1.1}$$

Moreover, the reflections

$$s_\alpha : V \to V,\ s_\alpha(v) = v - (v, \alpha^\vee)\alpha, \quad s_\alpha : V^\vee \to V^\vee,\ s_\alpha(v') = v' - (\alpha, v')\alpha^\vee, \quad \alpha \in R, \tag{2.1.2}$$

leave R and $R^\vee$ invariant, respectively. Let W be the subgroup of $GL(V)$ (respectively $GL(V^\vee)$) generated by $\{s_\alpha :\ \alpha \in R\}$. We assume that the root system Φ is reduced, meaning that $\alpha \in R$ implies $2\alpha \notin R$. We fix a choice of simple roots $\Pi \subset R$, and consequently, positive roots R^+ and positive coroots $R^{\vee,+}$. Often, we will write $\alpha > 0$ or $\alpha < 0$ in place of $\alpha \in R^+$ or $\alpha \in (-R^+)$, respectively. The complexifications of V and $V^\vee$ are denoted by $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$, respectively, and we denote by $^-$ the complex conjugations of $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$ induced by V and $V^\vee$, respectively. Notice that

$$\overline{(v, u)} = (\overline{v}, \overline{u}),\ \text{for all } v \in V_\mathbb{C},\ u \in V_\mathbb{C}^\vee. \tag{2.1.3}$$

Let $k : \Pi \to \mathbb{R}$ be a function such that $k_\alpha = k_{\alpha'}$ whenever $\alpha, \alpha' \in \Pi$ are W-conjugate. Let $\mathbb{C}[W]$ denote the group algebra of W and $S(V_\mathbb{C})$ the symmetric algebra over $V_\mathbb{C}$. The group W acts on $S(V_\mathbb{C})$ by extending the action on V. For every $\alpha \in \Pi$, denote the difference operator by

$$\Delta : S(V_\mathbb{C}) \to S(V_\mathbb{C}), \quad \Delta_\alpha(a) = \frac{a - s_\alpha(a)}{\alpha},\ \text{for all } a \in S(V_\mathbb{C}). \tag{2.1.4}$$

Definition 2.1.1. The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}(\Phi, k)$ is the unique associative unital algebra generated by $\mathbb{A} = S(V_{\mathbb{C}})$ and $\{t_w : w \in W\}$ such that

(i) the assignment $t_w a \mapsto w \otimes a$ gives an isomorphism $\mathbb{H} \cong \mathbb{C}[W] \otimes S(V_{\mathbb{C}})$ of $(\mathbb{C}[W], S(V_{\mathbb{C}}))$-bimodules;
(ii) $at_{s_\alpha} = t_{s_\alpha} s_\alpha(a) + k_\alpha \Delta_\alpha(a)$, for all $\alpha \in \Pi$, $a \in S(V_{\mathbb{C}})$.

The center of $\mathbb{H}$ is $S(V_{\mathbb{C}})^W$ ([Lu1]). By Schur's Lemma, the center of $\mathbb{H}$ acts by scalars on each irreducible $\mathbb{H}$-module. The central characters are parameterized by W-orbits in $V_{\mathbb{C}}^\vee$. If X is an irreducible $\mathbb{H}$-module, denote by $\mathsf{cc}(X) \in W \backslash V_{\mathbb{C}}^\vee$ its central character. By abuse of notation, we may also denote by $\mathsf{cc}(X)$ a representative in $V_{\mathbb{C}}^\vee$ of the central character of X.

If (π, X) is a finite dimensional $\mathbb{H}$-module and $\lambda \in V_{\mathbb{C}}^\vee$, denote

$$X_\lambda = \{x \in X : \text{ for every } a \in S(V_{\mathbb{C}}),\ (\pi(a) - (a, \lambda))^n x = 0, \text{ for some } n \in \mathbb{N}\}. \tag{2.1.5}$$

If $X_\lambda \neq 0$, call λ an $\mathbb{A}$-weight of X. Let $\Omega(X) \subset V_{\mathbb{C}}^\vee$ denote the set of $\mathbb{A}$-weights of X. If X has a central character, it is easy to see that $\Omega(X) \subset W \cdot \mathsf{cc}(X)$.

Definition 2.1.2 (Casselman's criterion)**.** Set

$$V^+ = \{\omega \in V : (\omega, \alpha^\vee) > 0, \text{ for all } \alpha \in \Pi\}.$$

An irreducible $\mathbb{H}$-module X is called tempered if

$$(\omega, \operatorname{Re}\lambda) \leq 0, \text{ for all } \lambda \in \Omega(X) \text{ and all } \omega \in V^+.$$

A tempered module is called a discrete series module if all the inequalities are strict.

When the root system Φ is semisimple, $\mathbb{H}$ has a particular discrete series module, the Steinberg module St. This is a one-dimensional module, on which W acts via the sgn representation, and the only $\mathbb{A}$-weight is $-\sum_{\alpha\in\Pi} k_\alpha \omega_\alpha^\vee$, where $\omega_\alpha^\vee$ is the fundamental coweight corresponding to α.

2.2. **An automorphism of $\mathbb{H}$.** Let w_0 denote the long Weyl group element. Define an assignment

$$\delta(t_w) = t_{w_0 w w_0},\ w \in W, \quad \delta(\omega) = -w_0(\omega),\ \omega \in V_{\mathbb{C}}. \tag{2.2.1}$$

Lemma 2.2.1. *Suppose $k_{\delta(\alpha)} = k_\alpha$, for all $\alpha \in \Pi$. The assignment δ from (2.2.1) extends to an involutive automorphism of $\mathbb{H}$. When w_0 is central in W, $\delta = \mathrm{Id}$.*

Proof. It is clear that δ is an automorphism of $\mathbb{C}[W]$ and it also extends to an automorphism on $S(V_{\mathbb{C}})$, so it remains to check the commutation relation in Definition 2.1.1:

$$\omega t_{s_\alpha} - t_{s_\alpha} s_\alpha(\omega) = k_\alpha(\omega, \alpha^\vee),\ \alpha \in \Pi,\ \omega \in V_{\mathbb{C}}. \tag{2.2.2}$$

Then

$$\begin{aligned}\delta(\omega)\delta(t_{s_\alpha}) =& \delta(\omega) t_{s_{\delta(\alpha)}} = t_{s_{\delta(\alpha)}} s_{\delta(\alpha)}(\delta(\omega)) + k_{\delta(\alpha)}(\delta(\omega), \delta(\alpha)^\vee) = \\ =& t_{s_{\delta(\alpha)}} s_{\delta(\alpha)}(\delta(\omega)) + k_\alpha(\omega, \alpha^\vee).\end{aligned}$$

Notice that we have used the fact that $\delta(\alpha) \in \Pi$ if $\alpha \in \Pi$. It is easy to see that $\delta(s_\alpha(\omega)) = s_{\delta(\alpha)}(\delta(\omega))$.

Since $w_0^2 = 1$, $\delta^2 = \mathrm{Id}$. □

Thus, one may define an extended graded Hecke algebra $\mathbb{H}' = \mathbb{H} \rtimes \langle\delta\rangle$.

2.3. **Star operations.**

Definition 2.3.1. Let $\kappa : \mathbb{H} \to \mathbb{H}$ be a conjugate linear involutive algebra anti-automorphism. An $\mathbb{H}$-module (π, X) is said to be κ-hermitian if X has a hermitian form $(\ ,\ )$ which is κ-invariant, i.e.,

$$(\pi(h)x, y) = (x, \pi(\kappa(h))y), \quad x, y \in X,\ h \in \mathbb{H}.$$

A hermitian module X is κ-unitary if the κ-hermitian form is positive definite.

Definition 2.3.2. Define

$$t_w^\star = t_{w^{-1}},\ w \in W, \quad \omega^\star = -t_{w_0} \cdot \overline{w_0(\omega)} \cdot t_{w_0},\ \omega \in V_{\mathbb{C}}, \tag{2.3.1}$$

and

$$t_w^\bullet = t_{w^{-1}},\ w \in W, \quad \omega^\bullet = \overline{\omega},\ \omega \in V_{\mathbb{C}}. \tag{2.3.2}$$

Lemma 2.3.3. *The operations $\star$ and $\bullet$ defined in (2.3.1) and (2.3.2), respectively, extend to conjugate linear algebra anti-involutions of $\mathbb{H}$.*

Proof. Straightforward by Lemma 2.2.1. □

Remark 2.3.4. The two star operations just defined are related as follows

$$h^\star = t_{w_0} \cdot \delta(h)^\bullet \cdot t_{w_0}, \quad h \in \mathbb{H}. \tag{2.3.3}$$

In particular, when w_0 is central in W, they are inner conjugate to each other.

Lemma 2.3.5. *For every $w \in W$, $\omega \in V_{\mathbb{C}}$,*

$$t_w \cdot \omega \cdot t_{w^{-1}} = w(\omega) + \sum_{\beta>0, w(\beta)<0} k_\beta(\omega, \beta^\vee) t_{s_{w(\beta)}}. \tag{2.3.4}$$

In particular,

$$\omega^\star = -\overline{\omega} + \sum_{\beta>0} k_\beta(\overline{\omega}, \beta^\vee) t_{s_\beta}. \tag{2.3.5}$$

Proof. This is [BM2, Theorem 5.6]. □

2.4. **Classification of involutions.** We define a filtration of $\mathbb{H}$ given by the degree in $S(V_{\mathbb{C}})$. Set $\deg t_w a = \deg_{S(V_{\mathbb{C}})} a$ for every $w \in W$, and homogeneous element $a \in S(V_{\mathbb{C}})$ and $F_i\mathbb{H} = \mathrm{span}\{h \in \mathbb{H} : \deg h \le i\}$. In particular, $F_0\mathbb{H} = \mathbb{C}[W]$. Set $F_{-1}\mathbb{H} = 0$. It is immediate from Definition 2.1.1 that the associated graded algebra $\overline{\mathbb{H}} = \oplus_{i \ge 0} \overline{\mathbb{H}}^i$, where $\overline{\mathbb{H}}^i = F_i\mathbb{H}/F_{i-1}\mathbb{H}$, is naturally isomorphic to the graded Hecke algebra for the parameter function $k_\alpha \equiv 0$.

Definition 2.4.1. An automorphism (respectively, anti-automorphim) κ of $\mathbb{H}$ is called *filtered* if $\kappa(F_i\mathbb{H}) \subset F_i\mathbb{H}$, for all $i \ge 0$. Notice that by Definition 2.1.1, this is equivalent with the requirement that $\kappa(F_i\mathbb{H}) \subset F_i\mathbb{H}$ for $i = 0, 1$. If, in addition, $\kappa(t_w) = t_w$ (resp., $\kappa(t_w) = t_{w^{-1}}$), we say that κ is *admissible*.

If κ is a filtered automorphism, then κ induces an automorphism of the associated graded algebra $\overline{\mathbb{H}}$ which preserves that grading, i.e., $\kappa(\overline{\mathbb{H}}^i) \subset \overline{\mathbb{H}}^i$.

Lemma 2.4.2. *Assume the root system Φ is simple. Let κ be an admissible involutive automorphism (or anti-automorphism) of $\overline{\mathbb{H}}$ which respects the grading $\kappa(\overline{\mathbb{H}}^i) \subset \overline{\mathbb{H}}^i$. Then $\kappa(\omega) = c_0\omega$, for all $\omega \in V_{\mathbb{C}}$, where c_0 is a constant equal to 1 or -1.*

Proof. We treat the case when κ is an automorphism, the other case is completely similar. By the assumptions on κ,

$$\kappa(\omega) = \sum_{y \in W} f_y(\omega) t_y, \ \omega \in V_{\mathbb{C}}, \tag{2.4.1}$$

where $f_y : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is a linear function, for every $y \in W$. Let α be a simple root. The commutation relation in $\overline{\mathbb{H}}$ is $t_{s_\alpha}\omega = s_\alpha(\omega) t_{s_\alpha}$. Applying κ to this relation, it follows, by a simple calculation, that

$$s_\alpha(f_{s_\alpha x}(\omega)) = f_{x s_\alpha}(s_\alpha(\omega)), \text{ for all } x \in W.$$

In particular, setting $x = s_\alpha$, we see that

$$s_\alpha(f_1(\omega)) = f_1(s_\alpha(\omega)). \tag{2.4.2}$$

Since the root system was assumed simple, this means that f_1 is a scalar function $f_1(\omega) = c_0\omega$, for some $c_0 \in \mathbb{C}$.

Now, we use that κ is an involution, $\kappa^2(\omega) = \omega$, which implies $\sum_{x,y \in W}(f_x \circ f_y)(\omega) t_{xy} = \omega$. Thus

$$\sum_{x \in W} f_x \circ f_{x^{-1}} = \mathrm{Id}, \text{ and } f_x \circ f_y = 0, \text{ if } x \neq y^{-1}. \tag{2.4.3}$$

Specializing $y = 1$ in the second relation, we see that $f_x = 0$ if $x \neq 1$. Then the first relation implies $c_0^2 = 1$, and this is the claim of the lemma. □

Proposition 2.4.3. *Assume the root system Φ is simple. If κ is an admissible involutive automorphism or anti-automorphism (in the sense of Definition 2.4.1), then*

$$\kappa(\omega) = \omega, \text{ for all } \omega \in V,$$

or

$$\kappa(\omega) = t_{w_0} \cdot \delta(\omega) \cdot t_{w_0}, \text{ for all } \omega \in V.$$

In particular, the only admissible conjugate linear involutive anti-automorphisms of $\mathbb{H}$ are $\star$ and $\bullet$ from Lemma 2.3.3.

Proof. Suppose κ is an admissible involutive automorphism. (The argument is identical if κ is an antiautomorphim). By the admissibility condition, κ induces an admissible involutive automorphism of $\overline{\mathbb{H}}$. Lemma 2.4.2 implies that $\kappa(\omega) \equiv c_0\omega$ mod $F_1\mathbb{H}$. Therefore, κ must be of the form:

$$\kappa(t_w) = t_w, \quad w \in W; \qquad \kappa(\omega) = c_0\omega + \sum_{y \in W} g_y(\omega) t_y, \quad \omega \in V_{\mathbb{C}},$$

where $g_y : V_{\mathbb{C}} \to \mathbb{C}$, $y \in W$, are linear.

Since κ has to preserve the commutation relation

$$t_{s_\alpha}\omega - s_\alpha(\omega) t_{s_\alpha} = k_\alpha(\omega, \alpha^\vee), \ \alpha \in \Pi, \omega \in V_{\mathbb{C}},$$

we find that

$$c_0 t_{s_\alpha}\omega - c_0 s_\alpha(\omega) t_{s_\alpha} + \sum_{y \in W} g_y(\omega) t_{s_\alpha y} - \sum_{x \in W} g_x(s_\alpha(\omega)) t_{x s_\alpha} = k_\alpha(\omega, \alpha^\vee),$$

or equivalently,

$$\sum_{y \in W} g_y(\omega) t_{s_\alpha y} - \sum_{x \in W} g_x(s_\alpha(\omega)) t_{x s_\alpha} = k_\alpha(1 - c_0)(\omega, \alpha^\vee). \tag{2.4.4}$$

This implies that

$$g_{s_\alpha y s_\alpha}(\omega) = g_y(s_\alpha(\omega)), \text{ for all } \alpha \in \Pi, y \in W, y \neq s_\alpha, \text{ and } \omega \in V_{\mathbb{C}}, \tag{2.4.5}$$

and

$$g_{s_\alpha}(\omega) - g_{s_\alpha}(s_\alpha(\omega)) = k_\alpha(1 - c_0)(\omega, \alpha^\vee),$$

from which one easily concludes that

$$g_{s_\alpha}(\alpha) = k_\alpha(1 - c_0), \ \alpha \in \Pi. \tag{2.4.6}$$

We first show that $g_y = 0$ unless $y = s_\beta$ for some positive root β. If $y = 1$, relation (2.4.5) shows that $g_y = 0$, so assume $y \neq 1$. The automorphism κ must also satisfy $\kappa(\omega_1)\kappa(\omega_2) = \kappa(\omega_2)\kappa(\omega_1)$ for all $\omega_1, \omega_2 \in V_{\mathbb{C}}$. This implies that

$$g_y(\omega_2)(\omega_1 - y^{-1}(\omega_1)) = g_y(\omega_1)(\omega_2 - y^{-1}(\omega_2)), \text{ for all } y \in W, \ \omega_1, \omega_2 \in V_{\mathbb{C}}. \tag{2.4.7}$$

If λ_1, λ_2 are eigenvalues of y^{-1}, then for $\omega_1 \in V_{\lambda_1}$ $\omega_2 \in V_{\lambda_2}$,

$$g_y(\omega_1)(1 - \lambda_2)\omega_2 = g_y(\omega_2)(1 - \lambda_1)\omega_1. \tag{2.4.8}$$

Set $\lambda_1 = 1$. Then

$$g_y(\omega_1)(1 - \lambda_2)\omega_2 = 0 \text{ for any } \omega_2 \in V_{\lambda_2}.$$

Because $y^{-1} \neq 1$, it has an eigenvalue $\lambda_2 \neq 1$, so g_y is 0 on the $1-$eigenspace of y^{-1}. Similarly, relation (2.4.8) implies that if $\lambda \neq 1$, any $\omega_1, \omega_2 \in V_\lambda$ must be multiples of each other. So $\dim V_\lambda \leq 1$ for any $\lambda \neq 1$.

Because y is an automorphism of the real space V, if λ is an eigenvalue, so is $\overline{\lambda}$. From relation (2.4.8), we see that unless $\lambda = \overline{\lambda}$, $g_y = 0$ on these eigenspaces. The only remaining case, when $g_y \neq 0$, is when y^{-1} has eigenvalues ± 1, and the $-1-$eigenspace has dimension 1. It follows that $g_y = 0$ unless $y = s_\beta$ for a root β.

In conclusion,

$$\kappa(\omega) = c_0\omega + \sum_{\beta > 0} g_{s_\beta}(\omega) t_{s_\beta}, \quad \text{where } c_0^2 = 1.$$

Now we use that $\kappa^2 = \mathrm{Id}$, which immediately implies that

$$\omega = \kappa^2(\omega) = \omega + (1 + c_0) \sum_{\beta > 0} g_{s_\beta}(\omega) t_{s_\beta}. \tag{2.4.9}$$

When $c_0 = 1$, we necessarily have $g_{s_\beta} = 0$, and therefore $\kappa(\omega) = \omega$.

Suppose now $c_0 = -1$. We wish to prove that, in this case, $\kappa(\omega) = t_{w_0} \cdot \delta(\omega) \cdot t_{w_0}$. Specialize in (2.4.7) $y = s_\beta$, for $\beta \in R^+$. Then

$$g_{s_\beta}(\omega_2)(\omega_1, \beta^\vee)\beta = g_{s_\beta}(\omega_1)(\omega_2, \beta^\vee)\beta, \ \omega_1, \omega_2 \in V_{\mathbb{C}},$$

and therefore $g_{s_\beta}(\omega) = c_\beta(\omega, \beta^\vee)$, for some $c_\beta \in \mathbb{C}$. When $\beta = \alpha \in \Pi$, (2.4.6) with $c_0 = -1$, implies that $c_\alpha = k_\alpha$. If β is not a simple root, we can use (2.4.5) inductively to check that $c_\beta = k_\beta$.

□

Remark 2.4.4. There may be many more (up to inner conjugation) filtered automorphisms κ that preserve, but are not the identity on W. Every filtered automorphism κ is, in particular, an automorphism of $\mathbb{C}[W]$, so a first question would be to classify the group of outer automorphisms of $\mathbb{C}[W]$, a subgroup of which is $\mathsf{Out}(W)$, and this can be nontrivial (e.g., when $W = S_6$, $\mathsf{Out}(S_6) = \mathbb{Z}/2\mathbb{Z}$). But if we require that κ preserves the root reflections, then κ is obtained from one of the

two automorphisms in Proposition 2.4.3 by composition with an automorphism of $\mathbb{H}$ coming from the root system.

3. Invariant hermitian forms

In this section, we study invariant hermitian forms for $\mathbb{H}$-modules with respect to the two star operations $\bullet$ and $\star$ from section 2.

3.1. **Relation between the forms.** The relation between $\bullet$ and $\star$ from (2.3.3) reflects into a relation between the invariant hermitian forms, when they exist, on a given simple $\mathbb{H}$-module X. This relation is more easily expressed in terms of the extended Hecke algebra $\mathbb{H}'$-modules.

