

CLOSURES OF K -ORBITS IN THE FLAG VARIETY FOR $SU^*(2n)$

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ABSTRACT. We characterize the Sp_{2n} -orbits in the flag variety for SL_{2n} with rationally smooth closure via a pattern avoidance criterion, also showing that the singular and rationally singular loci of such orbit closures coincide.

1. INTRODUCTION

Let G be a complex reductive group with Borel subgroup B and let $K = G^\theta$ be the fixed point subgroup of an involution of G . In this paper we continue the program begun in [M07] and continued in [MT08], using pattern avoidance to characterize the K -orbits in G/B with rationally smooth closure. Here we consider the case $G = SL(2n, \mathbb{C})$, $K = Sp(2n, \mathbb{C})$. We will adapt the techniques used in [B98] to study Schubert varieties for complex classical groups, focussing on the poset and graph structures of the set of orbits with closures contained in a given one.

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2. PRELIMINARIES

Set $G = GL(n, \mathbb{C})$, $K = Sp(2n, \mathbb{C})$. Let B be the subgroup of upper triangular matrices in G . The quotient G/B may be identified with the variety of complete flags $V_0 \subset V_1 \subset \dots \subset V_{2n}$ in $\mathbb{C}^{2n}$. The group K acts on this variety with finitely many orbits; these are parametrized by the set I_{2n} of involutions in the symmetric group S_{2n} without fixed points [MO88, RS90]. In more detail, let $\langle \cdot, \cdot \rangle$ be the standard nondegenerate skew form on $\mathbb{C}^{2n}$ with isometry group K . Then a flag $V_0 \subset \dots \subset V_{2n}$ lies in the orbit $\mathcal{O}_\pi$ corresponding to the involution π if and only if the rank of $\langle \cdot, \cdot \rangle$ on $V_i \times V_j$ equals the cardinality $\#\{k : 1 \leq k \leq i, \pi(k) \leq j\}$ for all $1 \leq i, j \leq 2n$.

We will be using a modified version of the usual notion of pattern avoidance for permutations. We say that $\pi = \pi_1 \dots \pi_{2n}$ (in one-line notation) includes the pattern $\mu = \mu_1 \dots \mu_{2n}$ if there are indices $i_1 < i_2 < \dots < i_{2n}$ permuted by π such that $\pi_{i_j} > \pi_{i_k}$ if and only if $\mu_{i_j} > \mu_{i_k}$ (the usual definition would omit the condition that π permute the i_j). We say that π avoids μ if it does not include it.

There are well-known poset- and graph-theoretic criteria for rational smoothness of complex Schubert varieties due to Carrell and Peterson [C94]. These have been extended by Hultman to our setting (or more generally to K -orbits in any flag variety G/B where the symmetric pair (G, K) corresponds to a real form G_0 of the reductive group G whose Cartan subgroups form a single conjugacy class [H07]). To state them we first recall that the standard partial order on K -orbits, by containment of their closures, corresponds to the (restriction of the) reverse Bruhat order on I_{2n} [RS90]. The poset I_{2n} equipped with this order is then graded via the

rank function

$$r(\pi) = n^2 - n - \sum_{i < \pi(i)} (\pi(i) - i - \#\{k \in \mathbb{N} : i < k < \pi(i), \pi(k) < i\})$$

where this quantity equals the difference in dimension between $\mathcal{O}_\pi$ and $\mathbb{O}_c$, the unique closed orbit, corresponding to the involution $w_0 = 2n \dots 1$) [RS90]. Let I_π be the interval consisting of all $\pi' \leq \pi$ in the reverse Bruhat order. Then Hultman has shown that $\bar{\mathcal{O}}_\pi$ is rationally smooth if and only if I_π is rank-symmetric in the sense of having the same number of elements of rank i as of rank $r - i$ for all i , where r is the rank of π ; equivalently, if and only if the rank generating function $P_\pi = \sum_{\pi' \leq \pi} q^{r(\pi')}$ is palindromic [H07]. If we transform I_π into a graph BG_π by decreeing that the vertices μ and ν are adjacent if and only if $\nu = t\mu t \neq \mu$ for some transposition t in S_{2n} , then $\bar{\mathcal{O}}_\pi$ is rationally smooth if and only if BG_π is regular of degree r . More generally, if $\mu < \pi$ and we make the reverse Bruhat interval $[\mu, \pi]$ into a graph $BG_{\mu, \pi}$ by the same recipe, then $\bar{\mathcal{O}}_\pi$ is rationally smooth at $\mathcal{O}_\mu$ if and only if $BG_{\mu, \pi}$ is regular of degree $r(\pi) - r(\mu)$ [H07]. (There is no corresponding poset-theoretic criterion for rational smoothness of $\bar{\mathcal{O}}_\pi$ at $\mathcal{O}_\mu$.)

