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ASYMPTOTIC BEHAVIOUR OF MATRIX COEFFICIENTS OF THE DISCRETE SERIES

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Introduction. Let G be a connected semisimple Lie group with finite center. The set of equivalence classes of square-integrable representations of G is called the discrete series of G. The discrete series play a crucial role in the representation theory of G, as can be seen from Harish-Chandra's work on the Plancherel formula [8].

In the following we suppose that G has nonempty discrete series. Therefore, by a result of Harish-Chandra, G has a compact Cartan subgroup H. Let K be a maximal compact subgroup of G containing H. The K-finite matrix coefficients of discrete series representations are analytic square-integrable functions on G. Moreover, Harish-Chandra has shown that they decay rapidly at infinity on the group. For various questions of harmonic analysis on G it is useful to have a better knowledge of their rate of decay. This problem was studied by P. C. Trombi and V. S. Varadarajan [16]. They have found a simple necessary condition on a discrete series representation having the K-finite matrix coefficients with certain rate of decay. Our main result is that their condition is sufficient too. In turn, this gives a precise characterization of discrete series representations whose K-finite matrix coefficients lie in $L^p(G)$ for $1 \leq p < 2$.

To describe this result we must recall Harish-Chandra's parametrisation of the discrete series representations. Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be the Lie algebras of G, K and H, and $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ their complexifications, respectively. Denote by Φ the root system of $(\mathfrak{g}, \mathfrak{h})$. A root $\alpha \in \Phi$ is called compact if its root subspace is contained in $\mathfrak{k}$ and noncompact otherwise. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and W_k its subgroup generated by the reflections with respect to the compact roots. The Killing form of $\mathfrak{g}$ induces an inner product $(\ |\ )$ on $i\mathfrak{h}_0^*$, the space of all linear forms on $\mathfrak{h}$ which assume imaginary values of $\mathfrak{h}_0$. An element λ of $i\mathfrak{h}_0^*$ is singular if it is orthogonal to at least one root in Φ, and nonsingular otherwise. The differentials of the characters of H form a lattice Λ in $i\mathfrak{h}_0^*$. Let ρ be the half-sum of positive roots in Φ, with respect to some ordering on $i\mathfrak{h}_0^*$. Then $\Lambda + \rho$ does not depend on the choice of this ordering.

Harish-Chandra has shown that to each nonsingular $\lambda \in \Lambda + \rho$ we can attach a class π_λ of discrete series representations, π_λ is equal to π_μ if and only if λ and μ are conjugate under W_k and the discrete series are exhausted in this way [8].

Roughly speaking, our result shows that the rate of decay of K-finite matrix

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coefficients of π_λ is determined by the distance of λ from the noncompact walls. More precisely, if we put

$$k(\alpha) = \tfrac{1}{4} \sum_{\beta \in \Phi} |(\alpha \mid \beta)|, \qquad \alpha \in \Phi,$$

and denote by Ξ and σ the K-biinvariant functions defined in [8], we have the following result (The implication (ii) $\Rightarrow$ (i) is the result of Trombi and Varadarajan aluded above).

THEOREM. *For nonsingular $\lambda \in \Lambda + \rho$ and $\kappa > 0$ the following conditions are equivalent*:

(i) $|(\lambda \mid \alpha)| \geq \kappa k(\alpha)$ *for every noncompact root* $\alpha \in \Phi$,

(ii) *for every K-finite matrix coefficient c of the discrete series representation π_λ there exist positive real numbers M and q such that*

$$|c(x)| \leq M\Xi(x)^{1+\kappa}(1 + \sigma(x))^q, \qquad x \in G.$$

This result, combined with a result of Trombi and Varadarajan, implies immediately that for $1 \leq p < 2$, all K-finite matrix coefficients of π_λ are in $L^p(G)$ if and only if

$$|(\lambda \mid \alpha)| > \left(\frac{2}{p} - 1\right)k(\alpha)$$

for every noncompact root $\alpha \in \Phi$. In the most important case, for $p = 1$, this result was proved by H. Hecht and W. Schmid [11].

Our argument was highly influenced by the ideas in the paper of Hecht and Schmid and a preprint of T. Enright considering the special case of the groups $SU(n, 1)$ and $SO(2n, 1)$ [2].

The contents of the paper can be described as follows. The first part of the paper contains certain quite technical generalizations of the results of Harish-Chandra on tempered distributions on G. By generalizing the construction of the Schwartz space $\mathcal{C}(G)$ of smooth rapidly decreasing functions on G [8], we construct a series of spaces $\mathcal{C}_\gamma(G)$ for positive real γ. All these spaces contain the space $C_0^\infty(G)$ of smooth compactly supported functions on G and are contained in $\mathcal{C}_0(G) = \mathcal{C}(G)$, and the inclusions are continuous. We define the rate of growth of an invariant eigendistribution on G, and show that an invariant eigendistribution has the rate of growth $\gamma \geq 0$ if and only if it extends to a continuous linear form on $\mathcal{C}_\gamma(G)$. This enables us to find the estimates of the Fourier coefficients of invariant eigendistributions with the rate of growth $\gamma \geq 0$, which generalize a result of Harish-Chandra stating that the Fourier coefficients of a tempered invariant eigendistribution satisfy the weak inequality [8].

The second part is completely independent of the first part. We study there the asymptotic behaviour of K-finite matrix coefficients of admissible representations of G of finite length. For a fixed minimal parabolic subgroup P of G, we associate to an admissible representation of finite length a finite set of leading exponents along P which characterize the asymptotic behaviour of its K-finite matrix coefficients. By a recent, unpublished result of W. Casselman [1] we can associate to each leading exponent a nontrivial infinitesimal intertwining map into a nonunitary principal series representation induced from P. Also, we can characterize the intertwining maps associated to the leading exponents as the "minimal" ones among all nontrivial intertwining maps into the nonunitary principal series representations induced from P. An immediate consequence is that the leading exponents depend only on the irreducible constituents of an admissible representation of finite length and not on its finer structure. As we understand this is an unpublished result of Casselman.

Finally, in the last part of the paper we study the asymptotic behaviour of K-finite matrix coefficients of discrete series representations. It is easy to see that the condition (i) from Theorem is equivalent to the condition that the character of π_λ has the rate of growth $-\kappa$. Therefore, our problem can be interpreted as the study of the relation between the rate of growth of characters and asymptotics of K-finite matrix coefficients of the discrete series. Although the study of this relation in the discrete series case looks quite awkward, as we see from the first part of the paper it is quite easy to get the right estimates for the representations with characters with positive rate of growth. To exploit this fact we consider the whole problem in the setting of admissible representations of finite length. The characters determine completely the irreducible constituents of such representations, hence they determine their leading exponents. The map from characters to leading exponents behaves very simply under the tensoring with finite-dimensional representations, which "increases" the rate of growth. This enables us to reduce the problem onto the estimates obtained in the first part of the paper.

Our argument can be considered as a natural extension of the argument of Hecht and Schmid [11]. The essential difference is the use of Corollary of Theorem II. 2.1. instead of Schmid's "complete reducibility" theorem [14], which obviously fails to be true for general admissible representations of finite length. Schmid's theorem follows easily from this result, as it is shown at the end of the paper.

We would like to take this opportunity to express our sincere appreciation to Wilfried Schmid for his constant encouragement and helpful suggestions during our stay at the Institute for Advanced Study, Princeton, New Jersey in the year 1975-76. A series of his talks on discrete series at the Institute served as a catalyst in various phases of our work. Also we would like to thank Professor Harish-Chandra for showing to us his unpublished work on differential equations and asymptotics of K-finite matrix coefficients [3], [4].

Part I. Rate of growth of eigendistributions.

1. *Definition of the rate of growth. The main estimate.*

Let G be a connected semisimple Lie group with finite center. Denote by

$\mathfrak{g}_0$ its Lie algebra, by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$ and by $\mathfrak{G}$ the enveloping algebra of $\mathfrak{g}$. As usual, we consider the elements of $\mathfrak{G}$ as left-invariant differential operators on G. Under this isomorphism the center $\mathfrak{Z}$ of $\mathfrak{G}$ is isomorphic to the algebra of all biinvariant differential operators on G.

Let $C_0^\infty(G)$ be the space of all smooth, compactly supported functions on G. A distribution T on G is said to be invariant if

$$T(f^x) = T(f)$$

for every $f \in C_0^\infty(G)$ and $x \in G$, where

$$f^x(y) = f(xyx^{-1}), \qquad y \in G.$$

A distribution T is $\mathfrak{Z}$-finite if the distributions zT, $z \in \mathfrak{Z}$, span a finite-dimensional space; if T is an eigendistribution of all $z \in \mathfrak{Z}$, we say simply that T is an eigendistribution on G.

Let l = rank G and let $D(x)$ be the coefficient of t^l in det $(t + 1 - Adx)$, $x \in G$, where t is an indeterminate. As usual, G' denotes the set of all regular points of G, i.e. the complement of the set of zeros of the function D.

By a fundamental result of Harish-Chandra [5, Theorem 2.], an invariant $\mathfrak{Z}$-finite distribution Θ on G is represented as integration against a locally summable function analytic on the regular set G'. By the abuse of language we shall identify Θ with this function in the following.

It is possible to say much more about the structure of invariant $\mathfrak{Z}$-finite distributions. To state the result we need, we must describe more precisely the structure of Cartan subgroups of G. Let B be a Cartan subgroup of G, $\mathfrak{b}_0$ its Lie algebra and $\mathfrak{b}$ the complexification of $\mathfrak{b}_0$. Denote by R the root system of $(\mathfrak{g}, \mathfrak{b})$, and for every $\alpha \in R$ let η_α be the corresponding quasicharacter of B. The intersection ${}^0B$ of kernels of all homomorphisms $b \mapsto |\eta_\alpha(b)|$, $\alpha \in R$, of B into the multiplicative group of positive real numbers is the maximal compact subgroup of B. Let $\mathfrak{b}_0^-$ be the subalgebra of $\mathfrak{b}_0$ consisting of all $H \in \mathfrak{b}_0$ such that all eigenvalues of adH are real. Then B is the direct product of ${}^0B$ with the maximal vector subgroup $B^- = \exp \mathfrak{b}_0^-$ of B.

Let R^- be the set of (restricted) roots of $(\mathfrak{g}_0, \mathfrak{b}_0^-)$. The multiplicity m_a of a restricted root $a \in R^-$ is Card $\{\alpha \in R; \alpha \mid \mathfrak{b}_0^- = a\}$. Fixing a Weyl chamber $\mathfrak{D}$ in $\mathfrak{b}_0^-$ determines an order on R^-, we put

$$\rho_{\mathfrak{D}} = \tfrac{1}{2} \sum_{a>0} m_a \cdot a.$$

The Killing form B of $\mathfrak{g}$ defines a norm on $\mathfrak{b}_0^-$ by $||H|| = B(H, H)^{1/2}$, $H \in \mathfrak{b}_0^-$.

By [5, Lemma 31] the function $b \mapsto |D(b)|^{1/2}|\Theta(b)|$ extends from $({}^0B \cdot \exp \mathfrak{D}) \cap G'$ to a continuous function on ${}^0B \cdot \exp \mathfrak{D}$. Another result of Harish-Chandra [20, 8.3.3.3] implies that this function can be majorized by an exponential function on ${}^0B \cdot \exp \mathfrak{D}$. All this shows that the following definition is natural.

We say that an invariant $\mathfrak{Z}$-finite distribution Θ on G has *the rate of growth* γ, $\gamma \in \mathbf{R}$, if for every Cartan subgroup B and Weyl chamber $\mathfrak{D}$ in $\mathfrak{b}_0^-$ there exist positive numbers M and m such that

$$|D(b_0 \cdot \exp H)|^{1/2} |\Theta(b_0 \cdot \exp H)| \leq M e^{\gamma \rho_{\mathfrak{D}}(H)} (1 + \|H\|)^m$$

for every regular $b = b_0 \exp H$, $b_0 \in {}^0B$ and $H \in \mathfrak{D}$.

By the finiteness of number of conjugacy classes of Cartan subgroups in G, it is obvious that M and m can be chosen independent of B and $\mathfrak{D}$.