Lemma 3.1.1. *An $\mathbb{H}'$-module (π, X) admits a $\bullet$-invariant form $\langle\ ,\ \rangle_\bullet$ if and only if it admits a $\star$-invariant form $\langle\ ,\ \rangle_\star$. In this case, the forms are related by*

$$\langle v_1, v_2\rangle_\star = \langle v_1, \pi(t_{w_0}\delta)v_2\rangle_\bullet. \tag{3.1.1}$$

Proof. Suppose $\langle\ ,\ \rangle_\bullet$ exists on X. We verify that the $\star$-form from (3.1.1) is indeed invariant. For $h \in \mathbb{H}$, we use (2.3.3):

$$\begin{aligned}\langle \pi(h)v_1, v_2\rangle_\star &= \langle \pi(h)v_1, \pi(w_0\delta)v_2\rangle_\bullet = \langle v_1, \pi(h^\bullet t_{w_0}\delta)v_2\rangle_\bullet \\ &= \langle v_1, \pi(t_{w_0}\delta(h^*)\delta)v_2\rangle_\bullet = \langle v_1, \pi(t_{w_0}\delta)\pi(h^*)v_2\rangle_\bullet \\ &= \langle v_1, \pi(h^*)v_2\rangle_\star.\end{aligned} \tag{3.1.2}$$

The invariance under $\pi(\delta)$ is immediate since $\delta^* = \delta$ and δ commutes with t_{w_0}. □

Suppose (π, X) is a simple $\mathbb{H}$-module. Define the δ-twist of X to be (π^δ, X^δ), where $X^\delta = X$ as vector spaces and $\pi^\delta(h) = \pi(\delta(h))$. Suppose X admits a $\bullet$-invariant form. Then, as in Lemma 3.1.1, we get a $\star$-invariant pairing between X^δ and X via

$$\langle\ ,\ \rangle_\star : X^\delta \times X \to \mathbb{C},\ \langle u, v\rangle_\star = \langle u, \pi(t_{w_0})v\rangle_\bullet,\ u \in X^\delta, v \in X. \tag{3.1.3}$$

This implies that, under the hypotheses, X admits also a $\star$-invariant form if and only if $X \cong X^\delta$. Notice that if there exists an $\mathbb{H}$-isomorphism $\tau_X^\delta : (\pi^\delta, X^\delta) \to (\pi, X)$, then X can be lifted to a simple $\mathbb{H}'$-module, where δ acts by τ_X^δ. In section 5.3, we will see that when $\mathbb{H}$ is of geometric type, these isomorphisms admit a canonical normalization. Then the $\bullet$ and $\star$-forms on X are related by

$$\langle v_1, v_2\rangle_\star = \langle v_1, \pi(t_{w_0})\tau_X^\delta(v_2)\rangle_\bullet. \tag{3.1.4}$$

The above analysis has an important application to the relation between the signatures of the form on W-isotypic components of X. Since δ acts by conjugation by w_0 on W, it is clear that $X^\delta|_W \cong X|_W$. Suppose μ is an irreducible W-representation, and let $X(\mu)$ denote the μ-isotypic component of μ in X. In particular, $X^\delta(\mu) \cong X(\mu)$. The pairing (3.1.3) descends to a W-invariant pairing

$$\langle\ ,\ \rangle_\star^\mu : X^\delta(\mu) \times X(\mu) \to \mathbb{C},\ \langle u, v\rangle_\star^\mu = \langle u, \pi(t_{w_0})v\rangle_\bullet,\ u, v \in X_\mu. \tag{3.1.5}$$

If $X^\delta \cong X$ as $\mathbb{H}$-modules, the $\mathbb{H}$-isomorphism τ_X^δ induces isomorphisms $\tau_X^\delta(\mu) : X^\delta(\mu) \to X(\mu)$, so composing with $\tau_X^\delta(\mu)$ in (3.1.5), we find a W-invariant pairing on $X(\mu)$. We have proved:

Lemma 3.1.2. *If (π, X) is a simple $\mathbb{H}$-module admitting a $\bullet$-invariant form then*

(1) *X admits also a $\star$-invariant form if and only if $X^\delta \cong X$, and in this case,*

(2) *the signatures of the two forms on a W-isotypic space $X(\mu)$, $\mu \in \widehat{W}$, are related by $\pi(t_{w_0}) \circ \tau_X^\delta(\mu)$, i.e., by the action of $t_{w_0}\delta$.*

3.2. **The elements R_w.** Let $\mathcal{O}(V_\mathbb{C})$ denote the ring of rational functions on $V_\mathbb{C}$, and consider the completion of $\mathbb{H}$

$$\hat{\mathbb{H}} = \mathbb{H} \otimes_{S(V_\mathbb{C})} \mathcal{O}(V_\mathbb{C}). \tag{3.2.1}$$

Following [Lu1, BM3], we define for every $\alpha \in \Pi$ the element of $\hat{\mathbb{H}}$

$$R_{s_\alpha} = t_{s_\alpha} \frac{\alpha}{k_\alpha - \alpha} - \frac{k_\alpha}{k_\alpha - \alpha}. \tag{3.2.2}$$

The reason for the normalization $k_\alpha - \alpha$ is so that for $k_\alpha > 0$, the intertwining operator has no poles when evaluating on a negative weight like $w_0\nu$. In that case $k_\alpha - \alpha(w_0\nu) > k_\alpha > 0$.

If $x \in W$ has a reduced expression $x = s_{\alpha_1} \cdot s_{\alpha_2} \cdot \dots \cdot s_{\alpha_k}$, set

$$R_x = R_{s_{\alpha_1}} R_{s_{\alpha_2}} \dots R_{s_{\alpha_k}}.$$

Notice that

$$R_x = \sum_{y \le x} t_y a_y,\ a_y \in \mathcal{O}(V_\mathbb{C}) \text{ and } a_x = \prod_{x^{-1}\alpha < 0} \frac{\alpha}{k_\alpha - \alpha}.$$

Lemma 3.2.1.

(1) *The element R_w, $w \in W$, does not depend on the choice of reduced expression for w.*

(2) *For every $a \in \mathcal{O}(V_\mathbb{C})$, $w \in W$*

$$a \cdot R_w = R_w \cdot w^{-1}(a). \tag{3.2.3}$$

(3) *For every $w \in W$,*

$$t_w \cdot R_{w_0} = (-1)^{\ell(w)} R_{w_0} \cdot \delta(t_w). \tag{3.2.4}$$

(4) $R_x R_y = R_{xy}$, $x, y \in W$.

Proof. Claims (1) and (2) are in [BM3, Lemma 1.6]. For (3), it is sufficient to verify that when $\alpha \in \Pi$, $t_{s_\alpha} \cdot R_{w_0} = -R_{w_0} t_{s_\beta}$, where $\beta = -w_0(\alpha)$. Write $w_0 = w s_\alpha = s_\beta w$. It follows that $R_{w_0} = R_w R_{s_\alpha} = R_{s_\beta} R_w$, and therefore $R_{w_0} R_{s_\alpha} = R_{s_\beta} R_{w_0}$. Then $(t_{s_\beta}\beta - k_\beta) R_{w_0} = R_{w_0}(t_{s_\alpha}\alpha - k_\alpha)$, and since $k_\alpha = k_\beta$, $R_{w_0} t_{s_\alpha}\alpha = t_{s_\beta}\beta R_{w_0} = t_{s_\beta} R_{w_0} w_0(\beta) = -t_{s_\beta} R_{w_0}\alpha$. Claim (4) follows immediately from $R_{s_\alpha}^2 = (t_{s_\alpha}\alpha - k_\alpha)^2 \frac{1}{k_\alpha^2 - \alpha^2} = 1$. □

Lemma 3.2.2. *The elements R_x satisfy*

(1) $R_x^\bullet = (-1)^{\ell(x)} R_{x^{-1}} \prod_{x^{-1}\alpha < 0} \frac{k_\alpha - \alpha}{k_\alpha + \alpha}$;

(2) $R_x^\star = (-1)^{\ell(x)} t_{w_0} R_{\delta(x)^{-1}} \left(\prod_{\delta(x)^{-1}\alpha < 0} \frac{k_\alpha - \alpha}{k_\alpha + \alpha} \right) t_{w_0}$.

Proof. Claim (2) follows from (1) by (2.3.3). For (1), we need to compute $R_{s_\alpha}^\bullet$. We have $R_{s_\alpha}^\bullet = [(t_{s_\alpha}\alpha - k_\alpha)(k_\alpha - \alpha)^{-1}]^\bullet = (k_\alpha - \alpha)^{-1}(\alpha t_{s_\alpha} - k_\alpha) = -(k_\alpha - \alpha)^{-1} R_{s_\alpha}(k_\alpha - \alpha) = -R_{s_\alpha} \frac{k_\alpha - \alpha}{k_\alpha + \alpha}$.

□

3.3. **Minimal principal series.** We wish to define invariant hermitian forms on irreducible $\mathbb{H}$-modules. It is instructive to consider first the case of minimal principal series. Every element $h \in \mathbb{H}$ can be written uniquely as $h = \sum_{w\in W} t_w a_w$, $a_w \in S(V_\mathbb{C})$. Define the $\mathbb{C}$-linear map

$$\epsilon_A : \mathbb{H} \to S(V_\mathbb{C}), \quad \epsilon_A(h) = a_1.$$

If $\nu \in V_\mathbb{C}^\vee$, let $\mathbb{C}_\nu$ denote the character of $S(V_\mathbb{C})$ given by evaluation at ν. For $a \in \mathbb{H}$, denote by $a(\nu)$ the evaluation of a at ν. The minimal principal series with parameter ν is $X(\nu) = \mathbb{H} \otimes_{S(V_\mathbb{C})} \mathbb{C}_\nu$.

If κ is any conjugate linear anti-involution of $\mathbb{H}$, and L, R are arbitrary elements of $\mathbb{H}$, and $\nu' \in V_\mathbb{C}^\vee$, the assignment

$$\langle h_1, h_2\rangle = \epsilon_A(L\kappa(h_2)h_1R)(\nu'), \quad h_1, h_2 \in \mathbb{H}, \tag{3.3.1}$$

defines a κ-invariant (not necessarily hermitian) pairing on $\mathbb{H}$ viewed as an $\mathbb{H}$-module under left multiplication. For such a form to descend to a κ-invariant hermitian form on $X(\nu)$, it must satisfy:

(H1) $\langle h_1 a, h_2\rangle = \underline{a(\nu)}\langle h_1, h_2\rangle$, for all $a \in S(V_\mathbb{C})$;
(H2) $\langle h_1, h_2 a\rangle = \overline{a(\nu)}\langle h_1, h_2\rangle$, for all $a \in S(V_\mathbb{C})$;
(H3) $\langle h_1, h_2\rangle = \overline{\langle h_2, h_1\rangle}$.

Of course, (H1) and (H3) imply (H2), but in practice it will be convenient for us to check (1) and (2) first, which will then reduce the verification of (3) on the basis $\{t_w \in W\}$ of $X(\nu)$.

We show this for $\kappa = \bullet$ and the pairing

$$\langle h_1, h_2\rangle_\bullet := \epsilon_A(t_{w_0} h_2^\bullet h_1 R_{w_0})(w_0\nu). \tag{3.3.2}$$

Let

$$\mathcal{R}_\alpha := t_\alpha \frac{\alpha}{k_\alpha + \alpha} - \frac{k_\alpha}{k_\alpha + \alpha},$$

and for $x = s_{\alpha_1} \dots s_{\alpha_k}$, define $\mathcal{R}_x = \prod \mathcal{R}_{\alpha_i}$. The $\mathcal{R}_x$ have the same properties as the R_x, except

$$\mathcal{R}_x^\bullet = (-1)^{\ell(x)} \mathcal{R}_{x^{-1}} \prod_{x^{-1}\alpha<0} \frac{k_\alpha + \alpha}{k_\alpha - \alpha}. \tag{3.3.3}$$

Let

$$V_{\mathsf{reg}}^\vee := \{\nu \in V_\mathbb{C}^\vee \ : \ (\alpha, \nu) \neq 0 \text{ for any } \alpha \in R^+\}.$$

For $\nu \in V_{\mathsf{reg}}^\vee$, a basis of $X(\nu)$ is given by

$$\{\mathcal{R}_x \otimes \mathbb{1}_\nu\}_{x\in W}. \tag{3.3.4}$$

Notice that $\mathcal{R}_x$ is not in $\mathbb{H}$, but in $\hat{\mathbb{H}}$. However it makes sense to express $\mathcal{R}_x = \sum t_y a_y^x$ with $a_y^x \in \mathcal{O}(V_\mathbb{C})$, and then evaluate at ν. The fact that $\nu \in V_{\mathsf{reg}}^\vee$ allows one to solve for the $t_x \otimes \mathbb{1}_\nu$ in terms of the $\mathcal{R}_x \otimes \mathbb{1}_\nu$; so indeed (3.3.4) is a basis. (Note that we have assumed that $k_\alpha > 0$.)

Lemma 3.3.1. *The vector $\mathcal{R}_x \otimes \mathbb{1}_\nu$ is an $\mathbb{A}$-weight vector of $X(\nu)$ with weight $x\nu$.*

Proof. Since $a \cdot \mathcal{R}_x = \mathcal{R}_x \cdot x^{-1}(a)$, $a \in S(V_\mathbb{C})$, it follows that in $X(\nu)$, $a \cdot (\mathcal{R}_x \otimes \mathbb{1}_\nu) = a(x\nu)(\mathcal{R}_x \otimes \mathbb{1}_\nu)$. □

We show that (H1)-(H3) hold for (3.3.4) and $\nu \in V^\vee_{\mathsf{reg}}$. Since the relations (and the change of basis matrices to the t_x) are rational in ν, and $V^\vee_{\mathsf{reg}}$ contains an open set in $V^\vee_{\mathbb{C}}$, they will hold in general.

The first identity holds by (3.2.3):

$$\langle h_1 a, h_2\rangle_\bullet = \langle h_1, h_2\rangle_\bullet a(\nu).$$

For the second identity,

$$\langle \mathcal{R}_x, \mathcal{R}_y a\rangle_\bullet = \langle \mathcal{R}_x, \mathcal{R}_y\rangle_\bullet (w_0 x^{-1} y)(a^\bullet)(w_0\nu) = \langle \mathcal{R}_x, \mathcal{R}_y\rangle_\bullet (x^{-1}y)(a^\bullet)(\nu).$$

Suppose $x = y$. Then this formula implies (H2) (with $h_1 = h_2 = \mathcal{R}_x$) if and only if $a^\bullet(\nu) = \overline{a(\nu)}$ which is equivalent to $\nu = \overline{\nu}$, i.e., $\nu \in V^\vee$.

Suppose $x \neq y$. We show that each of the two sides of (H2) are zero because $\epsilon_A(t_{w_0}\mathcal{R}_z R_{w_0}) = 0$ unless $z = 1$:

$$\epsilon_A\left(t_{w_0}(\mathcal{R}_y a)^\bullet \mathcal{R}_x R_{w_0}\right) = \epsilon_A\left(t_{w_0} a (-1)^{\ell(y)} \mathcal{R}_{y^{-1}} \prod_{y^{-1}\alpha<0} \frac{k_\alpha + \alpha}{k_\alpha - \alpha} \mathcal{R}_x R_{w_0}\right) =$$

$$= \epsilon_A\left(t_{w_0}\mathcal{R}_{y^{-1}x} R_{w_0}\right)\cdot (-1)^{\ell(y)} (w_0 x^{-1} y)(a) \prod_{y^{-1}\alpha<0} \frac{k_\alpha + w_0 x^{-1}\alpha}{k_\alpha - w_0 x^{-1}\alpha} = 0, \text{ and}$$

$$\epsilon_A\left(t_{w_0}\mathcal{R}_y^\bullet \mathcal{R}_x R_{w_0}\right) a = \epsilon_A\left(t_{w_0}\mathcal{R}_{y^{-1}x} R_{w_0}\right)\cdot (-1)^{\ell(x)} \prod_{y^{-1}\alpha<0} \frac{k_\alpha + w_0 x^{-1}\alpha}{k_\alpha - w_0 x^{-1}\alpha} a = 0.$$

So (H2) is verified.

We also record the formula

$$\begin{aligned}\langle \mathcal{R}_x \otimes \mathbb{1}_\nu, \mathcal{R}_x \otimes \mathbb{1}_\nu\rangle_\bullet &= (-1)^{\ell(x)} \left(\prod_{\alpha>0} \frac{\alpha}{k_\alpha - \alpha}\cdot \prod_{x^{-1}\alpha<0} \frac{k_\alpha - \delta(x^{-1}\alpha)}{k_\alpha + \delta(x^{-1}\alpha)}\right)(w_0\nu)\\ &= (-1)^{|R|} \prod_{\alpha>0} \frac{\langle\alpha,\nu\rangle}{\langle\alpha,\nu\rangle + k_\alpha} \prod_{x\alpha<0} \frac{\langle\alpha,\nu\rangle - k_\alpha}{\langle\alpha,\nu\rangle + k_\alpha}.\end{aligned} \tag{3.3.5}$$

The equivalence of the two formulas can be easily seen by the substitution $x^{-1}\alpha \mapsto \alpha$ in the second product. Notice that the factor $(-1)^{|R|}\prod_{\alpha>0}\frac{\langle\alpha,\nu\rangle}{\langle\alpha,\nu\rangle+k_\alpha}$ is independent of x, so we may divide the form uniformly by it. The resulting normalized hermitian form has the property that $\langle \mathcal{R}_1 \otimes \mathbb{1}_\nu, \mathcal{R}_1 \otimes \mathbb{1}_\nu\rangle = 1$.

When ν is dominant, $k_\alpha + \langle\alpha,\nu\rangle > 0$, so the denominator does not vanish, and it is always positive (we have assumed $k_\alpha > 0$).

The arguments also imply that $\langle h_2, h_1\rangle_\bullet = \overline{\langle h_1, h_2\rangle_\bullet}$ for $h_1, h_2 \in \{R_x \otimes \mathbb{1}_\nu\}_{x\in W}$, so also in general. In conclusion, we have proved the following result.

Proposition 3.3.2. *The form*

$$\langle h_1, h_2\rangle_\bullet := \epsilon_A(t_{w_0} h_2^\bullet h_1 R_{w_0})(w_0\nu)$$

defines a $\bullet$-invariant hermitian form on $X(\nu)$ if and only if $\overline{\nu} = \nu$, i.e., $\nu \in V^\vee$.

The case of $\star$ follows by formal manipulations. Set

$$\langle h_1, h_2\rangle_\star = \epsilon_A\left(h_2^\star h_1 R_{w_0}\right)(w_0\nu). \tag{3.3.6}$$

The relation between the forms is

$$\begin{aligned}\langle h_1, h_2\rangle_\star &= \epsilon_A\left(h_2^\star h_1 R_{w_0}\right)(w_0\nu) = \epsilon_A\left(t_{w_0}\delta(h_2)^\bullet t_{w_0} h_1 R_{w_0}\right)(w_0\nu)\\ &= \langle t_{w_0}h_1, \delta(h_2)\rangle_\bullet;\end{aligned} \tag{3.3.7}$$

compare with (3.1.4).

We also note the following formulas for the signatures.

Proposition 3.3.3. *Write $R_{w_0} = \sum_{w\in W} t_w a_w$.*

1. *The signature of $\langle\ ,\ \rangle_\bullet$ is given by the signature of the matrix $\left\{a_{x^{-1}yw_0}\right\}_{x,y\in W}$.*
2. *The signature of $\langle\ ,\ \rangle_\star$ is given by the signature of the matrix $\left\{a_{x^{-1}y}\right\}_{x,y\in W}$.*

Proof. Straightforward. □

Corollary 3.3.4. *For every $w \in W$,*

$$\epsilon_A\left(t_w R_{w_0}\right) = \epsilon_A\left(\delta(t_{w^{-1}})R_{w_0}\right).$$

Proof. The left hand side is

$$\epsilon_A\left(t_{w_0}t_{w_0}t_w R_{w_0}\right),$$

while the right hand side is

$$\epsilon_A\left(t_{w_0}t_{w^{-1}}t_{w_0}R_{w_0}\right)$$

Evaluating at $w_0\nu$, the left hand side is $\langle t_{w_0}, t_w\rangle_{\bullet,\nu}$ while the right hand side is $\langle t_w, t_{w_0}\rangle_{\bullet,\nu}$. The fact that the two are equal follows from the fact that $\langle\ ,\ \rangle_\bullet$ is symmetric for ν real. □

As a consequence of the relation (3.3.7) between $\bullet$ and $\star$ forms and Proposition 3.3.2, we have the following corollary.

