

NILPOTENT ORBITS AND THETA-STABLE PARABOLIC SUBALGEBRAS

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ABSTRACT. In this work, we present a new classification of nilpotent orbits in a real reductive Lie algebra $\mathfrak{g}$ under the action of its adjoint group. Our classification generalizes the Bala-Carter classification of the nilpotent orbits of complex semisimple Lie algebras. Our theory takes full advantage of the work of Kostant and Rallis on $\mathfrak{p}_{\mathbb{C}}$, the “complex symmetric space associated with $\mathfrak{g}$ ”. The Kostant-Sekiguchi correspondence, a bijection between nilpotent orbits in $\mathfrak{g}$ and nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$, is also used. We identify a fundamental set of *noticed* nilpotents in $\mathfrak{p}_{\mathbb{C}}$ and show that they allow us to recover all other nilpotents. Finally, we study the behaviour of a principal orbit, that is an orbit of maximal dimension, under our classification. This is not done in the other classification schemes currently available in the literature.

INTRODUCTION

Let $\mathfrak{g}_{\mathbb{C}}$ be a semisimple Lie algebra and $G_{\mathbb{C}}$ its adjoint group. We say that an element x of $\mathfrak{g}_{\mathbb{C}}$ is nilpotent if and only if, $ad_x : y \rightarrow [x, y]$ for all $y \in \mathfrak{g}_{\mathbb{C}}$, is a nilpotent endomorphism of $\mathfrak{g}_{\mathbb{C}}$. Kostant (see also Dynkin [Dy]) has shown, in his fundamental 1959 paper [Ko], that the number of nilpotent orbits of $G_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ is finite. The Bala-Carter classification can be expressed as follows:

There is a one-to-one correspondence between nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$ and conjugacy classes of pairs (m, p_m) , where m is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and p_m is a distinguished parabolic subalgebra of the semisimple algebra $[m, m]$. In this correspondence, the nilpotent $\mathfrak{g}_{\mathbb{C}}$ orbit comes from the Richardson orbit of P_m , the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra p_m , on the nilradical of p_m .

This work shows how a theory similar to the Bala-Carter classification can be used to parametrize nilpotent orbits of a semisimple real Lie algebra $\mathfrak{g}$ under the action of its adjoint group G . For a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ let $K_{\mathbb{C}}$ be the complexification of the connected subgroup K of G with Lie algebra $\mathfrak{k}$. Sekiguchi [Se] proves that there is a one to one correspondence between the G -orbits in $\mathfrak{g}$ and the $K_{\mathbb{C}}$ -orbits in $\mathfrak{p}_{\mathbb{C}}$. We call this correspondence the Kostant-Sekiguchi correspondence since it was first conjectured by Kostant. This allows us to exploit the rich theory of symmetric spaces [K-R].

A real reductive Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for a Cartan involution θ . Hence, by complexification we obtain $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$. Denote by σ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$.

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Our main classification theorem is proved in section 3. We prove that the orbits $K_{\mathbb{C}} \cdot e$ are in one-to-one correspondence with the triples of the form $(l, q_l, \mathfrak{w})$, where e is a non-zero nilpotent in $\mathfrak{p}_{\mathbb{C}}$, l is a minimal (θ, σ) -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e , q_l is a θ stable parabolic subalgebra of $[l, l]$ and $\mathfrak{w}$ is a certain $L \cap K_{\mathbb{C}}$ prehomogeneous subspace of $q_l \cap \mathfrak{p}_{\mathbb{C}}$ containing e [Theorem 3.2.4]. We note that Kawanaka has obtained related results. [Ka]

Complex semisimple Lie algebras regarded as real do not give rise to any new nilpotent orbits. Their compact real forms contain no non-zero nilpotent. Therefore we can limit our analysis to their non-compact real forms. Such simple real Lie algebras were classified by Cartan and can be found in Helgason [He]. Note that a complex conjugacy class of a real nilpotent can split under the action of the real adjoint group. For example the two matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are conjugate under the complex group $PSL_2(\mathbb{C})$ but not under the real group $PSL_2(\mathbb{R})$.

In section 1, we explain the nature of the Kostant-Sekiguchi correspondence. Given a nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$, we give a method for constructing a minimal θ -stable Levi subalgebra containing e [Proposition 1.1.3]. A KS-triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$, that is a normal triple in Kostant-Rallis' sense with the additional property that $\sigma(e) = f$, is associated to a θ -stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$. Several important Richardson-type theorems are also proved [Proposition 1.2.1 and Theorems 1.2.3, 1.2.6]. For example, let Q be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. If $\mathfrak{q} = l \oplus u$ is the Levi decomposition of $\mathfrak{q}$ and if e is even we have $\overline{Q \cap K_{\mathbb{C}} \cdot e} = u \cap \mathfrak{p}_{\mathbb{C}}$. This explains the fact that our theory is so close to that of Bala and Carter for even orbits. Define L to be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra l .

In section 2 we introduce the notion of *noticed* nilpotent element. Such an element e of $\mathfrak{p}_{\mathbb{C}}$ is characterized by the fact that the reductive centralizer $\mathfrak{k}_{\mathbb{C}}^{(x, e, f)}$ is trivial [Lemma 2.1.1]. In fact a nilpotent is always noticed in the minimal (θ, σ) -stable Levi subalgebra that contains it. Furthermore e is even and noticed if and only if $\mathfrak{q} = l \oplus u$ and $\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \overline{\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}}$ [Theorem 2.1.6]. This dimension criteria is very similar to the one that Bala and Carter give for their distinguished parabolic subalgebras. Finally we show that there are, in fact, non-even noticed elements whose associated θ -stable parabolic subalgebra does not satisfy the above dimension condition. This implies that our classification is different from the Bala-Carter classification, where all the distinguished nilpotent elements are even. A distinguished element in $\mathfrak{p}_{\mathbb{C}}$ is noticed but not vice versa.

In section 4 we give a description of the noticed orbits of the classical simple real Lie algebras in terms of signed Young diagrams. Among other results we show that the non-zero noticed nilpotent orbits of $\mathfrak{sl}_n(\mathbb{R})$ are parametrized by partitions of n with distinct parts. The noticed orbits of most of the real simple algebras are even. The classical algebras $\mathfrak{so}_{2n}^*$, $\mathfrak{su}_{2n}^*$, $\mathfrak{sp}(p, q)$ have no non-zero noticed nilpotent orbits [Theorem 4.2.1]. An exceptional simple real Lie algebra contains a non-zero noticed element if and only if it is quasi-split [Proposition 4.1.1].

In the last section we analyze the behaviour of a principal nilpotent element of $\mathfrak{p}_{\mathbb{C}}$, i.e. an element whose $K_{\mathbb{C}}$ -orbit has maximal dimension among all $K_{\mathbb{C}}$ -nilpotent orbits, under the classification. The main result is that a principal nilpotent element

e is regular in the minimal (θ, σ) -stable Levi subalgebra l of $\mathfrak{g}_{\mathbb{C}}$ containing it and the real form l_0 of l is quasi-split [Theorem 5.1.8].

Nilpotent orbits have been used extensively in Representation Theory. Their geometric structure is still being investigated by several researchers. Θ -stable parabolic subalgebras also play an important role in Representation Theory, specifically through the work of Zuckerman and Vogan on Cohomological Induction. Our classification relates the two concepts.

1. Θ -STABLE LEVI SUBALGEBRAS

1.1. Minimal θ -stable Levi subalgebras. Let $\mathfrak{g}$ be a real reductive Lie algebra with adjoint group G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Finally, let θ be the corresponding Cartan involution of $\mathfrak{g}$ and σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Denote by $K_{\mathbb{C}}$ the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$.

Definition. By a (σ, θ) -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ we shall mean a Levi subalgebra of a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ in Vogan's sense [Vo]. In other words if l is a (σ, θ) -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$, then there exists a θ -stable parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ with Levi decomposition $\mathfrak{q} = l \oplus u$ such that $\theta(l) = l$ and $\sigma(l) = l$. Every such l is of the form $l = \mathfrak{g}_{\mathbb{C}}^z$ for some $z \in i\mathfrak{k}$.

A triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ is called a standard triple if $[x, e] = 2e$, $[x, f] = -2f$ and $[e, f] = x$. If $x \in \mathfrak{k}_{\mathbb{C}}$, e and $f \in \mathfrak{p}_{\mathbb{C}}$, then (x, e, f) is a normal triple. It is a result of Kostant and Rallis [K-R] that any nilpotent e of $\mathfrak{p}_{\mathbb{C}}$ can be embedded in a standard normal triple (x, e, f) . Moreover e is $K_{\mathbb{C}}$ -conjugate to a nilpotent e' inside of a normal triple (x', e', f') with $\sigma(e') = f'$ [Se]. The triple (x', e', f') will be called a *Kostant-Sekiguchi* or *KS-triple*.

Every nilpotent E' in $\mathfrak{g}$ is G -conjugate to a triple (H, E, F) in $\mathfrak{g}$ with the property that $\theta(H) = -H$ and $\theta(E) = -F$ [Se]. Such a triple will be called a *KS-triple* also.

Define a map c from the set of KS-triples of $\mathfrak{g}$ to the set of normal triples of $\mathfrak{g}_{\mathbb{C}}$ as follows:

$$\begin{aligned} x &= c(H) = i(E - F), \\ e &= c(E) = \frac{1}{2}(H - i(E + F)), \\ f &= c(F) = \frac{1}{2}(H + i(E + F)). \end{aligned}$$

The triple (x, e, f) is called the Cayley transform of (H, E, F) . It is easy to verify that the triple (x, e, f) is a KS-triple and that $x \in i\mathfrak{k}$. The Kostant-Sekiguchi correspondence [Se] gives a one-to-one map between the set of G -conjugacy classes of nilpotents in $\mathfrak{g}$ and the $K_{\mathbb{C}}$ -conjugacy classes of nilpotents in $\mathfrak{p}_{\mathbb{C}}$. This correspondence sends the zero orbit to the zero orbit and the orbit through the nilpositive element of a KS-triple to the one through the nilpositive element of its Cayley transform. Recently, Michèle Vergne [Ve] has proved that there is in fact a K -invariant diffeomorphism between the G -conjugacy class and the $K_{\mathbb{C}}$ -conjugacy class associated by the Kostant-Sekiguchi correspondence.

The KS-triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ corresponds to a real KS-triple (H, E, F) in $\mathfrak{g}$ under the Kostant-Sekiguchi map and the reductive centralizer $\mathfrak{k}_{\mathbb{C}}^{(x, e, f)} = \mathfrak{k}^{(H, E, F)} \oplus i\mathfrak{k}^{(H, E, F)}$ is θ -stable and σ -stable.

Let $\mathfrak{s} = \mathfrak{k}_{\mathbb{C}}^e$ be the centralizer of e in $\mathfrak{k}_{\mathbb{C}}$. Call any subalgebra of $\mathfrak{k}_{\mathbb{C}}$ consisting of semisimple elements toral. Any toral subalgebra is commutative [H1]. A Cartan subalgebra of a Lie algebra is a self-normalizing Lie subalgebra [H1]. We shall need the following lemma:

Lemma 1.1.1. *Let $\mathfrak{t}$ be a maximal toral subalgebra of $\mathfrak{s}$. Then $\mathfrak{s}^{\mathfrak{t}}$ is a Cartan subalgebra of $\mathfrak{s}$, and $\mathfrak{t}$ consists exactly of the semisimple elements in $\mathfrak{s}^{\mathfrak{t}}$.*

Proof. This is part of the proof of Theorem 8.1.1 of [C-Mc]. $\square$

Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two maximal toral subalgebras of $\mathfrak{s}$. Then $\mathfrak{s}^{\mathfrak{t}_1}$ is $K_{\mathbb{C}}^e$ -conjugate to $\mathfrak{s}^{\mathfrak{t}_2}$ by a map that must send $\mathfrak{t}_1$ to $\mathfrak{t}_2$ because $K_{\mathbb{C}}^e$ preserves semisimplicity and nilpotence in $\mathfrak{k}_{\mathbb{C}}$. Hence any two maximal toral subalgebras of $\mathfrak{k}_{\mathbb{C}}^e$ are conjugate under $K_{\mathbb{C}}^e$.

Lemma 1.1.2. *If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$, then $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^e$.*

Proof. We know that $\mathfrak{k}_{\mathbb{C}}^e = \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \oplus u_e$ where u_e is an ad_x invariant nilpotent ideal of $\mathfrak{k}_{\mathbb{C}}^e$. Let $\mathfrak{t}'$ be a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^e$ such that $\mathfrak{t} \subsetneq \mathfrak{t}'$. Any element z of $\mathfrak{t}' \setminus \mathfrak{t}$ can be written in the form of

$$z = z_1 + z_2 \text{ with } z_1 \in \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \text{ and } z_2 \in u_e.$$

But

$$[\mathfrak{t}, z] = [\mathfrak{t}, z_1] \oplus [\mathfrak{t}, z_2] = 0.$$

Hence

$$[\mathfrak{t}, z_1] = 0 \text{ and } [\mathfrak{t}, z_2] = 0,$$

since $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ normalizes itself and $\mathfrak{g}_{\mathbb{C}}$.

Clearly z_1 is in $\mathfrak{t}$, for $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$. It follows that $z - z_1$ is in $\mathfrak{t}'$, that is, $z - z_1$ is semisimple. Since $z - z_1 = z_2$ and z_2 is nilpotent, $z_2 = 0$. Hence z must be in $\mathfrak{t}$. $\square$

Proposition 1.1.3. *If l is a minimal (σ, θ) -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing a nilpotent element e of $\mathfrak{p}_{\mathbb{C}}$, then $l = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$, where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^e$.*

Proof. Let (H, E, F) be a KS-triple in $\mathfrak{g}$. By definition $l = \mathfrak{g}_{\mathbb{C}}^z$ for some z in $i\mathfrak{k}$. Since $iz \in \mathfrak{k}$, we can find a maximal torus $\mathfrak{t}_0$ of $\mathfrak{k}^{(H,E,F)}$ containing iz . Therefore $\mathfrak{t} = \mathfrak{t}_0 \oplus i\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ containing z . From Lemma 1.1.2 it is also a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^e$. Moreover $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}} \subseteq \mathfrak{g}_{\mathbb{C}}^z = l$.

