Eigenvalues of contracted invariants

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Settings

G : real reductive Lie group in the Harish-Chandra class (noncompact type)

K : maximal compact subgroup $\leftrightarrow$ Cartan involution θ

$\mathfrak{g}$: complexified Lie algebra of G

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$\mathfrak{k} + \mathfrak{p}$: Cartan decomp. w.r.t. θ

X : $= G/K$, Riemannian symmetric space

$\mathcal{D}(X)$: algebra of G-invariant differential operators

Story: (1) A generator set of $\mathcal{D}(X)$ is explicitly given for some G. (2) Eigenvalues of generators are calculated by the generalized Harish-Chandra homomorphism.

Settings (continued)

$G = KAN$: Iwasawa decomp. ($\mathfrak{a} := \mathrm{Lie}(A) \subset \mathfrak{p}$)

W $:= N_K(\mathfrak{a})/M$ with $M := Z_K(\mathfrak{a})$

Σ : root system for $(\mathfrak{a}, \mathfrak{g})$

$\cup$

Σ^+: positive roots $\leftrightarrow$ $\mathfrak{n} := \mathrm{Lie}(N)$

$\mathfrak{g}$ $:= \mathfrak{a} + \mathfrak{m} + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, root space decomp.

ρ $:= \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}_\alpha) \alpha$.

2 types of Harish-Chandra's homo.

1st type:

$$\gamma : U(\mathfrak{g}) = U(\mathfrak{a}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k})$$
$$\xrightarrow{\text{projection}} U(\mathfrak{a}) \simeq S(\mathfrak{a}) \xrightarrow{(-\rho)\text{-shift}} S(\mathfrak{a})$$

2nd type:

$$\gamma^{\#} : U(\mathfrak{g}) = U(\mathfrak{a}) \oplus (\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$$
$$\xrightarrow{\text{projection}} U(\mathfrak{a}) \simeq S(\mathfrak{a}) \xrightarrow{\rho\text{-shift}} S(\mathfrak{a})$$

Both induce the same algebra isomorphism

$$\mathcal{D}(X) \simeq U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k} \xrightarrow{\sim} S(\mathfrak{a})^W.$$

Contracted K-invariants

Def:

V : finite-dimensional K-module with basis $\{v_1, \ldots, v_d\}$

V^* : the dual space with dual basis $\{v_1^*, \ldots, v_d^*\}$

$\Psi_1 \in \mathrm{Hom}_K(V, U(\mathfrak{g}))$

$\Psi_2 \in \mathrm{Hom}_K(V^*, U(\mathfrak{g}))$

The contracted invariant $C(\Psi_1, \Psi_2)$ for the pair (Ψ_1, Ψ_2) is defined by

$$C(\Psi_1, \Psi_2) = \sum_{i=1}^{d} \Psi_1[v_i]\Psi_2[v_i^*].$$

Simura systems

Suppose $X = G/K$ is an irreducible Hermitian symmetric space. $\Rightarrow$ $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ as K-module and

$$U(\mathfrak{p}_+) \simeq S(\mathfrak{p}_+) \simeq \bigoplus_{\mathbf{m}} V_{\mathbf{m}} \quad \text{(multiplicity free).}$$

$$\mathcal{L}_{\mathbf{m}} := C(\Psi_{\mathbf{m}}, \Psi_{\mathbf{m}}^*) \text{ for } \begin{cases} \Psi_{\mathbf{m}} : V_{\mathbf{m}} \hookrightarrow U(\mathfrak{p}_+) \\ \Psi_{\mathbf{m}}^* : V_{\mathbf{m}}^* \hookrightarrow U(\mathfrak{p}_-) \end{cases}$$

Thm(Shimura, 1990):

$\{\mathcal{L}_{(1,\ldots,1)}, \mathcal{L}_{(1,\ldots,1,0)}, \ldots \mathcal{L}_{(1,0,\ldots,0)}\}$ is an algebraically independent basis of $\mathbb{C}$-algebra $\mathcal{D}(X)$.

