CLOSURES OF K-ORBITS IN THE FLAG VARIETY FOR $U(p, q)$

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Abstract. We classify the $GL_p \times GL_q$-orbits in the flag variety for GL_{p+q} with rationally smooth closure, showing that they are all either already closed or are pullbacks from orbits with smooth closure in a partial flag variety.

1. Introduction

Let G be a complex reductive group with Borel subgroup B. The question of which Schubert varieties in the flag variety G/B are smooth has received a great deal of attention, particularly in recent years [BilLak00, BilPos05]. Less well studied, but very important for representation theory, are the closures of orbits in G/B under the action of the fixed point subgroup $K := G^\theta$ of G, where θ is an involutive automorphism of G [LusVog83]. Such orbit closures have been called symmetric varieties by Springer and are studied by him in [Spr92]. In this paper we use his techniques to decide which symmetric varieties are smooth in the special case $G = GL(p+q, \mathbb{C})$, $K = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$. We will give a pattern avoidance criterion for rational smoothness, along the lines of the well-known one for rational smoothness of Schubert varieties in type A. We will also show that all rationally smooth symmetric varieties in this case are either closed orbits or pullbacks of smooth varieties in partial flag varieties and so in particular are smooth.

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2. Preliminaries

Now let $G = GL(n, \mathbb{C})$, where $n = p + q$, and take θ to be conjugation by a diagonal matrix on G with p eigenvalues 1 and q eigenvalues -1, so that $K = G^\theta = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$. This group may also be viewed as the complexification of the maximal compact subgroup $U(p) \times U(q)$ of the real form $U(p, q)$ of G. Let B be the subgroup of upper triangular matrices in G. Recall that K-orbits in G/B are parametrized by clans, which are sequences $\gamma = (c_1, \ldots, c_n)$ of n symbols c_i, each either $+$ or $-$ or a natural number, such that every natural number occurs either exactly twice in γ or not at all [MatOsh88, Yam97]. We further require that the number of $+$ signs plus distinct numbers among the c_i be exactly p. We identify two clans if they have the same signs in the same positions and pairs of equal numbers in the same positions (so that for example $(1, +, 1, -)$ is identified with $(2, +, 2, -)$, but not with $(1, +, -, 1)$). We say that the clan $\gamma = (c_1, \ldots, c_n)$ *includes the pattern* $(d_1, \ldots, d_m)$ if there are indices $i_1 < \ldots < i_m$ such that the (possibly shorter) clans $(c_{i_1}, \ldots, c_{i_m})$ and $(d_1, \ldots, d_m)$ are identified. We say that γ *avoids* $(d_1, \ldots, d_m)$ if it does not include it. There is a similar parametrization of K-orbits in a partial flag variety G/Q (with Q a parabolic subgroup of G) by clans, except that the parabolic subgroup Q corresponds to a gathering of the n coordinates into blocks of consecutive coordinates (each block having only one coordinate if

$Q = B$) and we identify two clans whenever corresponding blocks of coordinates have the same signatures (number of $+$ signs plus pairs of equal numbers, and similarly for $-$ signs). For example, if $p = q = 2$ and Q corresponds to the middle simple root in the Dynkin diagram of G, then the clans $(1, +, -, 1)$ and $(1, 2, 2, 1)$ for G/Q are identified; if $p = q = 3$ and Q corresponds to the middle three roots in the Dynkin diagram of G, then the clans $(1, +, 2, -, 2, 1)$ and $(1, 2, 2, 3, 3, 1)$ are identified. The above notions of pattern inclusion and avoidance are motivated by the corresponding ones for permutations in one-line notation, which Lakshmibai and Sandhya used to characterize rationally smooth Schubert varieties in type A [LakSan90] and Billey later extended to the other classical types [Bil98].

K-orbits in G/B are partially ordered by containment of their closures. On the level of clans, this order includes the following operations: replace a pair of (not necessarily adjacent) opposite signs by a pair of equal numbers; or interchange a number with a sign so as to move the number farther away from its equal mate in the clan (and on the same side); or interchange a pair a, b of distinct numbers with a to the left of b, provided that the mate of a lies to the left of the mate of b ([RicSpr90, 5.12],[Yam97, 2.4]). Thus (the orbit corresponding to) $(1, +, 1, -)$ lies below $(1, 2, 1, 2)$ and $(1, +, -, 1)$, while $(1, 2, 1, 3, 2, 3)$ lies below $(1, 3, 1, 2, 2, 3)$) but not below $(1, 3, 1, 3, 2, 2)$. (These operations include but do *not* coincide with the ones generating the Matsuki-Oshima graph [MatOsh88].) In particular, the closed orbits are exactly those whose clans have only signs, while the open orbit has clan $(1, 2, \ldots, q, + \ldots, +, q, q-1, \ldots, 1)$, with $p - q$ plus signs, if $p > q$.

We will need a formula for the dimension of the orbit $\mathcal{O}_\gamma$ corresponding to the clan $\gamma = (c_1, \ldots, c_n)$ [Yam97, 2.3]. Set $d_{p,q} := \frac{1}{2}(p(p-1) + q(q-1))$. Then $\dim \mathcal{O}_\gamma$ is given by

$$d_{p,q} \ + \sum_{c_i = c_j \in \mathbb{N}, i<j} (j - i - \#\{k \in \mathbb{N} : c_s = c_t = k \text{ for some } s < i < t < j\})$$

In particular, the closed orbits all have the same dimension $d_{p,q}$.

