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Rings of regular functions on nilpotent orbits and their covers

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Summary. Let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. Let $\mathcal{O} \subset \mathfrak{g}$ be a nilpotent G-orbit, $R(\mathcal{O})$ its ring of regular functions. We derive a formula for $R(\mathcal{O})$ as a G-module and prove some partial results on $R(\tilde{\mathcal{O}})$, $\tilde{\mathcal{O}}$ a cover of $\mathcal{O}$. We then relate this formula to various existing multiplicity formulas for K-types in Harish-Chandra bimodules of G.

§ 1. Introduction

Let G be a complex semisimple algebraic group, $\mathfrak{g}$ its Lie algebra. In this paper, we study rings of regular functions on G-orbits of nilpotent elements in $\mathfrak{g}$, concentrating attention on their G-module structure. Although understanding such rings is important for studying algebro-geometric questions relating to the normality of orbit closures and the birationality of moment maps, our primary motivation comes from conjectures of Vogan in [V1] and [V2], which state that certain finitely-generated admissible Harish-Chandra bimodules over G having also the structure of associative algebras should be realizable as deformations of rings of regular functions. Such algebras are called Dixmier algebras in [M1] and [V2]; they are extensively studied in [M1] and [M2]. Vogan's conjectures actually apply only to Dixmier algebras corresponding to nilpotent orbits in some sense; such algebras are called "unipotent". (Their precise definition has not been standardized.) We focus here on the G-action rather than the algebra structure because Vogan's conjectures allow a wide latitude in the algebra structure of Dixmier algebras, but imply that their G-module structure must agree with that of the regular functions on some orbit cover.

In what follows T will denote a maximal torus of G with Lie algebra $\mathfrak{t}$, Δ^+ a choice of positive $\mathfrak{t}$-roots in $\mathfrak{g}$, W the Weyl group, $\varepsilon: W \to \pm 1$ the sign representation, and ρ the half-sum of the positive roots. If $\lambda \in \mathfrak{t}^*$ exponentiates to a character of T, we denote this character by e^λ. Let $\hat{G}$ denote the set of irreducible finite-dimensional representations of G; identify these with their corresponding

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modules. If V is any (possibly infinite) direct sum of elements of $\hat{G}$ and $E\in\hat{G}$ then we call $\dim_{\mathbb{C}}\operatorname{Hom}_G(E, V)$ the multiplicity of E in V and denote it by $[V:E]$. If $[V:E]\neq 0$ we call E a K-type of V. We extend this notation to representations of Levi subgroups of G.

§ 2. Two preliminary lemmas

Let $e\in\mathfrak{g}$ be nilpotent. We will describe $R(G\cdot e)$, the ring of regular functions on the orbit of e, in terms of algebraically induced building blocks. More precisely, let L be a Levi subgroup of G and F an element of $\hat{L}$. We define $\operatorname{Ind}_L^G(F)$ to be the unique sum V of elements in $\hat{G}$ such that $[V:E]=[E:F]$ for any $E\in\hat{G}$. We extend the definition of $\operatorname{Ind}_L^G(F)$ to F lying in the Grothendiek group generated by $\hat{L}$ in the obvious way, also allowing ourselves to write $\operatorname{Ind}_L^G(zF)$ and $z\operatorname{Ind}_L^G(F)$ for $z\in\mathbb{Z}$ and F in this Grothendieck group. The formula in the next section will express $R(G\cdot e)|_G$ as a finite weighted sum of various $\operatorname{Ind}_T^G(e^{\lambda})$. As motivation for using these particular building blocks, we may cite first a result of Kostant [K] that if e is principal, then $R(G\cdot e)\cong\operatorname{Ind}_T^G(e^0)$; the Weyl character formula, which implies that if $e=0$, then $R(G\cdot e)=\sum_{w\in W}\varepsilon(w)\operatorname{Ind}_T^G(e^{\rho-w\rho})$; and finally a very general result of Vogan [V3], which implies that for any e, $R(G\cdot e)$ must be expressible as a finite combination of $\operatorname{Ind}_T^G(e^{\lambda})$.