We shall now prove that $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$ is a (σ, θ) -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e . By minimality of l , this will complete the proof.

Since $\mathfrak{t}$ is (σ, θ) stable, so is the centralizer $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$. Let $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ be the root system relative to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{t} \subset \mathfrak{h}$. Then $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$ is the Levi subalgebra generated by $\mathfrak{h}$ and all the root spaces for the roots $\alpha \in \Delta$ such that $\alpha(\mathfrak{t}) = 0$. $\square$

We proceed to prove the following theorem.

Theorem 1.1.4. *Any two minimal (σ, θ) -stable Levi subalgebras of $\mathfrak{g}_{\mathbb{C}}$ containing a nilpotent element e of $\mathfrak{p}_{\mathbb{C}}$ are $K_{\mathbb{C}}^e$ conjugate.*

Proof. Let l_1 and l_2 be two minimal (σ, θ) -stable minimal Levi subalgebras of $\mathfrak{g}_{\mathbb{C}}$ containing e . Then $l_1 = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}_1}$ and $l_2 = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}_2}$, where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two maximal toral subalgebras of $\mathfrak{k}_{\mathbb{C}}$. Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are $K_{\mathbb{C}}^e$ -conjugate, so are l_1 and l_2 . $\square$

1.2. Some denseness theorems. Let (x, e, f) be a KS-triple with $x \in i\mathfrak{k}$. From the representation theory of $\mathfrak{sl}_2$, $\mathfrak{g}_{\mathbb{C}}$ has the following eigenspace decomposition:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(j)} \text{ where } \mathfrak{g}_{\mathbb{C}}^{(j)} = \{z \in \mathfrak{g}_{\mathbb{C}} \mid [x, z] = jz\}.$$

The subalgebra $\mathfrak{q} = \bigoplus_{j \in \mathbb{N}} \mathfrak{g}_{\mathbb{C}}^{(j)}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with a Levi part $l = \mathfrak{g}_{\mathbb{C}}^{(0)}$ and nilradical $u = \bigoplus_{j \in \mathbb{N}^*} \mathfrak{g}_{\mathbb{C}}^{(j)}$.

Call $\mathfrak{q}$ the Jacobson-Morosov parabolic subalgebra of e relative to the triple (x, e, f) . Our choice of the triple (x, e, f) forces $\mathfrak{q}$ to be θ -stable in Vogan's sense.

Retain the above notations. Let Q and L be the connected subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{q}$ and l respectively. Define $L \cap K_{\mathbb{C}}$ to be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $l \cap \mathfrak{k}_{\mathbb{C}}$. We shall prove some facts about some eigenspaces of x in $\mathfrak{p}_{\mathbb{C}}$.

Kostant and Rallis ([K-R, in proof of Lemma 4]) proved that $L \cap K_{\mathbb{C}} \cdot e$ is dense in $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$ which is therefore a prehomogeneous space in Sato's sense.

Let $\mathfrak{q}$ be the Jacobson-Morozov parabolic subalgebra of e relative to the normal triple (x, e, f) . Then

Proposition 1.2.1. *$Q \cap K_{\mathbb{C}} \cdot e$ is dense in $\bigoplus_{i \geq 2} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$. Moreover if e is even, that is $\mathfrak{g}_{\mathbb{C}}^{(i)} = 0$ for i odd, then $\overline{Q \cap K_{\mathbb{C}} \cdot e} = u \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. The proof can easily be obtained by modifying an argument of Carter [Ca, Proposition 5.7.3]. $\square$

Lemma 1.2.2. *With the above notation $\dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. See [No2]. $\square$

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Theorem 1.2.3. *$L \cap K_{\mathbb{C}}$ has finitely many orbits on each eigenspace $\mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. Let $S = \{1, \theta\}$ and define $H = L \rtimes_{\phi} S$ to be the semidirect product of L and S where ϕ is a homomorphism from S to $\text{Aut}(L)$ such that for $s \in S$ and $y \in L$, $\phi(s)(y) = s(y)$. If we identify L and S with $L \times 1$ and $1 \times S$ respectively, then they are closed subgroups of H . If (y_1, s_1) and (y_2, s_2) are two elements of H , then the group multiplication law on H is defined so that

$$(y_1, s_1) \times_{\phi} (y_2, s_2) = (y_1 \phi(s_1) y_2, s_1 s_2).$$

Observe that S is diagonalizable and since L is a subgroup of index 2 in H , it is normal in H . Moreover $\mathfrak{g}_{\mathbb{C}}$ is stable under the adjoint representation of H . Let $L^S = \{y \in L \mid sys^{-1} = y \text{ for every } s \in S\}$. It is obvious that $L^S = L \cap K_{\mathbb{C}}$ and that the (-1) weight space of S in $\mathfrak{g}_{\mathbb{C}}^{(i)}$ is $\mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$. The fact that L has a finite number

of orbits on $\mathfrak{g}_{\mathbb{C}}^{(i)}$ [R2, Theorem E] implies that

$$\mathfrak{g}_{\mathbb{C}}(i) \cap \mathfrak{p}_{\mathbb{C}} \subseteq \bigcup_{j=1}^n L.z_j \quad \text{for } z_j \in \mathfrak{g}_{\mathbb{C}}^{(i)}.$$

Our theorem now follows from a result of Richardson [R2, Theorem A] that if $z \in \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}$, then $L \cap K_{\mathbb{C}}$ has only a finite number of orbits on the intersection $(L.z) \cap (\mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}})$. $\square$

We shall now prove a theorem of great importance to us. We shall show that $L \cap K_{\mathbb{C}}$ has a finite number of orbits on $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$. But first, we give a Lie algebra version of a theorem of R.W. Richardson. See ([Röh, Theorem 1.4]).

Some definitions are needed. Let Φ be a root system of $\mathfrak{g}_{\mathbb{C}}$ and Δ the set of simple roots determined by our choice of a positive root system Φ^+ . It is a well known fact that any parabolic subalgebra is $G_{\mathbb{C}}$ -conjugate to a parabolic of the form $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, where

$$\mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Lambda \rangle} \mathfrak{g}_{\mathbb{C}}^{\alpha}$$

and

$$\mathfrak{u} = \sum_{\alpha \in \Phi^+ \setminus \langle \Lambda \rangle} \mathfrak{g}_{\mathbb{C}}^{\alpha}.$$

Here, $\Lambda \subseteq \Delta$, $\langle \Lambda \rangle$ denotes the subroot system of Φ generated by Λ and $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ relative to Δ . See [C-Mc].

Proposition 1.2.4. *Retaining the above notation*

1. u and $[u, u]$ are the direct sum of their 1-dimensional root spaces.
2. A root α of u is a root of $[u, u]$ if and only if it is the sum of two roots of u . Such roots are called decomposable.
3. A root in u is indecomposable if and only if it is the sum of one simple root not in Λ and various simple roots in Λ .

Proof. See Collingwood and McGovern [C-Mc, Proposition 8.2.7]. $\square$

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the Jacobson-Morozov parabolic subalgebra relative to the standard triple (x, e, f) of $\mathfrak{g}_{\mathbb{C}}$. Let L be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{l}$. We obtain:

Theorem 1.2.5 (Richardson). *L has only a finite number of orbits on $\frac{u}{[u, u]}$. In particular there exists a unique dense orbit and so $\dim \mathfrak{l} \geq \dim \frac{u}{[u, u]}$.*

Proof. See [No2], [R2, Theorem E]. $\square$

Now, let (x, e, f) be a normal triple and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ its Jacobson-Morozov parabolic. The next theorem is very important. It shows that $L \cap K_{\mathbb{C}}$ has a dense orbit on $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Theorem 1.2.6. *$L \cap K_{\mathbb{C}}$ has only a finite number of orbits on $\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.*

In particular there exists a unique dense $(L \cap K_{\mathbb{C}})$ -orbit and so $\dim \mathfrak{l} \cap \mathfrak{k}_{\mathbb{C}} \geq \dim \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.

Proof. Define S and H as in Theorem 1.2.3. Then $L \cap K_{\mathbb{C}} = L^S$. First we observe that $\frac{u \cap \mathfrak{p}_{\mathbb{C}} + [u, u]}{[u, u]}$ is included in $\left(\frac{u}{[u, u]}\right)_{-1}$, the -1 weight of S in $\frac{u}{[u, u]}$. It is a general fact (see [Br, Theorem 5.20]) that

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}} + [u, u]}{[u, u]} \cong \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u, u] \cap (u \cap \mathfrak{p}_{\mathbb{C}})}.$$

Furthermore

$$[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq [u, u] \cap (u \cap \mathfrak{p}_{\mathbb{C}}).$$

The fact that

$$[u, u] \subseteq [u \cap \mathfrak{k}_{\mathbb{C}}] + [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$$

implies that

$$[u, u] \cap (u \cap \mathfrak{p}_{\mathbb{C}}) \subseteq [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$$

because

$$[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq (u \cap \mathfrak{p}_{\mathbb{C}}) \quad \text{and} \quad (u \cap \mathfrak{k}_{\mathbb{C}}) \cap (u \cap \mathfrak{p}_{\mathbb{C}}) = 0.$$

Hence

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u, u] \cap (u \cap \mathfrak{p}_{\mathbb{C}})} = \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}.$$

It follows that

$$\frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]} \subseteq \left(\frac{u}{[u, u]}\right)_{-1}.$$

From the previous theorem $\frac{u}{[u, u]}$ is included in a finite union of L orbits. Each such orbit intersected with $\left(\frac{u}{[u, u]}\right)_{-1}$ is a finite union of $L \cap K_{\mathbb{C}}$ orbit by Richardson. [R2, Theorem A]. The desired result follows. $\square$

2. NOTICED NILPOTENT ORBITS IN SYMMETRIC SPACES

2.1. Noticed nilpotent elements and θ -stable parabolic subalgebras.

Definition. A nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$ (or its $K_{\mathbb{C}}$ -orbit) is *noticed* if the only (θ, σ) -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing e (or equivalently meeting $K_{\mathbb{C}} \cdot e$) is $\mathfrak{g}_{\mathbb{C}}$ itself.

A Levi subalgebra l contains e if and only if $[l, l]$ does. Thus if e is noticed in l , it is actually noticed in the semi-simple subalgebra $[l, l]$ and any nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ is noticed in any minimal (θ, σ) -stable Levi subalgebra l containing it.

Let (x, e, f) be a normal triple. Then from Proposition 1.1.3, e is noticed if and only if $\mathfrak{k}_{\mathbb{C}}^{(x, e, f)} = \{0\}$. Recall the $\mathbb{Z}$ -gradation

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(i)} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}.$$

Using this gradation it is easy to give a criterion for a nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ to be *noticed*.

Lemma 2.1.1. *Retain the above notations. Then a nilpotent element e in $\mathfrak{p}_{\mathbb{C}}$ is noticed if and only if $\dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. Dragomir Djoković proved the following result in section 12 of [D1]:

Let (x, e, f) be a normal triple of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{l}(e)$ be a Levi factor of $\mathfrak{k}_{\mathbb{C}}^e$. For any integer i let

$$\begin{aligned} N(0, i) &= \dim \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{k}_{\mathbb{C}}, \\ N(1, i) &= \dim \mathfrak{g}_{\mathbb{C}}^{(i)} \cap \mathfrak{p}_{\mathbb{C}}. \end{aligned}$$

Then $\dim \mathfrak{l}(e) = N(0, 0) - N(1, 2)$.

Note that $\mathfrak{k}_{\mathbb{C}}^{(x, e, f)}$ is a Levi factor of $\mathfrak{k}_{\mathbb{C}}^e$. Hence,

$$(2.1.2) \quad \dim \mathfrak{k}_{\mathbb{C}}^{(x, e, f)} = \dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} - \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}.$$

But e is *noticed* if and only if $\mathfrak{k}_{\mathbb{C}}^{(x, e, f)} = 0$. The lemma follows. $\square$

Later, we shall give a classification of the conjugacy classes of the nilpotent orbits of $K_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$. It is a generalization of the Bala-Carter classification for $G_{\mathbb{C}}$ -nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ [B-C1]. Bala and Carter use the notion of *distinguished* parabolic subalgebra in order to parametrize the nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$. Such parabolic algebras always contain a *distinguished* nilpotent element in $\mathfrak{g}_{\mathbb{C}}$.

Definition. A nilpotent element X in $\mathfrak{g}_{\mathbb{C}}$ (or its $G_{\mathbb{C}}$ -orbit) is *distinguished* if the only Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing X (or equivalently meeting $G_{\mathbb{C}} \cdot X$) is $\mathfrak{g}_{\mathbb{C}}$ itself.

Clearly any distinguished nilpotent belonging to $\mathfrak{p}_{\mathbb{C}}$ is automatically noticed. Furthermore Bala and Carter give the following criterion for a nilpotent $X \in \mathfrak{g}_{\mathbb{C}}$ to be distinguished:

Lemma 2.1.3. *Retain the above notations. Then X is distinguished if and only if $\dim \mathfrak{g}_{\mathbb{C}}^{(0)} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)}$.*

Proof. See Bala and Carter [B-C1] or Collingwood and McGovern [C-Mc, Lemma 8.2.1]. $\square$

Observe the similarity between the criterion for X to be distinguished and that for e to be noticed.

