GEOMETRIC LANGLANDS DUALITY AND REPRESENTATIONS OF ALGEBRAIC GROUPS OVER COMMUTATIVE RINGS

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1. Introduction

In this paper we give a geometric version of the Satake isomorphism [Sat]. As such, it can be viewed as a first step in the geometric Langlands program. The connected complex reductive groups have a combinatorial classification by their root data. In the root datum the roots and the coroots appear in a symmetric manner and so the connected reductive algebraic groups come in pairs. If G is a reductive group, we write $\check{G}$ for its companion and call it the dual group G. The notion of the dual group itself does not appear in Satake's paper, but was introduced by Langlands, together with its various elaborations, in [L1, L2] and is a corner stone of the Langlands program. It also appeared later in physics [MO, GNO]. In this paper we discuss the basic relationship between G and $\check{G}$.

We begin with a reductive G and consider the affine Grassmannian $\mathcal{G}\mathrm{r}$, the Grassmannian for the loop group of G. For technical reason we work with formal algebraic loops. The affine Grassmannian is an infinite dimensional complex space. We consider a certain category of sheaves, the spherical perverse sheaves, on $\mathcal{G}\mathrm{r}$. These sheaves can be multiplied using a convolution product and this leads to a rather explicit construction of a Hopf algebra, by what has come to be known as Tannakian formalism. The resulting Hopf algebra turns out to be the ring of functions on $\check{G}$. In this interpretation, the spherical perverse sheaves on the affine Grassmannian correspond to finite dimensional complex representations of $\check{G}$. Thus, instead of defining $\check{G}$ in terms of the classification of reductive groups, we provide a canonical construction of $\check{G}$, starting from G. We can carry out our construction over the integers. The spherical perverse sheaves are then those with integral coefficients, but the Grassmannian remains a complex algebraic object. The resulting $\check{G}$ turns out to be the Chevalley scheme over the integers, i.e., the unique split reductive group scheme whose root datum coincides with that of the complex $\check{G}$. Thus, our result can also be viewed as providing an explicit construction of the Chevalley scheme. Once we have a construction over the integers, we have one for every commutative ring and in particular for all fields. This provides another way of viewing our result: it provides a geometric interpretation of representation theory of algebraic groups over arbitrary rings. The change of rings on the representation theoretic side corresponds to change of coefficients of perverse sheaves, familiar from the universal coefficient theorem in algebraic topology. Note that for us it is crucial that we first prove our result for the integers (or p-adic integers) and then deduce the

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theorem for fields (of positive characteristic). We do not know how to argue the case of fields of positive characteristic directly.

One of the key technical points of this paper is the construction of certain algebraic cycles that turn out to give a basis, even over the integers, of the cohomology of the standard sheaves on the affine Grassmannian. This result is new even over the complex numbers. These cycles are obtained by utilizing semi-infinite Schubert cells in the affine Grassmannian. The semi-infinite Schubert cells can then be viewed as providing a perverse cell decomposition of the affine Grassmannian analogous to a cell decomposition for ordinary homology where the dimensions of all the cells have the same parity. The idea of searching for such a cell decomposition came from trying to find the analogues of the basic sets of [GM] in our situation.

The first work in the direction of geometrizing the Satake isomorphism is [Lu] where Lusztig introduces the key notions and proves the result in the characteristic zero case on a combinatorial level. Independently, Drinfeld had understood that geometrizing the Satake isomorphism is crucial for formulating the geometric Langlands correspondence. Following Drinfeld's suggestion, Ginzburg in [Gi], using [Lu], treated the characteristic zero case of the geometric Satake isomorphism. Our paper is self-contained in that it does not rely on [Lu] or [Gi] and provides some improvements and precision even in the characteristic zero case. However, we make crucial use of an idea of Drinfeld, going back to around 1990. He discovered an elegant way of obtaining the commutativity constraint by interpreting the convolution product of sheaves as a "fusion" product.

We now give a more precise version of our result. Let G be a reductive algebraic group over the complex numbers. We write $G_{\mathcal{O}}$ for the group scheme $G(\mathbb{C}[[z]])$ and $\mathcal{G}\mathrm{r}$ for the affine Grassmannian of $G(\mathbb{C}((z)))/G(\mathbb{C}[[z]])$; the affine Grassmannian is an ind-scheme, i.e., a direct limit of schemes. Let $\Bbbk$ be a Noetherian, commutative unital ring of finite global dimension. One can imagine $\Bbbk$ to be $\mathbb{C}$, $\mathbb{Z}$, or $\overline{\mathbb{F}_q}$, for example. Let us write $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ for the category of $G_{\mathcal{O}}$-equivariant perverse sheaves with $\Bbbk$-coefficients. Furthermore, let $\mathrm{Rep}_{\check{G}_{\Bbbk}}$ stand for the category of $\Bbbk$-representations of $\check{G}_{\Bbbk}$; here $\check{G}_{\Bbbk}$ denotes the canonical smooth split reductive group scheme over $\Bbbk$ whose root datum is dual to that of G. The goal of this paper is to prove the following:

(1.1) the categories $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ and $\mathrm{Rep}_{\check{G}_{\Bbbk}}$ are equivalent as tensor categories.

We do slightly more than this. We give a canonical construction of the group scheme $\check{G}_{\Bbbk}$ in terms of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$. In particular, we give a canonical construction of the Chevalley group scheme $\check{G}_{\mathbb{Z}}$ in terms of the complex group G. This is one way to view our theorem. We can also view it as giving a geometric interpretation of representation theory of algebraic groups over commutative rings. Although our results yield an interpretation of representation theory over arbitrary commutative rings, note that on the geometric side we work over the complex numbers and use the classical topology. The advantage of the classical topology is that one can work with sheaves with coefficients in arbitrary commutative rings, in particular, we can use integer coefficients. Finally, our work can be viewed as providing the unramified local geometric Langlands correspondence. In this context it is crucial that one works on the geometric side also over fields other than $\mathbb{C}$; this is easily done as the affine Grassmannian can be defined even over the

integers. The modifications needed to do so are explained in section 14. This can then be used to define the notion of a Hecke eigensheaf in the generality of arbitrary systems of coefficients.

We describe the contents of the paper briefly. Section 2 is devoted to the basic definitions involving the affine Grassmannian and the notion of perverse sheaves that we adopt. In section 3 we introduce our main tool, the weight functors. In this section we also give our crucial dimension estimates, use them to prove the exactness of the weight functors, and, finally, we decompose the global cohomology functor into a direct sum of the weight functors. The next section 4 is devoted to putting a tensor structure on the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$; here, again, we make use of the dimension estimates of the previous section. In section 5 we give, using the Beilinson-Drinfeld Grassmannian, a commutativity constraint on the tensor structure. In section 6 we show that global cohomology is a tensor functor and we also show that it is tensor functor in the weighted sense. Section 7 is devoted to the simpler case when $\Bbbk$ is a field of characteristic zero. The next section 8 treats standard sheaves and we show that their cohomology is given by specific algebraic cycles which provide a canonical basis for the cohomology. In the next section 9 we prove that the weight functors introduced in section 3 are representable. This, then, will provide us with a supply of projective objects. In section 10 we study the structure of these projectives and prove that they have filtrations whose associated graded consists of standard sheaves. In section 11 we show that $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ is equivalent, as a tensor category, to $\mathrm{Rep}_{\tilde{G}_{\Bbbk}}$ for some group scheme $\tilde{G}_{\Bbbk}$. Then, in the next section 12, we identify $\tilde{G}_{\Bbbk}$ with $\check{G}_{\Bbbk}$. A crucial ingredient in this section is the work of Prasad and Yu [PY]. We then briefly discuss in section 13 our results from the point of view of representation theory. In the final section 14 we briefly indicate how our arguments have to be modified to work in the étale topology.

Most of the results in this paper appeared in the announcement [MiV2]. Since our announcement was published, the papers [Br] and [Na] have appeared. Certain technical points that are necessary for us are treated in these papers. Instead of repeating the discussion here, we have chosen to refer to [Br] and [Na] instead. Finally, let us note that we have not managed to carry out the idea of proof proposed in [MiV2] for theorem 12.1 (theorem 6.2 in [MiV2]) and thus the paper [MiV2] should be considered incomplete. In this paper, as was mentioned above, we will appeal to [PY] to prove theorem 12.1.

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2. **Perverse sheaves on the affine Grassmannian**

We begin this section by recalling the construction and the basic properties of the affine Grassmannian $\mathcal{G}\mathrm{r}$. For proofs of these facts we refer to §4.5 of [BD]. See also, [BL1] and [BL2]. Then we introduce the main object of study, the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ of equivariant perverse sheaves on $\mathcal{G}\mathrm{r}$.

Let G be a complex, connected, reductive algebraic group. We write $\mathcal{O}$ for the formal power series ring $\mathbb{C}[[z]]$ and $\mathcal{K}$ for its fraction field $\mathbb{C}((z))$. Let $G(\mathcal{K})$ and $G(\mathcal{O})$ denote, as usual, the sets of the $\mathcal{K}$-valued and the $\mathcal{O}$-valued points of G, respectively. The affine Grassmannian is defined as the quotient $G(\mathcal{K})/G(\mathcal{O})$. The sets $G(\mathcal{K})$ and $G(\mathcal{O})$, and the quotient $G(\mathcal{K})/G(\mathcal{O})$ have an algebraic structure over the complex numbers. The space $G(\mathcal{O})$ has a structure of a group scheme, denoted by $G_{\mathcal{O}}$, over $\mathbb{C}$ and the spaces $G(\mathcal{K})$ and $G(\mathcal{K})/G(\mathcal{O})$ have structures of ind-schemes which we denote by $G_{\mathcal{K}}$ and $\mathcal{G}\mathrm{r} = \mathcal{G}\mathrm{r}_G$, respectively. For us an ind-scheme means a direct limit of family of schemes where all the maps are closed embeddings. The morphism $\pi : G_{\mathcal{K}} \to \mathcal{G}\mathrm{r}$ is locally trivial in the Zariski topology, i.e., there exists a Zariski open subset $U \subset \mathcal{G}\mathrm{r}$ such that $\pi^{-1}(U) \cong U \times G_{\mathcal{O}}$ and π restricted to $U \times G_{\mathcal{O}}$ is simply projection to the first factor. For details see for example [BL1, LS]. We write $\mathcal{G}\mathrm{r}$ as a limit

$$\mathcal{G}\mathrm{r} \;=\; \varinjlim \mathcal{G}\mathrm{r}_n\,, \tag{2.1}$$

where the $\mathcal{G}\mathrm{r}_n$ are finite dimensional schemes which are $G_{\mathcal{O}}$-invariant. The group $G_{\mathcal{O}}$ acts on the $\mathcal{G}\mathrm{r}_n$ via a finite dimensional quotient.

In this paper we consider sheaves in the classical topology, with the exception of section 14 where we use the etale topology. Therefore, it suffices for our purposes to consider the spaces $G_{\mathcal{O}}$, $G_{\mathcal{K}}$, and $\mathcal{G}\mathrm{r}$ as reduced ind-schemes. We will do so for the rest of the paper.

If $G = T$ is torus of rank r then, as a reduced ind-scheme, $\mathcal{G}\mathrm{r} \cong X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$, i.e., in this case the loop Grassmannian is discrete. Note that, because T is abelian, the loop Grassmannian is a group ind-scheme. Let G be a reductive group, write $Z(G)$ for the center of G and let $Z = Z(G)^0$ denote connected component of the center. Let us further set $\overline{G} = G/Z$. Then, as is easy to see, the map $\mathcal{G}\mathrm{r}_G \to \mathcal{G}\mathrm{r}_{\overline{G}}$ is a trivial covering with covering group $X_*(Z) = \mathrm{Hom}(\mathbb{C}^*, Z)$, i.e., $\mathcal{G}\mathrm{r}_G \cong \mathcal{G}\mathrm{r}_{\overline{G}} \times X_*(Z)$, non-canonically. Note also that the connected components of $\mathcal{G}\mathrm{r}$ are exactly parameterized by the component group of $G_{\mathcal{K}}$, i.e., by $G_{\mathcal{K}}/(G_{\mathcal{K}})^0$. This latter group is isomorphic to $\pi_1(G)$, the topological fundamental group of G.

The group scheme $G_{\mathcal{O}}$ acts on $\mathcal{G}\mathrm{r}$ with finite dimensional orbits. In order to describe the orbit structure, let us fix a maximal torus $T \subset G$. We write W for the Weyl group and $X_*(T)$ for the coweights $\mathrm{Hom}(\mathbb{C}^*, T)$. Then the $G_{\mathcal{O}}$-orbits on $\mathcal{G}\mathrm{r}$ are parameterized by the W-orbits in $X_*(T)$, and given $\lambda \in X_*(T)$ the $G_{\mathcal{O}}$-orbit associated to $W\lambda$ is $\mathcal{G}\mathrm{r}^\lambda = G_{\mathcal{O}} \cdot L_\lambda \subset \mathcal{G}\mathrm{r}$, where L_λ denotes the image of the point $\lambda \in X_*(T) \subseteq G_{\mathcal{K}}$ in $\mathcal{G}\mathrm{r}$. Note that the points L_λ are precisely the T-fixed points in the Grassmannian. To describe the closure relation between the $G_{\mathcal{O}}$-orbits, we choose a Borel $B \supset T$ and write N for the unipotent radical of B. We use the convention that the roots in B are the positive ones. Then, for dominant λ and μ we have

$$\mathcal{G}\mathrm{r}^\mu \subset \overline{\mathcal{G}\mathrm{r}^\lambda} \quad \text{if and only if} \quad \lambda - \mu \quad \text{is a sum of positive coroots}\,. \tag{2.2}$$

In a few arguments in this paper it will be important for us to consider a Kac-Moody group associated to the loop group $G_{\mathcal{K}}$. Let us write $\Delta = \Delta(G, T)$ for the root system of G with respect to T, and we write similarly $\check{\Delta} = \check{\Delta}(G, T)$ for the coroots. Let $\Gamma \cong \mathbb{C}^*$ denote the subgroup of automorphisms of $\mathcal{K}$ which acts by multiplying the parameter

$z \in \mathcal{K}$ by $s \in \mathbb{C}^* \cong \Gamma$. The group Γ acts on $G_{\mathcal{O}}$ and $G_{\mathcal{K}}$ and hence we can form the semi-direct product $\widetilde{G_{\mathcal{K}}} = G_{\mathcal{K}} \rtimes \Gamma$. Then $\widetilde{T} = T \times \Gamma$ is a Cartan subgroup of $\widetilde{G_{\mathcal{K}}}$. An affine Kac-Moody group $\widehat{G_{\mathcal{K}}}$ is a central extension, by the multiplicative group, of $\widetilde{G_{\mathcal{K}}}$; note that the root systems are the same whether we consider $\widetilde{G_{\mathcal{K}}}$ or $\widehat{G_{\mathcal{K}}}$. Let us write $\delta \in X^*(\widetilde{T})$ for the character which is trivial on T and identity on the factor $\Gamma \cong \mathbb{C}^*$ and let $\check{\delta} \in X_*(\widetilde{T})$ be the cocharacter $\mathbb{C}^* \cong \Gamma \subset T \times \Gamma = \widetilde{T}$. We also view the roots Δ as characters on $\widetilde{T}$, which are trivial on Γ. The $\widetilde{T}$-eigenspaces in $\mathfrak{g}_{\mathcal{K}}$ are given by

$$(\mathfrak{g}_{\mathcal{K}})_{k\delta+\alpha} \stackrel{def}{=} z^k \mathfrak{g}_\alpha, \ k \in \mathbb{Z}, \alpha \in \Delta \cup \{0\}, \tag{2.3}$$

and thus the roots of $G_{\mathcal{K}}$ are given by $\widetilde{\Delta} = \{\alpha + k\delta \in X^*(\widetilde{T}) \mid \alpha \in \Delta \cup \{0\}, k \in \mathbb{Z}\} - \{0\}$.

Furthermore, the orbit $G \cdot L_\lambda$ is isomorphic to the flag manifold G/P_λ, where P_λ, the stabilizer of L_λ in G, is a parabolic with a Levi factor associated to the roots $\{\alpha \in \Delta \mid \lambda(\alpha) = 0\}$. The orbit $\mathcal{G}\mathrm{r}^\lambda$ can be viewed as a G-equivariant vector bundle over G/P_λ. One way to see this is to observe that the varieties $G \cdot L_\lambda$ are the fixed point sets of the $\mathbb{G}_m$-action via the cocharacter $\check{\delta}$. In this language,

$$\mathcal{G}\mathrm{r}^\lambda \ = \ \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s \to 0} \check{\delta}(s)x \in G \cdot L_\lambda\} \tag{2.4}$$

In particular, the orbits $\mathcal{G}\mathrm{r}^\lambda$ are simply connected. If we choose a Borel B containing T and if we choose the parameter $\lambda \in X_*(T)$ of the orbit $\mathcal{G}\mathrm{r}^\lambda$ to be dominant, then $\dim(\mathcal{G}\mathrm{r}^\lambda) = 2\rho(\lambda)$, where $\rho \in X^*(T)$, as usual, is half the sum of positive roots with respect to B. Let us consider the map $\mathrm{ev}_0 : G_{\mathcal{O}} \to G$, evaluation at zero. We write $I = {\mathrm{ev}_0}^{-1}(B)$ for the Iwahori subgroup and $K = {\mathrm{ev}_0}^{-1}(1)$ for the highest congruence subgroup. The I-orbits are parameterized by $X_*(T)$, and because the I-orbits are also ${\mathrm{ev}_0}^{-1}(N)$-orbits, they are affine spaces. This way each $G_{\mathcal{O}}$-orbit acquires a cell decomposition as a union of I-orbits. The K-orbit $K \cdot L_\lambda$ is the fiber of the vector bundle $\mathcal{G}\mathrm{r}^\lambda \to G/P_\lambda$. Let us consider the subgroup ind-scheme $G^-_{\mathcal{O}}$ of $G_{\mathcal{K}}$ whose $\mathbb{C}$-points consist of $G(\mathbb{C}[z^{-1}])$. The $G^-_{\mathcal{O}}$-orbits are also indexed by W-orbits in $X_*(T)$ and the orbit attached to $\lambda \in X_*(T)$ is $G^-_{\mathcal{O}} \cdot L_\lambda$. The $G^-_{\mathcal{O}}$-orbits are opposite to the $G_{\mathcal{O}}$-orbits in the following sense:

$$G^-_{\mathcal{O}} \cdot L_\lambda \ = \ \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s \to \infty} \check{\delta}(s)x \in G \cdot L_\lambda\}. \tag{2.5}$$

The above description implies that

$$(G^-_{\mathcal{O}} \cdot L_\lambda) \cap \overline{\mathcal{G}\mathrm{r}^\lambda} \ = \ G \cdot L_\lambda \tag{2.6}$$

The group $G^-_{\mathcal{O}}$ contains a negative level congruence subgroup K_- which is the kernel of the evaluation map $G(\mathbb{C}[z^{-1}]) \to G$ at infinity. Just as for $G_{\mathcal{O}}$, the fiber of the projection $G^-_{\mathcal{O}} \cdot L_\lambda \to G/P_\lambda$ is $K_- \cdot L_\lambda$.

We will recall briefly the notion of perverse sheaves that we will use in this paper [BBD]. Let X be a complex algebraic variety with a fixed (Whitney) stratification $\mathcal{S}$. We also fix a commutative, unital ring $\Bbbk$. For simplicity of exposition we assume that $\Bbbk$ is Noetherian of finite global dimension. This has the advantage of allowing us to work with finite complexes and finitely generated modules instead of having to use more complicated notions of finiteness. With suitable modifications, the results of this paper hold for arbitrary $\Bbbk$. We denote by $\mathrm{D}_{\mathcal{S}}(X, \Bbbk)$ the bounded $\mathcal{S}$-constructible derived

category of $\Bbbk$-sheaves. This is the full subcategory of the derived category of $\Bbbk$-sheaves on X whose objects $\mathcal{F}$ satisfy the following two conditions:

$$\begin{aligned} &i)\ \ \mathrm{H}^k(X,\mathcal{F}) = 0 \quad \text{for } |k| > 0\,, \\ &ii)\ \ \mathrm{H}^k(\mathcal{F})\big|S \quad \text{is a local system of finitely generated } \Bbbk\text{-modules} \\ &\qquad \text{for all } S \in \mathcal{S}\,. \end{aligned}$$

As usual we define the full subcategory $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ of perverse sheaves as follows. An $\mathcal{F} \in \mathrm{D}_{\mathcal{S}}(X,\Bbbk)$ is perverse if the following two conditions are satisfied:

$$\begin{aligned} &i)\ \ \mathrm{H}^k(i^*\mathcal{F}) = 0 \quad \text{for } k > -\dim_{\mathbb{C}} S \ \text{ for any } \ i : S \hookrightarrow X\,, S \in \mathcal{S}\,, \\ &ii)\ \ \mathrm{H}^k(i^!\mathcal{F}) = 0 \quad \text{for } k < \dim_{\mathbb{C}} S \ \text{ for any } \ i : S \hookrightarrow X\,, S \in \mathcal{S}\,. \end{aligned}$$

As is explained in [BBD], perverse sheaves $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ form an abelian category and there is a cohomological functor

$$^p\mathrm{H}^0 : \mathrm{D}_{\mathcal{S}}(X,\Bbbk) \to \mathrm{P}_{\mathcal{S}}(X,\Bbbk)\,.$$

Given a stratum $S \in \mathcal{S}$ and M a finitely generated $\Bbbk$-module then Rj_*M and $j_!M$ belong in $\mathrm{D}_{\mathcal{S}}(X,\Bbbk)$. Following [BBD] we write $^pj_*M$ for $^p\mathrm{H}^0(Rj_*M) \in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ and $^pj_!M$ for $^p\mathrm{H}^0(j_!M) \in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$. We use this type of notation systematically throughout the paper. If $Y \subset X$ is locally closed and is a union of strata in $\mathcal{S}$ then, by abuse of notation, we denote by $\mathrm{P}_{\mathcal{S}}(Y,\Bbbk)$ the category $\mathrm{P}_{\mathcal{T}}(Y,\Bbbk)$, where $\mathcal{T} = \{S \in \mathcal{S} \ \big|\ S \subset Y\}$.

Let us now assume that we have an action of a connected algebraic group K on X, given by $a : K \times X \to X$. Fix a Whitney stratification $\mathcal{S}$ of X such that the action of K preserves the strata. Recall that an $\mathcal{F} \in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ is said to be K-equivariant if there exists an isomorphism $\phi : a^*\mathcal{F} \cong p^*\mathcal{F}$ such that $\phi\big|\{1\} \times X = \mathrm{id}$. Here $p : K \times X \to X$ is the projection to the second factor. If such an isomorphism ϕ exists it is unique. We denote by $\mathrm{P}_K(X,\Bbbk)$ the full subcategory of $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ consisting of equivariant perverse sheaves. In a few instances we also make use of the equivariant derived category $\mathrm{D}_K(X,\Bbbk)$, see [BL].

