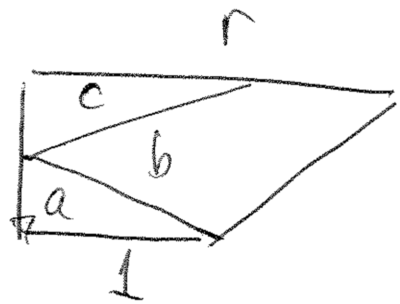
Monsky



$$\boxed{a r^2 - b r + c = 0}$$

if r satisfies with a, b, c pos integers,
then get equidissection

$2+\sqrt{2}$ ✓

$\sqrt{2}$?

$1+\sqrt{5}$?

$g \in \mathbb{R}[x]$. is alternating if the coeffs are all nonzero and alternate in sign.

$x^3 - 6x + 1$ no

$x^2 - 6x + 1$ ~~no~~ yes

x no.

FACTS (1) g_1, g_2 alternating $\Rightarrow$ $g_1 g_2$ alternating

(2) $f | g$, $f, g \in \mathbb{R}[x]$, g alternating $\overset{?}{\Rightarrow}$ f alternating

no $(x-1)(3x^2 + x + 3) = 3x^3 - 2x^2 + 2x - 3$.

Defn. g is strongly alternating if (2) holds; i.e. g is alternating if every $f \in \mathbb{R}[x]$ s.t. fg is alternating.

So if g is monic strongly alternating, g is a prod

$$g(x) = (x - c_i) \cdots (x^2 - a_i x + b_i) \cdots$$

$c > 0$ $\qquad$ $a, b > 0$. roots are $\frac{a \pm \sqrt{a^2 - 4b}}{2}$ ← positive real part.

Cor. g strongly alternating
$\Leftrightarrow$ all the real & cplx roots of g have positive real part.

Studied for a long time (stability of O.D.E over (real parts < 0))

degree 1, 2: strongly alt $\Leftrightarrow$ alternating

ex. degree 3: $s_4 x^3 - s_3 x^2 + s_2 x - s_1$ $\qquad s_i > 0$ (alternating)
is strongly alt $\Leftrightarrow$ $s_2 s_3 > s_1 s_4$.

degree 4: $s_5 x^4 - s_4 x^3 + s_3 x^2 - s_2 x + s_1$ $\qquad s_i > 0$ (alternating)
is strongly alternating $\Leftrightarrow$ $s_2 s_3 s_4 > s_4^2 s_1 + s_2^2 s_5$.

(becomes more and more complicated)

Theorem. Suppose g is strongly alternating, degree ≤ 4 and that r is a real root of g. (so $r > 0$ by the corollary above). Suppose further $g \in \mathbb{Z}[x]$. Then $T(r) =$  has an equidissection eg. $r^4-3r^3+3r^2-4r+1=0$

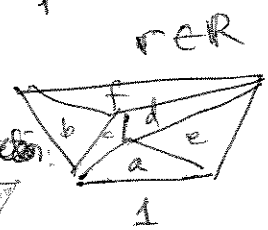
$T(r)$ has a equidissection into 16,536 triangles.

Can't do with <12. Is there something in between??

Rem. Seems like minimal # depends only on g

(not the individual root).

Look at a degree 3 picture.



Scale so that $a+b+c+d+e+f=1$.

equidissection

Such a picture exists $\Leftrightarrow$

(*) $S_4 r^3 - S_3 r^2 + S_2 r - S_1 = 0$.

where

$S_4 = a(1-b-f)$

$S_3 = c+e+af-ef+2ab$

$S_2 = a+b+af+2ef-ab$

$S_1 = f(1-a-e)$

N.B. $S_1+S_2+S_3+S_4 = 1$

$S_i > 0$; in fact $S_2S_3 > S_1S_4$.

So (*) is strongly alternating.

Now suppose you're handed

$$r^3-5r^2+3r-1=0$$

use picture to construct equidissection?

Let $s_4 = \frac{1}{10}$

$s_3 = \frac{5}{10}$

$s_2 = \frac{3}{10}$

$s_1 = \frac{1}{10}$.

So can we choose $a, b, c, d, e, f \in \mathbb{Q}^{>0}$ such that

(1) $a(1-b-f) = \frac{1}{10}$ (3) $c+e+af-ef+2ab = \frac{5}{10}$??

(2) $f(1-a-e) = \frac{1}{10}$ (4) $d+b+af+2efab = \frac{3}{10}$

Let a, f be arbitrary positive rationals.
Solve (1) for b, (2), ..., etc. They'll all be rational. So choose a, f carefully so they're all positive.

e.g. $a = \frac{1}{8} = 5/40$, $f = \frac{1}{8} = 5/40$.

$b = e = 3/40$

$c = \frac{16}{40}$ $d = \frac{8}{40}$.

So we can equidissect this trapezoid with 40 triangles of equal area.

Suppose you're given

$$s_4 r^3 - s_3 r^2 + s_2 r - s_1 \qquad s_i \in \mathbb{Q}^{>0}.$$

$$s_1 + s_2 + s_3 + s_4 = 1$$

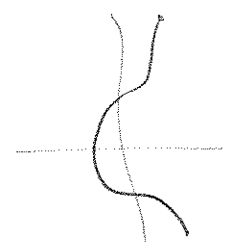
$$s_2 s_3 > s_1 s_4.$$

~~Thm~~ **Prop.** Can solve above equations in positive rationals.

Pf. not too hard. Set two zero, get a solution in 4 reals. Then wiggle around zero. get a real nonzero solution. Then iterate.

So this proves the above theorem for cubics.

Quartics? need elliptic curves.

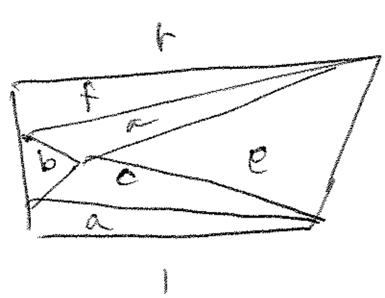


curve $y^2 = x^3 + x + 1$

rational pts? $(72, \pm 611)$, $(\frac{1}{4}, \pm\frac{9}{8})$, etc.. $(0, \pm 1)$,

dense in curve.

Eg



$a + b + c + d + e + f = 1.$

get quartic equation in r, strongly alternating.

Must solve some equation in rationals.

get into rational pts on elliptic curves.

(need to find some rational pt in a small interval).

<u>Theorem</u> If $T(r)$ has an equidissection into n triangles, then $\frac{n}{1+r}$ is algebraic.

<u>proof is complicated</u> ....

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