Recall that a non-zero nilpotent X of $\mathfrak{g}_{\mathbb{C}}$ is even if and only if $\mathfrak{g}_{\mathbb{C}}^{(i)} = 0$ whenever i is odd. Bala and Carter also proved that:

Theorem 2.1.4. *Any distinguished nilpotent $X \in \mathfrak{g}_{\mathbb{C}}$ is even.*

Proof. See Bala and Carter [B-C1] or Collingwood and McGovern [C-Mc, Theorem 8.2.3]. $\square$

The above evenness property is not shared by noticed nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$. We shall give some examples. But first we give the Bala-Carter characterization of *distinguished parabolic* subalgebras.

Let H, X, Y be a standard triple of $\mathfrak{g}_{\mathbb{C}}$ and let p be the Jacobson-Morozov parabolic subalgebra of X with Levi decomposition $p = m \oplus v$. Then we have:

Theorem 2.1.5. *X is distinguished if and only if $\dim m = \dim \frac{v}{[v, v]}$.*

Proof. See Bala and Carter [B-C1] or Collingwood and McGovern [C-Mc, Theorem 8.2.6]. $\square$

Therefore it makes sense to define an arbitrary parabolic subalgebra $p = m \oplus v$ to be *distinguished* if $\dim m = \dim \frac{v}{[v, v]}$.

If e is an even noticed nilpotent element of $\mathfrak{p}_\mathbb{C}$, then we can prove a theorem similar to Theorem 2.1.5.

Theorem 2.1.6. *Let e be an even nilpotent element of $\mathfrak{p}_\mathbb{C}$. Let $\mathfrak{q}$ be a θ -stable Jacobson-Morosov parabolic subalgebra of e relative to a triple (x, e, f) defined as above. Then $\mathfrak{q} = l \oplus u$ and $\dim l \cap \mathfrak{k}_\mathbb{C} = \dim \frac{u \cap \mathfrak{p}_\mathbb{C}}{[u \cap \mathfrak{k}_\mathbb{C}, u \cap \mathfrak{p}_\mathbb{C}]}$ if and only if e is noticed.*

Proof. We may assume that $\mathfrak{q}$ is defined as above. Let $u' = \bigoplus_{i \geq 4} \mathfrak{g}_\mathbb{C}^{(i)}$. Then

$$\dim l \cap \mathfrak{k}_\mathbb{C} = \dim \mathfrak{g}_\mathbb{C}^{(0)} \cap \mathfrak{k}_\mathbb{C},$$

while

$$\dim \mathfrak{g}_\mathbb{C}^{(2)} \cap \mathfrak{p}_\mathbb{C} = \dim u \cap \mathfrak{p}_\mathbb{C} - \dim u' \cap \mathfrak{p}_\mathbb{C}.$$

By definition we have

$$[u \cap \mathfrak{k}_\mathbb{C}, u \cap \mathfrak{p}_\mathbb{C}] \subset u' \cap \mathfrak{p}_\mathbb{C}.$$

By the representation theory of $\mathfrak{sl}_2$,

$$\text{if } z \in \mathfrak{g}_\mathbb{C}^{(i)} \cap \mathfrak{p}_\mathbb{C} \text{ and } i \geq 4, \text{ then } z = [e, z'] \text{ for some } z' \text{ in } \mathfrak{g}_\mathbb{C}^{(i-2)} \cap \mathfrak{k}_\mathbb{C} \subset u \cap \mathfrak{k}_\mathbb{C}.$$

Hence

$$u' \cap \mathfrak{p}_\mathbb{C} \subset [u \cap \mathfrak{k}_\mathbb{C}, u \cap \mathfrak{p}_\mathbb{C}].$$

The conclusion follows at once from Theorem 1.2.6 and Lemma 2.1.1. $\square$

For exceptional Lie algebras one obtains the following:

Proposition 2.1.7. *Let $\mathfrak{g}_\mathbb{C}$ be an exceptional simple complex Lie algebra. Let $\mathfrak{q} = l \oplus u$ be a θ -stable Jacobson-Morosov parabolic subalgebra of e relative to a normal triple (x, e, f) defined as above. If e is noticed, then $\dim l \cap \mathfrak{k}_\mathbb{C} = \dim \frac{u \cap \mathfrak{p}_\mathbb{C}}{[u \cap \mathfrak{k}_\mathbb{C}, u \cap \mathfrak{p}_\mathbb{C}]}$.*

Proof. For our purpose we use the tables obtained by Djoković [D1, D2] in order to isolate the orbits of interest. In view of the previous theorem, we only need to consider non-even orbits for which $\mathfrak{k}_\mathbb{C}^{(x, e, f)} = 0$. It turns out there are only five of them (see Table 1).

The details can be found in [No2]. Since $\mathfrak{q}$ is the Jacobson-Morosov parabolic subalgebra of e , we have

$$\frac{u \cap \mathfrak{p}_\mathbb{C}}{[u \cap \mathfrak{k}_\mathbb{C}, u \cap \mathfrak{p}_\mathbb{C}]} \cong \mathfrak{g}_\mathbb{C}^{(1)} \cap \mathfrak{p}_\mathbb{C} \oplus \frac{\mathfrak{g}_\mathbb{C}^{(2)} \cap \mathfrak{p}_\mathbb{C}}{[\mathfrak{g}_\mathbb{C}^{(1)} \cap \mathfrak{k}_\mathbb{C}, \mathfrak{g}_\mathbb{C}^{(1)} \cap \mathfrak{p}_\mathbb{C}]}$$

as $L \cap K_\mathbb{C}$ -modules.

Following Djoković we find that the only relevant cases come from the real forms $E_{7(7)}$, $E_{8(8)}$ and $E_{6(6)}$.

Computations using the software Lie on a Macintosh IIci reveal that

$$\begin{aligned} \dim \mathfrak{g}_\mathbb{C}^{(0)} \cap \mathfrak{k}_\mathbb{C} &= \dim \mathfrak{g}_\mathbb{C}^{(2)} \cap \mathfrak{p}_\mathbb{C}, \\ \dim \mathfrak{g}_\mathbb{C}^{(1)} \cap \mathfrak{k}_\mathbb{C} &= \dim [\mathfrak{g}_\mathbb{C}^{(1)} \cap \mathfrak{k}_\mathbb{C}, \mathfrak{g}_\mathbb{C}^{(1)} \cap \mathfrak{p}_\mathbb{C}] \end{aligned}$$

as indicated in the following table. $\square$

TABLE 1.¹

Algebra	orbits	$l \cap \mathfrak{k}_{\mathbb{C}}$	$\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}$	$\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}$	$\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$	[,]
$E_{7(7)}$	1111111	7	7	7	7	7
$E_{8(8)}$	11111111	8	8	8	8	8
$E_{8(8)}$	11111121	8	7	7	8	7
$E_{6(6)}$	1111	4	4	4	4	4
$E_{6(6)}$	1112	4	3	3	4	3

2.2. An important counter-example. It is not true in general that if e is noticed, then

$$\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}.$$

We shall give the following example.

Let $\mathfrak{g}$ be $\mathfrak{sl}(7, \mathbb{R})$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(7, \mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(7, \mathbb{C})$, and $\mathfrak{p}_{\mathbb{C}}$ is the space of 7×7 complex symmetric matrices. The Cartan involution θ is defined as $\theta(X) = -X^T$ for $X \in \mathfrak{g}$. Let

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$E = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Choose $F = E^T$. Then $\theta(H) = -H$, $\theta(E) = -F$. Hence (H, E, F) is a KS-triple, and $x = i(E - F)$ is in $\mathfrak{k}_{\mathbb{C}}$. In fact under the Kostant-Sekiguchi map (H, E, F) corresponds to a normal triple (x, e, f) , with

$$e = \frac{1}{2}(E + F + iH) \quad \text{and} \quad f = \frac{1}{2}(E + F - iH).$$

Next we compute the following eigenspaces of x .

A simple computation [No2] shows that $\dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} = 5$. Hence by Lemma 2.1.1, the triple (x, e, f) is noticed.

¹[,] stands for $[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]$.

From the theory of the classification of the real nilpotent orbits of $\mathfrak{sl}_n$ [C-Mc] and by Sekiguchi [Se]

$$G_{\mathbb{C}}.e = G_{\mathbb{C}}.E,$$

and the triple (x, e, f) can be associated to the partition $[4, 2, 1]$ of 7 and the corresponding weighted Dynkin diagram is

$$\begin{array}{cccccc} 2 & & 0 & & 1 & & 1 & & 0 & & 2 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$$

One sees that $\dim \mathfrak{g}_{\mathbb{C}}^{(1)} = 4$. Consequently $\dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} = 2$. But $\dim[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}] = 3$. So

$$\begin{aligned} \dim l \cap \mathfrak{k}_{\mathbb{C}} &= 5 \neq 2 + (5 - 3) = \dim \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}} \\ &+ \dim \frac{\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}}{[\mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^{(1)} \cap \mathfrak{p}_{\mathbb{C}}]} = \dim \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}. \end{aligned}$$

3. AN EXTENSION OF THE BALA-CARTER THEORY

3.1. The Bala-Carter correspondence. The Bala-Carter Classification can be expressed as follows:

Theorem 3.1.1 (Bala-Carter). *There is a one-to-one correspondence between nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$ and $G_{\mathbb{C}}$ -conjugacy classes of pairs (m, p_m) , where m is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and p_m is a distinguished parabolic subalgebra of the semisimple algebra $[m, m]$.*

Proof. Bala and Carter [B-C1] (Also see Carter [Ca, Theorem 5.9.5], Collingwood and McGovern [C-Mc, Theorem 8.2.12]). $\square$

3.2. Extension of the Bala-Carter correspondence. Let $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$. Then, $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_u \oplus i\mathfrak{g}_u$. Let κ be the Killing form on $\mathfrak{g}_{\mathbb{C}}$ and $\tau = \theta \circ \sigma$, the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}_u$. For X and Y in $\mathfrak{g}_{\mathbb{C}}$ define $\kappa'(X, Y) = -\kappa(X, \tau(Y))$. It is well known that κ' is a positive definite hermitian form on $\mathfrak{g}_{\mathbb{C}}$.

The next lemma is also known.

Lemma 3.2.1. *Let $\mathfrak{q} = \bigoplus_{j \geq 0} \mathfrak{g}_{\mathbb{C}}^{(j)}$ be the Jacobson-Morozov parabolic subalgebra associated with the triple (x, e, f) where $x \in i\mathfrak{g}_u$. Then for $j \neq k$, $\mathfrak{g}_{\mathbb{C}}^{(j)}$ is orthogonal to $\mathfrak{g}_{\mathbb{C}}^{(k)}$ relative to κ' .*

Proof. We can assume $k > 0$. Observe that $\tau(x) = -x$, since $x \in i\mathfrak{g}_u$. Let $y \in \mathfrak{g}_{\mathbb{C}}^{(k)}$, $z \in \mathfrak{g}_{\mathbb{C}}^{(j)}$ and $k \neq j$. Then

$$[x, \tau(z)] = \tau([\tau(x), z]) = \tau([-x, z]) = -j\tau(z).$$

Hence $\tau(z) \in \mathfrak{g}_{\mathbb{C}}^{(-j)}$. Using associativity of the Killing form κ , we have

$$\begin{aligned} \kappa'(y, z) &= -\kappa(y, \tau(z)) = -\frac{1}{k}\kappa([x, y], \tau(z)) \\ &= \frac{1}{k}\kappa(y, [x, \tau(z)]) = -\frac{j}{k}\kappa(y, \tau(z)) = \frac{j}{k}\kappa'(y, z). \end{aligned}$$

Hence

$$\kappa'(y, z) = 0.$$

□

We will need the following lemma.

Lemma 3.2.2. *For any normal triple (x, e, f) of $\mathfrak{g}_{\mathbb{C}}$, $[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}$.*

Proof. From the $\mathfrak{sl}(2, \mathbb{C})$ representation theory we know that

$$[\mathfrak{p}_{\mathbb{C}}^x, e] = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}.$$

Therefore, since $e \in \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$ we have

$$[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] \supseteq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}.$$

On the other hand, clearly,

$$[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] \subseteq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}.$$

The lemma follows. □

Let $\mathfrak{q} = l \oplus u$ be a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{m}$ be the orthogonal complement of $[u \cap \mathfrak{k}_{\mathbb{C}}, [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]]$ relative to κ' inside $u \cap \mathfrak{p}_{\mathbb{C}}$. Define $\mathfrak{w}$ to be an $L \cap K_{\mathbb{C}}$ module in $\mathfrak{m}$. Finally, let $\hat{\mathfrak{w}} = \mathfrak{w} \oplus [l \cap \mathfrak{p}_{\mathbb{C}}, \mathfrak{w}]$. Clearly, $\hat{\mathfrak{w}}$ is θ -stable.

Definition. Define $\mathcal{L}$ to be the set of triples $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ such that

1. $\mathfrak{w}$ has a dense $L \cap K_{\mathbb{C}}$ orbit: $(L \cap K_{\mathbb{C}}) \cdot \hat{e}$,
2. $\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{w}$
3. $L \cdot \hat{e}$ is dense in $\hat{\mathfrak{w}}$
4. $\hat{\mathfrak{w}} \perp [u, [u, u]]$
5. $[u, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}$
6. $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq [\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$.

Property 3 implies that $\hat{\mathfrak{w}}$ is an L -module for

$$\overline{L \cdot \hat{e}} = \hat{\mathfrak{w}}.$$

Hence $\hat{\mathfrak{w}}$ is L stable.

Let $\mathfrak{S}$ be the set of noticed KS-triples (x, e, f) of $\mathfrak{g}_{\mathbb{C}}$.

We have a map $\mathfrak{F}$ from $\mathfrak{S}$ to $\mathcal{L}$ which associates a triple (x, e, f) of $\mathfrak{S}$ to an element $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}_x, \mathfrak{w})$ of $\mathcal{L}$ where $\mathfrak{q}_x$ is the θ -stable Jacobson-Morosov parabolic subalgebra of (x, e, f) .