Let us now return to our situation. Denote the stratification induced by the $G_{\mathcal{O}}$-orbits on the Grassmannian $\mathcal{G}\mathrm{r}$ by $\mathcal{S}$. The closed embeddings $\mathcal{G}\mathrm{r}_n \subset \mathcal{G}\mathrm{r}_m$, for $n \leq m$ induce embeddings of categories $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_n,\Bbbk) \to \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_m,\Bbbk)$. This allows us to define the category of $G_{\mathcal{O}}$-equivariant perverse sheaves on $\mathcal{G}\mathrm{r}$ as

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk) \ =_{\mathrm{def}}\ \varinjlim \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_n,\Bbbk)\,.$$

Similarly we define $\mathrm{P}_{\mathcal{S}}(\mathcal{G}\mathrm{r},\Bbbk)$, the category of perverse sheaves on $\mathcal{G}\mathrm{r}$ which are constructible with respect to the $G_{\mathcal{O}}$-orbits. In our setting we have

2.1. **Proposition.** *The categories* $\mathrm{P}_{\mathcal{S}}(\mathcal{G}r,\Bbbk)$ *and* $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r,\Bbbk)$ *are naturally equivalent.*

We give a proof of this proposition in appendix A; the proof makes use of results of section 3.

Let us write $\mathrm{Aut}(\mathcal{O})$ for the group of automorphisms of the formal disc $\mathrm{Spec}(\mathcal{O})$. The group scheme $\mathrm{Aut}(\mathcal{O})$ acts on $G_{\mathcal{K}}$, $G_{\mathcal{O}}$, and $\mathcal{G}\mathrm{r}$. This action and the action of $G_{\mathcal{O}}$ on the affine Grassmannian extend to an action of the semidirect product $G_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathcal{O})$ on $\mathcal{G}\mathrm{r}$. In the appendix A we also prove

2.2. **Proposition.** *The categories* $\mathrm{P}_{G_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathcal{O})}(\mathcal{G}r, \Bbbk)$ *and* $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ *are naturally equivalent.*

2.3. **Remark.** *If* $\Bbbk$ *is field of characteristic zero then propositions 2.1 and 2.2 follow immediately from lemma 7.1.*

Finally, we fix some notation that will be used throughout the paper. Given a $G_{\mathcal{O}}$-orbit $\mathcal{G}\mathrm{r}_\lambda$, $\lambda \in X_*(T)$, and a $\Bbbk$-module M we write $\mathcal{I}_!(\lambda, M)$, $\mathcal{I}_*(\lambda, M)$, and $\mathcal{I}_{!*}(\lambda, M)$ for the perverse sheaves ${}^p j_!(M[\dim(\mathcal{G}\mathrm{r}^\lambda)])$, $j_{!*}(M[\dim(\mathcal{G}\mathrm{r}^\lambda)])$, and ${}^p j_*(M[\dim(\mathcal{G}\mathrm{r}^\lambda)])$, respectively; here $j : \mathcal{G}\mathrm{r}^\lambda \to \mathcal{G}\mathrm{r}$ denotes the inclusion.

3. **Semi-infinite orbits and weight functors**

Here we show that the global cohomology is a fiber functor for our tensor category. For $\Bbbk = \mathbb{C}$ this is proved by Ginzburg [Gi] and was treated earlier in [Lu], on the level of dimensions (the dimension of the intersection cohomology is the same as the dimension of the corresponding representation).

Recall that we have fixed a maximal torus T, a Borel $B \supset T$ and denoted by N the unipotent radical of B. Furthermore, we write $N_{\mathcal{K}}$ for the group ind-subscheme of $G_{\mathcal{K}}$ whose $\mathbb{C}$-points are $N(\mathcal{K})$. The $N_{\mathcal{K}}$-orbits on $\mathcal{G}\mathrm{r}$ are parameterized by $X_*(T)$; to each $\nu \in X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$ we associate the $N_{\mathcal{K}}$-orbit $S_\nu = N_{\mathcal{K}} \cdot L_\nu$. Note that these orbits are neither of finite dimension nor of finite codimension. We view them as ind-varieties, in particular, their intersection with any $\overline{\mathcal{G}\mathrm{r}^\lambda}$ is an algebraic variety. The following proposition gives the basic properties of these orbits. Recall that for $\mu, \lambda \in X_*(T)$ we say that $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of positive coroots.

3.1. **Proposition.** *We have*

(a) $\overline{S_\nu} = \cup_{\eta \leq \nu} S_\eta$.

(b) Inside $\overline{S_\nu}$*, the boundary of* S_ν *is given by a hyperplane section under an embedding of* $\mathcal{G}r$ *in projective space.*

Proof. Because translation by elements in $T_{\mathcal{K}}$ is an automorphism of the Grassmannian, it suffices to prove the claim on the identity component of the Grassmannian. Hence, we may assume that G is simply connected. In that case G is a product of simple factors and we may then furthermore assume that G is simple and simply connected.

For a positive coroot $\check{\alpha}$, there is T-stable $\mathbb{P}^1$ passing through $L_{\nu - \check{\alpha}}$ such that the remaining $\mathbb{A}^1$ lies in S_ν, constructed as follows. First observe that the one parameter subgroup U_ψ for an affine root $\psi = \alpha + k\delta$ fixes L_ν if $z^{k - \langle \alpha, \nu \rangle} \mathfrak{g}_\alpha$ fixes L_0, i.e., if $k \geq \langle \alpha, \nu \rangle$. So, for any integer $k < \langle \alpha, \nu \rangle$, $(\mathfrak{g}_{\mathcal{K}})_\psi$ does not fix L_ν, but $(\mathfrak{g}_{\mathcal{K}})_{-\psi}$ does. We conclude that for the SL_2-subgroup generated by the one parameter subgroups $U_{\pm\psi}$ the orbit through L_ν is a $\mathbb{P}^1$ and that $U_\psi \cdot L_\nu \cong \mathbb{A}^1$ lies in S_ν since $\alpha > 0$. The point at infinity is then $L_{s_\psi \nu}$ for the reflection s_ψ in the affine root ψ. For $k = \langle \alpha, \nu \rangle - 1$ this yields $L_{\nu - \check{\alpha}}$ as the point at infinity. Hence $S_{\nu - \check{\alpha}} \subseteq \overline{S_\nu}$ for any positive coroot $\check{\alpha}$ and therefore $\cup_{\eta \leq \nu} S_\eta \subset \overline{S_\nu}$.

To prove the rest of the proposition we embed the ind-variety $\mathcal{G}\mathrm{r}$ in an ind-projective space $\mathbb{P}(V)$ via an ample line bundle $\mathcal{L}$ on $\mathcal{G}\mathrm{r}$. For simplicity we choose $\mathcal{L}$ to be the positive generator of the Picard group of $\mathcal{G}\mathrm{r}$. The action of $G_{\mathcal{K}}$ on $\mathcal{G}\mathrm{r}$ only extends to

a projective action on the line bundle $\mathcal{L}$. To get an action on $\mathcal{L}$ we must pass to the Kac-Moody group $\widehat{G_{\mathcal{K}}}$ associated to $G_{\mathcal{K}}$, which was discussed in the previous section. The highest weight Λ_0 of the resulting representation $V = \mathrm{H}^0(\mathcal{G}\mathrm{r}, \mathcal{L})$ is zero on T and one on the central $\mathbb{G}_m$. Thus, we get a $G_{\mathcal{K}}$-equivariant embedding $\Psi : \mathcal{G}\mathrm{r} \hookrightarrow \mathbb{P}(V)$ which maps L_0 to the highest weight line V_{Λ_0}. In particular, the T-weight of the line $\Psi(L_0) = V_{\Lambda_0}$ is zero.

We need a formula for the T-weight of the line $\Psi(L_\nu) = \nu \cdot \Psi(L_0) = \nu \cdot V_{\Lambda_0}$. Now, $\nu \cdot V_{\Lambda_0} = V_{\widetilde{\nu}\cdot\Lambda_0}$, where $\widetilde{\nu}$ is any lift of the element $\nu \in X_*(T)$ to $\widehat{T_{\mathcal{K}}}$, the restriction to $\widetilde{T_{\mathcal{K}}}$ of the central extension $\widehat{G_{\mathcal{K}}}$ of $\widetilde{G_{\mathcal{K}}}$ by $\mathbb{G}_m$. For $t \in T$,

$$(\widetilde{\nu} \cdot \Lambda_0)(t) = \Lambda_0(\widetilde{\nu}^{-1} t \widetilde{\nu}) = \Lambda_0(\widetilde{\nu}^{-1} t \widetilde{\nu} t^{-1}) \,, \tag{3.1}$$

since $\Lambda_0(t) = 1$. The commutator $x, y \mapsto xyx^{-1}y^{-1}$ on $\widehat{T_{\mathcal{K}}}$ descends to a pairing of $T_{\mathcal{K}} \times T_{\mathcal{K}}$ to the central $\mathbb{G}_m$. The restriction of this pairing to $X_*(T) \times T \to \mathbb{G}_m$, can be viewed as a homomorphism $\iota : X_*(T) \to X^*(T)$, or, equivalently, as a bilinear form $(\ ,\ )_*$ on $X_*(T)$. Since Λ_0 is identity on the central $\mathbb{G}_m$ and since $\widetilde{\nu}^{-1} t \widetilde{\nu} t^{-1} \in \mathbb{G}_m$, we see that

$$(\widetilde{\nu} \cdot \Lambda_0)(t) \ = \ \widetilde{\nu}^{-1} t \widetilde{\nu} t^{-1} \ = \ (\iota\nu)(t)^{-1} \,, \tag{3.2}$$

i.e., $\widetilde{\nu} \cdot \Lambda_0 = -\iota\nu$ on T. We will now describe the morphism ι.

The description of the central extension of $\widetilde{\mathfrak{g}_{\mathcal{K}}}$, corresponding to $\widehat{G_{\mathcal{K}}}$, makes use of an invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$, see, for example, [PS]. From the basic formula for the coadjoint action of $\widehat{G_{\mathcal{K}}}$ (see, for example, [PS]), it is clear that the form $(\ ,\ )_*$ above is the restriction of $(\ ,\ )$ to $\mathfrak{t} = \mathbb{C} \otimes X_*(T)$. The form $(\ ,\ )$ is characterized by the property that the corresponding bilinear form $(\ ,\ )^*$ on $\mathfrak{t}^*$ satisfies $(\theta, \theta)^* = 2$ for the longest root θ. Now, for a root $\alpha \in \Delta$ we find that

$$\iota\check{\alpha} \ = \ \frac{2}{(\alpha,\alpha)^*}\alpha \ = \ \frac{(\theta,\theta)^*}{(\alpha,\alpha)^*}\alpha \in \{1,2,3\} \cdot \alpha \tag{3.3}$$

We conclude that $\iota(\mathbb{Z}\check{\Delta}) \cap \mathbb{Z}_+\Delta_+ = \iota(\mathbb{Z}_+\check{\Delta}_+)$, i.e.,

$$\nu < \eta \quad \text{is equivalent to} \quad \iota\nu < \iota\eta \qquad \text{for } \nu, \eta \in X_*(T) \,. \tag{3.4}$$

Let us write $V_{>-\iota\nu} \subseteq V_{\geq -\iota\nu}$ for the sum of all the T-weight spaces of V whose T-weight is bigger than (or equal to) $-\iota\nu$. Clearly the central extension of $N_{\mathcal{K}}$ acts by increasing the T weights, i.e., its action preserves the subspaces $V_{>-\iota\nu}$ and $V_{\geq -\iota\nu}$. This, together with (3.4), implies that $\cup_{\eta \leq \nu} S_\eta = \Psi^{-1}(\mathbb{P}(V_{\geq -\iota\nu}))$. In particular, $\cup_{\eta \leq \nu} S_\eta$ is closed. This, with $\cup_{\eta \leq \nu} S_\eta \subset \overline{S_\nu}$, implies that $\overline{S_\nu} = \cup_{\eta \leq \nu} S_\eta$, proving part (a) of the proposition.

To prove part (b), we first observe that $\cup_{\eta < \nu} S_\eta = \Psi^{-1}(\mathbb{P}(V_{>-\iota\nu}))$. The line $\Psi(L_\nu)$ lies in $V_{\geq -\iota\nu}$ but not in $V_{>-\iota\nu}$. Let us choose a linear form f on V which is non-zero on the line $\Psi(L_\nu)$ and which vanishes on all T-eigenspaces whose eigenvalue is different from $-\iota\nu$. Let us write H for the hyperplane $\{f = 0\} \subseteq V$. By construction, for $v \in \Psi(L_\nu)$, and any n in the central extension of $N_{\mathcal{K}}$, $nv \in \ \mathbb{C}^* \cdot v + \ V_{>-\iota\nu}$. So $v \neq 0$ implies $f(nv) \neq 0$, and we see that $S_\nu \cap H = \emptyset$. Since $\cup_{\eta < \nu} S_\eta \subset H$, we conclude that $\overline{S_\nu} \cap H = \cup_{\eta < \nu} S_\eta$, as required.

□

Let us also consider the unipotent radical N^- of the Borel B^- opposite to B. The $N^-_{\mathcal{K}}$-orbits on $\mathcal{G}\mathrm{r}$ are again parameterized by $X_*(T)$: to each $\nu \in X_*(T)$ we associate the orbit $T_\nu = N^-_{\mathcal{K}} \cdot L_\nu$. The orbits S_ν and T_ν intersect the orbits $\mathcal{G}\mathrm{r}^\lambda$ as follows:

3.2. **Theorem.** *We have*

a) The intersection $S_\nu \cap \mathcal{G}r^\lambda$ is non-empty precisely when $L_\nu \in \overline{\mathcal{G}r^\lambda}$ and then $S_\nu \cap \overline{\mathcal{G}r^\lambda}$ is of pure dimension $\rho(\nu+\lambda)$, if λ is chosen dominant.

b) The intersection $T_\nu \cap \mathcal{G}r^\lambda$ is non-empty precisely when $L_\nu \in \overline{\mathcal{G}r^\lambda}$ and then $T_\nu \cap \overline{\mathcal{G}r^\lambda}$ is of pure dimension $-\rho(\nu+\lambda)$, if λ is chosen anti-dominant.

3.3. **Remark.** *Note that, by* (2.2)*, $L_\nu \in \overline{\mathcal{G}r^\lambda}$ if and only if ν is a weight of the irreducible representation of $\check{G}_{\mathbb{C}}$ of highest weight λ; here $\check{G}$ is the complex Langlands dual group of G, i.e., the complex reductive group whose root datum is dual to that of G.*

Proof. It suffices to prove the statement a). Let the coweight $2\check{\rho} : \mathbb{G}_m \to T$ be the sum of positive coroots. When we act by conjugation by this coweight on $N_{\mathcal{K}}$, we see that for any element $n \in N_{\mathcal{K}}$, $\lim_{s\to 0} 2\check{\rho}(s)n = 1$. Therefore any point $x \in S_\nu$ satisfies $\lim_{s\to 0} 2\check{\rho}(s)x = L_\nu$. As the L_ν are the fixed points of the $\mathbb{G}_m$-action via $2\check{\rho}$, we see that

$$S_\nu \;=\; \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s\to 0} 2\check{\rho}(s)x = L_\nu\}\,. \tag{3.5}$$

Hence, if $x \in S_\nu \cap \mathcal{G}\mathrm{r}^\lambda$ then, because $\mathcal{G}\mathrm{r}^\lambda$ is T-invariant, we see that $L_\nu \in \overline{\mathcal{G}\mathrm{r}^\lambda}$. Thus, $S_\nu \cap \mathcal{G}\mathrm{r}^\lambda$ is non-empty precisely when $L_\nu \in \overline{\mathcal{G}\mathrm{r}^\lambda}$. Recall that, as was remarked above, we then conclude, by (2.2), that $S_\nu \cap \mathcal{G}\mathrm{r}^\lambda$ is non-empty precisely when ν is a weight of the irreducible representation of $\check{G}_{\mathbb{C}}$ of highest weight λ. Let us now assume that ν is such a wait.

We begin with two extreme cases. We claim:

$$S_\nu \cap \overline{\mathcal{G}\mathrm{r}^\nu} = \; N_{\mathcal{O}} \cdot L_\nu = \begin{cases} I \cdot L_\nu & \text{if } \nu \text{ is dominant} \\ \{L_\nu\} & \text{if } \nu \text{ is anti-dominant} \end{cases} \tag{3.6}$$

We see this as follows. We first observe that $N_{\mathcal{K}} = N_{\mathcal{O}} \cdot (N_{\mathcal{K}} \cap K_-)$. Then we can write

$$S_\nu \cap \overline{\mathcal{G}\mathrm{r}^\nu} = \; N_{\mathcal{O}} \cdot (N_{\mathcal{K}} \cap K_-) \cdot L_\nu \;\cap\; \overline{\mathcal{G}\mathrm{r}^\nu} \;=\; N_{\mathcal{O}} \cdot \left((N_{\mathcal{K}} \cap K_-) \cdot \nu \cap \overline{\mathcal{G}\mathrm{r}^\nu}\right)\,. \tag{3.7}$$

But now $(N_{\mathcal{K}} \cap K_-) \cdot L_\nu \subset K_- \cdot L_\nu$ and by (2.6) we know that $G^-_{\mathcal{O}} \cdot L_\nu \cap \overline{\mathcal{G}\mathrm{r}^\lambda} = G \cdot L_\nu$ and because $K_- \cdot L_\nu$ is the fiber of the projection $G^-_{\mathcal{O}} \cdot L_\nu \to G \cdot L_\lambda$, we get $K_- \cdot \nu \cap \overline{\mathcal{G}\mathrm{r}^\lambda} = L_\nu$. Thus we have proved the first equality in (3.6). If ν is antidominant, then $N_{\mathcal{O}}$ stabilizes L_ν. If ν is dominant then $N^-_{\mathcal{O}}$ stabilizes L_ν and then $I \cdot L_\nu = B_{\mathcal{O}} \cdot N^-_{\mathcal{O}} \cdot L_\nu = B_{\mathcal{O}} \cdot L_\nu = N_{\mathcal{O}} \cdot L_\nu$.

From (3.6) we conclude that the theorem holds in the extreme cases when $\nu = \lambda$ or $\nu = w_0 \cdot \lambda$, where w_0 is the longest element in the Weyl group. Let us now consider an arbitrary ν such that $L_\nu \in \overline{\mathcal{G}\mathrm{r}^\lambda}$, $\nu > w_0 \cdot \lambda$ and let C be an irreducible component

of $S_\nu \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$. We will now relate this component to the two extremal cases above and make use proposition 3.1.

Let us write C_0 for C, d for the dimension of C, and H_ν for the hyperplane of proposition 3.1 (b). Let us consider an irreducible component D of $\bar{C}_0 \cap H_\nu$. By proposition 3.1 the dimension of D is $d-1$ and $D \subset \cup_{\mu<\nu} S_\mu$. Hence there is an $\nu_1 < \nu = \nu_0$ such that $C_1 = D \cap S_{\nu_1}$ is open and dense in D. Of course $\dim C_1 = d-1$. Continuing in this fashion we produce a sequence of coweights ν_k, $k = 0, \dots, d$, such that $\nu_k < \nu_{k-1}$, and a corresponding chain of irreducible components C_k of $S_{\nu_k} \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$ such that $\dim C_k = d-k$. As dimension of C_d is zero, we conclude that $\nu_d \geq w_0\lambda$. Hence, we conclude that

$$\dim C = d \leq \rho(\nu - w_0 \cdot \lambda)\,. \tag{3.8}$$

We now start from the opposite end. Let us write $A_0 = S_\lambda \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$. Then, $\bar{A}_0 = \overline{\mathcal{G}\mathrm{r}^\lambda}$ and $\dim A_0 = 2\rho(\lambda)$. Let us proceed as before, and consider $\bar{A}_0 \cap H_\lambda$. As $C \subset \overline{\mathcal{G}\mathrm{r}^\lambda}$, we can find a component D of $\bar{A}_0 \cap H_\lambda$ such that $C \subset D$. Arguing just as above, there exists a $\mu < \lambda$ and a component A_1 of $S_\mu \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$ such that $\bar{A}_1 = D$. Of course, $\dim A_1 = 2\rho(\lambda) - 1$. Continuing in this manner we can produce a a sequence of coweights μ_k, $k = 0, \dots, e$, with $\mu_0 = \lambda$, $\mu_e = \nu$, such that $\mu_k < \mu_{k-1}$, and a corresponding chain of irreducible components A_k of $S_{\nu_k} \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$ such that $\dim A_k = 2\rho(\lambda) - k$ and $A_e = C$. From this we conclude that

$$\mathrm{codim}_{\overline{\mathcal{G}\mathrm{r}^\lambda}} C = e \leq \rho(\lambda - \nu)\,. \tag{3.9}$$

The fact that

$$\dim C + \mathrm{codim}_{\overline{\mathcal{G}\mathrm{r}^\lambda}} C \;=\; \dim \overline{\mathcal{G}\mathrm{r}^\lambda} \;=\; 2\rho(\lambda)\,, \tag{3.10}$$

together with the estimates (3.8) and (3.9) force

$$\dim C = \rho(\nu - w_0 \cdot \lambda) \quad \text{and} \quad \mathrm{codim}_{\overline{\mathcal{G}\mathrm{r}^\lambda}} C = \rho(\lambda - \nu)\,, \tag{3.11}$$

as was to be shown. □

The corollary below will be used to construct the convolution operation on perverse sheaves in the next section.

3.4. **Corollary.** *For any dominant $\lambda \in X_*(T)$ and any T-invariant closed subset $X \subset \overline{\mathcal{G}r^\lambda}$ we have $\dim(X) \leq \max_{L_\nu \in X^T} \rho(\lambda + \nu)$, where X^T stands for the set of T–fixed points of X.*

Proof. From the description (3.5) we see that $X \cap S_\nu$ is non-empty precisely when $L_\nu \in X$. As

$$X \;=\; \cup_{L_\nu \in X^T} X \cap S_\nu \;\subset\; \cup_{L_\nu \in X^T} \overline{\mathcal{G}\mathrm{r}^\lambda} \cap S_\nu\,, \tag{3.12}$$

we get our conclusion by appealing to the previous theorem. □

Let us write $\mathrm{Mod}_{\Bbbk}$ for the category of finitely generated $\Bbbk$-modules.

3.5. **Theorem.** *For all $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ we have a canonical isomorphism*

$$\mathrm{H}^k_c(S_\nu, \mathcal{A}) \xrightarrow{\sim} \mathrm{H}^k_{T_\nu}(\mathcal{G}r, \mathcal{A}) \tag{3.13}$$

and both sides vanish for $k \neq 2\rho(\nu)$.

In particular, the functors $F_\nu : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_\Bbbk$, defined by $F_\nu \stackrel{def}{=} \mathrm{H}^{2\rho(\nu)}_c(S_\nu, -) = \mathrm{H}^{2\rho(\nu)}_{T_\nu}(\mathcal{G}r, -)$, are exact.

