7 Induced Nilpotent Orbits

In Chapter 4, we constructed three canonical nilpotent orbits in any simple Lie algebra $\mathfrak{g}$, essentially from nothing. In this chapter we show how to construct new nilpotent orbits from old ones (in smaller algebras), following Lusztig and Spaltenstein. More precisely, given a nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we will produce a nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in $\mathfrak{g}$ called the orbit *induced* from $\mathcal{O}_{\mathfrak{l}}$. The definition of $\mathcal{O}_{\mathfrak{g}}$ seems to depend on a choice of parabolic subalgebra $\mathfrak{p}$ with Levi subalgebra $\mathfrak{l}$, but we will prove that $\mathcal{O}_{\mathfrak{g}}$ is actually independent of this choice. In §7.2 we show that every nilpotent orbit in $\mathfrak{sl}_n$ is induced from the 0 orbit in some Levi subalgebra and give a formula for the partition of any induced orbit. This formula is generalized in the next section to any classical algebra $\mathfrak{g}$. We also give a simple partition criterion due to Kempken and Spaltenstein for a classical orbit to be rigid (that is, not induced from any other orbit).

7.1 Basic Results

The results in this section are taken from [62]. Let $\mathfrak{p}$ be a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$ with Levi decomposition $\mathfrak{l} \oplus \mathfrak{n}$. Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in $\mathfrak{l}$. The basic idea is to generalize the construction of the principal orbit in Chapter 4. If $\mathfrak{p}$ happens to be a Borel subalgebra, then we know that $\mathcal{O}_{\mathfrak{l}}$ has to be the zero orbit. In this case we have seen that, although $G_{ad} \cdot \mathfrak{n}$ is not a single G_{ad}-orbit, it does admit a unique open dense suborbit. Motivated by that fact, we look for a nilpotent orbit in $\mathfrak{g}$ meeting $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$ in a dense set.

Theorem 7.1.1. *Retain the above notation and let P_{ad} be the connected Lie subgroup of G_{ad} with Lie algebra $\mathfrak{p}$. Then there is a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in $\mathfrak{g}$ meeting $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$ in an open dense set. We have $\dim \mathcal{O}_{\mathfrak{g}} = \dim \mathcal{O}_{\mathfrak{l}} + 2 \dim \mathfrak{n}$; in fact, $\mathcal{O}_{\mathfrak{g}}$ is also the unique orbit in $\mathfrak{g}$ of this dimension which meets $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$. The intersection $\mathcal{O}_{\mathfrak{g}} \cap (\mathcal{O}_{\mathfrak{l}} + \mathfrak{n})$ consists of a single P_{ad}-orbit.*

Proof. Let L_{ad} be the connected Lie subgroup of G_{ad} with Lie algebra $\mathfrak{l}$. By (3.8.1), there is a Borel subalgebra $\mathfrak{b}_{\mathfrak{l}}$ of $\mathfrak{l}$ such that $\mathfrak{b}_{\mathfrak{l}} + \mathfrak{n}$ is a Borel subalgebra $\mathfrak{b}$

of $\mathfrak{g}$ (in fact, any Borel subalgebra of $\mathfrak{l}$ has this property). Given $X \in \mathcal{O}_\mathfrak{l}$, we can conjugate it via L_{ad} to an element of the nilradical $\mathfrak{n}_\mathfrak{l}$ of $\mathfrak{b}_\mathfrak{l}$ (by (3.2.1) and (3.2.2)). Since L_{ad} stabilizes $\mathfrak{n}$, it follows that every element in the affine space $X + \mathfrak{n}$ is L_{ad}-conjugate to an element of the nilradical of $\mathfrak{b}$. Hence $\mathcal{U} := \mathcal{O}_\mathfrak{l} + \mathfrak{n} \subset \mathcal{N}$, the nilpotent cone in $\mathfrak{g}$. Since $\mathcal{N}$ is G_{ad}-stable, we also have $G_{ad} \cdot \mathcal{U} \subset \mathcal{N}$. Consequently $\mathcal{V} := G_{ad} \cdot \mathcal{U}$ is a finite union of orbits, at least one of which must have the same dimension as $\mathcal{V}$ itself, and thus be open. But now $\mathcal{O}_\mathfrak{l}, \mathcal{U}$, and $\mathcal{V}$ are all irreducible as varieties, the latter two because G_{ad} is irreducible. It follows that $\mathcal{V}$ has a *unique* suborbit $\mathcal{O}_\mathfrak{g}$, which is also dense. Since any other orbit meeting $\mathcal{U}$ lies in the closure of $\mathcal{O}_\mathfrak{g}$, this orbit is also the unique one of its dimension meeting $\mathcal{U}$. Assume for a moment that $\dim \mathcal{V} = d := \dim \mathcal{O}_\mathfrak{l} + 2\dim \mathfrak{n}$ and let $X \in \mathcal{O}_\mathfrak{g} \cap \mathcal{U}$. Then the dimension of the tangent space $[\mathfrak{g}, X]$ of $\mathcal{O}_\mathfrak{g}$ at X is the sum of the dimensions of $[\mathfrak{p}, X]$ and $[\bar{\mathfrak{n}}, X]$ (notation (3.8.6)). Since both $\mathfrak{p}$ and $\bar{\mathfrak{n}}$ are ad_X-stable, the former space $[\mathfrak{p}, X]$ sits inside the tangent space of $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ at X,while the latter one $[\bar{\mathfrak{n}}, X]$ has dimension bounded above by that of $\mathfrak{n}$. It follows at once that $\dim P_{ad} \cdot X = \dim \mathcal{U}$, whence $P_{ad} \cdot X$ is open in $\mathcal{U}$; it is also dense because $\mathcal{U}$ is irreducible. Hence $\mathcal{O}_\mathfrak{g} \cap \mathcal{U}$ is a single P_{ad}-orbit; there is no room to accommodate a second orbit.

It only remains to compute the dimension of $\mathcal{V}$. The above paragraph shows that $\dim \mathcal{V} \leq d$, so we need only prove the other inequality. If $X \in \mathcal{O}_\mathfrak{g} \cap \mathcal{U}$, then $G_{ad} \cdot X$ meets $\mathcal{U}$ in a dense set, so by differentiation $[\mathfrak{g}, X] \supset T_X \supset \mathfrak{n}$, where T_X is the tangent space of $\mathcal{U}$ at X. Since the centralizer $\mathfrak{g}^X$ is orthogonal to the tangent space $[\mathfrak{g}, X]$ under the Killing form, while $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are paired nondegenerately by this form, it follows that $\bar{\mathfrak{n}}^X = 0$. Consequently, there is a Zariski-open subset $\mathcal{O}_\mathfrak{g} \cap \mathcal{U}$ of $\mathcal{U}$ consisting of elements X with $\dim [\bar{\mathfrak{n}}, X] = \dim \bar{\mathfrak{n}}$. Now if we can pick out a dense subset of $\mathcal{O}_\mathfrak{g} \cap \mathcal{U}$ consisting of elements X such that $[\bar{\mathfrak{n}}, X]$ does not overlap with $[\mathfrak{p}, X]$, then it will follow that $\dim \mathcal{V} \geq d$, as required. To this end, let $\mathcal{U}'$ denote the set of $Y \in \mathcal{U}$ such that

$$[\bar{\mathfrak{n}}, Y] \cap ([\mathfrak{l}, \mathfrak{l}] + \mathfrak{n}) = 0.$$

Arguing as in the proof of (3.4.12), we see that $\mathcal{U}'$ is a Zariski-open subset of $\mathcal{U}$. It is not empty since an easy calculation shows that $Y + X_\alpha \in \mathcal{U}'$ whenever $Y \in \mathcal{O}_\mathfrak{l}$ and X_α is a simple root vector in $\mathfrak{n}$ (relative to $\mathfrak{b}$). Thus, the intersection $\mathcal{O}_\mathfrak{g} \cap \mathcal{U}'$ has the desired properties. This concludes the proof. □

Remark. One way to remember the dimension formula for $\mathcal{O}_\mathfrak{g}$ is to recall the discussion of symplectic manifolds in §1.4. The submanifold $\mathcal{O}_\mathfrak{g} \cap \mathcal{U}$ of $\mathcal{O}_\mathfrak{g}$ carries a symplectic form on each of its tangent spaces whose radical has dimension equal to that of $\mathfrak{n}$, by (7.1.1) and the definition (1.4.5) of the symplectic form. Thus, this submanifold needs to pick up $\dim \mathfrak{n}$ more dimensions before it can become the symplectic manifold $\mathcal{O}_\mathfrak{g}$. An easier way to remember the dimension formula is given in (7.1.4).

Following [62], we say that the orbit $\mathcal{O}_\mathfrak{g}$ is *induced from* $\mathcal{O}_\mathfrak{l}$ and denote it by

$$\mathrm{Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l}). \tag{7.1.2}$$

If $\mathcal{O}_\mathfrak{l} = 0$, then we call $\mathcal{O}_\mathfrak{g}$ a *Richardson* orbit; its existence and basic properties were proved in [72] before the paper [62] appeared. For example, it follows from (4.1.6) that the principal orbit $\mathcal{O}_{prin}$ is a Richardson orbit; it is induced from the zero orbit in any Cartan subalgebra of $\mathfrak{g}$. We have not quite completed the proof of (4.2.1), but you will readily observe that the subregular orbit $\mathcal{O}_{subreg}$ is also a Richardson orbit; in fact, our proof of its existence used this property in a crucial way.

Since we saw in §3.8 that $\mathfrak{g}$ has a large collection of nonconjugate parabolic subalgebras, induction of orbits is a powerful tool for constructing a large number of nilpotent orbits in $\mathfrak{g}$. Several questions immediately arise.

Questions. Which nilpotent orbits in $\mathfrak{g}$ are induced? Which orbits in $\mathfrak{g}$ are Richardson orbits? Is it possible for a nilpotent orbit to be induced in two different ways? Can we describe the weighted Dynkin diagram of an induced orbit in terms of the weighted diagram of the inducing orbit? Can we describe the partition of an induced orbit in terms of that of the inducing orbit?

We will ultimately discuss all these questions, but first we want to make the very important observation that the orbit induced from a particular $\mathcal{O}_\mathfrak{l}$ depends only on the Levi subalgebra $\mathfrak{l}$, not on the choice of parabolic subalgebra $\mathfrak{p}$ containing it.

Theorem 7.1.3. *Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{p}' = \mathfrak{l} \oplus \mathfrak{n}'$ be two parabolic subalgebras of $\mathfrak{g}$ with the same Levi subalgebra $\mathfrak{l}$, and let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$. Then, $\mathrm{Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l}) = \mathrm{Ind}_{\mathfrak{p}'}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$.*

Proof. Let $\mathcal{O}_\mathfrak{g} = \mathrm{Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$. It suffices to describe $\mathcal{O}_\mathfrak{g}$ in a way that does not involve $\mathfrak{n}$. We first claim that

$$\overline{\mathcal{O}}_\mathfrak{g} = \mathcal{O}^\mathfrak{g} := G_{ad} \cdot (\overline{\mathcal{O}}_\mathfrak{l} + \mathfrak{n}).$$

Indeed, since $\overline{\mathcal{O}}_\mathfrak{g}$ is G_{ad}-stable, it clearly contains $\mathcal{O}^\mathfrak{g}$. To see that equality holds, it is enough to show that $\mathcal{O}^\mathfrak{g}$ is closed in $\mathfrak{g}$. Now $\overline{\mathcal{O}}_\mathfrak{l} + \mathfrak{n}$ is clearly a closed cone in $\mathfrak{p}$, so it projects to a compact subvariety C of the projectivization $\mathbb{P}\mathfrak{g}$ of the vector space $\mathfrak{g}$. The homogeneous space G_{ad}/P_{ad} is compact [79, 7.2.6], and the projection of $\mathcal{O}^\mathfrak{g}$ to $\mathbb{P}\mathfrak{g}$ is the image under projection to the second coordinate of the closed subset $\{(gP_{ad}, \mathbb{C}x) \mid x \in gP_{ad} \cdot (\overline{\mathcal{O}}_\mathfrak{l} + \mathfrak{n}), g \in G_{ad}\}$ of the compact space $G_{ad}/P_{ad} \times \mathbb{P}\mathfrak{g}$. Hence, this projection is compact, and $\mathcal{O}^\mathfrak{g}$ is closed, as desired (cf. [79, 7.2.11(9)]).

Now let $\mathfrak{z}$ be the center of $\mathfrak{l}$. Our next claim is that

$$\overline{G_{ad} \cdot (\mathfrak{z} + \mathcal{O}_\mathfrak{l})} = G_{ad} \cdot (\mathfrak{z} + \overline{\mathcal{O}}_\mathfrak{l} + \mathfrak{n}).$$

The proof that $\mathcal{O}^{\mathfrak{g}}$ is closed also shows that the right side is closed. Choose a representative of $\mathcal{O}_{\mathfrak{l}}$ inside a fixed Borel subalgebra of $\mathfrak{l}$ and an element Z of $\mathfrak{z}$ such that $\alpha(Z) \neq 0$ if α is a root of $\mathfrak{n}$. Arguing as in the proof of Kostant's conjugacy theorem (3.4.10), we see that the left side, which contains $P_{ad}\cdot(\mathfrak{z}+\overline{\mathcal{O}}_{\mathfrak{l}})$, also contains $\mathfrak{z} + \overline{\mathcal{O}}_{\mathfrak{l}} + \mathfrak{n}$. Hence, the two sides coincide, as desired.

Now the result follows easily. We know that $\mathcal{O}_{\mathfrak{g}}$ is the unique open dense suborbit of $\overline{\mathcal{O}}_{\mathfrak{g}}$ (recall the introduction to §4), so it suffices to describe $\overline{\mathcal{O}}_{\mathfrak{g}}$ in a way that does not involve $\mathfrak{n}$. The proof of (7.1.1) shows that every element of $\overline{\mathcal{O}}_{\mathfrak{l}} + \mathfrak{n}$ is nilpotent. If conversely an element X of $\mathfrak{z} + \overline{\mathcal{O}}_{\mathfrak{l}} + \mathfrak{n}$ is nilpotent, then we see by repeatedly bracketing X with an appropriate simple root vector in $\mathfrak{n}$ that its component from $\mathfrak{z}$ must be 0. Thus we have $\overline{\mathcal{O}}_{\mathfrak{g}} = \mathcal{N} \cap G_{ad}\cdot(\overline{\mathfrak{z}+\mathcal{O}_{\mathfrak{l}}})$. This is the required description of $\overline{\mathcal{O}}_{\mathfrak{g}}$. □

The preceding argument is due to Borho (for the zero orbit it appeared earlier in [42]); the proof in [62] is longer and relies on some messy case-by-case calculations. As a consequence of this theorem, we may use the notations $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O})$ and $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O})$ interchangeably. The next proposition gives an easy way to remember the dimension formula for an induced orbit and shows that induction of orbits is transitive. Note that its proof completes the proof of (4.2.1).

Proposition 7.1.4. *Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be the Levi decomposition of a parabolic subalgebra of $\mathfrak{g}$ and $\mathcal{O}_{\mathfrak{l}}$ a nilpotent orbit in $\mathfrak{l}$.*

(i) $\mathrm{codim}_{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}}) = \mathrm{codim}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}))$.

(ii) *Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be two Levi subalgebras of $\mathfrak{g}$ with $\mathfrak{l}_1 \subset \mathfrak{l}_2$. Then* $\mathrm{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})) = \mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}_1})$.

Proof. Assertion (i) follows from a simple calculation:

$$\begin{aligned}\mathrm{codim}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})) &= \dim(\mathfrak{g}) - \dim(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}))\\ &= \dim(\mathfrak{l}) + 2\dim(\mathfrak{n}) - \dim(\mathcal{O}_{\mathfrak{l}}) - 2\dim(\mathfrak{n})\\ &= \mathrm{codim}_{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}}).\end{aligned}$$

To prove (ii), it suffices by (7.1.1) to show that the two sides both meet $\mathcal{O}_{\mathfrak{l}_1} + \mathfrak{n}_1$ and have the same dimension, where $\mathfrak{p}_1 = \mathfrak{l}_1 + \mathfrak{n}_1$ is a parabolic subalgebra with Levi subalgebra $\mathfrak{l}_1$. The first of these properties follows at once from the definition of an induced orbit; the equality of dimensions follows from (i). □

It is easy to exhibit nilpotent orbits that are induced in two essentially different ways. Indeed, such orbits arise whenever two nonconjugate parabolic subalgebras of $\mathfrak{g}$ have conjugate Levi subalgebras, by (7.1.3). For example, we saw in the proof of (4.2.1) that the subregular orbit is induced from the zero orbit $\mathcal{O}_0$ in any Levi subalgebra whose derived subalgebra has rank one; there are many nonconjugate parabolic subalgebras whose Levi subalgebras have this property (by (3.8.1)). Moreover, we will see in the next chapter that if $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two Levi subalgebras of $\mathfrak{g}$, then $\mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(\mathcal{O}_0) = \mathrm{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(\mathcal{O}_0)$ does not imply that $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are conjugate, or even that they are isomorphic.

The dimension formula implies that not every nilpotent orbit is induced; this answers another one of our questions. Any nilpotent orbit in $\mathfrak{g}$ that is not induced from any proper parabolic subalgebra is called *rigid*. It is clear that the zero orbit $\mathcal{O}_0$ is rigid. To see an example of a nonzero rigid orbit, take $\mathfrak{g} = \mathfrak{sp}_4$. By (4.3.5), the minimal orbit $\mathcal{O}_{min}$ has dimension 4. On the other hand, (7.1.4) implies that nontrivially induced orbits in $\mathfrak{g}$ have dimension 6 or 8. Thus $\mathcal{O}_{min}$ is rigid. In fact, the following stronger result is true.

Lemma 7.1.5. *Let $\mathfrak{g}$ be simple and not isomorphic to $\mathfrak{sl}_n$. Then $\mathcal{O}_{min}$ is rigid.*

Proof. Let $\alpha^{\#}$ be the highest root relative to a fixed positive system of roots Φ^+. It is enough to show that the equality

$$1 + \#|\{\alpha \in \Phi^+ \mid (\alpha, \alpha^{\#}) \neq 0\}| = \dim(\mathcal{O}) + 2\dim\mathfrak{n}$$

is never satisfied for any parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ and any nilpotent orbit $\mathcal{O}$ in $\mathfrak{l}$. We leave this calculation to you. □

Remark. The rigidity of the minimal orbit in $\mathfrak{sp}_{2n}$ may ultimately be used to show that a certain important unitary representation of Sp_{2n} attached to this orbit and called the *metaplectic representation* is also rigid; that is, it cannot be obtained by unitary (or even cohomological parabolic) induction from a unitary representation of a proper parabolic subgroup. Thus, this representation may be regarded as fundamental in some sense among unitary representations. For this reason (and many others), it has been studied extensively by both mathematicians and physicists.

We now address the question of which orbits are induced. The following theorem gives a powerful sufficient condition for an orbit to be induced and moreover gives a formula for the weighted Dynkin diagram of an induced orbit in terms of that of the inducing orbit in some important cases.

Theorem 7.1.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, $\mathfrak{h}$ a Cartan subalgebra contained in $\mathfrak{b}$. Let Φ^+, Δ be the respective choices of positive and simple roots corresponding to $\mathfrak{b}$ and $\mathfrak{h}$.*

(i) *Suppose that the weighted Dynkin diagram D of a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ has vertices $v_1, \ldots, v_t$ labeled 2. Then $\mathcal{O}$ takes the form $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$ for some orbit $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{l}$, where $\mathfrak{p} = \mathfrak{p}_\Theta = \mathfrak{l} + \mathfrak{n}$ is the parabolic subalgebra containing $\mathfrak{b}$ and indexed by the subset $\Theta = \Delta \setminus \{v_1, \ldots, v_t\}$ of Δ (notation (3.8)). Here of course $\mathfrak{l} + \mathfrak{n} = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$ is the Levi decomposition of $\mathfrak{p}$ relative to $\mathfrak{b}$.*

(ii) *Let D' be the subdiagram of D obtained by retaining only the vertices corresponding to roots in Θ. Suppose that D' is the weighted diagram of a nilpotent orbit $\mathcal{O}'$ in $\mathfrak{l}$. Then $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$.*

Proof. By (3.2.2) and (3.8.1) there is a representative $X \in \mathcal{O}$ of the form $Y + Z$, where Y lies in the intersection of $\mathfrak{l}$ and the nilradical of $\mathfrak{b}$ and $Z \in \mathfrak{n}$.

