# HARMONIC SPINORS ON HOMOGENEOUS SPACES

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Abstract. Let $G$ be a compact, semi-simple Lie group and $H$ a maximal rank reductive subgroup. The irreducible representations of $G$ can be constructed as spaces of harmonic spinors with respect to a Dirac operator on the homogeneous space $G/H$ twisted by bundles associated to the irreducible, possibly projective, representations of $H$. Here, we give a quick proof of this result, computing the index and kernel of this twisted Dirac operator using a homogeneous version of the Weyl character formula noted by Gross, Kostant, Ramond, and Sternberg, as well as recent work of Kostant regarding an algebraic version of this Dirac operator.

## 1. Introduction

Recently Gross, Kostant, Ramond, and Sternberg discovered a generalization of the Weyl character formula, describing the character of any irreducible representation $V_\lambda$ of a semi-simple Lie algebra $\mathfrak{g}$ in terms of the characters of a multiplet of irreducible representations $U_\mu$ of a maximal rank reductive Lie subalgebra $\mathfrak{h}$. Kostant then showed that these $\mathfrak{h}$-representations $U_\mu$ could be constructed as the kernel of a certain Dirac operator acting on $V_\lambda \otimes \mathbb{S}$, where $\mathbb{S}$ is the complex spin representation associated to the complement of $\mathfrak{h}$ in $\mathfrak{g}$. These results are summarized in Section 2, and a thorough exposition of the subject can be found in [6].

After a brief review of homogeneous differential operators in Section 3, we turn this argument on its head in Section 4. Let $G$ be the compact, simply connected Lie group with Lie algebra $\mathfrak{g}$, and let $H$ be the Lie subgroup with Lie algebra $\mathfrak{h}$. If the homogeneous space $G/H$ is a spin manifold, then we show that the index of a standard geometric Dirac operator on $G/H$ twisted by the homogeneous vector bundle induced by a representation $U_\mu$ of $H$ is, up to sign, the corresponding representation $V_\lambda$ of $G$. Furthermore, using the geometric version of Kostant's Dirac operator, we explicitly construct this representation $V_\lambda$ as the space of twisted harmonic spinors. If $G/H$ is not spin, or if the twisted spinors do not come from a suitable $\mathrm{Spin}^c$ structure on $G/H$, then we instead carry out these constructions using the analogous $\mathfrak{h}$-equivariant operators upstairs on the Lie group $G$.

## 2. The Homogeneous Weyl Formula

Let $\mathfrak{g}$ be a compact, semi-simple Lie algebra, and let $\mathfrak{h}$ be a reductive Lie subalgebra of maximal rank in $\mathfrak{g}$. Since $\mathfrak{h}$ has the same rank as $\mathfrak{g}$, any Cartan subalgebra of $\mathfrak{h}$ is likewise a Cartan subalgebra of $\mathfrak{g}$, and the roots of $\mathfrak{h}$ are then a subset of the roots of $\mathfrak{g}$. The Weyl group $W_\mathfrak{g}$ of $\mathfrak{g}$ acts simply transitively on the Weyl chambers for $\mathfrak{g}$, each of which is contained inside a Weyl chamber for $\mathfrak{h}$. Choosing a set of positive roots for $\mathfrak{g}$ also determines a positive root system for $\mathfrak{h}$, and we define

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$C \subset W_{\mathfrak{g}}$ to be the subset of elements that map the positive Weyl chamber for $\mathfrak{g}$ into the positive Weyl chamber for $\mathfrak{h}$.