Simura systems (continued)

Write $\underline{\mathbf{1}}^p = (\underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0)$.

Fact: Genkai Zhang calculated $\gamma(\mathcal{L}_{\underline{\mathbf{1}}^p})$ for $p = 1, \ldots, r$ where $r = \mathrm{rank}(X)$ (2001).

- Calculate first $\gamma(\mathcal{L}_{\underline{\mathbf{1}}^r})$ by Cayley-Capelli identity.
- General $\gamma(\mathcal{L}_{\underline{\mathbf{1}}^p})$ are obtained by reduction to Levi parts.

Other contracted invariants:

$G = SO_0(n,\ell) \curvearrowright V_{n+\ell} = V_n \oplus V_\ell$ (natural rep.).

$X = SO_0(n,\ell)/SO(n) \times SO(\ell)$.

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ {}^{\mathrm{t}}B & 0 \end{pmatrix} ; B \in \mathrm{Mat}_{n,\ell} \right\} \simeq V_n \otimes V_\ell.$$

Put $F_{ij} := E_{i,n+j} + E_{n+j,i}$ $(1 \le i \le n, 1 \le j \le \ell)$,

$V_p := \bigwedge^p V_n \otimes \bigwedge^p V_\ell$ $(p = 0, \ldots, \min(n,\ell))$.

$$\Psi_p : V_p \ni v_I \otimes v_J \quad \begin{pmatrix} I = \{1 \le i_1 < \cdots < i_p \le n\} \\ J = \{1 \le j_1 < \cdots < j_p \le \ell\} \end{pmatrix}$$

$$\longmapsto \det(F_{i_\mu j_\nu})_{\substack{1 \le \mu \le p \\ 1 \le \nu \le p}} \in U(\mathfrak{g})$$

$X = SO_0(n,\ell)/SO(n) \times SO(\ell)$. Put

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$$\Psi_p : V_p \ni v_I \otimes v_J \quad \begin{pmatrix} I = \{1 \le i_1 < \cdots < i_p \le n\} \\ J = \{1 \le j_1 < \cdots < j_p \le \ell\} \end{pmatrix}$$
$$\longmapsto \det(F_{i_\mu j_\nu})_{\substack{1 \le \mu \le p \\ 1 \le \nu \le p}} \in U(\mathfrak{g})$$

<ins>**Prop:**</ins> $C(\Psi_p, \Psi_p) = \sum_{\#I = \#J = p} \det(F_{i_\mu j_\nu})^2_{\substack{1 \le \mu \le p \\ 1 \le \nu \le p}}$ constitues an algebraically independent basis of $\mathcal{D}'(X)$ where $\mathcal{D}'(X)$ is the algebra of invariant operators under the whole isometries.

A result of Kostant

Suppose $\{v_1, \ldots, v_k\}$ and $\{v_1^*, \ldots, v_k^*\}$ are bases of V^M and $(V^*)^M$.
Then for $C(\Psi_1, \Psi_2) = \sum_{i=1}^d \Psi_1[v_i]\Psi_2[v_i^*]$ with $\Psi_1 \in \mathrm{Hom}_K(V, U(\mathfrak{g})),\ \Psi_2 \in \mathrm{Hom}_K(V^*, U(\mathfrak{g}))$,

$$\gamma\big(C(\Psi_1, \Psi_2)\big) = \sum_{i=1}^{k} \gamma^{\#}\big(\Psi_1[v_i]\big)\,\gamma\big(\Psi_2[v_i^*]\big).$$

Hua-Shimeno operators

Suppose $X = G/K$ is an irreducible Hermitian symmetric space and recall $\begin{cases} \Psi_{\underline{1}^p} : V_{\underline{1}^p} \hookrightarrow U(\mathfrak{p}_+), \\ \Psi^*_{\underline{1}^p} : V^*_{\underline{1}^p} \hookrightarrow U(\mathfrak{p}_-). \end{cases}$ Shimeno defined

$$\mathcal{H}_{\underline{1}^p} = \sum_{i=1}^{d} \Psi_{\underline{1}^p}[v_i] v_i^*$$

and calculated $\gamma^{\#}\big(\Psi_{\underline{1}^p}[v_i]\big)$ for $i = 1, \ldots, k$ (1999).