Corollary 3.3.5. *The pairing*

$$\langle h_1, h_2\rangle_\star = \epsilon_A\left(h_2^\star h_1 R_{w_0}\right)(w_0\nu)$$

defines a $\star$-invariant hermitian form on $X(\nu)$ if and only if $w_0\nu = -\overline{\nu}$.

3.4. **Parabolic subalgebras.** Let Π_M be a subset of simple roots of Π and R_M^+ the positive roots spanned by Π_M. Denote by W_M the parabolic subgroup of W generated by $\{s_\alpha : \alpha \in \Pi_M\}$ and by $w_{0,M}$ the long Weyl group element in W_M.

Let $\mathbb{H}_M$ be the subalgebra of $\mathbb{H}$ generated by $\{t_w : w \in W_M\}$ and $S(V_\mathbb{C})$. The star operations $\star_M$ and $\bullet_M$ as in Definition 2.3.2 for $\mathbb{H}_M$ are:

$$\begin{aligned} &t_w^{\star_M} = t_{w^{-1}},\ \ w\in W_M, \qquad \omega^{\star_M} = -t_{w_{0,M}}\cdot\overline{w_{0,M}(\omega)}\cdot t_{w_{0,M}},\ \omega\in V_\mathbb{C},\\ &t_w^{\bullet_M} = t_{w^{-1}},\ w\in W_M, \qquad\qquad\qquad\qquad \omega^{\bullet_M} = \overline{\omega},\ \omega\in V_\mathbb{C}.\end{aligned} \tag{3.4.1}$$

As before, from Definition 2.1.1, every element of $\mathbb{H}$ can be written uniquely as $h = \sum_{w\in W} t_w a_w$, where $a_w \in S(V_\mathbb{C})$. Denote

$$\mathcal{J}_M = \text{ the set of coset representatives of minimal length in } W/W(M); \tag{3.4.2}$$

recall that in every coset $xW(M)$ there exists a unique element of minimal length. Then, more generally, every $h \in \mathbb{H}$ can be written uniquely as

$$h = \sum_{w\in\mathcal{J}_M} t_w m_w, \quad m_w \in \mathbb{H}_M.$$

Define the $\mathbb{C}$-linear map

$$\epsilon_M : \mathbb{H} \to \mathbb{H}_M, \quad \epsilon_M(h) = m_1. \tag{3.4.3}$$

In particular, $\epsilon_M(hm) = \epsilon_M(h)m$, for all $m \in \mathbb{H}_M$. It is also easy to see that

$$\delta(\epsilon_M(h)) = \epsilon_{\delta(M)}(\delta(h)),\ h \in \mathbb{H}, \tag{3.4.4}$$

where $\Pi_{\delta(M)} = \delta(\Pi_M)$, and δ is the automorphism from Lemma 2.2.1. We need the relation between $\star$ and $\star_M$.

Proposition 3.4.1 ([BM3, Proposition 1.4])**.** *For every $h \in \mathbb{H}$, $\epsilon_M(h^\star) = \epsilon_M(h)^{\star_M}$.*

Corollary 3.4.2. *For every $h \in \mathbb{H}$, $\epsilon_{\delta(M)}(t_{w_0}h^\bullet t_{w_0}) = \delta(\epsilon_M(h))^{\star_{\delta(M)}}$.*

Proof. Since $t_{w_0}h^\bullet t_{w_0} = \delta(h)^\star$, the claim is immediate from Proposition 3.4.1 and (3.4.4). □

3.5. **Induced modules.** Let $\Pi_M \subset \Pi$ be given, and consider the subalgebra $\mathbb{H}_M$ of $\mathbb{H}$. If (σ, U_σ) is an $\mathbb{H}_M$-module, consider the induced module

$$X(M, \sigma) = \mathbb{H} \otimes_{\mathbb{H}_M} U_\sigma, \tag{3.5.1}$$

where $\mathbb{H}$ acts by left multiplication. The goal is to construct invariant hermitian forms on $X(M, \sigma)$ provided that σ admits such a form for $\mathbb{H}_M$. For this, we need to describe the $\mathbb{H}$-module structure π_σ on $X(M, \sigma)$ more explicitly.

A basis for $X(M, \sigma)$ is

$$\{t_x \otimes v_i\}, \quad x \in \mathcal{J}_M, \quad v_i \in \mathcal{B}(U_\sigma),$$

where $\mathcal{B}(U_\sigma)$ is a basis of U_σ.

Every $z \in W$ can be written uniquely

$$z = c(z) \cdot m(z), \tag{3.5.2}$$

where $c(z)$ is the element of $\mathcal{J}_M$ in the coset $zW(M)$ and $m(z) \in W(M)$.

Lemma 3.5.1. *The action π_σ on $X(M, \sigma)$ is given by*

$$\begin{aligned} &\pi(t_z)(t_x \otimes v) = t_{c(zx)} \otimes \sigma(m(zx))v; \\ &\pi(\omega)(t_x \otimes v) = t_x \otimes \sigma(x^{-1}(\omega))v + \sum_{\substack{\beta>0 \\ x^{-1}\beta<0}} (\omega, \beta^\vee)\ t_{c(s_\beta x)} \otimes \sigma(m(s_\beta x))v, \end{aligned} \tag{3.5.3}$$

for every $z \in W$ and $\omega \in V_{\mathbb{C}}$.

Proof. For $z \in W$,

$$\pi(t_z)(t_x \otimes v) = t_{zx} \otimes v = t_{c(zx)} \otimes \sigma(m(zx))v.$$

For $\omega \in V_{\mathbb{C}}$,

$$\pi(\omega)(t_x \otimes v) = \omega t_x \otimes v.$$

The claim follows from Lemma 2.3.5, i.e.,

$$\omega t_x = t_x x^{-1}(\omega) + \sum_{\substack{\beta>0 \\ x^{-1}\beta<0}} (\omega, \beta^\vee) t_{s_\beta x}.$$

□

3.6. **Action on the hermitian dual to an induced modules.** Let (σ, U_σ) be a module for $\mathbb{H}_M$ as in section 3.5. Let $(\sigma^\bullet, U_\sigma^h)$ be the hermitian dual of (σ, U_σ) and $(\pi_\sigma^\bullet, X(M,\sigma)^h)$, the hermitian dual of $(\pi_\sigma, X(M,\sigma))$ with respect to the star operation $\bullet$. A basis for the hermitian dual $X(M,\sigma)^h$ of $X(M,\sigma)$ is

$$\{t_x^h \otimes v_i^h\}, \text{ where } x \in \mathcal{J}_M \text{ and } v_i^h \in U_\sigma^h \text{ dual to the basis } \mathcal{B}(U_\sigma) = \{v_i\}. \quad (3.6.1)$$

We calculate the action $\pi_\sigma^\bullet$ of $\mathbb{H}$ on $X(M,\sigma)^h$. For $z \in W$,

$$\pi^\bullet(t_z)(t_x^h \otimes v_i^h)(t_y \otimes v_j) = (t_x^h \otimes v_i^h)(t_{z^{-1}x} \otimes v_j). \quad (3.6.2)$$

Then (3.6.2) is nonzero if and only if $c(z^{-1}y) = x$, so

$$z^{-1}y = xm(z^{-1}y), \text{ or equivalently, } zx = ym(z^{-1}y)^{-1}.$$

We conclude that $m(zx) = m(z^{-1}y)^{-1}$, and so

$$\pi^\bullet(t_z)(t_x^h \otimes v_i^h) = t_{c(zx)}^h \otimes \sigma^\bullet(m(zx))v_i^h. \quad (3.6.3)$$

For $\omega \in V_{\mathbb{C}}$,

$$\pi^\bullet(\omega)(t_x^h \otimes v_i^h)(t_y \otimes v_j) = (t_x^h \otimes v_i^h)(\omega t_y \otimes v_j).$$

Using Lemma 2.3.5

$$\omega t_y = t_y y^{-1}(\omega) - \sum_{\substack{\gamma>0 \\ y\gamma<0}} (\omega, y\gamma^\vee) t_{ys_\gamma} = t_y y^{-1}(\omega) + \sum_{\substack{\beta>0 \\ y^{-1}\beta<0}} (\omega, \beta^\vee) t_{s_\beta y},$$

we find that the expression is zero unless either $x = y$, or $c(s_\beta y) = x$. In this latter case,

$$s_\beta y = x \cdot m(s_\beta y), \text{ equivalently } s_\beta x = ym(s_\beta x), \quad \text{so} \quad m(s_\beta x) = m(s_\beta y)^{-1}.$$

The conclusion is

$$\pi^\bullet(\omega)(t_x^h \otimes v_i^h) = t_x^h \otimes \sigma^\bullet(x^{-1}(\omega))v_i^h - \sum_{\substack{\beta>0 \\ c(s_\beta x)^{-1}\beta<0}} (\omega, \beta^\vee)\ t_{c(s_\beta x)} \otimes \sigma^\bullet(m(s_\beta x))v_i^h.$$

Notice that since $y \in \mathcal{J}_M$, if $y^{-1}\beta < 0$, then in fact $y^{-1}\beta \in R^- \setminus R_M^-$. We show that

$$c(s_\beta x)^{-1}\beta < 0 \text{ if and only if } x^{-1}\beta \in R \setminus R_M.$$

We have $s_\beta x = ym$ for some $m \in W(M)$. Then $y^{-1} = mx^{-1}s_\beta$, and $x^{-1} = m^{-1}y^{-1}s_\beta$.

If $x^{-1}\beta \in R \setminus R_M$,

$$y^{-1}\beta = m^{-1}x^{-1}(-\beta) \in R^- \setminus R_M^-.$$

So $y^{-1}\beta < 0$.

If $y^{-1}\beta < 0$ then as observed earlier, $y^{-1}\beta \in R^- \setminus R_M^-$, so

$$x^{-1}\beta = m^{-1}y^{-1}(-\beta) \in m^{-1}(R^+ \setminus R_M^+) = R^+ \setminus R_M^+.$$

In conclusion, we have proved the following formulas for the action $\pi_\sigma^\bullet$.

Lemma 3.6.1. *The $\mathbb{H}$-module action on the $\bullet$-hermitian dual module $(\pi_\sigma^\bullet, X(M,\sigma)^h)$ is given by:*

$$\pi^\bullet(t_z)(t_x^h \otimes v_i^h) = t^h_{c(zx)} \otimes \sigma^\bullet(m(zx))v_i^h,$$
$$\pi^\bullet(\omega)(t_x^h \otimes v_i^h) = t_x^h \otimes \sigma^\bullet(x^{-1}(\omega))v_i^h - \sum_{\substack{\beta>0 \\ x^{-1}\beta\in R^+\setminus R_M^+}} (\omega,\beta^\vee)\ t_{c(s_\beta x)} \otimes \sigma^\bullet(m(s_\beta x))v_i^h. \tag{3.6.4}$$

3.7. **Hermitian dual of an induced module.** Retain the notation from the previous sections. In particular, write w_0 for the long Weyl group element of W, $w_{0,M}, w_{0,\delta(M)}$ for the corresponding long elements in the Levi components, and set

$$w_M^0 := w_0 w_{0,\delta(M)} = w_{0,M} w_0.$$

This element is minimal in the cosets $w_0 W_{\delta(M)}$, and $W_M w_0$.

Let (σ, U_σ) be an $\mathbb{H}_M-$module. Recall that $(\sigma^\bullet, U_\sigma^h)$ is the module on the hermitian dual with respect to the $\bullet$ action.

Lemma 3.7.1. *The map ϕ given by*

$$\phi(t_m) := t_{(w_M^0)^{-1} m w_M^0}, \quad m \in W_M,$$
$$\phi(\omega) := (w_M^0)^{-1}(\omega), \quad \omega \in V_{\mathbb{C}},$$

is an isomorphism between $\mathbb{H}_M$ and $\mathbb{H}_{\delta(M)}$ and it interchanges $\bullet_M$ with $\bullet_{\delta(M)}$.

Proof. Straightforward. □

Definition 3.7.2. In light of Lemma 3.7.1, to each $\mathbb{H}_M$-module (σ, U_σ), we associate the $\mathbb{H}_{\delta(M)}-$module $(a\sigma, U_{a\sigma})$ given by

$$U_{a\sigma} = U_\sigma, \text{ and } (a\sigma)(m') := \sigma(w_M^0 m' (w_M^0)^{-1}), m' \in \mathbb{H}_{\delta(M)}. \tag{3.7.1}$$

Proposition 3.7.3. *The element $x \in W$ is minimal in xW_M if and only if xw_M^0 is minimal in $xw_M^0 W_{\delta(M)}$.*

Proof. We observe that $w_0(R^\pm_{\delta(M)}) = R^\mp_M$. Then

$$w_0 w_{0,\delta(M)}(R^+_{\delta(M)}) = w_0(R^-_{\delta(M)}) = R^+_M.$$

The claim follows,

$$x(R_M^+) \subset R^+ \text{ if and only if } xw_M^0(R_{\delta(M)}) \subset R^+.$$

□

Corollary 3.7.4. *In the notation of (3.5.2):*

$$c_{\delta(M)}(xw_M^0) = c_M(x)w_M^0, \qquad m_{\delta(M)}(xw_M^0) = (w_M^0)^{-1} m_M(x) w_M^0,$$

for every $x \in W$.

Theorem 3.7.5. *The map*

$$\Phi(t_x^h \otimes v^h) := t_{x w_M^0} \otimes a v^h$$

is an $\mathbb{H}-$equivariant isomorphism between $\left(\pi_\sigma^\bullet, X(M,\sigma)^h\right)$ and $\left(\pi_\sigma, X(\delta(M), a\sigma^h)\right)$ where the action on σ^h is given by $\bullet_{\delta(M)}$.

Proof. Using Lemma 3.6.1, we have

$$\Phi[\pi^\bullet(t_z)(t_x^h\otimes v^h)]=\Phi[t^h_{c_M(zx)}\otimes\sigma^\bullet(m(zx))v^h]=t_{c_M(zx)w^0}\otimes a[\sigma^\bullet(m_M(zx))v^h],$$
$$\pi(t_z)\Phi[t_x^h\otimes v^h]=\pi(t_z)[t_{xw^0}\otimes av^h]=t_{c_{\delta(M)}(zxw^0)}\otimes(a\sigma)^\bullet[(m_{\delta(M)}(zxw^0))av^h].$$

Next

$$\begin{aligned}\Phi[\pi^\bullet(\omega)(t_x^h\otimes v^h)]&=t_{xw_M^0}\otimes\ a[\sigma^\bullet(x^{-1}\omega)v^h]-\\&-\sum_{\substack{\beta>0\\ x^{-1}\beta\in R^+\setminus R_M^+}}(\omega,\beta^\vee)t_{c_M(s_\beta x)w^0}\otimes a[\sigma^\bullet(m_M(s_\beta x))v^h]\\ \pi(\omega)\Phi[t_x^h\otimes v^h]&=\pi(\omega)[t_{xw^0}\otimes av^h]=t_{xw_M^0}\otimes(a\sigma)^\bullet((xw_M^0)^{-1}\omega)av^h+\\&+\sum_{\substack{\gamma>0\\ (xw^0)^{-1}\gamma\in R^+\setminus R^+_{\delta(M)}}}(\omega,\gamma^\vee)\ t_{c_{\delta(M)}(s_\beta xw_M^0)}\otimes(a\sigma)^\bullet(m_{\delta(M)}(s_\beta xw_M^0))av^h.\end{aligned}$$

The corresponding expressions are equal because of Corollary 3.7.4, and the fact that $w^0(R^+\setminus R_M^+)=R^-\setminus R^-_{\delta(M)}$. □

Example 3.7.6. A particular case of Theorem 3.7.5 is that of minimal principal series. The hermitian dual $(\pi^\bullet, X(\nu)^h)$ of a minimal principal series module identifies with $(\pi, X(w_0\overline{\nu}))$ via

$$\Phi(t_x^h\otimes\mathbb{1}_\nu)=t_{xw_0}\otimes\mathbb{1}_{w_0\overline{\nu}}.$$

In particular, this means that $X(\nu)$ admits an invariant $\bullet$ form if and only if $w_0\overline{\nu}$ is W-conjugate to ν, equivalently if $\overline{\nu}$ is W-conjugate to ν. Thus, for example, if w_0 is not central in W, $X(\nu)$ does not admit a $\bullet$-form for generic purely imaginary values of ν.

3.8. **Second form of Frobenius reciprocity.** As an application of Theorem 3.7.5, we obtain the following lemma, which is the $\mathbb{H}$-analogue of the second form of Frobenius reciprocity.

Lemma 3.8.1. *If $\mathbb{H}_M$ is a parabolic subalgebra of $\mathbb{H}$, V an $\mathbb{H}$-module and U an $\mathbb{H}_M$-module, then*

$$\mathrm{Hom}_{\mathbb{H}_M}[V|_{\mathbb{H}_M},U]=\mathrm{Hom}_{\mathbb{H}}[V,\mathbb{H}\otimes_{\mathbb{H}_{\delta(M)}}a(U)].\tag{3.8.1}$$

Proof. Theorem 3.7.5 computed the hermitian dual of a parabolically induced module. The same exact statement and proof hold of course for contragredient modules. We use here the same notation $V^\bullet$ to denote the contragredient (rather than the hermitian dual) with respect to the involution $\bullet$. We will also use twice the tautological isomorphism

$$\mathrm{Hom}[A,B^\bullet]=\mathrm{Hom}[B,A^\bullet].\tag{3.8.2}$$

We have:

$$\begin{aligned}\mathrm{Hom}_{\mathbb{H}_M}[V,U]&=\mathrm{Hom}_{\mathbb{H}_M}[V,(U^\bullet)^\bullet]=\mathrm{Hom}_{\mathbb{H}_M}[U^\bullet,V^\bullet]\\&=\mathrm{Hom}_{\mathbb{H}}[\mathbb{H}\otimes_{\mathbb{H}_M}U^\bullet,V^\bullet]\quad\text{(by first Frobenius reciprocity)}\\&=\mathrm{Hom}_{\mathbb{H}}[V,(\mathbb{H}\otimes_{\mathbb{H}_M}U^\bullet)^\bullet]\\&=\mathrm{Hom}_{\mathbb{H}}[V,\mathbb{H}\otimes_{\mathbb{H}_{\delta(M)}}a(U)]\qquad\text{(by Theorem 3.7.5).}\end{aligned}\tag{3.8.3}$$

□

3.9. **Sesquilinear Form.** A $\bullet$-invariant sesquilinear form on $X(M,\sigma)$ is equivalent to defining an $\mathbb{H}$-equivariant map

$$\mathcal{I} : (\pi, X(M,\sigma)) \longrightarrow (\pi^\bullet, X(M,\sigma)^h). \tag{3.9.1}$$

We call $\mathcal{I}$ hermitian if $\mathcal{I}^h = \mathcal{I}$ or equivalently $\mathcal{I}(v)(w) = \overline{\mathcal{I}(w)(v)}$, for all $v, w \in X(M,\sigma)$. Recall $M, \delta(M)$, and $w_0 = w_M^0 w_{0,\delta(M)} = w_{0,M} w_M^0$ with w_M^0 minimal in $w_0 W_{\widetilde{M}}$. To simplify notation, write $\widetilde{M} = \delta(M)$, and

$$w^0 = w_M^0,\ R^0 := R_{w_M^0}. \tag{3.9.2}$$

Furthermore,

$$\operatorname{Ad} w_M^0 : W_{\widetilde{M}} \longrightarrow W_M, \qquad \sigma(\widetilde{m}) = \sigma(w_M^0 \widetilde{m} (w^0)^{-1}).$$

If $x = c_M(x) m_M(x)$, then $x w_M^0 = c_M(x) w_M^0 (w_M^0)^{-1} m_M(x w_M^0)$, so

$$c_M(x) w_M^0 = c_{\widetilde{M}}(x w_M^0), \quad (w_M^0)^{-1} m_M(x) w_M^0 = m_{\widetilde{M}}(x w_M^0). \tag{3.9.3}$$

Assume that there is an $\mathbb{H}_M$-equivariant isomorphim

$$\iota : (\sigma, U_\sigma) \longrightarrow (\sigma^\bullet, U_\sigma^h)$$

defining a $\bullet$-invariant hermitian form on (σ, U_σ). The same map gives an isomorphism $\iota_a : (a\sigma, U_\sigma) \longrightarrow (a\sigma^\bullet, U_\sigma^h)$.