Embed e in a standard triple $\{h,e,f\}$ satisfying the bracket relations of the usual basis of $sl(2,\mathbb{C})$ by the Jacobson-Morozov theorem. Then h defines a $\mathbb{Z}$-gradation on $\mathfrak{g}$ via $\mathfrak{g}_i=i$-eigenspace of $\operatorname{ad} h$, as it is well known that all eigenvalues of $\operatorname{ad} h$ are integral (our standard triples have $[he]=2e$). Put $\mathfrak{q}=\sum_{i\geq 0}\mathfrak{g}_i$, the standard parabolic subalgebra attached to e, and let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be its Levi decomposition; then clearly $\mathfrak{l}=\mathfrak{g}_0$, $\mathfrak{u}=\sum_{i\geq 1}\mathfrak{g}_i$. Put $Q=LU=\exp\mathfrak{q}$, the corresponding parabolic subgroup of G. Since all standard triples with a fixed e are G-conjugate, the conjugacy class of Q is independent of the choice of standard triple [K]. Our first result relates Ind_L^G to sheaf cohomology of G/Q.

Lemma 2.1. $\sum_i(-1)^i H^i(G/Q, G\times_Q[S(\mathfrak{g}/\mathfrak{q})\otimes F])\cong\operatorname{Ind}_L^G(F)$ *for any* $F\in\hat{L}$, *extended to a* Q*-module by letting* U *act trivially. Here* $S(\mathfrak{g}/\mathfrak{q})$ *is the symmetric algebra on the vector space* $\mathfrak{g}/\mathfrak{q}$.

Proof. We have a standard identification $\operatorname{Ind}_L^G F\cong H^0(G/L, G\times_L F)$ realizing $\operatorname{Ind}_L^G F$ as a space of global sections of a vector bundle. The projection morphism $p\colon G/L\to G/Q$ is affine with fiber U. Hence $H^i(G/L, G\times_L F)\cong H^i(G/Q, p_*(G\times_L F))\cong H^i(G/Q, G\times_Q R(U)\otimes F)$, $R(U)$, the algebra of regular functions on U. Since G/L is affine, $H^i(G/L, G\times_L F)$ vanishes in positive degree, so $H^0(G/L, G\times_L F)$ is the Euler characteristic of $G\times_Q(R(U)\otimes F)$ over G/Q. The Q-modules $R(U)$ and $R(\mathfrak{u})$ are isomorphic when restricted to L($\mathfrak{u}$ being nilpotent), so by the additivity of Euler characteristic $H^0(G/L, G\times_L F)\cong\sum_i(-1)^i H^i(G/Q, G\times_Q R(\mathfrak{u})\otimes F)$. But $R(\mathfrak{u})=S(\mathfrak{u}^*)$ and the Killing form identifies $\mathfrak{u}^*$ with $\mathfrak{g}/\mathfrak{q}$, so the result follows. Q.E.D.

The above proof was communicated to me by Michel Brion. One can also give a computational proof by using Kostant's multiplicity formula for weights and the Bott-Borel-Weil theorem. The other result is a vanishing theorem which will require some notation. Put $V=\sum_{i\geq 2}\mathfrak{g}_i$; then V is a vector space which is just $\overline{Q\cdot e}$, by a result of Kostant [K]. Let $Z=G\times_Q V$. We have maps $\rho\colon Z\to G/Q$, $\pi\colon Z\to\overline{G\cdot e}$ given by projection to the first factor and the G-action, respectively. Let w_Z be the canonical line bundle on Z (the top exterior power of T^*Z). The vanishing result is

Lemma 2.2. $R^i\pi_* w_Z=0$ *for* $i>0$.

Proof. This is a special case of Satz 2.4 in [GR]. Q.E.D.