Let

$$\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}.$$

From Kostant and Rallis [K-R] we know that $L \cdot e$ (respectively $L \cap K_{\mathbb{C}} \cdot e$) is dense in $\mathfrak{g}_{\mathbb{C}}^{(2)}$ (respectively $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$). Also

$$\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}},$$

because e is noticed (see Lemma 2.1.1).

By definition $l \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^x$.

Since

$$[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}},$$

by Lemma 3.2.2, we must have

$$\hat{\mathfrak{w}} = \mathfrak{g}_{\mathbb{C}}^{(2)}.$$

Hence conditions 1, 2 and 3 in the above definition are verified. Furthermore, we know that

$$u = \bigoplus_{j \in \mathbb{N}^*} \mathfrak{g}_{\mathbb{C}}^{(j)}.$$

it follows that

$$\mathfrak{g}_{\mathbb{C}}^{(2)} \perp [u, [u, u]],$$

by Lemma 3.2.1.

To see that condition 5 holds it is enough to observe that

$$[\mathfrak{g}_{\mathbb{C}}^{(i)}, \mathfrak{g}_{\mathbb{C}}^{(j)}] \subseteq \mathfrak{g}_{\mathbb{C}}^{(i+j)}.$$

This observation, the definition of u and Lemma 3.2.1 imply that

$$[u, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}.$$

From the representation theory of $\mathfrak{sl}_2$ we have

$$[\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}] = u \cap \mathfrak{p}_{\mathbb{C}}.$$

Hence

$$[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq [\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}].$$

Therefore $\mathfrak{F}$ is well defined.

From a theorem of Kostant and Rallis [K-R], there is a bijection between the non-zero nilpotent $K_{\mathbb{C}}$ -orbits in $\mathfrak{p}_{\mathbb{C}}$ and the $K_{\mathbb{C}}$ -conjugacy classes of normal triples. Two normal noticed triples (x, e, f) and (x', e', f') are $K_{\mathbb{C}}$ conjugate if and only if their corresponding triples $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w})$ and $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}', \mathfrak{w}')$ are $K_{\mathbb{C}}$ conjugate. Hence, $\mathfrak{F}$ induces a one-to-one map from $K_{\mathbb{C}}$ -conjugacy classes of $\mathfrak{S}$ and the $K_{\mathbb{C}}$ -conjugacy classes of the triples of $\mathfrak{L}$. The next theorem tells us that such a map is also surjective.

If $q = l \oplus u$ is a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, then there exists $z \in i\mathfrak{l}$ such that

$$\begin{aligned} u &= \text{sum of eigenspaces of } ad_z \text{ for positive eigenvalues,} \\ l &= \text{eigenspace of } ad_z \text{ for eigenvalue 0,} \\ \bar{u} &= \text{sum of eigenspaces of } ad_z \text{ for negative eigenvalues [K-V].} \end{aligned}$$

Furthermore l is (θ, σ) -stable. Also u and $\bar{u}$ are both θ -stable. Finally

$$\mathfrak{g}_{\mathbb{C}} = \bar{u} \oplus l \oplus u,$$

$\tau(u) = \bar{u}$, and from Lemma 3.2.1 the spaces l , u and $\bar{u}$ are mutually orthogonal relative to κ' .

Theorem 3.2.3. *For any triple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w})$ of $\mathfrak{L}$ there exists a normal triple (x, e, f) in $\mathfrak{S}$ such that $\mathfrak{q}$ is the Jacobson-Morosov parabolic subalgebra for (x, e, f) and $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. Let $\tau(\mathfrak{w}) = \bar{\mathfrak{w}}$. First, we prove that

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l \cap \mathfrak{k}_{\mathbb{C}}.$$

Clearly $[\mathfrak{w}, \bar{\mathfrak{w}}] \subset \mathfrak{k}_{\mathbb{C}}$ since both $\mathfrak{w}$ and $\bar{\mathfrak{w}}$ are subsets of $\mathfrak{p}_{\mathbb{C}}$. Therefore it is enough to show that

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l.$$

But

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \perp u,$$

because for all $X \in \mathfrak{w}$, $Y \in \bar{\mathfrak{w}}$ and $U \in u$ we have

$$\begin{aligned} \kappa'(U, [X, Y]) &= -\kappa(U, \tau([X, Y])) = -\kappa(U, [\tau(X), \tau(Y)]) \\ &= \kappa(U, [\tau(Y), \tau(X)]) = -\kappa([\tau(Y), U], \tau(X)) = \kappa'([\tau(Y), U], X) = 0, \end{aligned}$$

since κ is invariant, $\tau(Y) \in \mathfrak{w}$ and $[u, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}$.

Furthermore

$$[\bar{u}, \bar{\mathfrak{w}}] \perp \bar{\mathfrak{w}},$$

because for all B and $C \in \bar{\mathfrak{w}}$ and $A \in \bar{u}$ we have

$$\kappa'([A, B], C) = -\kappa(\tau(C), \tau[\tau(A), \tau(B)]) = \kappa'(\tau(C), [\tau(A), \tau(B)]) = 0,$$

since $\tau(B), \tau(C) \in \mathfrak{w}$ and $\tau(A) \in u$ and $[u, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}$.

Moreover for all $X \in \mathfrak{w}$, $Y \in \bar{\mathfrak{w}}$ and $U \in \bar{u}$ we have

$$\begin{aligned} \kappa'(U, [X, Y]) &= -\kappa(U, \tau([X, Y])) = -\kappa(U, [\tau(X), \tau(Y)]) \\ &= -\kappa([U, \tau(X)], \tau(Y)) = \kappa'([U, \tau(X)], Y) = 0, \end{aligned}$$

since κ is invariant, $\tau(X) \in \bar{\mathfrak{w}}$ and $[\bar{u}, \bar{\mathfrak{w}}] \perp \bar{\mathfrak{w}}$. Hence $[\mathfrak{w}, \bar{\mathfrak{w}}] \perp \bar{u}$.

It follows that $[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l$, for l , u and $\bar{u}$ are mutually orthogonal relative to κ' . Therefore we must have

$$[\mathfrak{w}, \bar{\mathfrak{w}}] \subset l \cap \mathfrak{k}_{\mathbb{C}}.$$

Since $L \cap K_{\mathbb{C}}$ has a dense orbit in $\mathfrak{w}$ there exists $Z_{\mathfrak{w}}$ in $\mathfrak{w}$ such that

$$[l \cap \mathfrak{k}_{\mathbb{C}}, Z_{\mathfrak{w}}] = \mathfrak{w}.$$

We now claim that $\mathfrak{g}_{\mathbb{C}}^{Z_{\mathfrak{w}}} \cap \bar{\mathfrak{w}} = 0$. Indeed, let y be a non-zero element of $\bar{\mathfrak{w}}$ such that $[y, Z_{\mathfrak{w}}] = 0$. Then

$$\kappa(\mathfrak{w}, y) = \kappa([Z_{\mathfrak{w}}, l \cap \mathfrak{k}_{\mathbb{C}}], y) = -\kappa(l \cap \mathfrak{k}_{\mathbb{C}}, [Z_{\mathfrak{w}}, y]) = 0,$$

which contradicts the fact that $\mathfrak{w}$ and $\bar{\mathfrak{w}}$ are paired nondegenerately by the Killing form κ of $\mathfrak{g}_{\mathbb{C}}$; but

$$\mathfrak{g}_{\mathbb{C}}^{Z_{\mathfrak{w}}} \text{ is } \kappa\text{-orthogonal to } [\mathfrak{g}_{\mathbb{C}}, Z_{\mathfrak{w}}] \supset [l \cap \mathfrak{k}_{\mathbb{C}}, Z_{\mathfrak{w}}] = \mathfrak{w}.$$

Hence,

$$\dim[\bar{\mathfrak{w}}, Z_{\mathfrak{w}}] = \dim \bar{\mathfrak{w}} = \dim \mathfrak{w} = \dim l \cap \mathfrak{k}_{\mathbb{C}}$$

whence we have

$$[\bar{\mathfrak{w}}, Z_{\mathfrak{w}}] = l \cap \mathfrak{k}_{\mathbb{C}},$$

since $Z_{\mathfrak{w}} \in \mathfrak{w}$.

Fix a θ -stable positive root system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. To construct $\mathfrak{h}_{\mathbb{C}}$ one starts with a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}$ and adjoins a subspace $\mathfrak{a}$ of $\mathfrak{p}$ that is maximal

with respect to the properties of being abelian and commuting with $\mathfrak{t}$ (see Knapp and Vogan [K-V]). Then $\mathfrak{h}_{\mathbb{C}}$ is the complexification of

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}.$$

Furthermore since $\mathfrak{q}$ is a θ -stable parabolic subalgebra the subspaces u and $[u, u]$ have the properties mentioned in Proposition 1.2.4. If α is a weight of $\mathfrak{g}_{\mathbb{C}}$, then $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ denotes the corresponding weight space and $X_{\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ is called a weight vector.

Choose $x \in \mathfrak{k}_{\mathbb{C}}$ such that for any simple root β of $\mathfrak{g}_{\mathbb{C}}$,

$$\begin{aligned} \beta(x) &= 0 && \text{if } \mathfrak{g}_{\mathbb{C}}^{\beta} \subseteq l, \\ \beta(x) &= 2 && \text{if } \mathfrak{g}_{\mathbb{C}}^{\beta} \subseteq \hat{\mathfrak{w}}, \\ \beta(x) &= 1 && \text{otherwise.} \end{aligned}$$

The next step is to prove that $\mathfrak{w}$ is the 2-eigenspace of ad_x on $\mathfrak{p}_{\mathbb{C}}$. Observe that, since l , $\hat{\mathfrak{w}}$ and the set of simple roots of $\mathfrak{g}_{\mathbb{C}}$ whose root spaces are not in $\hat{\mathfrak{w}}$ are θ -stable, we have

$$\beta(x) = \theta\beta(x), \text{ for all simple roots } \beta \text{ of } \mathfrak{g}_{\mathbb{C}}.$$

Furthermore two different simple roots of $\mathfrak{g}_{\mathbb{C}}$ cannot restrict to the same weight of $\mathfrak{p}_{\mathbb{C}}$. (See [K-V], page 257). In other words $\mathfrak{t}$ -weights in $u \cap \mathfrak{p}_{\mathbb{C}}$ occur with multiplicity 1.

Define η and $\hat{\eta}$ such that

$$u \cap \mathfrak{p}_{\mathbb{C}} = [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \oplus \eta$$

and

$$u = [u, u] \oplus \hat{\eta}.$$

Clearly $\eta = \hat{\eta} \cap \mathfrak{p}_{\mathbb{C}}$.

Let ν be a weight of $\mathfrak{p}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}}^{\nu} \subseteq \hat{\mathfrak{w}}$. Then we only need to consider two cases.

Case 1. ν is the restriction of a simple root α and $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ lies in $\hat{\eta}$. Then $\alpha(x) = 2$ since $\hat{\eta} \subseteq \hat{\mathfrak{w}}$. Hence $\nu(x) = 2$.

Case 2. ν is a weight of $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$. Let X_{ν} be a non-zero vector in $[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ such that

$$[x, X_{\nu}] = \nu(x)X_{\nu}.$$

Then

$$X_{\nu} \in [u \cap \mathfrak{k}_{\mathbb{C}}, \eta] \oplus [u \cap \mathfrak{k}_{\mathbb{C}}, [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]].$$

Since $\hat{\mathfrak{w}} \perp [u, [u, u]]$ X_{ν} must be in $[u \cap \mathfrak{k}_{\mathbb{C}}, \eta]$. But

$$u \cap \mathfrak{k}_{\mathbb{C}} = [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{k}_{\mathbb{C}}] \oplus \mathfrak{v}.$$

Therefore

$$X_{\nu} \in [\mathfrak{v}, \eta],$$

for $\hat{\mathfrak{w}} \perp [u, [u, u]]$.

Now any weight α such that $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ lies in $\mathfrak{v}$ or η must be the restriction of a simple root of $\mathfrak{g}_{\mathbb{C}}$, otherwise X_{ν} would be in $[u, [u, u]]$.

Moreover, since

$$[u, \hat{\mathfrak{w}}] \cap \hat{\mathfrak{w}} = \{0\},$$

we must have

$$\alpha(x) = 1.$$

Hence

$$\nu(x) = 2.$$

Therefore $\mathfrak{w} \subseteq \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.

Next we show that $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} \subseteq \mathfrak{w}$.

If ν is a weight of $\mathfrak{m}$ but not of $\mathfrak{w}$, then either $X_{\nu} \notin [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ or $X_{\nu} \in [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]$ for

$$\mathfrak{m} \perp [u \cap \mathfrak{k}_{\mathbb{C}}, [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]].$$

In the first case we have $\nu(x) = 1$.

If $X_{\nu} \in [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \subseteq [\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$, then either $X_{\nu} \in [l \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$ or $X_{\nu} \in [u \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}]$. Since $\mathfrak{w}$ is an $L \cap K_{\mathbb{C}}$ -module, we must have

$$X_{\nu} \in [u \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{w}].$$

It follows that $\nu(x) \geq 3$.

Therefore if $\nu = \beta|_{\mathfrak{i}}$, then X_{β} must be in $\hat{\mathfrak{w}}$. Hence

$$\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}} \quad \text{and} \quad \hat{\mathfrak{w}} = \mathfrak{g}_{\mathbb{C}}^{(2)}.$$

Choose $e = Z_{\mathfrak{w}}$. Note that the fact that $[\mathfrak{w}, Z_{\mathfrak{w}}] = l \cap \mathfrak{k}_{\mathbb{C}}$ makes it possible to find f in $\mathfrak{w}$ such that $[e, f] = x$, but then since $\mathfrak{w}$ is the 2-eigenspace of x in $\mathfrak{p}_{\mathbb{C}}$, $\mathfrak{w}$ is the -2 -eigenspace. Therefore $[x, f] = -2f$. The triple (x, e, f) is normal. The nilpotent e is noticed since $\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{w}$. $\square$

The following theorem completes the classification.