Let K be a maximal compact subgroup of G. Denote by $\hat{K}$ the set of all equivalence classes of irreducible finite-dimensional representations of K. For each $\delta \in \hat{K}$, let ξ_δ be the character of δ and $d(\delta)$ the degree of δ. Put $\chi_\delta = d(\delta)\xi_\delta$. For a distribution T on G, the convolution $T_\delta = T * \chi_\delta$ is called the δth-Fourier component of T [20, 4.4.3.]. The Fourier components of invariant $\mathfrak{Z}$-finite distributions are K-finite and $\mathfrak{Z}$-finite, therefore they are analytic functions on G [20, 8.3.9].

Let Ξ and σ be the K-biinvariant functions on G as defined in [8]. Then we have the following theorem which relates the rate of growth of invariant $\mathfrak{Z}$-finite distributions and their Fourier components.

THEOREM 1. *Let Θ be an invariant $\mathfrak{Z}$-finite distribution on G with the rate of growth γ, $\gamma \geq 0$. Then for every $\delta \in \hat{K}$ the Fourier component Θ_δ of Θ is an analytic function on G satisfying*

$$|\Theta_\delta(x)| \leq C_\delta \Xi(x)^{1-\gamma} (1 + \sigma(x))^{r_\delta}, \qquad x \in G,$$

where C_δ and r_δ are positive real numbers.

The condition that Θ has the rate of growth 0 is equivalent to being tempered. Therefore the above result can be considered as a generalization of the result of Harish-Chandra stating that the Fourier components of tempered invariant $\mathfrak{Z}$-finite distributions satisfy the weak inequality. For invariant $\mathfrak{Z}$-finite distributions with negative rate of growth our theorem does not give more information than this result of Harish-Chandra.

The results of [5] imply easily that such distributions exist only in the case when rank G = rank K, and that their supports contain the set of all elliptic elements in G. Examples of such invariant eigendistributions are the discrete series characters. In fact, the problem of finding the asymptotic behaviour of K-finite matrix coefficients of the discrete series representations can be interpreted as finding an analogue of the above theorem for all real γ.

Obviously the above theorem is no longer true if we suppress the condition $\gamma \geq 0$. As was pointed out to us by W. Schmid the differences of certain discrete series characters for $SL(2, \mathbf{R})$ are supported on the elliptic set only. These invariant eigendistributions satisfy our definition for all real γ, but their Fourier components satisfy inequalities of the above type for negative γ with sufficiently small absolute values only.

We shall show in Part III that if we suppose that Θ is the character of an admissible representation of finite length, the analogue of the above result is true for all real γ. In fact, this will prove the sufficiency of the Trombi-Varadarajan's condition.

The proof of the above theorem is a generalization of Harish-Chandra's reasoning in the case of tempered invariant $\mathfrak{Z}$-finite distributions. In analogy with the Schwartz space $\mathcal{C}(G)$ of Harish-Chandra [8], we introduce a family of spaces $\mathcal{C}_\gamma(G)$, $\gamma \geq 0$, with the property that an invariant $\mathfrak{Z}$-finite distribution extends to a continuous linear form on $\mathcal{C}_\gamma(G)$ if and only if it has the rate of growth γ. This allows us to reduce the proof of Theorem 1 on the characterization of growth of K-finite and $\mathfrak{Z}$-finite distributions which extend continuously to $\mathcal{C}_\gamma(G)$.

The main vehicle for extending the arguments of Harish-Chandra onto our situation are some quite technical estimates collected in the next section. The fundamental role in this estimates is played by a function on G which attaches to each element of G its Hilbert-Schmidt norm in a suitable finite-dimensional representation of G. The use of this function was suggested to us by W. Schmid.

2. *Some simple estimates.* Let $\mathfrak{k}_0$ be the Lie algebra of the maximal compact subgroup K of G. It determines a Cartan involution ϑ of $\mathfrak{g}_0$, we denote by $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ the corresponding Cartan decomposition. The Cartan involution ϑ is the differential of an involutive automorphism of G which we denote by the same letter.

Let $\mathfrak{a}_0$ be a maximal abelian subalgebra in $\mathfrak{p}_0$, and $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{a}_0$. Denote by $\mathfrak{h}$ the complexification of $\mathfrak{h}_0$, and by R the root system of $(\mathfrak{g}, \mathfrak{h})$. Let R^- be the corresponding system of restricted roots of $(\mathfrak{g}_0, \mathfrak{a}_0)$. Fixing a Weyl chamber $\mathcal{C}$ in $\mathfrak{a}_0$ determines an ordering of roots in R^-, we equip R with a compatible ordering such that for $\alpha \in R$, $\alpha > 0$ and $\alpha \mid \mathfrak{a}_0 \neq 0$ imply that $\alpha \mid \mathfrak{a}_0 > 0$. Let ρ be the half-sum of the positive roots in R. Then $\rho \mid \mathfrak{a}_0 = \rho_{\mathcal{C}}$.

Let τ be the irreducible finite-dimensional representation of $\mathfrak{g}$ with the highest weight ρ. Since $G = K \cdot \exp \mathfrak{p}_0$, we can define a function ν on G by

$$\nu(k \cdot \exp X) = (\operatorname{tr}(\exp 2\tau(X)))^{1/2}, \qquad k \in K, \qquad X \in \mathfrak{p}_0 .$$

Obviously, ν is a positive, smooth, K-biinvariant function on G.

Passing eventually to a suitable covering of G, we can assume that G is acceptable [5]. Then there exist a complexification $G^{\mathbf{C}}$ of G and a holomorphic representation of $G^{\mathbf{C}}$ the differential of which is τ. We denote this representation by τ again. Denote by U the maximal compact subgroup of $G^{\mathbf{C}}$ with the Lie algebra $\mathfrak{k}_0 + i\mathfrak{p}_0$. There exists a unitary structure on the space W of the representation τ such that all operators $\tau(u)$, $u \in U$, are unitary. For a linear operator T on W, denote by T^* its adjoint. Then for $x = k \cdot \exp X$, $k \in K$, $X \in \mathfrak{p}_0$, we have

$$\tau(x)^* = \tau(\exp X)^* \tau(k)^* = \tau(\exp X)\tau(k^{-1}) = \tau(\vartheta(x)^{-1}).$$

If we denote by $T \mapsto |||T|||$ the Hilbert-Schmidt norm on the algebra of linear operators on W, we have finally

$$|||\tau(x)||| = (\mathrm{tr}\ (\tau(x)^*\tau(x)))^{1/2} = (\mathrm{tr}\ (\exp 2\tau(X)))^{1/2} = \nu(x)$$

for every $x \in G$. Hence, passing to a suitable covering, we can interpret ν as a norm of certain finite-dimensional representation.

As we remarked before, the use of function ν was suggested to us by W. Schmid. The following result relates the asymptotic behaviour of ν to that of Ξ and σ.

LEMMA 1. *There exist $M > 0$ and $d \geq 0$ such that*

$$\frac{1}{\nu(x)} \leq \Xi(x) \leq M\,\frac{(1 + \sigma(x))^d}{\nu(x)}\,,$$

for all $x \in G$.

Proof. If we denote by A^+ the closure of $\exp \mathfrak{C}$ in A, we have the well-known decomposition $G = KA^+K$. By the K-biinvariance of Ξ, σ and ν it is enough to prove the inequalities on A^+. Harish-Chandra has shown that there exist $c > 0$ and $d \geq 0$ such that

$$1 \leq e^{\rho_{\mathfrak{C}}(H)}\Xi(\exp H) \leq c(1 + ||H||)^d$$

for every $H \in \bar{\mathfrak{C}}$ [20, 8.3.7]. This result, combined with the obvious inequality

$$e^{\rho_{\mathfrak{C}}(H)} \leq \nu(\exp H) \leq (\dim W)^{1/2}e^{\rho_{\mathfrak{C}}(H)}\,, \qquad H \in \bar{\mathfrak{C}}$$

implies immediately our assertion. Q.E.D.

Another obvious consequence of the definition of ν in terms of the Hilbert-Schmidt norm is the following lemma.

LEMMA 2. *The function ν is submultiplicative on G,* i.e.

$$\nu(xy) \leq \nu(x)\nu(y)$$

for all x, $y \in G$.

The importance of the function ν in our study of invariant $\mathfrak{Z}$-finite distributions is based on its simple behaviour on the conjugacy classes in G.

For an element $x \in G$ let $C(x)$ be its conjugacy class in G.

LEMMA 3. *Let B be a ϑ-stable Cartan subgroup in G and $b \in B$. Then ν attains in b its minimal value on $C(b)$,* i.e.

$$\nu(b) = \min_{x \in C(b)} \nu(x).$$

Proof. The Cartan subgroup B is ϑ-stable, therefore $b \in B$ implies $\vartheta(b)^{-1} \in B$. The relation $\tau(x)^* = \tau(\vartheta(x)^{-1})$, $x \in G$, implies that $\tau(b)$ and $\tau(b)^*$ are elements of the Cartan subgroup $\tau(B)$ of $\tau(G)$. The group $\tau(G)$ is linear and its Cartan subgroups are abelian [20, 1.4.1.5]. Therefore, $\tau(b)$ commutes with $\tau(b)^*$, i.e. it is a normal operator on W.

To finish the proof we need to recall a simple result about the equivalence classes of normal operators on finite-dimensional Hilbert spaces. We include its proof for the sake of completeness.

LEMMA 4. *Let T be a normal operator on a finite-dimensional Hilbert space W. Then*

$$|||T||| \leq |||STS^{-1}|||$$

for every invertible linear operator S on W.

Proof. Let $C(T)$ be the equivalence class containing T. It is closed in the algebra of all linear operators on W. Therefore, there exist $T_0 \in C(T)$ such that $|||T_0||| \leq |||STS^{-1}|||$ for all invertible S. Obviously, this implies that the smooth function

$$F_R : t \mapsto |||e^{tR}T_0e^{-tR}|||$$

has a minimum for $t = 0$ for all linear operators R on W. By a direct calculation

$$0 = F_R'(0) = \text{tr}\ ([T_0\ ,\ T_0^*](R + R^*)).$$

This implies that $[T_0\ ,\ T_0^*] = 0$, hence T_0 is a normal operator. Since two normal operators are equivalent if and only if they are unitary equivalent it follows that $|||T||| = |||T_0|||$. Q.E.D.

Finally, we need a result about the behaviour of σ on conjugacy classes in G due to Harish-Chandra [9, Lemma 12.1]. Our proof is based on Lemma 3.

LEMMA 5. *Let B be a ϑ-stable Cartan subgroup in G. There exist $M > 0$ such that for every $b \in B$*

$$1 + \sigma(b) \leq M \min_{x \in C(b)} (1 + \sigma(x)).$$

Proof. For every $H \in \bar{\mathfrak{C}}$, $H \neq 0$, we have $\rho_{\mathfrak{C}}(H) > 0$. This implies that there exist $M_1\ ,\ M_2 > 0$ such that

$$\frac{1}{M_1}\ ||H|| \leq \rho_{\mathfrak{C}}(H) \leq M_2\ ||H||$$

for every $H \in \bar{\mathfrak{C}}$. Therefore, there exists $M > 0$ such that

$$1 + ||H|| \leq 1 + M_1 \log \nu(\exp H) \leq M(1 + ||H||),$$

for every $H \in \bar{\mathfrak{C}}$. Using K-biinvariance and the decomposition $G = KA^+K$ we have finally,

$$1 + \sigma(x) \leq 1 + M_1 \log \nu(x) \leq M(1 + \sigma(x))$$

for all $x \in G$. This inequality, combined with Lemma 3., implies our assertion. Q.E.D.

3. *The spaces* $\mathcal{C}_\gamma(G)$. Let $\mathfrak{D}$ be the algebra of all differential operators on G. Let $\mathfrak{D}_l$ and $\mathfrak{D}_r$ denote the subalgebras consisting of left-invariant and right-invariant operators respectively. Let $\mathfrak{D}_0$ be the subalgebra of $\mathfrak{D}$ generated by $\mathfrak{D}_l \cup \mathfrak{D}_r$.

