Opdam 5 August 2012 (Bonn)

$\mathbb{H}$ "quantization" of $S(V) \rtimes W_0$

natural expectation: K-theory is invariant under this deformation in multiplicity function k_α.

Look at $K_0(S(V)\rtimes W_0)$ = Grothendieck group of iso classes of projective modules.

Fundamental pairing

Grothendieck gp of finite length reps

$$K_0(S(V)\rtimes W_0) \times G(S(V)\rtimes W_0) \longrightarrow \mathbb{Z}$$

$$[\alpha, \sigma] = \mathrm{rank}(\sigma(p))$$

$\alpha = [p]$

p = idempotent in $M_n(S(V)\rtimes W_0)$

(1) Fact: pairing is nondegenerate on left:

$[\alpha,\sigma]=0 \quad \forall$ irred $\sigma \Rightarrow \alpha \equiv 0$.

(2) $\mathrm{Supp}(\alpha) = \left\{ W_0\lambda \in W_0\backslash V^* \;\middle|\; \exists\, \sigma \text{ irred module of } SV\rtimes W_0 \text{ with central char } W_0\lambda \text{ s.t. } [\alpha,\sigma]\neq 0 \right\}$

get filtration $F_i K_0(S(V)\rtimes W_0) = \{\alpha\in K_0 \mid \dim(\mathrm{supp}(\alpha))\le i\}$

$K_0 := K_0(S(V)\rtimes W_0)$

F_0K_0 = K_0-classes with zero dim support.

(3) If $\sigma\in \mathrm{Irr}(S(V)\rtimes W_0)$ with central ch $cc(\sigma)\neq 0$

$\Rightarrow$ σ is a member of family by varying central character. [Mackey Theory]

$\Rightarrow$ if $\alpha\in F_0K_0$, $cc(\sigma)\neq 0 \Rightarrow [\alpha,\sigma]=0$.

Hence $\alpha \in F_0 K_0 \Rightarrow \mathrm{supp}(\alpha) \subset \{0\}$.

Hence get left nondegenerate pairing

$$F_0 K_0 \times G(W_0) \longrightarrow \mathbb{Z}$$

(view any rep of W_0 as $S(V) \rtimes W_0$ - module with central character 0.)

(4) ~~If σ~~ Similarly, can further reduce to left nondegenerate pairing

$$F_0 K_0 \times ELL(W_0) \to \mathbb{C}.$$

(if induced rep of W, argument & proceed as in (3).)

CONCLUDE $\dim F_0 K_0 \leq \dim (ELL(W_0))$

In fact: have equality of dimensions. Why?:

~~get pairing~~

(8) Have $\gamma : \overset{G}{ELL}(W_0) \to K_0(S(V) \rtimes W_0)$

$\sigma \longmapsto$ Euler class of Koszul resolution of pullback of σ

$\sum (-1)^i [S(V) \otimes \Lambda^i(V) \otimes \sigma]$.

clear: $[\gamma(\sigma), \tau] = \langle \sigma, \tau \rangle_{\text{elliptic}}$ (of Reeder)

~~nondegenerate by Reeder~~

clear: if σ is induced from a parabolic, then $\gamma(\sigma) = 0$.

Hence get pairing $Ell(W_0) \times Ell(W_0) \to \mathbb{C}$

$$(\sigma, \tau) \longmapsto [\gamma(\sigma), \tau]$$

= Reeder's nondegenerate pairing

nondegeneracy
allows you to Conclude: γ has image in F_0K_0 and is an isomorphism $\gamma : \mathrm{Ell}(W_0) \xrightarrow{\sim} F_0K_0$

(7) Solleveld: periodic cyclic homology of $\mathbb{H}$

Thm: $HP_*(\mathbb{H}) = HP_*(S(V) \rtimes W_0)$

independent of parameter

chern character: $K\text{-theory} \xrightarrow{\sim}$ periodic cyclic homology.