Let l be a (θ, σ) -stable Levi Subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Define the set of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ to have the same properties as the triples of $\mathfrak{L}$, replacing $\mathfrak{g}_{\mathbb{C}}$ by l . Here $\mathfrak{q}_l$ is a θ -stable parabolic subalgebra of $[l, l]$. Then we have:

Theorem 3.2.4. *There is a one-to-one correspondence between nilpotent $K_{\mathbb{C}}$ -orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$ -conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ in $\mathfrak{L}$.*

Proof. We noted before that a Levi subalgebra l contains a nilpotent element $e \in \mathfrak{p}_{\mathbb{C}}$ if and only if $[l, l]$ does. Two Levi subalgebras are $K_{\mathbb{C}}$ -conjugate if and only if their derived subalgebras are conjugate. Each nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ can be put in a normal triple (x, e, f) inside of the minimal Levi subalgebra l containing e . By Theorem 1.1.4 two minimal Levi subalgebras containing e are conjugate under $K_{\mathbb{C}}^e$. Hence the theorem follows from Theorem 3.2.3 $\square$

3.3. Example. Let $\mathfrak{g}$ be $\mathfrak{sl}(3, \mathbb{R})$, the set of 3×3 real matrices of trace 0. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(3, \mathbb{C})$, and $\mathfrak{p}_{\mathbb{C}}$ is the space of 3×3 complex symmetric matrices. The Cartan involution θ is defined as $\theta(X) = -X^T$ for $X \in \mathfrak{g}$. Denote by $\bar{Y}$, the complex conjugate of a matrix $Y \in \mathfrak{g}_{\mathbb{C}}$.

The set of orthogonal matrices ($K_{\mathbb{C}}$) preserves the set of symmetric matrices ($\mathfrak{p}_{\mathbb{C}}$) under conjugation. The nilpotent orbits of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ are parametrized by the partitions of 3. Therefore, there are exactly two non-zero nilpotent classes since

the zero nilpotent class corresponds to the partition $[1, 1, 1]$. A computation shows that the following matrices

$$H_1 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$E_1 = \frac{1}{2} \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -1 \\ i & 1 & 0 \end{pmatrix}$$

generate the only θ -stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$ up to conjugacy. Let

$$\mathfrak{b} = \mathbb{C}H_1 \oplus \mathbb{C}E_3 \oplus \mathbb{C}H_2 \oplus \mathbb{C}E_1 \oplus \mathbb{C}E_2.$$

Of course $\mathfrak{b}$ is conjugate to the set of upper triangular matrices of $\mathfrak{sl}(3, \mathbb{C})$.

The only (σ, θ) stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$, other than $\mathfrak{g}_{\mathbb{C}}$, is $\mathfrak{b} = \mathbb{C}H_1 + \mathbb{C}H_2$. Let $\mathfrak{u} = \sum_i \mathbb{C}E_i$.

We see that $l \cap \mathfrak{k}_{\mathbb{C}} = \mathbb{C}H_1$ and $\mathfrak{m} = \mathbb{C}E_1 \oplus \mathbb{C}E_2$. Let $\mathfrak{w}_1 = \mathbb{C}E_1$ and $\mathfrak{w}_2 = \mathbb{C}E_2$. Clearly, $L \cap K_{\mathbb{C}}$ has a dense orbit on $\mathfrak{w}_1$ and $\mathfrak{w}_2$ respectively for $[H_1, E_1] = 2E_1$ and $[H_1, E_2] = E_2$. Also

$$\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{w}_1 = \dim \mathfrak{w}_2 = 1.$$

For each $\mathfrak{w}_i$ one verifies easily that the triple, $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_i)$, satisfies all the requirements specified above.

Thus we obtain the following correspondence between non-zero nilpotent orbits of $\mathfrak{p}_{\mathbb{C}}$ and elements of $\mathfrak{L}$:

$$(H_1, E_1, \bar{E}_1) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_1),$$

$$(2H_1, E_2, \bar{E}_2) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_2).$$

3.4. An important special case. In some special cases the θ -stable parabolic subalgebra in the triple $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ is characterized by a dimension criterion similar to that of the *distinguished* parabolic subalgebras that play an essential role in the Bala-Carter theory. One of these cases is very important because it classifies all the noticed orbits of the classical real Lie algebras of type B , C and D and those of the exceptional algebras of type $E_{6(2)}$, $F_{4(4)}$, $G_{2(2)}$. We shall say more about this in the next section.

Define $\bar{\mathfrak{L}}$ to be the set of triples $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ of $\mathfrak{L}$ such that

1. $\dim l \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{u \cap \mathfrak{p}_{\mathbb{C}}}{[u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]}$.
2. $u \cap \mathfrak{p}_{\mathbb{C}} = [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}] \oplus \mathfrak{w}$.

Let $\bar{\mathfrak{S}}$ be the set of KS-triples (x, e, f) of $\mathfrak{g}_{\mathbb{C}}$ such that e is even and noticed, then we have a map from $\bar{\mathfrak{S}}$ to $\bar{\mathfrak{L}}$ which associates a triple (x, e, f) of $\bar{\mathfrak{S}}$ to an element $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ of $\bar{\mathfrak{L}}$ where $\mathfrak{q}$ is the θ -stable Jacobson-Morozov parabolic subalgebra relative to (x, e, f) and $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}$. From the previous results (see Theorem 2.1.6) it is clear that such a map is well defined and induces a one-to-one map from $K_{\mathbb{C}}$ -orbits of $\bar{\mathfrak{S}}$ to the $K_{\mathbb{C}}$ -conjugacy classes of the triples of $\bar{\mathfrak{L}}$, which is also surjective by the following theorem.

Theorem 3.4.1. *For any triple $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ of $\bar{\mathfrak{L}}$ there exists a normal triple (x, e, f) in $\bar{\mathfrak{S}}$ such that $\mathfrak{q}$ is the θ -stable parabolic subalgebra relative to (x, e, f) and $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.*

Proof. From Theorem 3.2.3 $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ corresponds to a normal triple (x, e, f) of $\mathfrak{S}$ such that $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$. But from the definition of $\mathfrak{w}$ and from Theorem 2.1.6 the triple (x, e, f) belongs to $\overline{\mathfrak{S}}$. The theorem follows. $\square$

We may summarize the foregoing results as follows.

Theorem 3.4.2. *There is a one-to-one correspondence between even nilpotent $K_{\mathbb{C}}$ -orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$ conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ where l is a (σ, θ) -stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and*

0. $\mathfrak{q}_l = m_l \oplus u_l$ is a θ -stable parabolic subalgebra of $[l, l]$.
1. $\mathfrak{w}_l \subseteq u_l \cap \mathfrak{p}_{\mathbb{C}}$.
2. $\mathfrak{w}_l$ is $M_l \cap K_{\mathbb{C}}$ stable; (M_l connected Lie group of $G_{\mathbb{C}}$ with Lie algebra m_l).
3. $\mathfrak{w}_l$ has a dense $M_l \cap K_{\mathbb{C}}$ orbit.
4. $\dim m_l \cap \mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{w}_l$.
5. $\dim m_l \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{u_l \cap \mathfrak{p}_{\mathbb{C}}}{[u_l \cap \mathfrak{k}_{\mathbb{C}}, u_l \cap \mathfrak{p}_{\mathbb{C}}]}$.
6. $u_l \cap \mathfrak{p}_{\mathbb{C}} = [u_l \cap \mathfrak{k}_{\mathbb{C}}, u_l \cap \mathfrak{p}_{\mathbb{C}}] \oplus \mathfrak{w}_l$.

Proof. Similar to the proof of Theorem 3.2.4. $\square$

Remarks. It is possible to have more than two noticed elements associated to the same θ -stable parabolic subalgebra. For example, in $\mathfrak{sl}(7, \mathbb{R})$ the partitions $[7]$, $[6, 1]$, $[5, 2]$ and $[4, 3]$ are all associated to the same parabolic subalgebra, in this case a Borel subalgebra. Of course this shows that there can be more than two $\mathfrak{w}$'s associated with the same parabolic subalgebra. Moreover the correspondence does not necessarily associate a θ -stable parabolic to its Richardson nilpotent (see Humphreys [H2] for the definition of a Richardson element). For example, the parabolic subalgebra $\mathfrak{q}$ associated with the partition $[4, 2, 1]$ contains a representative X of $[5, 2]$. And X is the Richardson nilpotent for $\mathfrak{q}$. But since $\mathfrak{q}$ is not the Jacobson-Morosov for $[5, 2]$ this nilpotent is not assigned to $\mathfrak{q}$ by the correspondence.

4. DESCRIPTION OF NOTICED ORBITS IN SIMPLE REAL LIE ALGEBRAS

4.1. Simple exceptional real Lie algebras. Djoković [D1, D2] (see also [C-Mc]) has computed the reductive centralizer for all real nilpotent orbits in the case where $\mathfrak{g}$ is an exceptional simple real Lie algebra. The results are given in several tables, one for each algebra. Hence the noticed orbits can be easily identified from Djoković's tables. They are the ones for which $\mathfrak{k}_{\mathbb{C}}^{(x, e, f)} = 0$. The last two columns give the isomorphism type of $\mathfrak{k}_{\mathbb{C}}^{(x, e, f)}$ and $\mathfrak{g}^{(H, E, F)}$ respectively. A study of the tables reveals that

$$E_{6(2)}, E_{6(6)}, E_{7(7)}, E_{8(8)}, F_{4(4)}, G_{2(2)}$$

are the only exceptional simple real Lie algebras to admit noticed nilpotent orbits. They are *quasi-split*.

A real form, $\mathfrak{g}$, of $\mathfrak{g}_{\mathbb{C}}$ is called *quasi-split* if there is a subalgebra, $\mathfrak{b}$, of $\mathfrak{g}$ such that $\mathfrak{b}_{\mathbb{C}} = \mathfrak{b} + i\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

The following proposition characterizes quasi-split real forms.

Proposition 4.1.1 (Rothschild). *$\mathfrak{g}$ contains a regular nilpotent iff $\mathfrak{g}$ is quasi-split iff $\mathfrak{g}$ contains a regular semisimple H such that ad_H has all real eigenvalues.*

Proof. See Rothschild [Rot]. $\square$

An element z of $\mathfrak{g}$ is said to be regular if $\dim G \cdot z \geq \dim G \cdot y$ for all y in $\mathfrak{g}$.
Hence,

Theorem 4.1.2. *An exceptional simple real Lie algebra is quasi-split if and only if it contains a noticed nilpotent element.*

This is not the case in general, as we shall see below. Now we turn our attention to the classical algebras.

4.2. Simple classical real Lie algebras. Let $\mathfrak{g}$ be a real classical Lie algebra. It is known [S-S, B-Cu, C-Mc] that the nilpotent orbits of G in $\mathfrak{g}$ are parametrized by signed Young diagrams. Let (H, E, F) be a standard triple in $\mathfrak{g}$. Denote by V the standard representation of $\mathfrak{g}$, regarded as an $\mathfrak{sl}_2$ -module under the action of the real Lie algebra generated by (H, E, F) . Then

$$V = \bigoplus_{r \geq 0} M(r)$$

where each $M(r)$ is a direct sum of irreducible $(r+1)$ -dimensional $\mathfrak{sl}_2$ -modules. Let $[d_1, d_2, \dots, d_k]$ be a partition of $\dim_{\mathbb{R}} V$. Then the d_i 's are exactly the dimensions of the irreducible summands of V (see [C-Mc]). For $r \geq 0$, denote by $H(r)$ the highest weight space in $M(r)$. From the $\mathfrak{sl}_2$ theory we have

$$\dim H(r) = \text{mult}(\rho_r, M(r))$$

where ρ_r is the irreducible representation of $\mathfrak{sl}_2$ of highest weight r . If $\mathfrak{g}$ is not $\mathfrak{su}_{2n}^*$ or $\mathfrak{sl}_n(\mathbb{R})$, then V carries a $\mathfrak{g}$ -invariant form $\langle \cdot, \cdot \rangle$ which induces a nondegenerate bilinear form $(\cdot, \cdot)_r$ on $H(r)$ as follows:

$$\text{if } u, v \in H(r), \text{ then } (u, v)_r = \langle u, F^r v \rangle.$$

It turns out that the induced forms $(\cdot, \cdot)_r$ determine the ambient form $\langle \cdot, \cdot \rangle$ uniquely [C-Mc].

The reductive centralizer $C = \mathfrak{g}_{\mathbb{C}}^{(x, e, f)}$ can be described as a direct sum of simple complex Lie algebras [S-S, Ca, C-Mc]. The nilpotent orbits of $G_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ are also parametrized by certain partitions. Let r_i be the number of parts equal to i in the description of the partition associated to e .

For type A_n

$$C = \left(\bigoplus_i A_{r_i-1} \right) \oplus T_k$$

where $k = (\text{No. of } r_i) - 1$, and T_k is a torus of dimension k .

For type C_n

$$C = \left(\bigoplus_{i \text{ odd}} C_{r_i/2} \right) \oplus \left(\bigoplus_{i \text{ even}, r_i \text{ even}} D_{r_i/2} \right) \oplus \left(\bigoplus_{i \text{ even}, r_i \text{ odd}} B_{(r_i-1)/2} \right).$$

For type B_n and D_n

$$C = \left(\bigoplus_{i \text{ even}} C_{r_i/2} \right) \oplus \left(\bigoplus_{i \text{ odd}, r_i \text{ even}} D_{r_i/2} \right) \oplus \left(\bigoplus_{i \text{ odd}, r_i \text{ odd}} B_{(r_i-1)/2} \right).$$

In the above formulae D_1 must be interpreted as a 1-dimensional torus T_1 wherever it occurs.