3. MAIN RESULT

Now we can characterize the K -orbits with rationally smooth closure.

Theorem 1. *The orbit $\mathcal{O}_\pi$ has rationally smooth closure if and only if π avoids the 17 patterns 351624, 64827153, 57681324, 53281764, 43218765, 65872143, 21654387, 21563487, 34127856, 43217856, 34128765, 36154287, 21754836, 63287154, 54821763, 46513287, 21768435.*

Proof. Note first that this list of bad patterns is stable under the automorphism of the Dynkin diagram: the first nine patterns are fixed by this automorphism while the next four pairs of patterns are interchanged. Suppose first that π coincides with one of the bad patterns. Then one readily checks that the degree of the bottom vertex in BG_π is larger than $r(\pi)$. Now suppose that π contains a bad pattern. If $\pi \in I_{2n}$, then we will show by induction on n that BG_π is irregular at the vertex μ , where the one-line notation of μ is obtained from that of π by rewriting the indices in the bad pattern in decreasing order and leaving the other indices unchanged. Write $\pi = \pi' t$, $\mu = \mu' t$, where π' has the same bad pattern as π , μ' is obtained from π' as μ was from π , and t is a transposition of two indices fixed by both π' and μ' . We say that the two-cycle (a, d) flipping the indices a and d with $a < d$ encapsulates the two-cycle (b, c) with $b < c$ if $a < b < c < d$. Then the rank difference $r(\pi) - r(\mu)$ is given by $r(\pi') - r(\mu') + 2(n(\mu', \mu) - n(\pi, \pi'))$, where $n(\mu', \mu)$ is the number of pairs (c', c) of two-cycles c', c such that c' appears in μ' , c appears in μ but not μ' , and c' encapsulates c ; similarly $n(\pi, \pi')$ is the number of pairs (d, d') of two-cycles d, d' such that d' appears in π' , d appears in π but not π' , and d encapsulates d' . Now if t is not counted by $n(\mu', \mu)$ or by both $n(\mu', \mu)$ and $n(\pi, \pi')$, then π, μ have the same rank difference as π', μ' and one checks immediately that BG_π is irregular at μ . If instead t is counted by $n(\mu', \mu)$ but not $n(\pi, \pi')$ then one easily locates two transpositions counted by the degree of μ in π but not the degree of μ' in π' , so again BG_π is irregular at μ , as desired.

Conversely, suppose that π avoids all patterns in the above list. We will show that the rank generating polynomial P_π is palindromic, or more precisely it is the product of sums of the form $1 + q + \dots + q^t$ for various t . Let $\pi = \pi_1 \dots \pi_{2n}$ and

assume first that $2n - \pi_1 \leq \pi_{2n} - 1$ (i.e., 1 is closer to the end of $\pi_1 \dots \pi_{2n}$ than $2n$ is to the beginning). Set $\pi^{(1)} = t\pi t$, where t is the transposition interchanging π_1 and $\pi_1 + 1$, so that 1 appears one place further to the right in $\pi^{(1)}$ than in π . Define $\pi^{(2)}, \dots, \pi^{(2n-\pi_1)}$ similarly, so that 1 appears at the end of $\pi^{(2n-\pi_1)}$. If $\mu = \mu_1, \dots, \mu_{2n} < \pi$, then Proctor's criterion for the Bruhat order [P82] shows that $\mu_1 \geq \pi_1$. If $\mu_1 = \pi_1$, then one checks that $\mu' < \pi'$, where μ', π' are obtained from μ, π by omitting the indices 1 and μ_1 , replacing all indices i between 1 and μ_1 by $i - 1$, and replacing all indices $j > \mu_1$ by $j - 2$; moreover, π' continues to avoid all bad patterns. If instead $\mu_1 > \pi_1$, then we claim that $\mu \leq \pi^{(1)}$ and that $\pi^{(1)}$ continues to avoid all bad patterns. If this holds, then induction shows that $\mu \leq \pi^{(\mu_1-\pi_1)}$ and whence we may as above eliminate the indices 1 and μ_1 from μ and $\pi^{(\mu_1-\pi_1)}$ and repeat the above procedure. Using the formula for the rank function in I_{2n} , we deduce that P_π factors in the way claimed above, where the first factor is $1 + q + \dots + q^{2n-\pi_1}$.