By (4.1.4) there is a Zariski-open subset $\mathfrak{n}'$ of $\mathfrak{n}$ such that $Y + Z' \in \mathcal{O}$ whenever $Z' \in \mathfrak{n}'$. If g lies in the adjoint group L_{ad} of $\mathfrak{l}$, then $g \cdot Z \in \mathfrak{n}$ while $g \cdot Y$ lies in the orbit $\mathcal{O}_Y$ through Y in $\mathfrak{l}$. By (4.1.4) again, it follows that $\mathcal{O}$ meets $\mathcal{O}_Y + \mathfrak{n}$ in a dense set. Now (i) follows from (7.1.1). To prove (ii), it suffices to exhibit a representative $Y' + Z'$ of $\mathcal{O}$ with $Y' \in \mathcal{O}', Z \in \mathfrak{n}$. Let $H, H_\mathfrak{l} \in \mathfrak{h}$ be the neutral elements of standard triples in $\mathfrak{g}, \mathfrak{l}$, respectively, corresponding to the diagrams D, D'; recall from §3.5 that the elements $H, H_\mathfrak{l}$ are uniquely determined even though the standard triples are not. Now let $\{H, X, Y\}, \{H_\mathfrak{l}, X_\mathfrak{l}, Y_\mathfrak{l}\}$ run over all possible standard triples in $\mathfrak{g}, \mathfrak{l}$ with neutral elements $H, H_\mathfrak{l}$. Define the subspaces $\mathfrak{g}_2, \mathfrak{l}_2$ of $\mathfrak{g}, \mathfrak{l}$ as in §3.4. Since D' is a subdiagram of D, we have $\mathfrak{l}_2 = \mathfrak{g}_2 \cap \mathfrak{l}$ and $\mathfrak{g}_2 = \mathfrak{l}_2 \oplus (\mathfrak{g}_2 \cap \mathfrak{n})$. The proof of (3.4.12) shows that $X, X_\mathfrak{l}$ range over Zariski-open subsets $\mathcal{P}, \mathcal{P}_\mathfrak{l}$ of $\mathfrak{g}_2, \mathfrak{l}_2$. Since $\mathfrak{g}_2$ and $\mathfrak{l}_2$ are irreducible as varieties, there must be a standard triple $\{H, X, Y\}$ in $\mathfrak{g}$ such that X projects to an element Y' of $\mathcal{P}_\mathfrak{l}$ relative to the decomposition $\mathfrak{g}_2 = \mathfrak{l}_2 \oplus (\mathfrak{g}_2 \cap \mathfrak{n})$. Writing X as $Y' + Z'$ with $Z' \in \mathfrak{n}$, we arrive at a representative of $\mathcal{O}$ of the desired form. □

orollary 7.1.7. *Any even nilpotent orbit is a Richardson orbit. In particular, we have $\mathcal{O}_{prin} = Ind_\mathfrak{b}^\mathfrak{g}(\mathcal{O}_0)$, where $\mathcal{O}_{prin}$ is the principal orbit in $\mathfrak{g}$ and $\mathfrak{b}$ is any Borel subalgebra of $\mathfrak{g}$.*

Proof. Recall from (3.8.8) that an orbit is even if and only if its weighted diagram has only 0's and 2's. Now the result follows at once from (7.1.6). □

For your convenience, we formally restate a fact already observed.

emma 7.1.8. *Let $\mathfrak{g}$ be a simple Lie algebra. The subregular orbit $\mathcal{O}_{subreg}$ is a Richardson orbit; it is induced from the zero orbit of any Levi subalgebra whose derived subalgebra has rank one.*

We warn you that it is possible for an orbit to be nontrivially induced even if its weighted diagram has no 2's. (For example, we will see in the next section that every nilpotent orbit in $\mathfrak{sl}_n$ is a Richardson orbit.) Furthermore, even if the hypothesis of (7.1.6)(i) is satisfied for a given orbit $\mathcal{O}$ and some choice of vertices $v_1, \ldots, v_t$, the hypothesis (7.1.6)(ii) need not be; in that case there is no simple way to compute the weighted diagram of the orbit $\mathcal{O}_\mathfrak{l}$ in (i). It is an empirical fact, however, that if one removes *all* the vertices labeled 2 from the weighted diagram of a nilpotent orbit, one obtains the weighted diagram of another orbit. Thus, any orbit whose weighted diagram has a 2 may be realized as an induced orbit in a very explicit way.

7.2 Induced Nilpotent Orbits in Type A

In this section we show that every nilpotent orbit in $\mathfrak{sl}_n$ is a Richardson orbit. This is a result of Ozeki and Wakimoto [71]; we follow the exposition in Kraft's paper [55]. Kraft actually gives a formula for the partition of any Richardson

orbit in $\mathfrak{sl}_n$; he then observes that any nilpotent orbit can occur on the right-hand side of his formula. We are able to shorten the arguments of [55] at several points, thanks to our work in §6.2.

Recall from §5.2 that we may realize the root system of $\mathfrak{g} := \mathfrak{sl}_n$ as

$$\{e_i - e_j \mid 1 \leq i, j \leq n, \ i \neq j\}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. As a set of positive roots we take

$$\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}.$$

The corresponding set of simple roots is

$$\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}.$$

The diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$. The Borel subalgebra $\mathfrak{b}$ corresponding to $\mathfrak{h}$ and our choice of Φ^+ consists of the upper triangular matrices in $\mathfrak{g}$. Whenever $i \neq j$, the matrix unit $E_{i,j}$ defined in §5.2 is an $(e_i - e_j)$-root vector of $\mathfrak{h}$.

Now let $\mathbf{d} = [d_1, \ldots, d_n]$ be a partition of n. We begin by attaching a parabolic subalgebra $\mathfrak{p}(\mathbf{d})$ to $\mathbf{d}$. We take $\mathfrak{p}(\mathbf{d})$ to be the subalgebra containing $\mathfrak{b}$ and corresponding to the subset

$$\{e_1 - e_2, \ldots, e_{d_1-1} - e_{d_1}, e_{d_1+1} - e_{d_1+2}, \ldots, e_{d_1+d_2-1} - e_{d_1+d_2},$$
$$e_{d_1+d_2+1} - e_{d_1+d_2+2}, \ldots, e_{d_1+d_2+\cdots+d_n-1} - e_{d_1+d_2+\cdots+d_n}\}$$

of Δ. We can define $\mathfrak{p}(\mathbf{d})$ in a more elegant way, as follows. We may identify the span of the e_i with the standard representation V of $\mathfrak{g}$; then the vector e_i also has weight e_i (abusing notation). Define a flag $\{V_i\}$ of subspaces of V via

$$V_i = \bigoplus_{k=1}^{s_i} \mathbb{C}e_k \text{ where } s_i = \sum_{j=1}^{i} d_j, \tag{7.2.1}$$

and put $V_0 = 0$. Then one checks directly that

$$\begin{aligned} \mathfrak{p}(\mathbf{d}) &= \ \{X \in \mathfrak{sl}_n \mid X(V_i) \subset V_i, \ 1 \leq i \leq n\}, \\ \mathfrak{n}(\mathbf{d}) &= \ \{X \in \mathfrak{sl}_n \mid X(V_i) \subset V_{i-1}, \ 1 \leq i \leq n\}, \end{aligned}$$

where $\mathfrak{n}(\mathbf{d})$ is the nilradical of $\mathfrak{p}(\mathbf{d})$.

Our aim is now to compute the Richardson orbit attached to $\mathfrak{p}(\mathbf{d})$. Let $X \in \mathfrak{n}(\mathbf{d})$. Then the i^{th} matrix power X^i of X clearly kills V_i, whence we have

$$\operatorname{rank}(X^i) \leq n - \dim V_i = n - \sum_{j=1}^{i} d_j. \tag{7.2.2}$$

We claim that there is an $X' \in \mathfrak{n}(\mathbf{d})$ for which equality holds in (7.2.2) for all i. To see this put

$$W_i = \bigoplus_{j=s_{i-1}+1}^{s_i} e_j \text{ where } s_k = \sum_{i=1}^{k} d_i \text{ if } j > 0 \text{ and } s_0 = 0.$$

Then V is the direct sum of the W_i and we have $\dim W_i = d_i \geq \dim W_{i+1}$. Hence, there is a nilpotent matrix X' sending W_1 to 0 and embedding each W_i with $i > 0$ into W_{i-1}. We can easily check that X' has the required properties.

By (6.2.1) and (6.2.5), the orbit $\mathcal{O}_{X'}$ through X' strictly dominates any other nilpotent orbit meeting $\mathfrak{n}(\mathbf{d})$, so $\mathcal{O}_{X'}$ must be the Richardson orbit we are looking for. It only remains to compute its partition. Let $\mathbf{d^t} = [d_1^t, \ldots, d_n^t]$ be the transpose partition of $\mathbf{d}$. As noted in the last chapter, we have $d_i^t = |\{j \mid d_j \geq i\}|$ for all i. We compute directly that if X represents a nilpotent orbit $\mathcal{O}$ in $\mathfrak{sl}_n$ with partition $\mathbf{p}$, then the kernel of X^i has dimension equal to the sum of the first i parts of the transpose partition $\mathbf{p^t}$. In particular, the kernel of $(X')^i$ has the same dimension as that of the i^{th} power of any nilpotent matrix with partition $\mathbf{d^t}$. By (6.2.1), the partition of $\mathcal{O}_{X'}$ must be $\mathbf{d^t}$.

We summarize the above discussion in the following result.

Theorem 7.2.3 (Kraft, Ozeki, Wakimoto). *With notation as above, the partition of* $\operatorname{Ind}_{\mathfrak{p}(\mathbf{d})}^{\mathfrak{g}}(\mathcal{O}_0)$ *is* $\mathbf{d^t}$. *In particular, every nilpotent orbit in* $\mathfrak{sl}_n$ *is a Richardson orbit.*

Proof. The first assertion was proved above. The second follows since the map $\mathbf{d} \mapsto \mathbf{d^t}$ is an involution. Note that every *nonzero* nilpotent orbit in $\mathfrak{sl}_n$ is nontrivially induced; but to realize the zero orbit as a Richardson orbit, we must induce it from itself. □

Corollary 7.2.4. *The dimension of the orbit* $\mathcal{O}_{\mathbf{d}}$ *with partition* $\mathbf{d}$ *is* $n^2 - \sum_{j=1}^{n}(d_j^t)^2$.

Proof. We proved this formula in another way in §6.2, but now we can also combine the dimension formula from (7.1.1) with the above result and an easy computation of the dimension of $\mathfrak{n}(\mathbf{d^t})$. □

At first glance, it seems that our discussion of induced orbits in type A is not quite complete, for it is certainly false that every parabolic subalgebra $\mathfrak{p}$ containing our fixed Borel subalgebra $\mathfrak{b}$ takes the form $\mathfrak{p}(\mathbf{d})$ for some $\mathbf{d}$. Recall, however, from (7.1.3) that whenever two parabolic subalgebras $\mathfrak{p}_1, \mathfrak{p}_2$ have conjugate Levi subalgebras, the orbits induced from nilpotent orbits in these Levi subalgebras are exactly the same (even if $\mathfrak{p}_1, \mathfrak{p}_2$ are not themselves conjugate). Now it is an easy exercise to show that any parabolic subalgebra of $\mathfrak{g}$ has Levi subalgebra conjugate to that of $\mathfrak{p}(\mathbf{d})$ for a unique partition $\mathbf{d}$ (see (3.8.1)(v)). Thus (7.2.3) actually gives a formula for the partition of any Richardson orbit in $\mathfrak{sl}_n$. Moreover, thanks to the transitivity of induction (7.1.4)(iii), we can work out the partition of any induced orbit in $\mathfrak{sl}_n$. Our next result (taken from [50]) does this explicitly; we will need it in the next section. To state it, we need some notation. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. As observed previously, we may assume that $\mathfrak{p} = \mathfrak{p}(\mathbf{d})$ for some partition $\mathbf{d} = [d_1, \ldots, d_n]$ of n. Let $\mathcal{O}$ be a nilpotent orbit in a Levi subalgebra $\mathfrak{l} = \mathfrak{l}(\mathbf{d})$ of $\mathfrak{p}$. Let r be the largest index with $d_r \geq 1$. Then we may write $\mathcal{O} = \mathcal{O}_1 \oplus \cdots \oplus \mathcal{O}_r$ where each $\mathcal{O}_i$ is a nilpotent orbit in $\mathfrak{sl}_{d_i}$ and as such corresponds to a partition $\mathbf{p}(i) = [p_1^i, \ldots, p_n^i]$ of d_i. (In case $d_i = 1$, the corresponding partition is understood to be [1]. We assume without loss of generality that $\mathbf{p}(i)$ has n parts, by adding 0 parts to it as necessary.) Then we have

Lemma 7.2.5 (Kempken). *With notation as above, we have* $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{sl}_n}(\mathcal{O}) = \mathcal{O}_{\Sigma\mathbf{p}}$, *where* $\Sigma\mathbf{p}$ *denotes the partition with* i^{th} *part* $p_i^1 + p_i^2 + \cdots + p_i^r$.

Proof. By (7.2.3), we see that $\mathcal{O}$ is itself induced from the 0 orbit in the parabolic subalgebra $\mathfrak{p}'$ of $\mathfrak{l}$ indexed by the r-tuple

$$(\mathbf{p}(1)^{\mathbf{t}}, \ldots, \mathbf{p}(r)^{\mathbf{t}})$$

of partitions (as in the discussion before (7.2.1)). By the transitivity of induction, we have $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}) = \operatorname{Ind}_{\mathfrak{p}'}^{\mathfrak{g}}(\mathcal{O}_0)$. But now it is not difficult to check that if one concatenates the parts of all the $\mathbf{p}(i)^{\mathbf{t}}$, rearranges the resulting sequence of numbers in nonincreasing order, and then takes the transpose of the resulting partition of n, one obtains the partition $\Sigma\mathbf{p}$. The result follows. □

7.3 Induced and Rigid Orbits in the Classical Algebras

Following Kempken and Spaltenstein [50], [76, ch.II], we now generalize the results of the last section to give a formula for the partition of any induced orbit in a classical algebra $\mathfrak{g}$. We also derive a partition criterion for a classical nilpotent orbit to be rigid. (Recall that we showed in the proof of (7.2.3) that the only rigid nilpotent orbit in $\mathfrak{sl}_n$ is the zero orbit.)

In what follows we assume that $\mathfrak{g}$ is $\mathfrak{sp}_N$ or $\mathfrak{so}_N$; if $\mathfrak{g} = \mathfrak{sp}_N$, then we of course assume that N is even. Let X be the Cartan type of $\mathfrak{g}$, so that $X = B, C$, or D. Thanks once again to the transitivity of induction, to study induced orbits in $\mathfrak{g}$ we may restrict attention to maximal Levi subalgebras (that is, Levi subalgebras not contained in any other proper Levi subalgebra) of $\mathfrak{g}$. By (3.8.1), we check that any such Levi subalgebra takes the form

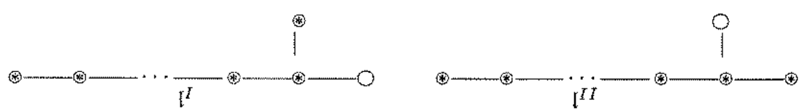
$$\mathfrak{l} \cong \mathfrak{gl}_\ell \oplus \mathfrak{g}', \tag{7.3.1}$$

where $\mathfrak{g}'$ is classical and of the same type as $\mathfrak{g}$. If the standard representation of $\mathfrak{g}$ has dimension N, then we have $2\ell + r = N$, where r is the dimension of the standard representation of $\mathfrak{g}'$. (If $\mathfrak{g} = \mathfrak{so}_N$ and N is even, then we have $r \neq 2$; otherwise, ℓ and r can be any pair of positive integers with $2\ell + r = N$.) For brevity we say that a Levi subalgebra of this form has *type* $(\ell; r)$. The proof of our main result rests primarily on the following one.

mma 7.3.2. *Regard $\mathfrak{g}$ as a subalgebra of $\mathfrak{sl}_N$ in the natural way.*

(i) *Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$ of type $(\ell; r)$. Assume for simplicity that either $r \neq 0$ or r=0 and the numeral of $\mathfrak{l}$ is I (using the terminology introduced in (ii) below; actually, a similar argument also handles the case where the numeral of $\mathfrak{l}$ is II). Then there is a parabolic subalgebra $\widetilde{\mathfrak{p}}$ of $\mathfrak{sl}_N$ such that $\mathfrak{p} = \widetilde{\mathfrak{p}} \cap \mathfrak{g}$. Furthermore, there is a Levi decomposition $\widetilde{\mathfrak{l}} + \widetilde{\mathfrak{n}}$ of $\widetilde{\mathfrak{p}}$ such that $\mathfrak{l} = \widetilde{\mathfrak{l}} \cap \mathfrak{g}, \mathfrak{n} = \widetilde{\mathfrak{n}} \cap \mathfrak{g}$. The subalgebra $\widetilde{\mathfrak{p}}$ is conjugate to $\mathfrak{p}(\mathbf{d})$, the parabolic subalgebra of $\mathfrak{sl}_N$ corresponding via §7.2 to the partition $\mathbf{d} = [\ell, \ell, r]$ (or rather to the partition obtained from $[\ell, \ell, r]$ by rearranging its parts in nonincreasing order).*

(ii) *Given an ordered pair of nonnegative integers (ℓ, r) with $2\ell + r = N$, there exists a unique conjugacy class of Levi subalgebras of type $(\ell; r)$, except when $\mathfrak{g}_\epsilon = \mathfrak{so}_{2n}$ and r=0, so that $\ell = n$. In that case, there are two conjugacy classes of Levi subalgebras of type $(\ell; 0)$, denoted $\mathfrak{l}^I$ and $\mathfrak{l}^{II}$. The simple roots appearing in $\mathfrak{l}^I$ and $\mathfrak{l}^{II}$ are starred below; we follow the conventions of §5.2 for the correspondence between nodes of a Dynkin diagram of type D and simple roots.*



Proof. Assertion (ii) is left to you; it follows easily from the theory of §3.8. To prove (i), we must study the way that $\mathfrak{g}$ sits inside $\mathfrak{sl}_N$. To do this it is convenient to make a *nonstandard* choice of positive roots in $\mathfrak{sl}_N$. Choose Cartan subalgebras and root vectors of $\mathfrak{sl}_N$ and $\mathfrak{g}$ as in §5.2. To avoid confusion we use e's to denote coordinates on the Cartan subalgebra dual of $\mathfrak{g}$ and E's to denote the analogous coordinates for $\mathfrak{sl}_N$. If $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$, then we choose a set of positive roots of $\mathfrak{sl}_N$ in such a way that the corresponding set of simple roots is

$$\{E_1 - E_2, E_2 - E_3, \ldots, E_{n-1} - E_n, E_n - E_{2n}, E_{2n} - E_{2n-1}, \ldots, E_{n+2} - E_{n+1}\}.$$

If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then we take

$$\{E_2 - E_3, \ldots, E_n - E_{n+1}, E_{n+1} - E_1, E_1 - E_{2n+1}, E_{2n+1} - E_{2n}, \ldots, E_{n+3} - E_{n+2}\}$$

as our set of simple roots for $\mathfrak{sl}_N$. Our assumption on $\mathfrak{p}$ implies that the set of simple roots in $\mathfrak{l}$ is

$$\{e_1 - e_2, \ldots, e_{\ell-1} - e_\ell, e_{\ell+1} - e_{\ell+2}, \ldots, e_{n-1} - e_n, 2e_n\}$$

if $\mathfrak{g} = \mathfrak{sp}_{2n}$, or

$$\{e_1 - e_2, \ldots, e_{\ell-1} - e_\ell, e_{\ell+1} - e_{\ell+2}, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$$

if $\mathfrak{g} = \mathfrak{so}_{2n}$, or

$$\{e_1 - e_2, \ldots, e_{\ell-1} - e_\ell, e_{\ell+1} - e_{\ell+2}, \ldots, e_{n-1} - e_n, e_n\}$$

if $\mathfrak{g} = \mathfrak{so}_{2n+1}$. (If $r = 0$, then the simple roots in $\mathfrak{l}$ are $\{e_1 - e_2, \ldots, e_{n-1} - e_n\}$, since $\mathfrak{l}$ has the numeral I if $\mathfrak{g}$ is of type D.) If $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$, then let $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{l}} + \widetilde{\mathfrak{n}}$ where $\widetilde{\mathfrak{l}}$ has the simple roots

$$\{E_1 - E_2, \ldots, E_{\ell-1} - E_\ell, E_{n+2} - E_{n+1}, E_{n+3} - E_{n+2}, \ldots, E_{n+\ell} - E_{n+\ell-1},$$
$$E_{n-s} - E_{n+1-s}, \ldots, E_{n-1} - E_n, E_n - E_{2n}, E_{2n} - E_{2n-1}, \ldots, E_{2n+1-s} - E_{2n-s}\}$$

and $s = \frac{r}{2} - 1$. (If $r = 0$, then all roots on the second line above are omitted.) If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then let $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{l}} + \widetilde{\mathfrak{n}}$ where $\widetilde{\mathfrak{l}}$ has the simple roots

$$\{E_2 - E_3, \ldots, E_\ell - E_{\ell+1}, E_{n+3} - E_{n+2}, \ldots, E_{n+\ell+1} - E_{n+\ell}, E_{n+2-s} - E_{n+1-s},$$
$$\ldots, E_n - E_{n+1}, E_1 - E_{2n+1}, E_{2n+1} - E_{2n}, \ldots, E_{2n+3-s} - E_{2n+2-s}\}$$

and $s = \frac{r+1}{2}$. (If $r = 1$, then all the roots on the second line and the last one on the first line are omitted.) Using the formulas in §5.2 for root vectors in $\mathfrak{g}$, we readily verify by a direct calculation that $\widetilde{\mathfrak{p}}$ has the required properties. □