To classify the noticed nilpotent $K_{\mathbb{C}}$ -orbits in $\mathfrak{p}_{\mathbb{C}}$ we shall proceed as follows: maintaining the above notations, for each real algebra we will determine which signed Young diagrams will force $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ to be zero. Since

$$\mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \oplus \mathfrak{p}_{\mathbb{C}}^{(x,e,f)},$$

we should be able to carry out our analysis on each real algebra separately by indentifying which signed Young diagrams produce a real centraliser $\mathfrak{g}^{(H,E,F)}$ in $\mathfrak{p}$. We are using Helgason's [He] realizations of the classical real algebras.

Type A_n .

Theorem 4.2.1. *The non-zero noticed nilpotent orbits of $\mathfrak{sl}_n(\mathbb{R})$ are parametrized by partitions of n with distinct parts. If such a partition is even, then it corresponds to two orbits labeled by I and II. The algebra $\mathfrak{su}_{2n}^*$ has no non-zero noticed nilpotent orbit. Also $\mathfrak{su}(p, p+1)$ has exactly one non-zero noticed nilpotent orbit, which is parametrized by a one-row signed Young diagram of signature $(p, p+1)$, $\mathfrak{su}(p, p)$ has exactly two non-zero noticed nilpotent orbits, each is parametrized by a one-row signed Young diagram of signature (p, p) . If $|p - q| \geq 2$, then $\mathfrak{su}(p, q)$ has no non-zero noticed nilpotent orbits.*

Proof.

1. $\mathfrak{sl}_n(\mathbb{R})$. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$. Then $\mathfrak{k} = \mathfrak{so}_n$ and $\mathfrak{p}$ = set of $n \times n$ symmetric real matrices. The nilpotent orbits of $\mathfrak{g}$ are parametrized by Young diagrams of size n , except that partitions having only even terms correspond to two orbits, denoted as usual by I and II. [C-Mc]. Here $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$ and its nilpotent orbits are also parametrized by partitions of n [C-Mc]. Moreover the Young diagram of the complexification of a real orbit of $\mathfrak{g}$ is obtained by omitting the signs and the numeral in the case of “even” orbits, that is, orbits parametrized by partitions with even parts only. Therefore for an orbit parametrized by a given partition $\vec{d}$ to be noticed, that is for $\mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$ to be trivial, all the parts of $\vec{d}$ must be distinct since otherwise C will have a summand of type A_m and any real form of A_m has a non-trivial compact part. Hence if e is noticed, then the associated partition has distinct parts and

$$\mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = T_k.$$

It remains to show that if the partition has distinct parts, then the torus part T_k of the centralizer C is in $\mathfrak{p}_{\mathbb{C}}$.

Observe that the partition $\vec{d} = [n]$ is distinguished [B-C1]. Hence it is noticed. Therefore we shall consider partitions made of two or more distinct parts. Let $\vec{d} = [d_1, d_2, \dots, d_{k+1}]$, $k \geq 1$, be the partition of n with distinct parts associated with e . Consider the set of $n \times n$ diagonal matrices D_i for $1 \leq i \leq k$ defined as follows:

Let $c_i = (\sum_{j=1}^i d_j) - d_i$. Then D_i is the $n \times n$ diagonal matrix with d_i 's consecutive 1's starting at row $c_i + 1$ and the last d_{k+1} entries are all equal to $\frac{-d_i}{d_{k+1}}$. For example

$$D_1 = \begin{pmatrix} I_{d_1} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-d_1}{d_{k+1}} & \vdots & 0 \\ 0 & 0 & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \frac{-d_1}{d_{k+1}} \end{pmatrix}.$$

Here I_{d_i} denotes the $d_i \times d_i$ identity matrix. The diagonal matrices D_i have trace zero and commute with (x, e, f) . Furthermore, by construction they are independent and generate T_k . Hence

$$\dim \mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = k.$$

Since all the matrices D_i described above belong to $\mathfrak{p}_{\mathbb{C}}$ we must have

$$\dim \mathfrak{p}_{\mathbb{C}}^{(x,e,f)} \geq k.$$

Thus by dimensionality considerations, we obtain

$$\mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = \mathfrak{p}_{\mathbb{C}}^{(x,e,f)}.$$

Hence, the non-zero noticed nilpotent orbits of $\mathfrak{sl}_n(\mathbb{R})$ are parametrized by partitions of n with distinct parts. If such a partition is even, then it corresponds to two orbits labeled by I and II.

2. $\mathfrak{su}(p, q)$. Let $\mathfrak{g} = \mathfrak{su}(p, q)$. Then (see Helgason [He])

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{u}(p), B \in \mathfrak{u}(q), \text{trace}(A + B) = 0 \right\}.$$

The nilpotent orbits of $\mathfrak{su}(p, q)$ are parametrized by signed Young diagrams of signature (p, q) [C-Mc]. The complexified complex orbits of these real orbits are parametrized by a subset of the set of partitions of $n = p + q$.

Observe that if $p = q$, then the one-row signed Young diagram of size $2p$ describes a nilpotent of $\mathfrak{su}(p, q)$. Similary if $|p - q| = 1$, then the one-row signed Young diagram of size $2p + 1$ describes a nilpotent of $\mathfrak{su}(p, q)$. In both cases the corresponding complex orbits are distinguished, hence noticed.

Let $\bar{d} = [d_1, d_2, \dots, d_{k+1}]$, $k \geq 1$, be a partition of n with distinct parts. Assume that $\bar{d}$ is the complexification of a nilpotent orbit of $\mathfrak{g}$. Then it is clear that the diagonal matrices of the form $\sqrt{-1}D_i$ where D_i is the matrix, defined above, generate T_k and all the independent matrices $\sqrt{-1}D_i$ belong to $\mathfrak{k}_{\mathbb{C}}$. Again in this case we have

$$\dim \mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = k.$$

But

$$\dim \mathfrak{k}_{\mathbb{C}}^{(x,e,f)} \geq k.$$

Thus by dimensionality considerations, we obtain

$$\mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = \mathfrak{k}_{\mathbb{C}}^{(x,e,f)}.$$

This shows that if e is noticed, $\mathfrak{k}$ must be zero which implies that e is distinguished. It follows that $\mathfrak{su}(p, p+1)$ has exactly one non-zero noticed nilpotent orbit, which is parametrized by a one-row signed Young diagram of signature $(p, p+1)$, $\mathfrak{su}(p, p)$ has

exactly two non-zero noticed nilpotent orbits, each is parametrized by a one-row signed Young diagram of signature (p, p) .

3. $\mathfrak{su}_{2n}^*$. Let $\mathfrak{g} = \mathfrak{su}_{2n}^*$. The nilpotent orbits of $\mathfrak{g}$ are parametrized by partitions of n . To obtain the partitions associated with their complexification we have to replace each row of their Young diagrams with two copies of itself [C-Mc]. Hence the complexified orbits are parametrized by partitions of $2n$ with no distinct parts. Therefore $\mathfrak{su}_{2n}^*$ has no non-zero noticed nilpotent orbit. $\square$

Types B_n and D_n .

Theorem 4.2.2. *The non-zero noticed nilpotent orbits of $\mathfrak{so}(p, q)$ are parametrized by Young diagrams of signature (p, q) such that:*

- (1) *all rows are odd and can be repeated atmost twice,*
- (2) *two rows of the same length must have their leftmost boxes labeled by different signs,*
- (3) *if all the rows have an even number of boxes labeled +, or all the rows have an even number of boxes labeled -, then one numeral I or II is attached.*

There are no non-zero noticed nilpotent orbits in $\mathfrak{so}_{2n}^$.*

Proof.

1. $\mathfrak{so}(p, q)$. Let $\mathfrak{g} = \mathfrak{so}(p, q)$. Then the nilpotent orbits in $\mathfrak{g}$ are parametrized by signed Young diagrams of signature (p, q) such that rows of even length occur with even multiplicity and have their leftmost boxes labeled +. Some of these diagrams get Roman numerals attached to them as follows. If all rows have even length, then two Roman numerals, each I or II are attached. If at least one row has odd length and all such rows have an even number of boxes labeled +, or all such rows have an even number of boxes labeled -, then one numeral I or II is attached [C-Mc]. Moreover $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{p+q}(\mathbb{C})$ and its nilpotent orbits are also parametrized by partitions of $p + q$ in which even parts appear with even multiplicity [C-Mc]. To obtain the Young diagram of the complexification of a real orbit of $\mathfrak{g}$ we omit the signs. If the associated partition is very even that is, every part is even and appears with even multiplicity, we omit the first numeral. If it is not very even we omit the numeral. For an orbit parametrized by a given partition $\bar{d}$ to be noticed, that is for $\mathfrak{p}_{\mathbb{C}}^{(x, e, f)}$ to be trivial, all the parts of $\bar{d}$ must be odd and repeated atmost twice. Otherwise the centralizer C will have a summand of type B_l or C_m or D_n with $n \geq 2$. Hence if e is noticed, then

$$\mathfrak{g}_{\mathbb{C}}^{(x, e, f)} = T_k.$$

Note that k is the number of parts with multiplicity 2. Clearly if $k = 0$, then e is distinguished.

We now show how to construct the torus part T_k of the centralizer C in $\mathfrak{p}_{\mathbb{C}}$. This will make it possible to identify which of the nilpotents whose partitions have all parts repeated atmost twice are noticed.

Let $\bar{d} = [d_1, d_2, \dots, d_l]$ be a partition of $p + q$ of signature (p, q) which consists of odd parts repeated atmost twice. Let (H, E, F) be a KS-triple associated with $\bar{d}$. Then the standard representation V of $\mathfrak{g}$ can be decomposed as a direct sum of irreducible $\langle H, E, F \rangle$ -modules of weight $r_i = d_i - 1$. The centralizer of $\mathfrak{g}$ preserves each $M(r_i)$. It also preserves the $\mathfrak{g}$ -invariant symmetric form, $\langle \cdot, \cdot \rangle$, carried by V . Since all the r_i 's are even (see [C-Mc]), $(\cdot, \cdot)_{r_i}$ is symmetric and the signature of $\langle \cdot, \cdot \rangle$

on $M(r_i)$ is obtained by starting with the signature of $(\cdot, \cdot)_{r_i}$ and replacing each $\pm$ sign by an alternating sequence of signs of length d_i . Therefore if d_i has multiplicity 1, there are only two possible labellings since the one-dimensional highest weight space is labelled either with a $+$ or a $-$. However if d_i is repeated twice, then we have three possible ways of labelling $M(r_i)$. It is useful to consider the following example:

Assume that we are trying to label $M(2)$ from the standard representation of $\mathfrak{so}(3, 3)$ with regard to the partition $[3, 3]$. Since symmetric real forms are equivalent up to signature, we can rearrange the basis of V such that

$$M(2) = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle.$$

It is understood that each basis vector denotes a one-dimensional weight space in an $\mathfrak{sl}_2$ irreducible module. In our notation (e_1, e_4) form a basis of $H(2)$. Clearly we can label $M(2)$ in three different ways:

$$\begin{array}{ccc} + & - & + \\ - & + & - \end{array} \quad \begin{array}{ccc} + & - & + \\ + & - & + \end{array} \quad \begin{array}{ccc} - & + & - \\ - & + & - \end{array}$$

The first row represents the labels of e_1, e_2 and e_3 . The second row represents the labels of e_4, e_5 and e_6 .

In general if the part d_i is represented twice in the partition $\vec{d}$, then $M(r_i)$ is the direct sum of two irreducible $\mathfrak{sl}_2$ -modules of highest weight r_i . Furthermore V has a basis where

$$M(r_i) = \langle e_k, e_{k+1}, \dots, e_{k+r_i} \rangle \oplus \langle e_{k+r_i+1}, e_{k+r_i+2}, \dots, e_{k+2r_i+1} \rangle.$$

Once again (e_k, e_{k+r_i+1}) is a basis for $H(r_i)$ and $(e_{k+r_i}, e_{k+2r_i+1})$ is a basis for $H(-r_i)$.

Assume that $M(r_i)$ is labelled as follows:

$$\begin{array}{ccccccc} + & - & + & - & + & \cdots & - & + \\ - & + & - & + & - & \cdots & + & - \end{array}$$

Let Z be a $(p+q) \times (p+q)$ matrix such that

$$Ze_i = e_{i+r_i+1} \quad \text{and} \quad Ze_{i+r_i+1} = e_i \quad \text{if} \quad k \leq i \leq k+r_i \quad \text{else} \quad Ze_i = 0.$$

Then

$$\begin{aligned} (e_k, Ze_k)_{r_i} &= (e_k, e_{k+r_i+1})_{r_i} = -(Ze_k, e_k)_{r_i} = 0, \\ (e_{k+r_i+1}, Ze_{k+r_i+1})_{r_i} &= (e_{k+r_i+1}, e_k)_{r_i} = -(Ze_{k+r_i+1}, e_{k+r_i+1})_{r_i} = 0, \\ (e_k, Ze_{k+r_i+1})_{r_i} &= (e_k, e_k)_{r_i} = -(e_{k+r_i+1}, e_{k+r_i+1})_{r_i} = -(Ze_k, e_{k+r_i+1})_{r_i}. \end{aligned}$$

Hence $(\cdot, \cdot)_{r_i}$ is invariant under Z and so is the ambient form $\langle \cdot, \cdot \rangle$. Furthermore it is easy to see that Z commutes with (H, E, F) . Therefore Z generates a one-dimensional factor of the centralizer $\mathfrak{g}^{(H, E, F)}$. By definition Z is a Hermitian matrix. So all its eigenvalues are real and it lies in the vector part of a Cartan subalgebra of $\mathfrak{g}$. We conclude that Z is G -conjugate to some element of $\mathfrak{p}$. (See Helgason [He].)