For a smooth function f on G, $\gamma \geq 0$, $D \in \mathfrak{D}_0$, and $r \in \mathbf{R}$, put

$$p_{D,r}{}^\gamma(f) = \sup_{x \in G} (1 + \sigma(x))^r \Xi(x)^{-(1+\gamma)} \, |(Df)(x)|.$$

Let $\mathcal{C}_\gamma(G)$ be the space of all smooth functions f such that $p_{D,r}{}^\gamma(f) < \infty$ for all $D \in \mathfrak{D}_0$ and $r \in \mathbf{R}$. We topologize $\mathcal{C}_\gamma(G)$ by means of the set of semi-norms $p_{D,r}{}^\gamma$, $D \in \mathfrak{D}_0$, $r \in \mathbf{R}$.

In the case $\gamma = 0$, the space $\mathcal{C}_0(G)$ is the Schwartz space $\mathcal{C}(G)$ of Harish-Chandra.

We list some of the properties of these spaces in the following theorem.

THEOREM 1. (i) *$\mathcal{C}_\gamma(G)$ is a Fréchet algebra under convolution.*

(ii) *The regular representations of G on $\mathcal{C}_\gamma(G)$ are differentiable.*

(iii) *The inclusion mapping of $C_0{}^\infty(G)$ into $\mathcal{C}_\gamma(G)$ is continuous and the image is dense in $\mathcal{C}_\gamma(G)$.*

(iv) *If $0 \leq \gamma_1 \leq \gamma_2$, $\mathcal{C}_{\gamma_2}(G)$ is a dense subspace of $\mathcal{C}_{\gamma_1}(G)$ and the inclusion mapping is continuous.*

The facts that $\mathcal{C}_\gamma(G)$ are Fréchet spaces and that inclusions in (iii) and (iv) are continuous are obvious. The density of $C_0{}^\infty(G)$ in $\mathcal{C}_\gamma(G)$ follows as in the case of $\mathcal{C}(G)$ [9, §15]. This completes the proof of (iii) and (iv). The statement (ii) follows by trivial modifications of the argument of Harish-Chandra in the case of $\mathcal{C}(G)$ [7, §10].

It remains to check that $\mathcal{C}_\gamma(G)$ are topological algebras under convolution. This is an immediate consequence of the following lemma, which generalizes Lemma 14.2 of [9] onto our situation.

We fix $r \geq 0$ such that

$$\int_G \Xi(x)^2 (1 + \sigma(x))^{-r} \, dx < \infty,$$

it exists by [8, §6].

LEMMA 1. *There exists $C > 0$ such that for any two real numbers p and q such that $|p| + q \leq -r - 2\gamma d$, we have*

$$\int_G \Xi(y)^{1+\gamma} \Xi(y^{-1}x)^{1+\gamma} (1 + \sigma(y))^p (1 + \sigma(y^{-1}x))^q \, dy \leq C \cdot \Xi(x)^{1+\gamma} (1 + \sigma(x))^{p+\gamma d}.$$

Proof. By Lemmas 2.1 and 2.2 we have

$$\Xi(y)^\gamma \Xi(y^{-1}x)^\gamma \leq M^{2\gamma} \frac{(1 + \sigma(y))^{\gamma d} (1 + \sigma(y^{-1}x))^{\gamma d}}{\nu(y)^\gamma \nu(y^{-1}x)^\gamma}$$

$$\leq M^{2\gamma} \frac{(1 + \sigma(y))^{\gamma d} (1 + \sigma(y^{-1}x))^{\gamma d}}{\nu(x)^\gamma} \leq M^{2\gamma} \Xi(x)^\gamma (1 + \sigma(y))^{\gamma d} (1 + \sigma(y^{-1}x))^{\gamma d}.$$

Therefore

$$\int_G \Xi(y)^{1+\gamma}\Xi(y^{-1}x)^{1+\gamma}(1+\sigma(y))^p(1+\sigma(y^{-1}x))^q\,dy$$

$$\leq M^{2\gamma}\Xi(x)^\gamma \int_G \Xi(y)\Xi(y^{-1}x)(1+\sigma(y))^{p+\gamma d}(1+\sigma(y^{-1}x))^{q+\gamma d}\,dy,$$

where $|p+\gamma d| + q + \gamma d \leq |p| + q + 2\gamma d \leq -r$. Our assertion follows by applying now Lemma 14.2 of [9]. Q.E.D.

The subspace of all K-biinvariant functions in $\mathcal{C}_\gamma(G)$ was studied closely in [15].

By Theorem 1 every continuous linear form on $\mathcal{C}_\gamma(G)$ defines a distribution on G. Conversely we say that a distribution is of type γ if it extends to a continuous linear form on $\mathcal{C}_\gamma(G)$.

A distribution T on G is said to be K-finite if both left and right translates of T by elements of K span a finite-dimensional space. We recall that a K-finite and $\mathfrak{Z}$-finite distribution is an analytic function on G [20, 8.3.9.].

We now characterize K-finite and $\mathfrak{Z}$-finite distributions of type γ. In the case $\gamma = 0$ this result reduces onto the result of Harish-Chandra that a K-finite and $\mathfrak{Z}$-finite distribution is tempered if and only if it satisfies the weak inequality [9, Theorem 14.1], [20, 8.3.8.7.]. Our argument is a simple modification of Harish-Chandra's.

THEOREM 2. *Let T be a K-finite and $\mathfrak{Z}$-finite distribution on G. Then the following conditions are equivalent*

(i) *T is of type γ,*

(ii) *there exist $M > 0$ and $m \geq 0$ such that*

$$|T(x)| \leq M\Xi(x)^{1-\gamma}(1+\sigma(x))^m, \qquad x \in G.$$

Proof. Obviously (ii) implies (i).

Suppose that T is of type γ. By repeating the reasoning of Harish-Chandra (see the proof of [9, Theorem 14.1] or [20, 8.3.8.7.]), we show that there exists a positive real number $m \geq \gamma d$ such that

$$c = \int_G |T(x)|\,\Xi(x)^{1+\gamma}(1+\sigma(x))^{-m+\gamma d}\,dx < \infty.$$

By Theorem 1 of [7] there exist a function $f \in C_0^\infty(G)$ such that $T = T * f$. Hence

$$|T(y)| = \left|\int_G T(yx)f(x^{-1})\,dx\right|$$

$$\leq c_1 \cdot \int_G |T(yx)|\,\Xi(x)^{1+\gamma}(1+\sigma(x))^{-m}\,dx,$$

where $c_1 = \sup_{x\in G} \Xi(x)^{-(1+\gamma)}(1+\sigma(x))^m\,|f(x^{-1})|$.

By Lemmas 2.1 and 2.2 it follows that

$$\frac{\Xi(y^{-1}x)}{(1+\sigma(y^{-1}x))^d} \leq \frac{M}{\nu(y^{-1}x)} \leq \frac{M\nu(y)}{\nu(x)} \leq M^2(1+\sigma(y))^d \frac{\Xi(x)}{\Xi(y)}, \qquad x, y \in G.$$

Therefore

$$|T(y)| \leq c_1 \cdot \int_G |T(x)|\, \Xi(y^{-1}x)^{1+\gamma}(1+\sigma(y^{-1}x))^{-m}\, dx$$

$$\leq c_2 \cdot (1+\sigma(y))^{\gamma d}\Xi(y)^{-\gamma} \int_G |T(x)|\, \Xi(y^{-1}x)\Xi(x)^{\gamma} \cdot (1+\sigma(y^{-1}x))^{-m+\gamma d}\, dx,$$

where $c_2 = c_1 \cdot M^{2\gamma}$. By the subadditivity of the function σ [7, Lemma 10.], we have

$$\frac{1}{1+\sigma(y^{-1}x)} \leq \frac{1+\sigma(y)}{1+\sigma(x)}, \qquad x, y \in G,$$

what implies

$$|T(y)| \leq c_2 \cdot (1+\sigma(y))^m \Xi(y)^{-\gamma} \int_G |T(x)|\, \Xi(y^{-1}x)\Xi(x)^{\gamma} \cdot (1+\sigma(x))^{-m+\gamma d}\, dx.$$

Now T is K-finite, so we can choose a finite subset $F \subset \hat{K}$ such that

$$T(x) = \int_K \overline{\chi_F(k)}\, T(k^{-1}x)\, dk$$

where $\chi_F = \sum_{\delta \in F} \chi_\delta$ and dk is the normalized Haar measure on K. Therefore,

$$|T(y)| \leq c_3(1+\sigma(y))^m \Xi(y)^{-\gamma} \int_{G\times K} |T(k^{-1}x)|\, \Xi(x)^{\gamma} \cdot (1+\sigma(x))^{-m+\gamma d}\Xi(y^{-1}x)\, dx\, dk$$

where $c_3 = c_2 \cdot \sup_{k\in K} |\chi_F(k)|$. Finally,

$$|T(y)| \leq c_3 \cdot (1+\sigma(y))^m \Xi(y)^{-\gamma} \int_G |T(x)|\, \Xi(x)^{\gamma}(1+\sigma(x))^{-m+\gamma d} \cdot \int_K \Xi(y^{-1}kx)\, dk\, dx,$$

and by using the doubling principle [9, §10]

$$|T(y)| \leq c_3(1+\sigma(y))^m \Xi(y)^{1-\gamma} \int_G |T(x)|\, \Xi(x)^{1+\gamma}(1+\sigma(x))^{-m+\gamma d}\, dx$$

$$= M(1+\sigma(y))^m \Xi(y)^{1-\gamma}$$

where $M = c_3 \cdot c$. Q.E.D.

Finally, we have the following important characterization of invariant $\mathfrak{Z}$-finite distributions of type γ.

THEOREM 3. *Let Θ be an invariant $\mathfrak{Z}$-finite distribution on G. Then the following conditions are equivalent for $\gamma \geq 0$*

(i) *Θ has the rate of growth γ,*

(ii) *there exist* $M > 0$ *and* $m \geq 0$ *such that*

$$|D(x)|^{1/2}\,|\Theta(x)| \leq M\Xi(x)^{-\gamma}(1 + \sigma(x))^m, \qquad x \in G',$$

(iii) Θ *is of type* γ.

Proof. Suppose that Θ has the rate of growth γ. By the definition, there exist $c > 0$ and $k \geq 0$ such that for every Cartan subgroup B of G and Weyl chamber $\mathfrak{D}$ in $\mathfrak{b}_0^-$, we have

$$|D(b_0 \cdot \exp H)|^{1/2}\,|\Theta(b_0 \cdot \exp H)| \leq c \cdot e^{\gamma \rho_{\mathfrak{D}}(H)}(1 + \|H\|)^k$$

for all regular $b = b_0 \cdot \exp H$, $b_0 \in {}^0B$, $H \in \mathfrak{D}$. We can choose a finite number of ϑ-stable Cartan subgroups B_i, $1 \leq i \leq p$, such that G' is the disjoint union of

$$G_i = \bigcup_{x \in G} x B_i' x^{-1}, \qquad 1 \leq i \leq p,$$

where $B_i' = B_i \cap G'$ [20, 1.4.1.7.]. Then above relations immediately imply that

$$|D(b)|^{1/2}\,|\Theta(b)| \leq c\nu(b)^{\gamma}(1 + \sigma(b))^k, \qquad b \in B_i',$$

and by Lemmas 2.3 and 2.5 there exists $c_1 > 0$ such that

$$|D(x)|^{1/2}\,|\Theta(x)| \leq c_1\nu(x)^{\gamma}(1 + \sigma(x))^k, \qquad x \in G'.$$

Now Lemma 2.1 implies that Θ satisfies (ii).

Since there exists $q \geq 0$ such that

$$\int_G |D(x)|^{-1/2}\,\Xi(x)(1 + \sigma(x))^{-q}\,dx < \infty$$

[20, 8.3.7.6.], an invariant $\mathfrak{Z}$-finite distribution satisfying (ii) is of type γ.

It remains to prove that Θ has the rate of growth γ if it is of type γ. The proof of this fact is a simple modification of the argument of Harish-Chandra proving that a tempered invariant $\mathfrak{Z}$-finite distribution has the rate of growth 0 [7, §19]. Passing eventually to a covering of G, we can suppose that G is acceptable. Reasoning as in the proof of Lemma 30 in [7] we can show that an invariant distribution of type γ is continuous with respect to the topology defined on $\mathcal{C}_\gamma(G)$ by the set of semi-norms $p_{D,r}{}^{\gamma}$, $D \in \mathfrak{D}_l$, $r \in \mathbf{R}$. Then we can modify the remaining part of Harish-Chandra's argument, as it was done on pages 273–274 in [16] (our situation corresponds to $\kappa = -\gamma$ in the argument of Trombi and Varadarajan, while they suppose that $\kappa > 0$, but this causes no difficulties). The modification being purely formal, we shall spare the details here. Q.E.D.