Write $R^0 = \sum t_z m_{\widetilde{z}}^0$ with $\widetilde{z}$ minimal in $\widetilde{z} W(\widetilde{M})$ and $m_{\widetilde{z}}^0 \in \mathbb{H}_{\widetilde{M}}$.

Define $\mathcal{I}$ to be the composition of the maps

$$X(M,\sigma) \xrightarrow{L_{R^0}} X(\widetilde{M}, a\sigma) \xrightarrow{\operatorname{Ind} \iota_a} X(\widetilde{M}, a\sigma^h) \xrightarrow{\Phi^{-1}} X(M,\sigma)^h.$$

where:

(i)

$$L_{R^0} : t_x \otimes v \mapsto t_x R^0 \otimes v = \sum t_{c_{\widetilde{M}}(x\widetilde{z})} \otimes a\sigma(m_{\widetilde{M}}(x\widetilde{z}) m_{\widetilde{z}}^0) v$$

(ii) $\operatorname{Ind} \iota_a$ maps it to

$$\operatorname{Ind} \iota_a \circ L_{R^0} : t_x \otimes v \mapsto \sum t_{c_{\widetilde{M}}(x\widetilde{z})} \otimes \iota_a[a\sigma(m_{\widetilde{M}}(x\widetilde{z}) m_{\widetilde{z}}^0) v].$$

(ii) Applying Φ^{-1} we get

$$\mathcal{I} : t_x \otimes v \mapsto \sum t^h_{c_{\widetilde{M}}(x\widetilde{z})(w^0)^{-1}} \otimes \iota_a[a\sigma[m_{\widetilde{M}}(x\widetilde{z}) m_{\widetilde{z}}^0 v] =$$
$$= \sum t^h_{c_M(x\widetilde{z}(w^0)^{-1})} \otimes \iota_a[a\sigma(m_{\widetilde{M}}(x\widetilde{z}) m_{\widetilde{z}}^0) v].$$

Thus

$$\langle t_x \otimes v_x, t_y \otimes v_y \rangle = \langle a\sigma(m_{\widetilde{M}}(x\widetilde{z}) m_{\widetilde{z}}^0) v_x, v_y \rangle$$

with $c_M(x\widetilde{z}(w^0)^{-1}) = y$. This equation gives

$$x\widetilde{z}(w^0)^{-1} = y \cdot m_M(x\widetilde{z}(w^0)^{-1}) \Leftrightarrow \widetilde{z} = x^{-1} y m_M(x\widetilde{z}(w^0)^{-1}) w^0 \Leftrightarrow \widetilde{z} = x^{-1} y w^0 m_{\widetilde{M}}(x\widetilde{z}) \Leftrightarrow$$
$$\Leftrightarrow x^{-1} y w^0 = \widetilde{z} m_{\widetilde{M}}(x\widetilde{z})^{-1} \Leftrightarrow \widetilde{z} = c_{\widetilde{M}}(x^{-1} y w^0),\ m_{\widetilde{M}}(x\widetilde{z}) = m_{\widetilde{M}}(x^{-1} y w^0)^{-1}.$$

The final answer is

$$\langle t_x \otimes v_x, t_y \otimes v_y \rangle = \langle a\sigma[m_{\widetilde{M}}(x^{-1} y w^0)^{-1} m^0_{c_{\widetilde{M}}(x^{-1} y w^0)}] v_x, v_y \rangle.$$

Compare and contrast this with

$$\epsilon_{\widetilde{M}}(t_{(w^0)^{-1}} t_{y^{-1}} t_x R^0) = \epsilon_{\widetilde{M}}(\sum t_{(w^0)^{-1}} t_{y^{-1}} t_x t_{\widetilde{z}} m_{\widetilde{z}}^0) = \widetilde{m} \cdot m_{\widetilde{z}}^0$$

where $(w^0)^{-1}y^{-1}x\widetilde{z} = \widetilde{m}$. So

$$x^{-1}yw^0 = \widetilde{z}\widetilde{m}^{-1} \Leftrightarrow \widetilde{z} = c_{\widetilde{M}}(x^{-1}yw^0),\ \widetilde{m} = m_{\widetilde{M}}(x^{-1}yw^0)^{-1}.$$

In conclusion, we have proved the following result.

Proposition 3.9.1. *Suppose (σ, U_σ) has a $\bullet$-invariant hermitian form $\langle\ ,\ \rangle_{\sigma,\bullet}$. The form*

$$\langle h_1 \otimes v_1, h_2 \otimes v_2\rangle_\bullet = \langle a\sigma[\epsilon_{\widetilde{M}}(t_{(w^0)^{-1}}h_2^\bullet h_1 R^0)]v_1, v_2\rangle_{\sigma,\bullet}$$

on $X(M,\sigma)$ is $\bullet$-invariant and sesquiliniar.

We prove in the next section that the form is also hermitian.

3.10. **Symmetry.** The parabolic Hecke subalgebra $\mathbb{H}_M$ of $\mathbb{H}$ is attached to the non-semisimple root system $(V, R_M, V^\vee, R_M^\vee)$. Let V_M be the $\mathbb{R}$-span of R_M in V, $V_M^\vee$ the $\mathbb{R}$-span of $R_M^\vee$ in $V^\vee$, and

$$V_M^\perp = \{v \in V : (v, \alpha^\vee) = 0,\ \text{for all } \alpha \in R_M\},$$
$$V_M^{\vee,\perp} = \{v^\vee \in V^\vee : (\alpha, v^\vee) = 0,\ \text{for all } \alpha \in R_M\}.$$

Then $V = V_M \oplus V_M^\perp$, $V^\vee = V_M^\vee \oplus V_M^{\vee,\perp}$. Let $\mathbb{H}_M^0$ denote the graded Hecke algebra attached to the semisimple root system $(V_M, R_M, V_M^\vee, R_M^\vee)$ by Definition 2.1.1. Then there is an algebra isomorphism

$$\mathbb{H}_M = \mathbb{H}_M^0 \otimes_{\mathbb{C}} S(V_M^\perp). \tag{3.10.1}$$

Assume $\sigma = \sigma_0 \otimes \mathbb{C}_\nu$, where σ_0 is an $\mathbb{H}_M^0$-module, and $\nu \in (V_M^{\perp,\vee})_{\mathbb{C}}$.

Lemma 3.10.1. *There is a set $V_{M,\mathsf{reg}}^{\perp,\vee}$ containing an open set of $V_M^{\perp,\vee}$ such that $\{\mathcal{R}_x \otimes v_i\}$ with x minimal in the coset $xW(M)$ and $\{v_i\}$ a basis of U_σ forms a basis of $X(M,\sigma)$.*

Proof. This follows from the formula

$$\mathcal{R}_x = t_x \prod_{x^{-1}\alpha<0} \frac{\alpha}{k_\alpha + \alpha} + \sum_{y<x} t_y m_y^x.$$

The leading term for R^0, $\sigma\left(\prod_{(w_M^0)^{-1}\alpha<0} \frac{\alpha}{k_\alpha+\alpha}\right)$, is invertible for generic ν. The claim follows from the fact that the expression of $\mathcal{R}_x$ is upper triangular in the t_y. □

Theorem 3.10.2. *The form in Proposition 3.9.1 is hermitian, and therefore, it gives a $\bullet$-invariant hermitian form on the induced module $X(M,\sigma)$.*

Proof. The claim follows (on $V_{M,\mathsf{reg}}^{\perp,\vee}$ first, and thus always) from the formula

$$\epsilon_{\widetilde{M}}(t_{(w^0)^{-1}}\mathcal{R}_y^\bullet \mathcal{R}_x R^0) = 0 \ \text{ unless } x = y.$$

□

As above, when $\nu \in (V_{M,\mathsf{reg}}^{\perp,\vee})_{\mathbb{C}}$, a basis of $X(M,\sigma)$ is given by $\{\mathcal{R}_x \otimes v\}$, where x ranges in $\mathcal{J}_M$, and v ranges over a basis of σ_0. In this case, one obtains a simpler formula for the signature of the $\bullet$-form.

Corollary 3.10.3. *When $\nu \in (V_{M,\mathsf{reg}}^{\perp,\vee})_{\mathbb{C}}$, the signature of the $\bullet$ form on $X(M,\sigma)$, $\sigma = \sigma_0 \otimes \mathbb{C}_\nu$, is given by*

$$\langle \mathcal{R}_x \otimes v_1, \mathcal{R}_y \otimes v_2 \rangle_\bullet = \begin{cases} 0, & x \neq y, \\ f(\mathsf{cc}(\sigma)) \left\langle \sigma \left(\prod_{\alpha>0, x\alpha<0} \frac{\alpha - k_\alpha}{\alpha + k_\alpha} \right) v_1, v_2 \right\rangle_{\sigma,\bullet}, & x = y, \end{cases} \tag{3.10.2}$$

where $x, y \in \mathcal{J}_M$, $v_1, v_2 \in U_\sigma$, and $f(\mathsf{cc}(\sigma)) = (-1)^{|R^+ \setminus R_M^+|} \prod_{\alpha \in R^+ \setminus R_M^+} \frac{\langle \alpha, \mathsf{cc}(\sigma) \rangle}{k_\alpha + \langle \alpha, \mathsf{cc}(\sigma) \rangle}$.

Proof. The first claim follows as in the proof of Theorem 3.10.2. For the second claim, one uses formula (3.3.3) for $\mathcal{R}_x^\bullet$ and the same substitution as in the second formula of (3.3.5). □

Since the factor $f(\mathsf{cc}(\sigma))$ is common for all x, it makes sense to normalize the hermitian form by dividing by it. The resulting form has the property that

$$\langle \mathcal{R}_1 \otimes v_1, \mathcal{R}_1 \otimes v_2 \rangle_\bullet = \langle v_1, v_2 \rangle_{\sigma,\bullet}. \tag{3.10.3}$$

Remark 3.10.4. In the particular case when $\sigma_0 = \mathsf{triv}$ (so that σ is the one-dimesional character $\mathbb{C}_\nu$) and ν is large, we recover a result of Opdam [Op2, Theorem 4.1]. In that case, the induced module $X(M,\nu) = \mathbb{H} \otimes_{\mathbb{H}_M} \mathbb{C}_\nu$ is $\mathbb{A}$-semisimple with a basis given by $\{\mathcal{R}_x \otimes \mathbb{1}_\nu : x \in \mathcal{J}_M\}$, and in the normalization (3.10.3), the form is

$$\langle \mathcal{R}_x \otimes \mathbb{1}_\nu, \mathcal{R}_y \otimes \mathbb{1}_\nu \rangle_\bullet = \delta_{x,y} \prod_{\alpha>0, x\alpha<0} \frac{\langle \alpha, \nu \rangle - k_\alpha}{\langle \alpha, \nu \rangle + k_\alpha} \tag{3.10.4}$$

It is easy to verify that this formula agrees (switching the between roots and coroots) with the one in [Op2, Theorem 4.1.(4)], after taking the scaling factor $a(\lambda, k) = \prod_{\alpha>0}(1 - k_\alpha/\widetilde{\lambda}(\alpha^\vee))$ in the notation therein.

3.11. We have analyzed the construction of induced $\bullet$-invariant forms. The same type of discussion works for $\star$-invariant forms, or otherwise, the result for $\star$-invariant forms can be deduced via formal manipulations as in section 3.1. We only state the result and skip more details. A similar result was obtained in [BM3, section 1.8].

Proposition 3.11.1. *Suppose (σ, U_σ) has a $\star$-invariant hermitian form $\langle\ ,\ \rangle_{\sigma,\star}$. The pairing*

$$\langle h_1 \otimes v_1, h_2 \otimes v_2 \rangle_\star = \langle a\sigma[\epsilon_{\widetilde{M}}(h_2^\star h_1 R^0)] v_1, v_2 \rangle_{\sigma,\star}$$

on $X(M,\sigma)$ is a hermitian, $\star$-invariant (sesquiliniar) form.

4. Langlands classification and $\mathbb{A}$-weights

We use Langlands classification to deduce certain results about the $\mathbb{A}$-weights of irreducible $\mathbb{H}$-modules. As a consequence, we show that every irreducible $\mathbb{H}$-module with real central character admits a $\bullet$-invariant hermitian form.

4.1. **Langlands quotient.** Retain the notation from section 3.10. The following form of Langlands classification is proved in [Ev].

Theorem 4.1.1. (i) *Let L be an irreducible $\mathbb{H}$-module. Then L is a quotient of $X(M,\nu) = \mathbb{H}\otimes_{\mathbb{H}_M}(\sigma\otimes\mathbb{C}_\nu)$, where σ is an irreducible tempered $\mathbb{H}_M^0$-module, and $\nu\in V_M^\perp$ such that $\operatorname{Re}\nu$ is dominant, i.e., $(\operatorname{Re}\nu,\alpha)>0$, for all $\alpha\in\Pi\setminus\Pi_M$.*

(ii) *If σ,ν are as in (i), then $\mathbb{H}\otimes_{\mathbb{H}_M}(\sigma\otimes\mathbb{C}_\nu)$ has a unique irreducible quotient $L(\sigma,\nu)$.*

(iii) *If $L(\sigma,\nu)\cong L(\sigma',\nu')$, then $M=M'$, $\sigma\cong\sigma'$, and $\nu=\nu'$.*

We need to review the construction of Π_M, σ and ν from L.

Let $\{\omega_1^\vee,\dots,\omega_n^\vee\}$ be the basis of $V^\vee$ consisting of fundamental coweights, i.e., the basis dual to $\Pi\subset V$. For every subset $F\subset\{1,2,\dots,n\}$, let

$$S_F=\{\sum_{j\notin F}c_j\omega_j^\vee-\sum_{i\in F}d_i\alpha_i^\vee : c_j>0, d_i\geq 0\}\subseteq V^\vee.$$

A lemma of Langlands, cf. [Ev, Lemma 2.3] says that for every $v\in V^\vee$, there exists a unique subset F such that $v\in S_F$. Denote this subset by $F(v)$. If $v=\sum_{j\notin F}c_j\omega_j^\vee-\sum_{i\in F}d_i\alpha_i^\vee$, then set

$$v^0=\sum_{j\notin F}c_j\omega_j^\vee.$$

On $V^\vee$ define the order relation $\geq$ by $v\geq v'$ if $v-v'\in\mathbb{R}_{\geq 0}\Phi^{\vee,+}$. Then, see for example [Ev, Lemma 2.4],

$$v_1\geq v_2 \text{ implies } v_1^0\geq v_2^0. \tag{4.1.1}$$

Choose $\lambda\in\Omega(L)$ such that $\operatorname{Re}\lambda$ is maximal with respect to $\geq$ among the real parts of weights of L. Then

$$\nu=\lambda|_{V_M^{\vee,\perp}}, \tag{4.1.2}$$

and σ is an irreducible $\mathbb{H}_M^0$-module such that $\sigma\otimes\mathbb{C}_\nu$ occurs in the restriction of L to $\mathbb{H}_M=\mathbb{H}_M^0\otimes S(V_M^\perp)$. Moreover the weights of σ are:

$$\Omega(\sigma)=\{\lambda'|_{V_M^\vee}:\lambda'\in\Omega(L),\ \lambda'|_{V_M^{\vee,\perp}}=\nu,\ F(\operatorname{Re}\lambda')=\Pi_M\}\subset V_M^\vee. \tag{4.1.3}$$

4.2. **Iwahori-Matsumoto involution.** The Iwahori-Matsumoto involution τ of $\mathbb{H}$ is defined on the generators of $\mathbb{H}$ by:

$$\tau(t_{s_\alpha})=-t_{s_\alpha},\ \alpha\in\Pi,\quad \tau(a)=-a,\ a\in V_{\mathbb{C}}. \tag{4.2.1}$$

It is immediate that this assignment extends to an algebra automorphism and therefore to a involution, denoted τ again on $\mathbb{H}$-modules. Notice that if X is an $\mathbb{H}$-module, then

$$\begin{aligned}&\tau(X)|_W\cong X|_W\otimes\mathsf{sgn},\\ &\Omega(\tau(X))=-\Omega(X),\quad \tau(X)_{-\lambda}\cong X_\lambda,\ \lambda\in\Omega(X).\end{aligned} \tag{4.2.2}$$

Lemma 4.2.1. *Assume $\mathbb{H}$ is semisimple. Suppose X is an irreducible tempered module such that $\tau(X)$ is also tempered. Then the central character χ of X is imaginary, i.e., $\chi\in\sqrt{-1}V$, and $X\cong X(\chi)$.*

Proof. Let $\lambda\in\Omega(X)$ be arbitrary. Since X is tempered, $(\omega,\operatorname{Re}\lambda)\leq 0$ for all dominant $\omega\in V$. If $\tau(X)$ is also tempered, $(\omega,-\operatorname{Re}\lambda)\leq 0$ as well, hence $(\omega,\operatorname{Re}\lambda)=0$ for all ω dominant in V. Thus $\operatorname{Re}\lambda=0$ and so $\chi\in\sqrt{-1}V$, which means $X\cong X(\chi)$, since at imaginary central character the minimal principal series is irreducible ([Ch], see [Op2, Theorem 1.3]). □

4.3. **$\mathbb{A}$-weights.** Let (π, X) be an irreducible $\mathbb{H}$-module, and $\Omega(X) \subset V_{\mathbb{C}}^{\vee}$ the set of $\mathbb{A} = S(V_{\mathbb{C}})$-weights of X. As noted before, $\Omega(X) \subset W \cdot \mathsf{cc}(X)$. Define the $\mathbb{A}$-character of X to be the formal sum:

$$\Theta_{\mathbb{A}}(X) = \sum_{\lambda \in \Omega_X} (\dim X_\lambda)\ e^{\lambda}, \tag{4.3.1}$$

where $X_\lambda = \{x \in X : \text{ for all } a \in \mathbb{A},\ (\pi(a) - \lambda)^n x = 0, \text{ for some } n\}$ is the generalized λ-weight space. Denote the multiplicity of λ in X by

$$m[\lambda : X] := \dim X_\lambda. \tag{4.3.2}$$

The following proposition is the graded Hecke algebra analogue of a result of Casselman for p-adic groups, and Evens-Mirković [EM, Theorem 5.5] for the geometric affine Hecke algebras.

Proposition 4.3.1. *Let X, X' be two irreducible $\mathbb{H}$-modules such that $\Omega(X) = \Omega(X')$. Then $X \cong X'$.*

Proof. By hypothesis $\mathsf{cc}(X) = \mathsf{cc}(X') = \chi$. Suppose X (and therefore also X') is not tempered. By Langlands classification, X is the unique irreducible quotient of $\mathbb{H} \otimes_{\mathbb{H}_M} (\sigma \otimes \mathbb{C}_\nu)$, where $\Pi_M \subsetneq \Pi$, ν and $\Omega(\sigma)$ are uniquely determined by $\Omega(X)$. Therefore, by induction of $|\Pi|$, the claim follows for nontempered X.

Now assume that X is tempered. We use the Iwahori-Matsumoto involution and Lemma 4.2.1: either $\tau(X)$ is not tempered and since $\Omega(\tau(X)) = \Omega(\tau(X'))$, we may finish as above, or else X (and also X') is the irreducible minimal principal series $X(\chi)$ with imaginary central character χ. □

As a consequence, we deduce indirectly that every irreducible module with real central character has a hermitian $\bullet$-invariant form.

Corollary 4.3.2. *Let X be an irreducible $\mathbb{H}$-module. Then X admits a $\bullet$-invariant hermitian form if and only if $\overline{\Omega(X)} = \Omega(X)$. In particular, if X has real central character then X admits a $\bullet$-invariant form.*

Proof. Since $a^\bullet = \overline{a}$ for all $a \in S(V_{\mathbb{C}})$, we have $\Omega(X^\bullet) = \overline{\Omega(X)}$, where $X^\bullet$ is the $\bullet$-hermitian dual of X. The claim follows at once from Proposition 4.3.1. □

4.4. **Linear independence.** Proposition 4.3.1 says that $\Theta_{\mathbb{A}}(X)$ uniquely determines X. We now prove the stronger statement that $\{\Theta_{\mathbb{A}}(X)\}$ is linearly independent.