Now let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$. Write

$$\mathcal{O}_\mathfrak{l} = \mathcal{O}_\mathbf{d} \oplus \mathcal{O}_\mathbf{f},$$

where $\mathcal{O}_\mathbf{d}$ is a nilpotent orbit in $\mathfrak{sl}_\ell$, $\mathcal{O}_\mathbf{f}$ is a nilpotent orbit in $\mathfrak{g}'$, and $\mathbf{d}$ and $\mathbf{f}$ are the respective partitions of $\mathcal{O}_\mathbf{d}$ and $\mathcal{O}_\mathbf{f}$. (Recall that if $\mathfrak{g}$ is of type D, then the orbit $\mathcal{O}_\mathbf{f}$ may also carry a I or II label.) Write

$$\widetilde{\mathcal{O}}_{\widetilde{\mathfrak{l}}} = \widetilde{\mathcal{O}}_\mathbf{d} \oplus \widetilde{\mathcal{O}}'_\mathbf{d} \oplus \widetilde{\mathcal{O}}_\mathbf{f}$$

for the orbit in $\tilde{\mathfrak{l}}$ corresponding to the ordered triple of partitions $(\mathbf{d}, \mathbf{d}, \mathbf{f})$. As in §7.2, we assume that $\mathbf{d}$ and $\mathbf{f}$ both have N parts, by adding 0 parts as necessary. Define a new partition $\mathbf{p} = [p_1, \ldots, p_N]$ of N via $p_i = 2d_i + f_i$, where d_i, f_i of course denote the i^{th} parts of $\mathbf{d}, \mathbf{f}$. The main result is

Theorem 7.3.3. *Retain the above notation. As above, let X be the Cartan type of $\mathfrak{g}$, so that X=B,C, or D. Put $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$.*

(i) *The partition of $\mathcal{O}$ is the X-collapse $\mathbf{p}_X$. If $\mathfrak{g} = \mathfrak{so}_{4n}$ and $\mathbf{p}_X$ is very even, then $\mathbf{p}_X = \mathbf{p}$.*

(ii) *If $\mathfrak{g} = \mathfrak{so}_{4n}$ and $\mathbf{p}_X = \mathbf{p}$ is very even, but $r \neq 0$, then the numeral of $\mathcal{O}$ is the same as that of $\mathcal{O}_{\mathbf{f}}$.*

(iii) *If $\mathfrak{g} = \mathfrak{so}_{4n}$, $\mathbf{p}_X = \mathbf{p}$ is very even, and $r = 0$, then the numeral of $\mathcal{O}$ is the same as that of $\mathfrak{l}$ if n is even but differs from it if n is odd.*

Proof. We will prove (i); (ii) and (iii) are left to you. We know that the partition of $\mathcal{O}$ is the unique largest one (under dominance) of any orbit in $\mathfrak{g}$ meeting $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$. By (7.2.5), the unique largest partition of any orbit in $\mathfrak{sl}_N$ meeting $\mathcal{O}_{\tilde{\mathfrak{l}}} + \tilde{\mathfrak{n}}$ is $\mathbf{p}$. Since $\mathcal{O}$ lives in $\mathfrak{g}$, it follows from (6.3.3) that its partition is dominated by $\mathbf{p}_X$. To show that its partition equals $\mathbf{p}_X$, it suffices to show that its dimension is the same as that of the orbit with partition $\mathbf{p}_X$. We refer to [50] for this calculation (or you may enjoy working it out for yourself); it amounts to an application of the dimension formula in §6.2 for orbits in classical algebras together with an analysis of how each reduction step in the procedure for constructing $\mathbf{p}_X$ from $\mathbf{p}$ affects the dimension of the corresponding orbit in $\mathfrak{sl}_N$. $\square$

We warn you that in applying this result to algebras of type B, you must index the unique orbit of the trivial algebra $\mathfrak{so}_1$ by the partition [1], not [0]. (You may wish to compare this with the remarks preceding (7.2.5).) If the inducing orbit is zero on the $\mathfrak{gl}_\ell$ subalgebra, we obtain the following corollary. For completeness we also include the case $\mathfrak{g} = \mathfrak{sl}_N$.

Corollary 7.3.4. *Let $\mathfrak{g}$ be a classical algebra and $\mathfrak{l} = \mathfrak{gl}_\ell \oplus \mathfrak{g}'$ a maximal Levi subalgebra, where $\mathfrak{g}'$ is of the same type as $\mathfrak{g}$. (Then $\mathfrak{l}$ is of type $(\ell; r)$ if $\mathfrak{g}$ is not of type A; if $\mathfrak{g} = \mathfrak{sl}_N$ and $\mathfrak{g}' \cong \mathfrak{sl}_r$, then $\ell + r = N$.) Let $\mathcal{O}_{\mathfrak{l}} = \mathcal{O}_0 \oplus \mathcal{O}_{\mathbf{f}}$ be a nilpotent orbit in $\mathfrak{l}$ whose component in the $\mathfrak{gl}_\ell$ factor is the zero orbit and whose component $\mathcal{O}_{\mathbf{f}}$ in the $\mathfrak{g}'$ factor has partition $\mathbf{f}$. Then the partition of $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$ is $\mathbf{p}$, where the partition $\mathbf{p}$ is obtained from $\mathbf{f}$ as follows.*

(i) *If $\mathfrak{g} = \mathfrak{sl}_N$, add 1 to the first ℓ parts of $\mathbf{f}$, adding zero parts to $\mathbf{f}$ as necessary.*

(ii) *If the Cartan type X of $\mathfrak{g}$ is B,C,or D, then add 2 to the first ℓ terms of $\mathbf{f}$, obtaining a partition $\tilde{\mathbf{f}}$ (again adding zeros as necessary), and then take the X-collapse of $\tilde{\mathbf{f}}$. If this collapse is nontrivial, it is obtained by subtracting 1 from the ℓ^{th} part of $\tilde{\mathbf{f}}$ and adding 1 to its $(\ell+1)^{th}$ part. If $\mathfrak{g} = \mathfrak{so}_{4n}$, $r \neq 0$ and the collapsed partition is very even, then $\mathbf{f}$ is also very even and*

the induced orbit inherits the I or II label of $\mathcal{O}_{\mathbf{f}}$; if r=0, then the orbit is labeled in accordance with (7.3.3)(iii).

Let $\epsilon = \pm 1$, as in §5.1. We now write down the partitions of the rigid orbits in $\mathcal{P}_\epsilon(N)$. Define $\mathcal{P}^*_\epsilon(N)$ to be the set of all partitions $[d_1, \ldots, d_N]$ in $\mathcal{P}_\epsilon(N)$ such that the following two conditions hold:

(i) $0 \leq d_{i+1} \leq d_i \leq d_{i+1} + 1$ for all i,

(ii) $|\{j \mid d_j = i\}| \neq 2$ if $\epsilon(-1)^i = -1$.

In other words, these two conditions single out partitions in which the difference between two successive parts (or between the last part and 0) is at most 1 and in which no even part occurs exactly twice (if $\epsilon = -1$), or no odd part occurs exactly twice (if $\epsilon = 1$). In particular, such partitions always have 1 as their smallest nonzero part, so that they cannot be very even.

Corollary 7.3.5. *The orbit corresponding to a partition in $\mathcal{P}_\epsilon(N)$ is rigid if and only if $\mathbf{d} \in \mathcal{P}^*_\epsilon(N)$.*

Proof. If $\mathbf{d} = [d_1, \ldots, d_N] \in \mathcal{P}_\epsilon(N)$ and $\mathcal{O}_{\mathbf{d}}$ is not rigid, then by (7.3.4) there are partitions $\mathbf{e} = [e_1, \ldots, e_\ell] \in \mathcal{P}(\ell)$ and $\mathbf{f} = [f_1, \ldots, f_r] \in \mathcal{P}_\epsilon(r)$ with $2\ell + r = N$ so that

$$\mathbf{d} = [2e_1 + f_1, \ldots, 2e_\ell + f_\ell, f_{\ell+1}, \ldots, f_N]_X$$

where X is the appropriate label B, C, or D. We now show that $\mathbf{d} \notin \mathcal{P}^*_\epsilon(N)$ by considering two cases. First, suppose that the partition $\mathbf{d}$ is already in $\mathcal{P}_\epsilon(N)$. Let s be maximal with $e_s > 0$. Then $2e_s + f_s \geq f_{s+1} + 2$, i.e., $d_s \geq d_{s+1} + 2$. This shows $\mathbf{d} \notin \mathcal{P}^*_\epsilon(N)$. In the second case, we assume $\mathbf{d} = [2e_1 + f_1, \ldots, 2e_\ell + f_\ell, f_{\ell+1}, \ldots, f_N] \notin \mathcal{P}^*_\epsilon(N)$. Then there must exist a pair (f_{s+1}, f_{s+2}) with $f_{s+1} = f_{s+2}, \epsilon(-1)^{f_{s+1}} = 1$. Then $e_{s+1} > e_{s+2}$ and the pair $(2e_{s+1} + f_{s+1}, 2e_{s+2} + f_{s+2})$ is replaced by $(2e_{s+1} + f_{s+1} - 1, 2e_{s+2} + f_{s+2} + 1)$ to obtain $\mathbf{d}$. If $e_{s+1} \geq e_{s+2} + 2$, then $2e_{s+1} + f_{s+1} - 1 \geq 2e_{s+2} + 4 + f_{s+2} - 1$; i.e., $d_{s+1} \geq d_{s+2} + 2$ and $\mathbf{d} \notin \mathcal{P}^*_\epsilon(N)$. If $e_{s+1} = e_{s+2} + 1$, then $2e_{s+1} + f_{s+1} - 1 = 2e_{s+2} + f_{s+2} + 1$ and $\epsilon(-1)^{2e_{s+1} + f_{s+1} - 1} = \epsilon(-1)^{f_{s+1} - 1} = -1$. Furthermore, we have $d_s > d_{s+1} = d_{s+2} > d_{s+3}$, so that $|\{i \mid d_i = 2e_{s+1} + f_{s+1}\}| = 2$. Thus $\mathbf{d} \notin \mathcal{P}^*_\epsilon(N)$, as desired.

Conversely, suppose that $\mathbf{d} \notin \mathcal{P}^*_\epsilon(N)$; then we consider two cases, each of which will imply that $\mathcal{O}_{\mathbf{d}}$ is not rigid.

First, suppose $\mathbf{d} = [d_1, \ldots, d_N] \in \mathcal{P}_\epsilon(N)$ with $d_j \geq d_{j+1} + 2$ for some j. Then $[1, \ldots 1] \in \mathcal{P}(j)$ and

$$[d_1 - 2, \ldots, d_j - 2, d_{j+1}, \ldots, d_N] \in \mathcal{P}_\epsilon(N - 2j)$$

is the partition of an orbit $\mathcal{O}_{\mathfrak{l}}$ in a Levi subalgebra $\mathfrak{l}$ of type $(j; N - 2j)$ with $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \mathcal{O}_{\mathbf{d}}$. Finally, suppose that $\mathbf{d} = [d_1, \ldots, d_N] \in \mathcal{P}_\epsilon(N)$ with $d_{s-1} > d_s = d_{s+1} > d_{s+2}$ and $\epsilon(-1)^{d_s} = -1$ for some s. Then $[1, \ldots, 1] \in \mathcal{P}(s)$ and

$$[d_1-2,\ldots,d_{s-1}-2,d_s-1,d_{s+1}-1,d_{s+2},\ldots,d_N]\in\mathcal{P}_\epsilon(N-2s)$$

is the partition of an orbit $\mathcal{O}_\mathfrak{l}$ in a Levi subalgebra $\mathfrak{l}$ of type $(s;N-2s)$ with $\mathrm{Ind}_\mathfrak{l}^{\mathfrak{g}_\epsilon}(\mathcal{O}_\mathfrak{l})=\mathcal{O}_\mathbf{d}$. □

Examples 7.3.6. (i) In $\mathfrak{sp}_6$, the rigid nilpotent orbits are $\mathcal{O}_{[1^6]}$ and $\mathcal{O}_{[2,1^4]}$.
(ii) In $\mathfrak{so}_7$, the rigid orbits are $\mathcal{O}_{[1^7]}$ and $\mathcal{O}_{[2^2,1^3]}$.
(iii) In $\mathfrak{so}_8$, the rigid orbits are $\mathcal{O}_{[1^8]}$, $\mathcal{O}_{[2^2,1^4]}$, and $\mathcal{O}_{[3,2^2,1]}$.

We conclude this chapter, as we did the last one, by making a few remarks about the exceptional algebras. Elashvili has computed exactly which orbits in the exceptional algebras are induced from which others. His tables are reproduced in [76, ch. II], where a reference is also given. We also mention a result of Lusztig linking the theory of this chapter with that of the preceding one: Any nilpotent orbit (in any semisimple algebra $\mathfrak{g}$) induced from a special one is again special [57]. In particular, any even orbit, or more generally any Richardson orbit, is special. Conversely, Spaltenstein has shown in unpublished work that the closure of any special classical orbit is the intersection of the closures of two Richardson orbits. Kempken gives a proof in [50]. (This result plays a key role in the Borho-Brylinski proof of the Irreducibility Theorem (10.2.2) in the classical integral case.) A slightly weaker result holds for special exceptional orbits (Spaltenstein, unpublished).

8 The Exceptional Cases and Bala-Carter Theory

In Chapter 5, we showed how to parametrize nilpotent orbits in classical Lie algebras (using partitions), but so far we have not seen how to do this for exceptional algebras. Of course, we know from Chapter 3 that nilpotent orbits correspond bijectively to their weighted Dynkin diagrams, but we still do not know which labelings of the nodes constitute weighted diagrams. In this chapter, we complete the program of Chapter 3 by showing how to write down all the nilpotent orbits in any semisimple Lie algebra $\mathfrak{g}$ in terms of data easily computed from its Dynkin diagram. We follow the approach of Bala and Carter in [2] and [3]. All the results we state are due to them, though in certain cases we have modified their proofs. The main idea is to look at nice subalgebras of $\mathfrak{g}$ meeting a given nilpotent orbit.

8.1 Levi Subalgebras Containing Nilpotent Elements

We begin by asking which proper subalgebras of $\mathfrak{g}$ contain a fixed nilpotent element X. Of course, we do not want to look at arbitrary subalgebras; there are far too many of them, even up to conjugacy. Since $\mathfrak{g}$ is (in particular) reductive, it is reasonable to start by restricting attention to its reductive subalgebras (which are more convenient to work with than the semisimple ones). But there are still far too many of these. To see why, recall from Chapter 3 that if we knew how to classify even the copies of $\mathfrak{sl}_2$ in $\mathfrak{g}$ up to conjugacy, the classification of nilpotent orbits would follow at once. Thus we really need to restrict to a class of reasonably visible reductive subalgebras of $\mathfrak{g}$. The first attempt to do this was made by Dynkin. He defines a proper subalgebra $\mathfrak{g}^\sharp$ of $\mathfrak{g}$ to be *regular* if it is spanned by a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ together with some (but not all) of its root spaces [27]. It is not difficult to check that such a $\mathfrak{g}^\sharp$ is reductive if and only if its set of $\mathfrak{h}$-roots $\Phi(\mathfrak{g}^\sharp,\mathfrak{h})$ is a subroot system of the root system $\Phi=\Phi(\mathfrak{g},\mathfrak{h})$ of $\mathfrak{g}$.

Now it is natural to divide nilpotent orbits of $\mathfrak{g}$ into two classes: those that meet regular reductive subalgebras and those that do not. The former can be classified by doing calculations in smaller algebras, and it turns out that there are sufficiently few of the latter to be tractable. This program is carried out in [27]; the first list of the weighted diagrams of the exceptional orbits appears there (but has mistakes). Unfortunately, the resulting classification is still somewhat mysterious; it does not yield a nice parametrization of the orbits. Bala and Carter's fundamental insight was that such a parametrization can be obtained if one repeats the above program using Levi subalgebras (which are regular and reductive) instead of arbitrary regular reductive subalgebras. Given an orbit $\mathcal{O}_X$, it turns out that there is a canonical conjugacy class $[\mathfrak{l}]$ of Levi subalgebras of $\mathfrak{g}$ meeting $\mathcal{O}_X$. The corresponding statement for regular subalgebras is false. Our aim in the rest of this section is to construct $[\mathfrak{l}]$; in the next one we parametrize the orbits in $\mathfrak{g}$ attached to a fixed $[\mathfrak{l}]$.

The choice of $[\mathfrak{l}]$ should come as no surprise. If a Levi subalgebra $\mathfrak{l}$ contains a nilpotent element X, then so too does any Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{l}$; so it is natural to look at minimal Levi subalgebras containing X. The main result is

Theorem 8.1.1. *Any two minimal Levi subalgebras containing a fixed nilpotent $X \in \mathfrak{g}$ are G_{ad}^X-conjugate.*

Proof. It follows easily from §§2.1,3.8 that the center $\mathfrak{c}$ of any Levi subalgebra $\mathfrak{l}$ is a subalgebra of a Cartan subalgebra of $\mathfrak{g}$, and so in particular consists of semisimple elements. We also have $\mathfrak{l} = \mathfrak{g}^{\mathfrak{c}}, \mathfrak{c} = \mathfrak{g}^{\mathfrak{l}}$. Call any subalgebra of $\mathfrak{g}^X$ consisting of semisimple elements *toral*. Then $\mathfrak{g}^{\mathfrak{t}}$ is a Levi subalgebra of $\mathfrak{g}$ containing X whenever $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{g}^X$. Thus, we get a one to one correspondence between maximal toral subalgebras of $\mathfrak{g}^X$ and minimal Levi subalgebras containing X. We are reduced to showing that any two maximal toral subalgebras of $\mathfrak{g}^X$ are G_{ad}^X-conjugate; note that the corresponding fact for $\mathfrak{g}$ was observed in (2.1.11). There is a standard generalization of this fact to arbitrary Lie algebras $\mathfrak{a}$. Recall that a *Cartan subalgebra* of $\mathfrak{a}$ is a nilpotent self-normalizing subalgebra; this definition agrees with the one in §2.1 if $\mathfrak{a}$ is reductive. The generalization states that any two Cartan subalgebras of $\mathfrak{a}$ are conjugate ([21, 1.9.4],[37, §16.4]). If $\mathfrak{s} := \mathfrak{g}^X$ were reductive, we would be done, but we saw in §3.7 that this cannot be the case unless $X = 0$. Nevertheless, we have

Claim. Let $\mathfrak{t}$ be a maximal toral subalgebra of $\mathfrak{s}$. Then $\mathfrak{s}^{\mathfrak{t}}$ is a Cartan subalgebra of $\mathfrak{s}$, and $\mathfrak{t}$ consists exactly of the semisimple elements in $\mathfrak{s}^{\mathfrak{t}}$.

Granting this for the moment, let $\mathfrak{t}_1, \mathfrak{t}_2$ be two maximal toral subalgebras of $\mathfrak{s}$. Then $\mathfrak{s}^{\mathfrak{t}_1}$ is G_{ad}^X-conjugate to $\mathfrak{s}^{\mathfrak{t}_2}$ by a map that must send $\mathfrak{t}_1$ to $\mathfrak{t}_2$ (since G_{ad}^X preserves semisimplicity and nilpotence in $\mathfrak{g}$). This is the desired result.