Proof. Let $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$. For any dominant η the restriction $\mathcal{A}\big|\mathcal{G}\mathrm{r}^\eta$ lies, as a complex of sheaves, in degrees $\leq -\dim(\mathcal{G}\mathrm{r}^\eta) = -2\rho(\eta)$, i.e., $\mathcal{A}\big|\mathcal{G}\mathrm{r}^\eta \in D^{\leq -2\rho(\eta)}(\mathcal{G}\mathrm{r}^\eta, \Bbbk)$. From the dimension estimates 3.2 and fact that $\mathrm{H}^k_c(S_\nu \cap \mathcal{G}\mathrm{r}^\eta, \Bbbk) = 0$, for $k > 2\dim(S_\nu \cap \mathcal{G}\mathrm{r}^\eta) = 2\rho(\nu + \eta)$ we conclude:

$$\mathrm{H}^k_c(S_\nu \cap \mathcal{G}\mathrm{r}^\eta, \mathcal{A}) \;=\; 0 \quad \text{if } k > 2\rho(\nu)\,. \tag{3.14}$$

A straightforward spectral sequence argument, filtering $\mathcal{G}\mathrm{r}$ by $\overline{\mathcal{G}\mathrm{r}^\eta}$, implies that $\mathrm{H}^*_c(S_\nu, \mathcal{A})$ can be expressed in terms of $\mathrm{H}^*_c(S_\nu \cap \mathcal{G}\mathrm{r}^\nu, \mathcal{A})$ and this implies the first of the statements below:

$$\begin{aligned} \mathrm{H}^k_c(S_\nu, \mathcal{A}) \;&=\; 0 \quad \text{if } k > 2\rho(\nu) \\ \mathrm{H}^k_{T_\nu}(\mathcal{G}\mathrm{r}, \mathcal{A}) \;&=\; 0 \quad \text{if } k < 2\rho(\nu)\,. \end{aligned} \tag{3.15}$$

The proof for the second statement is completely analogous.

It remains to prove (3.13). Recall that we have a $\mathbb{G}_m$-action on $\mathcal{G}\mathrm{r}$ via the cocharacter $2\check{\rho}$ whose fixed points are the points L_ν, $\nu \in X_*(T)$, and that

$$S_\nu \;=\; \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s\to 0} 2\check{\rho}(s)x = L_\nu\} \tag{3.16}$$

$$T_\nu \;=\; \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s\to \infty} 2\check{\rho}(s)x = L_\nu\}\,. \tag{3.17}$$

The statement (3.13) now follows from theorem 1 in [Br].

□

We will denote by $F : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk) \to \mathrm{Mod}_\Bbbk$ the sum of the functors F_ν, $\nu \in X_*(T)$.

3.6. **Theorem.** *We have a natural equivalence of functors*

$$\mathrm{H}^* \;\cong\; F \;=\; \bigoplus_{\nu \in X_*(T)} \mathrm{H}^{2\rho(\nu)}_c(S_\nu, -) \;:\; \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_\Bbbk\,.$$

Furthermore, the functors F_ν and this equivalence are independent of the choice of the pair $T \subset B$.

Proof. The Bruhat decomposition of $G_{\mathcal{K}}$ for the Borel subgroups $B_{\mathcal{K}}, B^-_{\mathcal{K}}$ gives decompositions $\mathcal{G}\mathrm{r} = \cup\, S_\nu = \cup\, T_\nu$ and hence two filtrations of $\mathcal{G}\mathrm{r}$ by closures of S_ν's and T_ν's. This gives two filtrations of the cohomology functor H^*, both indexed by $X_*(T)$. One is given by kernels of the morphisms of functors $\mathrm{H}^* \to \mathrm{H}^*_c(\overline{S_\nu}, -)$ and the other by the images of $\mathrm{H}^*_{\overline{T_\nu}}(\mathcal{G}\mathrm{r}, -) \to \mathrm{H}^*$. The vanishing statement in 3.5 implies that these filtrations are complementary. More precisely, in degree $2\rho(\nu)$ we get $\mathrm{H}^{2\rho(\nu)}_{\overline{T_\nu}}(\mathcal{G}\mathrm{r}, -) = \mathrm{H}^{2\rho(\nu)}_{T_\nu}(\mathcal{G}\mathrm{r}, -)$, $\mathrm{H}^{2\rho(\nu)}_c(\overline{S_\nu}, -) = \mathrm{H}^{2\rho(\nu)}_c(S_\nu, -)$, and the composition of the functors $\mathrm{H}^{2\rho(\nu)}_{T_\nu}(\mathcal{G}\mathrm{r}, -) \to \mathrm{H}^{2\rho(\nu)} \to \mathrm{H}^{2\rho(\nu)}_c(S_\nu, -)$ is the canonical equivalence in

3.5. Hence, the two filtrations of H^* split each other and provide the desired natural equivalence.

It remains to prove the independence of the equivalence and the functors F_ν of the choice of $T \subset B$. Let us fix a reference $T_0 \subset B_0$ and a $\nu \in X_*(T_0)$ which gives us the $S^0_\nu = (N_0)_{\mathcal{K}} \cdot \nu$. The choice of pairs $T \subset B$ is parameterized by the variety G/T_0. Note that there is a canonical isomorphism between T and T_0; they are both canonically isomorphic to the "universal" Cartan $B_0/N_0 = B/N$. Consider the following diagram

$$\begin{array}{ccccc} \mathcal{G}\mathrm{r} & \xleftarrow{p} & \mathcal{G}\mathrm{r} \times G/T_0 & \xleftarrow{j} & S \\ & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow \\ & & G/T_0 & = & G/T_0\,. \end{array} \tag{3.18}$$

Here p, q, r are projections and $S = \{(x, gT_0) \in \mathcal{G}\mathrm{r} \times G/T_0 \mid x \in gS_\nu\}$. For a point in G/T_0, i.e., for a choice of $T \subset B$ the fiber of r is precisely the set S_ν of the pair. Now, for any $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ the local system $Rq_* j_! j^* p^* \mathcal{A}$ is a sublocal system of $Rq_* p^* \mathcal{A}$. As the latter local system is trivial, so is the former and hence the functors F_ν are independent of the choice of $T \subset B$. □

3.7. **Corollary.** *The global cohomology functor* $\mathrm{H}^* = F : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_\Bbbk$ *is faithful and exact.*

Proof. The exactness follows from 3.5 and 3.6. If $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ is non-zero then there exists an orbit $\mathcal{G}\mathrm{r}^\lambda$ which is open in the support of $\mathcal{A}$. If we choose λ dominant then $T_\lambda \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$ is a point in $\mathcal{G}\mathrm{r}^\lambda$ and we see that $F_\lambda(\mathcal{A}) \neq 0$. As H^* does not annihilate non-zero objects it is faithful. □

3.8. **Remark.** *The decompositions for N and its opposite unipotent subgroup N^- are explicitly related by a canonical identification* $\mathrm{H}^k_{S_\nu}(\mathcal{G}r, \mathcal{A}) \cong \mathrm{H}^k_{T_{w_0 \cdot \nu}}(\mathcal{G}r, \mathcal{A})$*, given by the action of any representative of w_0, the longest element in the Weyl group.*

From the previous discussion we obtain the following criterion for a sheaf to be perverse:

3.9. **Lemma.** *For a sheaf $\mathcal{A} \in \mathrm{D}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$, the following statements are equivalent:*

1. *The sheaf $\mathcal{A}$ is perverse.*
2. *For all $\nu \in X_*(T)$ the cohomology group $\mathrm{H}^*_c(S_\nu, \mathcal{A})$ is zero except possibly in degree $2\rho(\nu)$.*
3. *For all ν group $\mathrm{H}^*_{S_\nu}(\mathcal{G}r, \mathcal{A})$ is concentrated in degree $-2\rho(\nu)$.*

Proof. By 3.5 and 3.6 and an easy spectral sequence argument one concludes that $\mathrm{H}^{2\rho(\nu)}_c(S_\nu, {}^p\mathrm{H}^k(\mathcal{A})) = \mathrm{H}^{2\rho(\nu)+k}_c(S_\nu, \mathcal{A})$. This forces $\mathcal{A}$ to be perverse. □

Finally, we use the results of this section to give a rather explicit geometric description of the cohomology of the standard sheaves $\mathcal{I}_!(\lambda, \Bbbk)$ and $\mathcal{I}_*(\lambda, \Bbbk)$.

3.10. **Proposition.** *There are canonical identifications*

$$F_\nu[\mathcal{I}_!(\lambda, \Bbbk)] \;\cong\; \Bbbk[Irr(\overline{\mathcal{G}r_\lambda} \cap S_\nu)] \;\cong\; F_\nu[\mathcal{I}_*(\lambda, \Bbbk)];$$

here $\Bbbk[Irr(\overline{\mathcal{G}r_\lambda} \cap S_\nu)]$ *stands for the free* $\Bbbk$*-module generated by the irreducible components of* $\overline{\mathcal{G}r_\lambda} \cap S_\nu$.

Proof. We will give the argument for $\mathcal{I}_!(\lambda, \Bbbk)$. The argument for $\mathcal{I}_*(\lambda, \Bbbk)$ is completely analogous. We proceed precisely the same way as in the beginning of the proof of 3.5. Let us write $\mathcal{A} = \mathcal{I}_!(\lambda, \Bbbk)$. Consider an orbit $\mathcal{G}\mathrm{r}^\eta$ in the boundary of $\mathcal{G}\mathrm{r}^\lambda$. Then $\mathcal{A}\big|\mathcal{G}\mathrm{r}^\eta \in D^{\leq -\dim(\mathcal{G}\mathrm{r}^\eta)-2}(\mathcal{G}\mathrm{r}^\eta, \Bbbk)$. The estimate 3.14 implies that $\mathrm{H}_c^k(S_\nu \cap \mathcal{G}\mathrm{r}^\eta, \mathcal{A}) = 0$ if $k > 2\rho(\nu) - 2$. Therefore, we conclude by using the spectral sequence associated to the filtration of $\mathcal{G}\mathrm{r}$ by $\overline{\mathcal{G}\mathrm{r}^\eta}$ that $\mathrm{H}_c^{2\rho(\nu)}(S_\nu, \mathcal{A}) \cong \mathrm{H}_c^{2\rho(\nu)}(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \mathcal{A})$. Finally,

$$\mathrm{H}_c^{2\rho(\nu)}(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \mathcal{A}) = \mathrm{H}_c^{2\rho(\nu+\lambda)}(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \Bbbk) = \mathrm{H}_c^{2\dim(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda)}(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \Bbbk)\,. \tag{3.19}$$

As the last cohomology group is the top cohomology group, it is a free $\Bbbk$-module with basis $Irr(\overline{\mathcal{G}\mathrm{r}^\lambda} \cap S_\nu)$. □

4. **The Convolution product**

In this section we will put a tensor category structure on $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$ via the convolution product. The idea that the convolution of perverse sheaves corresponds to tensor product of representations is due to Lusztig and the crucial proposition 4.1, for $\Bbbk = \mathbb{C}$, is easy to extract from [Lu]. In some of our constructions in this section and the next one we are lead to sheaves with infinite dimensional support. The fact that it is legitimate to work with such objects is explained in section 2.2 of [Na].

Consider the following diagram of maps

$$\mathcal{G}\mathrm{r} \times \mathcal{G}\mathrm{r} \xleftarrow{p} G_\mathcal{K} \times \mathcal{G}\mathrm{r} \xrightarrow{q} G_\mathcal{K} \times_{G_\mathcal{O}} \mathcal{G}\mathrm{r} \xrightarrow{m} \mathcal{G}\mathrm{r}\,. \tag{4.1}$$

Here $G_\mathcal{K} \times_{G_\mathcal{O}} \mathcal{G}\mathrm{r}$ denotes the quotient of $G_\mathcal{K} \times \mathcal{G}\mathrm{r}$ by $G_\mathcal{O}$ where the action is given on the $G_\mathcal{K}$-factor via right multiplication by an inverse and on the $\mathcal{G}\mathrm{r}$-factor by left multiplication. The p and q are projection maps and m is the multiplication map. We define the convolution product

$$\mathcal{A}_1 * \mathcal{A}_2 \;=\; Rm_*\widetilde{\mathcal{A}} \qquad \text{where } \; q^*\widetilde{\mathcal{A}} = \; p^*({}^p\mathrm{H}^0(\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2))\,. \tag{4.2}$$

To justify this definition, we note that the sheaf $p^*({}^p\mathrm{H}^0(\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2))$ on $G_\mathcal{K} \times \mathcal{G}\mathrm{r}$ is $G_\mathcal{O} \times G_\mathcal{O}$-equivariant with the first $G_\mathcal{O}$ acting on the left and the second $G_\mathcal{O}$ acting on the $G_\mathcal{K}$-factor via right multiplication by an inverse and on the $\mathcal{G}\mathrm{r}$-factor by left multiplication. As the second $G_\mathcal{O}$-action is free, we see that the unique $\widetilde{\mathcal{A}}$ in (4.1) exists.

4.1. **Lemma.** *If* $\Bbbk$ *is a field, or, more generally, if one of the factors* $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_i)$ *is flat over* $\Bbbk$*, then the outer tensor product* $\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2$ *is perverse.*

When $\Bbbk$ is a field this is obvious on general grounds. When $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_i)$ is flat over $\Bbbk$ one sees this by applying Lemma 3.9 to the Grassmannian $\mathcal{G}\mathrm{r} \times \mathcal{G}\mathrm{r}$ of $G \times G$. First, as $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_i)$ is flat, so are its direct summands $\mathrm{H}_c^{2\rho(\nu_i)}(S_{\nu_i}, \mathcal{A}_i)$. Now we have

$$\mathrm{H}_c^k(S_{\nu_1} \times S_{\nu_2}, \mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2) \;=\; \bigoplus_{k_1+k_2=k} \mathrm{H}_c^{k_1}(S_{\nu_1}, \mathcal{A}_1) \overset{L}{\otimes} \mathrm{H}_c^{k_2}(S_{\nu_2}, \mathcal{A}_2)\,. \tag{4.3}$$

By the flatness assumption the tensor product on the right has no derived functors. Hence, $\mathrm{H}^k_c(S_{\nu_1}\times S_{\nu_2}, \mathcal{A}_1\overset{L}{\boxtimes}\mathcal{A}_2)=0$ if $k\neq 2\rho(\nu_1+\nu_2)$. Therefore, by Lemma 3.9, $\mathcal{A}_1\overset{L}{\boxtimes}\mathcal{A}_2$ is perverse.

4.2. **Proposition.** *The convolution product $\mathcal{A}_1 * \mathcal{A}_2$ of two perverse sheaves is perverse.*

To prove this, let us introduce the notion of a stratified semi-small map. To this end, let us consider two complex stratified spaces $(Y,\mathcal{T})$ and $(X,\mathcal{S})$ and a map $f: Y\to X$. We assume that the two stratifications are locally trivial with connected strata and that f is a stratified with respect to the stratifications $\mathcal{T}$ and $\mathcal{S}$, i.e., that for any $T\in\mathcal{T}$ the image $f(T)$ is a union of strata in $\mathcal{S}$ and for any $S\in\mathcal{S}$ the map $f\big|f^{-1}(S): f^{-1}(S)\to S$ is locally trivial in the stratified sense. We say that f is a stratified semi-small map if

(4.4)
$$\begin{aligned}
&a)\ \text{for any } T\in\mathcal{T} \text{ the map } f\big|\overline{T} \text{ is proper}\\
&b)\ \text{for any } T\in\mathcal{T} \text{ and any } S\in\mathcal{S} \text{ such that } S\subset f(\overline{T}) \text{ we have}\\
&\quad \dim(f^{-1}(x)\cap\overline{T}) \leq \frac{1}{2}(\dim f(\overline{T}) - \dim S)\\
&\quad \text{for any (and thus all) } x\in S\,.
\end{aligned}$$

Let us also introduce the notion of a small stratified map. We say that f is a small stratified map if there exists a (non-trivial) open dense stratified subset W of Y such that

(4.5)
$$\begin{aligned}
&a)\ \text{for any } T\in\mathcal{T} \text{ the map } f\big|\overline{T} \text{ is proper}\\
&b)\ \text{the map } f\big|W: W\to f(W) \text{ is finite and } W=f^{-1}(f(W))\\
&c)\ \text{for any } T\in\mathcal{T} \text{ and any } S\in\mathcal{S} \text{ such that } S\subset f(\overline{T})-f(W)\\
&\quad \text{we have}\quad \dim(f^{-1}(x)\cap\overline{T}) < \frac{1}{2}(\dim f(\overline{T}) - \dim S)\\
&\quad \text{for any (and thus all) } x\in S\,.
\end{aligned}$$

The result below follows directly from dimension counting:

4.3. **Lemma.** *If f is a semi-small stratified map then $Rf_*\mathcal{A}\in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ for all $\mathcal{A}\in \mathrm{P}_{\mathcal{T}}(Y,\Bbbk)$. If f is a small stratified map then, with any W as above, and any $\mathcal{A}\in \mathrm{P}_{\mathcal{T}}(W,\Bbbk)$, we have $Rf_*j_{!*}\mathcal{A}=\widetilde{j}_{!*}f_*\mathcal{A}$, where $j: W\hookrightarrow Y$ and $\widetilde{j}: f(W)\hookrightarrow X$ denote the two inclusions.*

We apply the above considerations, in the semi-small case, to our situation. We take $Y=G_{\mathcal{K}}\times_{G_{\mathcal{O}}}\mathcal{G}\mathrm{r}$ and choose $\mathcal{T}$ to be the stratification whose strata are $\tilde{\mathcal{G}}\mathrm{r}^{\lambda,\mu}=p^{-1}(\mathcal{G}\mathrm{r}^{\lambda})\times_{G_{\mathcal{O}}}\mathcal{G}\mathrm{r}^{\mu}$, for $\lambda,\mu\in X_*(T)$. We also let $X=\mathcal{G}\mathrm{r}$, $\mathcal{S}$ the stratification by $G_{\mathcal{O}}$-orbits, and choose $f=m$. Note that the sheaf $\widetilde{\mathcal{A}}$ is constructible with respect to the stratification $\mathcal{T}$. To be able to apply 4.3 and conclude the proof of 4.2, we appeal to the following

4.4. **Lemma.** *The multiplication map $G_{\mathcal{K}}\times_{G_{\mathcal{O}}}\mathcal{G}r\xrightarrow{m}\mathcal{G}r$ is a stratified semi-small map with respect to the stratifications above.*

Proof. We need to check that for any $G_{\mathcal{O}}$-orbit $\mathcal{G}\mathrm{r}^{\nu}$ in $\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}$, the dimension of the fiber $m^{-1}L_\nu \cap \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}$ of $m : \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu} \to \overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}$ at L_ν, is not more than $\frac{1}{2}\mathrm{codim}_{\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}}\mathcal{G}\mathrm{r}_\nu$. We can assume that ν is anti-dominant since $\mathcal{G}\mathrm{r}^{w\cdot\eta} = \mathcal{G}\mathrm{r}^{\eta}$, $w \in W$. Since for any dominant η, $\dim \mathcal{G}\mathrm{r}^{\eta} = 2\rho(\eta)$, the codimension in question is:

$$\mathrm{codim}_{\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}}\mathcal{G}\mathrm{r}^{\nu} = 2\rho(\lambda+\mu) - 2\rho(w_0\cdot\nu) = 2\rho(\lambda+\mu+\nu).$$

Therefore, we need to show that

$$\dim(m^{-1}L_\nu \cap \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}) \leq \rho(\lambda+\mu+\nu)\,. \tag{4.6}$$

Let p be the projection $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}\mathrm{r} \to \mathcal{G}\mathrm{r}$ given by $(g, hG_{\mathcal{O}}) \mapsto gG_{\mathcal{O}}$, and consider the isomorphism $(p,m) : G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}\mathrm{r} \cong \mathcal{G}\mathrm{r} \times \mathcal{G}\mathrm{r}$. The mapping (p,m) carries the fiber $m^{-1}L_\nu$ to $\mathcal{G}\mathrm{r} \times L_\nu$. The set $p(m^{-1}L_\nu \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}})$ is T-invariant, and hence we can apply corollary 3.4 to compute its dimension. To do so, we need to find the T-fixed points in $p(m^{-1}L_\nu \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}}) \subset \overline{\mathcal{G}\mathrm{r}^{\lambda}}$. The T-fixed points in $m^{-1}L_\nu \cap \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}$ are precisely the points $(z^{\phi}, z^{\psi}G_{\mathcal{O}})$ such that ϕ and ψ are weights of $L(\lambda)$ and $L(\mu)$ and $\phi+\psi = \nu$. Hence, the set T-fixed points in $m^{-1}L_\nu \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}}$ consists of the points of the form $(z^{\phi}, z^{\psi}G_{\mathcal{O}})$ with $\phi+\psi = \nu$ and ϕ and ψ weights of irreducible representations $L(\lambda')$ and $L(\mu')$ for some dominant λ', μ' such that $\lambda' \leq \lambda$, $\mu' \leq \mu$. For ϕ, ψ, μ' as above, we have

$$\rho(\lambda+\phi) \leq \rho(\lambda+\phi) + \rho(\psi+\mu') = \rho(\lambda+\nu+\mu') \leq \rho(\lambda+\nu+\mu)\,.$$

Therefore,

$$\dim(p(m^{-1}L_\nu \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}})) \leq \max_{L_\phi \in p(m^{-1}L_\nu \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}})^T} (\rho(\lambda+\phi)) \leq \rho(\lambda+\nu+\mu)\,. \tag{4.7}$$

This implies (4.6) and concludes the proof.

□

In completely analogy with (4.2), we can define directly the convolution product of three sheaves, i.e., to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ we can associate a perverse sheaf $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3$. Furthermore, we get canonical isomorphisms $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3 \cong (\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3$ and $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3 \cong \mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$. This yields a functorial isomorphism $(\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3 \cong \mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$.

Thus we obtain:

4.5. **Proposition.** *The convolution product* (4.2) *on the abelian category* $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ *is associative.*

5. The commutativity constraint and the fusion product

In this section we show that the convolution product defined in the last section can be viewed as a "fusion" product. This interpretation allows one to provide the convolution product on $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ with a commutativity constraint, making $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ into an associative, commutative tensor category. The exposition follows very closely that in [MiV2]. The idea of interpreting the convolution product as a fusion product and obtaining the commutativity constraint in this fashion is due to Drinfeld and was communicated to us by Beilinson.

Let X be a smooth complex algebraic curve. For a closed point $x \in X$ we write $\mathcal{O}_x$ for the completion of the local ring at x and $\mathcal{K}_x$ for its fraction field. Furthermore, for a $\mathbb{C}$-algebra R we write $X_R = X \times \mathrm{Spec}(R)$, and $X_R^* = (X - \{x\}) \times \mathrm{Spec}(R)$. Using the results of [BL1, BL2, LS] we can now view the Grassmannian $\mathcal{G}\mathrm{r}_x = G_{\mathcal{K}_x}/G_{\mathcal{O}_x}$ in the following manner. It is the ind-scheme which represents the functor from $\mathbb{C}$-algebras to sets :

$$R \mapsto \{\mathcal{F} \text{ a } G\text{-torsor on } X_R,\ \nu : G \times X_R^* \to \mathcal{F}\big|_{X_R^*} \text{ a trivialization on } X_R^* \}\,.$$

Here the pairs $(\mathcal{F}, \nu)$ are to be taken up to isomorphism.

Following [BD] we globalize this construction and at the same time work over several copies of the curve. Denote the n fold product by $X^n = X \times \cdots \times X$ and consider the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, \ldots, x_n) \in X^n(R), \quad \mathcal{F} \text{ a } G\text{-torsor on } X_R\,, \\ \nu_{(x_1,\ldots,x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } X_R - \cup x_i \end{array} \right\}. \tag{5.1}$$

Here we think of the points $x_i : \mathrm{Spec}(R) \to X$ as subschemes of X_R by taking their graphs. This functor is represented by an ind-scheme $\mathcal{G}\mathrm{r}_{X^n}$. Of course $\mathcal{G}\mathrm{r}_{X^n}$ is an ind-scheme over X^n and its fiber over the point $(x_1, \ldots, x_n)$ is simply $\prod_{i=1}^k \mathcal{G}\mathrm{r}_{y_i}$, where $\{y_1, \ldots, y_k\} = \{x_1, \ldots, x_n\}$, with all the y_i distinct. We write $\mathcal{G}\mathrm{r}_{X^1} = \mathcal{G}\mathrm{r}_X$.