Let $\rho_{\mathfrak{g}}$ and $\rho_{\mathfrak{h}}$ denote half the sum of the positive roots of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Equivalently, $\rho$ is also the sum of the fundamental weights, which lie on the boundary of the positive Weyl chamber. For any dominant weight $\lambda$ of $\mathfrak{g}$, the weight $\lambda + \rho_{\mathfrak{g}}$ lies in the interior of the positive Weyl chamber for $\mathfrak{g}$. For any $c \in C$, the weight $c(\lambda + \rho_{\mathfrak{g}})$ then lies in the interior of the positive Weyl chamber for $\mathfrak{h}$, and so

$$c \bullet \lambda := c\,(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{h}}$$

is a dominant weight for $\mathfrak{h}$. The $\rho$-shift insures that each of the weights $c \bullet \lambda$ for $c \in C$ is distinct. Note that although every dominant weight of $\mathfrak{g}$ corresponds to a distinct multiplet of dominant weights of $\mathfrak{h}$, not every dominant weight of $\mathfrak{h}$ corresponds to a dominant weight of $\mathfrak{g}$. In particular, if $\mu$ is a dominant weight of $\mathfrak{h}$ such that $\mu + \rho_{\mathfrak{h}}$ lies on the boundary of a Weyl chamber for $\mathfrak{g}$, then $\mu$ is not of the form $c \bullet \lambda$ for any dominant weight $\lambda$ of $\mathfrak{g}$. Such orphan weights will behave as exceptional cases in the results of Section 4.

Putting an ad-invariant inner product on $\mathfrak{g}$, let $\mathfrak{p}$ denote the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. The adjoint action of $\mathfrak{g}$ then restricts to give an orthogonal action of $\mathfrak{h}$ on $\mathfrak{p}$, and this action lifts to the complex spin representation $\mathbb{S}$ associated to $\mathfrak{p}$. Furthermore, since $\mathfrak{h}$ is of maximal rank in $\mathfrak{g}$, the complement $\mathfrak{p}$ is even dimensional, and so the spin representation decomposes as the sum $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ of two distinct half-spin representations. To specify the sign convention, note that the weight space of $\mathbb{S}$ with highest weight $\rho_{\mathfrak{g}} - \rho_{\mathfrak{h}}$ is one dimensional, and take $\mathbb{S}^+$ to be the half-spin representation containing that highest weight space.

In [3], Gross, Kostant, Ramond, and Sternberg prove

**Theorem 1** (Homogeneous Weyl Formula)**.** *Let $V_\lambda$ be the irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$. The following identity holds in the representation ring $R(\mathfrak{h})$:*

$$V_\lambda \otimes \mathbb{S}^+ - V_\lambda \otimes \mathbb{S}^- = \sum_{c \in C} (-1)^c\, U_{c\bullet\lambda}, \tag{1}$$

*where $V_\lambda$ on the left side is viewed as a representation of $\mathfrak{h}$ by restriction, and $U_{c\bullet\lambda}$ denotes the irreducible representation of $\mathfrak{h}$ with highest weight $c \bullet \lambda$.*

If $\mathfrak{h}$ is a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, then $C$ is the full Weyl group $W_{\mathfrak{g}}$, which acts on weights as $w \bullet \lambda = w\,(\lambda + \rho_{\mathfrak{g}})$, and (1) becomes the familiar Weyl character formula

$$\chi(V_\lambda) \otimes \left(\mathbb{S}^+_{\mathfrak{g}/\mathfrak{t}} - \mathbb{S}^-_{\mathfrak{g}/\mathfrak{t}}\right) = \sum_{w \in W_{\mathfrak{g}}} (-1)^w\, w(e^{\lambda + \rho_{\mathfrak{g}}}). \tag{2}$$

The general form (1) of this identity can be derived from the Weyl character formula by dividing both sides of (2) by the character of the virtual spin representation $\mathbb{S}^+_{\mathfrak{h}/\mathfrak{t}} - \mathbb{S}^-_{\mathfrak{h}/\mathfrak{t}}$ associated to the complement of $\mathfrak{t}$ in $\mathfrak{h}$.