Lemma 4.4.1. *Suppose λ is a weight of the irreducible tempered $\mathbb{H}_M^0$-module σ and $X(\sigma, \nu)$ is a standard Langlands induced module. Then*

$$m[\lambda + \nu : L(\sigma, \nu)] = m[\lambda + \nu : X(\sigma, \nu)] = m[\lambda : \sigma].$$

Proof. By the construction of the Langlands quotient $L(\sigma, \nu)$, the restriction of $L(\sigma, \nu)$ to $\mathbb{H}_M$ contains the $\mathbb{H}_M$-module $\sigma \otimes \mathbb{C}_\nu$, hence $\mathrm{Hom}_{\mathbb{A}}[\sigma \otimes \mathbb{C}_\nu, L(\sigma, \nu)] \neq 0$, and therefore $m[\lambda + \nu : L(\sigma, \nu)] \geq m[\lambda : \sigma]$. Thus, it is sufficient to prove that $m[\lambda + \nu : X(\sigma, \nu)] = m[\lambda : \sigma]$.

By [BM2, Proposition 6.4], every weight in $X(\sigma, \nu)/(\sigma \otimes \mathbb{C}_\nu)$ is of the form $w(\lambda + \nu)$, where λ is a weight of σ, and $w \neq 1$ ranges over the set $\mathcal{J}_M$ of minimal length representatives of W/W_M.

We claim that if $w \neq 1$ is such a representative, then $w(\lambda+\nu) \neq \lambda'+\nu$, for every λ, λ' weights of σ. Let $F \subset \{1, 2, \ldots, n\}$ be such that $\lambda \in S_F$. Then, as before, write $\operatorname{Re}\lambda = -\sum_{i\in F} d_i\alpha_i^\vee$, $d_i \geq 0$ and $\operatorname{Re}\nu = \sum_{j\notin F} c_j\omega_j^\vee$.

Since $w\alpha_i^\vee \in R^{\vee,+}$, for all $i \in F$, we have $(w\operatorname{Re}\lambda, \omega_j^\vee) \leq 0$ for every $i \in F$, $j \notin F$. On the other hand, $w\beta_j < \beta_j$ for $j \notin F$, so $(w\operatorname{Re}\nu, \beta_j) < (\nu, \beta_j)$ for some $j \notin F$. This implies that $(w\operatorname{Re}(\lambda+\nu), \operatorname{Re}\nu) < (\operatorname{Re}(\lambda+\nu), \operatorname{Re}\nu) = (\operatorname{Re}\nu, \operatorname{Re}\nu)$.

□

Theorem 4.4.2. *The set $\{\Theta_{\mathbb{A}}(X)\}$ where X ranges over the set of (isomorphism classes of) simple $\mathbb{H}$-modules is $\mathbb{Z}$-linearly independent.*

Proof. Let

$$\sum_i c_i\Theta_{\mathbb{A}}(X_i) = 0 \tag{4.4.1}$$

be a finite linear combination of $\mathbb{A}$-characters, where $\{X_i\}$ are distinct simple modules. Without loss of generality, we may assume that all X_i have the same central character and moreover, that the central character is real. By Langlands classification, each X_i is the unique irreducible quotient $L(M_i, \sigma, \nu_i)$ of an induced $\mathbb{H}\otimes_{\mathbb{H}_{M_i}}(\sigma_i \otimes \mathbb{C}_{\nu_i})$.

Find λ a weight in the linear combination such that λ is maximal with respect to $\geq$ and no other λ' occuring in the linear combination satisfies $(\lambda')^0 > \lambda^0$, with the notation as in previous subsection. There exists a unique $F = F(\lambda)$ such that $\lambda \in S_F$, and write Π_M for the subset of Π corresponding to F, and $\nu = \lambda|_{V_M^{\vee,\perp}}$ accordingly. Then $\nu = \lambda^0$. Let $\sigma_1, \ldots, \sigma_k$ irreducible tempered $\mathbb{H}_M^0$-modules so that $L(M, \sigma_t, \nu)$ occurs in (4.4.1).

We claim that if λ_j is any weight of a σ_l, $l = 1, k$, then every time $\lambda_j + \nu$ occurs in (4.4.1), it occurs in a $\Theta_{\mathbb{A}}(L(M, \sigma_t, \nu))$ for some $t = 1, k$. To see this, suppose $\lambda_j + \nu$ appears in $L(M', \sigma', \nu')$. Then there exists an extremal weight λ' such that $(\lambda_j + \nu) \leq \lambda'$, but then $\nu = (\lambda_j + \nu)^0 \leq (\lambda')^0$, and by assumption $\nu = (\lambda')^0 = \nu'$, $M = M'$ and $\sigma' = \sigma_t$ for some t.

Combining this with Lemma 4.4.1, it follows that (4.4.1) implies

$$\sum_{j=1}^{k} c_j\Theta_{\mathbb{A}}(\sigma_j) = 0. \tag{4.4.2}$$

If $\Pi_M \neq \Pi$, we get $c_j = 0$ by induction and continue the same process with the remaining terms in (4.4.1). If $\Pi_M = \Pi$, i.e., the combination involves only $\mathbb{A}$-characters of irreducible tempered $\mathbb{H}$-modules, apply the Iwahori-Matsumoto involution and conclude as in the proof of Proposition 4.3.1.

□

5. Signature of hermitian forms and lowest W-types

In order to study the signature of $\bullet$-invariant forms, we need to construct explicitly the forms whose existence is guaranteed by Corollary 4.3.2. We use Langlands classification together with the explicit induced forms from section 3.10. To conclude certain results about signatures, we also make use of the geometric classification of $\mathbb{H}$-modules (for equal parameters).

5.1. **Tempered modules.**

Lemma 5.1.1. *Let X be an irreducible tempered $\mathbb{H}$-module. Then X is a submodule of a parabolically induced module $I(M, \sigma \otimes \mathbb{C}_\nu) = \mathbb{H} \otimes_{\mathbb{H}_M} (\sigma \otimes \mathbb{C}_\nu)$, where σ is a discrete series module of $\mathbb{H}_M^0$ and $\nu \in (V_M^{\perp,\vee})_{\mathbb{C}}$ with $\operatorname{Re}\nu = 0$.*

Proof. Let $\omega_i \in V$, $i = 1, n$, denote the fundamental weights of the root system. For every weight $\lambda \in \Omega(X)$, define

$$\mathcal{F}(\lambda) = \{i : (\omega_i, \operatorname{Re}\lambda) < 0\}. \tag{5.1.1}$$

Since X is assumed tempered, we necessarily have $(\omega_j, \operatorname{Re}\lambda) = 0$ for all $j \notin \mathcal{F}(\lambda)$. We assumed that the root system is semisimple, therefore,

$$\operatorname{Re}\lambda = -\sum_{i \in \mathcal{F}(\lambda)} d_i \alpha_i^\vee, \text{ where } d_i > 0.$$

Choose now $\lambda \in \Omega(X)$ such that $\mathcal{F} := \mathcal{F}(\lambda)$ is minimal with respect to set inclusion. Set

$$\Pi_M = \{\alpha_i : i \in \mathcal{F}\}.$$

Using the decomposition $V^\vee = V_M^\vee \oplus V_M^{\perp,\vee}$, let ν be the projection of λ onto $(V_M^{\perp,\vee})_{\mathbb{C}}$. Since $\operatorname{Re}\lambda \in V_M^\vee$ by construction, it follows that $\operatorname{Re}\nu = 0$.

Let Y be an irreducible consituent of the restriction of X to $\mathbb{H}_M$, such that $S((V_M^\perp)_{\mathbb{C}})$ acts on Y by ν. Then $Y \cong Y^0 \otimes \mathbb{C}_\nu$, where Y^0 is an irreducible $\mathbb{H}_M^{\mathbf{ss}}$-module. We claim that Y^0 is a discrete series $\mathbb{H}_M^0$-module. To see this, let $\mu \in (V_M^\vee)_{\mathbb{C}}$ be a $S((V_M)_{\mathbb{C}})$-weight of Y^0, write $\operatorname{Re}\mu = -\sum_{i\in\mathcal{F}} z_i \alpha_i^\vee$, and we want to prove that all $z_i > 0$. The sum $\mu + \nu$ is a weight of X and $\operatorname{Re}(\mu+\nu) = \operatorname{Re}(\mu) = -\sum_{i\in\mathcal{F}} z_i\alpha_i^\vee$, in particular, $z_i \geq 0$. Notice that if $j \notin \mathcal{F}$, then $j \notin \mathcal{F}(\mu+\nu)$, hence $\mathcal{F}(\mu+\nu) \subseteq \mathcal{F}$. By the minimality of $\mathcal{F}$, $\mathcal{F}(\mu+\nu) = \mathcal{F}$, and therefore $z_i > 0$ for all i. Setting $\sigma = Y^0$, the lemma is proved.

□

The following statement is well-known.

Proposition 5.1.2. *Every irreducible tempered $\mathbb{H}$-module is $*$-unitary.*

Sketch of proof. When the Hecke algebra $\mathbb{H}$ appears in the representation theory of p-adic groups (i.e., it is "geometric type" in the sense of Lusztig [Lu2]), the claim follows from the unitarizability of tempered representations of the p-adic group, see for example [BM1].

For Hecke algebras with arbitrary positive parameters, the statement is known from [Op1] in the setting of affine Hecke algebras, together with the fact that Lusztig's reduction from affine to graded affine Hecke algebras [Lu1] preserves temperedness and unitarity. □

Corollary 5.1.3. *Every irreducible tempered $\mathbb{H}$-module with real central character admits a $\bullet$-invariant hermitian form.*

Proof. Let X be an irreducible tempered module. If $X^\delta \cong X$, then we can define a $\bullet$-hermitian form, using the $*$-hermitian form $\langle\ ,\ \rangle_*$ from Proposition 5.1.2, as before, by setting $\langle x, y\rangle_\bullet = \langle \pi(t_{w_0})x, \delta(y)\rangle_*$, $x, y \in X$. We claim that $X^\delta \cong X$ for every irreducible tempered $\mathbb{H}$-module with real infinitesimal character. For this, we use that the restriction to W of the set of tempered modules with real central character is linearly independent in the Grothendieck group of W. When

the parameter function k of $\mathbb{H}$ is geometric in the sense of Lusztig, this (and more) follows from the geometric classification, see section 5.3. For arbitrary positive parameters k, this result is proved in [So].

If X is irreducible tempered with real central character, then X^δ is also tempered. This is because $\Omega(X^\delta) = -w_0(\Omega(X))$, and if ω_j is a fundamental weight, then so is $-w_0(\omega_j)$, hence the non-positivity conditions for weights are preserved.

Also $X|_W \cong X^\delta|_W$. By the W-linear independence mentioned above, $X \cong X^\delta$, as $\mathbb{H}$-modules. □

5.2. **Signature at infinity.** Assume $\langle \nu, \alpha \rangle > 0$ for all $\alpha \in R^+ \setminus R_M^+$ and denote

$$\sigma_t = \sigma_0 \otimes \mathbb{C}_{t\nu},\ t > 0.$$

We consider the signature of the form on $X(M, \sigma_t)$ as $t \to \infty$. We can use the basis $\{\mathcal{R}_x \otimes v_i\}$, so that the form is block-diagonal with respect to $x \in \mathcal{J}_M$. By Corollary 3.10.3, in the diagonal block (of the normalized form) for $\mathcal{R}_x$, we have

$$\langle \mathcal{R}_x \otimes v_1, \mathcal{R}_x \otimes v_2 \rangle_{\bullet,t} = \left\langle \sigma_t \left(\prod_{\alpha>0, x\alpha<0} \frac{\alpha - k_\alpha}{\alpha + k_\alpha} \right) v_1, v_2 \right\rangle_{\sigma_t, \bullet}.$$

As $t \to \infty$, the expression $\sigma_t \left(\prod_{x\alpha<0} \frac{\alpha - k_\alpha}{\alpha + k_\alpha} \right)$ goes to the identity, which means that

$$\lim_{t\to\infty} \langle \mathcal{R}_x \otimes v_1, \mathcal{R}_x \otimes v_2 \rangle_{\bullet,t} = \langle v_1, v_2 \rangle_{\sigma_0, \bullet}. \tag{5.2.1}$$

We have proved

Theorem 5.2.1. *The $\bullet-$signature of $X(M, \sigma_t)$ at ∞ is the induced signature of the $\bullet-$signature of (σ_0, U_{σ_0}).*

5.3. **Lowest W-types.** In this section, we assume that the graded Hecke algebra $\mathbb{H}$ has equal parameters. More generally, analogous results hold whenever the parameters of $\mathbb{H}$ are of geometric type, in the sense of [Lu2].

Suppose $\mathbb{H}$ is attached to a root system Ψ and constant parameter function k. Let $\mathfrak{g}$ be the reductive Lie algebra with root system Ψ. In particular, we identify a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with $V_{\mathbb{C}}^\vee$, so that the roots R live in $\mathfrak{h}^* \cong V_{\mathbb{C}}$. Let $\mathcal{N} \subset \mathfrak{g}$ denote the nilpotent cone. Let G be a complex connected Lie group with Lie algebra $\mathfrak{g}$; for our purposes, we may choose G to be the adjoint form. If $\mathcal{S}$ is a subset of $\mathfrak{g}$, denote by $Z_G(\mathcal{S})$ the mutual centralizer in G of the elements in $\mathcal{S}$ and $A(\mathcal{S})$ the group of components of $Z_G(\mathcal{S})$.

We summarize the results from [KL, Lu2] that we need for signatures.

One attaches a standard geometric $\mathbb{H}$-module $X(s, e, \psi)$ to every triple

$$(s, e, \psi), \quad s \in \mathfrak{g} \text{ semisimple},\ e \in \mathcal{N} \text{ such that } [s, e] = ke,\ \psi \in \widehat{A}(s, e)_0, \tag{5.3.1}$$

where $\widehat{A}(s, e)_0$ is the set of irreducible representations of $A(s, e)$ which appear in the permutation action on the top cohomology $H^{\text{top}}(\mathcal{B}_e^s, \mathbb{C})$. Here, $\mathcal{B}_e^s$ denotes the variety of Borel subalgebras of $\mathfrak{g}$ containing e and s. Morever,

$$X(s, e, \psi) \cong X(s', e', \psi') \text{ if and only if } g \cdot (s, e, \psi) = (s', e', \psi'),\ \text{for some } g \in G. \tag{5.3.2}$$

Consequently, we may assume, without loss of generality, that $s \in \mathfrak{h}$. Under the identification $\mathfrak{h} = V_{\mathbb{C}}^\vee$, write $s = s_0 + \sqrt{-1}s_1$ with $s_0, s_1 \in V^\vee$.

On the other hand, recall that the Springer correspondence realizes every irreducible W-representation as the ϕ-isotypic component

$$\mu(e,\phi) := \mathrm{Hom}_{A(e)}[\phi, H^{\mathrm{top}}(\mathcal{B}_e,\mathbb{C})] \tag{5.3.3}$$

of the top cohomology group of the Springer fiber $\mathcal{B}_e$. Denote by $\widehat{A}(e)_0$ the set of irreducible representations of $A(e)$ which appear in the action on $H^{\mathrm{top}}(\mathcal{B}_e,\mathbb{C})$. Moreover $\mu(e,\phi) \cong \mu(e',\phi')$ if and only if there exists $g \in G$ such that $g\cdot(e,\phi) = (e',\phi')$.

The inclusion $Z_G(s,e) \to Z_G(e)$ descends to an inclusion $A(s,e) \to A(e)$. The standard module $X(s,e,\psi)$ has the property that

$$\mathrm{Hom}_W[\mu(e,\phi) : X(s,e,\psi)] = \mathrm{Hom}_{A(s,e)}[\psi : \phi|_{A(s,e)}], \tag{5.3.4}$$

for all $\phi \in \widehat{A}(e)_0$.

Definition 5.3.1. We call $\mu(e,\phi)$ a lowest W-type of $X(s,e,\psi)$ if $\mathrm{Hom}_{A(s,e)}[\psi : \phi|_{A(s,e)}] \neq 0$.

Theorem 5.3.2 ([KL, Lu2])**.**

(1) *The standard module $X(s,e,\psi)$ where (s,e,ψ) is as in (5.3.1) has a unique composition factor $L(s,e,\psi)$ such that $L(s,e,\psi)$ contains every lowest W-type of $X(s,e,\psi)$ with full multiplicity $[\psi : \phi|_{A(s,e)}]$.*

(2) *The module $X(s,e,\psi)$ is tempered if and only if $s_0 = kh$ for a Lie triple (e,h,f) of e. In this case, $X(s,e,\psi) = L(s,e,\psi)$. The module $X(s,e,\psi)$ is a discrete series if in addition e is a distinguished nilpotent element.*

Notice that, in particular, there is a one-to-one correspondence between tempered $\mathbb{H}$-modules with real central character and (G-conjugacy classes) of pairs (e,ϕ) where $e \in \mathcal{N}$ and $\phi \in \hat{A}(e)_0$.

According to the parabolic Langlands classification recalled in Theorem 4.1.1, for every irreducible tempered $\mathbb{H}_M^0$ module σ_0 and every $\nu \in V_M^{\vee,\perp}$ such that ν is dominant, i.e., $(\alpha,\nu) > 0$ for all $\alpha \in \Pi \setminus \Pi_M$, the standard parabolically induced module

$$X(M,\sigma_0,\nu) = \mathbb{H} \otimes_{\mathbb{H}_M} (\sigma_0 \otimes \mathbb{C}_\nu), \tag{5.3.5}$$

has a unique irreducible quotient $L(M,\sigma_0,\nu)$.

The relation with the geometric classification is as follows. The tempered $\mathbb{H}_M^0$-module σ_0 is parameterized by a triple (s_M, e_M, ψ_M). Here $s_M \in (V_M^\vee)_{\mathbb{C}}$, e_M is a nilpotent element in the corresponding Levi subalgebra $\mathfrak{m} \subset \mathfrak{g}$ and ψ_M is a representation of $A_M(s_M,e_M)$. Set

$$s = s_M + \nu \in V_{\mathbb{C}} = \mathfrak{h},\ e = e_M.$$

Since ν commutes with s_M and $e = e_M$, $A_G(s,e) = A_G(s_M,e_M)$. The embedding $A_M(s_M,e_M) \to A_G(s_M,e_M) = A_G(s,e)$ induces a surjection $\widehat{A}_G(s,e) \to \widehat{A}_M(s_M,e_M)$, and let ψ the pull-back of ψ_M. Then

$$X(M,\sigma_0,\nu) \cong X(s,e,\psi) \text{ and } L(M,\sigma_0,\nu) \cong L(s,e,\psi). \tag{5.3.6}$$

Thus, we may speak of the lowest W-types of $X(M,\sigma_0,\nu)$ (and of $L(M,\sigma_0,\nu)$. Denote by $\mathsf{LWT}(M,\sigma_0)$ the set of lowest W-types of $X(M,\sigma,\nu)$. Since $A_G(s,e)$, $s = s_M + \nu$ does not change for all dominant ν, this set does not change with ν, hence the notation. Then Theorem 5.3.2(1) implies that $L(M,\sigma_0,\nu)$ contains all the W-types in $\mathsf{LWT}(M,\sigma_0)$ with full multiplicity.

Moreover, let μ_0 be the unique lowest W_M-type of σ_0. For every $\mu \in \mathsf{LWT}(M, \sigma_0)$, we have

$$\mathrm{Hom}_W[\mu, X(M, \sigma_0, \nu)] = \mathrm{Hom}_{W(M)}[\mu|_{W(M)}, \sigma_0] = \mathrm{Hom}_{W(M)}[\mu|_{W(M)}, \mu_0]. \tag{5.3.7}$$

The form $\langle\ ,\ \rangle_{\sigma_0,\bullet}$ can be normalized so that it is positive on the μ_0-isotypic component. Then Theorem 5.2.1 implies the following corollary.