The statement of Theorem 1.1 is an immediate consequence of Theorems 2 and 3.

Part II. Asymptotics of admissible representations of finite length.

1. *Category of admissible representations.*

It is the purpose of this section to collect various preliminaries about representations of semisimple Lie groups, and to establish some notation.

Let G be a connected semisimple Lie group with finite center. Let K be a maximal compact subgroup of G. Denote by $\hat{K}$ the set of all equivalence classes of finite-dimensional irreducible representations of K. For each $\delta \in \hat{K}$, let ξ_δ and $d(\delta)$ denote the character and the degree of the elements of δ respectively, and put $\chi_\delta = d(\delta)\xi_\delta$. We denote by dk the normalized Haar measure on K.

Let π be a continuous representation of G on a Banach space V. For every $\delta \in \hat{K}$, the map $P_\delta = \int_K \overline{\chi_\delta(k)}\pi(k)\, dk$ is a continuous projection in V. Its range V_δ is the space of vectors in V the linear span of whose K-orbit is finite-dimensional and splits into irreducible K-modules of type δ. We say that π is an admissible representation if all V_δ, $\delta \in \hat{K}$, are finite-dimensional.

Let $\mathfrak{g}_0$ be the Lie algebra of G, $\mathfrak{g}$ its complexification and $\mathfrak{G}$ the enveloping algebra of $\mathfrak{g}$. For an admissible representation π on a Banach space V denote by $\mathcal{V}$ the space of all K-finite vectors in V. Then π defines a structure of compatible $(\mathfrak{G}, K)$-module on $\mathcal{V}$ [13]. We denote the corresponding infinitesimal representation of $\mathfrak{G}$ on $\mathcal{V}$ also by π.

Let π_1 and π_2 be two admissible representations of G on Banach spaces V_1 and V_2, and let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the corresponding $(\mathfrak{G}, K)$-modules of K-finite vectors in V_1 and V_2 respectively. We denote by $\mathrm{Hom}_{(\mathfrak{G},K)}(\pi_1, \pi_2)$ the set of all $(\mathfrak{G}, K)$-morphisms of $\mathcal{V}_1$ into $\mathcal{V}_2$. Elements of $\mathrm{Hom}_{(\mathfrak{G},K)}(\pi_1, \pi_2)$ are infinitesimal intertwining operators of π_1 with π_2.

In the following we shall consider the category of admissible representations of G, whose objects are admissible representations and morphisms infinitesimal intertwining operators. If two representations are isomorphic in this category we say that they are infinitesimally equivalent.

An admissible representation π of V is of finite length if there exists a chain

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

of closed invariant subspaces of V such that the representations on V_j/V_{j-1}, $j = 1, 2, \cdots, n$, are irreducible.

Let P be a minimal parabolic subgroup of G. Denote by N its unipotent radical. If ϑ is the Cartan involution of G determined by K, $L = P \cap \vartheta(P)$ is a Levi subgroup of P and $P = LN$. If A is the split component of L and $M = L \cap K$, we have the Langlands decomposition $P = MAN$ and the Iwasawa decomposition $G = KAN$ [9, §4]. Let $\mathfrak{k}_0$, $\mathfrak{m}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$ be the Lie algebras of K, M, A, N respectively, and let $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ be their complexifications. Let Σ be the system of (restricted) roots of $(\mathfrak{g}_0, \mathfrak{a}_0)$. For a root $\alpha \in \Sigma$ put $\mathfrak{g}_0{}^\alpha = \{X \in \mathfrak{g}_0;\ [H, X] = \alpha(H)X, H \in \mathfrak{a}_0\}$. Denote by Σ^+ the set of positive roots $\alpha \in \Sigma$ such that $\mathfrak{g}_0{}^\alpha \subset \mathfrak{n}_0$. Let $\rho_P = \frac{1}{2}\,\mathrm{tr\ ad} \mid \mathfrak{n}_0$.

Let $\log : A \to \mathfrak{a}_0$ be the inverse of the exponential map $\exp : \mathfrak{a}_0 \to A$. Denote by $\mathfrak{a}^*$ the space of all complex linear forms on $\mathfrak{a}$. For every unitary representation σ of M on a finite-dimensional Hilbert space U and $\lambda \in \mathfrak{a}^*$ let $\mathcal{H}_{P,\sigma,\lambda}$ be the

Hilbert space of all classes of functions $f : G \to U$ such that

1) $f(manx) = \sigma(m)e^{(\lambda+\rho_P)(\log a)}f(x)$ for all $m \in M$, $a \in A$, $n \in N$ and $x \in G$,

2) $\int_K ||f(k)||^2 \, dk < \infty$

with the inner product

$$(f \mid g) = \int_K (f(k) \mid g(k)) \, dk.$$

We define a representation $\pi_{P,\sigma,\lambda}$ of G on $\mathcal{H}_{P,\sigma,\lambda}$ by

$$(\pi_{P,\sigma,\lambda}(x)f)(y) = f(yx), \qquad x, y \in G.$$

By the general results of Harish-Chandra it follows that $\pi_{P,\sigma,\lambda}$ are admissible representations of finite length.

Let $\hat{M}$ be the set of all equivalence classes of irreducible unitary representations of M. The representations $\pi_{P,\sigma,\lambda}$, $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}^*$, are called the nonunitary principal series (associated to P). They are unitary if and only if $a \mapsto e^{\lambda(\log a)}$ is a unitary representation of A, i.e. if λ attains pure imaginary values on $\mathfrak{a}_0$.

2. *Leading exponents and infinitesimal intertwining operators with nonunitary principal series.*

Let π be an admissible representation of finite length of G on a Banach space V. In this section we shall describe a close relationship between the asymptotic behaviour of the matrix coefficients of π and the infinitesimal intertwining operators of π with the nonunitary principal series.

To formulate this result we must recall some old, unpublished results of Harish-Chandra [4] which are recently reproduced in [20]. Although these results were formulated and proved for irreducible representations only, and we need to apply them on admissible representations of finite length, the extension is purely formal and we shall spare the proofs here. These results have been recently reinvestigated by W. Casselman in an unpublished work [1].

Let G, K and P be as in Section 1. We denote by $\mathcal{C}$ the "positive" Weyl chamber associated to Σ^+. Let $\mathcal{C}^- = -\mathcal{C}$ and A^- the closure of $\exp \mathcal{C}^-$ in A. Then we have the Cartan decomposition $G = KA^-K$.

If v is a K-finite vector in V and $\tilde{v}$ a continuous K-finite linear form on V, the matrix coefficient

$$c_{v,\tilde{v}}(x) = \langle \pi(x)v, \tilde{v} \rangle$$

is an analytic function on G. The asymptotic behaviour of $c_{v,\tilde{v}}$ is determined by the behaviour of its restriction to A^- because of the above decomposition.

Let $\mathfrak{Z}$ be the center of the enveloping algebra $\mathfrak{G}$. The finite length of π implies that $I = \ker(\pi \mid \mathfrak{Z})$ is an ideal of finite codimension in $\mathfrak{Z}$. If we consider, as usual, $\mathfrak{G}$ as the algebra of all left invariant differential operators on G, I becomes an ideal of finite codimension in the algebra of all biinvariant differential operators on G, and the matrix coefficients $c_{v,\tilde{v}}$ of π satisfy the system of differential equations $Dc_{v,\tilde{v}} = 0$, $D \in I$. Now it is possible to transform this system into a system of differential equations for restrictions $c_{v,\tilde{v}}$ to the interior of A^- [4], [20, 9.1]. By applying the results of Harish-Chandra on differential equations ([3], [20, Appendix 3], a refinement and considerable simplification of the proof is given recently by N. Wallach [18]) it is possible to obtain an expansion of $c_{v,\tilde{v}}$ in the interior of A^-, which we shall describe now.

Let L be the lattice generated by the roots of Σ in $\mathfrak{a}^*$. Denote by L^+ the elements of L which are sums of positive roots. We say that $\lambda, \mu \in \mathfrak{a}^*$ are integrally equivalent if $\lambda - \mu \in L$. Also we define an order relation $\gg$ on $\mathfrak{a}^*$ by $\lambda \gg \mu$ if $\lambda - \mu \in L^+$.

Then there exist linear forms $\mu_1, \cdots, \mu_r \in \mathfrak{a}^*$ which are mutually integrally nonequivalent, polynomials $p_1, \cdots, p_q$ on $\mathfrak{a}_0$ and complex numbers $c_{\mu_j+\lambda,k}$, for $j = 1, 2, \cdots, r$, $\lambda \in L^+$ and $k = 1, 2, \cdots, q$, all depending on v and $\tilde{v}$, such that

$$c_{v,\tilde{v}}(a) = e^{\rho_P(\log a)} \sum_{j,k} p_k(\log a) e^{\mu_j(\log a)} \sum_{\lambda \in L^+} c_{\mu_j+\lambda,k} e^{\lambda(\log a)}$$

in the interior of A^-. The inner sums converge absolutely on the interior of A^- and uniformly on the subsets $A_\epsilon^- = \{a \in A; \alpha(\log a) \leq -\epsilon, \alpha \in \Sigma^+\}$, $\epsilon > 0$. If we put, for every $\mu \in \mathfrak{a}^*$,

$$c_{v,\tilde{v},\mu}(a) = e^{\rho_P(\log a)} \sum_k c_{\mu,k} p_k(\log a) e^{\mu(\log a)},$$

in the interior of A^-, the functions $c_{v,\tilde{v},\mu}$, $\mu \in \mathfrak{a}^*$, are uniquely determined by $c_{v,\tilde{v}}$ and $c_{v,\tilde{v}} = \sum_\mu c_{v,\tilde{v},\mu}$. We say that μ is an exponent of $c_{v,\tilde{v}}$ along P if $c_{v,\tilde{v},\mu} \neq 0$.

Finally, denote by $\mathcal{E}_P(\pi)$ the union of all exponents along P of all K-finite matrix coefficients of π. We call the minimal elements $\mathcal{E}_P^0(\pi)$ with respect to the order $\ll$ in $\mathcal{E}_P(\pi)$ the *leading exponents of π along P*.

Two infinitesimally equivalent admissible representations of finite length have obviously the same set of K-finite matrix coefficients. Therefore their leading exponents along P are the same.

Let $\mathfrak{Z}_-$ be the center of the enveloping algebra of $\mathfrak{m} \oplus \mathfrak{a}$. Then there exists a natural homomorphism $\mu : \mathfrak{Z} \to \mathfrak{Z}_-$ and $\mathfrak{Z}_-$ is a free module of finite rank over $\mu(\mathfrak{Z})$ [20, 9.1.2]. Therefore $\mu(I)\mathfrak{Z}_-$ is an ideal of finite codimension in $\mathfrak{Z}_-$. Since the enveloping algebra $\mathfrak{A}$ of $\mathfrak{a}$ is contained in $\mathfrak{Z}_-$, we can consider $\mathfrak{Z}_-/\mu(I)\mathfrak{Z}_-$ as a finite-dimensional $\mathfrak{a}$-module. For $\lambda \in \mathfrak{a}^*$ denote by $(\mathfrak{Z}_-/\mu(I)\mathfrak{Z}_-)^{(\lambda)}$ the space of all $z \in \mathfrak{Z}_-/\mu(I)\mathfrak{Z}_-$ such that there exists a positive integer m such that $(H - \lambda(H))^m z = 0$ for all $H \in \mathfrak{a}$. Denote by S_I the finite set of all $\lambda \in \mathfrak{a}^*$ such that $(\mathfrak{Z}_-/\mu(I)\mathfrak{Z}_-)^{(\lambda)} \neq \{0\}$. Then a simple modification of the argument in [20, Vol. II, p. 310] proves the following statements:

(i) the set $\mathcal{E}_P^0(\pi)$ of leading exponents of π along P is contained in S_I,

(ii) every exponent along P of a K-finite matrix coefficient of π is contained in $\mathcal{E}_P^0(\pi) + L^+$.

The following theorem describes a precise relationship between the leading exponents of π and infinitesimal intertwining operators of π with nonunitary principal series.