§ 3. The multiplicity formula

Our main result is

Theorem 3.1. *With the above notations, we have*

$$R(G\cdot e)\cong_G\sum_i(-1)^i\operatorname{Ind}_L^G(\Lambda^i\mathfrak{g}_1)$$

where we decree that $\Lambda^0\mathfrak{g}_1=\mathbb{C}$ *even if* $\mathfrak{g}_1=0$.

Proof. I first claim that the fiber of π over any $x\in G\cdot e$ is a singleton. Indeed, if $x=h\cdot e$, then this fiber is $\{(g,v)\in G\times V\,|\,g\cdot v=h\cdot e\}/(g\cdot q, v)\sim(g, q\cdot v)\cong\{gQ\,|\,g^{-1}h\cdot e\in V\}\in G/Q$. Now $G^e\subset Q$ since $G^e=U^e G^s$, G^s the centralizer of the standard triple $\{h,e,f\}$ [BV3], so $\dim Q\cdot y\geq\dim Q\cdot e$ for any $y\in G\cdot e$. Hence $V-Q\cdot e$, the boundary of $Q\cdot e$, being a union of orbits of dimension less than that of $Q\cdot e$, does not meet $G\cdot e$. It follows that $\{gQ\,|\,g^{-1}h\cdot e\in V\}=\{gQ\,|\,g^{-1}h\cdot e\in Q\cdot e\}=\{gQ\,|\,h\in gQ\supset gG^e\}=\{hQ\}$, so the fiber is a singleton, as claimed. Thus $R(G\times_Q V)=$ normalization of $R(\overline{G\cdot e})=R(G\cdot e)$, since $G\cdot e$ and $G\times_Q V$ are smooth [H]; we also have $R(G\cdot e)=\Gamma(X,\mathcal{O}_X)$, where X is the normalization of Z, $\mathcal{O}_{(\cdot)}$ the structure sheaf, and Γ denotes global sections. We have a Koszul resolution of $R(V)$ as a Q-module:

$$S(\mathfrak{g}/\mathfrak{q})\otimes\Lambda^{d=\mathrm{top}}\mathfrak{g}_1\to S(\mathfrak{g}/\mathfrak{q})\otimes\Lambda^{d-1}\mathfrak{g}_1\to\dots$$
$$\to S(\mathfrak{g}/\mathfrak{q})\otimes\Lambda^0\mathfrak{g}_1\to R(V)$$

where we note that $\mathfrak{u}/V\cong\mathfrak{g}_1$. From general nonsense $w_Z=\Lambda^{\mathrm{top}}V^*\otimes w_{G/Q}$, so corresponds to the character $\Lambda^{\mathrm{top}}(\sum_{i\leq -2}\mathfrak{g}_i)\otimes\Lambda^{\mathrm{top}}(\sum_{j\geq 1}\mathfrak{g}_j)\cong\Lambda^{\mathrm{top}}\mathfrak{g}_1$ of Q. We thus get a resolution of $\rho_* w_Z=G\times_Q(R(V)\otimes\Lambda^{\mathrm{top}}\mathfrak{g}_1^*)$ by replacing each term W in the above resolution by $G\times_Q(W\otimes\Lambda^{\mathrm{top}}\mathfrak{g}_1^*)$. Since the Euler characteristic is additive, we deduce that

$$\sum_i(-1)^i H^i(G/Q,\, p_* w_Z)=\sum_j(-1)^j\sum_i(-1)^i H^i(G/Q,\, G\times_Q[S(\mathfrak{g}/\mathfrak{q})\otimes\Lambda^j\mathfrak{g}_1\otimes\Lambda^{\mathrm{top}}\mathfrak{g}_1^*].$$