We will now extend the diagram of maps (4.1), which was used to define the convolution product, to the global situation, i.e., to a diagram of ind-schemes over X^2:

$$\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X \xleftarrow{p} \widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X} \xrightarrow{q} \mathcal{G}\mathrm{r}_X \widetilde{\times} \mathcal{G}\mathrm{r}_X \xrightarrow{m} \mathcal{G}\mathrm{r}_{X^2} \xrightarrow{\pi} X^2\,. \tag{5.2}$$

Here, $\widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X}$ denotes the ind-scheme representing the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R);\ \mathcal{F}_1, \mathcal{F}_2 \text{ } G\text{-torsors on } X_R;\ \nu_i \text{ a trivialization of} \\ \mathcal{F}_i \text{ on } X_R - x_i, \text{ for } i = 1, 2;\ \ \mu_1 \text{ a trivialization of } \mathcal{F}_1 \text{ on } \widehat{(X_R)}_{x_2} \end{array} \right\}, \tag{5.3}$$

where $\widehat{(X_R)}_{x_2}$ denotes the formal neighborhood of x_2 in X_R. The "twisted product" $\mathcal{G}\mathrm{r}_X \widetilde{\times} \mathcal{G}\mathrm{r}_X$ is the ind-scheme representing the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R);\ \mathcal{F}_1, \mathcal{F} \text{ } G\text{-torsors on } X_R;\ \nu_1 \text{ a trivialization} \\ \text{of } \mathcal{F}_1 \text{ on } X_R - x_1;\ \eta : \mathcal{F}_1\big|_{(X_R - x_2)} \xrightarrow{\simeq} \mathcal{F}\big|_{(X_R - x_2)} \end{array} \right\}. \tag{5.4}$$

It remains to describe the morphisms p, q, and m in (5.2). Because all the spaces in (5.2) are ind-schemes over X^2, and all the functors involve the choice of the same point $(x_1, x_2) \in X^2(R)$, we omit it in the formulas below. The morphism p simply forgets the choice of μ_1, the morphism q is given by the natural transformation

$$(\mathcal{F}_1, \nu_1, \mu_1; \mathcal{F}_2, \nu_2) \mapsto (\mathcal{F}_1, \nu_1, \mathcal{F}, \eta),$$

where $\mathcal{F}$ is the G-torsor gotten by gluing $\mathcal{F}_1$ on $X_R - x_2$ and $\mathcal{F}_2$ on $\widehat{(X_R)}_{x_2}$ using the isomorphism induced by $\nu_2 \circ \mu_1^{-1}$ between $\mathcal{F}_1$ and $\mathcal{F}_2$ on $(X_R - x_2) \cap \widehat{(X_R)}_{x_2}$. The morphism m is given by the natural transformation

$$(\mathcal{F}_1, \nu_1, \mathcal{F}, \eta) \mapsto (\mathcal{F}, \nu)\,,$$

where $\nu = (\eta \circ \nu_1)\big|(X_R - x_1 - x_2)$.

The global analogue of $G_{\mathcal{O}}$ is the group-scheme $G_{X^n,\mathcal{O}}$ which represents the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, \dots, x_n) \in X^n(R), \ \ \mathcal{F} \text{ the trivial } G\text{-torsor on } X_R\,, \\ \mu_{(x_1,\dots,x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } \widehat{(X_R)}_{(x_1 \cup \dots \cup x_n)} \end{array} \right\}. \tag{5.5}$$

Proceeding as in section 4 we define the convolution product of $\mathcal{B}_1, \mathcal{B}_2 \in \mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}\mathrm{r}_X, \Bbbk)$ by the formula

$$\mathcal{B}_1 *_X \mathcal{B}_2 \ = \ Rm_* \ \widetilde{\mathcal{B}} \qquad \text{where} \ \ q^*\widetilde{\mathcal{B}} = p^*({}^p\mathrm{H}^0(\mathcal{B}_1 \overset{L}{\boxtimes} \mathcal{B}_2))\,. \tag{5.6}$$

To make sense of this definition, we have to explain how the group scheme $G_{X,\mathcal{O}}$ acts on various spaces. First, to see that it acts on $\mathcal{G}\mathrm{r}_X$, we observe that we can rewrite the functor in (5.1), when $n = 1$, as follows:

$$R \mapsto \left\{ \begin{array}{l} x \in X(R), \ \ \mathcal{F} \text{ a } G\text{-torsor on } \widehat{(X_R)}_x\,, \\ \nu_x \text{ a trivialization of } \mathcal{F} \text{ on } \widehat{(X_R)}_x - x \end{array} \right\}. \tag{5.7}$$

Thus we see that $G_{X,\mathcal{O}}$ acts on $\mathcal{G}\mathrm{r}_X$ by altering the trivialization in (5.7) and hence we can define the category $\mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}\mathrm{r}_X, \Bbbk)$. As to $\widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X}$, two actions of $G_{X,\mathcal{O}}$ are relevant to us. First, let us view $G_{X,\mathcal{O}}$ as group scheme on X^2 by pulling it back for the second factor. Then $G_{X,\mathcal{O}}$ acts by altering the trivialization μ_1 in (5.3). This action is free and exhibits $p : \widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X} \to \mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X$ as a $G_{X,\mathcal{O}}$ torsor. To describe the second action we rewrite the definition of $\widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X}$ in the same fashion as we did for $\mathcal{G}\mathrm{r}_X$, i.e., $\widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X}$ can also be viewed as representing the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R); \text{ for } i = 1, 2 \ \mathcal{F}_i \ \text{ is a } G\text{-torsor on } \widehat{(X_R)}_{x_i}, \\ \nu_i \text{ a trivialization of } \mathcal{F}_i \text{ on } \widehat{(X_R)}_{x_i} - x_i, \\ \ \text{and} \ \ \mu_1 \text{ is a trivialization of } \mathcal{F}_1 \text{ on } \widehat{(X_R)}_{x_2} \end{array} \right\}, \tag{5.8}$$

We again view $G_{X,\mathcal{O}}$ as a group scheme on X^2 by pulling it back from the second factor. Then we can define the second action of $G_{X,\mathcal{O}}$ on $\widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X}$ by letting $G_{X,\mathcal{O}}$ act by altering both of the trivializations μ_1 and ν_2. This action is also free and exhibits $q : \widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X} \to \mathcal{G}\mathrm{r}_X \widetilde{\times} \mathcal{G}\mathrm{r}_X$ as a $G_{X,\mathcal{O}}$ torsor. Thus, we conclude that the sheaf $\widetilde{\mathcal{B}}$ in (5.6) exists and is unique.

Let us note that the map m is a stratified small map – regardless of the stratification on X. To see this, let us denote by $\Delta \subset X^2$ the diagonal and set $U = X^2 - \Delta$. Then we can take, in definition 4.5, as W the locus of points lying over U. That m is small now follows as m is an isomorphism over U and over points of Δ the map m coincides with its analogue in section 4 which is semi-small by proposition 4.4.

We will now construct the commutativity constraint. For simplicity we specialize to the case $X = \mathbb{A}^1$. The advantage is that we can once and for all choose a global coordinate. Then the choice of a global coordinate on $\mathbb{A}^1$, trivializes $\mathcal{G}\mathrm{r}_X$ over X; let us

write $\tau : \mathcal{G}\mathrm{r}_X \to \mathcal{G}\mathrm{r}$ for the projection. By restricting $\mathcal{G}\mathrm{r}_{X^{(2)}}$ to the diagonal $\Delta \cong X$ and to U, and observing that these restrictions are isomorphic to $\mathcal{G}\mathrm{r}_X$ and to $(\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X)\big|U$, respectively, we get the following diagram

$$\begin{array}{ccccc} \mathcal{G}\mathrm{r}_X & \xrightarrow{\ i\ } & \mathcal{G}\mathrm{r}_{X^2} & \xleftarrow{\ j\ } & (\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X)\big|U \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X^2 & \longleftarrow & U \end{array} \quad . \tag{5.9}$$

Let us denote $\tau^o = \tau^*[1] : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk) \to \mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}\mathrm{r}_X, \Bbbk)$ and $i^o = i^*[-1] : \mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}\mathrm{r}_X, \Bbbk) \to \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$. For $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ we have:

$$\begin{aligned} &\text{a)} \qquad \tau^o\mathcal{A}_1 *_X \tau^o\mathcal{A}_2 \ \cong\ j_{!*}\left({}^p\mathrm{H}^0(\tau^o\mathcal{A}_1 \overset{L}{\boxtimes} \tau^o\mathcal{A}_2)\big|U\right) \\ &\text{b)} \qquad \tau^o(\mathcal{A}_1 * \mathcal{A}_2) \ \cong\ i^o(\tau^o\mathcal{A}_1 *_X \tau^o\mathcal{A}_2)\,. \end{aligned} \tag{5.10}$$

Part a) follows from smallness of m and lemma 4.3, and part b) follows directly from definitions.

Utilizing the the statements above yields the following sequence of isomorphisms:

$$\begin{aligned} \tau^o(\mathcal{A}_1 * \mathcal{A}_2) &\cong i^o j_{!*}\left({}^p\mathrm{H}^0(\tau^o\mathcal{A}_1 \overset{L}{\boxtimes} \tau^o\mathcal{A}_2)\big|U\right) \\ &\cong i^o j_{!*}({}^p\mathrm{H}^0(\tau^o\mathcal{A}_2 \overset{L}{\boxtimes} \tau^o\mathcal{A}_1)\big|U) \cong \tau^o(\mathcal{A}_2 * \mathcal{A}_1)\,. \end{aligned} \tag{5.11}$$

Specializing this isomorphism to (any) point on the diagonal yields a functorial isomorphism between $\mathcal{A}_1 * \mathcal{A}_2$ and $\mathcal{A}_2 * \mathcal{A}_1$. This gives us a commutativity constraint making $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ into a tensor category. In the next section we modify this commutativity constraint slightly. The modified commutativity constraint will be used in the rest of the paper.

5.1. **Remark.** *One can avoid having to specialize to the case $X = \mathbb{A}^1$ here, as well as in the next section. This can be done, for example, following* [BD] *and dealing with all choices of a local coordinate at all points of the curve X. This gives rise to the* $\mathrm{Aut}(\mathcal{O})$*-torsor $\hat{X} \to X$. The functor $\tau^o : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}r_X, \Bbbk)$ is constructed by noting that $\mathcal{G}r_X \to X$ is the fibration associated to the* $\mathrm{Aut}(\mathcal{O})$*-torsor $\hat{X} \to X$ and the* $\mathrm{Aut}(\mathcal{O})$*-action on $\mathcal{G}r$. By proposition 2.2, sheaves in $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ are* $\mathrm{Aut}(\mathcal{O})$*-equivariant and hence we can transfer them to sheaves on $\mathcal{G}r_X$.*

6. **Tensor functors**

In this section we show that our functor

$$\mathrm{H}^* \ \cong\ F \ =\ \bigoplus_{\nu \in X_*(T)} \mathrm{H}_c^{2\rho(\nu)}(S_\nu, -) \ :\ \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk) \to \mathrm{Mod}_{\Bbbk} \tag{6.1}$$

is a tensor functor. In the case when $\Bbbk$ is not a field, the argument is slightly more complicated and we have to make use of some results from section 10. However, the results of this present section are used in section 7 only in the case when $\Bbbk$ is a field and not in full generality till chapter 11.

Let us write $\mathrm{Mod}^{\epsilon}_{\Bbbk}$ for the tensor category of finitely generated $\mathbb{Z}/2\mathbb{Z}$ - graded (super) modules over $\Bbbk$. Let us consider the global cohomology functor as a functor $\mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}^{\epsilon}_{\Bbbk}$; here we only keep track of the parity of the grading on global cohomology. Then:

6.1. **Lemma.** *The functor* $\mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}^{\epsilon}_{\Bbbk}$ *is a tensor functor with respect to the commutativity constraint of the previous section.*

Proof. We use the interpretation of the convolution product as a fusion product, explained in the previous section. Let us recall that we write $\pi : \mathcal{G}\mathrm{r}_{X^2} \to X^2$ for the projection and again set $X = \mathbb{A}^1$. The lemma is an immediate consequence of the following statements:

$$R\pi_*(\tau^0(\mathcal{A}_1) *_X \tau^0(\mathcal{A}_2))|U \text{ is the constant sheaf } \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1) \otimes \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2)\,. \tag{6.2a}$$

$$R\pi_*(\tau^0(\mathcal{A}_1) *_X \tau^0(\mathcal{A}_2))|\Delta = \tau^0(\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1 * \mathcal{A}_2))\,. \tag{6.2b}$$

$$\text{the sheaves } \quad R^k\pi_*(\tau^0(\mathcal{A}_1) *_X \tau^0(\mathcal{A}_2)) \text{ are constant } . \tag{6.2c}$$

From (5.10) we immediately conclude (6.2b) in general and (6.2a) when $\Bbbk$ is field. To prove (6.2a) in general, we must show:

$$\mathrm{H}^*(\mathcal{G}\mathrm{r} \times \mathcal{G}\mathrm{r}, {}^p\mathrm{H}^0(\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2)) \;=\; \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1) \otimes \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2)\,. \tag{6.3}$$

We will argue this point last and deal with (6.2c) next. Let us write $\tilde{\pi} : \mathcal{G}\mathrm{r}_X \widetilde{\times} \mathcal{G}\mathrm{r}_X \to X^2$ for the natural projection. Then $\tilde{\pi} = \pi \circ m$. Thus, in order to prove (6.2c) it suffices to show:

$$R^k\tilde{\pi}_*\tilde{\mathcal{B}} \text{ is constant;} \tag{6.4}$$

recall that here $q^*\tilde{\mathcal{B}} = p^*(\tau^0(\mathcal{A}_1) \overset{L}{\boxtimes} \tau^0(\mathcal{A}_2))$. To do so, we will show that the stratification underlying the sheaf $\tilde{\mathcal{B}}$ is smooth over X^2. Recall that by a choice of a global coordinate on $X = \mathbb{A}^1$ we get an isomorphism $\mathcal{G}\mathrm{r}_X \cong \mathcal{G}\mathrm{r} \times X$. Thus, the sheaves $\tau^0(\mathcal{A}_1)$ and $\tau^0(\mathcal{A}_2)$ are constructible with respect to the stratification $\mathcal{G}\mathrm{r}^{\lambda}_X$ which correspond to $\mathcal{G}\mathrm{r}^{\lambda} \times X$ under the above isomorphism; here, as usual, $\lambda \in X_*(T)$. These strata are smooth over the base X by construction. Thus, we conclude that the sheaf $\tilde{\mathcal{B}}$ is constructible with respect to the strata $\mathcal{G}\mathrm{r}^{\lambda}_X \widetilde{\times} \mathcal{G}\mathrm{r}^{\mu}_X$, for $\lambda, \mu \in X_*(T)$, which are uniquely described by the following property:

$$q^{-1}(\mathcal{G}\mathrm{r}^{\lambda}_X \widetilde{\times} \mathcal{G}\mathrm{r}^{\mu}_X) \;=\; p^{-1}(\mathcal{G}\mathrm{r}^{\lambda}_X \times \mathcal{G}\mathrm{r}^{\mu}_X)\,. \tag{6.5}$$

In other words, the strata $\mathcal{G}\mathrm{r}^{\lambda}_X \widetilde{\times} \mathcal{G}\mathrm{r}^{\mu}_X$ are quotients of $p^{-1}(\mathcal{G}\mathrm{r}^{\lambda}_X \times \mathcal{G}\mathrm{r}^{\mu}_X)$ by the second $G_{X,\mathcal{O}}$ action on $\widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X}$ defined in section 5 which makes $q : \widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X} \to \mathcal{G}\mathrm{r}_X \widetilde{\times} \mathcal{G}\mathrm{r}_X$ a $G_{X,\mathcal{O}}$ torsor. As such, the $\mathcal{G}\mathrm{r}^{\lambda}_X \widetilde{\times} \mathcal{G}\mathrm{r}^{\mu}_X$ are smooth. Furthermore, the projection morphism $\tilde{\pi}_{\lambda,\mu} : \mathcal{G}\mathrm{r}^{\lambda}_X \widetilde{\times} \mathcal{G}\mathrm{r}^{\mu}_X \to X^2$ is smooth. This can be verified either by a direct inspection or concluded by general principles from the fact that all the fibers of $\tilde{\pi}_{\lambda,\mu}$ are smooth and equidimensional. This, then, lets us conclude (6.2c).

It remains to argue (6.3). Let us first assume that one of the factors $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_i)$ is flat over $\Bbbk$. Then, by Lemma (4.1), the sheaf $\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2$ is perverse. Then, again using

the flatness of $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_i)$, we get

$$\mathrm{H}^*(\mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}, {}^p\mathrm{H}^0(\mathcal{A}_1\overset{L}{\boxtimes}\mathcal{A}_2)) \;=\; \mathrm{H}^*(\mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}, \mathcal{A}_1\overset{L}{\boxtimes}\mathcal{A}_2) \;= \tag{6.6}$$
$$\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1)\overset{L}{\otimes}\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2) \;=\; \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1)\otimes\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2)\,.$$

To argue the general case we make use of Corollary 9.2 and Proposition 10.1. Corollary 9.2 allows us to write any $\mathcal{A}\in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ as a quotient of a projective $\mathcal{P}\in \mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ and Proposition 10.1 tells us that $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{P})$ is free over $\Bbbk$; here Z is any $G_{\mathcal{O}}$-invariant finite dimensional subvariety of $\mathcal{G}\mathrm{r}$ which contains the support of $\mathcal{A}$. Let us consider a resolution of $\mathcal{A}_1$ by such projectives:

$$\mathcal{Q}\to\mathcal{P}\to\mathcal{A}_1\to 0\,. \tag{6.7}$$

As the functor $\mathcal{A}\mapsto {}^p\mathrm{H}^0(\mathcal{A}\overset{L}{\boxtimes}\mathcal{A}_2)$ is right exact, we get an exact sequence

$${}^p\mathrm{H}^0(\mathcal{Q}\overset{L}{\boxtimes}\mathcal{A}_2)\to {}^p\mathrm{H}^0(\mathcal{P}\overset{L}{\boxtimes}\mathcal{A}_2)\to {}^p\mathrm{H}^0(\mathcal{A}_1\overset{L}{\boxtimes}\mathcal{A}_2)\to 0\,. \tag{6.8}$$

Because cohomology is a an exact functor and making use of the fact that we have already proved (6.3) for the first two terms, we get an exact sequence

$$\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{Q})\otimes\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2)\to \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{P})\otimes\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2)\to \tag{6.9}$$
$$\to \mathrm{H}^*(\mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}, {}^p\mathrm{H}^0(\mathcal{A}_1\overset{L}{\boxtimes}\mathcal{A}_2))\to 0\,.$$

Comparing this exact sequence to the one we get by tensoring the exact sequence

$$\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{Q})\to \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{P})\to \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1)\to 0 \tag{6.10}$$

with $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2)$ concludes the proof. □

6.2. **Remark.** *The statements in* (6.2) *hold for an arbitrary curve X. This can be seen by utilizing the* $\mathrm{Aut}(\mathcal{O})$*-torsor $\hat{X}\to X$ of remark 5.1 and proposition 2.2; for details see* [Na].

Let $\mathrm{Mod}_{\Bbbk}$ denote the category of finite dimensional vector spaces over $\Bbbk$. To make $\mathrm{H}^*: \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)\to \mathrm{Mod}_{\Bbbk}$ into a tensor functor we alter, following Beilinson and Drinfeld, the commutativity constraint of the previous section slightly. We consider the constraint from section 5 on the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)\otimes\mathrm{Mod}_{\Bbbk}^{\epsilon}$ and restrict it to a subcategory that we identify with $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$. Divide $\mathcal{G}\mathrm{r}$ into unions of connected components $\mathcal{G}\mathrm{r} = \mathcal{G}\mathrm{r}_+\cup\mathcal{G}\mathrm{r}_-$ so that the dimension of $G_{\mathcal{O}}$-orbits is even in $\mathcal{G}\mathrm{r}_+$ and odd in $\mathcal{G}\mathrm{r}_-$. This gives a $\mathbb{Z}_2$-grading on the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ hence a new $\mathbb{Z}_2$-grading on $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)\otimes\mathrm{Mod}_{\Bbbk}^{\epsilon}$. The subcategory of even objects is identified with $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ by forgetting the grading. Hence, we conclude from the previous lemma:

6.3. **Proposition.** *The functor* $\mathrm{H}^*: \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)\to \mathrm{Mod}_{\Bbbk}$ *is a tensor functor with respect to the above commutativity constraint.*

Let us write $\mathrm{Mod}_{\Bbbk}(X_*(T)))$ for the (tensor) category of finitely generated $\Bbbk$-modules with a $X_*(T)$-grading. We can view $F=\oplus_{\nu\in X_*(T)}F_\nu$ as a functor from $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ to $\mathrm{Mod}_{\Bbbk}(X_*(T)))$. Then we have the following generalization of the previous proposition:

6.4. **Proposition.** *The functor* $F : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_{\Bbbk}(X_*(T)))$ *is a tensor functor.*

Proof. The notion of the subspaces S_ν and T_ν can be extended to the situation of families, i.e., to the global Grassmannians $\mathcal{G}\mathrm{r}_{X^n}$. Recall that the fiber of the projection $r_n : \mathcal{G}\mathrm{r}_{X^n} \to X^n$ over the point $(x_1, \ldots, x_n)$ is simply $\prod_{i=1}^k \mathcal{G}\mathrm{r}_{y_i}$, where $\{y_1, \ldots, y_k\} = \{x_1, \ldots, x_n\}$, with all the y_i distinct. Attached to the coweight $\nu \in X_*(T)$ we associate the ind-subscheme

$$\prod_{\nu_1 + \cdots + \nu_k = \nu} S_{\nu_i} \ \subset \ \prod_{i=1}^k \mathcal{G}\mathrm{r}_{y_i} = r_n^{-1}(x_1, \ldots, x_n) \tag{6.11}$$

These ind-schemes altogether form an ind-subscheme $S_\nu(X^n)$ of $\mathcal{G}\mathrm{r}_{X^n}$. This is easy to see for $n = 1$ by choosing a global parameter, for example. By the same argument we see that outside of the diagonals $S_\nu(X^n)$ form a subscheme. It is now not difficult to check that the closure of this locus lies inside $S_\nu(X^n)$. Similarly, we define the ind-subschemes $T_\nu(X^n)$. Let us write s_ν and t_ν for the inclusion maps of $S_\nu(X^n)$ and $T_\nu(X^n)$ to $\mathcal{G}\mathrm{r}_{X^n}$, respectively. We have the action of $\mathbb{G}_m$ on $\mathcal{G}\mathrm{r}_{X^n}$ via the cocharacter $2\check{\rho}$. The fixed point set of this action consists of the locus of products of the fixed points in the individual affine Grassmannians, i.e., above the point $(x_1, \ldots, x_n)$ where $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_k\}$, with all the y_i distinct, the fixed points are of of the form

$$(L_{\nu_1}, \ldots, L_{\nu_k}) \in \prod_{i=1}^k \mathcal{G}\mathrm{r}_{y_i}; \tag{6.12}$$

recall that we write L_ν for the point in $\mathcal{G}\mathrm{r}$ corresponding to the cocharacter $\nu \in X_*(T)$. We write C_ν for the subset of the fixed point locus lying inside $S_\nu(X^n)$, i.e.,

$$C_\nu \cap r_n^{-1}(x_1, \ldots, x_n) \ = \ \bigcup_{\nu_1 + \cdots + \nu + k = \nu} \{(L_{\nu_1}, \ldots, L_{\nu_k})\}\,. \tag{6.13}$$