It deduces the eigenvalues of Shimura's invariants:

$$\gamma\big(C(\Psi_{\underline{1}^p}, \Psi^*_{\underline{1}^p})\big) = \sum_{i=1}^{k} \gamma^{\#}\big(\Psi_{\underline{1}^p}[v_i]\big)\, \gamma\big(\Psi^*_{\underline{1}^p}[v_i^*]\big).$$

Next...

We give a more general method to calculate $\gamma^{\#}\big(\boldsymbol{\Psi_1}[v_i]\big)$ and $\gamma\big(\boldsymbol{\Psi_2}[v_i^*]\big)$.

The K-modules $V_{\underline{1}^\ell}$ for the Hermitian cases and $V_p = \bigwedge^p V_n \otimes \bigwedge^p V_\ell$ for $SO_0(n,\ell)$ are single-petaled, in which cases we can use the generalized Harish-Chandra homomorphism.

Single-petaled K-modules

Σ_1 $:= \Sigma \setminus 2\Sigma$ (Recall Σ is the root system)

$\mathcal{R}$: a subset of Σ_1 such that $W\mathcal{R} = \Sigma_1$

X_α : a fixed root vector for each $\alpha \in \mathcal{R}$

$B(\cdot,\cdot)$: Killing form of $\mathfrak{g}$

Def: A K-type V is called single-petaled iff for any $\alpha \in \mathcal{R}$ and $v \in V^M$

$$(X_\alpha + \theta X_\alpha)\big((X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha)\big)v = 0.$$

Examples of Single-petaled K-modules

1: Suppose G is complex. Identifying K-modules with holomorphic reps of G, the notion of single-petaled K-modules coincides with that of small reps of G in the sense of Broer.

2: Suppose G is split. Then a petite K-type is single-petaled, but not vice versa.

3: Suppose G is arbitrary. Then the trivial K-type and each constituent of $\mathfrak{p}$ are single-petaled.

Degenerate affine Hecke algebras

Let Π be the set of simple roots ($\Pi \subset \Sigma^+ \cap \Sigma_1$).

Define the multiplicity function $\mathrm{k} : \Sigma_1 \longrightarrow \mathbb{C}$ by

$$\mathrm{k}(\alpha) = \dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha}.$$

Theorem (Lusztig): $\exists!$ an algebra H_k satisfying

(1) $S(\mathfrak{a}), \mathbb{C}[W] \subset \mathrm{H}_\mathrm{k}$ are subalgebras with the same 1.

(2) $\mathrm{H}_\mathrm{k} \simeq S(\mathfrak{a}) \otimes \mathbb{C}[W]$ by multiplication.

(3) For $\forall \alpha \in \Pi$ and $\xi \in \mathfrak{a}$,

$$s_\alpha \cdot \xi = s_\alpha(\xi) \cdot s_\alpha - k(\alpha)\alpha(\xi).$$

Modification of the W-module $S(\mathfrak{a})$

Define the left $\mathrm{H}_{\mathbf{k}}$-module

$$
\begin{aligned}
S_{\mathrm{H}_{\mathbf{k}}}(\mathfrak{a}) &:= \mathrm{H}_{\mathbf{k}} \Big/ \sum_{w \in W} \mathrm{H}_{\mathbf{k}}(w-1) \\
&= S(\mathfrak{a}) \otimes \mathbb{C}[W] \Big/ S(\mathfrak{a}) \otimes \sum_{w \in W} \mathbb{C}[W](w-1) \\
&\simeq S(\mathfrak{a}).
\end{aligned}
$$

Then clearly $S_{\mathrm{H}_{\mathbf{k}}}(\mathfrak{a}) \simeq S(\mathfrak{a})$ as left $S(\mathfrak{a})$-modules,

while $S_{\mathrm{H}_{\mathbf{k}}}(\mathfrak{a}) \not\simeq S(\mathfrak{a})$ as left W-modules,

although $S_{\mathrm{H}_{\mathbf{k}}}(\mathfrak{a})^W = S(\mathfrak{a})^W$.