Now the standard symplectic form on $T_*(G\cdot e)$ pairs $\mathfrak{g}_1$ with itself, so $\mathfrak{g}_1$ is even-dimensional. It follows that the right side above corresponds to that of the theorem, by Lemma 2.1, so we need only show the same is true of the left side. There is a Grothendieck spectral sequence [G]

$$H^p(G/Q,\, R^q\rho_* w_Z)\Rightarrow H^{p+q}(Z, w_Z).$$

The direct images R^q with $q>0$ are 0 since ρ is affine, so the spectral sequence collapses and we get an isomorphism

$$H^p(G/Q,\, G\times_Q(R(V)\otimes\Lambda^{\mathrm{top}}\mathfrak{g}_1^*))\cong H^p(Z, w_Z).$$

Similarly, there is a spectral sequence

$$H^p(\overline{G\cdot e},\, R^q w_Z)\Rightarrow H^{p+q}(Z, w_Z)$$

and now since $\overline{G\cdot e}$ is affine, the cohomology $H^p(\cdot)$ with $p>0$ vanishes. Hence there is an isomorphism

$$H^0(\overline{G\cdot e},\, R^p w_Z)\cong\Gamma R^p w_Z\cong H^p(Z, w_Z)\cong H^p(G/Q,\, G\times_Q(R(V)\otimes\Lambda^{\mathrm{top}}\mathfrak{g}_1^*)).$$

By Lemma 2.2, the $H^p(Z, w_Z)$ vanish in positive degree, so we need only show that $H^0(Z, w_Z)\cong\Gamma(X, \mathcal{O}_X)$, since we observed above that $\Gamma(X, \mathcal{O}_X)\cong R(G\cdot e)$. Now it follows from what we have shown so far that w_Z has a nonzero G-invariant section ϕ. We have maps $J\colon \mathcal{O}_Z\to w_Z$, $j\colon\mathcal{O}_X\to w_Z$, $\Gamma J\colon\Gamma(Z,\mathcal{O}_Z)\to\Gamma(Z, w_Z)$, $\Gamma j\colon\Gamma(X,\mathcal{O}_X)\to\Gamma(Z, w_Z)$, the last two defined respectively by $F\to F\cdot\phi$, $f\to f\cdot\phi$. Now ΓJ is obviously injective, so Γj is. To see that Γj is surjective, note that since ϕ is not the zero section, any element of $\Gamma(Z, w_Z)$ is of the form $g\cdot\phi$, where g is a rational function (possibly with poles) on X. We must show that g has no poles. By the G-invariance of ϕ, g has no poles on the open G-invariant preimage of $G\cdot e$ on X (under the composition of the normalization map with π). But the complement of this set is a union of orbits of codimension at least 2, so by the normality of X, g has no poles there either. Q.E.D.

I do not know whether the map J occurring in the above proof is an isomorphism. If so, we would obtain a vanishing result conjectured in [M1]: $H^i(G/Q,\, G\times_Q R(V))=0$ for $i>0$. I would like to thank Eckart Viehweg for bringing certain crucial ideas in the above proof to my attention.

This formula for $R(G\cdot e)$ has a number of nice combinatorial properties. We may rewrite it (as promised above) in terms of the $\mathrm{Ind}_T^G(e^\lambda)$:

Corollary 3.2. $R(G\cdot e)\cong_G \mathrm{Ind}_T^G\Big(\sum_{\substack{\alpha\in\Delta\\ \mathfrak{g}_\alpha\subseteq\mathfrak{g}_0+\mathfrak{g}_1}}(e^0-e^\alpha)\Big)$, *where $\mathfrak{g}_\alpha$ denotes the α-root space of $\mathfrak{g}$.*

Proof. It is clear that the formal character of $\Sigma(-1)^i\Lambda^i\mathfrak{g}_1$ is given by $\prod_{\mathfrak{g}_\alpha\subseteq\mathfrak{g}_1}(e^0-e^\alpha)$. The above formula then follows immediately from this formula, the Weyl denominator formula for L, and a corollary of the Weyl character formula enabling one to rewrite any $\mathrm{Ind}_L^G(F)$ in terms of $\mathrm{Ind}_T^G(e^\lambda)$. Q.E.D.