Let us write $i_\nu : S_\nu(X^n) \to \mathcal{G}\mathrm{r}_{X^n}$ and $k_\nu : T_\nu(X^n) \to \mathcal{G}\mathrm{r}_{X^n}$ for the inclusions. By the same argument as in the proof of theorem 3.2 we see that

$$S_\nu(X^n) \ = \ \{z \in \mathcal{G}\mathrm{r}_{X^n} \mid \lim_{s \to 0} 2\check{\rho}(s) z \in C_\nu\} \tag{6.14}$$

and

$$T_\nu(X^n) \ = \ \{z \in \mathcal{G}\mathrm{r}_{X^n} \mid \lim_{s \to \infty} 2\check{\rho}(s) z \in C_\nu\}\,. \tag{6.15}$$

Let us write $p_\nu : S_\nu(X^n) \to C_\nu$ and $q_\nu : T_\nu(X^n) \to C_\nu$ for the retractions:

$$p_\nu(z) \ = \ \lim_{s \to 0} 2\check{\rho}(s) z \qquad \text{for} \ \ z \in S_\nu(X^n) \tag{6.16}$$

$$q_\nu(z) \ = \ \lim_{s \to \infty} 2\check{\rho}(s) z \qquad \text{for} \ \ z \in T_\nu(X^n)\,. \tag{6.17}$$

Furthermore,

$$C_\nu \ = \ S_\nu(X^n) \cap T_\nu(X^n)\,. \tag{6.18}$$

By Theorem 1 of [Br] we conclude that

$$i_\nu^! s_\nu^* \mathcal{B} \ = \ k_\nu^* t_\nu^! \mathcal{B} \qquad \text{for} \ \ \mathcal{B} \in \mathrm{P}_{G_{X^n, \mathcal{O}}}(\mathcal{G}\mathrm{r}_{X^n}, \Bbbk)\,. \tag{6.19}$$

Rephrasing this result in terms of the contractions p_ν and q_ν gives:

$$R(p_\nu)_! s_\nu^* \mathcal{B} \;=\; R(q_\nu)_* t_\nu^! \mathcal{B} \qquad \text{for } \; \mathcal{B} \in \mathrm{P}_{G_{X^n,\mathcal{O}}}(\mathcal{G}\mathrm{r}_{X^n}, \Bbbk)\,. \tag{6.20}$$

Let us now, for simplicity, choose $X = \mathbb{A}^1$. Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$. We write $\mathcal{B}_1 = \tau^o \mathcal{A}_1$ and $\mathcal{B}_2 = \tau^o \mathcal{A}_2$ and form the convolution product $\mathcal{B}_1 * \mathcal{B}_2 = Rm_* \widetilde{\mathcal{B}}$. By statement (6.2c) we see:

$$\begin{gathered} \text{The sheaf } \; R^k \pi_* R m_* \widetilde{\mathcal{B}} \quad \text{on } X^2 \text{ is constant with fiber} \\ \mathrm{H}^k(\mathcal{G}\mathrm{r}, \mathcal{A}_1 * \mathcal{A}_2) \;=\; \bigoplus_{k_1+k_2=k} \mathrm{H}^{k_1}(\mathcal{G}\mathrm{r}, \mathcal{A}_1) \otimes \mathrm{H}^{k_2}(\mathcal{G}\mathrm{r}, \mathcal{A}_2)\,. \end{gathered} \tag{6.21}$$

Let us now consider the sheaves

$$\mathcal{L}_\nu^k(\mathcal{A}_1, \mathcal{A}_2) = R^k \pi_* R(p_\nu)_! s_\nu^* R m_* \widetilde{\mathcal{B}} = R^k \pi_* R(q_\nu)_* t_\nu^! R m_* \widetilde{\mathcal{B}}\,; \tag{6.22}$$

here we have used (6.20) to identify the last two sheaves. Let us calculate the stalks of this sheaf. First, by definition,

$$\mathcal{L}_\nu^k(\mathcal{A}_1, \mathcal{A}_2)_{(x_1,x_2)} = \begin{cases} \mathrm{H}_c^k(S_\nu, \mathcal{A}_1 * \mathcal{A}_2) & \text{if } x_1 = x_2 \\ \oplus_{\nu_1+\nu_2=\nu} \mathrm{H}_c^k(S_{\nu_1} \times S_{\nu_2}, {}^p\mathrm{H}^0(\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2)) & \text{if } x_1 \neq x_2\,. \end{cases} \tag{6.23}$$

Arguing in the same way as in the proof of (6.3) we see that

$$\mathrm{H}_c^*(S_{\nu_1} \times S_{\nu_2}, {}^p\mathrm{H}^0(\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2)) \;=\; \mathrm{H}_c^*(S_{\nu_1}, \mathcal{A}_1) \otimes \mathrm{H}_c^*(S_{\nu_2}, \mathcal{A}_2)\,. \tag{6.24}$$

We conclude that

$$\mathcal{L}_\nu^k(\mathcal{A}_1, \mathcal{A}_2)_{(x_1,x_2)} \;=\; 0 \qquad \text{if } k \neq 2\rho(\nu)\,, \tag{6.25a}$$

and

$$\mathcal{L}_\nu^{2\rho(\nu)}(\mathcal{A}_1, \mathcal{A}_2)_{(x_1,x_2)} = \begin{cases} \mathrm{H}_c^{2\rho(\nu)}(S_\nu, \mathcal{A}_1 * \mathcal{A}_2) & \text{if } x_1 = x_2 \\ \oplus_{\nu_1+\nu_2=\nu} \mathrm{H}_c^{2\rho(\nu_1)}(S_{\nu_1}, \mathcal{A}_1) \otimes \mathrm{H}_c^{2\rho(\nu_2)}(S_{\nu_2}, \mathcal{A}_2) & \text{if } x_1 \neq x_2\,. \end{cases} \tag{6.25b}$$

We now proceed as in the proof of theorem 3.6. Let us consider the closures $\overline{S_\nu}(X^n)$ and $\overline{T_\nu}(X^n)$ of the ind-subschemes $S_\nu(X^n)$ and $T_\nu(X^n)$ and let us write $\overline{i_\nu} : \overline{S_\nu}(X^n) \to \mathcal{G}\mathrm{r}_{X^n}$ and $\overline{k_\nu} : \overline{T_\nu}(X^n) \to \mathcal{G}\mathrm{r}_{X^n}$ for the inclusions. Let us write $\mathcal{B} = Rm_* \widetilde{\mathcal{B}}$. Then we have the following canonical morphisms

$$R\pi_* \mathcal{B} = R\pi_! \mathcal{B} \to R\pi_! \overline{s_\nu}^* \mathcal{B} \tag{6.26a}$$

$$R\pi_* \mathcal{B} \leftarrow R\pi_* t_\nu^! \mathcal{B}\,. \tag{6.26b}$$

These morphisms give us two filtrations of $R^k \pi_* R m_* \widetilde{\mathcal{B}}$, one by kernels of the the morphisms

$$R^k \pi_* \mathcal{B} \to R^k \pi_! \overline{s_\nu}^* \mathcal{B} \tag{6.27}$$

and the other by images of the morphisms

$$R^k \pi_* \overline{t_\nu}^! \mathcal{B} \to R^k \pi_* \mathcal{B}\,. \tag{6.28}$$

By the discussion above, these filtrations are complementary and hence yield the following canonical isomorphism

$$R^k\pi_* Rm_*\widetilde{\mathcal{B}} \;=\; \bigoplus_{2\rho(\nu)=k} \mathcal{L}^k_\nu(\mathcal{A}_1,\mathcal{A}_2)\,. \tag{6.29}$$

By (6.21) the sheaf on the left hand side is constant. Therefore the sheaves $\mathcal{L}^k_\nu(\mathcal{A}_1,\mathcal{A}_2)$ must be constant. Appealing to (6.25) completes the proof.

□

7. **The case of a field of characteristic zero**

In this section we treat the case when the base ring $\Bbbk$ is a field of characteristic zero. This case was treated already in [Gi] when $\Bbbk = \mathbb{C}$. Here we make use of Tannakian formalism, using [DM] as a general reference. In section 11, where we work over an arbitrary base ring $\Bbbk$, we carry out the constructions explicitly without referring to the general Tannakian formalism.

7.1. **Lemma.** *If $\Bbbk$ is a field of characteristic zero then the category $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r,\Bbbk)$ is semisimple. In particular, the sheaves $\mathcal{I}_!(\lambda,\Bbbk)$, $\mathcal{I}_{!*}(\lambda,\Bbbk)$, and $\mathcal{I}_*(\lambda,\Bbbk)$ are isomorphic.*

Proof. The parity vanishing of the stalks of $\mathcal{I}_{!*}(\lambda,\Bbbk)$, proved in [Lu], section 11, and the fact that the orbits $\mathcal{G}\mathrm{r}^\lambda$ are simply connected implies immediately that there are no extensions between the simple objects in $\mathrm{P}_\mathcal{S}(\mathcal{G}\mathrm{r},\Bbbk)$. Thus, there are no extensions in the full subcategory $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$ (which then, obviously, coincides with $\mathrm{P}_\mathcal{S}(\mathcal{G}\mathrm{r},\Bbbk)$). □

7.2. **Remark.** *The use of the above lemma can be avoided. One must then ignore this section and first go through the rest of the paper in the case when $\Bbbk$ is a field of characteristic zero. The arguments of section 12, in a greatly simplified form, then give theorem 7.3.*

The constructions above and the properties we established suffice for verifying the conditions of the proposition 1.20 in [DM] and then also the conditions of the theorem 2.11 in [DM], which are summarized by the phrase "$(\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk),*,F)$ is a *neutralized Tannakian category*". Hence, by theorem 2.11 of [DM], we conclude:

$$\begin{gathered}\text{there is a group scheme } \widetilde{G} \text{ over } \Bbbk \text{ such that}\\ \text{the category of finite dimensional } \Bbbk\text{-representations of } \widetilde{G}\\ \text{is equivalent to } \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk), \text{ as tensor categories}\,.\end{gathered} \tag{7.1}$$

We will now identify the group $\widetilde{G}$. Let us write $\check{G}$ for the dual group of G, i.e., $\check{G}$ is the split reductive group over $\Bbbk$ whose root datum is dual to that of G.

7.3. **Theorem.** *The category of finite dimensional $\Bbbk$-representations of $\check{G}$ is equivalent to $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r,\Bbbk)$, as tensor categories.*

Before giving a proof of this theorem we discuss it briefly from the point of view of representation theory. We can view the theorem as giving us a geometric interpretation of representation theory of $\check{G}$. First of all, as we use global cohomology as fiber functor, it follows that the representation space for the representation $V_\mathcal{F}$, associated to $\mathcal{F} \in$

$\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$, is the global cohomology $\mathrm{H}^*(\mathcal{G}\mathrm{r},\mathcal{F})$. As the proof of the theorem will show, $\check{T}$, the dual torus of T, is a maximal torus in $\check{G}$. The decomposition of $V_{\mathcal{F}}$ into its $\check{T}$-weight spaces is then given by theorem 3.6:

$$\mathrm{H}^*(\mathcal{G}\mathrm{r},\mathcal{F}) \;\cong\; \bigoplus_{\nu\in X^*(\check{T})} \mathrm{H}_c^{2\rho(\nu)}(S_\nu,\mathcal{F})\,. \tag{7.2}$$

Given a dominant $\lambda \in X_*(T) = X^*(\check{T})$ we can associate to it both the highest weight representation $L(\lambda)$ of $\check{G}$ and the sheaf $\mathcal{I}_{!*}(\lambda,\Bbbk) \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$. Obviously, $V_{\mathcal{I}_{!*}(\lambda,\Bbbk)}$ is irreducible and by the formula above we see that it is of highest weight λ. Hence, $V_{\mathcal{I}_{!*}(\lambda,\Bbbk)} = L(\lambda)$. Combining this discussion with lemma 7.1 and proposition 3.10 gives:

7.4. **Corollary.** *The ν-weight space $L(\lambda)_\nu$ of $L(\lambda)$ can be canonically identified with the $\Bbbk$-vector space spanned by the irreducible components of $\overline{\mathcal{G}r_\lambda}\cap S_\nu$. In particular, the dimension of $L(\lambda)_\nu$ is given by the number of irreducible components of $\overline{\mathcal{G}r_\lambda}\cap S_\nu$.*

The rest of this section is devoted to the proof of theorem 7.3. We begin with an observation:

(7.3) the group scheme $\widetilde{G}$ is a split connected reductive algebraic group

To see that $\widetilde{G}$ is algebraic, we observe that it has a tensor generator. Let $\lambda_1,\dots,\lambda_r$ be a set of generators for the dominant weights in $X_*(T)$. As a generator we can then take $\oplus\mathcal{I}_{!*}(\lambda_i,\Bbbk)$. It is tensor generator because for any dominant λ the sheaf $\mathcal{I}_{!*}(\lambda,\Bbbk)$ appears as a direct summand in the product

$$\mathcal{I}_{!*}(\lambda_1,\Bbbk)^{*k_1} * \cdots * \mathcal{I}_{!*}(\lambda_r,\Bbbk)^{*k_r}\,; \tag{7.4}$$

here $\lambda = \sum k_1\lambda_1 + \cdots + k_r\lambda_r$. Thus, by [DM], proposition 2.20, $\widetilde{G}$ is an algebraic group. As there is no tensor subcategory of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ whose objects are direct sums of finitely many fixed irreducible objects the group $\widetilde{G}$ is connected by [DM], corollary 2.22. Finally, as $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ is semisimple, $\widetilde{G}$ is reductive, by [DM], proposition 2.23. To see that $\widetilde{G}$ is split, we exhibit a split maximal torus in $\widetilde{G}$. By proposition 6.4 the fiber functor $F = \mathrm{H}^*$ factors as follows:

$$F = \mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk) \to \mathrm{Mod}_\Bbbk(X_*(T))) \to \mathrm{Mod}_\Bbbk\,. \tag{7.5}$$

This gives us a homomorphism $\check{T}\to\widetilde{G}$; here $\check{T}$ is the torus dual to T. As any character $\lambda\in X^*(\check{T}) = X_*(T)$ appears as the direct summand $F_\lambda(\mathcal{I}_{!*}(\lambda,\Bbbk))$ in $F(\mathcal{I}_{!*}(\lambda,\Bbbk))$ we conclude that $\check{T}$ is a split torus in $\widetilde{G}$. It is clearly maximal as the representation ring of $\widetilde{G}$ is of the same rank as $\check{T}$.

It now remains to identify the root datum of $\widetilde{G}$ with the dual of the root datum of G. Recall that we have also fixed a choice of positive roots, i.e., a Borel B such that $T\subset B\subset G$. The root datum of G is then given as $(X^*(T), X_*(T), \Delta(G,T), \check{\Delta}(G,T))$, where $\Delta(G,T)\subset X^*(T)$ are the roots and $\check{\Delta}(G,T)\subset X_*(T)$ are the coroots of G with respect to T. Because $X^*(\check{T}) = X_*(T)$ and $X_*(\check{T}) = X^*(T)$, it suffices to show that

$$\Delta(\widetilde{G},\check{T}) = \check{\Delta}(G,T) \quad\text{ and }\quad \check{\Delta}(\widetilde{G},\check{T}) = \Delta(G,T)\,. \tag{7.6}$$

To this end we note that theorem 3.2, corollary 3.3, and proposition 3.10 imply that:

(7.7) The irreducible representations of $\widetilde{G}$ are parameterized by dominant coweights $\lambda \in X_*(T)$.

and

(7.8) The $\check{T}$-weights of the irreducible representation $L(\lambda)$ associated to λ are the same as the $\check{T}$-weights of the irreducible representation of $\check{G}$ associated to λ.

We now argue using the pattern of the weights. For clarity we spell out this familiar structure. From (7.8) we conclude:

(7.9a) The weights of $L(\lambda)$ are symmetric under the Weyl group W.

(7.9b) For α a simple positive root of G, with the corresponding coroot $\check{\alpha}$, the weights of $L(\lambda)$ on the line segment between $s_{\check{\alpha}}(\lambda)$ and λ are the weights on that segment which are of the form $\lambda - k\check{\alpha}$, with k an integer.

Note that the choice of a Borel subgroup of $\widetilde{G}$ is equivalent to a consistent choice of a line, the highest weight line, in each irreducible representation of $\widetilde{G}$. The choice $F_\lambda(\mathcal{I}_{!*}(\lambda,\Bbbk))$ in $F(\mathcal{I}_{!*}(\lambda,\Bbbk))$ for all dominant $\lambda \in X_*(T)$ yields a Borel subgroup $\widetilde{B}$ of $\widetilde{G}$ such that the dominant weights of $\widetilde{G}$ in $X^*(\check{T})$ coincide with the dominant coweights of G in $X_*(T)$. This implies that the simple coroot directions of the triple $(\check{T},\widetilde{B},\widetilde{G})$ coincide with the simple root directions of (T,B,G). The statements (7.9) above now imply that the simple roots of the triple $(\check{T},\widetilde{B},\widetilde{G})$ coincide with the simple coroots of (T,B,G). This, finally gives (7.6).

8. **Standard sheaves**

In this section we prove some basic results about standard sheaves which will be crucial for us later. Let us write $\mathbb{D}$ for the Verdier duality functor.

8.1**. Proposition.** *We have*

(a) $$\mathcal{I}_!(\lambda,\Bbbk) \;\cong\; \mathcal{I}_!(\lambda,\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$$

(b) $$\mathcal{I}_*(\lambda,\Bbbk) \;\cong\; \mathcal{I}_*(\lambda,\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$$

(c) $$\mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk) \;\cong \mathcal{I}_*(\lambda,\Bbbk)\,.$$

Proof. The proofs of (a) and (b) are analogous and hence we will only prove (a). Because

(8.1) $$\mathrm{H}^*_c(S_\nu, \mathcal{I}_!(\lambda,\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk) \;=\; \mathrm{H}^*_c(S_\nu, \mathcal{I}_!(\lambda,\mathbb{Z})) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$$

and, by proposition 3.10, $\mathrm{H}^*_c(S_\nu, \mathcal{I}_!(\lambda,\mathbb{Z}))$ is a free abelian group in degree $2\rho(\nu)$ we conclude that $\mathrm{H}^*_c(S_\nu, \mathcal{I}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk)$ is nonzero only in degree $2\rho(\nu)$. Hence, by lemma 3.9, we see that $\mathcal{I}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$ is perverse. There is a canonical map

$$\mathcal{I}_!(\lambda,\Bbbk) \xrightarrow{\ \iota\ } \mathcal{I}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk\,, \tag{8.2}$$

which is an isomorphism when restricted to $\mathcal{G}\mathrm{r}^\lambda$. Therefore, applying the functor F_ν to the morphism ι and using the proposition 3.10 yields an isomorphism

$$F_\nu(\mathcal{I}_!(\lambda,\Bbbk)) = \Bbbk[Irr(\overline{\mathcal{G}\mathrm{r}_\lambda}\cap S_\nu)] \xrightarrow{F_\nu(\iota)} \mathbb{Z}[Irr(\overline{\mathcal{G}\mathrm{r}_\lambda}\cap S_\nu)]\underset{\mathbb{Z}}{\otimes}\Bbbk = F_\nu(\mathcal{I}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk)\,. \tag{8.3}$$

By corollary 3.7 the functor $F = \oplus F_\nu$ is faithful and thus we conclude that ι is an isomorphism.

The proof of part (c) proceeds in a similar fashion. First we observe that

$$\mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}, \mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk)) \cong \mathbb{D}(\mathrm{H}^*_c(T_\nu, \mathcal{I}_!(\lambda,\Bbbk))) = \mathbb{D}(\mathrm{H}^*_c(S_{w_0\cdot\nu}, \mathcal{I}_!(\lambda,\Bbbk)))\,. \tag{8.4}$$

Because $\mathrm{H}^*_c(S_{w_0\cdot\nu}, \mathcal{I}_!(\lambda,\Bbbk))$ is a free $\Bbbk$-module concentrated in degree $2\rho(w_0\cdot\nu)$, we conclude that $\mathbb{D}(\mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}, \mathcal{I}_!(\lambda,\Bbbk)))$ is a concentrated in degree $-2\rho(w_0\cdot\nu) = 2\rho(\nu)$. Thus, we conclude that $\mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk)$ is perverse. Furthermore, we note that

$$\begin{array}{ccc} \mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}, \mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk)) & \xrightarrow{\ \cong\ } & \mathbb{D}(\mathrm{H}^*_c(T_\nu, \mathcal{I}_!(\lambda,\Bbbk))) \\ \big\downarrow & & \big\downarrow\cong \\ \mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}^\lambda, \mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk)) & \xrightarrow{\ \cong\ } & \mathbb{D}(\mathrm{H}^*_c(T_\nu\cap\mathcal{G}\mathrm{r}^\lambda, \mathcal{I}_!(\lambda,\Bbbk)))\,, \end{array} \tag{8.5}$$

which implies that the left hand arrow is also an isomorphism. We have a canonical map

$$\mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk) \xrightarrow{\ \iota\ } \mathcal{I}_*(\lambda,\Bbbk) \tag{8.6}$$

To show that this map is an isomorphism it suffice to show that the maps $F_\nu(\iota)$ are isomorphisms. Restricting to $\mathcal{G}\mathrm{r}^\lambda$ gives us the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}, \mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk)) & \xrightarrow{F_\nu(\iota)} & \mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}, \mathcal{I}_*(\lambda,\Bbbk)) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}^\lambda, \mathbb{D}\,\mathcal{I}_!(\lambda,\Bbbk)) & \xrightarrow{\ \cong\ } & \mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r}^\lambda, \mathcal{I}_*(\lambda,\Bbbk))\,. \end{array} \tag{8.7}$$

In this diagram the bottom arrow is an isomorphism because ι restricted to $\mathcal{G}\mathrm{r}^\lambda$ is an isomorphism, the left vertical arrow is an isomorphism by (8.5), and finally, the right vertical arrow is an isomorphism by proposition 3.10 (or, rather, by the proof thereof). This shows that $F_\nu(\iota)$ is an isomorphism.