Denote by $I_P(\pi)$ the set of $\lambda \in \mathfrak{a}^*$ such that $\mathrm{Hom}_{(\mathfrak{G},K)}(\pi, \pi_{P,\sigma,\lambda}) \neq \{0\}$ for some $\sigma \in \hat{M}$.

THEOREM 1. *Let π be an admissible representation of finite length. Then the set $\mathcal{E}_P{}^0(\pi)$ of leading exponents of π along P is equal to the set of minimal elements in $I_P(\pi)$.*

We shall prove this theorem in Section 3.

The connection between the asymptotics of representations and infinitesimal intertwinings with nonunitary principal series was observed firstly by Harish-Chandra in a proof of his celebrated "subquotient theorem" ([4], [20, 5.5.1.5]). Recently, Casselman has given a very simple argument showing in fact that $\mathcal{E}_P{}^0(\pi)$ is contained in $I_P(\pi)$. As a corollary it gives the existence of infinitesimal embeddings of irreducible admissible representations into the nonunitary principal series [1], sharpening the original result of Harish-Chandra.

Our theorem has a very straightforward, but important consequence, which is, as we understand, an unpublished result of Casselman. In the moment of writing we are unaware of his argument.

Let π be an admissible representation of finite length on a Banach space V. Take a maximal chain

$$\{0\} = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} V_m = V$$

of closed invariant subspaces, and denote by π_j the irreducible representation on the quotient V_j/V_{j-1}, $j = 1, 2, \cdots, m$. The representation $\pi_{ss} = \pi_1 \oplus \cdots \oplus \pi_m$ is called the semisimplification of π. By [20, 4.5.6.2] π_{ss} is (up to the infinitesimal equivalence) independent of the choice of the composition series. Every K-finite matrix coefficient of π_{ss} is a K-finite matrix coefficient of π, what implies that $\mathcal{E}_P(\pi_{ss}) \subset \mathcal{E}_P(\pi)$. Also, it is obvious that $I_P(\pi) \subset I_P(\pi_{ss})$. This immediately implies the following result.

COROLLARY. *The set of leading exponents $\mathcal{E}_P{}^0(\pi)$ of π along P is equal to the set of leading exponents $\mathcal{E}_P{}^0(\pi_{ss})$ of π_{ss} along P.*

This result implies that the asymptotic behaviour of K-finite matrix coefficients of admissible representations of finite length is completely controlled by their irreducible constituents. It is a basic fact in our study of the relation between the rate of growth of characters and the asymptotics of representations in Part III. It gives also a very simple proof of Schmid's complete reducibility theorem for representations whose irreducible constituents are the discrete series representations [14, Theorem 1.6], as we shall show at the end of this paper.

3. *Proof of Theorem* 2.1. As we remarked before, W. Casselman has given, in an unpublished work announced in [1], a simple argument showing that $\mathcal{E}_P{}^0(\pi)$

$\subset I_P(\pi)$ for every admissible representation π of finite length. Although only the corollary of this result, the fact that every irreducible admissible representation embeds infinitesimally into a nonunitary principal series representation is stated in [1], it is easy to reconstruct the argument indicated there and see that it really proves the above statement. This reduces the proof of Theorem 2.1 onto the characterization of $\mathcal{E}_P{}^0(\pi)$ as the set of minimal elements of $I_P(\pi)$.

The essential point in the proof of this statement is the following theorem which is a refinement of a result of Langlands [12, Lemma 3.13], although the proof is essentially the same.

Let $(\lambda, \mu) \mapsto (\lambda \mid \mu)$ be the inner product in $\mathfrak{a}_0$ defined by the Killing form of $\mathfrak{g}_0$.

THEOREM 1. *Let $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}^*$ be such that* $\operatorname{Re}(\lambda \mid \alpha) < 0$ *for all $\alpha \in \Sigma^+$. Then the nonunitary principal series representation $\pi_{P,\sigma,\lambda}$ has a unique irreducible subrepresentation $\nu_{P,\sigma,\lambda}$ and λ is a leading exponent of $\nu_{P,\sigma,\lambda}$ along P.*

The statement of Theorem 2.1 for $\lambda \in I_P(\pi)$ satisfying $\operatorname{Re}(\lambda \mid \alpha) < 0$ for all $\alpha \in \Sigma^+$ is in fact an immediate consequence of the above theorem. By the tensoring with finite-dimensional representations the general case can be easily reduced to this situation as we shall see later. Firstly we shall prove Theorem 1.

Define the analytic maps $\kappa : G \to K$ and $H : G \to \mathfrak{a}_0$ such that for every $x \in G$ we have $x = \kappa(x) \exp H(x)n$ with $n \in N$. Let $\bar{P} = \vartheta(P)$ be the parabolic subgroup opposite to P, the Langlands decomposition of $\bar{P}$ is $\bar{P} = MA\bar{N}$ where $\bar{N} = \vartheta(N)$. We fix a Haar measure $d\bar{n}$ on $\bar{N}$ by the condition

$$\int_{\bar{N}} e^{-2\rho_P(H(\bar{n}))}\, d\bar{n} = 1.$$

If $\operatorname{Re}(\lambda \mid \alpha) < 0$ for all $\alpha \in \Sigma^+$ by a standard result from the theory of intertwining operators [19] we know that the operator $J_{\sigma,-\bar{\lambda}}$ defined by

$$(J_{\sigma,-\bar{\lambda}}f)(x) = \int_{\bar{N}} f(\bar{n}x)\, d\bar{n}, \qquad x \in G,$$

is a continuous intertwining operator of $\pi_{P,\sigma,-\bar{\lambda}}$ with $\pi_{\bar{P},\sigma,-\bar{\lambda}}$.

In the following, it is convenient to identify the spaces $\mathcal{H}_{Q,\sigma,\lambda}$ for a minimal parabolic subgroup Q, $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}^*$ with the Hilbert space $\mathcal{H}_\sigma$ of all square-integrable functions $g : K \to U$ satisfying $g(mk) = \sigma(m)g(k)$, $m \in M$, $k \in K$, by the mapping $f \mapsto f \mid K$. This map is an isomorphism of K-modules if we consider $\mathcal{H}_\sigma$ as a K-module induced from σ. We denote the induced action of K on $\mathcal{H}_\sigma$ by $(k, f) \mapsto k \cdot f$, $k \in K$, $f \in \mathcal{H}_\sigma$.

The main step in the proof of Theorem 1 is the following lemma.

LEMMA 1. *Let* $\operatorname{Re}(\lambda \mid \alpha) < 0$ *for every $\alpha \in \Sigma^+$. For K-finite functions f, $g \in \mathcal{H}_\sigma$, $f \neq 0$, the following assertions are equivalent*

(i) *λ is a leading exponent of the matrix coefficient $c_{k_1 \cdot f, k_2 \cdot g}$ of $\pi_{P,\sigma,\lambda}$ for some k_1, $k_2 \in K$,*

(ii) $J_{\sigma,-\bar{\lambda}}g \neq 0$.

Proof. In order to prove the lemma we shall use the following relation. For every K-finite functions f, $g \in \mathcal{H}_\sigma$ and $H \in \mathfrak{C}^-$ we have

$$\lim_{t\to\infty} e^{-t(\lambda+\rho_P)(H)}(\pi_{P,\sigma,\lambda}(\exp tH)f \mid g) = (f(1) \mid (J_{\sigma,-\bar\lambda}g)(1)).$$

This is (up to trivial modifications) a classical formula due to Harish-Chandra [4], [20, 9.1.6.1]. We sketch the argument for the convenience of the reader.

Put $a_t = \exp tH$. Then

$$(\pi_{P,\sigma,\lambda}(a_t)f \mid g) = \int_K e^{-(\lambda+\rho_P)(H(a_t^{-1}k))}(f(\kappa(a_t^{-1}k)^{-1}) \mid g(k^{-1}))\, dk.$$

Using the standard integral formulas we can transfer the integration over K to the integration over $\bar N \times M$ and using simple properties of the functions κ and H we get

$$\lim_{t\to\infty} e^{-t(\lambda+\rho_P)(H)}(\pi_{P,\sigma,\lambda}(a_t)f \mid g)$$
$$= \lim_{t\to\infty} \int_{\bar N} e^{-(\lambda+\rho_P)(H(a_t^{-1}\bar n a_t))} e^{(\lambda-\rho_P)(H(\bar n))}(f(\kappa(a_t^{-1}\bar n a_t)^{-1}) \mid g(\kappa(\bar n)^{-1}))\, d\bar n.$$

After checking the conditions of Lebesgue's dominated convergence theorem as in [20, 9.1.6.1], we can interchange the limit and the integration what finally eads to

$$\lim_{t\to\infty} e^{-t(\lambda+\rho_P)(H)}(\pi_{P,\sigma,\lambda}(a_t)f \mid g) = \int_{\bar N} e^{(\lambda-\rho_P)(H(\bar n))}(f(1) \mid g(\kappa(\bar n)^{-1}))\, d\bar n$$
$$= (f(1) \mid (J_{\sigma,-\bar\lambda}g)(1)).$$

This finishes the proof of the above relation. It implies immediately that λ is a leading exponent of $c_{k_1\cdot f, k_2\cdot g}$ if and only if $(f(k_1) \mid (J_{\sigma,-\bar\lambda}g)(k_2)) \neq 0$. Hence (i) implies (ii). In order to prove the converse it remains to see that $(f(k_1) \mid (J_{\sigma,-\bar\lambda}g)(k_2)) = 0$ for all k_1, $k_2 \in K$ implies $J_{\sigma,-\bar\lambda}g = 0$. Suppose that this is not true. Then, there exists a K-finite function $g \in \mathcal{H}_\sigma$ such that $J_{\sigma,-\bar\lambda}g \neq 0$ and $(f(k_1) \mid (J_{\sigma,-\bar\lambda}g)(k_2)) = 0$ for all k_1, $k_2 \in K$. Choosing k_1 and k_2 so that $f(k_1) \neq 0$ and $(J_{\sigma,-\bar\lambda}g)(k_2) \neq 0$, we get $(\sigma(m)f(k_1) \mid (J_{\sigma,-\bar\lambda}g)(k_2)) = 0$ for all $m \in M$, what contradicts the irreducibility of σ. Q.E.D.

Now we can prove Theorem 1. Denote by $\mathcal{J}_{\sigma,\lambda}$ the orthogonal complement of the kernel of $J_{\sigma,-\bar\lambda}$. Obviously the kernel of $J_{\sigma,-\bar\lambda}$ is an invariant subspace for $\pi_{P,\sigma,-\bar\lambda}$. The relation

$$(\pi_{P,\sigma,\lambda}(x)f \mid \pi_{P,\sigma,-\bar\lambda}(x)g) = (f \mid g), \qquad x \in G,\ f,\ g \in \mathcal{H}_\sigma,$$

and the fact that $J_{\sigma,-\bar\lambda} \neq 0$ implies that $\mathcal{J}_{\sigma,\lambda}$ is a closed invariant subspace for $\pi_{P,\sigma,\lambda}$ different from zero. Denote by $\nu_{P,\sigma,\lambda}$ the subrepresentation of $\pi_{P,\sigma,\lambda}$ on $\mathcal{J}_{\sigma,\lambda}$. It remains to show that it has the properties stated in Theorem 1.

Let $\mathcal{H}'$ be a closed $\pi_{P,\sigma,\lambda}$-invariant subspace in $\mathcal{H}_\sigma$. Let g be a K-finite function in $\mathcal{H}_\sigma$ orthogonal to $\mathcal{H}'$. Then for every K-finite function f in $\mathcal{H}'$ the matrix coefficient $c_{k_1 \cdot f, k_2 \cdot g}$, k_1, $k_2 \in K$, of $\pi_{P,\sigma,\lambda}$ is equal to zero. By Lemma 1. this implies that g is in the kernel of $J_{\sigma,-\bar{\lambda}}$. Therefore, $\mathcal{H}'$ contains $\mathcal{J}_{\sigma,\lambda}$ and $\nu_{P,\sigma,\lambda}$ is the unique irreducible subrepresentation of $\pi_{P,\sigma,\lambda}$. Applying again Lemma 1 we see that for each nonzero f, $g \in \mathcal{J}_{\sigma,\lambda}$ there exist k_1, $k_2 \in K$ such that λ is a leading exponent of the matrix coefficient $c_{k_1 \cdot f, k_2 \cdot g}$ of $\nu_{P,\sigma,\lambda}$. This finishes the proof of Theorem 1.