To prove the claim, let $Z \in \mathfrak{s}^{\mathfrak{t}}$. By (1.1.1), the semisimple and nilpotent parts Z_s, Z_n of Z also lie in $\mathfrak{s}^{\mathfrak{t}}$, whence $Z_s \in \mathfrak{t}$ by maximality. Since $\mathfrak{t}$ obviously acts trivially on $\mathfrak{s}^{\mathfrak{t}}$ (under the adjoint action), it follows that Z acts on $\mathfrak{s}^{\mathfrak{t}}$ by the nilpotent endomorphism Z_n. By Engel's Theorem [37, §3.3], $\mathfrak{s}^{\mathfrak{t}}$ is nilpotent. Since toral subalgebras of $\mathfrak{s}$ or $\mathfrak{g}$ are abelian [37, §8.1], we have $\mathfrak{t} \subset \mathfrak{s}^{\mathfrak{t}}$, and $\mathfrak{t}$ consists of semisimple elements. Then the argument above to show that $Z_s \in \mathfrak{t}$ also shows that all semisimple elements of $\mathfrak{s}^{\mathfrak{t}}$ lie in $\mathfrak{t}$.

It only remains to show that $\mathfrak{s}^{\mathfrak{t}}$ is self-normalizing. Via the adjoint representation, $\mathfrak{t}$ acts on the normalizer $\tilde{\mathfrak{n}}$ of $\mathfrak{s}^{\mathfrak{t}}$ in $\mathfrak{s}$, whence this normalizer is the direct sum of its $\mathfrak{t}$-weight spaces. But the 0-weight space $\tilde{\mathfrak{n}}_0$ is just $\mathfrak{s}^{\mathfrak{t}}$ itself, and the independence of weight spaces shows that no others can occur. This completes the proof. □

We warn you that if $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{s}$, it does not follow that $\mathfrak{t}$ is the center of $\mathfrak{s}^{\mathfrak{t}}$.

8.2 Distinguished Nilpotent Elements and Parabolic Subalgebras

You will notice that we never used the Jacobson-Morozov Theorem in the proof of (8.1.1). The theory of standard triples comes into play when we study nilpotent orbits attached via (8.1.1) to a fixed conjugacy class $[\mathfrak{l}]$ of Levi subalgebras of $\mathfrak{g}$. It is intuitively clear that the hardest orbits to classify are those for which $[\mathfrak{l}] = [\mathfrak{g}] = \mathfrak{g}$. We therefore define a nilpotent element X or its orbit $\mathcal{O}_X$ to be *distinguished* if the only Levi subalgebra of $\mathfrak{g}$ containing X (or equivalently meeting $\mathcal{O}_X$) is $\mathfrak{g}$ itself. Since Levi subalgebras of Levi subalgebras of $\mathfrak{g}$ are again Levi subalgebras of $\mathfrak{g}$, any nilpotent $X \in \mathfrak{g}$ is distinguished in any minimal Levi subalgebra containing it. Thus, if we can understand distinguished orbits, we can understand arbitrary orbits. Note also that a Levi subalgebra $\mathfrak{l}$ contains X if and only if its semisimple part $[\mathfrak{l}, \mathfrak{l}]$ does, by (1.1.6) and the discussion before (1.1.7). Thus if X is distinguished in $\mathfrak{l}$, it is actually distinguished in the semisimple algebra $[\mathfrak{l}, \mathfrak{l}]$.

Now we are ready to apply the theory of Chapter 3. Let $X \in \mathfrak{g}$ be a nonzero nilpotent element, $\{H, X, Y\}$ a standard triple. Recall the $\mathbb{Z}$-gradation $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ introduced in §3.4. Using this gradation, it is easy to give a criterion for X to be distinguished.

Lemma 8.2.1. *Retain the above notations. Then $X \in \mathfrak{g}$ is distinguished if and only if $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$.*

Proof. Suppose first that $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the Jacobson-Morozov parabolic subalgebra of X relative to $\{H, X, Y\}$ defined in §3.8, so that

$$\mathfrak{l} = \mathfrak{g}_0, \mathfrak{u} = \bigoplus_{i>0} \mathfrak{g}_i. \tag{8.2.2}$$

Since $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$, it follows from $\mathfrak{sl}_2$ theory that $\mathfrak{g}^X = \mathfrak{u}^X$ consists of nilpotent elements. Hence, the only toral subalgebra of $\mathfrak{g}^X$ is 0, so that X is distinguished, by the remarks at the beginning of the proof of (8.1.1).

Now suppose that $\dim \mathfrak{g}_0 \neq \dim \mathfrak{g}_2$. Appealing to $\mathfrak{sl}_2$ theory once again, we get that $\dim \mathfrak{g}_0 > \dim \mathfrak{g}_2$ and the reductive subalgebra $\mathfrak{g}^\phi$ of $\mathfrak{g}^X$ defined in §3.7 is nonzero. If the semisimple part $[\mathfrak{g}^\phi, \mathfrak{g}^\phi]$ of this subalgebra is nonzero, then it certainly contains nontrivial toral subalgebras, so that X is non-distinguished, as desired. So assume that $\mathfrak{g}^\phi$ is abelian. We showed that $\mathfrak{g}^\phi$ is reductive in §3.7 by showing that the Killing form on $\mathfrak{g}$ restricts to a nondegenerate form on $\mathfrak{g}^\phi$. If $\mathfrak{g}^\phi$ consisted entirely of nilpotent elements, the Killing form would restrict to 0 on it, by a calculation we have seen several times (e.g., in the proof of (1.3.9)). Since $\mathfrak{g}^\phi$ is closed under the Jordan decomposition in $\mathfrak{g}$ by (1.1.1), it must in any event contain nonzero semisimple elements. But then $\mathfrak{g}^X$ has a nontrivial toral subalgebra, whence X is nondistinguished. □

What really gets the theory of distinguished orbits off the ground is the following result.

Theorem 8.2.3. *Any distinguished orbit in $\mathfrak{g}$ is even.*

Proof. Let $X \in \mathfrak{g}$ be distinguished with Jacobson-Morozov parabolic $\mathfrak{q}$ relative to a standard triple $\{H, X, Y\}$. We saw in §7.1 that there is a unique nilpotent orbit $\mathcal{O} = \mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{g}}(\{0\})$ in $\mathfrak{g}$ whose intersection with $\mathfrak{u}$ is dense in the latter; i.e., $\mathcal{O}$ is the Richardson orbit attached to $\mathfrak{q}$. If Q is the connected Lie subgroup of G_{ad} with Lie algebra $\mathfrak{q}$ and $Z \in \mathcal{O} \cap \mathfrak{u}$, then we also showed that, in fact,

$$Q \cdot Z \text{ is dense in } \mathfrak{u}. \tag{8.2.4}$$

Now suppose that $\mathfrak{g}_1 \neq 0$. Write

$$Z = \sum_{i>0} Z_i$$

with $Z_i \in \mathfrak{g}_i$. By (8.2.4), $[\mathfrak{q}, Z]$ is a subspace of $\mathfrak{u}$ of full dimension; so $[\mathfrak{q}, Z] = \mathfrak{u}$. Using $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and the definition of $\mathfrak{u}$, we deduce that

$$[\mathfrak{g}_0, Z_1 + Z_2] + [\mathfrak{g}_1, Z_1] = \mathfrak{g}_1 + \mathfrak{g}_2. \tag{8.2.5}$$

If $Z_1 = 0$, then obviously $[\mathfrak{g}_1, Z_1] = 0$; if $Z_1 \neq 0$, then $\dim\,[\mathfrak{g}_1, Z_1] = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_1^{Z_1} < \dim \mathfrak{g}_1$ since $Z_1 \in \mathfrak{g}_1^{Z_1}$. Thus, in any event $\dim\,[\mathfrak{g}_1, Z_1] < \dim \mathfrak{g}_1$. On the other hand, we have

$$\dim\,[\mathfrak{g}_0, Z_1 + Z_2] \leq \dim \mathfrak{g}_0 = \dim \mathfrak{g}_2.$$

Thus (8.2.5) leads to a contradiction, and we get $\mathfrak{g}_1 = 0$, as desired. □

This proof is due to Jantzen (see [18]); Bala and Carter's original proof was longer and involved some case-by-case computations. It follows from (7.1.7) that any distinguished orbit $\mathcal{O}_X$ is uniquely determined by the Jacobson-Morozov parabolic subalgebra $\mathfrak{q}$ of any of its representatives: It is the Richardson orbit attached to $\mathfrak{q}$. (This fails for arbitrary orbits $\mathcal{O}$ because the Jacobson-Morozov parabolic subalgebra tells only which nodes are labeled 0 in the weighted diagram of $\mathcal{O}$.) Thus, it is natural to rewrite the distinguishedness criterion (8.2.1) in terms of $\mathfrak{q}$. One obtains

Theorem 8.2.6. *An even nilpotent orbit $\mathcal{O}_X$ is distinguished if and only if its Jacobson-Morozov parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ satisfies $\dim \mathfrak{l} = \dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$.*

Proof. We may assume that $\mathfrak{q}$ is chosen as in (8.2.1). Put $\mathfrak{u}' = \oplus_{i \geq 4} \mathfrak{g}_i$. Then obviously $\dim \mathfrak{g}_0 = \dim \mathfrak{l}$, while $\dim \mathfrak{g}_2 = \dim \mathfrak{u} - \dim \mathfrak{u}'$. Since $\mathfrak{u}$ is the sum of eigenspaces $\mathfrak{g}_i$ with $i \geq 2$, we have $[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}'$; conversely, if $Z \in \mathfrak{g}_i$ and $i \geq 4$, then we can write $Z = [X, Z']$ for some $Z' \in \mathfrak{g}_{i-2} \subset \mathfrak{u}$, by $\mathfrak{sl}_2$ theory. Hence, $\mathfrak{u}' \subset [\mathfrak{u}, \mathfrak{u}]$. The conclusion follows at once. □

We therefore define an arbitrary parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ to be *distinguished* if $\dim \mathfrak{l} = \dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$. For example, a Borel subalgebra is always distinguished (and turns out to correspond to the principal orbit). The last two results show that there is an injective map from distinguished orbits to conjugacy classes of distinguished parabolic subalgebras; it sends an orbit $\mathcal{O}_X$ to the class $[\mathfrak{q}]$ of any Jacobson-Morozov parabolic $\mathfrak{q}$ of X. The classification is completed by showing that this map is a bijection. To do this we need a combinatorial fact about parabolic subalgebras.

Proposition 8.2.7. *Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, as in §3.8, and adopt the notation of that section. The nilradical $\mathfrak{n}_\Theta$ of the subalgebra $\mathfrak{p}_\Theta$ corresponding to the subset Θ of simple roots satisfies the following properties.*

(i) *$\mathfrak{n}_\Theta$ and $[\mathfrak{n}_\Theta, \mathfrak{n}_\Theta]$ are the direct sums of their 1-dimensional root spaces.*

(ii) *A root α of $\mathfrak{n}_\Theta$ is a root of $[\mathfrak{n}_\Theta, \mathfrak{n}_\Theta]$ if and only if it is a sum of two roots of $\mathfrak{n}_\Theta$ (we call such a root decomposable).*

(iii) *A root α of $\mathfrak{n}_\Theta$ is indecomposable (i.e., not decomposable) if and only if it is the sum of one simple root not in Θ and various simple roots in Θ.*

Proof. Assertions (i) and (ii) are clear. Since every root of $\mathfrak{n}_\Theta$ is a sum of simple roots at least one of which is not in Θ, it follows that any root α satisfying the criterion of (iii) is indecomposable. Let V an $\mathfrak{l}_\Theta$-stable complement to $[\mathfrak{n}_\Theta, \mathfrak{n}_\Theta]$ in $\mathfrak{n}_\Theta$. Then it is easy to see that V is uniquely defined; it must be the sum of the indecomposable root spaces. Suppose (iii) fails; then there is an indecomposable root α, which is the sum of at least two simple roots not in Θ and various simple roots in Θ. By [37, §10.2], one can repeatedly subtract simple roots from α to obtain a sequence of positive roots of $\mathfrak{g}$ followed by 0. By the indecomposability of α, every root in this sequence must also be an indecomposable root that is the sum of at least two simple roots not in Θ and various simple roots in Θ. Since the last (nonzero) root in the sequence must be simple, this is a contradiction. □

Now we can show that our map is surjective.

Theorem 8.2.8. *Any distinguished parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is the Jacobson-Morozov parabolic subalgebra of a distinguished nilpotent element X (relative to a suitable standard triple).*

Proof. Choose a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ of $\mathfrak{g}$ contained in $\mathfrak{q}$ and adopt once more the notation of §3.8. Then $\mathfrak{q} = \mathfrak{q}_\Theta$ for some $\Theta \subset \Delta$. Let V be the subspace of $\mathfrak{u} = \mathfrak{n}_\Theta$ defined in the proof of the last proposition. If α, β are roots in V and $\alpha - \beta$ is a root, then (8.2.7)(iii) shows that $\alpha - \beta$ must be a root in $\mathfrak{l}_\Theta$. Hence, if $\overline{\mathfrak{q}} = \overline{\mathfrak{q}}_\Theta = \mathfrak{l}_\Theta \oplus \overline{\mathfrak{n}}_\Theta$ is the opposite parabolic subalgebra to $\mathfrak{q}$, and if we define $\overline{V} \subset \overline{\mathfrak{n}}_\Theta$ in the same way as we did $V \subset \mathfrak{n}_\Theta$, then we have

$$[\overline{V}, V] \subset \mathfrak{l}. \tag{8.2.9}$$

The proof of (8.2.3) shows that if X exists at all, its orbit must be the Richardson orbit $\mathcal{O}_\mathfrak{q}$ attached to $\mathfrak{q}$, so let $Z \in \mathcal{O}_\mathfrak{q} \cap \mathfrak{u}$. As in the proof of (8.2.3), we see that $[\mathfrak{q}, Z] = \mathfrak{u}$. Write $Z = Z_V + Z_{[\mathfrak{u},\mathfrak{u}]}$, where $Z_V \in V$ and $Z_{[\mathfrak{u},\mathfrak{u}]} \in [\mathfrak{u}, \mathfrak{u}]$. Then one easily checks that

$$[\mathfrak{l}, Z_V] = V. \tag{8.2.10}$$

We now claim that $\mathfrak{g}^{Z_V} \cap \overline{V} = 0$. Indeed, it follows from (8.2.7)(iii) that V and $\overline{V}$ are paired nondegenerately by the Killing form κ of $\mathfrak{g}$; but we saw in the proof of (1.3.9) that $\mathfrak{g}^{Z_V}$ is κ-orthogonal to $[\mathfrak{g}, Z_V] \supset [\mathfrak{l}, Z_V] = V$. Hence,

$$\dim[\overline{V}, Z_V] = \dim \overline{V} = \dim V = \dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] = \dim \mathfrak{l}$$

whence we get

$$[\overline{V}, Z_V] = \mathfrak{l} \tag{8.2.11}$$

by (8.2.9), since $Z_V \in V$. Now choose $H \in \mathfrak{h}$ so that $\alpha(H) = 0$ if $\alpha \in \Theta$ and $\alpha(H) = 2$ if $\alpha \in \Delta \setminus \Theta$. By (8.2.7)(iii), we have $\beta(H) = 2$ for all roots β in V, while $\gamma(H) = -2$ for all roots γ in $\overline{V}$. Now put $X = Z_V$ and choose $Y \in \overline{V}$ so that $[X, Y] = H$, as is possible by (8.2.11). It follows that $\{H, X, Y\}$ is a standard triple. Then X is distinguished by (8.2.1) and its corresponding Jacobson-Morozov parabolic is $\mathfrak{q}$, as desired. □

We may summarize the foregoing results as follows.

Theorem 8.2.12. *There is a natural one-to-one correspondence between nilpotent orbits of $\mathfrak{g}$ and G_{ad}-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_\mathfrak{l}$ is a distinguished parabolic subalgebra of the semisimple algebra $[\mathfrak{l}, \mathfrak{l}]$.*

Proof. We noted before (8.2.1) that a Levi subalgebra $\mathfrak{l}$ contains a nilpotent $X \in \mathfrak{g}$ if and only if $[\mathfrak{l}, \mathfrak{l}]$ does, and it is easy to check that two Levi subalgebras are conjugate if and only if their derived subalgebras are. The theorem follows at once from (8.2.3), (8.2.6), and (8.2.8). □

Example 8.2.13. We illustrate this theorem by computing the number of nilpotent orbits in type G_2. Let $\Delta = \{\alpha, \beta\}$ be a set of simple roots for an algebra $\mathfrak{g}$ of type G_2. Assume that α is short and β is long. Then it is well known and easy to check that the set of positive roots is $\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$. By (3.8.1), $\mathfrak{g}$ has exactly four conjugacy classes of parabolic subalgebras (corresponding to the four subsets of Δ), and one computes that $\mathfrak{g}$ also has four conjugacy classes of Levi subalgebras. (The only nontrivial point is to verify that the Levi subalgebras corresponding to the subsets $\{\alpha\}$ and $\{\beta\}$ are nonconjugate, and this follows since α and β are not conjugate under the Weyl group, since they have different lengths; cf. (3.8.1).) Note that the Levi subalgebra $\mathfrak{l}$ of a proper parabolic subalgebra has derived algebra either $\{0\}$ or isomorphic to $\mathfrak{sl}_2$, whence $\mathfrak{l}$ has a unique distinguished orbit. This is the 0 orbit for a Cartan subalgebra $\mathfrak{h}$ and the principal orbit for the other two nonconjugate proper Levi subalgebras containing $\mathfrak{h}$. Thus, we have accounted for three orbits in $\mathfrak{g}$ so far, and it only remains to count the distinguished parabolic subalgebras. (Note however that we cannot read off the weighted diagrams of the two nonzero orbits without doing some further computations.) A Borel subalgebra is distinguished while $\mathfrak{g}$ itself is not (these last facts hold for any semisimple $\mathfrak{g}$). So it suffices to look at the parabolics $\mathfrak{p}_\alpha, \mathfrak{p}_\beta$ containing a fixed Borel subalgebra and corresponding to the subsets $\{\alpha\}, \{\beta\}$ of Δ. The first of these has the roots $\beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$ in its nilradical, whence the subspace V in the proof of (8.2.7) contains all these roots except the last. Since $\dim \mathfrak{l}_\alpha = 4$, we see that $\mathfrak{p}_\alpha$ is distinguished. The corresponding orbit $\mathcal{O}$ has the node corresponding to α labeled 0 and the node corresponding to β labeled 2 in its weighted diagram. By the proof of (8.2.8), it is represented by a sum of at most four root vectors. In fact, $\mathcal{O}$ is represented by a sum of two root vectors; it meets a regular subalgebra of $\mathfrak{g}$ of type $A_1 + A_1$ even though it is distinguished. (It is the subregular orbit of $\mathfrak{g}$, and it arose in the proof of (4.2.1).) Conversely, the parabolic subalgebra $\mathfrak{p}_\beta$ has the roots $\alpha, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$ in its nilradical; only the first two of these lie in the subspace V. Thus $\mathfrak{p}_\beta$ is not distinguished. It is not surprising that $\dim V < \dim \mathfrak{l}_\beta$, for one can adapt the proof of (8.2.8) to show that $\dim V \leq \dim \mathfrak{l}$ for any parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Thus, $\mathfrak{g}$ has exactly 5 nilpotent orbits in all.

We conclude this section by remarking that there are easier ways to classify the distinguished orbits than appealing to (8.2.6) and the definition of distinguished parabolic subalgebra. Indeed, if the ambient Lie algebra $\mathfrak{g}$ is of classical type, then we showed in §§5.1,6.1 how to compute the reductive part of the centralizer of any nilpotent X from its partition. Even without using distinguished parabolics, it is easy to decide exactly when this centralizer fails to contain nonzero semisimple elements, so that X is distinguished. We obtain

heorem 8.2.14.

(i) *If $\mathfrak{g}$ is of type A, then the only distinguished orbit is principal.*

(ii) *If $\mathfrak{g}$ is of type B, C, or D, then an orbit is distinguished if and only if its partition has no repeated parts. Thus, the partition of a distinguished orbit in types B, D has only odd parts, each occurring once, while the partition of a distinguished orbit in type C has only even parts, each again occurring once.*

An immediate consequence is that none of the orbits in type D_{2n} corresponding to a very even partition is distinguished. The point of this observation is that it implies that two pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l}), (\mathfrak{l}, \mathfrak{p}'_\mathfrak{l})$ are G_{ad}-conjugate if and only if the parabolics $\mathfrak{p}_\mathfrak{l}, \mathfrak{p}'_\mathfrak{l}$ are L_{ad}-conjugate. Equivalently, if $\mathfrak{l}$ is a minimal Levi subalgebra of $\mathfrak{g}$ meeting a given orbit $\mathcal{O}$, then it meets $\mathcal{O}$ in a single L_{ad}-orbit. This fact seems not to have been observed by Bala and Carter. (Here L_{ad} of course denotes the adjoint group of $\mathfrak{l}$.)