To prove the claim, set $\pi_1 = k, \pi_{k+1} = i$ and suppose that there is μ with $\mu < \pi, \mu \not\leq \pi^{(1)}$, and $\mu_i > \pi_1$. Set $j := \mu_k$. There are two cases. If $i < k$, then the conditions $\mu < \pi, \mu \not\leq \pi^{(1)}$ force $\pi_j > k + 1$, whence π contains the pattern $p := 465132$. The assumption $2n - \pi_1 \leq \pi_{2n} - 1$ implies that $\pi \neq p$, so that π is the product of three two-cycles forming the pattern p and at least one more two-cycle. Now one checks that no matter how one chooses this two-cycle to force $2n - \pi_1 \leq \pi_{2n} - 1$ we get a bad pattern in π , a contradiction. Similarly, if instead $i > k + 1$, then one must again have $\pi_j > k + 1$. In this case π contains the pattern 361542, and once again the assumption $2n - \pi_1 \leq \pi_{2n} - 1$ forces π to contain a bad pattern.

If instead $\pi_{2n} - 1 < 2n - \pi_1$, then one repeats the above argument, replacing 1 by $2n$ and moving $2n$ to the left instead of 1 to the right. Thus we define $\pi^{(1)}, \pi^{(2)}$, and so on, so that $2n$ appears one place to the left in $\pi^{(1)}$ than it does in π ; if $\mu \leq \pi$ then we must have $\mu_{2n} \leq \pi_{2n}$, and if $\mu_{2n} < \pi_{2n}$, then we must have $\mu \leq \pi^{(1)}$, lest π contain a bad pattern.

Finally, we must check in both cases that $\pi^{(1)}$ avoids all bad patterns whenever π does. This follows by a similar argument: if for example $2n - \pi_1 \leq \pi_{2n} - 1$ and $\pi^{(1)}$ contains a bad pattern, then one checks that π has one also (and recall that the list of bad patterns is stable under the Dynkin diagram automorphism). $\square$

4. SMOOTHNESS AND THE BOTTOM VERTEX

We now consider reverse Bruhat intervals $[\mu, \pi]$ and their graphs $BG_{\mu, \pi}$. We will find (as for Schubert varieties in type A) that it is only necessary to test one vertex in this graph to determine whether or not $\bar{\mathcal{O}}_\pi$ is (rationally) smooth at $\bar{\mathcal{O}}_\mu$.

Theorem 2. *f $\mu < \pi$ and the degree of μ in $BG_{\mu, \pi}$ equals $r(\pi) - r(\mu)$, then $\bar{\mathcal{O}}_\pi$ is smooth along $\bar{\mathcal{O}}$. In particular, the singular and rationally singular loci of $\bar{\mathcal{O}}$ coincide.*

Proof. Assume first that $\mathcal{O}_\mu = \mathcal{O}_c$, the closed orbit. Fix a basis (e_i) of $\mathbb{C}^{2n}$ such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i < j, i + j = 2n + 1 \\ -1 & \text{if } i > j, i + j = 2n + 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the skew form. Let (a_{ij}) be a family of complex parameters indexed by ordered pairs (i, j) satisfying either $i < n < j$ and $i + j \neq 2n + 1$, or $n < i < j$. We assume that $a_{ij} = -a_{2n+1-j, 2n+1-i}$ if $i > 2n + 1 - j$ but otherwise put no restrictions on the a_{ij} . Define a basis (b_i) of $\mathbb{C}^{2n}$ via

$$b_i = \begin{cases} e_i + \sum_{j=n+1}^{2n} a_{ij} e_j & \text{if } i \leq n \\ e_i + \sum_{j=i+1}^{2n} a_{ij} e_j & \text{otherwise} \end{cases}$$

Then the Gram matrix $G := (g_{ij} = (\langle b_i, b_j \rangle))$ of the b_i relative to the form satisfies

$$g_{ij} = \begin{cases} 2a_{i, 2n+1-j} & \text{if } i < j \leq n \\ -g_{ji} & \text{if } j < i \leq n \\ a_{ij} & \text{if } i < n < j < 2n + 1 - i \\ 1 & \text{if } i < n < j = 2n + 1 - i \\ -g_{ji} & \text{if } j < n < i \\ 0 & \text{otherwise} \end{cases}$$