To prove Theorem 2.1 we need some preparation on the tensoring with finite-dimensional representations. Let π be an admissible representation of G of finite length on a Banach space V. Let τ be an irreducible finite-dimensional representation of G with the lowest restricted weight $-\mu$. The tensor product $\pi \otimes \tau$ is again an admissible representation of finite length. The next two lemmas relate $I_P(\pi)$ and $\mathcal{E}_P{}^0(\pi)$ to $I_P(\pi \otimes \tau)$ and $\mathcal{E}_P{}^0(\pi \otimes \tau)$ respectively.

LEMMA 2. $I_P(\pi) - \mu \subset I_P(\pi \otimes \tau)$.

Proof. Let W be the space of τ. Denote by W_ν the weight subspace corresponding to a restricted weight ν. Denote by $P : W \to W_{-\mu}$ the projection associated to the decomposition $W = W_{-\mu} + \sum_{\nu \neq -\mu} W_\nu$. Let ω be the representation of M on $W_{-\mu}$. Then obviously

$$P\tau(man) = e^{-\mu(\log a)}\omega(m)P$$

for every $m \in M$, $a \in A$ and $n \in N$.

For $f \in \mathcal{H}_{P,\sigma,\lambda}$ and $w \in W$ define $\varphi(f, w) : G \to U \otimes W_{-\mu}$ by

$$\varphi(f, w)(x) = f(x) \otimes P\tau(x)w, \qquad x \in G.$$

Then

$$\begin{aligned}\varphi(f, w)(manx) &= f(manx) \otimes P\tau(manx)w \\ &= e^{(\lambda-\mu+\rho_P)(\log a)}(\sigma(m)f(x)) \otimes (\omega(m)P\tau(x)w) \\ &= e^{(\lambda-\mu+\rho_P)(\log a)}(\sigma \otimes \omega)(m)\varphi(f, w)(x),\end{aligned}$$

for all $m \in M$, $a \in A$, $n \in N$ and $x \in G$. It follows that φ is a continuous bilinear map from $\mathcal{H}_{P,\sigma,\lambda} \times W$ into $\mathcal{H}_{P,\sigma\otimes\omega,\lambda-\mu}$. Denote by Φ the corresponding linear map from $\mathcal{H}_{P,\sigma,\lambda} \otimes W$ into $\mathcal{H}_{P,\sigma\otimes\omega,\lambda-\mu}$. Then it is easy to see that Φ intertwines $\pi_{P,\sigma,\lambda} \otimes \tau$ with $\pi_{P,\sigma\otimes\omega,\lambda-\mu}$.

Suppose now that $\lambda \in I_P(\pi)$ and let T be a nontrivial infinitesimal intertwining operator of π with $\pi_{P,\sigma,\lambda}$. Put $S = \Phi \circ (T \otimes 1)$, then S is an infinitesimal intertwining operator of $\pi \otimes \tau$ with $\pi_{P,\sigma\otimes\omega,\lambda-\mu}$. Suppose $S = 0$. Then for every K-finite vector $v \in V$ and $w \in W$ we have

$$0 = (S(v \otimes w))(x) = (\Phi(Tv \otimes w))(x) = (Tv)(x) \otimes P\tau(x)w, \qquad x \in G.$$

This clearly implies that $T = 0$, contradicting to our assumption. Therefore S is nontrivial. If $\sigma \otimes \omega$ is the direct sum of irreducible representations $\sigma_1, \cdots, \sigma_k$ of M, $\pi_{P,\sigma\otimes\omega,\lambda-\mu}$ is the direct sum of $\pi_{P,\sigma_j,\lambda-\mu}$, $j = 1, 2, \cdots, k$. By the above result there exists at least one j, $1 \leq j \leq k$, such that

$$\mathrm{Hom}_{(\mathfrak{G},K)}\,(\pi \otimes \tau,\ \pi_{P,\sigma_j,\lambda-\mu}) \neq \{0\}.$$

Therefore $\lambda - \mu \in I_P(\pi \otimes \tau)$. Q.E.D.

LEMMA 3. $\mathcal{E}_P{}^0(\pi) - \mu = \mathcal{E}_P{}^0(\pi \otimes \tau)$.

Proof. It is obvious that the set of exponents of τ coincides with the set of all restricted weights of τ shifted by $-\rho_P$. Hence $-\mu - \rho_P$ is the only leading exponent of τ along P. Because all K-finite matrix coefficients of $\pi \otimes \tau$ are sums of products of K-finite matrix coefficients of π with matrix coefficients of τ the relation

(a) $\mathcal{E}_P(\pi \otimes \tau) \subset \mathcal{E}_P{}^0(\pi) - \mu + L^+$ is obvious.

Also, if $\lambda \in \mathcal{E}_P{}^0(\pi)$ there exist K-finite v and $\tilde{v}$ such that the matrix coefficient $c_{v,\tilde{v}}$ of π has λ as one of its exponents, and w and $\tilde{w}$ such that the matrix coefficient $c_{w,\tilde{w}}$ of τ has $-\mu - \rho_P$ as its exponent. This implies that the matrix coefficient $c_{v\otimes w,\tilde{v}\otimes\tilde{w}} = c_{v,\tilde{v}} \cdot c_{w,\tilde{w}}$ of $\pi \otimes \tau$ has $\lambda - \mu$ as one of its exponents, i.e. the relation

(b) $\mathcal{E}_P{}^0(\pi) - \mu \subset \mathcal{E}_P(\pi \otimes \tau)$

holds. Now (a) and (b) imply immediately that $\mathcal{E}_P{}^0(\pi) - \mu \subset \mathcal{E}_P{}^0(\pi \otimes \tau)$. Let $\lambda - \mu \in \mathcal{E}_P{}^0(\pi \otimes \tau)$. By (a) there exists $\nu \in \mathcal{E}^0(\pi)$ such that $\nu - \mu \ll \lambda - \mu$, and by (b) we have $\nu - \mu \in \mathcal{E}_P(\pi \otimes \tau)$. By the minimality of $\lambda - \mu$ this implies $\lambda = \nu$, i.e. $\lambda \in \mathcal{E}_P{}^0(\pi)$. Therefore $\mathcal{E}_P{}^0(\pi \otimes \tau) \subset \mathcal{E}_P{}^0(\pi) - \mu$. Q.E.D.

Now we can finish the proof of Theorem 2.1. Let $\lambda \in I_P(\pi)$. We can choose τ such that $\operatorname{Re}(\lambda - \mu \mid \alpha) < 0$ for all $\alpha \in \Sigma^+$. By Lemma 2. $\lambda - \mu \in I_P(\pi \otimes \tau)$, therefore there exists a nontrivial infinitesimal intertwining operator of $\pi \otimes \tau$ with $\pi_{P,\sigma,\lambda-\mu}$ for some $\sigma \in \hat{M}$. By Theorem 1 the representation $\nu_{P,\sigma,\lambda-\mu}$ is infinitesimally equivalent to an irreducible constituent of $\pi \otimes \tau$. This implies that $\lambda - \mu$ is an exponent of $\pi \otimes \tau$. Hence there exist $\nu \ll \lambda$ such that $\nu - \mu \in \mathcal{E}_P{}^0(\pi \otimes \tau)$. By Lemma 3., $\nu \in \mathcal{E}_P{}^0(\pi)$ and Theorem 2.1 is proved.

Part III. Characters and asymptotics.

1. *Characters and leading exponents.*

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let $C_0{}^\infty(G)$ be the convolution algebra of smooth functions with compact support on G. Denote by $C_*{}^\infty(G)$ the dense subalgebra of K-finite functions in $C^\infty(G)$. If π is an admissible representation of G on a Banach space we can associate to every $f \in C_0{}^\infty(G)$ a bounded linear operator

$$\pi(f) = \int_G f(x)\pi(x)\,dx,$$

where dx is a Haar measure on G. The mapping $f \mapsto \pi(f)$ is a representation of $C_0{}^\infty(G)$. For every $f \in C_*{}^\infty(G)$ the operator $\pi(f)$ is of finite rank. Therefore the linear form $f \mapsto \operatorname{tr} \pi(f)$ is well-defined on $C_*{}^\infty(G)$, we shall call it the character of π and denote by Θ_π.

By Casselman's "subrepresentation theorem" [1] every irreducible admissible representation is infinitesimally equivalent to a subrepresentation of a non-unitary principal series representation, hence it is infinitesimally equivalent to an irreducible admissible representation on a Hilbert space. Since the characters of infinitesimally equivalent admissible representations are equal, the character of an irreducible admissible representation extends to an invariant eigendistribution on G [20, 4.5.8.6]. If π is an admissible representation of finite length, its character is the sum of the characters of its irreducible constituents. Therefore, it extends to an invariant $\mathfrak{Z}$-finite distribution on G.

By the previous discussion to every object in the category of admissible representations of finite length of G we can associate its character which is an invariant $\mathfrak{Z}$-finite distribution. Two admissible representations of finite length have the same character if and only if their semisimplifications are infinitesimally equivalent [20, 4.5.8.1]. Combining this result with Corollary of Theorem II.2.1 it follows that admissible representations of finite length with the same character have the same leading exponents along any minimal parabolic subgroup of G. Therefore, it should be possible, at least in principle to read off the leading exponents of admissible representations of finite length from their characters. Although in this stage this looks like a formidable task, the following result shows that growth conditions on the characters are equivalent to simple conditions on the leading exponents.

Let P be a minimal parabolic subgroup of G. As in Part II denote by $\mathfrak{C}^-$ the corresponding "negative" Weyl chamber in $\mathfrak{a}_0$, and by $\mathfrak{F}$ the set of all complex linear forms μ on $\mathfrak{a}$ such that Re $\mu(H) \leq 0$ for every $H \in \mathfrak{C}^-$.

THEOREM 1. *Let π be an admissible representation of finite length of G and $\gamma \in \mathbf{R}$. Then the following conditions are equivalent*

(i) *the character Θ_π of π has the rate of growth γ,*

(ii) *every leading exponent of π along P lies in $-\gamma\rho_P + \mathfrak{F}$.*

The proof of this theorem will be given in the next section.

By an unpublished result of Harish-Chandra [3, Theorem 4] the asymptotic behaviour of K-finite matrix coefficients is controlled by the leading exponents. Hence, the implication (i) $\Rightarrow$ (ii) represents an extension of Theorem I.1.1 holding for all real γ, but under the assumption that the invariant $\mathfrak{Z}$-finite distribution is the character of an admissible representation of finite length.

Obviously, Theorem 1 reduces the problem of asymptotic behaviour of K-finite matrix coefficients of discrete series representations onto the determination of the rate of growth of their characters. This will be considered in the last section.

2. *Proof of Theorem* 1.1. The main idea in the proof of Theorem 1.1 is that it is enough to prove its assertion for sufficiently large positive γ. This reduction is established by the following simple lemma.

Without any loss of generality we can suppose that G is acceptable. Denote by τ the finite-dimensional representation of G introduced in Section I.2 and by τ^n its n-th tensor power.

LEMMA 1. *Let π be an admissible representation of finite length of G. Then, for $n \in \mathbf{N}$,*

(i) *the character Θ_π of π has the rate of growth γ if and only if the character of $\pi \otimes \tau^n$ has the rate of growth $\gamma + n$,*

(ii) $\mathcal{E}_P{}^0(\pi \otimes \tau^n) = \mathcal{E}_P{}^0(\pi) - n\rho_P$.

These assertions are obvious consequences of the definitions and Lemma II.3.3.

Suppose now that $M \geq 0$ and that Theorem 1.1 holds for all $\gamma \geq M$. For every real γ we can choose $n \in \mathbf{N}$ such that $\gamma + n \geq M$. Since by Lemma 1 the assertions (i) and (ii) for (π, γ) and for $(\pi \otimes \tau^n, \gamma + n)$ are equivalent, our theorem holds for all real γ.