Corollary 5.3.3. *The $\bullet$-signature on the lowest $W-$types of the standard module $X(M, \sigma_0, \nu)$ is given by Theorem 5.2.1 for* all *dominant ν. In particular, the $\bullet$-form on $L(M, \sigma_0, \nu)$ can be normalized so that the signature is positive definite on all lowest W-types.*

Remark 5.3.4. Suppose $L = L(M, \sigma_0, \nu)$ also carries a $\star$-invariant hermitian form. Since ν is real, this is the case precisely when $\delta(M) = M$ and $w_0\nu = -\nu$. (We have $L(M, \sigma_0, \nu)^\delta = L(\delta(M), \sigma_0, \delta(\nu))$.) As in section 3.1, in order to compare $\star$-signatures with $\bullet$-signatures, we need first to choose an isomorphism $\tau_L^\delta : L^\delta \to L$. It is an empirical fact that always L has one lowest W-type that appears with multiplicity 1, and we normalize τ_L^δ to be $+1$ on the isotypic space of this lowest W-type.

From Corollary 5.3.3 and Lemma 3.1.2, we see that the $\star$-signature on each isotypic space $L(\mu)$ of a lowest W-type μ of $L(M, \sigma_0, \nu)$ is also independent of (dominant) ν. Moreover, this signature is given by the action of $t_{w_0} \circ \tau_L^\delta$ on $L(\mu)$. In particular, if w_0 is central in W (so $\delta = 1$) or if $\dim L(\mu) = 1$, the $\star$-form can be normalized so that the $\star$-signature on $L(\mu)$ equals

$$(-1)^{h(\mu)} \dim L(\mu),$$

where $h(\mu)$ is the lowest degree in which μ occurs in harmonic polynomials on V.

In fact, when the root system is simple, the only case when there exists a lowest W-type μ such that $\dim L(\mu) > 1$ is as follows. The root system is of type E_6 and the standard module is $X(M, \sigma_0, \nu)$, where M is of type D_4 and σ_0 is the subregular discrete series of D_4. In Theorem 5.3.2, this corresponds to a nilpotent element e of type $D_4(a_1)$ in E_6, whose centralizer has component group $A(e) = S_3$. The standard module $X(M, \sigma_0, \nu)$ has three lowest W-types denoted 80_s, 90_s, and 20_s with multiplicities 1, 2, and 1, respectively. One can compute the $\star$-form on the two-dimensional isotypic component of 90_s and find that the signature is $(1, -1)$, cf. [Ci2, page 458].

6. Jantzen filtration and hermitian Kazhdan-Lusztig polynomials

6.1. **Jantzen filtration.** We follow [Vo, section 3]. Let E be a complex vector space endowed with an analytic family $\langle\ ,\ \rangle_t$ of hermitian forms, such that $\langle\ ,\ \rangle_t$ are nondegenerate for $t \neq t_0$, close to t_0. The Jantzen filtration of E ([Ja]) is a filtration of vector subspaces

$$E = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_N = 0,$$

defined as follows. For every $n \geq 0$, $x \in E$ is in E_n if and only if there exists $\epsilon > 0$ and a polynomial function $f_x : (t_0 - \epsilon, t_0 + \epsilon) \to E$ with the properties:

(i) $f_x(t_0) = x$;
(ii) $\langle f_x(t), y\rangle_t$ vanishes at least to order n at $t = t_0$.

Set

$$\langle x, y\rangle^n = \lim_{t\to t_0} \frac{1}{(t-t_0)^n}\langle f_x(t), f_y(t)\rangle_t; \tag{6.1.1}$$

this definition is independent of f_x, f_y.

Theorem 6.1.1 (Jantzen [Ja, 5.1], cf. Vogan [Vo, Theorem 3.2, Corollary 3.6])**.** *The pairing $\langle\ ,\ \rangle^n$ is a hermitian form on E_n with radical E_{n+1}. In particular,*

(a) *Rad$\langle\ ,\ \rangle_0 = E_1$;*
(b) *$\langle\ ,\ \rangle^n$ is a nondegenerate hermitian form on E_n/E_{n+1}.*

Suppose (p_n, q_n) is the signature of $\langle\ ,\ \rangle^n$ on E_n/E_{n+1}. If (p^+, q^+) is the signature of $\langle\ ,\ \rangle_t$ for $t > t_0$ and (p^-, q^-) is the signature of $\langle\ ,\ \rangle_t$ for $t < t_0$, then

(c) *$p^+ = p^- + \sum_{n\ odd} p_n - \sum_{n\ odd} q_n$ and $q^+ = q^- + \sum_{n\ odd} q_n - \sum_{n\ odd} p_n$.*

Let $X = X(M, \sigma, \nu)$ be a standard module as in Theorem 4.1.1 with Langlands quotient $\overline{X} = L(M, \sigma, \nu)$. Consider a polynomial in t family of parameters ν_t, such that $\nu_1 = \nu$ and $X_t = X(M, \sigma, \nu_t)$ is irreducible for $t \neq 1$ in some small interval centered at 1. Suppose σ is a tempered module with real central character, and ν is real. By Corollary 5.1.3, every X_t admits a $\bullet$-invariant nondegenerate form $\langle\ ,\ \rangle_{t,\bullet}$ that we assume, as we may by Corollary 5.3.3, to be positive definite the lowest W-types of X_t. Notice that the W-structure and lowest W-types of X_t are independent on t. Therefore, we may think of the modules X_t as being realized on the same vector space E with the analytic family of hermitian forms $\langle\ ,\ \rangle_{t,\bullet}$, and the previous discussion applies. We have the Jantzen filtration of X:

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_N = 0, \tag{6.1.2}$$

with the following properties, cf. [Vo, Theorem 3.8]:

(a) the filtration (6.1.2) is a filtration by $\mathbb{H}$-modules;
(b) X_0/X_1 is the Langlands quotient $\overline{X}$;
(c) The form $\langle\ ,\ \rangle^n_\bullet$ on X_n/X_{n+1} is nondegenerate and $\bullet$-invariant. Let (p_n, q_n) be its signature. If $(p^\pm, q^\pm)$ is the signature of the $\langle\ ,\ \rangle_{\bullet,t}$ for $t < 1$, respectively $t > 1$, then $p^+ = p^- + \sum_{n\ \mathrm{odd}} p_n - \sum_{n\ \mathrm{odd}} q_n$ and $q^+ = q^- + \sum_{n\ \mathrm{odd}} q_n - \sum_{n\ \mathrm{odd}} p_n$.

6.2. **Kazhdan-Lusztig polynomials.** We recalled in Theorem 5.3.2 the geometric classification of standard and simple $\mathbb{H}$-modules. We record now the known results about the composition factors of a standard module. Retain the notation from section 5.3. In particular, let $s \in \mathfrak{h}$ be the semisimple parameter, and let $\mathsf{Irr}_s\mathbb{H}$ denote the irreducible $\mathbb{H}$-modules with central character $W \cdot s$. Denote

$$G(s) = \{g \in G : \mathrm{Ad}(g)s = s\}, \quad \mathfrak{g}_1(s) = \{x \in \mathfrak{g} : [s, x] = x\}. \tag{6.2.1}$$

It is well-known that $G(s)$ acts on $\mathfrak{g}_1(s)$ with finitely many orbits. Let $\mathcal{C}(s)$ denote the set of orbits. Theorem 5.3.2 can be rephrased as saying that there is a natural bijection:

$$\mathsf{Irr}_s\mathbb{H} \leftrightarrow \{(\mathcal{O}, \mathcal{L}) : \mathcal{O} \in \mathcal{C}(s),\ \mathcal{L} \text{ irr. local system of Springer type supported on } \mathcal{O}\}. \tag{6.2.2}$$

Let $L(\mathcal{O}, \mathcal{L})$ denote the irreducible $\mathbb{H}$-module and $X(\mathcal{O}, \mathcal{L})$ the corresponding standard module. In this setting, the Kazhdan-Lusztig conjectures take the following form.

Theorem 6.2.1 ([Lu2, Theorem 8.5])**.** *In the Grothendieck group of $\mathbb{H}$-modules,*

$$X(\mathcal{O},\mathcal{L}) = \sum_{(\mathcal{O}',\mathcal{L}')} P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(1)\ L(\mathcal{O}',\mathcal{L}'),$$

where

$$P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(q) = \sum_{i\geq 0} [\mathcal{L} : \mathcal{H}^{2i} IC(\overline{\mathcal{O}'},\mathcal{L}')|_{\mathcal{O}}] \cdot q^i; \tag{6.2.3}$$

here $\mathcal{H}^\bullet IC(\ )$ denote the cohomology groups of the intersection cohomology complex.

The polynomials $P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(q)$ can be computed using the algorithms in [Lu3]. In fact, [Lu3] computes the related v-polynomials

$$c_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(v) = v^{\dim \mathcal{O}' - \dim \mathcal{O}} \cdot P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}\left(\frac{1}{v^2}\right). \tag{6.2.4}$$

These polynomials enter in the Jantzen conjecture for $\mathbb{H}$.

Conjecture 6.2.2 (Jantzen conjecture)**.** *Let $X_0 \supset X_1 \supset \dots$ be the Jantzen filtration (6.1.2) of $X = X(\mathcal{O},\mathcal{L})$.*

(a) *For every $n \geq 0$, the $\mathbb{H}$-module X_n/X_{n+1} is semisimple.*

(b) *The multiplicity of the irreducible module $L(\mathcal{O},\mathcal{L})$ in X_n/X_{n+1} equals the coefficient of v^n in the polynomial $c_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(v)$ defined in (6.2.4), or equivalently,*

(b')
$$P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(q) = \sum_{n\geq 0} m_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n)\ q^{\frac{\dim \mathcal{O}' - \dim \mathcal{O} - n}{2}}, \tag{6.2.5}$$
where $m_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n)$ denotes the multiplicity of the irreducible module $L(\mathcal{O}',\mathcal{L}')$ in X_n/X_{n+1}.

6.3. **Hermitian Kazhdan-Lusztig polynomials.** As in the previous subsection, let $X = X(\mathcal{O},\mathcal{L})$ be a standard module with Jantzen filtration $X = X_0 \supset X_1 \supset \dots$. Suppose in addition that s is real, i.e., $s \in \mathfrak{h}_{\mathbb{R}}$.

Let

$$\mathsf{gr}X = \bigoplus_{n\geq 0} X_n/X_{n+1}$$

denote the associated graded $\mathbb{H}$-module. In section 6, we have defined a nondegenerate $\bullet$-invariant form $\langle\ ,\ \rangle_\bullet^n$ on each X_n/X_{n+1}. Let $\langle\ ,\ \rangle_\bullet^X$ be the direct sum form $\bigoplus_{n\geq 0}\langle\ ,\ \rangle_\bullet^n$ on $\mathsf{gr}X$.

By Corollary 5.3.3, every irreducible module $L(\mathcal{O}',\mathcal{L}')$ has a canonical $\bullet$-invariant form $\langle\ ,\ \rangle_\bullet^{(\mathcal{O}',\mathcal{L}')}$ which is positive definite on every lowest W-type. Fix such a form for every $L(\mathcal{O}',\mathcal{L}')$. Assuming the truth of Conjecture 6.2.2(a), the form $\langle\ ,\ \rangle_\bullet^n$ on X_n/X_{n+1} induces a nondegenerate form on the isotypic component of $L(\mathcal{O}',\mathcal{L}')$ in X_n/X_{n+1} whose signature is

$$(p_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n), q_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n));$$

of course, $p_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n) + q_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n) = m_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n)$. With this notation, we have

$$(X_n/X_{n+1}, \langle\ ,\ \rangle_\bullet^n) = \sum_{(\mathcal{O}',\mathcal{L}')} (p_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n) - q_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n)) \left(L(\mathcal{O}',\mathcal{L}'), \langle\ ,\ \rangle_\bullet^{(\mathcal{O}',\mathcal{L}')}\right). \tag{6.3.1}$$

Definition 6.3.1. Analogous to [ALTV], define the hermitian Kazhdan-Lusztig polynomials

$$P^h_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(q) = \sum_{n\geq 0}(p_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n) - q_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(n))\ q^{\frac{\dim \mathcal{O}'-\dim \mathcal{O}-n}{2}}. \quad (6.3.2)$$

From the definition, it is clear that

$$(\mathsf{gr}X, \langle\ ,\ \rangle^X_\bullet) = \sum_{(\mathcal{O}',\mathcal{L}')} P^h_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(1)\ \left(L(\mathcal{O}',\mathcal{L}'), \langle\ ,\ \rangle_\bullet^{(\mathcal{O}',\mathcal{L}')}\right). \quad (6.3.3)$$

The question is to compute the polynomials $P^h_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(q)$. We make the following conjecture, motivated by the main theorem of [ALTV].

Conjecture 6.3.2. *For every $(\mathcal{O},\mathcal{L})$, there exists an orientation number $\epsilon(\mathcal{O},\mathcal{L}) \in \{\pm 1\}$, such that*

$$P^h_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(q) = \epsilon(\mathcal{O},\mathcal{L})\epsilon(\mathcal{O}',\mathcal{L}')\ P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}(-q).$$

In the rest of the section, we present some examples in support of this conjecture and determine the explicit form of the orientation number in some cases. In particular, we prove Conjecture 6.3.2 in the case of regular central character, see Proposition 6.6.1.

6.4. **Regular central character.** Let $\mathbb{H}$ be a graded Hecke algebra with parameter function k. Recall the minimal principal series $X(\nu)$ with real parameter $\nu \in V^\vee$. Suppose ν is dominant, i.e., $(\alpha,\nu) > 0$ for all $\alpha \in R^+$. A basis of $X(\nu)$ is given by the $\mathbb{A}$-weight vectors $\{\mathcal{R}_x \otimes \mathbb{1}_\nu\}_{x\in W}$ from (3.3.4), and every $\mathbb{A}$-weight space has multiplicity 1. In particular, this means that every irreducible subquotient of $X(\nu)$ occurs with multiplicity 1.

If we normalize the form $\langle\ ,\ \rangle_\bullet$ on $X(\nu)$ so that

$$\langle \mathcal{R}_1 \otimes \mathbb{1}_\nu, \mathcal{R}_1 \otimes \mathbb{1}_\nu\rangle = 1,$$

by (3.3.5), we have

$$\langle \mathcal{R}_x \otimes \mathbb{1}_\nu, \mathcal{R}_x \otimes \mathbb{1}_\nu\rangle_\bullet = \prod_{\beta>0, x\beta<0} \frac{(\beta,\nu)-k_\beta}{(\beta,\nu)+k_\beta}. \quad (6.4.1)$$

In particular, one gets the following well-known result:

Lemma 6.4.1. *If ν is dominant, $X(\nu)$ is reducible if and only if there exists $\beta > 0$ such that $(\beta,\nu) = k_\beta$.*

Moreover, (6.4.1) allows us to determine easily the levels of the Jantzen filtration of $X(\nu)$. For every $x \in W$, set

$$\tau(x,\nu) = \{\beta > 0 : x\beta < 0 \text{ and } (\beta,\nu) = k_\beta\}. \quad (6.4.2)$$

Lemma 6.4.2. *Suppose ν is dominant. The n-th level in the Jantzen filtration (6.1.2) of $X(\nu)$ is*

$$X(\nu)_n = span\{\mathcal{R}_x \otimes \mathbb{1}_\nu : \tau(x,\nu) \geq n\},$$

where $\tau(x,\nu)$ is as in (6.4.2).

Proof. This is immediate from (6.4.1), since the order of zero of $\langle \mathcal{R}_x \otimes \mathbb{1}_\nu, \mathcal{R}_x \otimes \mathbb{1}_\nu\rangle_\bullet = \tau(x,\nu)$ and the form $\langle\ ,\ \rangle_\bullet$ is diagonal in the basis $\{\mathcal{R}_x \otimes \mathbb{1}_\nu\}$. □

6.5. Now suppose that the parameter function for the Hecke algebra is constant $k = 1$. We analyze first the case $\nu = \rho^\vee$. Consider the one-parameter family $X(\nu_t)$, $\nu_t = t\rho^\vee$, t close to 1. For every positive root β, the positive integer $(\beta, \rho^\vee)$ is the height of β. We have $(\beta, \rho^\vee) = 1$ if and only if β is a simple root. Then

$$\langle \mathcal{R}_x \otimes \mathbb{1}_{\nu_t}, \mathcal{R}_x \otimes \mathbb{1}_{\nu_t} \rangle_\bullet = \left(\frac{t-1}{t+1}\right)^{\tau_0(x)} \prod_{\beta>0, x\beta<0, (\beta,\rho^\vee)>1} \frac{t(\beta,\rho^\vee)-1}{t(\beta,\rho^\vee)+1} > 0. \quad (6.5.1)$$

where

$$\tau_0(x) = \{\alpha \text{ simple root} : x\alpha < 0\}. \quad (6.5.2)$$

This implies that the n-th level of the associated graded of the Jantzen filtration at $\rho^\vee$ is given by

$$X_n/X_{n+1} = \mathrm{span}\{\mathcal{R}_x \otimes \mathbb{1}_{\rho^\vee} : \tau_0(x) = n\}, \quad (6.5.3)$$

and n ranges from 0 to $|\Pi|$, the number of simple roots. The classification of simple $\mathbb{H}$-modules with central character $\rho^\vee$ is well-known: there are $2^{|\Pi|}$ simple $\mathbb{H}$-modules, one for each subset of the simple roots, and each one occurs with multiplicity 1 in $X(\rho^\vee)$. Formula (6.5.1) implies that each irreducible module contributes $+1$ to the $\bullet$-form in the level of $X(\rho^\vee)$ where it occurs.

One can analyze similarly the Jantzen filtration at $\rho^\vee$ for every standard module. Notice that the standard modules at $\rho^\vee$ are precisely of the form $\mathrm{Ind}_{\mathbb{H}_J}^{\mathbb{H}}(\mathsf{St} \otimes \mathbb{C}_{\nu_J})$, where $\nu_J = \rho^\vee - \rho_J^\vee$.

This is consistent with the geometric picture at $\rho^\vee$. There are $2^{|\Pi|}$ orbits of $G(\rho^\vee)$ on $\mathfrak{g}_1(\rho)$, each orbit is of the form $\oplus_{\alpha\in J}\mathbb{C}^\times \cdot X_\alpha$, for a unique $J \subset \Pi$; here X_α denote root vectors for $\alpha \in \Pi$. In particular, the closure relations of orbits coincide with the inclusion of subsets J, and the KL polynomials are $P_{J,J'}(q) = 1$ if $J \subset J'$, and 0 otherwise. In conclusion, at central character $\rho^\vee$, we have

$$P^h_{J,J'}(q) = P_{J,J'}(q) = \begin{cases} 1, & J \subset J', \\ 0, & \text{otherwise.} \end{cases} \quad (6.5.4)$$

6.6. Now suppose that s is an arbitrary regular dominant central character. The structure of the composition series at s reduces to a parabolic subalgebra as follows. Let

$$\Delta_s = \{\beta \in R^+ : (\beta, s) = 1\}.$$

Theorem 5.3.2 implies in this case that the simple $\mathbb{H}$-modules with central character s are in one-to-one correspondence with $G(s) = H$-orbits on $\mathfrak{g}_1(s) = \{x \in \mathfrak{g} : [s, x] = x\} = \mathrm{span}\{x_\beta : \beta \in \Delta_s\}$, where x_β is a root vector for β. There exists $w \in W$ such that $w\Delta_S \subset \Pi$, i.e., a subset of simple roots, so denote $w\Delta_s = \Pi_M$, for some Levi subgroup M.

Set $s' = w^{-1}s$. Then $s' = \rho_M^\vee + \nu$, where $(\alpha, \nu) = 0$ for all $\alpha \in \Pi_M$. It is equivalent to determine $G(s') = H$-orbits on $\mathfrak{g}_1(s) = \mathrm{span}\{x_\alpha : \alpha \in \Pi_M\}$, but this reduces the problem to thecase of composition series at $\rho_M^\vee$ in $\mathbb{H}_M$. Thus the orbits are in one-to-one correspondence with

$$\{J \subset \Pi_M\} \leftrightarrow \sum_{\alpha\in J} \mathbb{C}^\times \cdot x_\alpha =: \mathcal{O}^M(J).$$

Since every $\mathcal{O}^M(J)$ has smooth closure, as before, all KL polynomials are 0 or 1 depending on inclusion $J' \subset J$.