Wasserman: isomorphism!

get

$$\boxed{\begin{aligned} &\cancel{K_0(\mathbb{H})} \\ &HP_0(\mathbb{H}) = HP_0(S(V) \rtimes W_0) \xrightarrow[ch]{\sim} K_0(S(V) \rtimes W_0) \\ &\text{and } ch(F_0K_0) \cong F_0(HP_0(\mathbb{H})) \end{aligned}}$$

(passing between through chern character, so filtration is preserved.)

affine Hecke alg, arbitrary parameter.

$$\mathrm{Ell}^{temp}_{pos,\,real}(\mathcal{H}) \xrightarrow{\sim} \mathrm{Ell}^{temp}(\mathbb{H})$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$F_0K_0(\mathcal{S})_{pos,\,real} \leftarrow \text{schwartz algebra completion}$$

Reference: Opdam–Solleveld

ICM lecture

$$\text{\rotatebox{90}{$\cong$}}$$

schwartz algebra

$$F_0K_0(\mathcal{S}_{k\equiv 1} = \mathcal{S} \text{ of affine Weyl gr})_{pos\,real}$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$\mathrm{Ell}(C^\infty(V) \rtimes W_0) = \mathrm{Ell}(S(V) \rtimes W_0) \cong F_0K_0.$$

$\tilde{=}$ ~~Ell~~ In any case, we expect to find an isomorphism

$$(\dagger) \quad \boxed{\mathrm{Ell}(W_0) \xrightarrow{\sim} \mathrm{Ell}^{\mathrm{ten}}(\mathbb{H})} \quad \text{known.}$$

'~~quantization~~' of ~~g~~ γ.

$$\begin{array}{ccc} \mathrm{Ell}(W_0) & & \mathrm{Ell}^{\mathrm{ten}}(\mathbb{H}) \\ \wr\wr & & \wr\wr \\ F_0K_0 & & F_0K_0(\mathbb{H}) \end{array}$$

($\dagger$) is natural conclusion of the "F_0K_0 is indep of parameter" philosophy

Hope: make ($\dagger$) explicit using index theory.
i.e. get map

$$\mathrm{Ell}(W_0) \longrightarrow \text{tempered reps of } \mathbb{H}$$
$$\sigma \longmapsto \mathrm{Index}(D_\sigma)$$

*some Dirac operator

establish ($\dagger$) explicit.

II Question: ① If σ induced $\Rightarrow \mathrm{Ind}(D_\sigma) = 0$?

② $[\mathrm{Index}(D_\sigma), \tau] = \text{elliptic pair of } (\sigma, \tau|_W)$.

*finite length $\mathbb{H}$-module

III. Experiments?

Labesse reference in Eva's ICM talk.

Opdam 8 August (Bonn).

$K_0 \mathbb{H}$:= grothendieck gp of finitely gen. proj. modules
(finite length repn of $\mathbb{H}$)

rk pairing $[\,,\,]$: $K_0(\mathbb{H}) \times R(\mathbb{H}) \longrightarrow \mathbb{Z}$.

$$H_0 K := K_0\mathbb{H} \Big/ \bigcap_{\pi \in R(\mathbb{H})} \ker[\cdot, \pi] \overset{?}{=} K_0\mathbb{H}$$

$\shortparallel$ $F_{-1}(K_0)$ (no support).

So H_0K is compatible with all dimension filtrations.

[§8 ICM Talk: should use $\underline{\underline{H_0\mathbb{H}}}$ $\underline{\underline{\text{instead}}}$ of $K_0\mathbb{H}$.]

Ex. $\mathbb{H} \otimes_W \sigma$ projective modules (all of them?)

X fd $\mathbb{H}$-module

exercise: $[\mathbb{H}\otimes_W \sigma, X] := \dim \mathrm{Hom}_W(\sigma, X|_W)$

Index map: $\alpha : R(\tilde{W}')_{gen} \longrightarrow K_0(\mathbb{H})$ or $H_0(\mathbb{H})$

$\tilde{\sigma} \longmapsto [\mathbb{H}\otimes_{\tilde{W}'}(S^+\otimes\tilde{\sigma})] - [\mathbb{H}\otimes_{\tilde{W}'}(S^-\otimes\tilde{\sigma})]$

$\shortparallel$ $P_{S^+\otimes\sigma}$ $\shortparallel$ $P_{S^-\otimes\sigma}$

Recall $\mathbb{H}$-module maps

Dirac: $P_{S^-\otimes\sigma} \xrightarrow{D^-_{\mathbb{H},\tilde{\sigma}}} P_{S^+\otimes\sigma}$. $D^+_{\mathbb{H},\tilde{\sigma}} : \mathrm{Hom}_{\mathbb{H}}(D_{\mathbb{H},\tilde{\sigma}},$

"Fredholm index"

Cor $[\alpha(\tilde{\sigma}), X] = \dim\ker D^+_{X,\tilde{\sigma}} - \dim\operatorname{coker} D^+_{X,\sigma}$.