Of course there is no counterpart to (8.2.14) if $\mathfrak{g}$ is exceptional, for then nilpotent orbits do not correspond to partitions. Nevertheless, Bala and Carter use various tricks in [2] (which we do not describe here) to compute the list of distinguished parabolic subalgebras of exceptional algebras without too much labor. They also use some earlier computations of Dynkin in [27]. The number of distinguished orbits in types G_2 resp. F_4, E_6, E_7, E_8 is 2 resp. 4,3,6,11.

3 Connections with Induction

We used a very naive method in the last section to pass from orbits $\mathcal{O}_\mathfrak{l}$ in a Levi subalgebra of $\mathfrak{g}$ to orbits $\mathcal{O}_\mathfrak{g}$ in $\mathfrak{g}$. Indeed, we just took $\mathcal{O}_\mathfrak{g}$ to be the G_{ad}-saturation of $\mathcal{O}_\mathfrak{l}$; that is, $\mathcal{O}_\mathfrak{g} = G_{ad} \cdot \mathcal{O}_\mathfrak{l}$. We could have used the theory of the last chapter to produce an orbit in $\mathfrak{g}$ from $\mathcal{O}_\mathfrak{l}$ in a much more sophisticated way, namely, by inducing $\mathcal{O}_\mathfrak{l}$ from $\mathfrak{l}$ to $\mathfrak{g}$. Unfortunately, this more sophisticated operation (though it of course has many nice properties) is substantially *less* useful than the naive one if we want to classify the orbits. More precisely, suppose that we took rigid orbits, rather than distinguished ones, as base cases in the classification. We would then try to give an *a priori* parametrization of rigid orbits in $\mathfrak{g}$ and hope to show that any orbit is uniquely induced in some sense from a rigid one. The problem is that both these goals are unrealistic. There is no nice description of the rigid orbits for exceptional $\mathfrak{g}$ and, furthermore, no uniqueness theorem for induced orbits. For example, in type E_8, more than one-fourth of the orbits are rigid, and their weighted diagrams exhibit no particular pattern.

In type F_4, let $\mathfrak{l}_1, \mathfrak{l}_2$ be two nonconjugate Levi subalgebras with semisimple parts of type $A_1 + A_2$, and let $\mathfrak{l}_3$ be a Levi subalgebra with semisimple part of type B_2. Then it turns out that the Richardson orbits attached to the $\mathfrak{l}_i$ (or their corresponding parabolics) all coincide, even though no two of the $\mathfrak{l}_i$ are conjugate and $\mathfrak{l}_3$ is not even isomorphic to either $\mathfrak{l}_1$ or $\mathfrak{l}_2$.

Nevertheless, there is a beautiful connection between Bala-Carter theory and induction that is realized by the duality of §6.3. More precisely, one has

Theorem 8.3.1 (Barbasch, Vogan, Spaltenstein). *Suppose that the nilpotent orbit $\mathcal{O}_\mathfrak{g}$ of $\mathfrak{g}$ meets a proper Levi subalgebra $\mathfrak{l}$ in the orbit $\mathcal{O}_\mathfrak{l}$. Then $d(\mathcal{O}_\mathfrak{g}) = Ind_\mathfrak{l}^\mathfrak{g} d(\mathcal{O}_\mathfrak{l})$.*

We omit the proof, which amounts to a fairly tedious case-by-case computation (see [6, Appendix] or [76, III.11.7]), but we do make a couple of remarks. First, the converse statement fails; that is, if an orbit $\mathcal{O}$ is induced, then its dual orbit need not meet a proper Levi subalgebra. For example, the orbit in $\mathfrak{sp}_6$ with partition $[2^2, 1^2]$ is induced, but the dual orbit has the partition $[4, 2]$ and is distinguished. Second, Spaltenstein originally defined the duality of §6.3 in [76] by a list of axioms, one of which was Theorem 8.3.1 in the special case where $\mathcal{O}_\mathfrak{l}$ is principal and $d(\mathcal{O}_\mathfrak{l}) = 0$. Finally, we note that the induction construction *is* crucial to the classification of certain important unions of (not necessarily nilpotent) orbits called sheets [8].

8.4 Tables

We conclude the chapter by giving a list of the weighted diagrams of the exceptional nilpotent orbits. We follow the notation of Bala and Carter. An orbit corresponding to (the conjugacy class of) the ordered pair $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$ via (8.2.12) is given the label $X_N(a_i)$, where X_N is the Cartan type of the semisimple part of $\mathfrak{l}$ and i is the number of simple roots in any Levi subalgebra of $\mathfrak{p}_\mathfrak{l}$. If $i = 0$, one writes X_N rather than $X_N(a_0)$. In case there are two orbits with the same X_N and the same value of i, we choose one of them arbitrarily and label it $X_N(a_i)$; the other gets the label $X_N(b_i)$. One has to be a bit careful dealing with conjugacy classes of Levi subalgebras; in most cases, two Levi subalgebras are conjugate if and only if they are isomorphic, but this is not always the case. If $\mathfrak{g}$ has two root lengths and a simple component of a Levi subalgebra $\mathfrak{l}$ involves short roots, then we place a tilde over its Cartan label. Thus, if $\mathfrak{g}$ is of type F_4, one of its Levi subalgebras is labeled $A_1 + \tilde{A}_2$ and another is labeled $\tilde{A}_1 + A_2$. These two subalgebras are not conjugate, since any conjugacy must respect root lengths; cf. (3.8.1). There is one further difficulty. If $\mathfrak{g}$ is of type E_7, then it has only one root length, but it turns out to have three pairs of nonconjugate isomorphic Levi subalgebras. One subalgebra in each pair is chosen arbitrarily and labeled with a prime; the other receives a double prime. This difficulty turns out to occur only in type E_7 (for exceptional $\mathfrak{g}$).

In the tables below (taken from [18]), we give four pieces of information about every nilpotent orbit: its Bala-Carter label, weighted diagram, dimension, and fundamental group. The notation kX_N in the leftmost column of course denotes k copies of X_N. In the rightmost column, we indicate whether the orbit is special in the sense of §6.3. One can also read off the G_{ad}-equivariant fundamental group of the orbit from the table, as follows. This is the same as the fundamental group in types G_2, F_4, and E_8. In types E_6 and E_7, one obtains $A(\mathcal{O})$ from $\pi_1(\mathcal{O})$ by omitting a direct factor $\mathbb{Z}/d$, if one is present. (Here $d = 2$ or 3.) In particular, the total number of nilpotent orbits in types G_2 resp. F_4, E_6, E_7, E_8 is 5 resp. 16,21,45,70. For the Hasse diagram of nilpotent orbit closures and the order-reversing involution on special orbits see [18] or [76]. As noted in Chapter 6, the order-reversing surjection from arbitrary nilpotent orbits to special orbits is completely determined by its restriction to the special orbits.

Nilpotent Orbits in Type G_2

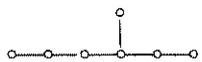
Label	Diagram		$\dim\mathcal{O}$	$\pi_1(\mathcal{O})$	Special
0	0	0	0	1	yes
A_1	1	0	6	1	no
$\widetilde{A}_1$	0	1	8	1	no
$G_2(a_1)$	2	0	10	S_3	yes
G_2	2	2	12	1	yes

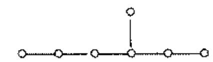
Nilpotent Orbits in Type F_4

Label	Diagram				$\dim\mathcal{O}$	$\pi_1(\mathcal{O})$	Special
0	0	0	0	0	0	1	yes
A_1	1	0	0	0	16	1	no
$\widetilde{A}_1$	0	0	0	1	22	S_2	yes
$A_1+\widetilde{A}_1$	0	1	0	0	28	1	yes
A_2	2	0	0	0	30	S_2	yes
$\widetilde{A}_2$	0	0	0	2	30	1	yes
$A_2+\widetilde{A}_1$	0	0	1	0	34	1	no
B_2	2	0	0	1	36	S_2	no
$\widetilde{A}_2+A_1$	0	1	0	1	36	1	no
$C_3(a_1)$	1	0	1	0	38	S_2	no
$F_4(a_3)$	0	2	0	0	40	S_4	yes
B_3	2	2	0	0	42	1	yes
C_3	1	0	1	2	42	1	yes
$F_4(a_2)$	0	2	0	2	44	S_2	yes
$F_4(a_1)$	2	2	0	2	46	S_2	yes
F_4	2	2	2	2	48	1	yes

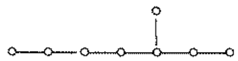
Nilpotent Orbits in Type E_6

Label	Diagram	$\dim\mathcal{O}$	$\pi_1(\mathcal{O})$	Special
0	$\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$	0	1	yes
A_1	$\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$	22	1	yes
$2A_1$	$\begin{smallmatrix} & & 0 & & \\ 1 & 0 & 0 & 0 & 1 \end{smallmatrix}$	32	1	yes
$3A_1$	$\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}$	40	1	no
A_2	$\begin{smallmatrix} & & 2 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$	42	S_2	yes
A_2+A_1	$\begin{smallmatrix} & & 1 & & \\ 1 & 0 & 0 & 0 & 1 \end{smallmatrix}$	46	1	yes
$2A_2$	$\begin{smallmatrix} & & 0 & & \\ 2 & 0 & 0 & 0 & 2 \end{smallmatrix}$	48	$\mathbb{Z}/3\mathbb{Z}$	yes
A_2+2A_1	$\begin{smallmatrix} & & 0 & & \\ 0 & 1 & 0 & 1 & 0 \end{smallmatrix}$	50	1	yes
A_3	$\begin{smallmatrix} & & 2 & & \\ 1 & 0 & 0 & 0 & 1 \end{smallmatrix}$	52	1	yes
$2A_2+A_1$	$\begin{smallmatrix} & & 0 & & \\ 1 & 0 & 1 & 0 & 1 \end{smallmatrix}$	54	$\mathbb{Z}/3\mathbb{Z}$	no
A_3+A_1	$\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 0 & 1 & 0 \end{smallmatrix}$	56	1	no
$D_4(a_1)$	$\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 2 & 0 & 0 \end{smallmatrix}$	58	S_3	yes
A_4	$\begin{smallmatrix} & & 2 & & \\ 2 & 0 & 0 & 0 & 2 \end{smallmatrix}$	60	1	yes
D_4	$\begin{smallmatrix} & & 2 & & \\ 0 & 0 & 2 & 0 & 0 \end{smallmatrix}$	60	1	yes
A_4+A_1	$\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 0 & 1 & 1 \end{smallmatrix}$	62	1	yes
A_5	$\begin{smallmatrix} & & 1 & & \\ 2 & 1 & 0 & 1 & 2 \end{smallmatrix}$	64	$\mathbb{Z}/3\mathbb{Z}$	no
$D_5(a_1)$	$\begin{smallmatrix} & & 2 & & \\ 1 & 1 & 0 & 1 & 1 \end{smallmatrix}$	64	1	yes
$E_6(a_3)$	$\begin{smallmatrix} & & 0 & & \\ 2 & 0 & 2 & 0 & 2 \end{smallmatrix}$	66	$S_2\times\mathbb{Z}/3\mathbb{Z}$	yes
D_5	$\begin{smallmatrix} & & 2 & & \\ 2 & 0 & 2 & 0 & 2 \end{smallmatrix}$	68	1	yes
$E_6(a_1)$	$\begin{smallmatrix} & & 2 & & \\ 2 & 2 & 0 & 2 & 2 \end{smallmatrix}$	70	$\mathbb{Z}/3\mathbb{Z}$	yes
E_6	$\begin{smallmatrix} & & 2 & & \\ 2 & 2 & 2 & 2 & 2 \end{smallmatrix}$	72	$\mathbb{Z}/3\mathbb{Z}$	yes

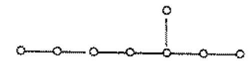
Nilpotent Orbits in Type E_7				
Label	Diagram 	$\dim \mathcal{O}$	$\pi_1(\mathcal{O})$	Special
0	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&0&0 \end{smallmatrix}$	0	1	yes
A_1	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&0&1 \end{smallmatrix}$	34	1	yes
$2A_1$	$\begin{smallmatrix} &&&0&& \\ 0&1&0&0&0&0 \end{smallmatrix}$	52	1	yes
$(3A_1)''$	$\begin{smallmatrix} &&&0&& \\ 2&0&0&0&0&0 \end{smallmatrix}$	54	$\mathbb{Z}/2\mathbb{Z}$	yes
$(3A_1)'$	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&1&0 \end{smallmatrix}$	64	1	no
A_2	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&0&2 \end{smallmatrix}$	66	S_2	yes
$4A_1$	$\begin{smallmatrix} &&&1&& \\ 1&0&0&0&0&0 \end{smallmatrix}$	70	1	no
A_2+A_1	$\begin{smallmatrix} &&&0&& \\ 0&1&0&0&0&1 \end{smallmatrix}$	76	S_2	yes
A_2+2A_1	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&0&0 \end{smallmatrix}$	82	1	yes
A_3	$\begin{smallmatrix} &&&0&& \\ 0&1&0&0&0&2 \end{smallmatrix}$	84	1	yes
$2A_2$	$\begin{smallmatrix} &&&0&& \\ 0&2&0&0&0&0 \end{smallmatrix}$	84	1	yes
A_2+3A_1	$\begin{smallmatrix} &&&2&& \\ 0&0&0&0&0&0 \end{smallmatrix}$	84	$\mathbb{Z}/2\mathbb{Z}$	yes
$(A_3+A_1)''$	$\begin{smallmatrix} &&&0&& \\ 2&0&0&0&0&2 \end{smallmatrix}$	86	$\mathbb{Z}/2\mathbb{Z}$	yes
$2A_2+A_1$	$\begin{smallmatrix} &&&0&& \\ 0&1&0&0&1&0 \end{smallmatrix}$	90	1	no
$(A_3+A_1)'$	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&0&1 \end{smallmatrix}$	92	1	no
$D_4(a_1)$	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&2&0 \end{smallmatrix}$	94	S_3	yes
A_3+2A_1	$\begin{smallmatrix} &&&0&& \\ 1&0&1&0&0&1 \end{smallmatrix}$	94	$\mathbb{Z}/2\mathbb{Z}$	no
D_4	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&2&2 \end{smallmatrix}$	96	1	yes
$D_4(a_1)+A_1$	$\begin{smallmatrix} &&&1&& \\ 1&0&0&0&1&0 \end{smallmatrix}$	96	$S_2\times\mathbb{Z}/2\mathbb{Z}$	yes
A_3+A_2	$\begin{smallmatrix} &&&0&& \\ 0&1&0&1&0&0 \end{smallmatrix}$	98	S_2	yes
A_4	$\begin{smallmatrix} &&&0&& \\ 0&2&0&0&0&2 \end{smallmatrix}$	100	S_2	yes
$A_3+A_2+A_1$	$\begin{smallmatrix} &&&0&& \\ 0&0&2&0&0&0 \end{smallmatrix}$	100	$\mathbb{Z}/2\mathbb{Z}$	yes
$(A_5)''$	$\begin{smallmatrix} &&&0&& \\ 2&2&0&0&0&2 \end{smallmatrix}$	102	1	yes

Nilpotent Orbits in Type E_7 (continued)				
Label	Diagram 	$\dim \mathcal{O}$	$\pi_1(\mathcal{O})$	Special
D_4+A_1	$\begin{smallmatrix} &&&1&& \\ 1&0&0&0&1&2 \end{smallmatrix}$	102	$\mathbb{Z}/2\mathbb{Z}$	no
A_4+A_1	$\begin{smallmatrix} &&&0&& \\ 0&1&0&1&0&1 \end{smallmatrix}$	104	S_2	yes
$D_5(a_1)$	$\begin{smallmatrix} &&&0&& \\ 0&1&0&1&0&2 \end{smallmatrix}$	106	S_2	yes
A_4+A_2	$\begin{smallmatrix} &&&0&& \\ 0&0&0&2&0&0 \end{smallmatrix}$	106	1	yes
$(A_5)'$	$\begin{smallmatrix} &&&0&& \\ 0&2&0&1&0&1 \end{smallmatrix}$	108	1	no
A_5+A_1	$\begin{smallmatrix} &&&0&& \\ 2&1&0&1&0&1 \end{smallmatrix}$	108	$\mathbb{Z}/2\mathbb{Z}$	no
$D_5(a_1)+A_1$	$\begin{smallmatrix} &&&0&& \\ 0&0&2&0&0&2 \end{smallmatrix}$	108	$\mathbb{Z}/2\mathbb{Z}$	yes
$D_6(a_2)$	$\begin{smallmatrix} &&&1&& \\ 2&0&1&0&1&0 \end{smallmatrix}$	110	$\mathbb{Z}/2\mathbb{Z}$	no
$E_6(a_3)$	$\begin{smallmatrix} &&&0&& \\ 0&2&0&0&2&0 \end{smallmatrix}$	110	S_2	yes
D_5	$\begin{smallmatrix} &&&0&& \\ 0&2&0&0&2&2 \end{smallmatrix}$	112	1	yes
$E_7(a_5)$	$\begin{smallmatrix} &&&0&& \\ 2&0&0&2&0&0 \end{smallmatrix}$	112	$S_3\times\mathbb{Z}/2\mathbb{Z}$	yes
A_6	$\begin{smallmatrix} &&&0&& \\ 0&2&0&2&0&0 \end{smallmatrix}$	114	1	yes
D_5+A_1	$\begin{smallmatrix} &&&1&& \\ 0&1&1&0&1&2 \end{smallmatrix}$	114	$\mathbb{Z}/2\mathbb{Z}$	yes
$D_6(a_1)$	$\begin{smallmatrix} &&&1&& \\ 2&0&1&0&1&2 \end{smallmatrix}$	114	$\mathbb{Z}/2\mathbb{Z}$	yes
$E_7(a_4)$	$\begin{smallmatrix} &&&0&& \\ 2&0&0&2&0&2 \end{smallmatrix}$	116	$S_2\times\mathbb{Z}/2\mathbb{Z}$	yes
D_6	$\begin{smallmatrix} &&&1&& \\ 2&2&1&0&1&2 \end{smallmatrix}$	118	$\mathbb{Z}/2\mathbb{Z}$	no
$E_6(a_1)$	$\begin{smallmatrix} &&&0&& \\ 0&2&0&2&0&2 \end{smallmatrix}$	118	S_2	yes
E_6	$\begin{smallmatrix} &&&0&& \\ 0&2&0&2&2&2 \end{smallmatrix}$	120	1	yes
$E_7(a_3)$	$\begin{smallmatrix} &&&0&& \\ 2&2&0&2&0&2 \end{smallmatrix}$	120	$S_2\times\mathbb{Z}/2\mathbb{Z}$	yes
$E_7(a_2)$	$\begin{smallmatrix} &&&2&& \\ 2&0&2&0&2&2 \end{smallmatrix}$	122	$\mathbb{Z}/2\mathbb{Z}$	yes
$E_7(a_1)$	$\begin{smallmatrix} &&&2&& \\ 2&2&2&0&2&2 \end{smallmatrix}$	124	$\mathbb{Z}/2\mathbb{Z}$	yes
E_7	$\begin{smallmatrix} &&&2&& \\ 2&2&2&2&2&2 \end{smallmatrix}$	126	$\mathbb{Z}/2\mathbb{Z}$	yes