In [4], Kostant constructs a Dirac operator $\partial\!\!\!/ : V_\lambda \otimes \mathbb{S}^+ \to V_\lambda \otimes \mathbb{S}^-$. Since the domain and range are finite dimensional, the $\mathfrak{h}$-index of any such operator is automatically given by (1). However, Kostant's Dirac operator is unique in that it also respects the sign decomposition given by the right side of (1), satisfying

$$\operatorname{Ker} \partial\!\!\!/ = \sum_{(-1)^c = +1} U_{c\bullet\lambda}, \quad \operatorname{Ker} \partial\!\!\!/^* = \sum_{(-1)^c = -1} U_{c\bullet\lambda}. \tag{3}$$

This Dirac operator is formally self-adjoint, so the adjoint of the Dirac operator $\partial\!\!\!/^* : V_\lambda \otimes \mathbb{S}^- \to V_\lambda \otimes \mathbb{S}^+$ can be viewed as the same operator acting on the opposite half-spin representation. This operator thus provides a mechanism by which to extract the multiplet of $\mathfrak{h}$-representations $U_{c\bullet\lambda}$ directly from the associated $\mathfrak{g}$-representation $V_\lambda$.

In its most abstract form, Kostant's Dirac operator can be viewed as an element of the non-abelian Weil algebra $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ (see [1]), where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, and $\mathrm{Cl}(\mathfrak{p})$ is the Clifford algebra of $\mathfrak{p}$. Choosing a basis $\{X_i\}$ of $\mathfrak{p}$ and letting $\{X_i^*\}$ be the dual basis satisfying $\langle X_i, X_j^* \rangle = \delta_{ij}$, Kostant defines his Dirac operator to be the element

$$\partial\!\!\!/ := \sum_i X_i \otimes X_i^* + 1 \otimes v, \tag{4}$$

where $v \in \mathrm{Cl}(\mathfrak{p})$ is the image of the fundamental 3-form $\omega \in \Lambda^3(\mathfrak{p}^*)$,

$$\omega(X, Y, Z) = \langle X, [Y, Z] \rangle, \tag{5}$$

under the Chevalley identification $\Lambda^*(\mathfrak{p}^*) \to \mathrm{Cl}(\mathfrak{p})$. Now, any representation $r$ of $\mathfrak{g}$ on $V_\lambda$ extends to a homomorphism $r : U(\mathfrak{g}) \to \mathrm{End}(V_\lambda)$, and the Clifford action on the spin representation yields a homomorphism $c : \mathrm{Cl}(\mathfrak{p}) \to \mathrm{End}(\mathbb{S})$ with the odd part of the Clifford algebra interchanging $\mathbb{S}^+$ and $\mathbb{S}^-$. Combining these maps gives a representation of the non-abelian Weil algebra on the tensor product,

$$r \otimes c : U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}) \to \mathrm{End}(V_\lambda \otimes \mathbb{S}),$$

which takes the Dirac element $\partial\!\!\!/$ to an operator in $\mathrm{Hom}(V_\lambda \otimes \mathbb{S}^+, V_\lambda \otimes \mathbb{S}^-)$ as desired.

To compute the kernel of this Dirac operator, Kostant expressed its square as a sum of quadratic Casimir operators for the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. When restricted to a subspace of $V_\lambda \otimes \mathbb{S}$ transforming like the representation $U_\mu$ under the diagonal right $\mathfrak{h}$-action, the square of the Dirac operator becomes

$$\partial\!\!\!/^2|_{U_\mu} = \|\lambda + \rho_\mathfrak{g}\|^2 - \|\mu + \rho_\mathfrak{h}\|^2. \tag{6}$$

Since the Weyl group acts by isometries, all of the weights $\mu$ of $\mathfrak{h}$ satisfying $\mu + \rho_\mathfrak{h} = c(\lambda + \rho_\mathfrak{g})$ must have the same $\rho$-shifted norm as $\lambda$. In fact, the weights $\mu = c \bullet \lambda$ are precisely those for which the expression (6) vanishes. Furthermore, Kostant showed that each of the representations $U_{c\bullet\lambda}$ occurs exactly once in the decomposition of $V_\lambda \otimes \mathbb{S}$, and it follows that the kernel of the Dirac operator must be of the form (3).