Fix $J \subset \Pi_M$. Suppose that we have a standard module $X(J,\nu_J) = \mathrm{Ind}_{\mathbb{H}_J}^{\mathbb{H}}(\mathsf{St}_J \otimes \mathbb{C}_{\nu_J})$ with Langlands quotient $L(J,\nu_J)$. We want to know the level and orientation number of $L(J,\nu_J)$. Since all $\mathbb{A}$-weights have multiplicity one, $L(J,\nu_J)$ is uniquely determined by the $\mathbb{A}$-weight

$$\lambda_{L(J,\nu_J)} := -\rho_J^\vee + \nu_J;$$

here ν_J is dominant with respect to $\Pi \setminus J$. Inside the minimal principal series $X(s)$, the $\mathbb{A}$-weight vector with weight $-\rho_J + \nu_J$ is of the form $\mathcal{R}_x \otimes \mathbb{1}_s$. Since $\mathcal{R}_x \otimes \mathbb{1}_s$ has $\mathbb{A}$-weight xs, it follows that

$$xs = \lambda_{L(J,\nu_J)}. \tag{6.6.1}$$

By (6.4.1), the form on $\mathcal{R}_x \otimes \mathbb{1}_s$ is

$$\prod_{\beta>0, x\beta<0} \frac{(\beta,s)-1}{(\beta,s)+1}.$$

The contribution of $\mathcal{R}_x \otimes \mathbb{1}_s$ to the hermitian form in the associated graded module for $X(s)$ is obtained by replacing s with st, $0<t<1$, and taking $\lim_{t\to 1}$. The sign is

$$\epsilon(L(J,\nu_J)) := (-1)^{\ell_0(x)}, \text{ where } \ell_0(x) = \#\{\beta>0 : 0<(\beta,s)<1 \text{ and } x\beta<0\}, \tag{6.6.2}$$

or equivalently,

$$\ell_0(x) = \#\{\beta>0 : x\beta<0 \text{ and } 0<(x\beta, \lambda_{L(J,\nu_J)})<1\} \tag{6.6.3}$$

In order to establish the truth of Conjecture 6.3.2 at regular central character, it remains to verify that the normalization of $\bullet$-form on $L(J,\nu_J)$ is given by the requirement that $\mathcal{R}_x \otimes \mathbb{1}_s$ be positive. This is indeed the case as follows. The canonical $\bullet$-form on a simple $\mathbb{H}$-module is normalized so that it is positive definite on all W-types. For $L(J,\nu_J)$ this is equivalent with the normalization which as $\nu_J \to \infty$ has the form positive definite on all of $L(J,\nu_J)$. But by Corollary 3.10.3, this is the normalization where the $\mathbb{A}$-weight vector corresponding to the leading weight $\lambda_{L(J,\nu_J)} = -\rho_J^\vee + \nu_J$ is positive. Thus:

Proposition 6.6.1. *Conjecture 6.3.2 holds in the case of regular central character with the orientation numbers given by (6.6.2).*

6.7. **Subregular orbit in B_2.** Consider the semisimple element $s=(1,0)$ in type B_2. There are three $G(s)$-orbits in $\mathfrak{g}_1(s)$, which we denote by 0, A_1, and $\widetilde{A}_1$ (the notation is compatible with the labeling of their G-saturations). The orbits have dimension 0, 2, and 3, respectively, and the closure ordering is the obvious total order. The local systems that enter are trivial for 0 and A_1, so we drop them from notation, and there are two local systems $\mathcal{L}_{\mathsf{triv}}$ and $\mathcal{L}_{\mathsf{sgn}}$ for $\widetilde{A}_1$. The matrix of polynomials P, computed in [Ci1] using the algorithms of Lusztig [Lu3], is in Table 6.7.

We only need to compute the Jantzen filtration and signatures for $X(A_1)$ and $X(0)$. For this, we do a computation with the intertwining operators and the W-structure of standard modules. There are 5 W-types, with the notation in terms of bipartitions as in [Ca]. The W-structure of the standard and the irreducible

TABLE 1. KL polynomials: $B_2,\ s=(1,0)$.

Dim.	0	2	3	3
Orbits	0	A_1	$(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})$	$(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}})$
0	1	1	1	q
A_1	0	1	1	0
$(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})$	0	0	1	0
$(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}})$	0	0	0	1

modules at $s=(1,0)$ is as follows (the $*$ indicates the lowest W-type):

$$\begin{aligned} &X(0)=\mathbb{C}[W(B_2)], && L(0)=2\times 0^*+1\times 1;\\ &X(A_1)=11\times 0+1\times 1+0\times 11, && L(A_1)=11\times 0^*;\\ &X(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})=L(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})=1\times 1^*+0\times 11;\\ &X(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}})=L(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}})=0\times 2^*. \end{aligned} \tag{6.7.1}$$

For the case A_1, we consider the induced module $X(A_1,(-1/2+\nu,1/2+\nu))=\mathrm{Ind}_{A_1}^{B_2}(\mathsf{St}\otimes\mathbb{C}_\nu)$, $\nu>0$, whose central character is $(-1/2+\nu,1/2+\nu)$. A direct calculation with the intertwining operator shows that at $\nu=1/2$, the Jantzen filtration is given by $L(A_1)$ at level 0, and $L(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})$ at level 1. The signature of the $\bullet$-form on each W-type for $0\le\nu<1/2$ is given by the parity of the lowest harmonic degree, and thus it is $+$ for 11×0 and $-$ for 1×1. The normalization of the $\bullet$-forms implies then that at $\nu=1/2$, the forms on level 1 are related by:

$$(X(A_1)_1,\langle\ ,\ \rangle_\bullet^1)=(L(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}}),\langle\ ,\ \rangle_\bullet^{(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})}), \tag{6.7.2}$$

and thus $P^h_{(A_1),(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})}(q)=q^{\frac{3-2-1}{2}}=1$.

For the case 0, we consider the minimal principal series $X(\nu_1,\nu_2)$, $0=\nu_2\le\nu_1\le 1$. The levels of the Jantzen filtration at $(1,0)$ are given by the order of zeros of the intertwining operator as follows: $L(0)$ in level 0, $L(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}})$ at level 1, $L(A_1)$ at level 2, and $L(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})$ at level 3. Using again that the signature of W-types for $0\le\nu_1<1$ is given by the parity of the lowest harmonic degree, we see that forms on levels 1–3 are related by:

$$\begin{aligned} (X(0)_1/X(0)_2,\langle\ ,\ \rangle_\bullet^1)&=-(L(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}}),\langle\ ,\ \rangle_\bullet^{(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}})});\\ (X(0)_2/X(0)_3,\langle\ ,\ \rangle_\bullet^2)&=(L(A_1),\langle\ ,\ \rangle_\bullet^{A_1});\\ (X(0)_3,\langle\ ,\ \rangle_\bullet^3)&=(L(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}}),\langle\ ,\ \rangle_\bullet^{(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})}). \end{aligned} \tag{6.7.3}$$

Thus, $P^h_{(0),(A_1)}=q^{\frac{2-0-2}{2}}=1$, $P^h_{(0),(\widetilde{A}_1,\mathcal{L}_{\mathsf{triv}})}=q^{\frac{3-0-3}{2}}=1$, and $P^h_{(0),(\widetilde{A}_1,\mathcal{L}_{\mathsf{sgn}})}=(-1)q^{\frac{3-0-1}{2}}=-q$.

In conclusion, for the subregular s in B_2, $P^h_{(\mathcal{O},\mathcal{L}),(\mathcal{O}'\mathcal{L}')}(q)=P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}'\mathcal{L}')}(-q)$.

6.8. **Subregular orbit in G_2.** We choose simple roots for G_2: $\alpha_s=\frac{1}{3}(2,-1,-1)$ and $\alpha_l=(-1,1,0)$ and fundamental coweights $\omega_1^\vee=(1,1,-2)$ and $\omega_2^\vee=(0,1,-1)$. Let s_1 and s_2 be the simple reflections corresponding to α_s and α_l, respectively. There are 6 irreducible Weyl group representations, which we label as 1_1 (the

trivial), 1_2 (the sign), 1_3 ($s_1 = 1$, $s_2 = -1$), 1_4 ($s_1 = -1$, $s_2 = 1$), 2_1 (the reflection representation), and $2_2 = 2_1 \otimes 1_3$.

Let s be one half of a neutral element for the subregular nilpotent orbit in G_2. In our coordinates, we choose $s = (0, 1, -1)$.

There are four $G(s)$-orbits on $\mathfrak{g}_1(s)$, labeled 0, A_1^l, A_1^s, $G_2(a_1)$, of dimensions 0, 2, 3, and 4, respectively. The local systems that enter for 0, A_1^l, A_1^s are trivial, but there are two local systems of Springer type for $G_2(a_1)$, that we denote $\mathcal{L}_{\mathsf{triv}}$ and $\mathcal{L}_{\mathsf{refl}}$. The matrix of polynomials P, computed in [Ci1], is in Table 6.8.

TABLE 2. KL polynomials: G_2, subregular s.

Dim.	0	2	3	4	4
Orbits	0	A_1^l	A_1^s	$(G_2(a_1), \mathcal{L}_{\mathsf{triv}})$	$(G_2(a_1), \mathcal{L}_{\mathsf{refl}})$
0	1	1	$q+1$	1	q
A_1^l	0	1	1	1	0
A_1^s	0	0	1	1	1
$(G_2(a_1), \mathcal{L}_{\mathsf{triv}})$	0	0	0	1	0
$(G_2(a_1), \mathcal{L}_{\mathsf{refl}})$	0	0	0	0	1

The W-structure of the standard and the irreducible modules at $s = (0, 1, -1)$ is as follows (the $*$ indicates the lowest W-type):

$$\begin{aligned}
&X(0) = \mathbb{C}[W(G_2)], && L(0) = 1_1^* + 2_1;\\
&X(A_1^l) = 1_3^* + 2_1 + 2_2 + 1_2, && L(A_1^l) = 1_3^*;\\
&X(A_1^s) = 2_2^* + 2_1 + 1_4 + 1_2, && L(A_1^s) = 2_2^*;\\
&X(G_2(a_1), \mathcal{L}_{\mathsf{triv}}) = L(G_2(a_1), \mathcal{L}_{\mathsf{triv}}) = 2_1^* + 1_2;\\
&X(G_2(a_1), \mathcal{L}_{\mathsf{refl}}) = L(G_2(a_1), \mathcal{L}_{\mathsf{refl}}) = 1_4^*.
\end{aligned} \tag{6.8.1}$$

For the case A_1^s, we consider the standard induced module $\mathrm{Ind}_{A_1^s}^{G_2}(\mathsf{St} \otimes \mathbb{C}_\nu)$, of central character $-\frac{1}{2}(2, -1, -1) + \nu(0, 1, -1)$. The relevant reducibility point is $\nu = \frac{1}{2}$. We can analyze the Jantzen filtration and signature of the forms in the same way as for B_2 and find:

$$\begin{aligned}
&(X(A_1^s)_0/X(A_1^s)_1, \langle\ ,\ \rangle_\bullet^0) = (L(A_1^s), \langle\ ,\ \rangle_\bullet^{(A_1^s)});\\
&(X(A_1^s)_1, \langle\ ,\ \rangle_\bullet^1) = (L(G_2(a_1), \mathcal{L}_{\mathsf{triv}}), \langle\ ,\ \rangle_\bullet^{(G_2(a_1), \mathcal{L}_{\mathsf{triv}})}) + (L(G_2(a_1), \mathcal{L}_{\mathsf{refl}}), \langle\ ,\ \rangle_\bullet^{(G_2(a_1), \mathcal{L}_{\mathsf{refl}})}),
\end{aligned} \tag{6.8.2}$$

and so $P^h_{(A_1^s), (G_2(a_1), \mathcal{L}_{\mathsf{triv}})}(q) = P^h_{(A_1^s), (G_2(a_1), \mathcal{L}_{\mathsf{refl}})}(q) = q^{\frac{4-3-1}{2}} = 1$.

For the case A_1^l, we consider the standard induced module $\mathrm{Ind}_{A_1^l}^{G_2}(\mathsf{St} \otimes \mathbb{C}_\nu)$, of central character $-\frac{1}{2}(-1, 1, 0) + \nu(1, 1, -2)$. The relevant reducibility point is $\nu = \frac{1}{2}$, where we find:

$$\begin{aligned}
&(X(A_1^l)_0/X(A_1^l)_1, \langle\ ,\ \rangle_\bullet^0) = (L(A_1^l), \langle\ ,\ \rangle_\bullet^{(A_1^l)});\\
&(X(A_1^l)_1/X(A_1^l)_2, \langle\ ,\ \rangle_\bullet^1) = (L(A_1^s), \langle\ ,\ \rangle_\bullet^{(A_1^s)});\\
&(X(A_1^l)_2, \langle\ ,\ \rangle_\bullet^2) = (L(G_2(a_1), \mathcal{L}_{\mathsf{triv}}), \langle\ ,\ \rangle_\bullet^{(G_2(a_1), \mathcal{L}_{\mathsf{triv}})}),
\end{aligned} \tag{6.8.3}$$

and so and so $P^h_{(A_1^l), (A_1^s)}(q) = q^{\frac{3-2-1}{2}} = 1$, and so $P^h_{(A_1^l), (G_2(a_1), \mathcal{L}_{\mathsf{triv}})}(q) = q^{\frac{4-2-2}{2}} = 1$, and $P^h_{(A_1^l), (G_2(a_1), \mathcal{L}_{\mathsf{refl}})}(q) = 0$.

Finally, we have the case 0, where we consider the minimal principal series with central character $\nu(0,1,-1)$. The relevant reducibility point is $\nu=1$. We compute the Janzten filtration and the signature of the forms using the normalized long intertwining operator on W-types. The only case where more care is needed is the W-type 2_2 appearing with multiplicity 2 which corresponds to the factor $L(A_1^s)$. The 2×2 matrix giving the operator on this isotypic space has determinant $\frac{(\frac{1}{2}-\nu)(1-\nu)^4}{(\frac{1}{2}+\nu)(1+\nu)^4}$ and trace $\frac{(1-\nu)(1+3\nu^2)}{(1+\nu)^3(\frac{1}{2}+\nu)}$. In particular, this implies that one copy of 2_2 (and hence of $L(A_1^s)$) occurs in level 1 of the Jantzen filtration and the other copy in level 3. For the signatures, we analyze the eigenvalues. We find the following structure of the filtration together with signatures:

$$\begin{aligned}
&(X(0)_0/X(0)_1, \langle\ ,\ \rangle_\bullet^0) = (L(0), \langle\ ,\ \rangle_\bullet^{(0)});\\
&(X(0)_1/X(0)_2, \langle\ ,\ \rangle_\bullet^1) = -(L(A_1^s), \langle\ ,\ \rangle_\bullet^{(A_1^s)});\\
&(X(0)_2/X(0)_3, \langle\ ,\ \rangle_\bullet^2) = (L(A_1^l), \langle\ ,\ \rangle_\bullet^{(A_1^l)}) - (L(G_2(a_1), \mathcal{L}_{\mathsf{refl}}), \langle\ ,\ \rangle_\bullet^{(G_2(a_1),\mathcal{L}_{\mathsf{refl}})});\\
&(X(0)_3/X(0)_4, \langle\ ,\ \rangle_\bullet^3) = (L(A_1^s), \langle\ ,\ \rangle_\bullet^{(A_1^s)});\\
&\qquad (X(0)_4, \langle\ ,\ \rangle_\bullet^4) = (L(G_2(a_1), \mathcal{L}_{\mathsf{triv}}), \langle\ ,\ \rangle_\bullet^{(G_2(a_1),\mathcal{L}_{\mathsf{triv}})}).
\end{aligned} \tag{6.8.4}$$

The corresponding hermitian KL polynomials are: $P^h_{(0),(A_1^l)} = q^{\frac{2-0-2}{2}} = 1$, $P^h_{(0),(A_1^s)} = (-1)q^{\frac{3-0-1}{2}} + q^{\frac{3-0-3}{2}} = 1-q$, $P^h_{(0),(G_2(a_1),\mathcal{L}_{\mathsf{triv}})} = q^{\frac{4-0-4}{2}} = 1$, $P^h_{(0),(G_2(a_1),\mathcal{L}_{\mathsf{refl}})} = (-1)q^{\frac{4-0-2}{2}} = -q$.

In conclusion, for the subregular s in G_2, $P^h_{(\mathcal{O},\mathcal{L}),(\mathcal{O}'\mathcal{L}')}(q) = P_{(\mathcal{O},\mathcal{L}),(\mathcal{O}'\mathcal{L}')}(-q)$, and Conjecture 6.3.2 is verified in this case.

7. Bernstein's projective modules

In this section, we explain how the $\bullet$-form for affine Hecke algebras appears naturally when the Iwahori-Hecke algebras are viewed as endomorphism algebras of the Bernstein projective modules [Be], see also [He]. The notation in this section is independent of the previous sections.

7.1. **Sesquilinear Forms.** Let V be a complex vector space,

$$V^h := \left\{\lambda : V \longrightarrow \mathbb{C}\ :\ \lambda(\alpha_1 v_1 + \alpha_2 v_2) = \overline{\alpha}_1\lambda(v_1) + \overline{\alpha}_2\lambda(v_2)\right\}.$$

A sesquilinear form is a bilinear form $\langle\cdot,\cdot\rangle$ which is linear in the first variable, conjugate linear in the second variable. This is the same as a complex linear map $\lambda : V \longrightarrow V^h$. The relation is

$$\langle v, w\rangle_\lambda = \lambda(v)(w).$$

Such a form is called nondegenerate if λ is injective. To any sesquilinear form λ there is associated $\lambda^h : V \subset (V^h)^h \longrightarrow V^h$, $\lambda^h(v)(w) := \overline{\lambda(w)(v)}$. The form is called symmetric, if $\lambda = \lambda^h$. A symmetric form is an inner product if $\lambda(v)(v) \geq 0$, with equality if and only if $v = 0$.

Let G be a reductive p-adic group. If (π, V) is a representation of G, then (π^h, V^h) is the representation defined as

$$\big(\pi^h(g)\lambda\big)(v) := \lambda(\pi(g^{-1})v).$$

7.2. **The projective $\mathcal{P}$.** Let M be a Levi subgroup of G. Denote by M_0 the intersection of the kernels of all the unramified characters of M. Let $\widetilde{\sigma}$ be a relative supercuspidal representation of M, σ_0 a supercuspidal constituent of $\widetilde{\sigma}\mid_{M_0}$.

Define

$$\begin{aligned} &\big(\sigma,\ V_\sigma = \mathrm{Ind}_{M_0}^M \sigma_0\big)_c && \text{induction with compact support,}\\ &\big(\Pi,\ \mathcal{P} = \mathrm{Ind}_P^G V_\sigma\big) && \text{normalized induction.}\end{aligned}$$

A typical element of σ is $\delta_{mM_0,v}$ with $m \in M/M_0$ and $v \in V_{\sigma_0}$. This is the *delta*-function supported on the coset mM_0 taking constant value v.

A typical element of $\mathcal{P}$ is given by $\delta_{UxP,\delta_{mM_0,v}}$ where $U \in G/P$ is a neighborhood of the identity, the function satisfies the appropriate transformation law under P on the right, and the value at x is $\delta_{mM_0,v}$.

If $\psi \in \mathrm{Hom}_G[\mathcal{P},\mathcal{P}]$, then $\psi^h \in \mathrm{Hom}_G[\mathcal{P}^h,\mathcal{P}^h]$. But $\mathcal{P}$ admits a $G-$invariant positive definite hermitian form, so while $\mathcal{P} \neq \mathcal{P}^h$, nevertheless there is an inclusion $\iota : \mathcal{P} \longrightarrow \mathcal{P}^h$. More precisely, if $\mathcal{P} = \mathrm{Ind}_P^G \sigma$, then the hermitian dual $\mathcal{P}^h$ is naturally isomorphic to $\mathrm{Ind}_P^G \sigma^h$. If $\lambda : G \longrightarrow V_\sigma^h$ is such that $\lambda(xp) = \sigma^h(p^{-1})\lambda(g)$, and $f : G \longrightarrow V$ is such that $f(gp) = \sigma(p^{-1})f(x)$, then the pairing is

$$\langle \lambda, f\rangle := \int_{G/P} \lambda(x)(f(x))dx.$$

When σ is unitary (or just has a nondegenerate form so that $\sigma \subset \sigma^h$), we get $\mathcal{P} \subset \mathcal{P}^h$ via

$$g \in \mathcal{P} \mapsto \lambda_g \in \mathcal{P}^h, \quad \lambda_g(f) = \int_{G/P} \langle f(x), g(x)\rangle dx \text{ for } f \in \mathcal{P}.$$

7.3. **Inner Product.** We recall two classical results.