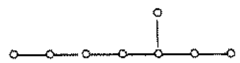
Nilpotent Orbits in Type E_8

Label	Diagram 	$\dim \mathcal{O}$	$\pi_1(\mathcal{O})$	Special
0	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&0&0&0&0 \end{smallmatrix}$	0	1	yes
A_1	$\begin{smallmatrix} &&&&0&& \\ 1&0&0&0&0&0&0 \end{smallmatrix}$	58	1	yes
$2A_1$	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&0&0&0&1 \end{smallmatrix}$	92	1	yes
$3A_1$	$\begin{smallmatrix} &&&&0&& \\ 0&1&0&0&0&0&0 \end{smallmatrix}$	112	1	no
A_2	$\begin{smallmatrix} &&&&0&& \\ 2&0&0&0&0&0&0 \end{smallmatrix}$	114	S_2	yes
$4A_1$	$\begin{smallmatrix} &&&&1&& \\ 0&0&0&0&0&0&0 \end{smallmatrix}$	128	1	no
A_2+A_1	$\begin{smallmatrix} &&&&0&& \\ 1&0&0&0&0&0&1 \end{smallmatrix}$	136	S_2	yes
A_2+2A_1	$\begin{smallmatrix} &&&&0&& \\ 0&0&1&0&0&0&0 \end{smallmatrix}$	146	1	yes
A_3	$\begin{smallmatrix} &&&&0&& \\ 2&0&0&0&0&0&1 \end{smallmatrix}$	148	1	yes
A_2+3A_1	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&0&0&1&0 \end{smallmatrix}$	154	1	no
$2A_2$	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&0&0&0&2 \end{smallmatrix}$	156	S_2	yes
$2A_2+A_1$	$\begin{smallmatrix} &&&&0&& \\ 0&1&0&0&0&0&1 \end{smallmatrix}$	162	1	no
A_3+A_1	$\begin{smallmatrix} &&&&0&& \\ 1&0&1&0&0&0&0 \end{smallmatrix}$	164	1	no
$D_4(a_1)$	$\begin{smallmatrix} &&&&0&& \\ 0&2&0&0&0&0&0 \end{smallmatrix}$	166	S_3	yes
D_4	$\begin{smallmatrix} &&&&0&& \\ 2&2&0&0&0&0&0 \end{smallmatrix}$	168	1	yes
$2A_2+2A_1$	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&1&0&0&0 \end{smallmatrix}$	168	1	no
A_3+2A_1	$\begin{smallmatrix} &&&&0&& \\ 1&0&0&0&0&1&0 \end{smallmatrix}$	172	1	no
$D_4(a_1)+A_1$	$\begin{smallmatrix} &&&&1&& \\ 0&1&0&0&0&0&0 \end{smallmatrix}$	176	S_3	yes
A_3+A_2	$\begin{smallmatrix} &&&&0&& \\ 0&0&1&0&0&0&1 \end{smallmatrix}$	178	S_2	yes
A_4	$\begin{smallmatrix} &&&&0&& \\ 2&0&0&0&0&0&2 \end{smallmatrix}$	180	S_2	yes
$A_3+A_2+A_1$	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&0&1&0&0 \end{smallmatrix}$	182	1	no
D_4+A_1	$\begin{smallmatrix} &&&&1&& \\ 2&1&0&0&0&0&0 \end{smallmatrix}$	184	1	no
$D_4(a_1)+A_2$	$\begin{smallmatrix} &&&&2&& \\ 0&0&0&0&0&0&0 \end{smallmatrix}$	184	S_2	yes

Nilpotent Orbits in Type E_8 (continued)

Label	Diagram 	$\dim \mathcal{O}$	$\pi_1(\mathcal{O})$	Special
A_4+A_1	$\begin{smallmatrix} &&&&0&& \\ 1&0&1&0&0&0&1 \end{smallmatrix}$	188	S_2	yes
$2A_3$	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&1&0&0&1 \end{smallmatrix}$	188	1	no
$D_5(a_1)$	$\begin{smallmatrix} &&&&0&& \\ 2&0&1&0&0&0&1 \end{smallmatrix}$	190	S_2	yes
A_4+2A_1	$\begin{smallmatrix} &&&&0&& \\ 1&0&0&0&1&0&0 \end{smallmatrix}$	192	S_2	yes
A_4+A_2	$\begin{smallmatrix} &&&&0&& \\ 0&0&2&0&0&0&0 \end{smallmatrix}$	194	1	yes
A_5	$\begin{smallmatrix} &&&&0&& \\ 1&0&1&0&0&0&2 \end{smallmatrix}$	196	1	no
$D_5(a_1)+A_1$	$\begin{smallmatrix} &&&&0&& \\ 2&0&0&0&1&0&0 \end{smallmatrix}$	196	1	yes
$A_4+A_2+A_1$	$\begin{smallmatrix} &&&&0&& \\ 0&0&1&0&0&1&0 \end{smallmatrix}$	196	1	yes
D_4+A_2	$\begin{smallmatrix} &&&&2&& \\ 2&0&0&0&0&0&0 \end{smallmatrix}$	198	S_2	yes
$E_6(a_3)$	$\begin{smallmatrix} &&&&0&& \\ 0&2&0&0&0&0&2 \end{smallmatrix}$	198	S_2	yes
D_5	$\begin{smallmatrix} &&&&0&& \\ 2&2&0&0&0&0&2 \end{smallmatrix}$	200	1	yes
A_4+A_3	$\begin{smallmatrix} &&&&0&& \\ 0&1&0&0&1&0&0 \end{smallmatrix}$	200	1	no
A_5+A_1	$\begin{smallmatrix} &&&&0&& \\ 1&0&0&0&1&0&1 \end{smallmatrix}$	202	1	no
$D_5(a_1)+A_2$	$\begin{smallmatrix} &&&&0&& \\ 1&0&1&0&0&1&0 \end{smallmatrix}$	202	1	no
$D_6(a_2)$	$\begin{smallmatrix} &&&&1&& \\ 0&1&0&0&0&1&0 \end{smallmatrix}$	204	S_2	no
$E_6(a_3)+A_1$	$\begin{smallmatrix} &&&&0&& \\ 0&1&0&1&0&0&1 \end{smallmatrix}$	204	S_2	no
$E_7(a_5)$	$\begin{smallmatrix} &&&&0&& \\ 0&0&1&0&1&0&0 \end{smallmatrix}$	206	S_3	no
D_5+A_1	$\begin{smallmatrix} &&&&0&& \\ 2&1&0&1&0&0&1 \end{smallmatrix}$	208	1	no
$E_8(a_7)$	$\begin{smallmatrix} &&&&0&& \\ 0&0&0&2&0&0&0 \end{smallmatrix}$	208	S_5	yes
A_6	$\begin{smallmatrix} &&&&0&& \\ 0&0&2&0&0&0&2 \end{smallmatrix}$	210	1	yes
$D_6(a_1)$	$\begin{smallmatrix} &&&&1&& \\ 2&1&0&0&0&1&0 \end{smallmatrix}$	210	S_2	yes
A_6+A_1	$\begin{smallmatrix} &&&&0&& \\ 0&0&1&0&1&0&1 \end{smallmatrix}$	212	1	yes
$E_7(a_4)$	$\begin{smallmatrix} &&&&0&& \\ 2&0&1&0&1&0&0 \end{smallmatrix}$	212	S_2	yes
[illegible]	$\begin{smallmatrix} &&&&0&& \\ \text{[illegible]} \end{smallmatrix}$	[illegible]	[illegible]	[illegible]

Nilpotent Orbits in Type E_8 (continued)

Label	Diagram	$\dim \mathcal{O}$	$\pi_1(\mathcal{O})$	Special
				
D_5+A_2	$\begin{smallmatrix}&&&&0&&\\2&0&0&2&0&0&0\end{smallmatrix}$	214	S_2	yes
D_6	$\begin{smallmatrix}&&&&1&&\\2&1&0&0&0&1&2\end{smallmatrix}$	216	1	no
E_6	$\begin{smallmatrix}&&&&0&&\\2&2&2&0&0&0&2\end{smallmatrix}$	216	1	yes
$D_7(a_2)$	$\begin{smallmatrix}&&&&0&&\\1&0&1&0&1&0&1\end{smallmatrix}$	216	S_2	yes
A_7	$\begin{smallmatrix}&&&&0&&\\0&1&1&0&1&0&1\end{smallmatrix}$	218	1	no
$E_6(a_1)+A_1$	$\begin{smallmatrix}&&&&0&&\\2&0&1&0&1&0&1\end{smallmatrix}$	218	S_2	yes
$E_7(a_3)$	$\begin{smallmatrix}&&&&0&&\\2&0&1&0&1&0&2\end{smallmatrix}$	220	S_2	yes
$E_8(b_6)$	$\begin{smallmatrix}&&&&0&&\\2&0&0&0&2&0&0\end{smallmatrix}$	220	S_3	yes
$D_7(a_1)$	$\begin{smallmatrix}&&&&0&&\\2&0&0&2&0&0&2\end{smallmatrix}$	222	S_2	yes
E_6+A_1	$\begin{smallmatrix}&&&&0&&\\2&2&1&0&1&0&1\end{smallmatrix}$	222	1	no
$E_7(a_2)$	$\begin{smallmatrix}&&&&1&&\\2&2&0&1&0&1&0\end{smallmatrix}$	224	1	no
$E_8(a_6)$	$\begin{smallmatrix}&&&&0&&\\0&2&0&0&2&0&0\end{smallmatrix}$	224	S_3	yes
D_7	$\begin{smallmatrix}&&&&1&&\\1&0&1&1&0&1&2\end{smallmatrix}$	226	1	no
$E_8(b_5)$	$\begin{smallmatrix}&&&&0&&\\2&2&0&0&2&0&0\end{smallmatrix}$	226	S_3	yes
$E_7(a_1)$	$\begin{smallmatrix}&&&&1&&\\2&2&0&1&0&1&2\end{smallmatrix}$	228	1	yes
$E_8(a_5)$	$\begin{smallmatrix}&&&&0&&\\0&2&0&0&2&0&2\end{smallmatrix}$	228	S_2	yes
$E_8(b_4)$	$\begin{smallmatrix}&&&&0&&\\2&2&0&0&2&0&2\end{smallmatrix}$	230	S_2	yes
E_7	$\begin{smallmatrix}&&&&1&&\\2&2&2&1&0&1&2\end{smallmatrix}$	232	1	no
$E_8(a_4)$	$\begin{smallmatrix}&&&&0&&\\2&0&2&0&2&0&2\end{smallmatrix}$	232	S_2	yes
$E_8(a_3)$	$\begin{smallmatrix}&&&&0&&\\2&2&2&0&2&0&2\end{smallmatrix}$	234	S_2	yes
$E_8(a_2)$	$\begin{smallmatrix}&&&&2&&\\2&2&0&2&0&2&2\end{smallmatrix}$	236	1	yes
$E_8(a_1)$	$\begin{smallmatrix}&&&&2&&\\2&2&2&2&0&2&2\end{smallmatrix}$	238	1	yes
E_8	$\begin{smallmatrix}&&&&2&&\\2&2&2&2&2&2&2\end{smallmatrix}$	240	1	yes

9 Real Nilpotent Orbits

So far we have worked exclusively over $\mathbb{C}$ (except for brief excursions into arbitrary algebraically closed basefields in a couple of remarks). We now indicate what happens over $\mathbb{R}$. Although the theory is neither as elegant nor as well developed as in the complex case, many of the ideas and techniques carry over. Moreover, real nilpotent orbits come up quite frequently in the representation theory of real Lie groups (just as complex orbits do in the representation theory of complex Lie groups; see the next chapter).

9.1 Survey of Real Simple Algebras

We begin by recalling (without proof) Cartan's well-known list of the real simple Lie algebras. Basic references for this material are [83, pp.134-35] and [33, §§III,X]; the latter derives the classification in detail (using Kač's classification of automorphisms of finite order) and gives much explicit information about the real forms.

There are two basic types of real simple Lie algebras: complex simple Lie algebras, regarded as real; and real forms of complex simple algebras. Algebras of the first type obviously yield no new nilpotent orbits, so we restrict attention to real forms. Recall, as noted in §4.4, that a fixed complex simple algebra $\mathfrak{g}_{\mathbb{C}}$ admits a unique compact form up to conjugacy. Its noncompact forms correspond bijectively to conjugacy classes of involutions (automorphisms of order 2) in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ (*not* $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})^o$). Any element of a compact form is skew-adjoint with respect to a negative definite form (the Killing form). Thus, it is semisimple with pure imaginary eigenvalues and cannot be nilpotent unless it is 0. We will therefore concentrate on noncompact forms. We begin with the ones of classical type.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_{2n}\mathbb{C}$, then it has $n+2$ noncompact real forms (if $n \geq 2$). The most obvious one is $\mathfrak{sl}_{2n}\mathbb{R}$, the set of traceless real $2n \times 2n$ matrices. Then there is a series $\mathfrak{su}_{p,2n-p}$ in which each algebra consists of the skew-adjoint matrices relative to a Hermitian form of signature $(p, 2n-p)$ on $\mathbb{C}^{2n}$. Here $n \leq p \leq 2n-1$.

Finally, one has $\mathfrak{su}^*_{2n} \cong \mathfrak{sl}_n\mathbb{H}$, which consists of the $n \times n$ matrices over $\mathbb{H}$ of pure quaternionic trace. ("Pure quaternionic" means a linear combination of i, j, k in $\mathbb{H}$.) If $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_{2n+1}\mathbb{C}$, then one gets the same forms with $2n$ replaced by $2n+1$, except that $\mathfrak{su}^*_{2n+1}$ is not defined.

The situation is simpler for $\mathfrak{g}_\mathbb{C} = \mathfrak{so}_{2n+1}\mathbb{C}$. There are just n noncompact real forms, namely the algebras $\mathfrak{so}_{p,2n+1-p}$ of skew-adjoint matrices relative to a symmetric form of signature $(p, 2n+1-p)$ on $\mathbb{R}^{2n+1}$; here $n+1 \le p \le 2n$. If $\mathfrak{g}_\mathbb{C} = \mathfrak{so}_{2n}\mathbb{C}$, then there is one additional real form (as in type A). This is $\mathfrak{so}^*_{2n}$, the algebra of skew-adjoint matrices relative to a skew-Hermitian form on $\mathbb{H}^n$. One can also realize $\mathfrak{so}^*_{2n}$ as an algebra of $2n \times 2n$ complex matrices; these are required to be skew-adjoint relative to *two* forms: a symmetric one and a skew-Hermitian one. (The forms must be compatible in a suitable sense to establish the identification with $\mathfrak{so}^*_{2n}$; see [33, §X] or [83].) The last classical case, $\mathfrak{g}_\mathbb{C} = \mathfrak{sp}_{2n}\mathbb{C}$, is analogous to $\mathfrak{so}_{2n}\mathbb{C}$. Now, however, the isolated form is the split one $\mathfrak{sp}_{2n}\mathbb{R}$, and the series $\mathfrak{sp}_{2n-2p,2p}$ consists of $n \times n$ matrices over $\mathbb{H}$ skew-adjoint relative to a Hermitian form of signature $(n-p, p)$. Here $1 \le p \le [n/2]$. (This last algebra is usually denoted by $\mathfrak{sp}_{n-p,p}$, but we have chosen the above notation to be consistent with our notation for $\mathfrak{g}_\mathbb{C}$.) Once again, there is a realization of $\mathfrak{sp}_{2n-2p,2p}$ by complex $2n \times 2n$ matrices, this time skew-adjoint relative to a compatible pair of forms, one symplectic and the other Hermitian (of signature $(2n-2p, 2p)$).

We now give a table of the noncompact exceptional real forms, each listed with the Cartan type of its complexified maximal compact subalgebra $\mathfrak{k}_\mathbb{C}$ (which is reductive); clarifying remarks follow.

Exceptional Real Forms

Form	Type of $\mathfrak{k}_\mathbb{C}$
$E_6(-26)$	F_4
$E_6(-14)$	$D_5 + \mathbb{C}$
$E_6(2)$	$A_5 + A_1$
$E_6(6)$	C_4
$E_7(-25)$	$E_6 + \mathbb{C}$
$E_7(-5)$	$D_6 + A_1$
$E_7(7)$	A_7
$E_8(-24)$	$E_7 + A_1$
$E_8(8)$	D_8
$F_4(-20)$	B_4
$F_4(4)$	$C_3 + A_1$
$G_2(2)$	$A_1 + A_1$

When interpreting this table, it is helpful to recall some structural facts. Any real simple algebra $\mathfrak{g}_\mathbb{R}$ admits a *Cartan decomposition* $\mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$ in which $\mathfrak{k}_\mathbb{R}$ is a maximal compact subalgebra and $\mathfrak{p}_\mathbb{R}$ is a $\mathfrak{k}_\mathbb{R}$-module such that $[\mathfrak{p}_\mathbb{R}, \mathfrak{p}_\mathbb{R}] \subset \mathfrak{k}_\mathbb{R}$. Thus, there is also a *Cartan involution* θ of $\mathfrak{g}_\mathbb{R}$ (or its complexification $\mathfrak{g}_\mathbb{C}$) defined to be 1 on $\mathfrak{k}_\mathbb{R}$ and -1 on $\mathfrak{p}_\mathbb{R}$. Any two Cartan decompositions or involutions are conjugate. The Killing form is negative definite on $\mathfrak{k}_\mathbb{R}$ and positive definite on $\mathfrak{p}_\mathbb{R}$. Following Cartan, it is standard to denote a real form of an exceptional algebra of type X_N by $X_N(s)$, where s is the common value of $\dim \mathfrak{p}_\mathbb{R} - \dim \mathfrak{k}_\mathbb{R}$ for all Cartan decompositions $\mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$. (Of course, we could also use this notation for classical real forms, but it is ambiguous in that context: It is possible for two forms of the same classical $\mathfrak{g}_\mathbb{C}$ to have the same value of s.)

9.2 The Jacobson-Morozov Theorem Revisited

We now develop the theory of real nilpotent orbits. Except for one remark about real orthogonal groups in the next section, we will confine attention to orbits under the adjoint group $G_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$. This group is defined exactly as in the complex case; see §1.2. As in the complex case, the first step is to establish a tight link between nilpotent elements and copies of $\mathfrak{sl}_2\mathbb{R}$, or equivalently with standard triples of elements in $\mathfrak{g}_\mathbb{R}$. The main result should come as no surprise, but its proof will take a little more work in this setting.

Theorem 9.2.1. *Let $\mathfrak{g}_\mathbb{R}$ be a real semisimple Lie algebra, $X \in \mathfrak{g}_\mathbb{R}$ a nonzero nilpotent element. Then there is a standard triple $\{H, X, Y\}$ in $\mathfrak{g}_\mathbb{R}$ with X nilpositive.*

Proof. The argument we gave in the complex case actually carries over word for word once we have the theory of Jordan decompositions over non-algebraically-closed basefields (of characteristic 0). We prefer, however, not to assume or develop this theory, and so we take another path.

Clearly, X is also nilpotent in the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{R}$, and $\mathfrak{g}_\mathbb{C}$ is semisimple (though it need not be simple even if $\mathfrak{g}_\mathbb{R}$ is simple). So we get a standard triple $\{H_\mathbb{R} + iH'_\mathbb{R}, X, Y_\mathbb{R} + iY'_\mathbb{R}\}$ in $\mathfrak{g}_\mathbb{C}$ in which every element with an $\mathbb{R}$ subscript belongs to $\mathfrak{g}_\mathbb{R}$. The triple $\{H_\mathbb{R}, X, Y_\mathbb{R}\}$ lies in $\mathfrak{g}_\mathbb{R}$ and clearly satisfies two of the defining equations of a standard triple: $[H_\mathbb{R}, X] = 2X$ and $[X, Y_\mathbb{R}] = H_\mathbb{R}$. Unfortunately, we need not have $[H_\mathbb{R}, Y_\mathbb{R}] = -2Y_\mathbb{R}$. The following lemma cures this difficulty and completes the proof of the theorem. □

Lemma 9.2.2 (Jacobson). *Let $H, X, Y' \in \mathfrak{g}_\mathbb{R}$ satisfy the relations $[H, X] = 2X$ and $[X, Y'] = H$. Then there is $Y \in \mathfrak{g}_\mathbb{R}$ such that $\{H, X, Y\}$ is a standard triple.*

Proof. We begin by applying the Jacobi identity three times. First, ad_X maps the generalized λ-eigenspace of ad_H in $\mathfrak{g}_\mathbb{C}$ to the generalized $\lambda+2$-eigenspace, for any $\lambda \in \mathbb{C}$. Thus, X is nilpotent. Second, ad_H stabilizes $\mathfrak{g}^X_\mathbb{R}$. Third, X centralizes $[H, Y'] + 2Y'$. Suppose now that we can show that ad_H acts on $\mathfrak{g}^X_\mathbb{R}$ without eigenvalue -2. Then $\mathrm{ad}_H + 2$ acts nonsingularly on $\mathfrak{g}^X_\mathbb{R}$, so that there is $Z \in \mathfrak{g}^X_\mathbb{R}$ with $(\mathrm{ad}_H + 2)(Z) = -[H, Y'] - 2Y'$. Replacing Y' by $Y := Y' + Z$, we see that $\{H, X, Y\}$ is a standard triple, as desired.