## 3. Homogeneous Differential Operators

Looking at the discussion of the previous section from a geometric, rather than algebraic viewpoint, let $G$ be a compact, semi-simple Lie group, and let $H$ be a reductive Lie subgroup of maximal rank. Without loss of generality, also assume that $G$ is simply connected. The Hilbert space $L^2(G)$ of functions on $G$ admits a $G \times G$ action, with left and right components

$$l(h)f : g \mapsto f(h^{-1}g), \quad r(h)f : g \mapsto f(gh) \tag{7}$$

for any function $f \in L^2(G)$ and elements $g, h \in G$. At the Lie algebra level, the infinitesimal left and right actions on functions are given by differentiation with respect to the right-invariant and left-invariant vector fields respectively. More precisely, for any $X \in \mathfrak{g}$ we have

$$r(X)f : g \mapsto \partial_t f(g\, e^{tX})|_{t=0} = (X_L f)(g), \tag{8}$$

where $X_L$ is the left-invariant vector field taking value $X$ at the identity. Similarly, we obtain $l(X)f = X_R f$, where $X_R$ is the corresponding right-invariant vector field.

Given an $\mathfrak{h}$-representation $M$ with $\mathfrak{h}$-action $m$, we view the Hilbert space $L^2(G)\otimes M$ of sections of the trivial bundle $G\times M$ as a representation of $\mathfrak{g}\oplus\mathfrak{h}$, taking the left $\mathfrak{g}$-action $l\otimes 1$ on the $L^2(G)$ component and the diagonal right $\mathfrak{h}$-action $r\otimes 1 + 1\otimes m$. This apparent asymmetry between the left and right actions is a consequence of the convention of trivializing vector bundles by left translation. We say that a linear differential operator $D : L^2(G)\otimes M \to L^2(G)\otimes N$ on $G$ is *homogeneous* relative to $\mathfrak{h}$ if it commutes with the $\mathfrak{g}\oplus\mathfrak{h}$ actions on its domain and range.

If the $\mathfrak{h}$-actions on $M$ and $N$ exponentiate to give single-valued actions of the Lie group $H$, then the representations $M$ and $N$ induce equivariant $G$-bundles $G\times_H M$ and $G\times_H N$ over the homogeneous space $G/H$. The sections of these bundles correspond to the right $H$-equivariant functions on $G$ taking values in $M$ and $N$ respectively, giving

$$L^2(G\times_H M) \cong \big(L^2(G)\otimes M\big)^H, \quad L^2(G\times_H N) \cong \big(L^2(G)\otimes N\big)^H.$$

If $D$ is homogeneous relative to $H$, then it preserves the $H$-invariance of both its domain and range, and so it restricts to give a $G$-equivariant operator

$$D_0 : L^2(G\times_H M) \to L^2(G\times_H N).$$

In general, such a $G$-equivariant linear differential operator on $G/H$ is called a *homogeneous differential operator*. However, the notion of homogeneous operators given above is more flexible, as it is not limited to bundles that descend to the quotent $G/H$, and we are free to work upstairs on $G$. In fact, a homogeneous operator on $G$ defines not just one but rather an entire family of twisted homogeneous operators on $G/H$ indexed by the irreducible representations $U_\mu$ of $H$, obtained by restricting to the subspaces of the domain and range transforming like the dual representations $U_\mu^*$. Specifically, the family of operators $D_\mu$ are given by

$$\begin{aligned} D_\mu : \operatorname{Hom}_{\mathfrak{h}}\big(U_\mu^*, L^2(G)\otimes M\big) &\longrightarrow \operatorname{Hom}_{\mathfrak{h}}\big(U_\mu^*, L^2(G)\otimes N\big) \\ L^2\big(G\times_H (M\otimes U_\mu)\big) &\longrightarrow L^2\big(G\times_H (N\otimes U_\mu)\big). \end{aligned}$$

Also, even if the $\mathfrak{h}$-actions on $M$ and $N$ fail to exponentiate to the Lie group $H$, the operators $D_\mu$ nevertheless descend to well-defined operators on $G/H$ provided that the tensor products $M\otimes U_\mu$ and $N\otimes U_\mu$ are indeed true representations of $H$. In other words, twisting by a projective representation with the opposite cocycle kills the obstruction.