A similar analysis can be carried out for the other two labellings. In both cases one defines Z as follows:

$$Ze_i = e_{i+r_i+1} \quad \text{and} \quad Ze_{i+r_i+1} = -e_i \quad \text{if} \quad k \leq i \leq k+r_i \quad \text{else} \quad Ze_i = 0.$$

It is easy to verify that Z preserves $(\cdot, \cdot)_{r_i}$. Since Z is a skew-hermitian matrix, all its eigenvalues are imaginary. Therefore Z is G conjugate to some element of $\mathfrak{k}$. (See Helgason [He].)

Observe that the matrices Z described above commute and are independent. Also there are exactly k such matrices for each orbit. Furthermore, without loss of generality we can assume that (x, e, f) is the Cayley transform of the KS-triple (H, E, F) . Therefore all the matrices Z commute with (x, e, f) and generate T_k . Hence

$$\dim \mathfrak{g}_{\mathbb{C}}^{(x, e, f)} = k.$$

In the first case all the Z 's are in $\mathfrak{p}_{\mathbb{C}}$. Hence

$$\dim \mathfrak{p}_{\mathbb{C}}^{(x, e, f)} \geq k.$$

Thus by dimensionality considerations, we obtain

$$\mathfrak{g}_{\mathbb{C}}^{(x, e, f)} = \mathfrak{p}_{\mathbb{C}}^{(x, e, f)}.$$

In the two other cases all the Z 's are in $\mathfrak{k}_{\mathbb{C}}$. Hence

$$\mathfrak{g}_{\mathbb{C}}^{(x, e, f)} = \mathfrak{k}_{\mathbb{C}}^{(x, e, f)}.$$

2. $\mathfrak{so}_{2n}^*$. Let $\mathfrak{g} = \mathfrak{so}_{2n}^*$. Then the nilpotent orbits are parametrized by signed Young diagrams of size n and any signature in which rows of odd length have their leftmost boxes labeled $+$ [C-Mc]. Therefore we only need to consider partitions with distinct odd parts. Let $\bar{d} = [d_1, d_2, \dots, d_l]$ be such a partition of n . Now $\mathfrak{so}_{(2n)}^*$ acts on a $2n$ -dimensional complex space V and can be defined as the subalgebra of $\mathfrak{so}(2n, \mathbb{C})$ that leaves invariant a skew-hermitian form $\langle \cdot, \cdot \rangle$ on V . With the above notation V can be seen as a direct sum of M_{r_i} 's and each of them is made of two irreducible modules of dimension d_i . Since all the r_i 's are even (see [C-Mc]), $(\cdot, \cdot)_{r_i}$ is a skew-hermitian. Define $Z.e_k = \sqrt{-1}e_k$ for $e_k \in M_{r_i}$ and zero elsewhere. Then clearly Z preserves H_{r_i} and belongs to $g^{(E, H, F)}$. But Z is a skew-hermitian matrix and consequently is G -conjugate to some element of $\mathfrak{k}$. Consequently

$$\dim \mathfrak{k}_{\mathbb{C}}^{(x, e, f)} \geq k.$$

Thus by dimensionality considerations, we obtain

$$\mathfrak{g}_{\mathbb{C}}^{(x, e, f)} = \mathfrak{k}_{\mathbb{C}}^{(x, e, f)}.$$

The theorem follows. $\square$

Type C_n . Let $\mathfrak{g} = \mathfrak{sp}(p, q)$. Then the nilpotent orbits in $\mathfrak{g}$ are parametrized by signed Young diagrams of signature (p, q) such that rows of even length have their leftmost boxes labeled $+$ [C-Mc].

Nilpotent orbits of $\mathfrak{sp}_n(\mathbb{R})$ are parametrized by signed Young diagrams of size $2n$ and any signature in which odd rows appear with even multiplicity and begin with $+$ [C-Mc].

Also $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}_n(\mathbb{C})$ and its nilpotent orbits are parametrized by a partition of $2n$ in which odd parts occur with even multiplicity.

A discussion similar to the one given in the previous section gives the next theorem.

Theorem 4.2.3. *The non-zero noticed nilpotent orbits of $\mathfrak{sp}_n(\mathbb{R})$ are parametrized by signed Young diagrams of size $2n$ such that*

- (1) *all rows are even and can be repeated at most twice,*

- (2) *two rows of the same length must have their leftmost boxes labeled with different signs.*

There are no non-zero noticed nilpotent orbits in $\mathfrak{sp}(p, q)$. $\square$

We have seen that the only exceptional simple real Lie algebras admitting nilpotent orbits are the quasi-split ones. This is not true in the case of the classical simple real Lie algebras, as the following example indicates.

Let $\mathfrak{g} = \mathfrak{so}(6, 3)$. Then from Theorem 4.2.2, the orbit parametrized by the signed Young diagram:

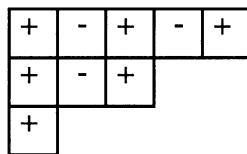


FIGURE 1

is noticed but $\mathfrak{g}$ is not quasi-split.

Remark. We note that the noticed orbits of the simple real Lie algebras of type B_n , C_n and D_n are even. This can be seen through an analysis of the weighted Dynkin diagram associated with their complex counterparts (see [C-Mc]). Therefore our classification, when restricted to the above algebras, is similar to the Bala-Carter classification.

Noticed nilpotent orbits Types and Chromosomes. The above description of noticed nilpotent orbits will be translated in the language of “Type” (see [B-Cu] for the definition of Type). Except for $\mathfrak{sl}_n(\mathbb{R})$ our description will use the work of Djoković [D3] as a reference. Djoković used “Chromosomes” which are roughly speaking signed and unsigned Young diagrams to describe nilpotent orbits of the classical Lie algebras and then translated his description into the language of Type.

A *gene*, as Djoković defines it, can be interpreted as a row of a Young diagram. The rank of a gene is the size of the row. In the case of signed Young diagrams, if a row of length n ends with a $+$, we write $g^+(n)$ to denote the corresponding gene. Similarly $g^-(n)$ corresponds to the row of length n ending with $-$. For unsigned Young diagrams we use $g(n)$. The signature (r^+, r^-) of a gene is the signature of the corresponding row. A *chromosome* is a non-negative integral linear combination of genes. The signature of a chromosome is the sum of the signatures of its constituent genes. For example the chromosome $g^+(5) + g^+(3) + g^+(1)$ corresponds to the Young tableau in Figure 1. The “Type” representation of this orbit is $\Delta_4^+(0) + \Delta_2^+(0) + \Delta_0^+(0)$. The description is given in the following table where $\epsilon = \pm$. Simple algorithms to find ϵ are given in [D3].

Remarks. In the case of $\mathfrak{sl}_n(\mathbb{R})$ if all the k ’s are odd, then we have two orbits ([B-Cu, page 355]). For $\mathfrak{so}(p, q)$ and $\mathfrak{sp}_n(\mathbb{R})$ $2\Delta_k^\epsilon(0) = \Delta_k^+(0) + \Delta_k^-(0)$. Furthermore, in the case of $\mathfrak{so}(p, q)$ if for all the genes r^+ is even or if for all the genes r^- is even, then we have two orbits.

TABLE 2

ALGEBRA	TYPES	CHROMOSOMES	
$\mathfrak{sl}_n(\mathbb{R})$	$\sum a_k \Delta_k(0)$	$\sum a_k g(k+1)$	$0 \leq a_k \leq 1$
$\mathfrak{su}(p, q)$	$\Delta_{2p-1}^+(0), \Delta_{2p-1}^-(0)$	$g^+(2p), g^-(2p)$	$q = p$
	$\Delta_{2p}^-(0)$	$g^-(2p+1)$	$q = p+1$
$\mathfrak{so}(p, q)$	$\sum_{k:\text{even}} a_k \Delta_k^\epsilon(0)$	$\sum_{k:\text{even}} a_k g^\epsilon(k+1)$	$0 \leq a_k \leq 2$
$\mathfrak{sp}_n(\mathbb{R})$	$\sum_{k:\text{odd}} a_k \Delta_k^\epsilon(0)$	$\sum_{k:\text{odd}} a_k g^\epsilon(k+1)$	$0 \leq a_k \leq 2$

5. NOTICED PRINCIPAL ORBITS IN SIMPLE REAL LIE ALGEBRAS

5.1. Principal nilpotent element. Let e be in $\mathfrak{p}_\mathbb{C}$. Then e is principal if and only if $K_\mathbb{C} \cdot e$ is a maximal $K_\mathbb{C}$ -orbit in $\mathfrak{p}_\mathbb{C}$ [K-R] that is if and only if

$$\dim K_\mathbb{C} \cdot e \geq \dim K_\mathbb{C} \cdot e' \quad \text{for all } e' \in \mathfrak{p}_\mathbb{C}.$$

If the orbit $G \cdot \lambda_\mathbb{R}$ corresponds to the orbit $K_\mathbb{C} \cdot \lambda_\mathbb{C}$, then

$$\dim_\mathbb{C} K_\mathbb{C} \cdot \lambda_\mathbb{C} = \frac{1}{2} \dim_\mathbb{R} G \cdot \lambda_\mathbb{R} = \frac{1}{2} \dim_\mathbb{C} G_\mathbb{C} \cdot \lambda_\mathbb{C}.$$

Kostant and Rallis give several characterizations of a principal nilpotent of $\mathfrak{p}_\mathbb{C}$. We will use the following criterion due to them. We will say that e is principal if and only if

$$\dim K_\mathbb{C} \cdot e = \dim \mathfrak{p}_\mathbb{C} - \dim \mathfrak{a}_\mathbb{C},$$

where $\mathfrak{a}_\mathbb{C}$ is the complexification of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is called the *real rank* of $\mathfrak{g}$ (see Helgason [He]). The following theorem gives a characterization of the *quasi-split* simple real Lie algebras $\mathfrak{g}$ in terms of the noticed principal elements.

Theorem 5.1.1. *Let $\mathfrak{g}$ be a simple real Lie algebra. Then $\mathfrak{g}$ is quasi-split if and only there exists a nilpotent element $\lambda_\mathbb{C}$ of $\mathfrak{p}_\mathbb{C}$ such that $\lambda_\mathbb{C}$ is noticed and principal.*

Proof. If $\mathfrak{g}$ is an exceptional simple real Lie algebra, then an analysis of Djoković's tables [D2, D1] shows that the principal nilpotent orbit is noticed if and only if $\mathfrak{g}$ is quasi-split. Now assume that $\mathfrak{g}$ is a classical real Lie algebra. If $\mathfrak{g}$ is equal to $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{su}(p, q)$, or $\mathfrak{sp}_n(\mathbb{R})$, then Theorems 4.2.1 and 4.2.3 tell us that the principal nilpotent is noticed because it is also regular in each case. Moreover, since $\mathfrak{sp}(p, q)$, $\mathfrak{su}_{2n}^*$ and $\mathfrak{so}_{2n}^*$ have no non-zero noticed elements, we only need to give a proof for the case where $\mathfrak{g} = \mathfrak{so}(p, q)$. First we will give a general description of the principal orbit of $\mathfrak{g}$ which corresponds to a maximal $K_\mathbb{C}$ -orbit in $\mathfrak{p}_\mathbb{C}$ under the Kostant-Sekiguchi correspondence, and then we shall prove that such an orbit is noticed if and only if $\mathfrak{g}$ is quasi-split. In our case the principal orbit is characterized by its dimension $pq - q$. The dimension of the complex nilpotent orbits of type B_n and D_n are given by

$$(5.1.2) \quad 2n^2 + n - \frac{1}{2} \sum_i s_i^2 + \frac{1}{2} \sum_{i \text{ odd}} r_i$$

and

$$(5.1.3) \quad 2n^2 - n - \frac{1}{2} \sum_i s_i^2 + \frac{1}{2} \sum_{i \text{ odd}} r_i,$$

respectively, where r_i and s_i are defined as follows: if $[d_1, d_2, \dots, d_l]$ is the partition associated with the nilpotent orbit. Put $r_i = |\{j | d_j = i\}|$ and $s_i = |\{j | d_j \geq i\}|$. (See [C-Mc].)

We consider two cases:

Case 1. $p = q$.

Consider the orbit $G \cdot \lambda_{\mathbb{R}}$ parametrized by the following Young diagram:

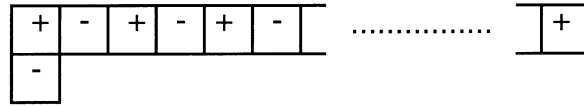


FIGURE 2

If p is even, then there are two such orbits and the Young diagram should be labeled with Roman numerals I or II (see [C-Mc]).

The associated complex orbit $G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}}$ is parametrized by $[2p-1, 1]$. Then $r_1 = 1$, $r_{2p-1} = 1$, $s_1 = 2$, and $s_i = 1$ for $2 \leq i \leq 2p-1$. Using formula 5.1.3 with $n = p$ we find that

$$\dim G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} = 2p^2 - p - \frac{1}{2}(2p+2) + 1 = 2p^2 - 2p.$$

Hence

$$\dim K_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} = p^2 - p.$$

Therefore the above orbit is principal. From Theorem 4.2.2 it is also noticed. Finally $\mathfrak{so}(p, p)$ is quasi-split.

Case 2. $p > q$.

Consider the orbit $G \cdot \lambda_{\mathbb{R}}$ parametrized by the following Young diagram:

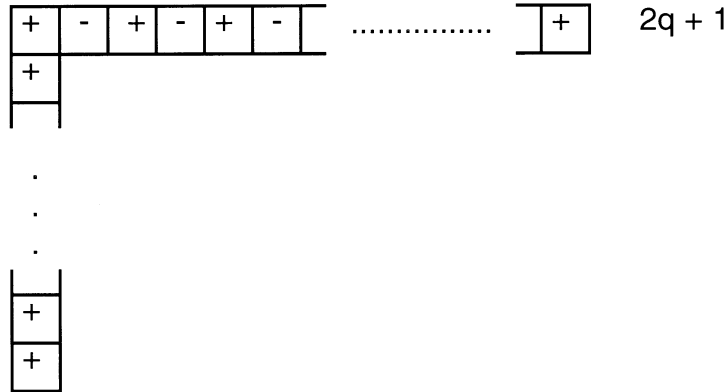


FIGURE 3

where the first row has length $2q + 1$ and the remaining $p - q - 1$ rows have length 1. This orbit is noticed by Theorem 4.2.2.