We can also describe the L-action on $\mathfrak{g}_1$. By a result of Dynkin, we may assume that each simple root space (relative to Δ^+) belongs to $\mathfrak{g}_0, \mathfrak{g}_1$, or $\mathfrak{g}_2$; then $\mathfrak{g}_1$ is generated as on L-module by the simple root spaces in $\mathfrak{g}_1$. It is easy to see that these root spaces are L-lowest weight spaces (so they are all the L-lowest weight spaces). From the classification of Dynkin diagrams, we see that the corresponding highest weights are sums of fundamental dominant weights for simple components of $[\mathfrak{l},\mathfrak{l}]$. The multiples are usually 1 and the weights are usually minimal in the standard partial order in $\mathfrak{t}^*$, so the exterior powers are quite small and easy to decompose over L.

§ 4. Covers of $G\cdot e$

Let $G\tilde{\cdot}e$ be the simply connected cover of $G\cdot e$; recall that $\pi_1(G\cdot e)$ is just the component group G^e/G^e_0 of G^e if G is simply connected (assume this henceforth for simplicity). The largest and most interesting Dixmier algebras should correspond to orbit covers rather than orbits, so $R(G\tilde{\cdot}e)$ is more interesting than $R(G\cdot e)$ in some sense. There is a formula for $R(G\tilde{\cdot}e)$ having the same general shape as the formula in Theorem 3.1. More precisely, we have

Theorem 4.1. *With the above notations, there are finite dimensional representations $F_1, F_2, \ldots$ of L (not necessarily irreducible) such that*

$$R(G\tilde{\cdot}e)\cong\sum_i(-1)^i\mathrm{Ind}_L^G(\Lambda^i\mathfrak{g}_1\otimes F_i).$$

Proof. The first step is to realize $G\tilde{\cdot}e$ as a (noncoadjoint) orbit in its own right. As noted above, we have $G^e=U^eG^s$, where G^s is the centralizer of the standard triple $\{h,e,f\}$. By Chevalley's lemma, we can find a finite dimensional representation W' of G and $v\in W'$ such that the isotropy subgroup G^v of v is UG^s_0. Then the orbit of (e,v) in $\mathfrak{g}\oplus W'$ is isomorphic to $G\tilde{\cdot}e$. The orbit closure $\tilde{V}=\overline{Q\cdot(e,v)}$ is a finite cover of the vector space $V=\overline{Q\cdot e}$, and just as in the proof of Theorem 3.1, we get a proper map $G\times_Q\tilde{V}\to G\tilde{\cdot}e$ which is birational over the open orbit. Now $R(\tilde{V})$ is a finite $R(V)$-algebra, by virtue of the pullback of the natural map $\tilde{V}\to V$. As such it is graded, since $\tilde{V}$ injects into $G\tilde{\cdot}e$ and $G\cdot e$ has a natural $\mathbb{C}^*$-action (which lifts to $G\tilde{\cdot}e$ via a finite cover of $\mathbb{C}^*$). By the Hilbert chain-of-syzygies theorem, any free resolution of $R(\tilde{V})$ may be terminated after $\dim(\mathfrak{g}_1)$ steps. We obtain one such resolution by starting with the resolution for $R(V)$ and then inductively replacing each term by a free module of larger rank whose canonical basis maps onto a Q-invariant set of generators for the next term to the right. Now a typical term of our new resolution looks like $S(\mathfrak{g}/\mathfrak{q})\otimes\Lambda^i\mathfrak{g}_i\otimes F_i$. Lemma 2.1 carries over immediately to this new resolution, so it suffices to show that the analogue of Lemma 2.2 carries over. This follows easily from the proof of Satz 2.4 in [GR]; it is deduced from Satz 2.3, which carries over to generically finite maps (see [Ko, Thm. 2.1(ii)]). This concludes the proof. Q.E.D.