In [2], Bott showed that the index of a homogeneous differential operator, like that of a finite dimensional operator, depends only on the domain and range and not on the operator itself. Since the domain and range are now infinite dimensional representations of $G$, Bott viewed them as elements of the completion $\hat{R}(G)$ of the representation ring of $G$, consisting of all possibly infinite formal sums $\sum_\lambda a_\lambda [V_\lambda]$ with integer coefficients indexed by the equivalence classes of irreducible representations of $G$. In this notation, the space of sections of a homogeneous vector bundle induces the class

$$\begin{aligned} \big[L^2(G\times_H M)\big] &= \sum_\lambda [V_\lambda] \dim \operatorname{Hom}_G\big(V_\lambda, L^2(G\times_H M)\big) \\ &= \sum_\lambda [V_\lambda] \dim \operatorname{Hom}_H(V_\lambda, M), \end{aligned} \tag{9}$$

where the second line follows from the first by Frobenius reciprocity.

**Theorem 2** (Bott). *If $D : L^2(G\times_H M) \to L^2(G\times_H N)$ is an elliptic homogeneous differential operator on $G/H$, then its $G$-index is the element of $\hat{R}(G)$ given by*

$$\operatorname{Index}_G D = [L^2(G \times_H M)] - [L^2(G \times_H N)].$$

*Furthermore, this difference is actually a finite element in $R(G) \subset \hat{R}(G)$.*

Bott's theorem follows from the Peter-Weyl decomposition of the space of $L^2$ functions on $G$ into the Hilbert space direct sum

$$L^2(G) \cong \widehat{\bigoplus}_\lambda V_\lambda \otimes V_\lambda^*$$

with respect to the natural action (7) of $G \times G$ on functions. For sections of a homogeneous vector bundle, the Peter-Weyl decomposition becomes

$$\begin{aligned} L^2(G \times_H M) &\cong \widehat{\bigoplus}_\lambda V_\lambda \otimes (V_\lambda^* \otimes M)^H \\ &\cong \widehat{\bigoplus}_\lambda V_\lambda \otimes \operatorname{Hom}_H(V_\lambda, M), \end{aligned} \tag{10}$$

which shows that the expression (9) for the class of $L^2(G\times_H M)$ in $\hat{R}(G)$ completely characterizes this space of sections as a representation of $G$. Since the operator $D$ is equivariant with respect to the $G$-actions on its domain and range, it can be written in block diagonal form as $D = \bigoplus_\lambda D|_{V_\lambda}$, where each of the operators

$$D|_{V_\lambda} : V_\lambda \otimes \operatorname{Hom}_H(V_\lambda, M) \to V_\lambda \otimes \operatorname{Hom}_H(V_\lambda, N)$$

is finite dimensional and must therefore have $G$-index

$$\operatorname{Index}_G D|_{V_\lambda} = [V_\lambda]\left(\dim \operatorname{Hom}_H(V_\lambda, M) - \dim \operatorname{Hom}_H(V_\lambda, N)\right).$$

The total $G$-index of $D$ is then the sum $\operatorname{Index}_G D = \sum_\lambda \operatorname{Index}_G D|_{V_\lambda}$, and all but finitely many of these summands must vanish since $D$ is Fredholm.

## 4. The Geometric Dirac Operator

Returning to Kostant's Dirac operator (4), when applied to spinors $L^2(G) \otimes \mathbb{S}$ with $\mathfrak{g}$ acting on $L^2(G)$ by the right action (8), it becomes the operator