If q is even, then there are two such orbits and the Young diagram should be labeled with Roman numerals I or II (see [C-Mc]).

The associated complex orbit $G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}}$ is parametrized by $[2q + 1, 1^{p-q-1}]$. Thus $r_1 = p - q - 1$, $r_{2q+1} = 1$, $s_1 = p - q$, and $s_i = 1$ for $2 \leq i \leq 2q + 1$. Using formula 5.1.3 with $n = \frac{p+q}{2}$ we find that

$$\dim G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} = \frac{(p+q)^2}{2} - \frac{(p+q)}{2} - \frac{1}{2}((p-q)^2 + 2q) + \frac{1}{2}(p-q) = 2pq - 2q.$$

Using formula 5.1.3 with $n = \frac{p+q-1}{2}$ we have

$$\begin{aligned} \dim G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} &= \frac{(p+q)^2 - 2(p+q) + 1}{2} \\ &\quad - \frac{(p+q-1)}{2} - \frac{1}{2}((p-q)^2 + 2q) + \frac{1}{2}(p-q) = 2pq - 2q. \end{aligned}$$

In either case we obtain

$$\dim K_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} = pq - q.$$

Again the orbit $K_{\mathbb{C}} \cdot \lambda_{\mathbb{C}}$ is principal and noticed if and only if $[2q + 1, 1^{p-q-1}]$ dominates every other admissible partition that is, if and only if $|p - q| \leq 2$ ([C-Mc]), that is, if and only if $\mathfrak{g}$ is quasi-split. The result follows. $\square$

We shall need the following lemma.

Lemma 5.1.4. *Let l be a minimal (σ, θ) -stable Levi subalgebra containing a principal nilpotent e of $\mathfrak{p}_{\mathbb{C}}$. Then l is a minimal Levi subalgebra containing e .*

Proof. Let (x, e, f) be a KS-triple containing e . We have proved [Proposition 1.1.3] that $l = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{t}}$ where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}_{\mathbb{C}}^e$. In fact $\mathfrak{t} \subseteq \mathfrak{k}_{\mathbb{C}}^{(x,e,f)}$. From King [Ki] we know that if e is principal, then

$$\mathfrak{g}_{\mathbb{C}}^{(x,e,f)} = \mathfrak{k}_{\mathbb{C}}^{(x,e,f)}.$$

Hence $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}_{\mathbb{C}}^e$ and l is a minimal Levi subalgebra containing e [C-Mc]. $\square$

We wish to thank David Vogan from MIT and Roger Carter from the University of Warwick (England) for their suggestions about the proof of the following proposition. The result was also mentioned by Collingwood and McGovern [C-Mc]. But they did not provide a proof.

The Bala-Carter theory associates each $G_{\mathbb{C}}$ -nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$ to pair (m, p_m) where m is a Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and p_m is a distinguished parabolic of the semi-simple algebra $[m, m]$ (see [Ca]). Maintaining the above notations we have:

Proposition 5.1.5. *Using the above notations (m, p_m) and (m, q_m) are conjugate under $G_{\mathbb{C}}$ if and only p_m and q_m are conjugate under M , the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra m .*

Proof. If p_m and q_m are M -conjugate, then they are necessarily $G_{\mathbb{C}}$ -conjugate since $M \subseteq G_{\mathbb{C}}$. Hence (m, p_m) and (m, q_m) are $G_{\mathbb{C}}$ -conjugate.

Suppose that (m, p_m) and (m, q_m) are conjugate under $G_{\mathbb{C}}$. Let $g \in G_{\mathbb{C}}$ be a conjugating element. Then g induces an automorphism on the semisimple part $[m, m]$ of m . Consequently to prove that p_m and q_m are conjugate under M it is enough

to show that $\frac{N(m)}{M}$ acts trivially on conjugacy classes of distinguished parabolic subalgebras, where $N(m)$ denote the normalizer of m in $G_{\mathbb{C}}$. Let $\frac{Aut(m)}{Int(m)}$ denote the outer automorphisms of m . It is isomorphic to the group of automorphisms of the Dynkin diagram of m (see [Kn]). $Int(m)$ should be understood as the set of inner automorphisms of m .

We have $M \subseteq Ker(\Pi \circ \varphi)$ where Π and φ are the projection and inclusion maps respectively, as defined in the following sequence:

$$N(m) \xrightarrow{\varphi} Aut(m) \xrightarrow{\Pi} \frac{Aut(m)}{Int(m)} \longrightarrow 1$$

giving

$$\frac{N(m)}{Ker(\Pi \circ \varphi)} \subseteq \frac{Aut(m)}{Int(m)}.$$

and

$$(5.1.6) \quad 1 \longrightarrow \frac{Ker(\Pi \circ \varphi)}{M} \longrightarrow \frac{N(m)}{M} \longrightarrow \frac{N(m)}{Ker(\Pi \circ \varphi)} \longrightarrow 1.$$

Hence, the only important elements of $\frac{N(m)}{M}$ are those outside of $\frac{Ker(\Pi \circ \varphi)}{M}$.

We shall prove that the outer automorphisms of m do not change the conjugacy classes of distinguished parabolic subalgebras. It is known that the group of automorphisms of the Dynkin diagram is a cyclic group of order 2 for types $A_l (l \geq 2)$, $D_l (l > 4)$, and E_6 . It is $\mathfrak{S}_3$, the permutation group on three letters, for D_4 . Otherwise it is trivial (see [He]).

Bala and Carter [B-C2] give a description of the classes of semisimple subalgebras of parabolic type. In every case, all but atmost one simple component has type A_l . Bala and Carter [B-C1] also give a description of the weighted Dynkin diagram of each class of distinguished simple parabolic subalgebras. In type A_l , the only distinguished parabolic subalgebras are the Borel subalgebras. Thus, if all the simple parts are of type $A_l (l \geq 2)$, then p_m and q_m are conjugate under M . Suppose there is a simple component not of type A_l . A careful analysis of the weighted Dynkin classes of distinguished simple parabolic subalgebras shows that they are invariant under their nontrivial diagram automorphisms. Thus they are invariant under the outer automorphisms of m . Hence they are invariant under $\frac{N(m)}{M}$. It follows that p_m and q_m must be M -conjugate. $\square$

Moreover we have:

Proposition 5.1.7. *Let e be a nilpotent element of $\mathfrak{g}_{\mathbb{C}}$ and let l be a minimal Levi subalgebra containing e . Then $G_{\mathbb{C}} \cdot e \cap l = L \cdot e$, where L is the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra l .*

Proof. Since $L \subseteq G_{\mathbb{C}}$ it is clear that

$$L \cdot e \subseteq G_{\mathbb{C}} \cdot e \cap l.$$

Let e' be a nilpotent element of $G_{\mathbb{C}} \cdot e \cap l$. From the Bala-Carter theory we can associate a pair (l, p_l) to $G_{\mathbb{C}} \cdot e$ where p_l is a distinguished parabolic subalgebra of $[l, l]$. Similarly we can associate a pair $(l', p_{l'})$ to $G_{\mathbb{C}} \cdot e'$. Since e and e' are $G_{\mathbb{C}}$ -conjugate the two pairs (l, p_l) and $(l', p_{l'})$ are also conjugate under $G_{\mathbb{C}}$. In particular l is $G_{\mathbb{C}}$ -conjugate to l' . Therefore l is a minimum Levi subalgebra containing e' . Then the Bala-Carter theory tells us that we can find a pair (l, q_l) , where q_l is

a distinguished parabolic subalgebra of $[l, l]$ containing e' , which is $G_{\mathbb{C}}$ -conjugate to (l, p_l) . From Proposition 5.1.5 we know that p_l and q_l are conjugate under L . From the Bala-Carter theory we know that e and e' are Richardson elements of p_l and q_l respectively. Since the Levi parts of p_l and q_l are L -conjugate they are “associated” in Johnston’s and Richardson’s sense (see [J-R], [H2]). Thus $e' \in L \cdot e$. The proposition follows. $\square$

Now we are ready to prove the main result on principal nilpotents in $\mathfrak{p}_{\mathbb{C}}$.

Theorem 5.1.8. *Let e be a nilpotent principal element of $\mathfrak{p}_{\mathbb{C}}$. Then for any normal triple (x, e, f) corresponding to a triple $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ as described in the classification Theorem 3.2.4:*

1. e is regular in $l \cap \mathfrak{p}_{\mathbb{C}}$,
2. the real form l_0 of l is quasi-split,
3. if $\mathfrak{q}_l = m \oplus v$ is a Levi decomposition of $\mathfrak{q}_l$ then $\dim m \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{v \cap \mathfrak{p}_{\mathbb{C}}}{[v \cap \mathfrak{k}_{\mathbb{C}}, v \cap \mathfrak{p}_{\mathbb{C}}]}$ and q_l is a Borel subalgebra of l ,
4. $\mathfrak{g}$ and l_0 have the same real rank.

Proof. From Kostant and Rallis [K-R] e lies in the closure of a principal orbit of $L \cap K_{\mathbb{C}}$ on $l \cap \mathfrak{p}_{\mathbb{C}}$. Therefore we can find a nilpotent $e' \in l \cap \mathfrak{p}_{\mathbb{C}}$ such that

$$e \in \overline{L \cap K_{\mathbb{C}} \cdot e'} \subseteq \overline{K_{\mathbb{C}} \cdot e'}.$$

Thus

$$K_{\mathbb{C}} \cdot e \subseteq K_{\mathbb{C}} \cdot e'.$$

Since e is principal

$$K_{\mathbb{C}} \cdot e = K_{\mathbb{C}} \cdot e'.$$

It follows that e' is $G_{\mathbb{C}}$ -conjugate to e . From Lemma 5.1.4 and Proposition 5.1.7

$$G_{\mathbb{C}} \cdot e \cap l = L \cdot e.$$

Thus e and e' are conjugate under L . But

$$\dim L \cap K_{\mathbb{C}} \cdot e = \dim L \cap K_{\mathbb{C}} \cdot e' = \frac{1}{2} \dim L \cdot e.$$

This implies that $L \cap K_{\mathbb{C}} \cdot e$ is a principal orbit in $l \cap \mathfrak{p}_{\mathbb{C}}$.

Moreover, e is noticed in l by definition. By Theorem 5.1.1 the real form l_0 of l is quasi-split. This implies that e is regular in l , and so is distinguished. Therefore q_l is a Borel subalgebra of l and since e is an even nilpotent

$$\dim m \cap \mathfrak{k}_{\mathbb{C}} = \dim \frac{v \cap \mathfrak{p}_{\mathbb{C}}}{[v \cap \mathfrak{k}_{\mathbb{C}}, v \cap \mathfrak{p}_{\mathbb{C}}]} \quad \text{from Theorem 2.1.6.}$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. By the Kostant-Sekiguchi correspondence there is a real KS-triple (H, E, F) in $\mathfrak{g}$ such that $l_0 = \mathfrak{k}^{\mathfrak{t}}$ where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}^{(H, E, F)}$. Thus we have

$$\mathfrak{k}^{(H, E, F)} \subseteq \mathfrak{k}^H = \mathfrak{m}.$$

Hence $\mathfrak{a} \subseteq l_0$. This implies that the real rank of l_0 is equal to the dimension of the subspace $\mathfrak{a}$ which is also the real rank of $\mathfrak{g}$ by definition. The desired result follows. $\square$

The previous theorem and some results of Bala and Carter [B-C2, B-C1] allow us to describe the type of the semi-simple part of the quasi-split real form of l . The next table gives the type of $[l, l]$ and $[l_0, l_0]$ for the non-quasi-split cases.

TABLE 3

Algebra	$[l, l]$	$[l_0, l_0]$	
$\mathfrak{su}(p, q)$	$\mathfrak{sl}_{2q+1}(\mathbb{C})$	$\mathfrak{su}(q+1, q)$	
$\mathfrak{su}_{2n}^*$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	
$\mathfrak{so}(p, q)$	$\mathfrak{so}_{2q+1}(\mathbb{C})$	$\mathfrak{so}(q+1, q)$	$p+q$ even
$\mathfrak{so}(p, q)$	$\mathfrak{so}_{2(q+1)}(\mathbb{C})$	$\mathfrak{so}(q+2, q)$	$p+q$ odd
$\mathfrak{so}_{2n}^*$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{su}(\frac{n}{2}, \frac{n}{2})$	n even
		$\mathfrak{su}(\frac{n+1}{2}, \frac{n-1}{2})$	n odd
$\mathfrak{sp}(p, q)$	$\mathfrak{sl}_{2q+1}(\mathbb{C})$	$\mathfrak{su}(q+1, q)$	$p > q$
$\mathfrak{sp}(q, q)$	$\mathfrak{sl}_{2q}(\mathbb{C})$	$\mathfrak{su}(q, q)$	
$E_{6(-14)}$	$\mathfrak{sl}_5(\mathbb{C})$	$\mathfrak{su}(3, 2)$	
$E_{6(-26)}$	$\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$	$\mathfrak{sl}_3(\mathbb{C})$	
$E_{7(-5)}$	E_6	$E_{6(2)}$	
$E_{7(-25)}$	$\mathfrak{sl}_6(\mathbb{C})$	$\mathfrak{su}(3, 3)$	
$E_{8(-24)}$	E_6	$E_{6(2)}$	
$F_{4(-20)}$	$\mathfrak{sl}_3(\mathbb{C})$	$\mathfrak{su}(2, 1)$	

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