Unfortunately, it is far from clear just what the modules F_i ought to be; we cannot expect them to be naturally constructible from the L-modules $\mathfrak{g}_i$. As an example, we mention the following

Conjecture 4.2. *If $G\cdot e$ is principal, then $R(G\tilde{\cdot}e)\cong \mathrm{Ind}_T^G(\Sigma e^\lambda)$, where the λ run over the $\mathfrak{t}$-dominant integral weights that are minimal in the standard partial order.*

This is reasonable because it can be shown that $R(G\tilde{\cdot}e)$ contains $\mathrm{Ind}_T^G(\Sigma e^\lambda)$ by using first-order deformations, which we now introduce. Let $x\in\mathfrak{g}$ and let V be any finite-dimensional G-module. Define a map $\exp\infty x$ from the Grassmanian of V to itself, as follows: $(\exp\infty x)\cdot S=\lim_{n\to\infty}(\exp nx)\cdot S$. This limit always exists in the Grassmannian of V (and has the same dimension as S) provided that no two distinct eigenvalues of $\exp x$ acting on V have the same absolute value [H]. The map $\exp\infty x$ is called a first-order deformation; we will actually use it only in the case where x is nilpotent, where it always exists. We can even give an explicit formula for $\exp\infty x\cdot V$ if $V=\mathbb{C}v$ is one-dimensional; it is just $\mathbb{C}x^k\cdot v$, where $x^k\in U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$, and k is the largest integer such that $x^k\cdot v\neq 0$. The most important property of first-order deformations for our purposes is the following one, valid for $\mathfrak{s}$ in the Grassmannian of $\mathfrak{g}$, S in that of V: $[(\exp\infty x)\cdot\mathfrak{s}]\cdot[(\exp\infty x)\cdot S]\subset(\exp\infty x)\ (\mathfrak{s}\cdot S)$. This follows immediately from the definition. Equality need not hold, as we see from the following example: $\mathfrak{g}=\mathfrak{sl}(2)$, $V=\mathfrak{g}$, $x=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\mathfrak{s}=\mathbb{C}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, $S=\mathbb{C}x$. Our main result on deformations is the following

Proposition 4.3. *Let $e\in\mathfrak{g}$ be nilpotent. Suppose that the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$ is a deformation (= image under a finite sequence of first-order deformations) of a reductive subalgebra $\mathfrak{m}$. Then $R(G\tilde{\cdot}e)\supset\mathrm{Ind}_{M=\exp\mathfrak{m}}^G(\mathbb{C})$.*

Proof. By Frobenius reciprocity (as pointed out in [K]) we have $R(G\tilde{\cdot}e)=\mathrm{Ind}_{G_0^e}^G(\mathbb{C})=\mathrm{Ind}_{\mathfrak{g}^e}^{\mathfrak{g}}(\mathbb{C})$, where the last two terms are defined in the obvious way. Then the result follows immediately from the property of first-order deformations just mentioned. Q.E.D.

To prove the above assertion about $R(G\tilde{\cdot}e)$ when e is principal, we need a slight refinement of this proposition.

Proposition 4.4. *Retain the hypotheses of Proposition 4.3. Suppose in addition that the deformation mapping $\mathfrak{m}$ into $\mathfrak{g}^e$ maps some subalgebra $\mathfrak{m}'$ of $\mathfrak{m}$ into $Z(\mathfrak{g}^s)$, the center of $\mathfrak{g}^s=\mathrm{Lie}\,G^s$. Then $R(G\tilde{\cdot}e)\supset\overline{\mathrm{Ind}}_{\mathfrak{m}}^{\mathfrak{g}}(\Sigma V_\lambda)$, where the V_λ run over the one-dimensional representations of $\mathfrak{m}$ on which $\mathfrak{m}'$ acts trivially and* $\overline{\mathrm{Ind}}$, *the "flattened induction functor", is defined as follows:*