$$\partial\!\!\!/ = \sum_i c(X_i^*)\, r(X_i) + c(v), \tag{11}$$

where $c$ is the Clifford action on the spin representation $\mathbb{S}$, and $v \in \mathrm{Cl}(\mathfrak{p})$ corresponds to the fundamental 3-form (5). Recalling that the infinitesimal right action of $\mathfrak{g}$ on functions is the same as differentiation with respect to the left-invariant vector fields, we see that this Dirac operator is a linear differential operator. Furthermore, since the expression (11) for the Dirac operator is written entirely in terms of the right action on $L^2(G)$ and assorted endomorphims of the $\mathbb{S}$ component, it must automatically commute with the left action $l \otimes 1$ of $\mathfrak{g}$ on $L^2(G)\otimes\mathbb{S}$. A quick computation at the level of the non-abelian Weil algebra then shows that $\partial\!\!\!/$ commutes with the diagonal action $r\otimes 1+1\otimes\widetilde{\mathrm{ad}}$ of $\mathfrak{h}$ on $V\otimes\mathbb{S}$ for any representation $V$ of $\mathfrak{g}$. Therefore, the Dirac operator $\partial\!\!\!/ : L^2(G) \otimes \mathbb{S} \to L^2(G) \otimes \mathbb{S}$ is homogeneous with respect to $\mathfrak{h}$.

For reasons that will soon become evident, rather than decomposing $\mathbb{S}$ into the two half-spin representations $\mathbb{S}^+$ and $\mathbb{S}^-$ as usual, we instead take their dual representations $\mathbb{S}_+^*$ and $\mathbb{S}_-^*$. This modification simply introduces an overall sign factor, since the total spin representation itself is self-dual. As for the half-spin representations, they are self-dual when $\frac{1}{2}\dim\mathfrak{p}$ is even, and they are dual to each other when $\frac{1}{2}\dim\mathfrak{p}$ is odd.

**Theorem 3.** *Given an irreducible representation $U_\mu$ of $\mathfrak{h}$ with highest weight $\mu$ such that the tensor product $\mathbb{S}\otimes U_\mu$ is a true representation of $H$, then the $G$-equivariant index of the twisted Dirac operator*

$$\partial\!\!\!/_\mu : L^2\big(G\times_H(\mathbb{S}_+^*\otimes U_\mu)\big)\to L^2\big(G\times_H(\mathbb{S}_-^*\otimes U_\mu)\big)$$

*is* $\operatorname{Index}_G \partial\!\!\!/_\mu = (-1)^w[V_{w(\mu+\rho_H)-\rho_G}]$ *if there exists a Weyl group element $w\in W_G$ such that the weight $w(\mu+\rho_H)-\rho_G$ is dominant for $G$. If no such element $w$ exists, then* $\operatorname{Index}_G \partial\!\!\!/_\mu = 0$.

*Proof.* For such a choice of $U_\mu$, the operator $\partial\!\!\!/_\mu$ descends to give a homogeneous differential operator on $G/H$. The symbol of the Dirac operator is Clifford multiplication, which is invertible, so the operator is elliptic. We may therefore apply Theorem 2 to compute its index, and by (9) we have

$$\begin{aligned}\operatorname{Index}_G \partial\!\!\!/_\mu &= \sum_\lambda [V_\lambda]\big(\dim\operatorname{Hom}_H(V_\lambda,\,\mathbb{S}_+^*\otimes U_\mu)-\dim\operatorname{Hom}_H(V_\lambda,\,\mathbb{S}_-^*\otimes U_\mu)\big)\\ &= \sum_\lambda [V_\lambda]\big(\dim\operatorname{Hom}_H(V_\lambda\otimes\mathbb{S}^+,\,U_\mu)-\dim\operatorname{Hom}_H(V_\lambda\otimes\mathbb{S}^-,\,U_\mu)\big)\\ &= \sum_\lambda\sum_{c\in C}(-1)^c\,[V_\lambda]\,\delta_{c\bullet\lambda,\,\mu},\end{aligned}$$

using the homogeneous Weyl formula (1) in the final line to decompose the virtual representation $V_\lambda\otimes\mathbb{S}^+ - V_\lambda\otimes\mathbb{S}^-$ into irreducible representations of $\mathfrak{h}$. Finally, we have $\mu = c\bullet\lambda = c(\lambda+\rho_G)-\rho_H$ if and only if $\lambda = w(\mu+\rho_H)-\rho_G$ for $w=c^{-1}$. ☐

Note that even if $\mathbb{S}\otimes U_\mu$ is only a projective representation of $H$, rather than a true representation, the operator $\partial\!\!\!/_\mu$ is nevertheless Fredholm and $G$-equivariant. Although it no longer descends to give an elliptic operator on $G/H$, the proofs of both Theorem 2 and Theorem 3 continue to hold, and the statement of Theorem 3 for the $G$-index of $\partial\!\!\!/_\mu$ is unchanged.