□

8.2. **Proposition.** *The canonical map* $\mathcal{I}_!(\lambda,\mathbb{Z})\to \mathcal{I}_{!*}(\lambda,\mathbb{Z})$ *is an isomorphism.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{I}_!(\lambda,\mathbb{Z}) & \xrightarrow{\alpha} & \mathcal{I}_*(\lambda,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{I}_!(\lambda,\mathbb{Q}) & \longrightarrow & \mathcal{I}_*(\lambda,\mathbb{Q})\,. \end{array} \tag{8.8}$$

The bottom map is an isomorphism by lemma 7.1. Let us apply the functor F_ν to this diagram. The columns become inclusions by proposition 8.1 and the bottom arrow is an isomorphism as we just observed. Therefore, $F_\nu(\alpha)$ is an inclusion and then so is α. This implies that the canonical surjection $\mathcal{I}_!(\lambda,\mathbb{Z}) \twoheadrightarrow \mathcal{I}_{!*}(\lambda,\mathbb{Z})$ is an isomorphism. □

9. **Representability of the weight functors**

In section §3 we showed that the functors $F_\nu \stackrel{def}{=} \mathrm{H}_c^{2\rho(\nu)}(S_\nu,\ ) \cong\ \mathrm{H}^*_{T_\nu}(\mathcal{G}\mathrm{r},-)[2\rho(\nu)]$, from $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$ to $\mathrm{Mod}_\Bbbk$, for $\nu \in X_*(T)$, are exact. Hence, one would expect them to be (pro) representable. Here we prove that this is indeed the case:

9.1**. Proposition.** *Let $Z \subset \mathcal{G}r$ be a closed subset which is a finite union of $G_\mathcal{O}$-orbits. The functor F_ν restricted to $\mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$ is represented by a projective object $P_Z(\nu,\Bbbk)$ of $\mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$.*

Proof. We make use of the induction functors. Let us recall their construction. For more details see, for example, [MiV1]. Let A be an algebraic group acting on a variety Y and let B be a subgroup of A. The forgetful functor $\mathcal{F}_B^A:\ D_A(Y,\Bbbk) \to D_B(Y,\Bbbk)$ has a left adjoint $\gamma_B^A:\ D_B(Y,\Bbbk) \to D_A(Y,\Bbbk)$ which can be constructed as follows. Consider the diagram

$$Z \xleftarrow{\ p\ } A \times Z \xrightarrow{\ q\ } A \times_B Z \xrightarrow{\ a\ } Z\,. \tag{9.1}$$

The maps p and q are projections, and a is the action map. The group $A \times B$ acts on the leftmost copy of Z via the factor B, on $A \times Z$ and $A \times_B Z$ by the formula $(a,b)\cdot(a',z) = (a\cdot a'\cdot b^{-1}, b\cdot z)$, and on the leftmost copy of Z via the factor A. The left adjoint γ_B^A is now given by

$$\gamma_B^A(\mathcal{A}) \ =\ a_!\tilde{\mathcal{A}} \ \text{ where } \ \tilde{\mathcal{A}} \ \text{ is defined via } \ q^!\tilde{\mathcal{A}} \ \cong\ p^!\mathcal{A}\,; \text{for } \ \mathcal{A} \in D_B(Y,\Bbbk) \tag{9.2}$$

Let $\mathcal{O}_n = \mathcal{O}/z^{n+1}$ and let us write $G_{\mathcal{O}_n}$ for the algebraic group whose $\mathbb{C}$-points are $G(\mathcal{O}_n)$. We use analogous notation for other groups. Now choose $n >> 0$ so that the $G_\mathcal{O}$-action on Z factors through the action of $G_{\mathcal{O}_n}$. We write $P_Z(\nu,\Bbbk) = {}^p\mathrm{H}^0(\gamma_{\{e\}}^{G_{\mathcal{O}_n}}(\Bbbk_{T_\nu \cap Z}[-2\rho(\nu)]))$. We claim:

$$\text{the functor } F_\nu : \mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk) \to \mathrm{Mod}_\Bbbk \text{ is represented by } P_Z(\nu,\Bbbk) \tag{9.3}$$

To see this, let $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$, and then:

$$F_\nu(\mathcal{A}) \;=\; \mathrm{H}^{2\rho(\nu)}_{T_\nu}(Z,\mathcal{A}) \;= \tag{9.4}$$
$$\mathrm{Ext}^0_{\mathrm{D}(Z,\Bbbk)}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)],\mathcal{F}^{G_{\mathcal{O}_n}}_{\{e\}}\,\mathcal{A}) \;\cong$$
$$\mathrm{Ext}^0_{\mathrm{D}_{G_{\mathcal{O}_n}}(Z,\Bbbk)}(\gamma^{G_{\mathcal{O}_n}}_{\{e\}}\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)],\;\mathcal{A})\,.$$

Let us write $\mathcal{F}=\gamma^{G_{\mathcal{O}_n}}_{\{e\}}\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)]$. Then

$$\text{the sheaf }\;\mathcal{F}\;\text{ lies in }\;{}^p\mathrm{D}^{\leq 0}_{G_{\mathcal{O}_n}}(Z,\Bbbk)\,. \tag{9.5}$$

To see this, let us write d for the largest integer such that ${}^p\mathrm{H}^d(\mathcal{F})\neq 0$. Then, we see, as in (9.4), that

$$0\neq \mathrm{Hom}_{\mathrm{D}_{G_{\mathcal{O}_n}}(Z,\Bbbk)}(\mathcal{F},{}^p\mathrm{H}^d(\mathcal{F})[-d]) = \tag{9.6}$$
$$\mathrm{Ext}^{-d}_{\mathrm{D}_{G_{\mathcal{O}_n}}(Z,\Bbbk)}(\mathcal{F},{}^p\mathrm{H}^d(\mathcal{F}))\cong \mathrm{H}^{2\rho(\nu)-d}_{T_\nu}(Z,{}^p\mathrm{H}^d(\mathcal{F}))\,.$$

According to theorem 3.5 this forces $d=0$ and we have proved (9.5). This immediately implies (9.3).

□

9.2. **Corollary.** *The category* $\mathrm{P}_{G_{\mathcal{O}}}(Z,\Bbbk)$ *has enough projectives.*

Proof. Let $\mathcal{A}\in\mathrm{P}_{G_{\mathcal{O}}}(Z,\Bbbk)$. Choose finitely generated $\Bbbk$-projective covers $f_\nu : P_\nu\to F_\nu(\mathcal{A})$. Then

$$\mathrm{Hom}(P_\nu\otimes_\Bbbk P_Z(\nu,\Bbbk),\mathcal{A}) \;\cong\; \mathrm{Hom}_\Bbbk[P_\nu,\mathrm{Hom}(P_Z(\nu,\Bbbk),\mathcal{A})] \;\cong\; \mathrm{Hom}_\Bbbk[P_\nu,F_\nu(\mathcal{A})]\,. \tag{9.7}$$

By construction, the map $p_\nu\in\mathrm{Hom}(P_\nu\otimes_\Bbbk P_Z(\nu,\Bbbk),\mathcal{A})$ corresponding to the $\Bbbk$-projective cover $f_\nu : P_\nu\to F_\nu(\mathcal{A})$ satisfies $F_\nu(p_\nu)=f_\nu$. Since $\oplus_\nu F_\nu = F$ is exact and faithful, $\oplus_\nu\, P_\nu\otimes_\Bbbk P_Z(\nu,\Bbbk)$ is a projective cover of $\mathcal{A}$.

□

We can describe the sheaf $P_Z(\nu,\Bbbk)={}^p\mathrm{H}^0(\gamma^{G_{\mathcal{O}_n}}_{\{e\}}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)]))$ rather explicitly as follows. Let us consider the following diagram:

$$\begin{array}{ccccc} Z & \xleftarrow{\;p\;} & G_{\mathcal{O}_n}\times Z & \xrightarrow{\;a\;} & Z \\ {\scriptstyle i}\uparrow & & \uparrow & & \| \\ T_\nu\cap Z & \xleftarrow{\;r\;} & G_{\mathcal{O}_n}\times(T_\nu\cap Z) & \xrightarrow{\;\widetilde{a}\;} & Z \end{array} \tag{9.8}$$

and use it to calculate $\gamma^{G_{\mathcal{O}_n}}_{\{e\}}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)])$:

$$\gamma^{G_{\mathcal{O}_n}}_{\{e\}}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)]) = Ra_!p^!i_*\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)] = \tag{9.9}$$
$$R\widetilde{a}_!r^!\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)] = R\widetilde{a}_!\Bbbk_{G_{\mathcal{O}_n}\times(T_\nu\cap Z)}[2\dim(G_{\mathcal{O}_n})-2\rho(\nu)]\,.$$

Let us consider a point $L_\eta\in Z$, where we again choose $\eta\in X_*(T)$ dominant. Then

9.3. **Lemma.** *The dimension of the fiber $\widetilde{a}^{-1}(L_\eta)$ is $\dim G_{\mathcal{O}_n} - \rho(\nu+\eta)$ if $\eta \geq \nu$, otherwise it is empty.*

Proof. We have

$$\widetilde{a}^{-1}(L_\eta) \;=\; \{(g,z) \in G_{\mathcal{O}_n} \times (T_\nu \cap Z) \mid g\cdot z = \eta\} \;= \\ \{(g,z) \in G_{\mathcal{O}_n} \times (T_\nu \cap \mathcal{G}\mathrm{r}^\eta) \mid g\cdot z = \eta\}\,. \tag{9.10}$$

By theorem 3.2, we see that $\widetilde{a}^{-1}(L_\eta)$ is empty unless $\eta \geq \nu$. Furthermore, from theorem 3.2, we see that if $\eta \geq \nu$ then

$$\dim \widetilde{a}^{-1}(L_\eta) \;= \\ \rho(\eta-\nu) + \dim((G_{\mathcal{O}_n})_\eta) \;=\; \rho(\eta-\nu) + \dim(G_{\mathcal{O}_n}) - \dim \mathcal{G}\mathrm{r}^\eta \;= \\ \rho(\eta-\nu) + \dim(G_{\mathcal{O}_n}) - 2\dim\rho(\eta) \;=\; \dim(G_{\mathcal{O}_n}) - \rho(\eta+\nu)\,. \tag{9.11}$$

□

In other words, $P_Z(\nu,\Bbbk)$ is the zeroth perverse cohomology of the !-image of a (shift of) a constant sheaf under an "essentially semi-small" map. Here "essentially semi-small" means that the the dimensions of fibers have the correct increment but the generic fiber is not finite.

10. **The structure of projectives that represent weight functors**

In this section we analyze the projective $P_Z(\Bbbk) = \oplus_\nu\, P_Z(\nu,\Bbbk)$ which represents the fiber functor F on $\mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$. As in the previous section, Z is closed subset of $\mathcal{G}\mathrm{r}$ which is a union of finitely many $G_\mathcal{O}$-orbits.

10.1. **Proposition.** *(a) Let $Y \subset Z$ be a closed subset consisting of $G_\mathcal{O}$-orbits. Then*

$$P_Y(\Bbbk) = \;{}^pH^0(P_Z(\Bbbk)\big|Y)\,,$$

and there is a canonical surjection

$$P_Z(\Bbbk) \to P_Y(\Bbbk)\,.$$

(b) The projective $P_Z(\Bbbk)$ has a filtration such that the the associated graded

$$Gr(P_Z(\Bbbk)) \;=\; \bigoplus_{\mathcal{G}r^\lambda \subseteq Z} F[\mathcal{J}_*(\lambda,\Bbbk)]^* \otimes_\Bbbk \mathcal{J}_!(\lambda,\Bbbk)\,.$$

In particular, $F(P_Z(\Bbbk))$ is free over $\Bbbk$.

(c) $P_Z(\nu,\Bbbk) \cong \; P_Z(\nu,\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$.

Proof. We begin with (a). We write $i :\; Y \hookrightarrow Z$ for the inclusion. The identity $\mathrm{Hom}(P_Z(\Bbbk), i_*-) = \mathrm{Hom}(i^*P_Z(\Bbbk), -)$ shows that the complex $P_Z(\Bbbk)\big|Y \in {}^pD^{\leq 0}_{G_\mathcal{O}}(Y,\Bbbk)$, represents F on the subcategory $\mathrm{P}_{G_\mathcal{O}}(Y,\Bbbk)$, and hence so does ${}^pH^0(P_Z(\Bbbk)\big|Y) \in \mathrm{P}_{G_\mathcal{O}}(Y,\Bbbk)$. Thus, $P_Y(\Bbbk) = \;{}^pH^0(P_Z(\Bbbk)\big|Y)$. For any $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(Y,\Bbbk)$ we have the identity $\mathrm{Hom}(P_Z(\Bbbk),\mathcal{A}) = \mathrm{Hom}(P_Y(\Bbbk),\mathcal{A})$. This gives a canonical surjection $P_Z(\Bbbk) \to P_Y(\Bbbk)$.

Now we will prove parts (b) and (c) simultaneously. We will argue (b) by induction on the number of $G_{\mathcal{O}}$-orbits in Z. Let us assume that $\mathcal{G}\mathrm{r}^{\lambda}$ is an open $G_{\mathcal{O}}$-orbit in Z and let $Y = Z - \mathcal{G}\mathrm{r}^{\lambda}$. Let us consider the following exact sequence:

$$0 \to K \to P_Z(\Bbbk) \to P_Y(\Bbbk) \to 0\,, \tag{10.1}$$

where K is simply the kernel of the canonical map from (a). Let M be a $\Bbbk$-module and let us take RHom of the exact sequence above to $\mathcal{I}_*(\lambda, M)$:

$$\begin{aligned} 0 \to \mathrm{Hom}(P_Y(\Bbbk), \mathcal{I}_*(\lambda, M)) \to \mathrm{Hom}(P_Z(\Bbbk), \mathcal{I}_*(\lambda, M)) \to \\ \mathrm{Hom}(K, \mathcal{I}_*(\lambda, M)) \to \mathrm{Ext}^1(P_Y(\Bbbk), \mathcal{I}_*(\lambda, M))\,. \end{aligned} \tag{10.2}$$

By adjunction the first and last terms are zero and so we get, again by adjunction,

$$\begin{aligned} \mathrm{Hom}(K|_{\mathcal{G}\mathrm{r}^{\lambda}}, M_{\mathcal{G}\mathrm{r}^{\lambda}}[2\rho(\lambda)]) \cong \mathrm{Hom}(K, \mathcal{I}_*(\lambda, M)) \cong \\ \mathrm{Hom}(P_Z(\Bbbk), \mathcal{I}_*(\lambda, M)) = F(\mathcal{I}_*(\lambda, M))\,. \end{aligned} \tag{10.3}$$

We can view (10.3) as a functor from $\Bbbk$-modules to $\Bbbk$-modules

$$M \mapsto F(\mathcal{I}_*(\lambda, M))\,. \tag{10.4}$$

This functor is, by the results in §8 represented by the free $\Bbbk$-module $F(\mathcal{I}_*(\lambda, \Bbbk))^*$. As it is also represented by $K|_{\mathcal{G}\mathrm{r}^{\lambda}}$, we conclude:

$$K|_{\mathcal{G}\mathrm{r}^{\lambda}} \;=\; F(\mathcal{I}_*(\lambda, \Bbbk))^* \otimes_{\Bbbk} \Bbbk_{\mathcal{G}\mathrm{r}^{\lambda}}[2\rho(\lambda)]\,. \tag{10.5}$$

Now we claim:

$$K \;=\; F(\mathcal{I}_*(\lambda, \Bbbk))^* \otimes_{\Bbbk} \mathcal{I}_!(\lambda, \Bbbk) = \mathcal{I}_!(\lambda, F(\mathcal{I}_*(\lambda, \Bbbk))^*)\,. \tag{10.6}$$

To prove this claim, let us consider the following exact sequence

$$0 \to K' \to F(\mathcal{I}_*(\lambda, \Bbbk))^* \otimes_{\Bbbk} \mathcal{I}_!(\lambda, \Bbbk) \to K \to C \to 0\,. \tag{10.7}$$

The kernel K' and the cokernel C are supported on Y. If we take RHom of the exact sequence (10.1) to C, we get

$$0 \to \mathrm{Hom}(P_Y(\Bbbk), C) \xrightarrow{\alpha} \mathrm{Hom}(P_Z(\Bbbk), C) \to \mathrm{Hom}(K, C) \to \mathrm{Ext}^1(P_Y(\Bbbk), C)\,. \tag{10.8}$$

Because C is supported on Y the map α is an isomorphism and the last term vanishes. Therefore C must be zero. To prove that K' is zero, we first assume that $\Bbbk = \mathbb{Z}$. Then, by 8.2, $\mathcal{I}_!(\lambda, \mathbb{Z}) = \mathcal{I}_{!*}(\lambda, \mathbb{Z})$. As $\mathcal{I}_{!*}(\lambda, \mathbb{Z})$ has no subobjects supported on Y, $K' = 0$. Using (10.1) and (10.6), and proceeding by induction on the number of $G_{\mathcal{O}}$-orbits, we obtain (b) when $\Bbbk = \mathbb{Z}$.

We will now prove (c). Because $\Bbbk \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{I}_!(\lambda, \mathbb{Z}) \cong \mathcal{I}_!(\lambda, \Bbbk)$ and because (b) holds for $\Bbbk = \mathbb{Z}$, we see that $\Bbbk \otimes_{\mathbb{Z}} P_Z(\mathbb{Z})$ is perverse. By formula (2.6.7) in [KS], we see that for

any $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ we have

$$\mathrm{Hom}_{\Bbbk}(\Bbbk \otimes_{\mathbb{Z}} P_Z(\mathbb{Z}), \mathcal{A}) \;=\; \mathrm{Hom}_{\mathbb{Z}}(P_Z(\mathbb{Z}), \mathrm{R}\,Hom(\Bbbk, \mathcal{A})) \;= \\ \mathrm{Hom}_{\mathbb{Z}}(P_Z(\mathbb{Z}), \mathcal{A}) \;=\; \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A})\,. \tag{10.9}$$

Thus, $\Bbbk \otimes_{\mathbb{Z}} P_Z(\mathbb{Z})$ represents the functor F on $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ and hence we must have $\Bbbk \otimes_{\mathbb{Z}} P_Z(\mathbb{Z}) = P_Z(\Bbbk)$.

Finally to get statement (b) for an arbitrary ring $\Bbbk$, it suffices to use (c), (b) for the case $\Bbbk = \mathbb{Z}$, and the fact that $\Bbbk \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{I}_!(\lambda, \mathbb{Z}) \cong \mathcal{I}_!(\lambda, \Bbbk)$.

□

Let us write $\mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(Z, \Bbbk)$ for the subcategory of $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ consisting of sheaves $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ such that $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A})$ is $\Bbbk$-projective. Note that, by lemma 3.9, the category $\mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(Z, \Bbbk)$ is closed under Verdier duality. Because $P_Z(\Bbbk) \in \mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(Z, \Bbbk)$, its dual $I_Z(\Bbbk) = \mathbb{D}(P_Z(\Bbbk))$ also belongs in $\mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(Z, \Bbbk)$. The sheaf $I_Z(\Bbbk)$ is an injective object in the subcategory $\mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(Z, \Bbbk)$. Note that the abelianization of the exact category $\mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ is precisely $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$.

11. **Construction of the group scheme**

In this section we construct a group scheme $\widetilde{G}_{\Bbbk}$ such that $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ is the category of its representations. We proceed by Tannakian formalism, see, for example, [DM]. Unlike [DM] we work over an arbitrary commutative ring $\Bbbk$. This is made possible by the fact that $F(P_Z(\Bbbk))$ is free over $\Bbbk$, 10.1.

11.1**. Proposition.** *There is a group scheme $\widetilde{G}_{\Bbbk}$ over $\Bbbk$ such that the tensor category of representations, finitely generated over $\Bbbk$, is equivalent to $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$. Furthermore, the coordinate ring $\Bbbk[\widetilde{G}_{\Bbbk}]$ is free over $\Bbbk$ and $\widetilde{G}_{\Bbbk} = \mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{Z})} \widetilde{G}_{\mathbb{Z}}$.*

Proof. We view $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ as a direct limit $\varinjlim_{Z} \mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$; here Z runs through finite dimensional $G_{\mathcal{O}}$-invariant closed subsets of the affine Grassmannian $\mathcal{G}\mathrm{r}$. Let us write $A_Z(\Bbbk)$ for the $\Bbbk$-algebra $\mathrm{End}(P_Z(\Bbbk)) = F(P_Z(\Bbbk))$. The algebra $A_Z(\Bbbk)$ is free of finite rank over $\Bbbk$. Let us write $\mathrm{Mod}_{A_Z(\Bbbk)}$ for the category of $A_Z(\Bbbk)$-modules which are finitely generated over $\Bbbk$. Because $P_Z(\Bbbk)$ is a projective generator of $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$, we see that the restriction of the functor F to $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk) \to \mathrm{Mod}_{\Bbbk}$ lifts to an equivalence of abelian categories:

(11.1) the categories $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ and $\mathrm{Mod}_{A_Z(\Bbbk)}$ are equivalent as abelian categories.

As $A_Z(\Bbbk)$ is free of finite rank over $\Bbbk$, its $\Bbbk$-dual $B_Z(\Bbbk)$ is naturally a co-algebra. Furthermore, let us consider a $\Bbbk$-module V. Because

$$\mathrm{Hom}_{\Bbbk}(A_Z(\Bbbk) \otimes_{\Bbbk} V, V) \;=\; \mathrm{Hom}_{\Bbbk}(V, B_Z(\Bbbk) \otimes_{\Bbbk} V)\,, \tag{11.2}$$

we see that it is equivalent to give to V a structure of an $A_Z(\Bbbk)$-module or to give it a structure of a $B_Z(\Bbbk)$-comodule. Let us write $\mathrm{Comod}_{B_Z(\Bbbk)}$ for the category of $B_Z(\Bbbk)$-comodules which are finitely generated over $\Bbbk$.

From the previous discussion we conclude:

$$\text{the categories } \mathrm{P}_{G_{\mathcal{O}}}(Z,\Bbbk) \text{ and } \mathrm{Comod}_{B_Z(\Bbbk)} \text{ are equivalent as abelian categories.} \tag{11.3}$$

Let us write $I_Z(\Bbbk)$ for the Verdier dual of $P_Z(\Bbbk)$. Now,

$$B_Z(\Bbbk) \;=\; A_Z(\Bbbk)^* \;=\; \mathrm{H}^*(\mathcal{G}\mathrm{r}, P_Z(\Bbbk))^* \;=\; \mathrm{H}^*(\mathcal{G}\mathrm{r}, I_Z(\Bbbk)) \;=\; F(I_Z(\Bbbk))\,. \tag{11.4}$$

If $Z \subset Z'$ are both closed and $G_{\mathcal{O}}$-invariant then the canonical morphism $P_{Z'}(\Bbbk) \to P_Z(\Bbbk)$ gives rise to a morphism $I_Z(\Bbbk) \to I_{Z'}(\Bbbk)$ and this, in turn, gives a map of co-algebras $B_Z(\Bbbk) \to B_{Z'}(\Bbbk)$. Hence we can form the coalgebra

$$B(\Bbbk) \;=\; \varinjlim B_Z(\Bbbk)\,, \tag{11.5}$$

and we get:

(11.6) the categories $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ and $\mathrm{Comod}_{B(\Bbbk)}$ are equivalent as abelian categories.