$$[\overline{\mathrm{Ind}}_{\mathfrak{m}}^{\mathfrak{g}}(\Sigma V_\lambda):E]=\begin{cases}[E:V_0=\mathbb{C}] & \text{if } [E:V_0]\neq 0\\ 1 & \text{if } [E:V_0]=0 \text{ but } [E:V_\lambda]\neq 0 \quad \text{for some } \lambda\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We need only show that any E with $[E:V_\lambda]\neq 0$ for some λ appears in $R(G\tilde{\cdot}e)$. Let w span a copy of V_λ in E; then G_0^e acts on the image of $\mathbb{C}w$. Its reductive part G^s acts trivially since both $Z(G^s)$ and $[G^s,G^s]$ do. Its unipotent radical U^e acts unipotently and thus trivially also. The conclusion follows. Q.E.D.

We also need the following useful criterion for realizing a centralizer $\mathfrak{g}^e$ as a deformation.

Proposition 4.5. *In the notation of § 2, let $\mathfrak{s}=\mathrm{Span}_{\mathbb{C}}(h,e,f)$. Let $\mathfrak{g}=\oplus\mathfrak{g}^i$ be the decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{s}$-modules. Suppose that there is a subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{m}\cap\mathfrak{g}^i$ is one-dimensional for each i. Then* $(\exp\infty e)\cdot\mathfrak{m}\supset\mathfrak{g}^e$.

Proof. This follows immediately from the definition of $\exp\infty e$, together with the above formula for $\exp\infty e$ of a one-dimensional space. Q.E.D.

This proposition applies to any even e (one for which all the h-eigenspaces $\mathfrak{g}_i$ of § 2 are 0 for i odd), for then we may take $\mathfrak{m}=\mathfrak{g}_0=\mathfrak{l}$. Now we can prove one containment of Conjecture 4.2. It is well known that for any $F\in\hat{G}$ we have $[F:e^\lambda]\neq 0$ for a unique minimal λ (regarding F as T-module); let F_λ denote the λ-weight space of F. Then Gupta has shown [Gu] that $(\exp\infty e)\cdot F_\lambda$ is the sum of $\exp\infty e$ applied to certain one-dimensional subspaces of F_λ (recall that e is now principal). The proof of Propositions 4.4 and 4.5 (with $\mathfrak{m}=\mathfrak{t}$) now combine to show that $\mathfrak{g}^e$ kills $(\exp\infty e)\cdot F_\lambda$, whence $R(G\tilde{\cdot}e)\supset\mathrm{Ind}_T^G(\Sigma e^\lambda)$.

Returning to the case of general e, we remark that Proposition 4.5 often leads to a formula for a (hopefully large) piece of $R(G\tilde{\cdot}e)$ that looks quite different from 3.1 or 4.1. For example, if $\mathfrak{g}=\mathfrak{sl}(n)$ and e corresponds to the partition $(p_1,\dots,p_r)$ of n via the Jordan form, then it turns out that 4.5 applies to a certain Levi subalgebra $\mathfrak{m}$ corresponding to the partition $(q_1,\dots,q_s)$ conjugate to $(p_1,\dots,p_r)$. We may construct $\mathfrak{m}$ as follows. Assume that the standard triple $\{h,e,f\}$ has h diagonal and e in Jordan form. Then $\mathfrak{m}$ is spanned by all matrix units E_{ij} having all entries 0 except for a 1 in the (i,j)-position, satisfying the following properties: (a) $i\neq j$; (b) h_i-h_j is 0 or ± 1; and (c) if $h_i-h_j\neq 0$, then $\max(h_i,h_j)$ is even. As the center Z of G in this case surjects onto $\pi_1(G\cdot e)$, it can be seen that $\exp\infty e$ applied to the $\mathfrak{m}$-fixed vectors in any G-module V is actually G^e-fixed, not merely G_0^e-fixed. Hence $R(G\cdot e)\supset\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(\mathbb{C})$. Since $G\cdot e$ is induced from the 0 orbit of $\mathfrak{m}$ in the sense of Lusztig and Spaltenstein, it follows from [LS] that equality actually holds. (The situation for other classical groups is unfortunately much more complicated, but 4.5 still applies to some orbits.) The importance of these observations will become clear in the next section.