Thus the matrix G is skew-symmetric and has zeroes below the antidiagonal from lower left to upper right. The antidiagonal entries are all ± 1 . Now one checks that the set $\mathcal{F}$ of all flags $V_0 \subset \dots \subset V_{2n}$ where (b_i) runs through all bases obtained as above from the a_{ij} and V_i is the span of $b_1, \dots, b_i$ is a slice in the sense of Brion to $\mathcal{O}_c$ at the flag f_c corresponding to the basis (e_i) [Br99, 2.1]. Intersecting $\mathcal{F}$ with $\bar{\mathcal{O}}_\pi$ we get another slice to $\mathcal{O}_c$ at f_c .

By hypothesis there are $n^2 - n - r(\pi)$ distinct conjugates $c = tw_0t$ of w_0 by a transposition t such that $c \not\leq \pi$. Writing c as $c_1 \dots c_{2n}$, let i be the smallest index such that $\pi_1 \dots \pi_i \not\leq c_1 \dots c_i$ in the standard partial order on sequences used to characterize the Bruhat order. Thus if $\pi_1 \dots \pi_i$ is rearranged in increasing order as $\pi'_1 \dots \pi'_i$ and similarly $c_1 \dots c_i$ is rearranged as $c'_1 \dots c'_i$, then $\pi'_j > c'_j$ for some j . Then for some k there are more indices $\ell \leq i$ with $c_\ell < k$ than indices $m \leq i$ with $\pi_m < k$. Equating all minors of the appropriate size lying in the first i rows of the Gram matrix to 0, we arrive at $n^2 - n - r(\pi)$ polynomials vanishing on $\bar{\mathcal{O}}_\pi \cap \mathcal{F}$, each involving a distinct variable raised to the first power with coefficient ± 1 . Then the Jacobian matrix of these polynomials has rank $n^2 - n - r(\pi)$, whence by the Jacobian criterion both $\mathcal{F} \cap \bar{\mathcal{O}}_\pi$ and $\bar{\mathcal{O}}_\pi$ are smooth at $\mathcal{O}_c$, as desired [Br99, 2.1]

If $\mathcal{O}_c$ is replaced by any orbit $\mathcal{O}_\mu \subset \bar{\mathcal{O}}_\pi$, then let G_μ be the matrix whose ij -entry is 1 if $j = \mu_i > i$, -1 if $j = \mu_i < i$, and 0 otherwise. This is the Gram matrix of a basis (b_i) obtained by suitably rearranging the basis (e_i) ; let f_μ be the corresponding flag. Now consider the set of all Gram matrices G whose ij -entries agree with those of G_μ if $j \geq \mu_i$ and whose other possibly nonzero entries are determined as follows. There are $n^2 - n - r(\mu)$ distinct conjugates of μ by transpositions with $c > \mu$. Write each c as $c_1 \dots c_{2n}$ and let i be the smallest index with $c_i < \mu_i$. Then the other possibly nonzero entries of G appear in the positions (i, c_i) together with their transposes (c_i, i) . All other entries of G are 0; they are otherwise unrestricted apart of course from being skew-symmetric. This set of Gram matrices stands in bijection to a set $\mathcal{F}_\mu$ of flags which is a slice to $\bar{\mathcal{O}}_\mu$ at f_μ . Then one shows as above that if the degree of μ in $BG_{\mu, \pi}$ equals the difference $r(\pi) - r(\mu)$, then there are

$n^2 - n - r(\pi)$ polynomials vanishing on $\bar{\mathcal{O}}_\pi \cap \mathcal{F}_\mu$ whose Jacobian matrix has rank $n^2 - n - r(\pi)$, whence again $\bar{\mathcal{O}}_\pi$ is smooth along $\mathcal{O}_\mu$, as desired. $\square$

There are two other symmetric pairs (G, K) of complex reductive groups satisfying the hypothesis of [H07] (that all Cartan subgroups in the corresponding real form G_0 of G are conjugate), namely $(SO(2n, \mathbb{C}), SO(2n - 2, \mathbb{C}) \times SO(2, \mathbb{C}))$ and (E_6, F_4) . In the first case all K -orbits in G/B have smooth closure. In the second case, eleven out of the forty-five K -orbits have rationally singular closure. Hultman has checked in each case the bottom vertex of the Bruhat graph detects the rational singularity. It is not known whether smoothness and rational smoothness are equivalent for these orbit closures.

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