We are therefore reduced to studying the eigenvalues of ad_H on $\mathfrak{g}^X_\mathbb{R}$. We will actually show that they all lie in $\mathbb{N}$ (note that this *must* be the case if our desired element Y exists, by $\mathfrak{sl}_2$ theory). To this end, we define a filtration $\{\mathfrak{g}_i\}$ on $\mathfrak{g}_\mathbb{R}^X$ via $\mathfrak{g}_i = \mathfrak{g}_\mathbb{R}^X \cap \mathrm{ad}_X^i(\mathfrak{g}_\mathbb{R})$. We have $\mathfrak{g}_i = 0$ for large i since X is nilpotent. Now let $Z = \mathrm{ad}_X^i(W) \in \mathfrak{g}_i$. If $i = 0$, we get

$$\mathrm{ad}_H(Z) = \mathrm{ad}_X \mathrm{ad}_{Y'}(Z) - \mathrm{ad}_{Y'} \mathrm{ad}_X(Z) = \mathrm{ad}_X \mathrm{ad}_{Y'}(Z) \in \mathfrak{g}_1,$$

since $Z \in \mathfrak{g}_\mathbb{R}^X$. If $i > 0$, then we have

$$\begin{aligned} \mathrm{ad}_H(Z) &= \mathrm{ad}_X \mathrm{ad}_{Y'} \mathrm{ad}_X^i(W) \\ &= \mathrm{ad}_X \left(\textstyle\sum_{j=0}^{i-1} \mathrm{ad}_X^j(-\mathrm{ad}_H)\mathrm{ad}_X^{i-1-j} W \right) + \mathrm{ad}_X^{i+1}\mathrm{ad}_{Y'} W \end{aligned}$$

by the product rule for derivations, since $\mathrm{ad}_{Y'} X = -H$. Now move every occurrence of $-\mathrm{ad}_H$ to the extreme left of its term. Every time $-\mathrm{ad}_H$ moves past ad_X, we introduce an "error term" involving $-\mathrm{ad}_X(H) = 2X$. We get

$$\mathrm{ad}_H Z = \mathrm{ad}_X^{i+1}\mathrm{ad}_{Y'} W - \sum_{j=0}^{i-1} \mathrm{ad}_H \mathrm{ad}_X^i W + \sum_{j=0}^{i-1} 2(j+1)\mathrm{ad}_X^i W$$

and thus

$$(i+1)\mathrm{ad}_H Z = \mathrm{ad}_X^{i+1}\mathrm{ad}_{Y'} W + i(i+1)Z.$$

It follows that ad_H acts on the vector space $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ by the scalar i. Since $\mathfrak{g}_i = 0$ for large i, we deduce that ad_H acts (diagonally) on $\mathfrak{g}_\mathbb{R}^X$ with all eigenvalues in $\mathbb{N}$, as claimed. □

Remark. The result generalizes immediately to semisimple Lie algebras $\mathfrak{g}_\mathbf{k}$ over any field $\mathbf{k}$ of characteristic 0. To see this, let $\overline{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$ and let $\mathbf{k}'$ be a $\mathbf{k}$-vector space complement of $\mathbf{k}$ in $\overline{\mathbf{k}}$. Identify $\mathfrak{g}_\mathbf{k}$ with its canonical image in $\mathfrak{g}_{\overline{\mathbf{k}}} = \mathfrak{g}_\mathbf{k} \otimes_\mathbf{k} \overline{\mathbf{k}}$, and write $\mathfrak{g}_{\mathbf{k}'} = \mathbf{k}'\mathfrak{g}_\mathbf{k} \subset \mathfrak{g}_{\overline{\mathbf{k}}}$. Then one need only replace $\mathfrak{g}_\mathbb{C}$ in the proof by $\mathfrak{g}_{\overline{\mathbf{k}}}$ and $i\mathfrak{g}_\mathbb{R}$ by $\mathfrak{g}_{\mathbf{k}'}$. If $\mathbf{k}$ has characteristic $p > 0$, then one must place some restrictions on p to make the theorem hold over $\mathbf{k}$.

Now it is natural to ask to what extent the conjugacy theorems of §3.4 hold for $\mathfrak{g}_\mathbb{R}$. Kostant's conjugacy theorem (3.4.10) carries over at once.

Theorem 9.2.3. *Any two standard triples $\{H, X, Y\}, \{H', X, Y'\}$ in $\mathfrak{g}_\mathbb{R}$ with the same nilpositive element X are conjugate under $G_\mathbb{R}^X$, the centralizer of X in the adjoint group $G_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$.*

The proof is the same as that of (3.4.10). Mal'cev's conjugacy theorem (3.4.12) is a different matter, for it fails already in $\mathfrak{sl}_2\mathbb{R}$. The two matrices $aE_{1,2}, bE_{1,2}$ in $\mathfrak{sl}_2\mathbb{R}$ are conjugate under $SL_2\mathbb{R}$ or $PSL_2\mathbb{R}$ if and only if $b = ac^2$ for some $c \in \mathbb{R}$. (Here $E_{1,2}$ denotes the elementary matrix having a 1 as its (1,2)-entry

and zeros elsewhere, as in §5.2.) Thus, there are two nonconjugate standard triples in $\mathfrak{sl}_2\mathbb{R}$ with the same neutral element (and infinitely many nonconjugate standard triples in say $\mathfrak{sl}_2\mathbb{Q}$ with the same neutral element!). In §9.4 we will produce a substitute for H that does determine the conjugacy class of a standard triple $\{H, X, Y\}$, but the theory developed so far is already sufficient to classify nilpotent orbits in any $\mathfrak{g}_\mathbb{R}$ of classical type. This we do in the next section.

9.3 Nilpotent Orbits in Classical Algebras

The results in this section are due to Springer and Steinberg [80]; they are also stated somewhat more explicitly in [17]. Let $\mathfrak{g}_\mathbb{R}$ belong to the list of classical real forms given in §9.1. Let $X \in \mathfrak{g}_\mathbb{R}$ be a nonzero nilpotent element and $\{H, X, Y\}$ a standard triple. The first step is to decompose the standard representation V of $\mathfrak{g}_\mathbb{R}$ over the span $\mathfrak{a}$ of H, X, and Y, just as we did in the complex case in §5.1. (Note the basefield of V need not be $\mathbb{R}$, even though $\mathfrak{a}$ is only the real span of H, X, Y. Nevertheless, the action of $\mathfrak{a}$ commutes with scalar multiplication by the basefield, so that the irreducible summands are indeed subspaces.) We note that if $\mathfrak{g}_\mathbb{R}$ is not $\mathfrak{su}_{2n}^*$ or $\mathfrak{sl}_n(\mathbb{R})$, then V carries a nondegenerate form $\langle\cdot,\cdot\rangle$ that is $\mathfrak{g}_\mathbb{R}$-invariant. Write

$$V = \bigoplus_{r \geq 0} M(r)$$

as in (5.1.8), so that each $M(r)$ is a direct sum of irreducible $(r+1)$-dimensional $\mathfrak{a}$-modules. Let $H(r)$ denote the highest weight space of $M(r)$, as in §5.1. Assume for definiteness that the ambient form $\langle\cdot,\cdot\rangle$ exists. Again, we get an induced form $(\cdot,\cdot)_r$ on $H(r)$ defined by

$$(v, w)_r = \langle v, Y^r \cdot w\rangle.$$

Corresponding to (5.1.10) and (5.1.14), we have the following result.

Lemma 9.3.1. *If $\langle\cdot,\cdot\rangle$ is symmetric, then $(\cdot,\cdot)_r$ is symmetric or symplectic according as r is even or odd. The signature of $\langle\cdot,\cdot\rangle$ on $M(r)$ has the same number of + as − signs if r is odd. If r is even, then this signature is obtained by starting with the signature of $(\cdot,\cdot)_r$ and then replacing each ± sign s by an alternating sequence of signs of length $r+1$, beginning with s. The same results hold if $\langle\cdot,\cdot\rangle$ is Hermitian, replacing "symmetric" by "Hermitian" and "symplectic" by "skew-Hermitian" throughout. The same results hold if $\langle\cdot,\cdot\rangle$ is skew-Hermitian, interchanging the roles of even and odd r, and again replacing "symmetric" by "Hermitian", "symplectic" by "skew-Hermitian". Finally, the same results hold if $\langle\cdot,\cdot\rangle$ is symplectic, interchanging the roles of even and odd r.*

This is proved by a direct calculation. Furthermore, the proofs of (5.1.11) and (5.1.17) carry over to show that the induced forms $(\cdot,\cdot)_r$ are nondegenerate and determine the ambient form $\langle\cdot,\cdot\rangle$ uniquely. We now recall some standard facts from linear algebra.

Proposition 9.3.2. *Let W be a $\mathbf{k}$-vector space of dimension n, where $\mathbf{k}$ is one of the three fields $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.*

(i) *If $\mathbf{k} = \mathbb{C}$, then W admits a unique nondegenerate symmetric form up to equivalence. It also admits a nondegenerate symplectic form, unique up to equivalence, provided that n is even; if n is odd, there is no such form. Nondegenerate skew-Hermitian forms on W are in one-to-one correspondence with nondegenerate Hermitian forms under multiplication by i, the imaginary unit. Equivalence classes of either of these are parametrized by their signatures, which can be any combination of n signs s_j, each either $+$ or $-$.*

(ii) *If $\mathbf{k} = \mathbb{R}$, then W admits a nondegenerate symplectic form if and only if n is even, in which case it is unique up to equivalence (just as for $\mathbb{C}$). Equivalence classes of nondegenerate symmetric forms are again parametrized by their signatures, which take the same form as in (i).*

(iii) *If $\mathbf{k} = \mathbb{H}$, then W admits a unique nondegenerate skew-Hermitian form up to equivalence (regardless of the parity of n). Equivalence classes of nondegenerate Hermitian forms are parametrized by their signatures, as in (i). There are no nonzero symmetric or symplectic forms on W (and indeed no nonzero bilinear forms).*

(The last assertion in (iii) is a consequence of the noncommutativity of $\mathbb{H}$.)

We are now ready to give the classification of the classical orbits. We state it in terms of signed Young diagrams, which we now introduce. Recall from §6.3 that a Young diagram is a left-justified array of rows of empty boxes arranged so that no row is shorter than the one below it. There is an obvious correspondence between these and partitions. We now define a *signed Young diagram* to be a Young diagram in which every box is labeled with a $+$ or $-$ sign in such a way that signs alternate across rows (they need not alternate down columns). Two signed diagrams are regarded as equivalent if and only if one can be obtained from the other by interchanging rows of equal length. The *signature* of a signed diagram is the ordered pair (m, n), where m is the number of boxes labeled $+$ and n is the number of boxes labeled $-$. We now run through the list of classical real forms in §9.1.

Theorem 9.3.3. *Nilpotent orbits in $\mathfrak{su}_{p,q}$ are parametrized by signed Young diagrams of signature (p, q). Nilpotent orbits in $\mathfrak{su}^*_{2n}$ are parametrized by partitions of n (not $2n$). Nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$ are parametrized by partitions of n, except that "even" partitions having only even terms (not necessarily with even multiplicity) correspond to two orbits, denoted as usual by I and II.*

Proof. The previous discussion shows that nilpotent $U_{p,q}$-orbits in $\mathfrak{su}_{p,q}$ are parametrized by signed Young diagrams of signature (p,q), where $U_{p,q}$ of course denotes the isometry group of a Hermitian form on $\mathbb{C}^{p+q}$ of signature (p,q). But the orbits of $U_{p,q}$ clearly coincide with those of this group modulo its center, which identifies with the quotient of $SU_{p,q}$ modulo *its* center. The first assertion follows. To prove the second, we just use the Jordan normal form for elements in $\mathfrak{sl}_n\mathbb{H}$; since $\mathbb{H}$ is closed under m^{th} roots for any m, there is again no difference between $GL_n\mathbb{H}$-orbits and $SL_n\mathbb{H}$-orbits. The situation is slightly different for $\mathfrak{sl}_n\mathbb{R}$. If two direct sums of Jordan block matrices are conjugate by a matrix of negative determinant and at least one of the block sizes is odd, then one can multiply the conjugating matrix by an appropriate diagonal matrix to make its determinant 1. If, however, all block sizes are even, then we see by looking at centralizers (as in the proof of (5.1.4)) that there is no way to replace the conjugating matrix by one of positive determinant. The theorem follows. □

Theorem 9.3.4. *Nilpotent orbits in $\mathfrak{so}_{p,q}$ are parametrized by "orthogonal signed Young diagrams" of signature (p,q) with numerals; that is, by signed Young diagrams of signature (p,q) such that rows of even length occur with even multiplicity and have their leftmost boxes labeled $+$. Some of these diagrams get Roman numerals attached to them, as follows. If all rows have even length, then two Roman numerals, each I or II, are attached. If at least one row has odd length and all such rows have an even number of boxes labeled $+$, or all such rows have an even number of boxes labeled $-$, then one numeral I or II is attached. Otherwise no numeral is attached. Nilpotent orbits in $\mathfrak{so}^*_{2n}$ are parametrized by signed Young diagrams of size n and any signature in which rows of odd length have their leftmost boxes labeled $+$.*

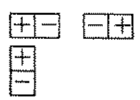
Proof. We first note that orthogonal signed Young diagrams of signature (p,q) *without* numerals parametrize nilpotent $O_{p,q}$-orbits (where $O_{p,q}$ is defined in the obvious way). The point of requiring rows of even length to begin with a $+$ is simply to account for the uniqueness (up to equivalence) of a nondegenerate symplectic form on an even-dimensional real vector space (cf. (9.3.2)(ii)); the signatures work out properly by (9.3.1). The proof that the rules for attaching Roman numerals account for the way in which $O_{p,q}$-orbits split into $SO^o_{p,q}$-orbits is left an exercise; it requires the observation that $O_{p,q}$ has four connected components. Finally, the last assertion follows from the discussion before (9.3.3) together with the realization of $\mathfrak{so}^*_{2n}$ in §1. Again, we require that odd rows begin with $+$ because skew-Hermitian forms are unique up to equivalence, by (9.3.2)(iii). We don't have to keep track of the numbers of $+$ and $-$ signs because skew-Hermitian forms don't have signatures; furthermore, orbits under the adjoint group coincide with those under the isometry group. □

orem 9.3.5. *Nilpotent orbits in $\mathfrak{sp}_{2p,2q}$ are parametrized by signed Young diagrams of signature (p,q) in which even rows begin with $+$. Nilpotent orbits in $\mathfrak{sp}_{2n}\mathbb{R}$ are parametrized by signed Young diagrams of size $2n$ and any signature in which odd rows begin with $+$ and occur with even multiplicity.*

Proof. This follows from the above discussion together with the realization of $\mathfrak{sp}_{2p,2q}$ in §1. Once again, there is no difference between orbits under the adjoint and isometry groups. □

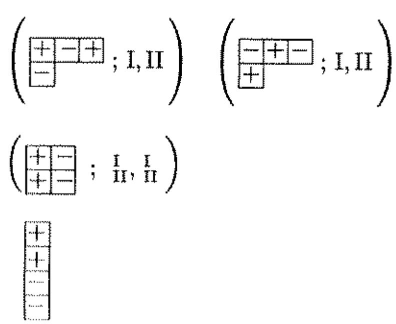
For every classical real form other than $\mathfrak{su}^*_{2n}, \mathfrak{so}^*_{2n}$, and $\mathfrak{sp}_{2p,2q}$, the Young diagram of the complexification $\mathcal{O}_\mathbb{C}$ of a real orbit $\mathcal{O}_\mathbb{R}$ is obtained from that of $\mathcal{O}_\mathbb{R}$ by omitting the signs. (For $\mathfrak{so}_{p,q}$, one also omits the numeral if the partition of $\mathcal{O}_\mathbb{R}$ is not very even; if it is very even, one omits only the first numeral. For $\mathfrak{sl}_n\mathbb{R}$, one also omits the numeral.) For $\mathfrak{su}^*_{2n}, \mathfrak{so}^*_{2n}$, and $\mathfrak{sp}_{2p,2q}$, one obtains the Young diagram for $\mathcal{O}_\mathbb{C}$ from that of $\mathcal{O}_\mathbb{R}$ by omitting the signs and replacing every row by two copies of itself. We now give some examples.

Example 9.3.6. There is a well-known isomorphism $\mathfrak{su}_{1,1} \cong \mathfrak{sl}_2\mathbb{R}$. Applying (9.3.3), we see that orbits in the former algebra correspond to the diagrams



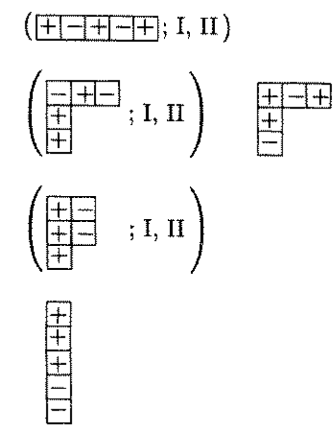
while those in the latter correspond to the partitions ([2];I), ([2];II), and $[1^2]$. The parametrizing sets match up in an obvious way; note that there are two principal orbits in this case, in contrast to (4.1.6). Of course, these orbits are conjugate under the complex group.

Example 9.3.7. There is a slightly less well-known isomorphism $\mathfrak{so}_{2,2} \cong \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$. Since we observed in the last example that $\mathfrak{sl}_2\mathbb{R}$ has three nilpotent orbits, it follows that $\mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$ has nine such orbits. By (9.3.4), $\mathfrak{so}_{2,2}$ also has nine nilpotent orbits, corresponding to the diagrams

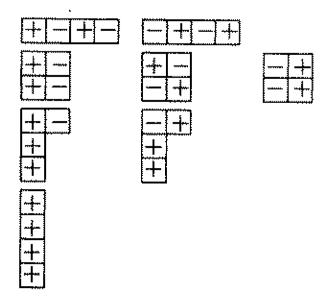


These again match up with the orbits in $\mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$ in a straightforward way. There are four principal orbits, obtained by taking products of principal orbits in $\mathfrak{sl}_2\mathbb{R}$. The four orbits whose diagrams are given in the second row are obtained by taking the product of a principal orbit in one $\mathfrak{sl}_2\mathbb{R}$ factor and the 0 orbit in the other.

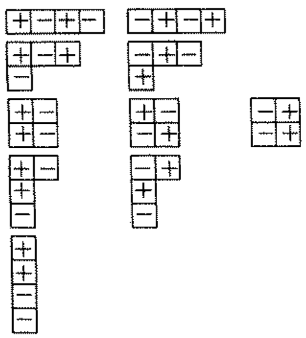
Example 9.3.8. We have an isomorphism of split forms $\mathfrak{so}_{3,2} \cong \mathfrak{sp}_4\mathbb{R}$. The diagrams parametrizing the orbits in the former algebra are



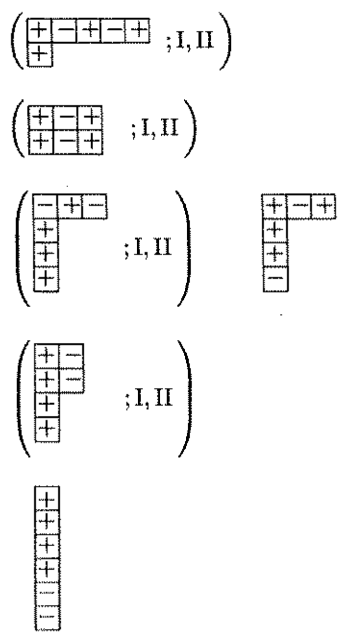
while the corresponding diagrams for the latter algebra are



Example 9.3.9. Finally, we consider the isomorphism $\mathfrak{su}_{2,2} \cong \mathfrak{so}_{4,2}$, which arises in relativity theory. The relevant diagrams for the first algebra are



while those for the latter algebra are



You may enjoy matching up the parametrizations given by Theorems 9.3.3 through 9.3.5 for the following three pairs of isomorphic algebras: $\mathfrak{so}_{4,1} \cong \mathfrak{sp}_{1,1}$; $\mathfrak{su}_4^* \cong \mathfrak{so}_{5,1}$; $\mathfrak{so}_8^* \cong \mathfrak{so}_{6,2}$. We conclude this section by remarking that Djoković has completely described the order relation on real classical nilpotent orbits given by containment of closures; see [22].

9.4 Cayley and Normal Triples: Basic Conjugacy Results

We now return to the case of a general real semisimple algebra $\mathfrak{g}_\mathbb{R}$ and develop the theory of standard triples further. Fix a Cartan decomposition $\mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$ and let θ be the corresponding Cartan involution. Given a standard triple $\{H, X, Y\}$, our goal is to produce a semi-simple element whose conjugacy class determines that of the triple. Since Cartan involutions play such a prominent role in the theory of real simple algebras, it is reasonable to expect this semisimple element to have something to do with θ. Now Theorem 9.2.1 immediately suggests a connection. The span $\mathfrak{a}$ of $\{H, X, Y\}$ is isomorphic to $\mathfrak{sl}_2\mathbb{R}$, so it has a Cartan involution θ' in its own right, which is usually taken to be the negative transpose map. Thus $\theta'(H) = -H, \theta'(X) = -Y, \theta'(Y) = -X$ (cf. the formulas for H, X, Y as images of matrices in $\mathfrak{sl}_2\mathbb{R}$ in §3.2). The obvious way to bring θ into the picture would be to have it stabilize $\mathfrak{a}$ and agree with θ' there. Actually, it is not reasonable to expect this, as there is nothing canonical about our choice of θ; it is unique only up to conjugation. The most to be hoped for is that $\{H, X, Y\}$ is *conjugate* to a triple whose span is θ-stable. This is indeed the case.