Generalization of γ and $\gamma^{\#}$

For the multiplicity function $-\mathbf{k}$, the Hecke algebra $\mathbf{H}_{-\mathbf{k}}$ and the $\mathbf{H}_{-\mathbf{k}}$-module $S_{\mathbf{H}_{-\mathbf{k}}}(\mathfrak{a})$ are similarly defined.

Def: For a single-petaled K-module V, define

$$\Gamma_V^{\#} : \mathrm{Hom}_K(V, U(\mathfrak{g})) \ni \Psi \longmapsto \psi \in \mathrm{Hom}(V^M, S_{\mathbf{H}_{-\mathbf{k}}}(\mathfrak{a}))$$

so that ψ is given by

$$\psi : V^M \hookrightarrow V \xrightarrow{\Psi} U(\mathfrak{g}) \xrightarrow{\gamma^{\#}} S(\mathfrak{a}) \simeq S_{\mathbf{H}_{-\mathbf{k}}}(\mathfrak{a}).$$

Generalization of Harish-Chandra's theorem

Thm: We have

$$\Gamma_V\big(\mathrm{Hom}_K(V, U(\mathfrak{g}))\big) \subset \mathrm{Hom}_W(V^M, S_{\mathrm{H}_{\mathfrak{k}}}(\mathfrak{a})),$$

$$\Gamma_V^{\#}\big(\mathrm{Hom}_K(V, U(\mathfrak{g}))\big) \subset \mathrm{Hom}_W(V^M, S_{\mathrm{H}_{-\mathfrak{k}}}(\mathfrak{a}))$$

and the following exact sequences:

$$0 \to \mathrm{Hom}_K(V, U(\mathfrak{g})\mathfrak{k}) \to \mathrm{Hom}_K(V, U(\mathfrak{g})) \xrightarrow{\Gamma_V} \mathrm{Hom}_W(V^M, S_{\mathrm{H}_{\mathfrak{k}}}(\mathfrak{a})) \to 0,$$

$$0 \to \mathrm{Hom}_K(V, U(\mathfrak{g})\mathfrak{k}) \to \mathrm{Hom}_K(V, U(\mathfrak{g})) \xrightarrow{\Gamma_V^{\#}} \mathrm{Hom}_W(V^M, S_{\mathrm{H}_{-\mathfrak{k}}}(\mathfrak{a})) \to 0.$$

$W_{(BC)_n}$-invariants

Suppose $\{e_1, \ldots, e_n\}$ is an orthogonal basis of $\mathfrak{a}^*$ such that $\Pi = \{e_1 - e_2, \ldots, e_{n-1} - e_n\} \cup \{ae_n\}$ with $a = 1$ or 2. Let $\{H_1, \ldots, H_n\} \subset \mathfrak{a}$ be its dual.

For $I = \{\, 1 \le i_1 < \cdots < i_p \le n\}$, put

$$\zeta_{\mathbf{k},I} = \prod_{\mu=1}^{p} \bigl(H_{i_\mu} - \rho(H_p - H_\mu + H_{i_\mu})\bigr).$$

Then $U_{\mathbf{k}}^p := \sum_{\#I=p} \mathbb{C}\, \zeta_{\mathbf{k},I} \in S_{\mathrm{H}_{\mathbf{k}}}(\mathfrak{a})$ is a W-subspace.

$W_{(BC)_n}$-invariants (continued)

$U_{\mathbf{k}}^p := \sum_{\# I = p} \mathbb{C}\, \zeta_{\mathbf{k},I} \in S_{\mathrm{H}_{\mathbf{k}}}(\mathfrak{a})$ with

$$\zeta_{\mathbf{k},I} = \prod_{\mu=1}^{p} \bigl(H_{i_\mu} - \rho(H_p - H_\mu + H_{i_\mu})\bigr).$$

$S_{\mathrm{H}_{\mathbf{k}}}(\mathfrak{a})$ has natural filtration $\{S_{\mathrm{H}_{\mathbf{k}}}^{(d)}(\mathfrak{a})\}$.