Now we shall show that the implication (i) $\Rightarrow$ (ii) for $\gamma \geq 0$ is an easy consequence of the results of Part I. Let π be an admissible representation of finite length of G whose character Θ_π has the rate of growth $\gamma \geq 0$. Let $\pi_1, \pi_2, \cdots, \pi_k$ be the pairwise infinitesimally inequivalent irreducible admissible representations of G on Banach spaces $V_1, V_2 . \cdots, V_k$ respectively, such that every irreducible constituent of π is infinitesimally equivalent to some π_j, $1 \leq j \leq k$. Denote by Θ_j the character of π_j, $1 \leq j \leq k$. Then

$$\Theta_\pi = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_k\Theta_k ,$$

where $n_j \in \mathbf{N}$ is the multiplicity of π_j in π for $1 \leq j \leq k$.

Fix $\delta \in \hat{K}$. Let $C_\delta{}^\infty(G)$ be the subalgebra of $C_0{}^\infty(G)$ consisting of those f which verify the relation $f = \bar{\chi}_\delta * f * \bar{\chi}_\delta$. Obviously, the spaces $V_{j,\delta}$, $1 \leq j \leq k$, are invariant under the action of $\pi_j(f)$, $f \in C_\delta{}^\infty(G)$. Denote the corresponding representations of $C_\delta{}^\infty(G)$ by $\pi_{j,\delta}$, $1 \leq j \leq k$. By [20, 4.5.1] the nontrivial representations $\pi_{j,\delta}$ are irreducible and pairwise inequivalent. Hence, there exist $f_{j,\delta} \in C_\delta{}^\infty(G)$, $1 \leq j \leq k$, such that

$$\pi_{j,\delta}(f_{j,\delta}) = 1, \qquad \pi_{j,\delta}(f_{i,\delta}) = 0, \quad \text{for} \quad i \neq j.$$

Put $g_{j,\delta}(x) = f_{j,\delta}(x^{-1})$, $x \in G$. Then

$$\Theta_\pi * g_{j,\delta} = n_j\Theta_j * g_{j,\delta} = n_j\Theta_{j,\delta} ,$$

for every $1 \leq j \leq k$. By Theorem I.3.3. Θ_π is an invariant $\mathfrak{Z}$-finite distribution of type γ. This implies immediately that $\Theta_{j,\delta}$ is a K-finite and $\mathfrak{Z}$-finite distribution of type γ for $1 \leq j \leq k$, $\delta \in K$.

A trivial modification of the argument in [7, p. 53] implies now that all K-finite matrix coefficients of π_j, $1 \leq j \leq k$, are K-finite and $\mathfrak{Z}$-finite distributions of type γ. Therefore, Theorem I.3.2 implies that for every K-finite matrix coefficient c of the semisimplification π_{ss} of π there exist $M > 0$ and $p > 0$, such that

$$|c(x)| \leq M\Xi(x)^{1-\gamma}(1 + \sigma(x))^p, \qquad x \in G.$$

Now, the asymptotic of Ξ [20, 8.3.7.4] implies the existence of $N > 0$ and $q \geq 0$ such that

$$|c(a)| \leq Ne^{(1-\gamma)\rho_P(\log a)}(1 + ||\log a||)^q, \qquad a \in A^-.$$

This obviously implies that all leading exponents of π_{ss} along P are contained in $-\gamma\rho_P + \mathfrak{F}$. Our assertion follows now from Corollary of Theorem II.2.1.

It remains to prove the implication (ii) $\Rightarrow$ (i). In this case we shall suppose that $\gamma \geq 1$. Also we can assume that π is an irreducible admissible representation. As we shall see in a moment the following estimate implies easily our assertion.

LEMMA 2. *Let π be an irreducible admissible representation of G. Suppose that all its leading exponents along P lie in $-\gamma\rho_P + \mathfrak{F}$, $\gamma \geq 1$. Then there exist $C > 0$ and $s \geq 0$ such that the Fourier components $\Theta_{\pi,\delta}$ of its character Θ_π satisfy*

$$|\Theta_{\pi,\delta}(x)| \leq C\cdot d(\delta)^3\Xi(x)^{1-\gamma}(1 + \sigma(x))^s, \qquad x \in G,$$

for all $\delta \in \hat{K}$.

Assuming the lemma for a moment we shall now finish the proof of Theorem 1.1.

Let $X_1, X_2, \cdots, X_r$ be a basis of $\mathfrak{k}_0$ orthonormal with respect to $-B$ restricted to $\mathfrak{k}_0$, where B is the Killing form of $\mathfrak{g}$, and put

$$\Omega = 1 - (X_1^2 + X_2^2 + \cdots + X_r^2).$$

Obviously Ω is an element of the center of the enveloping algebra of $\mathfrak{k}$. Now the results of §3 of [7] imply that for every $\delta \in \hat{K}$ there exists a real number $c(\delta) \geq 1$ such that

$$\Omega\Theta_{\pi,\delta} = c(\delta)\Theta_{\pi,\delta}$$

and $m \in \mathbf{N}$ such that

$$c_0 = \sum_{\delta\in\hat{K}} \frac{d(\delta)^3}{c(\delta)^m} < \infty.$$

Then for $f \in C_0^\infty(G)$ we have

$$\Theta_\pi(f) = \sum_{\delta\in\hat{K}} \int_G \Theta_{\pi,\delta}(x)f(x)\,dx = \sum_{\delta\in\hat{K}} \frac{1}{c(\delta)^m}\int_G (\Omega^m\Theta_{\pi,\delta})(x)f(x)\,dx$$
$$= \sum_{\delta\in\hat{K}} \frac{1}{c(\delta)^m}\int_G \Theta_{\pi,\delta}(x)(\Omega^m f)(x)\,dx.$$

Therefore

$$|\Theta_\pi(f)| \leq c_1\cdot\int_G |(\Omega^m f)(x)|\,\Xi(x)^{1-\gamma}(1 + \sigma(x))^s\,dx,$$

where $c_1 = C\cdot c_0$. If we choose $r \geq 0$ such that

$$c_2 = \int_G \Xi(x)^2(1 + \sigma(x))^{-r}\,dx < \infty,$$

we have finally

$$|\Theta_\pi(f)| \leq M p_{\mathfrak{Q}^m, s+r}{}^{\gamma}(f)$$

where $M = c_1 \cdot c_2$, what implies that Θ_π is of type γ. Now, Theorem I.3.3. implies that Θ_π has the rate of growth γ, what finishes the proof of Theorem 1.1.

It remains to prove Lemma 2. The leading exponents of π are contained in $\mathfrak{F}$ if and only if π is tempered. In this case by the main result of [17] or Lemma 4.10 of [12] π is a unitary representation and the assertion is obvious. Therefore we can assume that π is nontempered. In this case we shall use the classification of such representations due to Langlands [12] to obtain our estimate.

Let P_1 be a parabolic subgroup of G containing P. Denote by $P_1 = M_1 A_1 N_1$ its Langlands decomposition. Obviously $M \subset M_1$, $A \supset A_1$ and $N \supset N_1$. Let $\mathfrak{m}_{01}$, $\mathfrak{a}_{01}$ and $\mathfrak{n}_{01}$ be the Lie algebras of M_1, A_1 and N_1 and $\mathfrak{m}_1$, $\mathfrak{a}_1$ and $\mathfrak{n}_1$ their complexifications respectively. Denote by ${}^*\mathfrak{a}_0 = \mathfrak{m}_{01} \cap \mathfrak{a}_0$ and ${}^*\mathfrak{a} = \mathfrak{m}_1 \cap \mathfrak{a}$. Obviously $\mathfrak{a} = {}^*\mathfrak{a} \oplus \mathfrak{a}_1$. We shall consider the linear forms on $\mathfrak{a}_1$ as linear forms on $\mathfrak{a}$ trivial on ${}^*\mathfrak{a}$.

Define the analytic functions $\mu : G \to \exp(\mathfrak{m}_{01} \cap \mathfrak{p}_0)$ and $H_1 : G \to \mathfrak{a}_{01}$ such that $x = \kappa(x)\mu(x) \exp H_1(x) n$, $n \in N_1$, for all $x \in G$. It is easy to see that

$$H(x) - H_1(x) \in {}^*\mathfrak{a}_0, \qquad x \in G.$$

Finally, we define $\rho_1 = \rho_{P_1} \in \mathfrak{a}_1^*$ by $\rho_1(H) = \frac{1}{2} \operatorname{tr}(adH \mid \mathfrak{n}_1)$, $H \in \mathfrak{a}_1$.

Let σ be an irreducible unitary representation of M_1 on a Hilbert space U and $\mu \in \mathfrak{a}_1^*$. As in II.1. we define the Hilbert space $\mathcal{H}_{\sigma,\mu}$ of classes of functions $f : G \to U$ satisfying

(i) $f(manx) = \sigma(m) e^{(\mu+\rho_1)(\log a)} f(x)$ for all $m \in M_1$, $a \in A_1$, $n \in N_1$ and $x \in G$,

(ii) $\displaystyle\int_K ||f(k)||^2 \, dk < \infty$,

with the inner product $(f \mid g) = \int_K (f(k) \mid g(k)) dk$, and the representation $\pi_{\sigma,\mu}$ of G on $\mathcal{H}_{\sigma,\mu}$ by

$$(\pi_{\sigma,\mu}(x) f)(y) = f(yx), \qquad x, y \in G.$$

Let $\mathfrak{a}_-^* = \{\lambda \in \mathfrak{a}^*; \operatorname{Re}(\lambda \mid \alpha) \leq 0, \alpha \in \Sigma^+\}$.

By the Langlands' theory [12], for a nontempered irreducible representation π satisfying the conditions of Lemma 2, there exist a parabolic subgroup P_1 containing P, an irreducible tempered representation σ of M_1 and $\mu \in \mathfrak{a}_1^*$ such that

(i) μ is contained in $(-\gamma\rho_P + \mathfrak{F}) \cap \mathfrak{a}_-^*$,

(ii) π is infinitesimally equivalent to a subrepresentation of $\pi_{\sigma,\mu}$.

This reduces our problem onto the estimates of matrix coefficients of $\pi_{\sigma,\mu}$.

Take $f \in \mathcal{H}_{\sigma,\mu}$ such that $||f|| = 1$ and its K-orbit spans an irreducible K-module of type δ. Choose an orthonormal basis $f_1, f_2, \cdots, f_{d(\delta)}$ for this module such that $f = f_1$; and denote its matrix coefficients in this basis by a_{ij}, $1 \leq i, j \leq d(\delta)$.

Then

$$f(kh) = \sum_{j=1}^{d(\delta)} a_{j1}(h) f_j(k), \qquad k, h \in K.$$

Therefore, by the use of orthogonality relations

$$1 = \|f\|^2 = \int_K \|f(kh)\|^2 \, dh = \sum_{i,j=1}^{d(\delta)} (f_i(k) \mid f_j(k)) \cdot \int_K a_{i1}(h)\overline{a_{j1}(h)} \, dh$$
$$= \frac{1}{d(\delta)} \sum_{i=1}^{d(\delta)} \|f_i(k)\|^2, \qquad k \in K.$$

This implies that

$$\|f(k)\| \leq \sqrt{d(\delta)}, \qquad k \in K.$$

Now we can estimate the function $x \mapsto (\pi_{\sigma,\mu}(x)f \mid f)$. It is obvious that

$$(\pi_{\sigma,\mu}(x)f \mid f) = \int_K (f(kx) \mid f(k)) \, dk$$
$$= \int_K e^{-(\mu+\rho_1)(H_1(x^{-1}k))} (\sigma(\mu(x^{-1}k)^{-1}) f(\kappa(x^{-1}k)^{-1}) \mid f(k^{-1})) \, dk,$$

for all $x \in G$. Therefore, the fact that σ is a unitary representation and the above estimate imply immediately that

$$|(\pi_{\sigma,\mu}(x)f \mid f)| \leq d(\delta) \int_K e^{-(\mathrm{Re}\,\mu+\rho_1)(H_1(x^{-1}k))} \, dk$$
$$= d(\delta) \int_K e^{-(\mathrm{Re}\,\mu+\rho_1)(H(x^{-1}k))} \, dk, \qquad x \in G.$$

Denote by φ_λ the zonal spherical function

$$\varphi_\lambda(x) = \int_K e^{(\lambda-\rho_P)(H(xk))} \, dk, \qquad x \in G.$$

Then by [20, 6.2.2.1]

$$|(\pi_{\sigma,\mu}(x)f \mid f)| \leq d(\delta) \varphi_{\mathrm{Re}\,\mu+\rho_1-\rho_P}(x), \qquad x \in G.$$

Applying Lemma 3.6 of [12] and the conditions on μ and γ we easily see that there exist $c > 0$ and $p \geq 0$ such that

$$\varphi_{\mathrm{Re}\,\mu+\rho_1-\rho_P}(a) \leq c e^{(1-\gamma)\rho_P(\log a)} (1 + \|\log a\|)^p, \qquad a \in A^-.$$

Now, the asymptotic of Ξ [20, 8.3.7.4] implies the existence of $C > 0$ and $s \geq 0$ such that

$$|(\pi_{\sigma,\mu}(x)f \mid f)| \leq C d(\delta) \Xi(x)^{1-\gamma} (1 + \sigma(x))^s, \qquad x \in G.$$

Since the multiplicity of δ in $\pi_{\sigma,\mu}$ is less or equal to $d(\delta)$, this estimate immediately implies the assertion of Lemma 2.