Pf. From exercise & EP for this 4 term sequence

$0 \to \ker \to \quad \longrightarrow \operatorname{coker} \to 0$

Hom(

Cor. $[\alpha(\tilde{\sigma}), X] = \dim \mathrm{Hom}_{\tilde{W}}(\tilde{\sigma}, S^+\otimes X) - \dim \mathrm{Hom}_{\tilde{W}}(\tilde{\sigma}, S^-\otimes X)$

$= \dim \mathrm{Hom}_{\tilde{W}'}(\tilde{\sigma}, I(X)) = \dim \mathrm{Hom}_{\tilde{W}'}(\tilde{\sigma}, H_D^+(X) - H_D^-(X))$

Cor. View $\alpha : R(\tilde{W}')_g \longrightarrow H_0(\mathbb{H})$

Then $\alpha(\tilde{\sigma}) = 0$ if $\tilde{\sigma}$ is a contraction of induced modules.

Hence α factors to a map

$$\alpha : \overline{R_g}(\tilde{W}') \longrightarrow H_0(\mathbb{H})$$

Image is in $F_0 H_0$ (since central characters of image are limited to finitely many possibilities by Vogan's conj.)

Conclude: $\alpha : \overline{R_g}(\tilde{W}') \longrightarrow F_0 H_0(\mathbb{H})$

Get pairing $\tilde{R}_g(\tilde{W}') \cancel{\otimes} R\mathbb{H} \longrightarrow \mathbb{C}$

$$(\tilde{\sigma}', X) \longmapsto [\alpha(\tilde{\sigma}), X]$$

Also clear, that ~~[illegible]~~ Induced from proper parabolics are in right radical (since you can deform them).

So get pairing

$$\overline{R_g}(\tilde{W}') \times \overline{R}(\mathbb{H}) \longrightarrow \mathbb{C}$$

elliptic

Compose with i: $\overline{R}(W) \xrightarrow{i} \overline{R_g}(\tilde{W}) \xrightarrow{\alpha}$, ~~get~~ $F_0 H_0 \mathbb{H}$

$$\sigma \longmapsto \sigma \otimes (S^+ - S^-) \longrightarrow \alpha(i(\sigma))$$

$i(\sigma) :=$

β : $R(W) \times R\mathbb{H} \to \mathbb{C}$.

Have

$$[\beta(\sigma), X] = 2e_W(\sigma, X|_W)$$

By Solleveld: all Weyl group characters are in span of $X|_W$ as X runs over $R(\mathbb{H})$

By nondegeneracy of elliptic pairing (Reeder) get ~~injective map~~ that β is injective:

$$\beta : \overline{R}(W) \longrightarrow F_0 H_0 \mathbb{H}.$$

One can also show β is surjective. (elementary argument?)
(one proof: pass to affine Hecke algebra, use [OS], check compatibility with induction & Lusztig passing fr $\mathbb{H}$ to $\mathcal{H}$ — see Solleveld's thesis)

So get

$$\beta : \overline{R}(W) \longrightarrow F_0 H_0 \mathbb{H}.$$

this is the main map.

and nondegenerate pairing

$$\overline{R}(W) \times \overline{R}\mathbb{H} \to \mathbb{C}.$$

Also nondegenerate pairing

$$EP : \overline{R}\mathbb{H} \times \overline{R}\mathbb{H} \longrightarrow \mathbb{C}$$

$$\{X, Y\} = \Sigma(-1)\, \mathrm{Ext}^i_{\mathbb{H}}(X, Y)$$

get natural identification

$$F_0 H_0(\mathbb{H}) \xrightarrow{\sim} \overline{R}(\mathbb{H}).$$