It now remains to give the coalgebra $B(\Bbbk)$ the structure of an algebra and to give an inverse in its coalgebra structure. We will start by giving $B(\Bbbk)$ an algebra structure. To this end, let us consider the filtration of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ by the subcategories $\mathrm{P}_{G_{\mathcal{O}}}(\lambda,\Bbbk) = \mathrm{P}_{G_{\mathcal{O}}}(\overline{\mathcal{G}\mathrm{r}_\lambda},\Bbbk)$ indexed dominant coweights λ. In the discussion that is to follow we use the following convention. When we substitute $\overline{\mathcal{G}\mathrm{r}_\lambda}$ for the subvariety Z we use the following shorthand notation $A_\lambda(\Bbbk) = A_{\overline{\mathcal{G}\mathrm{r}^\lambda}}(\Bbbk)$, $P_\lambda(\Bbbk) = P_{\overline{\mathcal{G}\mathrm{r}^\lambda}}(\Bbbk)$ etc. This filtration is compatible with the convolution product in the sense that $\mathrm{P}_{G_{\mathcal{O}}}(\lambda,\Bbbk) * \mathrm{P}_{G_{\mathcal{O}}}(\mu,\Bbbk) \subseteq \mathrm{P}_{G_{\mathcal{O}}}(\lambda+\mu,\Bbbk)$. We have:

$$\mathrm{Hom}[P_{\lambda+\mu}(\Bbbk), P_\lambda(\Bbbk) * P_\mu(\Bbbk)] \cong F[P_\lambda(\Bbbk) * P_\mu(\Bbbk)] \cong F[P_\lambda(\Bbbk)] \underset{\Bbbk}{\otimes} F[P_\mu(\Bbbk)] = A_\lambda(\Bbbk) \otimes_\Bbbk A_\mu(\Bbbk)\,. \tag{11.7}$$

The element $1 \otimes 1 \in A_\lambda(\Bbbk) \otimes_\Bbbk A_\mu(\Bbbk)$ gives rise to a morphism $P_{\lambda+\mu}(\Bbbk) \to P_\lambda(\Bbbk) * P_\mu(\Bbbk)$. Dualizing this gives a morphism $I_\lambda(\Bbbk) * I_\mu(\Bbbk) \to I_{\lambda+\mu}(\Bbbk)$ and by applying the functor F a morphism

$$B_\lambda(\Bbbk) \otimes_\Bbbk B_\mu(\Bbbk) \to B_{\lambda+\mu}(\Bbbk)\,. \tag{11.8}$$

Passing to the limit gives $B(\Bbbk)$ a structure of a commutative $\Bbbk$-algebra; the associativity and the commutativity of the multiplication come from the associativity and commutativity of the tensor product. To summarize, we have constructed an affine monoid $\widetilde{G}_\Bbbk = \mathrm{Spec}(B(\Bbbk))$ such that

$$\mathrm{Rep}_{\widetilde{G}_\Bbbk} \quad \text{is equivalent to} \quad \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk) \quad \text{as tensor categories}\,; \tag{11.9}$$

here $\mathrm{Rep}_{\widetilde{G}_\Bbbk}$ denotes the category of representations of $\widetilde{G}_\Bbbk$ which are finitely generated as $\Bbbk$-modules.

We will show next that $\widetilde{G}_\Bbbk$ is a group scheme, i.e., that it has inverses. To do so, we first observe that while the ind-scheme is $G_{\mathcal{K}}$ is not of ind-finite type it is a torsor for the the pro-algebraic group $G_{\mathcal{O}}$, over two ind-finite type schemes $G_{\mathcal{K}}/G_{\mathcal{O}} = \mathcal{G}\mathrm{r}$ and $G_{\mathcal{O}}\backslash G_{\mathcal{K}}$. This gives notions of two kinds of equivariant perverse sheaves on $\mathcal{G}\mathrm{r}$ that come with equivalences $\mathrm{P}_{G_{\mathcal{O}}\times 1}(G_{\mathcal{K}},\Bbbk) \cong \mathrm{P}(G_{\mathcal{O}}\backslash G_{\mathcal{K}},\Bbbk)$ and $\mathrm{P}_{1\times G_{\mathcal{O}}}(G_{\mathcal{K}},\Bbbk) \cong \mathrm{P}(\mathcal{G}\mathrm{r},\Bbbk)$. In particular one obtains two notions of full subcategories $\mathrm{P}_{G_{\mathcal{O}}\times G_{\mathcal{O}}}(G_{\mathcal{K}},\Bbbk)$ of

$\mathrm{P}_{G_{\mathcal{O}}\times 1}(G_{\mathcal{K}},\Bbbk)$ and $\mathrm{P}_{1\times G_{\mathcal{O}}}(G_{\mathcal{K}},\Bbbk)$, but these are easily seen to coincide. Let us recall that we write $\mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ for the subcategory of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ consisting of sheaves $\mathcal{A}\in\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ such that $\mathrm{H}^*(\mathcal{G}\mathrm{r},\mathcal{A})$ is $\Bbbk$-projective. The inversion map $i: G_{\mathcal{K}}\to G_{\mathcal{K}}$, $i(g)=g^{-1}$ exchanges $\mathrm{P}_{G_{\mathcal{O}}\times 1}(G_{\mathcal{K}},\Bbbk)$ and $\mathrm{P}_{1\times G_{\mathcal{O}}}(G_{\mathcal{K}},\Bbbk)$ and defines an autoequivalence of $\mathrm{P}_{G_{\mathcal{O}}\times G_{\mathcal{O}}}(G_{\mathcal{K}},\Bbbk)$ which we can view as $i^*:\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)\to\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$. Now we can define an anti-involution on $\mathrm{P}^{\Bbbk-proj}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ by

$$\mathcal{A}\mapsto\mathcal{A}^*=\mathbb{D}(i^*\mathcal{A})\,. \tag{11.10}$$

This involution makes $\mathrm{Rep}^{\Bbbk-proj}_{\widetilde{G}_\Bbbk}$, the category of representations of $\widetilde{G}_\Bbbk$ on finitely generated projective $\Bbbk$-modules, a rigid tensor category. By [Sa] II.3.1.1, I.5.2.2, and I.5.2.3, we conclude that $\widetilde{G}_\Bbbk=\mathrm{Spec}(B(\Bbbk))$ is a group scheme.

The statement $\widetilde{G}_\Bbbk=\mathrm{Spec}(\Bbbk)\times_{\mathrm{Spec}(\mathbb{Z})}\widetilde{G}_{\mathbb{Z}}$, i.e., that $B(\Bbbk)=\Bbbk\otimes_{\mathbb{Z}}B(\mathbb{Z})$, now follows from Proposition 10.1 part (c). The algebra $B(\Bbbk)$ is free over $\Bbbk$ by construction.

12. The identification of $\widetilde{G}_\Bbbk$ with the dual group of G

In this section we identify the group scheme $\widetilde{G}_\Bbbk$. As $\widetilde{G}_\Bbbk=\mathrm{Spec}(\Bbbk)\times_{\mathrm{Spec}(\mathbb{Z})}\widetilde{G}_{\mathbb{Z}}$, by 11.1, it suffices to do so when $\Bbbk=\mathbb{Z}$. Recall that there exists a unique split reductive group scheme, the Chevalley group scheme, over $\mathbb{Z}$ associated to any root datum ([SGA 3],[Dem]). Let $\check{G}_{\mathbb{Z}},\check{T}_{\mathbb{Z}}$ be such schemes associated to the root data dual to that of G,T and denote $\check{G}_\Bbbk=\check{G}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk,\check{T}_\Bbbk=\check{T}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ for any $\Bbbk$. We claim:

12.1. **Theorem.** *The group scheme $\widetilde{G}_{\mathbb{Z}}$ is the split reductive group scheme over $\mathbb{Z}$ whose root datum is dual to that of G.*

The rest of this section is devoted to the proof of this theorem. We first recall that we have shown in section 7 that the above statement holds at the generic point, i.e., that $\widetilde{G}_{\mathbb{Q}}$ is split reductive reductive group whose root datum is dual to that of G. By the uniqueness of the Chevalley group scheme, [Dem], and the fact that $\widetilde{G}_{\mathbb{Q}}$ is a split reductive reductive group whose root datum is dual to that of G it suffices to show:

(12.1a) The group scheme $\widetilde{G}_{\mathbb{Z}}$ is smooth over $\mathrm{Spec}(\mathbb{Z})$

(12.1b) At each geometric point $\mathrm{Spec}(\kappa)$, $\kappa=\bar{\mathbb{F}}_p$, the group scheme $\widetilde{G}_\kappa$ is reductive.

(12.1c) The dual (split) torus $\check{T}_{\mathbb{Z}}$ is a maximal torus of $\widetilde{G}_{\mathbb{Z}}$.

The properties (12.1a) and (12.1b) together with the fact that $\widetilde{G}_{\mathbb{Z}}$ is affine amount to the definition of a reductive group. The last statement (12.1c) says that $\widetilde{G}_{\mathbb{Z}}$ is split. Note that it is not necessary to check in (12.1b) that $\widetilde{G}_\kappa$ has root datum dual to that of G; that is automatic because it holds for $\widetilde{G}_{\mathbb{Q}}$. However, to prove the fact that $\widetilde{G}_{\mathbb{Z}}$ is smooth over $\mathrm{Spec}(\mathbb{Z})$ and to deal with the fact that we do not yet know that $\widetilde{G}_{\mathbb{Z}}$ is of finite type we end up having to calculate the root data of the $\widetilde{G}_\kappa$.

In what follows, we will make crucial use of results in [PY]. We will first recall their theorem 1.5:

(12.2) An affine flat group scheme over the integers with all its fibers connected reductive algebraic groups is a reductive group .

As $\mathbb{Z}[\widetilde{G}_{\mathbb{Z}}]$ is free over $\mathbb{Z}$, by proposition 11.1, we see that $\widetilde{G}_{\mathbb{Z}}$ is flat. By section 7 all of the groups $\widetilde{G}_{\mathbb{Q}_p}$ are split with their root datum dual to that of G. Thus, to prove that $\widetilde{G}_{\mathbb{Z}}$ is reductive, it suffices to show:

(12.3) The groups $\widetilde{G}_{\mathbb{Z}_p}$ are reductive .

In order to prove this, we will make use of the maximal torus, so we will now make a start at proving (12.1c). We begin by exhibiting $\check{T}_{\mathbb{Z}}$ as a sub torus of $\widetilde{G}_{\mathbb{Z}}$. We proceed as in section 7. The functor

$$F = \mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \mathbb{Z}) \to \mathrm{Mod}_{\mathbb{Z}}(X_*(T))) \tag{12.4}$$

gives us a homomorphism $\check{T}_{\mathbb{Z}} \to \widetilde{G}_{\mathbb{Z}}$. This makes $\check{T}_{\mathbb{Z}}$ a sub torus of $\widetilde{G}_{\mathbb{Z}}$ because for any cocharacter $\nu \in X_*(T)$, the ν-weight space $F_\nu(\mathcal{J}_!(\lambda, \mathbb{Z})) = H^*_c(S_\nu, \mathcal{J}_!(\lambda, \mathbb{Z}))$ is non-zero.

Let us now write $\kappa = \overline{\mathbb{F}}_p$, for p a prime, and $(\widetilde{G}_\kappa)_{red}$ for the reduced subscheme of $\widetilde{G}_\kappa$. We note that, just as in 7 we see that the group scheme $\widetilde{G}_\kappa$ is connected because $\widetilde{G}_\kappa$ has no finite quotients – there is no non-trivial tensor subcategory of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \overline{\mathbb{F}}_p)$ supported on finitely many $G_{\mathcal{O}}$-orbits. To complete the proof of Theorem 12.1 we thus must argue, in addition to (12.3), that:

(12.5) The torus $\check{T}_\kappa$ is maximal in $\widetilde{G}_\kappa$.

We will argue these two points simultaneously. A crucial ingredient will be theorem 1.2 of [PY], which we now state in a form useful to us. The formulation below has much stronger hypotheses than in [PY]. We inserted these stronger hypotheses as they are hold in case at hand to simplify the formulation. By theorem 1.2 of [PY], the group $\widetilde{G}_{\mathbb{Z}_p}$ is reductive if the following conditions hold:

(12.6a) $\widetilde{G}_{\mathbb{Z}_p}$ is affine and flat

(12.6b) $\widetilde{G}_{\mathbb{Q}_p}$ is connected and reductive

(12.6c) $(\widetilde{G}_\kappa)_{red}$ is a connected reductive group of the same type as $\widetilde{G}_{\mathbb{Q}_p}$.

As has been observed before, the first two hypotheses above are satisfied. Thus, it remains to prove (12.6c). To summarize, we are reduced to showing::

(12.7a) the torus $\check{T}_\kappa$ is maximal in $(\widetilde{G}_\kappa)_{red}$ and

(12.7b) the group scheme $(\widetilde{G}_\kappa)_{red}$ is reductive with root datum dual to that of G .

The two statements above are statements about the fiber at $\kappa = \overline{\mathbb{F}}_p$. In the rest of this section we will be working at this fiber, but before doing so, we make one more argument using the entire flat family. The flatness of $\widetilde{G}_{\mathbb{Z}_p}$ implies that

$$\dim G = \dim \widetilde{G}_{\mathbb{Q}_p} \geq \dim(\widetilde{G}_\kappa)_{red}\,. \tag{12.8}$$

To see this, let us choose algebraically independent elements in the coordinate ring of $(\widetilde{G}_{\mathbb{F}_p})_{red}$ and lift them to the coordinate ring of $\widetilde{G}_{\mathbb{Z}_p}$. By flatness, $\mathbb{Z}_p[\widetilde{G}_{\mathbb{Z}_p}] \subset \mathbb{Q}_p[\widetilde{G}_{\mathbb{Q}_p}]$ and hence these lifts remain algebraically independent in the coordinate ring of $\widetilde{G}_{\mathbb{Q}_p}$. This gives (12.8). Note that we do not a priori have an equality in (12.8), as we do not yet know that $\widetilde{G}_{\mathbb{Z}_p}$ is of finite type.

Now we are ready to work at κ. We proceed at the beginning in the same way as we did in section 7. First, we have:

$$\begin{gathered}\text{The irreducible representations of } \widetilde{G}_\kappa \text{ are} \\ \text{parameterized by dominant weights } \lambda \in X^*(\check{T}_\kappa) = X_*(T)\,.\end{gathered} \tag{12.9}$$

We see this, just as in section 7, by noting that the irreducible objects in $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \overline{\mathbb{F}}_p)$ are given by the $\mathcal{I}_{!*}(\lambda, \overline{\mathbb{F}}_p)$, for $\lambda \in X_*(T)$ dominant. Let us write, as in section 7, $L(\lambda)$ for the irreducible representation of $\widetilde{G}_\kappa$ associated to λ. First of all, because of proposition 3.10, we see that the weights of the representation $W(\lambda, \Bbbk)$ corresponding to $\mathcal{I}_!(\lambda, \Bbbk)$ are independent of $\Bbbk$. Hence, those weights are precisely the weights of the irreducible representation of $\check{G}_{\mathbb{C}}$ of highest weight λ. On the other hand, we can write $\mathcal{I}_{!*}(\lambda, \overline{\mathbb{F}}_p)$ in the Grothendieck group as a sum involving $\mathcal{I}_!(\lambda, \overline{\mathbb{F}}_p)$ and terms $\mathcal{I}_!(\mu, \overline{\mathbb{F}}_p)$, for $\mu < \lambda$. Hence, we conclude:

$$\begin{gathered}\text{The } \check{T}_\kappa\text{-weights of the irreducible representation } L(\lambda) \text{ are contained} \\ \text{in the } \check{T}\text{-weights of the irreducible representation of } \check{G}_{\mathbb{C}} \\ \text{associated to } \lambda, \text{ and } \lambda \text{ is the highest weight in } L(\lambda)\,.\end{gathered} \tag{12.10a}$$

$$\text{The weights of } L(\lambda) \text{ are symmetric under the Weyl group } W. \tag{12.10b}$$

$$\begin{gathered}\text{For } \alpha \text{ a simple positive root of } G, \text{ with the corresponding coroot} \\ \check{\alpha}, \text{ the weights of } L(\lambda) \text{ on the line segment between} \\ s_{\check{\alpha}}(\lambda) \text{ and } \lambda \text{ are all of the form } \lambda - k\check{\alpha}, \text{ with } k \text{ an integer.}\end{gathered} \tag{12.10c}$$

Note that, contrary to the case of characteristic zero, not all the weights between λ and $\lambda - k\check{\alpha}$ occur as weights of $L(\lambda)$.

Next, we approximate $\widetilde{G}_\kappa$ by finite type quotients $\widetilde{G}^*_\kappa$. For any group scheme H let us write Irr_H for the set of irreducible representations of H. We choose a quotient group scheme $\widetilde{G}^*_\kappa$ of $\widetilde{G}_\kappa$ with the following properties:

$$\widetilde{G}^*_\kappa \text{ is of finite type} \tag{12.11a}$$

(12.11b) the canonical map $\mathrm{Irr}_{\widetilde{G}^*_\kappa} \to \mathrm{Irr}_{\widetilde{G}_\kappa}$ is a bijection

(12.11c) $$(\widetilde{G}^*_\kappa)_{red} = (\widetilde{G}_\kappa)_{red}$$

It is possible to satisfy the first and third conditions because any group scheme is a projective limit of group schemes of finite type and by (12.8) the group scheme $(\widetilde{G}_\kappa)_{red}$ is of finite type. To ensure that (12.11b) is satisfied, it is enough to choose $\widetilde{G}^*_\kappa$ sufficiently large so that the irreducible representations $L(\lambda)$ associated to a finite set of generators λ of the semigroup of dominant cocharacters, are pull-backs of representations of $\widetilde{G}^*_\kappa$. For any dominant coweights λ, μ, the sheaf $\mathcal{I}_{!*}(\lambda+\mu, \overline{\mathbb{F}}_p)$ is a subquotient of the convolution product $\mathcal{I}_{!*}(\lambda, \overline{\mathbb{F}}_p) * \mathcal{I}_{!*}(\mu, \overline{\mathbb{F}}_p)$ since the support of the convolution is $\overline{\mathcal{G}\mathrm{r}}_{\lambda+\mu}$. This shows that all irreducible representations $L(\lambda)$ of $\widetilde{G}_\kappa$ come from $\widetilde{G}^*_\kappa$.

Let us write R for the reductive quotient of $(\widetilde{G}_\kappa)_{red} = (\widetilde{G}^*_\kappa)_{red}$ and note that, of course, $\check{T}_\kappa$ lies naturally in R. As any irreducible representation of $(\widetilde{G}^*_\kappa)_{red}$ is trivial on the unipotent radical we have:

(12.12) The canonical map $\mathrm{Irr}_R \to \mathrm{Irr}_{(\widetilde{G}^*_\kappa)_{red}}$ is a bijection.

We now argue that in order to prove (12.7) it suffices to show that:

(12.13a) the torus $\check{T}_\kappa$ is maximal in R

(12.13b) the root datum of R with respect to $\check{T}_\kappa$ is dual to that of G.

Let us, then, assume (12.13). We conclude immediately that $\dim(R) = \dim(G)$, and, together with (12.8), this gives

(12.14) $$\dim G = \dim \tilde{G}_{\mathbb{Q}} \geq \dim(\widetilde{G}^*_\kappa)_{red} \geq \dim R = \dim G\,.$$

Thus, we must have $(\widetilde{G}^*_\kappa)_{red} = R$ and hence $(\widetilde{G}^*_\kappa)_{red}$ is reductive with its root datum dual to that of G. Since this holds for each of approximation $\widetilde{G}^*_\kappa$ of $\widetilde{G}_\kappa$ we see that $(\widetilde{G}_\kappa)_red$ coincides with R, has $\check{T}_\kappa$ as its maximal torus, and its root datum dual to that of G. This gives (12.7).

The proof of (12.13) will be based on relating representations of $\widetilde{G}_\kappa$ and R (i.e., of $\widetilde{G}^*_\kappa$ and $(\widetilde{G}^*_\kappa)_{red}$), by considering the n^{th} powers of the Frobenius maps between the κ-scheme $\widetilde{G}^*_\kappa$ and its n^{th} Frobenius twist $(\widetilde{G}^*_\kappa)^{(n)}$, as depicted in the diagram below:

(12.15)
$$\begin{array}{ccc} \widetilde{G}^*_\kappa & \xrightarrow{\mathrm{Fr}^n_{\widetilde{G}^*_\kappa}} & (\widetilde{G}^*_\kappa)^{(n)} \\ \uparrow & & \uparrow \\ (\widetilde{G}^*_\kappa)_{red} & \xrightarrow{\mathrm{Fr}^n_{\widetilde{G}^*_\kappa}} & (\widetilde{G}^*_\kappa)^{(n)}_{red}. \end{array}$$

Since $\widetilde{G}^*_\kappa$ is of finite type we see, by [DG, corollary III.3.6.4], that it is isomorphic to $(\widetilde{G}^*_\kappa/(\widetilde{G}^*_\kappa)_{red}) \times (\widetilde{G}^*_\kappa)_{red}$ as a scheme with the right multiplication action by $(\widetilde{G}^*_\kappa)_{red}$, and

that the coordinate ring of $\widetilde{G}^*_\kappa/(\widetilde{G}^*_\kappa)_{red}$ is of the form $\kappa[X_1,...,X_n]/\langle X_1^{p_1},...,X_n^{p_n}\rangle$ for some powers $p_i = p^{e_i}$ of p. Hence, for $n \geq e_i$,

$$\text{the Frobenius map } \mathrm{Fr}^n : \widetilde{G}^*_\kappa \to (\widetilde{G}^*_\kappa)^{(n)} \text{ factors through } (\widetilde{G}^*_\kappa)^{(n)}_{red}\,. \tag{12.16}$$

This implies:

$$\begin{gathered}\text{the n}^{th}\text{ Frobenius twists of irreducible representations}\\ \text{of } (\widetilde{G}^*_\kappa)_{red} \text{ extend to } \widetilde{G}^*_\kappa\ .\end{gathered} \tag{12.17}$$

From the previous discussion we conclude, first of all, that

$$\text{We have an injection } \mathrm{Irr}_{R^{(n)}} \hookrightarrow \mathrm{Irr}_{\widetilde{G}_\kappa}\,. \tag{12.18}$$

This means that the torus $\check{T}_\kappa$ must be maximal in R, as the statement above bounds the size of the lattice of irreducible representations of R. Furthermore, we also conclude that

$$L^{\widetilde{G}_\kappa}(p^n\lambda) \;=\; L^R(p^n\lambda) \quad \text{for all } \lambda \in X_*(T) \text{ dominant}\,. \tag{12.19}$$

We will now argue the second part of (12.13), i.e., that :

$$\Delta(R,\check{T}_\kappa) = \check{\Delta}(G,T) \quad \text{ and } \quad \check{\Delta}(R,\check{T}_\kappa) = \Delta(G,T)\ . \tag{12.20}$$

The argument here is a bit more involved than the argument in characteristic zero in section 7, but the basic idea is the same: the pattern of the weights of irreducible representations determines the root datum.

The statement (12.10) expresses the patterns of weights of the representations $L^{\widetilde{G}_\kappa}(\lambda)$. The pattern of weights of the $L^R(\lambda)$ has a similar description, as R is a reductive group. Comparing these patterns for $p^n\lambda$, we conclude that that the walls of the Weyl chambers of the root systems of (G,T) and $(R,\check{T}_\kappa)$ coincide in $X_*(T)$, and furthermore, that the simple root directions of R coincide with the simple coroot directions of G. Recall that in characteristic zero we obtained an equality of simple roots of R and the simple coroots of G, but we cannot immediately conclude this fact here, as we only have a containment in (12.10c). Hence, we must argue further.