§ 5. Comparison to multiplicity formulas in Dixmier algebras

If unipotent Dixmier algebras A are to be realizable as deformations of rings of regular functions R, then the G-module structures of A and R must be isomorphic. That is, it must be possible to rewrite the above formulas for $R(G\cdot e)$ and $R(G\tilde{\cdot}e)$ so that they look like character formulas for complex groups. We therefore make the following

Conjecture 5.1. *For each nilpotent e, there is formula for $R(G\cdot e)$ expressing it as a linear combination of $\operatorname{Ind}_T^G(\lambda - w\lambda)$ for a fixed λ and $w\in W$. The same holds if $G\cdot e$ is replaced by any cover.*

Probably this can be verified case by case without too much difficulty; it would be especially nice if the coefficient of $\operatorname{Ind}_T^G(\lambda - w\lambda)$ turned out to be a class function of $w\in W$, for every existing character formula for unipotent representations of complex groups has this property [BV1, 2, 3]. Now both 3.2 and 4.3 provide philosophical support for Conjecture 5.1, the former because it can be interpreted as a ratio of products of Weyl denominators, and the latter because it allows us to take $\lambda=\rho_m$ whenever $\operatorname{Ind}_m^{\mathfrak{g}}(\mathbb{C})$ fills out $R(G\tilde{\cdot}e)$ or $R(G\cdot e)$ (Here ρ_m is the half-sum of the positive roots of $\mathfrak{m}$; note also in this case that the coefficient of $\operatorname{Ind}_T^G(\lambda - w\lambda)$ is just $\varepsilon(w)$ or 0.)

Conjecture 5.1 allows us to attach elements of $\mathfrak{t}^*/W$ (which we may think of as infinitesimal characters) to nilpotent orbits. Barbasch and Vogan have already done this for the classical groups [BV1, 2], but they were motivated by very different (and rather more complicated) considerations than ours. It is therefore quite striking that their recipe often turns out to work well in our situation. For example, if $\mathfrak{g}$ is of type A_n, then we have already observed that $R(G\cdot e)\cong \operatorname{Ind}_m^{\mathfrak{g}}(\mathbb{C})$ for some $\mathfrak{m}$, and 4.4 provides a reasonable conjectural formula for $R(G\tilde{\cdot}e)$. Then the Barbasch-Vogan infinitesimal character attached to $G\cdot e$ is a twist $\tilde{\rho}_m$ of ρ_m, $\tilde{\rho}_m - W\tilde{\rho}_m$ meets the root lattice in the same set as $\rho_m - W\rho_m$ does, and $\tilde{\rho}_m - W\tilde{\rho}_m$ meets the weight lattice in exactly the set of weights λ such that $\operatorname{Ind}_T^G(e^{\lambda})$ appears in the conjectural formula for $R(G\tilde{\cdot}e)$ [M1, BV1]. If $\mathfrak{g}$ is of type G_2, then it turns out that 5.1 rapidly leads to a unique infinitesimal character attached to every orbit and that these characters play an important role in [BV2]. Finally, if e is principal (for any $\mathfrak{g}$), then the Barbasch-Vogan character attached to $G\cdot e$ is $\lambda=\frac{1}{n+1}\sum_{i=1}^{n}\lambda_i$, where the λ_i are exactly the λ of 4.2 [V3]. In [M1] it is shown that $\lambda - W\lambda$ meets the weight lattice in $\{\lambda_1, \dots, \lambda_n\}$ and the root lattice in $\{0\}$ (cf. 4.2). It is not difficult to show that λ is uniquely specified up to W-conjugacy by these properties.

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