3. *Applications to the discrete series.* In this section we shall apply our results to the discrete series representations.

Firstly we shall review some results of Harish-Chandra on the discrete series. Without any loss of generality we can suppose that G is acceptable. We fix a compact Cartan subgroup H of G such that $H \subset K$. Let $\mathfrak{h}_0$ be the Lie algebra of H and $\mathfrak{h}$ its complexification. Denote by Φ the root system of $(\mathfrak{g}, \mathfrak{h})$ and by Φ_c and Φ_n the sets of compact and noncompact roots respectively. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and W_k the subgroup generated by reflections with respect to the compact roots. The Killing form of $\mathfrak{g}$ induces an inner product $(\ |\ )$ on $i\mathfrak{h}_0{}^*$.

Denote by Λ the lattice of differentials of characters of H. An element $\lambda \in \Lambda$ is nonsingular if $(\lambda \mid \alpha) \neq 0$ for every $\alpha \in \Phi$. Denote by Λ' the set of nonsingular elements in Λ. To each $\lambda \in \Lambda'$ a discrete series representation π_λ is attached and $\pi_\lambda = \pi_\mu$ if and only if there is $w \in W_k$ such that $w\lambda = \mu$.

For every root $\alpha \in \Phi$ we put

$$k(\alpha) = \tfrac{1}{4} \sum_{\beta \in \Phi} |(\alpha \mid \beta)|.$$

Then our main result can be stated as follows.

THEOREM 1. *Let $\kappa > 0$. For every $\lambda \in \Lambda'$ the following conditions are equivalent*

(i) $|(\lambda \mid \alpha)| \geq \kappa k(\alpha)$ *for every noncompact root* α,

(ii) *for every K-finite matrix coefficient c of the discrete series representation π_λ there exist $M > 0$ and $q \geq 0$ such that*

$$|c(x)| \leq M\Xi(x)^{1+\kappa}(1 + \sigma(x))^q, \qquad x \in G.$$

By Theorem 7.5 of [16] the above result implies immediately the following corollary.

COROLLARY. *Let $1 \leq p < 2$. For every $\lambda \in \Lambda'$ the following conditions are equivalent*

(i) $|(\lambda \mid \alpha)| > (2/p - 1)k(\alpha)$ *for every noncompact root* α,

(ii) *every K-finite matrix coefficient of the discrete series representation π_λ is in $L^p(G)$.*

The implications (ii) $\Rightarrow$ (i) in the above results are due to Trombi and Varadarajan [16]. Our argument is different from theirs. The converse implication of Corollary, for the most interesting case $p = 1$, was proved by Hecht and Schmid [11]. In the special case of the groups $SO(2n, 1)$ and $SU(n, 1)$ these results were obtained by T. Enright [2]; however, his proof appears to be incomplete.

Let Θ_λ be the character of the discrete series representation π_λ for $\lambda \in \Lambda'$. The main step in the reduction of Theorem 1. to Theorem 1.1. is the following result which shows that the condition (i) is essentially a growth condition on the character Θ_λ.

THEOREM 2. *Let $\kappa > 0$. Then for every $\lambda \in \Lambda'$ the following conditions are equivalent*
(i) $|(\lambda \mid \alpha)| \geq \kappa k(\alpha)$ *for every noncompact root* α,
(ii) *the invariant eigendistribution* Θ_λ *has the rate of growth* $-\kappa$.

Assuming this result for a moment we can finish the proof of Theorem 1. Theorem 2 and Theorem 1.1 imply that the following assertions are equivalent for $\kappa > 0$, $\lambda \in \Lambda'$ and a minimal parabolic subgroup P:
(i) $|(\lambda \mid \alpha)| \geq \kappa k(\alpha)$ for every noncompact root α,
(ii) all leading exponents of π_λ along P lie in $\kappa\rho_P + \mathfrak{F}$.
By an unpublished result of Harish-Chandra [3, Theorem 4] it follows that (ii) is equivalent to
(ii′) for every K-finite matrix coefficient c of π_λ there exist $N > 0$ and $s \geq 0$ such that

$$|c(a)| \leq Ne^{(1+\kappa)\rho_P(\log a)}(1 + \|\log a\|)^s, \qquad a \in A^-.$$

Now, the asymptotics of Ξ [20, 8.3.7.3, 8.3.7.4] implies that (ii′) is equivalent to the condition (ii) of Theorem 1.

It remains ro prove Theorem 2. It is a consequence of deep results of Harish-Chandra on the structure of Θ_λ [6]. The implication (ii) $\Rightarrow$ (i) is an immediate consequence of the formula for Θ_λ on the sets of regular elements of Cartan subgroups with one-dimensional split components, as was explained in [16, p. 275].

The implication (i) $\Rightarrow$ (ii) is much harder to establish, because we must control the expressions for Θ_λ on all noncompact Cartan subgroups. Our argument is based on remarkable "coherence" properties of the invariant eigendistributions Θ_λ , this type of argument was used extensively by Hecht and Schmid in their work on the Blattner's conjecture [10].

Firstly, from the expressions for Θ_λ on noncompact Cartan subgroups of G given in Lemma 57 of [6] (more specifically from the fact that the constants $c_{b^*}(s : t : A^+)$ appearing there do not depend on λ but only on the Weyl chamber containing λ), the following lemma immediately follows.

LEMMA 1. *Let $\kappa > 0$ and $n \in \mathbf{N}$. The following assertions are equivalent*
(i) Θ_λ *has the rate of growth* $-\kappa$,
(ii) $\Theta_{n\lambda}$ *has the rate of growth* $-n\kappa$.

Now we shall use Lemma 1 to reduce the proof on the case of linear groups. This reduction is inessential, but it enables us to use directly the results of Hecht and Schmid [10].

Let Z be the center of G. Obviously $Z \subset H$ and by Lemma 51 of [6] we have

$$\Theta_\lambda(zx) = \eta_{\lambda-\rho}(z)\Theta_\lambda(x), \qquad z \in Z, \quad x \in G',$$

where η_μ is the character of H whose differential is $\mu \in \Lambda$. Let $Z_1 \subset Z$ be the intersection of the kernels of all finite-dimensional representations of G. Then G/Z_1 is a linear group [20, Vol. I, p. 195]. Also, $\eta_\rho(z) = 1$ for every $z \in Z_1$.

Therefore, if n is the order of Z_1,

$$\Theta_{n\lambda}(zx) = \Theta_{n\lambda}(x), \qquad z \in Z_1, \quad x \in G,$$

and we can consider $\Theta_{n\lambda}$ as an invariant eigendistribution on G/Z_1. By Lemma 1 we see that it is enough to establish our assertion for the discrete series characters of G/Z_1.

Therefore, in the following we can assume that G is a linear group which lies in its simply connected complexification. In this situation Λ is the lattice of weights of Φ.

Let $\lambda \in \Lambda'$ and $\kappa > 0$ be such that $|(\lambda \mid \alpha)| \geq \kappa k(\alpha)$ for every $\alpha \in \Phi_n$. We fix a set Ψ of positive roots in Φ such that λ is dominant with respect to Ψ. Following Hecht and Schmid [10] we can construct a family $\Theta(\Psi, \mu)$, $\mu \in \Lambda$, of invariant eigendistributions with the following properties:

(i) $\Theta(\Psi, \mu)$ depend "coherently" on $\mu \in \Lambda$ (see **§2** of [10]),

(ii) $\Theta(\Psi, \mu) = \Theta_\mu$ for all $\mu \in \Lambda'$ dominant with respect to Ψ.

The basic fact about $\Theta(\Psi, \mu)$ on which our argument is based is

(iii) if $(\mu \mid \alpha) \geq 0$ for all $\alpha \in \Phi_n \cap \Psi$, $\Theta(\Psi, \mu)$ is a tempered invariant eigendistribution [10, Lemma 3.1].

Now we can finish the proof of Theorem 2. Let ρ be the half sum of the elements in Ψ. Then

$$(\lambda \mid \alpha) \geq \kappa k(\alpha) \geq \kappa(w\rho \mid \alpha), \qquad \alpha \in \Phi_n \cap \Psi,$$

for all $w \in W$. Therefore we see that

$$(\lambda - \kappa w\rho \mid \alpha) \geq 0, \qquad w \in W, \quad \alpha \in \Phi_n \cap \Psi.$$

Take a positive rational number $r = p/q$ such that $r \leq \kappa$ and $(\lambda - rw\rho \mid \beta) \neq 0$ for all $w \in W, \beta \in \Phi$. Then $(\lambda - rw\rho \mid \alpha) > 0$ for all $w \in W$ and $\alpha \in \Phi_n \cap \Psi$. Therefore for all $w \in W$,

(a) $q\lambda - pw\rho \in \Lambda'$,

and by (iii),

(b) $\Theta(\Psi, q\lambda - pw\rho)$ is a tempered invariant eigendistribution.

It is easy to see from the expression (2.5) of [10] for $\Theta(\Psi, \mu)$ that (a) and (b) imply that $\Theta(\Psi, q\lambda) = \Theta_{q\lambda}$ has the rate of growth $-p$. By Lemma 1 it follows that Θ_λ has the rate of growth $-r$. Since we can choose r arbitrary close to κ we see that Θ_λ has the rate of growth $-\kappa$. This ends the proof of Theorem 2.

At the end we want to compare our argument with the argument of Hecht and Schmid [10]. As remarked before, our idea is similar to theirs, in the following way. They tensor a discrete series representation π_λ for λ "sufficiently far" of noncompact walls with a suitable finite-dimensional representation so that all irreducible constituents of the tensor product remain infinitesimally equivalent to discrete series representations. Then by a result of Schmid [14, Theorem 1.6] this tensor product is completely reducible. Therefore it is a tempered representation and all its K-finite matrix coefficients satisfy the weak inequality, what gives the right estimates for K-finite matrix coefficients of the discrete series representation π_λ.

Instead of reducing to the tempered case, what is impossible in our situation, we have extended Harish-Chandra's results to the eigendistributions with positive rate of growth, and used Corollary of Theorem II.2.1 instead of Schmid's complete reducibility theorem.

Although in the situation when all irreducible constituents are infinitesimally equivalent to discrete series representations Corollary of Theorem II.2.1 is apparently weaker then Schmid's result, it actually implies this result. Compared to Schmid's original argument the following argument uses only very weak results about discrete series representations.

THEOREM 3. (Schmid, [14]) *Let π be an admissible representation of finite length. If all irreducible constituents of π are infinitesimally equivalent to discrete series representations, π is infinitesimally equivalent to its semisimplification.*

Proof. By assumption, there exists $\kappa > 0$ such that the character of π has the rate of growth $-\kappa$. As in the proof of Theorem 1 and its corollary this implies that all K-finite matrix coefficients of π are square-integrable. This fact can be also deduced from the theory of constant term [9] and Corollary of Theorem II.2.1 without any use of results on characters.

Let $\mathcal{V}$ be the space of K-finite vectors of π. It is easy to see that there exist K-finite linear forms $\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n$ on $\mathcal{V}$ such that

$$(v \mid w) = \sum_{j=1}^{n} \int_G c_{v,\tilde{v}_j}(x)\overline{c_{w,\tilde{v}_j}(x)}\, dx$$

is an inner product on $\mathcal{V}$. It is obviously $(\mathfrak{G}, K)$-invariant, hence $\mathcal{V}$ is completely reducible as a $(\mathfrak{G}, K)$-module, what proves our assertion.

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