Prop: $U_{\mathbf{k}}^p$ is a unique irreducible W-subspace of $\{S_{\mathrm{H}_{\mathbf{k}}}^{(p)}(\mathfrak{a})\}$ of type $((n-p), (\underbrace{1, \ldots, 1}_{p}))$.

Hermitian cases

Recall $\begin{cases} \Psi_{\underline{1}^p} : V_{\underline{1}^p} \hookrightarrow U(\mathfrak{p}_+) \in \mathrm{Hom}_K(V_{\underline{1}^p}, U(\mathfrak{g})), \\ \Psi^*_{\underline{1}^p} : V^*_{\underline{1}^p} \hookrightarrow U(\mathfrak{p}_-) \in \mathrm{Hom}_K(V^*_{\underline{1}^p}, U(\mathfrak{g})). \end{cases}$

We can observe $V_{\underline{1}^p}$ and $V^*_{\underline{1}^p}$ are single-petaled.

$V^M_{\underline{1}^p} \simeq (V^*_{\underline{1}^p})^M \simeq U^p_{\mathbf{k}}$ as W-module, and

$$\begin{aligned} \Gamma_V\left(\Psi^*_{\underline{1}^p}\right) &\in \mathrm{Hom}_W((V^*_{\underline{1}^p})^M, S^{(p)}_{\mathbf{H}_\mathbf{k}}(\mathfrak{a})) \\ &\simeq \mathrm{Hom}_W(U^p_{\mathbf{k}}, S^{(p)}_{\mathbf{H}_\mathbf{k}}(\mathfrak{a})) \\ &\simeq \mathbb{C} \text{ (the injection map of } U^p_{\mathbf{k}} \hookrightarrow S^{(p)}_{\mathbf{H}_\mathbf{k}}(\mathfrak{a})) \end{aligned}$$

Hermitian cases (final result)

Thm:

$$\gamma\Big(C(\Psi_{\underline{1}^p}, \Psi^*_{\underline{1}^p})\Big)$$

$$= C \sum_{1 \le i_1 < \cdots < i_p \le n} \prod_{\mu=1}^{p} \Big(H_{i_\mu}^2 - \big(\rho(H_p - H_\mu + H_{i_\mu})\big)^2 \Big)$$

for some non-zero constant C, which is easily determined by a symbol calculation.

Rem: The argument for case $X = SO_0(n, \ell)/SO(n)$ is quite similar and the same result holds.

Case $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n$

Def (generalized Capelli operators): For $t \in \mathbb{C}$ and two sequences of indices $I = \{1 \le i_1 < \cdots < i_p \le n\}$, $J = \{1 \le j_1 < \cdots < j_p \le n\}$, put

$$D_{IJ}(t) = \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn}\sigma \left(E_{i_{\sigma(1)} j_1} + \left(\tfrac{p-1}{2} - t\right)\delta_{i_{\sigma(1)} j_1}\right) \cdots \cdots \left(E_{i_{\sigma(p)} j_p} + \left(-\tfrac{p-1}{2} - t\right)\delta_{i_{\sigma(p)} j_p}\right) \in U(\mathfrak{g}).$$

Case $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n$ (final result)

For $p = 1, \ldots, n$, the eigenvalue of contracted invariants

$$C_p(t) = \sum_{\substack{I=\{1\le i_1<\cdots<i_p\le n\}\\ J=\{1\le j_1<\cdots<j_p\le n\}}} D_{IJ}(t)\, D_{IJ}(-t\,)$$

is

$$\sum_{1\le i_1<\cdots<i_p\le n} \prod_{\mu=1}^{p} \left(H_{i_\mu}^2 - \left(t + \tfrac{n-p}{2} + \mu - i_\mu\right)^2\right)$$

<u>**Rem:**</u> The operator $C_p(t)$ with a special parameter value is recently used in a paper by Lee, Nishiyama, and Wachi.