Theorem 7.3.1 (Frobenius reciprocity, [Cas, Theorem 3.2.4])**.**

$$\mathrm{Hom}_G[V,\mathcal{P}] \cong \mathrm{Hom}_M[V_N, \sigma\delta_P^{-1}].$$

Theorem 7.3.2 (Second adjointness, [Be, Theorem 20])**.**

$$\mathrm{Hom}_G[\mathcal{P},V] \cong \mathrm{Hom}_M[\delta_{\overline{P}}^{-1}\sigma, V_{\overline{N}}] \cong \mathrm{Hom}_{M_0}[\sigma_0, \delta_P^{-1}V_{\overline{N}}].$$

Let $\overline{\mathcal{P}}$ be the module induced from σ from the opposite parabolic $\overline{P} := M\overline{N}$. The (second) adjointness theorem gives

$$\begin{aligned}\mathrm{Hom}_G[\mathcal{P},\mathcal{P}] &= \mathrm{Hom}_M[\delta_P^{-1}\sigma, \mathcal{P}_{\overline{N}}] = \mathrm{Hom}_{M_0}[\sigma_0, \delta_P^{-1}\mathcal{P}_{\overline{N}}],\\ \mathrm{Hom}_G[\overline{\mathcal{P}},\mathcal{P}] &= \mathrm{Hom}_M[\delta_{\overline{P}}^{-1}\sigma, \mathcal{P}_N] = \mathrm{Hom}_{M_0}[\sigma_0, \delta_P\mathcal{P}_N].\end{aligned}$$

Assume P and $\overline{P}$ are conjugate, and let $w_0 \in W$ be the shortest Weyl group element taking P to $\overline{P}$, stabilizing M and taking N to $\overline{N}$. Assume also that there is an

isomorphism $\tau_0 : (\sigma_0, V_{\sigma_0}) \longrightarrow (w_0 \circ \sigma_0, V_{\sigma_0})$. Extend it to $\tau : V_\sigma \longrightarrow V_\sigma$ by $\tau(\delta_{mM_0,v}) = \delta_{w_0(m)M_0,\tau(v)}$. Write $\widetilde{\tau}$ for the isomorphism

$$\widetilde{\tau} : \overline{\mathcal{P}} \longrightarrow \mathcal{P},$$
$$\widetilde{\tau}(f)(x) = \tau(f(xw_0)).$$

Thus given $\Phi, \Psi \in \mathrm{Hom}_G[\mathcal{P}, \mathcal{P}]$, then $\overline{\Phi} := \Phi \circ \widetilde{\tau} \in \mathrm{Hom}_G[\overline{\mathcal{P}}, \mathcal{P}]$, and they give rise to

$$\overline{\phi} \in \mathrm{Hom}_{M_0}[\sigma_0, \mathcal{P}_{\overline{N}}]$$
$$\psi \in \mathrm{Hom}_{M_0}[\sigma_0, \mathcal{P}_N].$$

According to Casselmann [Cas, Proposition 4.2.3], there is a nondegenerate pairing $\langle\ ,\ \rangle_{N,\overline{N}}$ between $\mathcal{P}_N$ and $\mathcal{P}_{\overline{N}}$. Given $v_1, v_2 \in V_{\sigma_0}$, we can form

$$\langle v_1, v_2 \rangle_{\Phi,\Psi} := \langle \overline{\phi}(v_1), \psi(v_2) \rangle_{N,\overline{N}}.$$

This pairing is invariant and sesquilinear, so there is a constant $m_{\Phi,\Psi}$ such that

$$\langle v_1, v_2 \rangle_{\Phi,\Psi} = m_{\Phi,\Psi} \langle v_1, v_2 \rangle_{\sigma_0}. \tag{7.3.1}$$

We define a sesquilinear pairing

$$\langle \Phi, \Psi \rangle := m_{\Phi,\Psi}. \tag{7.3.2}$$

7.4. We make the form (7.3.2) precise. Let K_ℓ be an open compact subgroup with an Iwasawa decomposition compatible with P, *i.e.* $K_\ell = K_\ell^- \cdot K_\ell^0 \cdot K_\ell^+$, invariant by w_0.

Let $x_0, y_0 \in V_{\sigma_0}^{K_\ell^0}$, and $x := \delta_{M_0,x_0}, y := \delta_{M_0,y_0}$. Then $\delta_{K_\ell^+\overline{P},x} \in \overline{\mathcal{P}}$ and $\delta_{K_\ell^- P,y} \in \mathcal{P}$. The isomorphism $\widetilde{\tau}$ takes $\delta_{K_\ell^+\overline{P},x}$ to $\delta_{K_\ell^+ w_0 P,\tau(x)}$. So

$$(x_0, y_0)_{\Phi,\Psi} := \langle \overline{\Phi}_{\overline{N}}\big(\delta_{K_\ell^+ w_0 P,\tau(x)}\big), \Psi_N\big(\delta_{K_\ell^- P,y}\big)\rangle_{\overline{N},N} = m_{\Phi,\Psi}\langle x_0, y_0\rangle_{\sigma_0}.$$

Here $\overline{\Phi}_{\overline{N}}$ and Ψ_N are the projection maps onto $\mathcal{P}_{\overline{N}}$ and $\mathcal{P}_N$ respectively.

Let $\Lambda \in A := Z(M)$ be such that it is regular on N and *contracts it.* Let $a(\Lambda)$ and $a(-\Lambda)$ be the K_ℓ double cosets of Λ and its inverse.

By Casselman [Cas, section 4] and Bernstein [Be, chapter III.3],

$$\mathcal{P}^{a(-\Lambda),K_\ell} \cong \mathcal{P}_{\overline{N}}^{K_\ell^0},$$
$$\mathcal{P}^{a(\Lambda),K_\ell} \cong \mathcal{P}_N^{K_\ell^0},$$

because $a(\Lambda)$ contracts K_ℓ^+. We conclude that

$$\delta_{K_\ell P,x} \in \mathcal{P}^{a(-\Lambda),K_\ell}, \qquad \text{so} \quad \Phi(\delta_{K_\ell P,x}) \in \mathcal{P}^{a(-\Lambda),K_\ell} \cong \mathcal{P}_{\overline{N}}^{K_\ell^0},$$
$$\delta_{K_\ell w_0 P,\tau_0 y} \in \mathcal{P}^{a(\Lambda),K_\ell}, \qquad \text{so} \quad \Psi(\delta_{K_\ell w_0 P,\tau_0 y}) \in \mathcal{P}^{a(\Lambda),K_\ell} \cong \mathcal{P}_N^{K_\ell^0}.$$

Proposition 7.4.1. *With the notation as in (7.3.1), $m_{\Phi,\Psi} = \overline{m_{\Psi,\Phi}}$. In other words, the sesquilinear form (7.3.2) is hermitian.*

Proof. Assume $\tau_0 \neq -Id$, or else use $-\tau_0$. Thus there is x_0 such that $\tau_0 x_0 = x_0$. Let $f_{w_0} := \delta_{K_l w_0 K_l}$. Then $f_{w_0}^* = f_{w_0}$, and

$$\Pi(f_{w_0})\delta_{K_l w_0 P,x} = \delta_{K_\ell P,x},$$
$$\Pi(f_{w_0})\delta_{K_\ell P,x} = \delta_{K_\ell w_0 P,x}.$$

Then

$$\begin{aligned} m_{\Phi,\Psi} < x_0, x_0 > &=< \Phi(\delta_{K_\ell P,x}), \Psi(\delta_{K_\ell w_0 P,x}) >= \\ &=< \Phi(\Pi(f_{w_0})\delta_{K_\ell w_0 P,x}), \Psi(\delta_{K_\ell w_0 P,x}) >= \\ &=< \Phi(\delta_{K_\ell w_0 P,x}), \Psi(\Pi(f_{w_0})\delta_{K_\ell w_0 P,x}) >= \\ &=< \Phi(\delta_{K_\ell w_0 P,x}), \Psi(\delta_{K_\ell P,x}) >= \\ &= \overline{< \Psi(\delta_{K_\ell P,x}), \Phi(\delta_{K_\ell w_0 P,x}) >} = \\ &= \overline{m_{\Psi,\Phi}} < x_0, x_0 > . \end{aligned}$$

□

7.5. For $a \in A$, let $\Theta_a \in \mathrm{Hom}_G[\mathcal{P}, \mathcal{P}]$ be given by

$$\Theta_a(\delta_{K_\ell g P,x}) = \delta_{K_\ell g P,\theta_a(x)}, \qquad \theta_a(x) := \theta_a(\delta_{mM_0,x_0}) = \delta_{maM_0,x_0}. \tag{7.5.1}$$

Proposition 7.5.1.

$$< \Phi, \Psi \circ \Theta_a >=< \Phi \circ \Theta_a, \Psi > .$$

Proof. There is $f_a \in \mathcal{H}(K_\ell \backslash G / K_\ell)$ (namely $\delta_{K_\ell a K_\ell}$) such that $\Theta_a(\delta_{K_\ell P,x}) = \Pi(f_a)(\delta_{K_\ell P,x})$. Then use the fact that $f_a^* = f_{a^{-1}}$ for $a \in A^+$ *dominant*. □

7.6. **Digression about the intertwining operator.** Let $J : \mathcal{P} \longrightarrow \mathcal{P}$ be given by the formula

$$Jf(x) := \int_N \tau_0 f(xnw_0)\, dn = \int_{\overline{N}} \tau_0 f(xw_0\overline{n})\, d\overline{n}. \tag{7.6.1}$$

This should be considered as a formal expression. When you specialize to a value $\nu \in \widehat{A}$, the split part of the center of M, J will have poles.

Recall the inner product on $\mathcal{P}$,

$$\langle f_1, f_2 \rangle := \int_{K_0} \langle f_1(k), f_2(k) \rangle\, dk.$$

Proposition 7.6.1.

$$< Jf_1, f_2 >=< f_1, Jf_2 > .$$

Proof.

$$< f_1, Jf_2 >= \int_{K_0} < f_1(k), \int_{\overline{N}} \tau_0 f_2(kw_0\overline{n})\, d\overline{n} >\ dk. \tag{7.6.2}$$

We can move w_0 and τ_0 to the other side:

$$< f_1, Jf_2 >= \int_{K_0} < \tau_0 f_1(kw_0), \int_{\overline{N}} f_2(k\overline{n})\, d\overline{n} >\ dk. \tag{7.6.3}$$

Write $\overline{n} = \kappa(\overline{n}) \cdot n(\overline{n}) \cdot m(\overline{n})$. So

$$\begin{aligned} &< f_1, Jf_2 >= \int_{K_0} < \tau_0 f_1(kw_0), \int_{\overline{N}} \sigma(m(\overline{n})^{-1}) f_2(k\kappa(\overline{n}))\, d\overline{n} >\ dk = \\ &= \int_{K_0} < \int_{\overline{N}} \sigma(m(\overline{n})) \tau_0 f_1(k\kappa(\overline{n})^{-1} w_0)\, d\overline{n}, f_2(k) >\ dk. \end{aligned} \tag{7.6.4}$$

Since $\kappa(\overline{n}) = \overline{n} \cdot m(\overline{n})^{-1} \cdot n(\overline{n})^{-1}$, we conclude $\kappa(\overline{n})^{-1} = n(\overline{n}) \cdot m(\overline{n}) \cdot \overline{n}^{-1}$. So

$$\begin{aligned} &< f_1, Jf_2 > = \\ &\quad = \int_{K_0} < \int_{\overline{N}} \sigma(m(\overline{n}))\tau_0 f_1(kn(\overline{n})m(\overline{n})\overline{n}^{-1}w_0)\, d\overline{n}, f_2(k) > \, dk = \\ &\quad = \int_{K_0} < \int_{\overline{N}} \tau_0 f_1(kn(\overline{n}))\, d\overline{n}, f_2(k) > \, dk \end{aligned} \tag{7.6.5}$$

because $\sigma(m(\overline{n}))$ is conjugated by w_0 but then flipped back by τ_0, and then cancels $\sigma(m(\overline{n}))$. Finally

$$Jf_1 = \int_{\overline{N}} \tau_0 f_1(kn(\overline{n})w_0)\, d\overline{n}$$

follows from the fact that $\overline{n} \mapsto w_0 n(\overline{n}) w_0^{-1}$ is an isomorphism with trivial Jacobian. □

7.7. Assume from now on that G is a split p-adic group. Let $P = B = AN$ be a Borel subgroup. Let K_0 be the hyperspecial maximal compact subgroup, and $K_1 \subset \mathcal{I} \subset K_0$ be an Iwahori subgroup. It has an Iwasawa decomposition $\mathcal{I} = \mathcal{I}^- \cdot A_0 \cdot \mathcal{I}^+$. Furthermore, $G = KB = \cup \mathcal{I} w B$ disjoint union where $w \in W$.

We consider the case of the trivial representation of $A_0 := K_0 \cap A$, $\sigma_0 = \mathsf{triv}$, i.e, this is the case of representations with $\mathcal{I}-$fixed vectors. Let $\mathcal{H} = \mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ be the Iwahori-Hecke algebra of compactly supported smooth $\mathcal{I}$-biinvariant functions with convolution with respect to a Haar measure.

Proposition 7.7.1. *In the Iwahori-spherical case, the algebra* $\mathrm{Hom}[\mathcal{P}, \mathcal{P}]$ *is naturally isomorphic to the opposite algebra to* $\mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$.

Proof. Recall

$$\mathrm{Hom}_G[\mathcal{P}, \mathcal{P}] \cong \mathrm{Hom}_{A_0}[\sigma_0, \mathcal{P}_{\overline{N}}] \cong \mathcal{P}_{\overline{N}}^{A_0} \cong \mathcal{P}^{\mathcal{I}}.$$

The element $\phi_1 = \delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}$ is in $\mathcal{P}^{\mathcal{I}}$, and it generates $\mathcal{P}$. So any $\Phi \in \mathrm{Hom}_G[\mathcal{P}, \mathcal{P}]$ is determined by its value on ϕ_1. Furthermore, $\Phi(\phi_1) \in \mathcal{P}^I$.

Conversely, $\phi \in \mathrm{Hom}_{A_0}[\sigma_0, \mathcal{P}_{\overline{N}}] \cong \mathcal{P}_{\overline{N}}^{A_0} \cong \mathcal{P}^{\mathcal{I}}$ gives rise to $\Phi \in \mathrm{Hom}_G[\mathcal{P}, \mathcal{P}]$ by the relation

$$\Phi(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}) = \phi.$$

The map

$$h \in \mathcal{H} \mapsto \Pi(h)\big(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}\big)$$

is an isomorphism between $\mathcal{H}$ and $\mathcal{P}^{\mathcal{I}}$. Let $h_\psi \in \mathcal{I}$ be the element in $\mathcal{H}$ corresponding to ψ. Then if $\Phi(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}) = \phi$,

$$\Phi[\psi] = \Phi[\Pi(h_\psi)(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}})] = \Pi(h_\phi)\Phi[\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}] = \Pi(h_\psi)\phi.$$

Now let $\phi_1, \phi_2 \in \mathcal{P}_{\overline{N}}^{A_0}$. Then

$$\begin{aligned} &(\Phi_1 \circ \Phi_2)(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}) = \Phi_1[\Pi(h_{\phi_2})(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}})] = \Pi(h_{\phi_2})\Phi_1[\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}] = \\ &= \Pi(h_{\phi_2})\Pi(h_{\phi_1})(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}) = \Pi(h_{\phi_1}) \cdot \Pi(h_{\phi_2})(\delta_{\mathcal{I}^- B, \delta_{A_0, \mathbb{1}}}). \end{aligned}$$

□

Remark 7.7.2. The opposite algebra to the Iwahori-Hecke algebra is isomorphic to itself, e.g.,

$$T \circ_{opp} \theta = \theta^{-1} \circ_{opp} T + (q-1)\frac{\theta - \theta^{-1}}{1-\theta_{-\alpha}}$$

is equivalent to

$$\theta \cdot T = T \cdot \theta^{-1} + (q-1)\frac{\theta - \theta^{-1}}{1-\theta_{-\alpha}}.$$

7.8. The operators J_α are defined analogously to J for each simple root, integration is along the root subgroup N_α. The operators satisfy the formula analogous to 7.6.1. By specializing to $\nu \in \widehat{A}$ unramified, we can prove the following result. Define

$$F(\Theta) = (q-1)\frac{1}{1-\Theta^{-1}}, \tag{7.8.1}$$

and write F_α for $F(\Theta_\alpha)$.

Theorem 7.8.1.

$$T_\alpha := J_\alpha - F_\alpha \in \mathrm{Hom}_G[\mathcal{P},\mathcal{P}]. \tag{7.8.2}$$

T_α and Θ_α form a set of generators of $\mathrm{Hom}[\mathcal{P},\mathcal{P}]$ *and satisfy the defining relations in the Bernstein-Lusztig presentation (*[Lu1]*) for the Iwahori-Hecke algebra.*

Sketch of proof. Because the group is split, this reduces to a calculation in $SL(2)$. The operator J has a term which is a rational function in Θ_α with $1-\Theta_{-\alpha}$ in the denominator, and subtracting F_α removes the singularity.

□

Remark 7.8.2. For a classical p-adic group G and any Bernstein projective module $\mathcal{P}$, it is shown in [He] that a generalization of Theorem 7.8.1 holds, namely, $\mathrm{End}_G[\mathcal{P}]$ is naturally isomorphic to an extended affine Hecke algebra with unequal parameters.

Proposition 7.8.3. *There is $f_\alpha \in \mathcal{H}(K_\ell \backslash G / K_\ell)$ and $\tau_\alpha : \sigma \longrightarrow \sigma$ such that*

$$< \Phi(\delta_{K_\ell w_0 P,x}), \Psi(T_\alpha(\delta_{K_\ell P,y})) > = < \Phi(\delta_{K_\ell w_0 P,x}), \Psi\big(\Pi(f_\alpha)(\delta_{K_\ell P,\tau_\alpha(y)})\big) > . \tag{7.8.3}$$

Proof. This follows from the formula of J_α as an integral. We want $T_\alpha(\delta_{K_\ell P,y}) = \Pi(f_\alpha)(\delta_{K_\ell P,y})$.

For $SL(2)$, let K_ℓ be the usual congruence subgroup. Let $a := \begin{bmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{bmatrix}$. Then $\mathcal{I}B = \mathcal{I}^- B$, and $a^{-\ell} K_\ell B a^\ell = \mathcal{I}^- B$. Thus

$$\Pi(a^{-\ell})(\delta_{K_\ell B,\alpha}) = \delta_{\mathcal{I}a^{-\ell}B,a^\ell\alpha} = \Pi(\delta_{\mathcal{I}a^{-\ell}\mathcal{I}})\delta_{\mathcal{I}B,\alpha}.$$

T_α commutes with $\Pi(\delta_{\mathcal{I}a^{-\ell}\mathcal{I}})$ and $\Pi(a^{-\ell})$, and is computable on $\delta_{\mathcal{I}B,\alpha}$. it can be written as convolution with a $\mathcal{I}-$biinvariant function. The conclusion of the calculation is that $T_\alpha(\delta_{K_\ell B,\alpha})$ can be expressed as convolution with an element $\mathcal{T}_\alpha \in \mathcal{H}(\mathcal{I}\backslash G/\mathcal{I})$ and composition with a $\Pi(a^{\pm\ell})$. We can then argue as in Proposition 7.5.1 to conclude that

$$\langle \Phi, \Psi \circ T_\alpha \rangle = \langle \Phi \circ T_\alpha, \Psi \rangle. \tag{7.8.4}$$

□

We summarize the results.

Theorem 7.8.4. *In the case of Iwahori fixed vectors, unramified principal series,* $\mathcal{H} := \mathrm{Hom}[\mathcal{P}, \mathcal{P}]$ *inherits a natural star operation* $\bullet$ *from the unitary structure of* $\mathcal{P}$ *satisfying*

$$\langle \Phi, \Psi \circ \mathcal{R} \rangle = \langle \Phi \circ \mathcal{R}^\bullet, \Psi \rangle, \quad \Phi, \Psi, \mathcal{R} \in \mathcal{H}.$$

In particular,

$$T_\alpha^\bullet = T_\alpha, \quad \Theta^\bullet = \Theta.$$

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