By taking the index of the Dirac operator on $G/H$ twisted by an irreducible representation $U_\mu$ of $\mathfrak{h}$, we have effectively inverted the construction of Gross, Kostant, Ramond, and Sternberg. Instead of using the homogeneous Weyl formula to extract a multiplet of $\mathfrak{h}$-representations associated to a given representation of $\mathfrak{g}$, we can now start with a single representation of $\mathfrak{h}$ and use this index to determine the unique $\mathfrak{g}$-representation from which it can be obtained. In fact, we can be even more precise. In Theorem 3, the index of the Dirac operator depends only on the domain and range and not on the operator itself. However, we have been using a particular choice of Dirac operator, which Kostant constructed in [4] specifically for the properties of its kernel and cokernel. Using Kostant's results, we obtain a short proof of the following theorem of Slebarski (see [5]):

**Theorem 4** (Slebarski)**.** *Given an $\mathfrak{h}$-representation $U_\mu$ as in the statement of Theorem 3, the space of harmonic spinors for the twisted Dirac operator*

$$\partial\!\!\!/_\mu : L^2\big(G\times_H(\mathbb{S}\otimes U_\mu)\big)\to L^2\big(G\times_H(\mathbb{S}\otimes U_\mu)\big)$$

*is* $\operatorname{Ker}\partial\!\!\!/_\mu = V_{w(\mu+\rho_H)-\rho_G}$ *if there exists a Weyl element $w\in W_G$ satisfying the conditions of Theorem 3, and* $\operatorname{Ker}\partial\!\!\!/_\mu = 0$ *otherwise.*

*Proof.* As in the proof of Theorem 2, we use the homogeneous form (10) of the Peter-Weyl theorem to decompose the kernel of $\partial\!\!\!/_\mu$ as the direct sum

$$\operatorname{Ker} \partial\!\!\!/_\mu = \widehat{\bigoplus}_\lambda \operatorname{Ker} \partial\!\!\!/_\mu|_{V_\lambda},$$

over the finite dimensional Dirac operators $\partial\!\!\!/_\mu|_{V_\lambda}$ acting on the spaces

$$V_\lambda \otimes \operatorname{Hom}_H(V_\lambda, \mathbb{S} \otimes U_\mu) \cong V_\lambda \otimes \operatorname{Hom}_H(U_\mu, V_\lambda \otimes \mathbb{S}),$$

where all but finitely many of these kernels vanish since the operator $\partial\!\!\!/_\mu$ is Fredholm. Ignoring the signs of the half-spin representations, equation (3) for the kernel of $\partial\!\!\!/$ implies that the kernel of $\partial\!\!\!/_\mu|_{V_\lambda}$ is $V_\lambda$ if $\mu = c\bullet\lambda$ for some $c \in C$ and 0 otherwise. □

This result provides an explicit construction for any representation $V_\lambda$ of $G$ as a space of twisted harmonic spinors on $G/H$, giving a harmonic induction map from the irreducible representations of $H$ to the irreducible representations of $G$. In particular, if $H$ is a maximal torus in $G$, then this theorem becomes a version of the Borel-Weil-Bott theorem expressed not in its customary form involving holomorphic sections and Dolbeault cohomology, but rather in terms of spinors and the Dirac operator. Again, note that if $\mathbb{S}\otimes U_\mu$ is not a true representation of $H$, the statement and proof of Theorem 4 still hold, but the twisted spinors and Dirac operator no longer descend to the homogeneous space $G/H$. Also note that we can recover the sign factor $(-1)^c = (-1)^w$ in the homogeneous Weyl formula (1) and the index of the twisted Dirac operator by splitting the spin representation into its dual half-spin representations as we did in Theorem 3. The representations with positive sign then appear in the kernel of the Dirac operator, while the negative ones appear in the kernel of its adjoint, as in (3).

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