As the next step, we prove two inclusions of lattices:

$$\mathbb{Z}\cdot\Delta(R,\check{T}_\kappa) \;\subset\; \mathbb{Z}\cdot\check{\Delta}(G,T) \;\text{ and }\; \mathbb{Z}\cdot\Delta(G,T) \;\subset\; \mathbb{Z}\cdot\check{\Delta}(R,\check{T}_\kappa)\,. \tag{12.21}$$

We do so by analyzing the centers. First of all, note that the center of the reductive group R can be identified with the group scheme

$$\mathrm{Hom}(X^*(\check{T}_\kappa)/\mathbb{Z}\cdot\Delta(R,\check{T}_\kappa),\mathbb{G}_{m,\kappa})\,. \tag{12.22}$$

On the other hand, the tensor category $\mathcal{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\overline{\mathbb{F}}_p)$ is naturally equipped with a grading by the abelian group $\pi_0(\mathcal{G}\mathrm{r}) \cong \pi_1(G) = X_*(T)/\mathbb{Z}\cdot\check{\Delta}(G,T)$, the group of connected components of $\mathcal{G}\mathrm{r}$. Note that this grading is compatible with the tensor structure. This grading exhibits the group scheme

$$\mathrm{Hom}(X_*(T)/\mathbb{Z}\cdot\check{\Delta}(G,T),\mathbb{G}_{m,\kappa}) \tag{12.23}$$

as a group subscheme of the center of $\widetilde{G}_\kappa$. Since this group subscheme lies in $\check{T}_\kappa$ it also lies in the center of R. This inclusion of centers corresponds to a natural surjection

$$X^*(\check{T}_\kappa)/\mathbb{Z}\cdot\Delta(R,\check{T}_\kappa) \quad \twoheadrightarrow \quad X_*(T)/\mathbb{Z}\cdot\check{\Delta}(G,T)\,. \tag{12.24}$$

This gives the first inclusion in (12.21). Since the lattices have bases of simple (co)roots which have the same directions the second inclusion follows from $\langle\check{\alpha},\alpha\rangle = 2$. Let us note that by the same reasoning the proof of (12.20) can now be reduced to proving either of the equalities in (12.21). We will do so by building up from special cases.

First, let us observe that we are done when G is of adjoint type. In that case $\mathbb{Z}\cdot\Delta(G,T) = X^*(T)$ and so we must have $\mathbb{Z}\cdot\Delta(G,T) = \mathbb{Z}\cdot\check{\Delta}(R,\check{T}_\kappa)$. At this point we no longer have need to argue in terms of R. In what follows we will prove directly that $\widetilde{G}_\kappa$ is reductive with its root datum dual to that of G.

Next, assume that G is semi-simple and write

$$G \twoheadrightarrow G_{\mathrm{ad}}\,, \quad \text{where } G_{\mathrm{ad}} \text{ is the adjoint quotient of } G\,. \tag{12.25}$$

We have already shown that $(\widetilde{G}_{\mathrm{ad}})_\kappa \cong (\check{G}_{\mathrm{ad}})_\kappa$. Just as above, we note that we can provide the category $\mathrm{P}_{G_{\mathrm{ad},\mathcal{O}}}(\mathcal{G}\mathrm{r}_{G_{\mathrm{ad}}},\overline{\mathbb{F}}_p)$ with a grading, as a tensor category, by the finite group $\pi_1(G_{\mathrm{ad}})/\pi_1(G)$. With this grading the tensor subcategory $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}_G,\overline{\mathbb{F}}_p)$ of $\mathrm{P}_{G_{\mathrm{ad},\mathcal{O}}}(\mathcal{G}\mathrm{r}_{G_{\mathrm{ad}}},\overline{\mathbb{F}}_p)$ corresponds to the identity coset $\pi_1(G)$. Thus, we obtain a surjective homomorphism

$$(\check{G}_{\mathrm{ad}})_\kappa = (\widetilde{G}_{\mathrm{ad}})_\kappa \twoheadrightarrow \widetilde{G}_\kappa \tag{12.26}$$

with a finite central kernel, given precisely by $\mathrm{Hom}(\pi_1(G_{\mathrm{ad}})/\pi_1(G),\mathbb{G}_{m,\kappa})$. This implies that $\widetilde{G}_\kappa$ is reductive. Let us write T_{ad} for the maximal torus in G_{ad}. Then from (12.25) we see that the roots and the coroots of the pairs (G,T) and $(G_{\mathrm{ad}},T_{\mathrm{ad}})$ coincide under the surjection $T \to T_{\mathrm{ad}}$ and similarly, from (12.26), we conclude that the roots and coroots of the pairs $(\widetilde{G}_\kappa,\check{T}_\kappa)$ and $((\check{G}_{\mathrm{ad}})_\kappa,(\check{T}_{\mathrm{ad}})_\kappa)$ coincide under the surjection $(\check{T}_{\mathrm{ad}})_\kappa \to \check{T}_\kappa$. Thus, as we know the result for the adjoint group, we conclude that the root datum of $(\widetilde{G}_\kappa,\check{T}_\kappa)$ is dual to that of (G,T).

Finally, consider the case of a general reductive G. Let us write $S = Z(G)^0$ for the connected component of the center of G. Then we have an exact sequence

$$1 \to S \to G \to G_{\mathrm{der}} \to 1\,, \tag{12.27}$$

where the derived group G_{der} of G is semisimple. This gives maps:

$$\mathcal{G}\mathrm{r}_S \xrightarrow{\ i\ } \mathcal{G}\mathrm{r}_G \xrightarrow{\ \pi\ } \mathcal{G}\mathrm{r}_{G_{\mathrm{der}}}\,, \tag{12.28}$$

which exhibit $\mathcal{G}\mathrm{r}_G$ as a trivial cover of $\mathcal{G}\mathrm{r}_{G_{\mathrm{der}}}$ with fiber $\mathcal{G}\mathrm{r}_S$. By taking pushfowards of sheaves this gives us the following sequence of functors:

$$\mathrm{P}_{S_\mathcal{O}}(\mathcal{G}\mathrm{r}_S,\overline{\mathbb{F}}_p) \xrightarrow{\ \omega\ } \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}_G,\overline{\mathbb{F}}_p) \xrightarrow{\ \gamma\ } \mathrm{P}_{G_{\mathrm{der}\mathcal{O}}}(\mathcal{G}\mathrm{r}_{G_{\mathrm{der}}},\overline{\mathbb{F}}_p)\,, \tag{12.29}$$

where ω is clearly an embedding and γ is essentially surjective because of the triviality of the cover. This, in turn, gives the following exact sequence of group schemes:

$$1 \to (\widetilde{G}_{\mathrm{der}})_\kappa \to \widetilde{G}_\kappa \to \widetilde{S}_\kappa \to 1 \tag{12.30}$$

The fact that we have exactness at both ends follows from the fact ω is an embedding and γ is essentially surjective. To see the exactness in the middle, let us consider the

quotient $\widetilde{G}_\kappa/(\widetilde{G}_{\mathrm{der}})_\kappa$. The representations of this groups scheme are given by objects in $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_G,\kappa)$ whose push-forward under π to $\mathcal{G}\mathrm{r}_{G_{\mathrm{der}}}$ consists of direct sums of trivial representations. But these constitute precisely $\mathrm{P}_{S_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_S,\kappa)$. Thus, $\widetilde{G}_\kappa$ is reductive. Arguing just as in the previous step, using the fact that the roots and coroots of G and G_{der} on one hand and those of $\widetilde{G}_\kappa$ and $(\widetilde{G}_{\mathrm{der}})_\kappa$ on the other, coincide, we conclude that the root datum of $\widetilde{G}_\kappa$ is dual to that of G.

13. **Representations of reductive groups**

The point of view we have taken in this paper so far is that of giving a canonical, geometric construction of the dual group. In this section we turn things around and view our work as giving a geometric interpretation of representation theory of split reductive groups.

As before, $\Bbbk$ is commutative ring, noetherian and of finite global dimension. Recall the content of our main theorem 12.1:

$$\mathrm{Rep}_{\check{G}_\Bbbk} \quad \text{is equivalent to} \quad \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk) \quad \text{as tensor categories;} \tag{13.1}$$

here $\mathrm{Rep}_{\check{G}_\Bbbk}$ stand for the category of $\Bbbk$-representations of $\check{G}_\Bbbk$, finitely generated over $\Bbbk$, and $\check{G}_\Bbbk$ stands for the canonical split group scheme associated to the root datum dual to that of the complex group G. This way we get a geometric interpretation of representation theory of $\check{G}_\Bbbk$. The case when $\mathrm{Char}(\Bbbk)=0$ was discussed in section 7.

Following our previous discussion we have $\check{T}_\Bbbk \subset \check{B}_\Bbbk \subset \check{G}_\Bbbk$, a maximal torus and a Borel in $\check{G}_\Bbbk$. Associated to a weight $\lambda \in X_*(\check{T})$ there are two standard representations of highest weight λ. Let us describe these representations. We extend λ to a character on $\check{B}_\Bbbk$ so that it is trivial on the unipotent radical of $\check{B}_\Bbbk$ and then induce this character to a representation of $\check{G}_\Bbbk$. We call the resulting representation the Schur module and denote it by $S(\lambda)$. As a module it is free over $\Bbbk$. The other representation associated to λ is the Weyl module $W(\lambda) = S(-w_0\lambda)^*$, where w_0 is the longest element in the Weyl group. There is a canonical morphism $W(\lambda) \to S(\lambda)$ which is the identity on the λ-weight space. We have:

13.1. **Proposition.** *Under the equivalence* (13.1) *the diagrams* $W(\lambda) \to S(\lambda)$ *and* $\mathcal{J}_!(\lambda) \to \mathcal{J}_*(\lambda)$ *correspond to each other.*

Proof. The modules $S(\lambda)$ and $W(\lambda)$ can also be characterized in the following manner. Let us write $\mathrm{Rep}^{\leq\lambda}_{\check{G}_\Bbbk}$ for the full subcategory of $\mathrm{Rep}_{\check{G}_\Bbbk}$ consisting of representations whose $\check{T}_\Bbbk$-weights are all $\leq \lambda$. Then the representations $S(\lambda)$ and $W(\lambda)$ satisfy the following universal properties:

$$\text{for } V \in \mathrm{Rep}^{\leq\lambda}_{\check{G}_\Bbbk} \quad \text{we have} \quad \mathrm{Hom}_{\check{G}_\Bbbk}(V, S(\lambda)) = \mathrm{Hom}_\Bbbk(V_\lambda, \Bbbk) \tag{13.2a}$$

and

$$\text{for } V \in \mathrm{Rep}^{\leq\lambda}_{\check{G}_\Bbbk} \quad \text{we have} \quad \mathrm{Hom}_{\check{G}_\Bbbk}(W(\lambda), V) = \mathrm{Hom}_\Bbbk(\Bbbk, V_\lambda) \tag{13.2b}$$

On the geometric side the category $\mathrm{Rep}^{\leq\lambda}_{\check{G}_\Bbbk}$ corresponds to the category $\mathrm{P}_{G_{\mathcal{O}}}(\lambda,\Bbbk) = \mathrm{P}_{G_{\mathcal{O}}}(\overline{\mathcal{G}\mathrm{r}_\lambda},\Bbbk)$. Obviously, the sheaves $\mathcal{J}_*(\lambda)$ and $\mathcal{J}_!(\lambda)$ belong $\mathrm{P}_{G_{\mathcal{O}}}(\lambda,\Bbbk)$ and satisfy the universal properties (13.2), proving the proposition. □

As a corollary, proposition 3.10 gives:

13.2. **Corollary.** *The ν-weight spaces $S(\lambda)_\nu$ and $W(\lambda)_\nu$ of $S(\lambda)$ and $W(\lambda)$, respectively, can both be canonically identified with the $\Bbbk$-vector space spanned by the irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\nu$. In particular, the dimensions of these weight spaces are given by the number of irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\nu$.*

Finally, let us assume that $\Bbbk$ is a field. Then we also have an irreducible representation $L(\lambda)$ associated to λ. Under the correspondence (13.1) the irreducible representation $L(\lambda)$ corresponds to the irreducible sheaf $\mathcal{I}_{!*}(\lambda, \Bbbk)$.

An important motivation for our work was its potential application to representation theory of algebraic groups. To this end, we would like to propose the following:

13.3. **Conjecture.** *The stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ are free.*

14. Variants and the geometric Langlands program

As was stated before, in this paper we have worked with $\mathbb{C}$-schemes because in this case we have a good sheaf theory for sheaves with coefficients in any commutative ring, in particular, the integers. It is also possible to work with other topologies. This is important for certain applications, for example for the geometric Langlands program, since our results can be viewed as providing the unramified local geometric Langlands correspondence.

We will explain briefly the modifications necessary to work in the etale topology and over an arbitrary algebraically closed base field K. To this end, let us view the group G as a split reductive group over the integers. All the geometric constructions made in this paper go through over the integers, in particular, our Grassmannian $\mathcal{G}\mathrm{r}$ is defined over $\mathbb{Z}$. We write $\mathcal{G}\mathrm{r}_K$ for the affine Grassmannian over the base field K. In a few places in the paper we have argued using $\mathbb{Z}$ as coefficients, for instance, in section 12. When we work in the etale topology, we simply replace $\mathbb{Z}$ by $\mathbb{Z}_\ell$, where $\ell \neq Char(K)$.

For completeness, we state here a version of our theorem for $\mathcal{G}\mathrm{r}_K$:

14.1. **Theorem.** *There is an equivalence of tensor categories*

$$\mathrm{P}_{G(\mathcal{O})}(\mathcal{G}r_K, \Bbbk) \cong Rep(\check{G}_\Bbbk).$$

where we can take $\Bbbk$ to be any ring for which the left hand side is defined and which can be obtained by base change from $\mathbb{Z}_\ell$, for example, $\Bbbk$ could be $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, $\mathbb{Z}/\ell^n\mathbb{Z}$, $\overline{\mathbb{F}_\ell}$.

14.2. **Remark.** *The previous theorem allows one to extend the notion of Hecke eigensheaves in the geometric Langlands program from the case of characteristic zero coefficients to coefficients in an arbitrary field. This is used in* [G] *which gives a proof of de Jong's conjecture.*

Appendix A. Categories of perverse sheaves

In this appendix we prove propositions 2.1 and 2.2, i.e., we will show that

A.1. **Proposition.** *The categories $\mathrm{P}_{G_\mathcal{O} \rtimes \mathrm{Aut}(\mathcal{O})}(\mathcal{G}r, \Bbbk)$, $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk)$, and $\mathrm{P}_\mathcal{S}(\mathcal{G}r, \Bbbk)$ are naturally equivalent.*

We first note that $\mathrm{P}_{G_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathcal{O})}(\mathcal{G}\mathrm{r}, \Bbbk)$ is a full subcategory of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ and $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ is a full subcategory of $\mathrm{P}_{\mathcal{S}}(\mathcal{G}\mathrm{r}, \Bbbk)$; this follows from the fact that stabilizers of points are connected for the actions of $G_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathcal{O})$ and $G_{\mathcal{O}}$ on $\mathcal{G}\mathrm{r}$. The reductive quotients of $G_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathcal{O})$, $G_{\mathcal{O}}$, and $\mathrm{Aut}(\mathcal{O})$ are $G \times \mathbb{G}_m$, G, and $\mathbb{G}_m$, respectively. Hence, it suffices to show that for any $G_{\mathcal{O}}$-invariant finite dimensional subvariety we have:

(A.1) the categories $\mathrm{P}_{\mathcal{S},G}(Z, \Bbbk)$ and $\mathrm{P}_{\mathcal{S},\mathbb{G}_m}(Z, \Bbbk)$ are equivalent to $\mathrm{P}_{\mathcal{S}}(Z, \Bbbk)$;

here $\mathrm{P}_{\mathcal{S},G}(Z, \Bbbk)$ and $\mathrm{P}_{\mathcal{S},\mathbb{G}_m}(Z, \Bbbk)$ denote subcategories of $\mathrm{P}_{\mathcal{S}}(Z, \Bbbk)$ consisting of sheaves which are G and $\mathbb{G}_m$- equivariant, respectively.

We will proceed by induction in the following manner. Obviously, the statement above holds if Z is a $G_{\mathcal{O}}$-orbit. Hence, by induction, it suffices to prove (A.1) for a $G_{\mathcal{O}}$-invariant subset Z under the following hypotheses:

$$\begin{gathered} \text{for some dominant } \lambda, \text{ the orbit } \mathcal{G}\mathrm{r}^{\lambda} \text{ is closed in } Z \\ \text{and (A.1) holds for the open set } U = Z - \mathcal{G}\mathrm{r}^{\lambda}. \end{gathered} \tag{A.2}$$

To prove the above statement, we use the gluing construction of [MV], [V] for perverse sheaves. Let us recall the construction. We write $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F_1, G_1 : \mathcal{A} \to \mathcal{B}$ two functors, F_1 right exact, F_2 left exact, and $T : F_1 \to F_2$ a natural transformation. We define a category $\mathcal{C}(F_1, F_2; T)$ as follows. The objects of $\mathcal{C}(F_1, F_2; T)$ consist of pairs of objects $(A, B) \in \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ together with a factorization $F_1(A) \xrightarrow{m} B \xrightarrow{n} F_2(A)$ of $T(A)$, i.e., $n \circ m = T(A)$. The morphisms of $\mathcal{C}(F_1, F_2; T)$ are given by pairs of morphisms $(f, g) \in \mathrm{Mor}(\mathcal{A}) \times \mathrm{Mor}(\mathcal{B})$ which make the appropriate prism commute. The category $\mathcal{C}(F_1, F_2; T)$ is abelian.

We use this formalism in various situations. To begin with, let us write $j : U \hookrightarrow Z$ for the inclusion and set:

$$\begin{aligned} \mathcal{A} &= \mathrm{P}_{\mathcal{S}}(U, \Bbbk) \\ \mathcal{B} &= \mathrm{Mod}_{\Bbbk} \\ F_1 &= F_\lambda \circ {}^p j_! \\ F_2 &= F_\lambda \circ {}^p j_* \\ T &= F_\lambda({}^p j_! \to {}^p j_*). \end{aligned} \tag{A.3}$$

We have a functor

$$E : \mathrm{P}_{\mathcal{S}}(Z, \Bbbk) \to \mathcal{C}(F_1, F_2; T) \tag{A.4}$$

which sends $\mathcal{F} \in \mathrm{P}_{\mathcal{S}}(Z, \Bbbk)$ to $A = \mathcal{F}\big|U$, $B = F_\lambda(\mathcal{F})$ and the factorization $F_1(A) \xrightarrow{m} B \xrightarrow{n} F_2(A)$ is the one gotten by applying F_λ to ${}^p j_!(\mathcal{F}\big|U) \to \mathcal{F} \to {}^p j_*(\mathcal{F}\big|U)$. By Proposition 1.2 in [V] the functor E is an embedding. Two remarks are in order. First, in [V] we work over a field, but this is not used in the proof. Secondly, the functor E is actually an equivalence of categories.

Let us now bring in the group G. We write $\widetilde{\mathcal{S}}$ for the stratification of $G \times Z$ by subvarieties $G \times \mathcal{G}\mathrm{r}^{\lambda}$. We write $a : G \times Z \to Z$ for the action map and $p : G \times Z \to Z$ for the projection. Let $\mathcal{F} \in \mathrm{P}_{\mathcal{S}}(Z, \Bbbk)$. We have an isomorphism $\phi : p^*\mathcal{F}|G \times U \cong a^*\mathcal{F}|G \times U$ such that $\phi|\{e\} \times U = \mathrm{id}$. We are now to extend the ϕ to $G \times Z$. To this end we first construct a functor $\widetilde{F}_\lambda : \mathrm{P}_{\widetilde{\mathcal{S}}}(G \times Z, \Bbbk) \to \widetilde{\mathcal{B}}$, where $\widetilde{\mathcal{B}}$ stands for the category of $\Bbbk$-local

systems on G. Let us write $\widetilde{S}_\lambda = G \times S_\lambda$, denote by $i : \widetilde{S}_\lambda \hookrightarrow G \times Z$ the inclusion, and write $\pi : G \times Z \to G$ for the projection. Then $\widetilde{F}_\lambda = R\pi_! i^*$. Furthermore, we write $j : G \times U \hookrightarrow G \times G$ for the inclusion and set:

$$\begin{aligned} \widetilde{\mathcal{A}} &= \mathrm{P}_{\mathbb{S}}(G \times U, \Bbbk) \\ \widetilde{\mathcal{B}} &= \{\Bbbk\text{-local systems on } G\} \\ \widetilde{F}_1 &= \widetilde{F}_\lambda \circ {}^p\widetilde{j}_! \\ \widetilde{F}_2 &= \widetilde{F}_\lambda \circ {}^p\widetilde{j}_* \\ \widetilde{T} &= \widetilde{F}_\lambda({}^p\widetilde{j}_! \to {}^p\widetilde{j}_*)\,. \end{aligned} \tag{A.5}$$

As before, we have a functor

$$\widetilde{E} : \mathrm{P}_{\widetilde{\mathbb{S}}}(G \times Z, \Bbbk) \to \mathcal{C}(\widetilde{F}_1, \widetilde{F}_2; \widetilde{T}) \tag{A.6}$$

which sends $\widetilde{\mathcal{F}} \in \mathrm{P}_{\mathbb{S}}(G \times Z, \Bbbk)$ to $\widetilde{A} = \widetilde{\mathcal{F}}\big|G \times U$, $\widetilde{B} = \widetilde{F}_\lambda(\widetilde{\mathcal{F}})$ and the factorization $\widetilde{F}_1(\widetilde{A}) \xrightarrow{m} \widetilde{B} \xrightarrow{n} \widetilde{F}_2(\widetilde{A})$ is the one gotten by applying $\widetilde{F}_\lambda$ to ${}^p\widetilde{j}_!(\widetilde{\mathcal{F}}\big|G \times U) \to \widetilde{\mathcal{F}} \to {}^p\widetilde{j}_*(\widetilde{\mathcal{F}}\big|G \times U)$. By the same reasoning as above, the functor $\widetilde{E}$ is an equivalence of categories.

We will now apply $\widetilde{E}$ to $p^*\mathcal{F}$ and to $a^*\mathcal{F}$. For $\widetilde{E}(p^*\mathcal{F})$ we get the data of $E(\mathcal{F})$ at $\{e\} \times Z$ extended across G as the constant local system. For $\widetilde{E}(a^*\mathcal{F})$ we also get the extension data of $E(\mathcal{F})$ at $\{e\} \times Z$. Because, by theorem 3.6, the functors F_ν are independent of the data $T \subset B$ used in defining them, we see that the extension data for $a^*\mathcal{F}$ restricted to $\{g\} \times Z$, for any $g \in G$, is canonically identified with the extension data of $a^*\mathcal{F}$ at $\{e\} \times Z$. This gives us an identification of $\widetilde{E}(a^*\mathcal{F})$ with $\widetilde{E}(p^*\mathcal{F})$ and hence an isomorphism between $a^*\mathcal{F}$ and $p^*\mathcal{F}$. This shows the first part of A.1.

The case of the group $\mathbb{G}_m$ is even a bit simpler. Here we use the fact that $\mathbb{G}_m$-action preserves the variety S_λ and hence all the $\mathbb{G}_m$-translates of the functor F_λ are identical to F_λ.

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