We conclude this section by recalling and slightly generalizing the well-known derived functor module construction on the level of K-orbits. For this purpose let G be any complex reductive group and K its fixed points under an involution θ. Let Q be any θ-stable parabolic subgroup of G, containing the θ-stable Borel subgroup B. If $\mathfrak{q}$ is the Lie algebra of Q, then the orbit $K \cdot \mathfrak{q}$ identifies with a closed orbit in the partial flag variety G/Q [RicSpr90, 2.5]. Its preimage $\pi^{-1}(K \cdot \mathfrak{q})$ in G/B under the natural projection $\pi : G/B \to G/Q$ is the support of a derived functor module; we call the open orbit in this preimage a derived functor orbit [Tra05, §1]. Its closure fibers smoothly via π over $K \cdot \mathfrak{q}$ with fiber the flag variety Q/B of Q, so is smooth. More generally, let $\mathcal{O}_Q$ be any K-orbit in G/Q with smooth closure $\bar{\mathcal{O}}_Q$. The preimage $\pi^{-1}(\bar{\mathcal{O}}_Q)$ fibers smoothly over $\bar{\mathcal{O}}_Q$ with fiber Q/B, so is again smooth. We also call the open orbit $\mathcal{O}$ in this preimage a derived functor orbit; to avoid trivialities, we assume that $Q \neq B$ in the more general setting, unless $\mathcal{O}_B$ is already closed in G/B.

3. Main result

Now we can characterize the K-orbits with rationally smooth closure.

Theorem. *If the clan $\gamma = (c_1, \ldots, c_n)$ includes one of the patterns $(1,+,-,1)$, $(1,-,+,1)$, or $(1,2,1,2)$, then the orbit $\mathcal{O}_\gamma$ does not have rationally smooth closure. Otherwise $\mathcal{O}_\gamma$ is a derived functor orbit, so that its closure is smooth.*

Proof. Suppose first that γ includes one of the above patterns. If it includes $(1,+,-,1)$, replace the 1s by $-$ and $+$, in that order; if it includes $(1,-,+,1)$, replace the 1s by $+$ and $-$, in that order; if it includes $(1,2,1,2)$, replace these four symbols by $(+,-,-,+)$. In all three cases, continue by replacing every pair $a, \ldots, a$ of equal numbers in γ by $+, \ldots, -$. We obtain a clan corresponding to a closed orbit $\mathcal{O}$ below $\mathcal{O}_\gamma$ in the partial order. Now Springer has defined an action of the noncompact root reflections in the Weyl group S_n on the closed orbits, sending each such orbit to a higher orbit whose clan has exactly two numbers; more precisely, two opposite signs in the clan of the closed orbit are replaced by a pair of equal numbers [Spr92, 3.1,4.1]. One easily checks that more than $\dim \mathcal{O}_\gamma - d_{p,q}$ of these reflections send $\mathcal{O}$ to an orbit lying between it and $\mathcal{O}_\gamma$, whence $\mathcal{O}_\gamma$ is not rationally smooth, as claimed [Spr92, 3.2,3.3].

Now suppose that γ avoids the above patterns. Then the intervals $[s,t]$ of indices s,t with $c_s = c_t \in \mathbb{N}$ are such that any two of them are one contained in the other or disjoint. Moreover, all signs lying between any pair of equal numbers in γ are the same. If γ has a sign not lying between a pair of equal numbers, let Q be the parabolic subgroup of G corresponding to the simple roots not involving the coordinate of the sign. Deleting this sign from γ, we obtain one or two clans, one consisting of the coordinates of γ to the left of this sign, the other of the coordinates of γ to the right of it. By induction on the number of coordinates of γ, we may assume that the orbits corresponding to these clans have smooth closure. One checks that the image of $\mathcal{O}_\gamma$ under π identifies with the closed orbit $K \cdot \mathfrak{q}$ in G/Q (where $\mathfrak{q}$ is the Lie algebra of Q), with the same clan γ, whence $\bar{\mathcal{O}}_\gamma$ fibers smoothly over this orbit via the projection $\pi : G/B \to G/Q$ with smooth fiber. Hence $\bar{\mathcal{O}}_\gamma$ is smooth, as desired. If γ consists of at least two blocks of coordinates, each flanked by pairs of equal numbers not lying between any other pair of equal numbers, with no signs lying between the blocks, then a similar argument shows that $\mathcal{O}_\gamma$ has smooth closure, taking Q to be the parabolic subgroup corresponding to the simple roots whose coordinates lie in the same block. Otherwise the first and last coordinates of γ are a pair of equal numbers. Taking Q to be the parabolic subgroup of G corresponding to the remaining coordinates, we now find that the image of $\bar{\mathcal{O}}_\gamma$ under π is the full flag variety G/Q, again with the same clan γ, and the restriction of π to this orbit closure has smooth fibers, as usual. Hence in all cases this orbit closure is smooth. If $\mathcal{O}_\gamma$ is not already closed, let $c_i = c_j = a$ be a pair of equal numbers in γ with no numbers between them. Let Q be the parabolic subgroup of G corresponding to the coordinates weakly between i and j. Then the image of $\bar{\mathcal{O}}_\gamma$ under π is smooth in G/Q and $\mathcal{O}_\gamma$ is the derived functor orbit obtained from this image, as desired. □

In future work we hope to find similar pattern avoidance criteria for rational smoothness of K-orbit closures in the flag varieties of other classical groups. There are two nonsmooth orbit closures for $GL(4,\mathbb{R})$, none for $SU^*(4)$, and one for $SU^*(6)$.

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