Theorem 9.4.1. *Given a Cartan involution θ, any standard triple $\{H, X, Y\}$ in $\mathfrak{g}_\mathbb{R}$ is conjugate to another one $\{H', X', Y'\}$ such that $\theta(H') = -H', \theta(X') = -Y', \theta(Y') = -X'$. (Triples with this last property are called Cayley triples.)*

Proof. By a result of Mostow [69, Thm.6] (see also [33, Ch.VI,ex.8]), any Cartan involution of a semisimple subalgebra of $\mathfrak{g}_\mathbb{R}$ extends to one on $\mathfrak{g}_\mathbb{R}$ itself. The result follows from the conjugacy of Cartan involutions. □

We now see (philosophically) why H does not determine the conjugacy class of $\{H, X, Y\}$: both H and $X + Y$ lie in the noncompact part of a Cartan decomposition, so neither one can be regarded as distinguished. By contrast, $X - Y$ is the unique compact element in $\mathfrak{a}$, up to conjugation and scalar multiples, so it should be the semisimple element we are looking for. We will prove this below, but first we need to introduce an important auxiliary standard triple attached to any Cayley triple. Let $\mathfrak{k}_\mathbb{C}, \mathfrak{p}_\mathbb{C}$ be the complexifications of $\mathfrak{k}_\mathbb{R}, \mathfrak{p}_\mathbb{R}$ and let $\{H, X, Y\}$ be a Cayley triple. We need to construct a second triple which not only spans a θ-stable subalgebra but actually consists of θ-eigenvectors. Set $\{H', X', Y'\} = \{i(X-Y), \frac{1}{2}(X+Y+iH), \frac{1}{2}(X+Y-iH)\}$. We can check directly that $\{H', X', Y'\}$ is indeed a standard triple; it is called the *Cayley transform* of $\{H, X, Y\}$. Of course it lives in $\mathfrak{g}_\mathbb{C}$, not $\mathfrak{g}_\mathbb{R}$; more precisely, its neutral element lies in $\mathfrak{k}_\mathbb{C}$ while the other two lie in $\mathfrak{p}_\mathbb{C}$. Any standard triple in $\mathfrak{g}_\mathbb{C}$ with this last property is called *normal.*

Now we must digress for a moment to study the conjugacy properties of normal triples. Fortunately, the main facts accord precisely with what we would expect from the results in §3.4. More precisely, we have the following analogues of (3.3.1), (3.4.10), and (3.4.12).

rem 9.4.2. *Any nonzero nilpotent element $X \in \mathfrak{p}_\mathbb{C}$ is the nilpositive element of a normal triple.*

rem 9.4.3. *Any two normal triples $\{H, X, Y\}, \{H', X, Y'\}$ with the same nilpositive element X are $K_\mathbb{C}^X$-conjugate, where $K_\mathbb{C}^X$ denotes the centralizer of X in the adjoint group $K_\mathbb{C}$ of $\mathfrak{k}_\mathbb{C}$.*

rem 9.4.4. *Any two normal triples $\{H, X, Y\}, \{H, X', Y'\}$ with the same neutral element H are $K_\mathbb{C}^H$-conjugate.*

These results are all taken from a fundamental paper [54] of Kostant and Rallis that studies normal triples independently of their applications to nilpotent orbits. They all essentially follow easily from the proofs of the corresponding results in Chapter 3. For example, to prove (9.4.2), let $X \in \mathfrak{p}_\mathbb{C}$ be nilpotent and nonzero. As in the argument before (3.3.5), we get $H' \in \mathfrak{g}_\mathbb{C}$ with $[H', X] = 2X$. Replacing H' by its component in $\mathfrak{k}_\mathbb{C}$, we obtain $H'' \in \mathfrak{k}_\mathbb{C}$ with $[H'', X] = 2X$. Replacing H'' by its semisimple component in the Jordan decomposition in $\mathfrak{k}_\mathbb{C}$, we may assume that H'' is semisimple. Then the proof is completed by induction on the dimension of $\mathfrak{g}_\mathbb{C}$, as in (3.3.1). To prove (9.4.3), we argue as in the proof of (3.4.10), replacing the "positive weight" part $\mathfrak{u}^X$ of $\mathfrak{g}_\mathbb{C}^X$ by its projection to $\mathfrak{k}_\mathbb{C}$. Finally, to prove (9.4.4), we replace the subspace $\mathfrak{g}_2$ occurring in the proof of (3.4.12) by its projection $\mathfrak{p}_2$ to $\mathfrak{p}_\mathbb{C}$, define a Zariski-open subset $\mathcal{P}'$ of $\mathfrak{p}_2$ as in (3.4.12), and then use a connectedness argument to show that $\mathcal{P}'$ is a single $K_\mathbb{C}^H$-orbit.

Before we can state and prove the analogue of (3.4.12) for $\mathfrak{g}_\mathbb{R}$, we need one more structural fact.

ma 9.4.5. *Any two elements of $\mathfrak{k}_\mathbb{R}$ or $\mathfrak{p}_\mathbb{R}$ are $G_\mathbb{R}$-conjugate if and only if they are $K_\mathbb{R}$-conjugate.*

Proof. Suppose for definiteness that $W, g \cdot W \in \mathfrak{k}_\mathbb{R}$ (the other case is similar) with $g \in G_\mathbb{R}$. By the Cartan decomposition for reductive groups [33, VI.1.1], we may write $g = (\text{Exp } P)k$ for some $k \in K_\mathbb{R}$ and $P \in \mathfrak{p}_\mathbb{R}$. Replacing W by $W' := k \cdot W$, we may assume that $g = $Exp P. Now ad_P acts semisimply on $\mathfrak{g}_\mathbb{R}$ with real eigenvalues, so that ad_P^2 acts semisimply on $\mathfrak{k}_\mathbb{R}$ and $\mathfrak{p}_\mathbb{R}$ with nonnegative real eigenvalues. From the power series expansion of Exp $P \cdot W'$, we see that its component in $\mathfrak{p}_\mathbb{R}$ cannot vanish unless $[P, W'] = 0$. Thus $g \cdot W' = W'$, whence W and $g \cdot W$ are already conjugate under $k \in \mathfrak{k}_\mathbb{R}$, as claimed. □

We are finally ready for our main result.

orem 9.4.6 (Rao). *Any two standard triples $\{H, X, Y\}, \{H', X', Y'\}$ in $\mathfrak{g}_\mathbb{R}$ with $X - Y = X' - Y'$ are conjugate under $G_\mathbb{R}^{X-Y}$, the centralizer of $X - Y$ in $G_\mathbb{R}$.*

Proof. A direct attack along the lines of (3.4.12) seems to lead to a blind alley (one problem is that the centralizer $G_\mathbb{R}^{X-Y}$, unlike the centralizer G_{ad}^H in (3.4.12), need not be connected). So we give a more roundabout argument using the above ideas. By (9.4.1), there are $g, h \in G_\mathbb{R}$ such that $g \cdot \{H, X, Y\}$ and $h \cdot \{H', X', Y'\}$ are both Cayley. Since $g \cdot (X - Y)$ and $h \cdot (X' - Y')$ are elements of $\mathfrak{k}_\mathbb{R}$ conjugate under $G_\mathbb{R}$, the proof of (9.4.5) shows that we may replace $h \cdot \{H', X', Y'\}$ by a Cayley triple $h' \cdot \{H', X', Y'\}$ with $h' \cdot (X' - Y') = g \cdot (X - Y)$. Thus, the triple $\{H'', X'', Y''\} := g^{-1}h' \cdot \{H', X', Y'\}$ satisfies $X'' - Y'' = X - Y$ and both $\{H, X, Y\}$ and $\{H'', X'', Y''\}$ are Cayley triples with respect to the *same* Cartan decomposition, which we may assume is our fixed one $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$. Replacing $\{H, X, Y\}, \{H'', X'', Y''\}$ by their Cayley transforms, we get two normal triples in $\mathfrak{g}_\mathbb{C}$ with the same neutral element $i(X - Y)$. By (9.4.4), these normal triples are conjugate by some $k \in K_\mathbb{C}^{X-Y}$, which also conjugates one of the Cayley triples to the other. Since $\mathfrak{k}_\mathbb{C}$ is reductive and $X - Y$ is semisimple, (2.1.2) implies that the algebra centralizer $\mathfrak{k}_\mathbb{C}^{X-Y}$ is reductive. Applying the proof of (9.4.5) to the Cartan decomposition $\mathfrak{k}_\mathbb{R}^{X-Y} + i\mathfrak{k}_\mathbb{R}^{X-Y}$ of $\mathfrak{k}_\mathbb{C}^{X-Y}$, we see that we may replace k by an element of $K_\mathbb{R}^{X-Y}$. Thus, $\{H, X, Y\}$ and $\{H', X', Y'\}$ are conjugate under $G_\mathbb{R}^{X-Y}$, as desired. □

9.5 Sekiguchi's Bijection and Weighted Dynkin Diagrams

Now we are ready to state our main result on nilpotent orbits in $\mathfrak{g}_\mathbb{R}$. Its proof combines the basic conjugacy theory of Cayley and normal triples via the Cayley transform. The key observation is that the groups $G_\mathbb{R}$ and $K_\mathbb{C}$ share a common maximal compact subgroup, namely $K_\mathbb{R}$.

Theorem 9.5.1 (Sekiguchi [74]). *There is a natural one-to-one correspondence between nilpotent $G_\mathbb{R}$-orbits in $\mathfrak{g}_\mathbb{R}$ and nilpotent $K_\mathbb{C}$-orbits in $\mathfrak{p}_\mathbb{C}$. This correspondence sends the zero orbit to the zero orbit and the orbit through the nilpositive element of a Cayley triple to the one through the nilpositive element of its Cayley transform.*

Proof. We first show that the correspondence is well defined. We have already shown that any standard triple in $\mathfrak{g}_\mathbb{R}$ is conjugate to a Cayley triple. Any two Cayley triples $\{H, X, Y\}, \{H', X, Y'\}$ with the same nilpositive element are $G_\mathbb{R}^X$-conjugate by (9.2.3). In particular, $X - Y$ is conjugate to $X - Y'$, whence these two elements are actually $K_\mathbb{R}$-conjugate, by (9.4.5). Thus the Cayley transforms of $\{H, X, Y\}, \{H', X, Y'\}$ have $K_\mathbb{R}$-conjugate neutral elements. By (9.4.4), these normal triples are $K_\mathbb{C}$-conjugate, whence so are their nilpositive elements, as desired.

Now we show that the correspondence is a bijection by constructing an explicit inverse for it. Given a normal triple $\{H_\mathbb{C}, X_\mathbb{C}, Y_\mathbb{C}\}$ in $\mathfrak{g}_\mathbb{C}$, we must first show that it is $K_\mathbb{C}$-conjugate to a Cayley transform of a Cayley triple. This is the hardest step. Let $\mathfrak{a}_\mathbb{C}$ be the complex span of $H_\mathbb{C}, X_\mathbb{C}$, and $Y_\mathbb{C}$. Let ψ be the unique *conjugate* linear automorphism of $\mathfrak{a}_\mathbb{C}$ (regarded as a real Lie algebra) fixing $X_{\mathbb{C}} - Y_{\mathbb{C}}$ and sending $H_{\mathbb{C}}, X_{\mathbb{C}} + Y_{\mathbb{C}}$ to their negatives. Then ψ is a Cartan involution of $\mathfrak{a}_{\mathbb{C}}$; note that it commutes with the restriction of θ to $\mathfrak{a}_{\mathbb{C}}$. By the result of Mostow quoted above, we know that ψ extends to a Cartan involution θ' of $\mathfrak{g}_{\mathbb{C}}$ (regarded as a real Lie algebra). The proof of this result in [69] shows that we may assume that θ' commutes with θ. Now let θ_c be the Cartan involution of $\mathfrak{g}_{\mathbb{C}}$ corresponding to its compact real form $\mathfrak{k}_{\mathbb{R}}+i\mathfrak{p}_{\mathbb{R}}$. By [33, III.7.2] there is $Z \in \mathfrak{g}_{\mathbb{C}}$ such that the automorphism $g =$Exp Z commutes with θ and conjugates θ' into θ_c; then we must have $g \in \mathfrak{k}_{\mathbb{C}}$. One computes that $g(H_{\mathbb{C}}) \in \mathfrak{k}_{\mathbb{C}} \cap (i\mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}) = i\mathfrak{k}_{\mathbb{R}}$; similarly $g(X_{\mathbb{C}} + Y_{\mathbb{C}}) \in \mathfrak{p}_{\mathbb{R}}$ and $g(X_{\mathbb{C}} - Y_{\mathbb{C}}) \in \mathfrak{p}_{\mathbb{R}}$. Thus, $g \cdot \{H_{\mathbb{C}}, X_{\mathbb{C}}, Y_{\mathbb{C}}\}$ is the Cayley transform of a Cayley triple and $g \in \mathfrak{k}_{\mathbb{C}}$, as desired.

Given a nilpotent $X_{\mathbb{C}} \in \mathfrak{p}_{\mathbb{C}}$, we know that it is the nilpositive element of a normal triple and any two such triples are $K_{\mathbb{C}}$-conjugate, by (9.4.2) and (9.4.3). If two such triples are conjugated so as to be Cayley transforms of Cayley triples, then their inverse Cayley transforms $\{H, X, Y\}, \{H', X', Y'\}$ give rise to $K_{\mathbb{C}}$-conjugate elements $X - Y, X' - Y'$ of $i\mathfrak{k}_{\mathbb{R}}$. Again, by (9.4.5) applied to $\mathfrak{k}_{\mathbb{C}}$, we see that $X - Y$ and $X' - Y'$ are $K_{\mathbb{R}}$-conjugate. It follows from (9.4.6) that $\{H, X, Y\}, \{H', X', Y'\}$ are $G_{\mathbb{R}}$-conjugate. Thus, the correspondence has a well-defined two-sided inverse. □

Remark 9.5.2. If the $G_{\mathbb{R}}$-orbit $G_{\mathbb{R}} \cdot \lambda_{\mathbb{R}}$ corresponds to the $K_{\mathbb{C}}$-orbit $K_{\mathbb{C}} \cdot \lambda_{\mathbb{C}}$, then we also have the following properties, whose proofs we omit.

(i) $G_{\mathbb{C}} \cdot \lambda_{\mathbb{R}} = G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}}$;

(ii) $\dim_{\mathbb{C}}(K_{\mathbb{C}} \cdot \lambda_{\mathbb{C}}) = \frac{1}{2} \dim_{\mathbb{R}}(G_{\mathbb{R}} \cdot \lambda_{\mathbb{R}}) = \frac{1}{2} \dim_{\mathbb{C}}(G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}})$;

(iii) the centralizers $G_{\mathbb{R}}^{\lambda_{\mathbb{R}}}, K_{\mathbb{C}}^{\lambda_{\mathbb{C}}}$ have a common maximal compact subgroup, namely the centralizer $K_{\mathbb{R}}^{\lambda_{\mathbb{R}}, \lambda_{\mathbb{C}}}$ of the span of $\lambda_{\mathbb{R}}$ and $\lambda_{\mathbb{C}}$ in $K_{\mathbb{R}}$.

We also note that Sekiguchi's bijection is known to preserve the closure order on orbits (defined in the introduction to Chapter 4) for classical algebras $\mathfrak{g}_{\mathbb{R}}$ [70]. It is not known whether this is still true if $\mathfrak{g}_{\mathbb{R}}$ is exceptional. Finally, we note that Djoković gave an independent proof of (9.5.1) in [23].

We conclude this section by giving a brief treatment of weighted Dynkin diagrams of nilpotent orbits in the simple real forms. Given a nonzero nilpotent orbit $G_{\mathbb{R}} \cdot X_{\mathbb{R}} = \mathcal{O}_{\mathbb{R}}$, let $K_{\mathbb{C}} \cdot X_{\mathbb{C}}$ be the orbit in $\mathfrak{p}_{\mathbb{C}}$ corresponding to it via (9.5.1). Let $\{H_{\mathbb{C}}, X_{\mathbb{C}}, Y_{\mathbb{C}}\}$ be a normal triple with nilpositive element $X_{\mathbb{C}}$. We may conjugate $\{H_{\mathbb{C}}, X_{\mathbb{C}}, Y_{\mathbb{C}}\}$ so that its neutral element $H_{\mathbb{C}}$ lies in a fixed dominant Weyl chamber of a fixed Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. As in §3.5, this neutral element then determines the orbit $K_{\mathbb{C}} \cdot X_{\mathbb{C}}$ uniquely. Label each node of the Dynkin diagram of $\mathfrak{k}_{\mathbb{C}}$ by the eigenvalue of the corresponding simple root on $H_{\mathbb{C}}$. This labeled Dynkin diagram is called the weighted diagram of $G_{\mathbb{R}} \cdot X_{\mathbb{R}}$ or $K_{\mathbb{C}} \cdot X_{\mathbb{C}}$. It is an invariant of the orbit. If $\mathfrak{g}_{\mathbb{R}}$ is not a Hermitian symmetric real form, the weighted diagram is a complete invariant; but if $\mathfrak{g}_{\mathbb{R}}$ is Hermitian symmetric, then $\mathfrak{k}_{\mathbb{C}}$ has a one-dimensional center and we must also keep track of the component of $H_{\mathbb{C}}$ in the center. This can be done by extending the weighted diagram as follows. From the classification of Hermitian symmetric real forms, it follows that we may obtain a set of simple roots of $\mathfrak{k}_{\mathbb{C}}$ from a corresponding set for $\mathfrak{g}_{\mathbb{C}}$ by omitting one root α. We extend a typical weighted diagram for $\mathfrak{g}_{\mathbb{R}}$ by adding a label equal to the eigenvalue of $\mathrm{ad}_{H_{\mathbb{C}}}$ on the α-weight space. The resulting weighted diagram is denoted $\Delta_K(\mathcal{O}_{\mathbb{R}})$. If $\mathcal{O}_{\mathbb{R}}$ is the trivial orbit, we adopt the convention that its diagram has every node labeled zero.

Just as in the complex case, the labels of the weighted diagram are nonnegative integers (except that the additional label in a typical *extended* weighted diagram need not be nonnegative; we will say more about it below). In the complex case, all labels are 0,1, or 2, but this need not hold in the real case. To see why, we review some of the theory in the complex case. Once a Cartan subalgebra $\mathfrak{h}$ of a complex semisimple algebra $\mathfrak{g}$ has been fixed, choosing a dominant chamber in $\mathfrak{h}$ amounts to choosing a set of positive roots of $\mathfrak{h}$ in $\mathfrak{g}$, or equivalently choosing a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$. Suppose that this has been done and let $\{H, X, Y\}$ be a standard triple in $\mathfrak{g}$ with H in the fixed dominant chamber. Then X must lie in $\mathfrak{n}$, the nilradical of $\mathfrak{b}$. If $X_{-\alpha}$ is a root vector corresponding the negative of a simple root α, then either $[X_{-\alpha}, X] = 0$ (in which case the label of α must be 0) or $[X_{-\alpha}, X]$ is a nonzero element of $\mathfrak{b}$ (in which case the label of α can also be 1 or 2). Now let's try to repeat this argument in the real case. Given a normal triple $\{H, X, Y\}$ with H in a fixed dominant chamber of a fixed Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$, and a root vector $X_{-\alpha}$ in $\mathfrak{k}_{\mathbb{C}}$ corresponding to the negative of a simple root, the first problem is that $[X_{-\alpha}, X]$ lies in $\mathfrak{p}_{\mathbb{C}}$, not $\mathfrak{k}_{\mathbb{C}}$. We can cure this difficulty by bracketing again with X; $[[X_{-\alpha}, X], X]$ does lie in $\mathfrak{k}_{\mathbb{C}}$. If this is 0, the label of α must be 0 or 1. But if it is not, then there is a further difficulty, which turns out to be insuperable: The double bracket $[[X_{-\alpha}, X], X]$ need not lie in the bracket of $X_{-\alpha}$ and the fixed Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$, in general (the bracket operation is not associative). Thus, we can almost, but not quite, show that all labels are 0,1,2,3, or 4. In fact (at least if $\mathfrak{g}_{\mathbb{R}}$ is exceptional), all labels are 0,1,2,3,4, or 8, except for the additional label in an extended weighted diagram. This label turns out be an integer bounded between -10 and 2.

It is not clear from the arguments of the last two paragraphs that there are only finitely many nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$. To see this, we consider a slightly different kind of labeled diagram. Fix a dominant chamber of a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ as above and let $\{H, X, Y\}$ be a normal triple with neutral element in this dominant chamber. Now label every node of the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ (not $\mathfrak{k}_{\mathbb{C}}$) with the eigenvalue of the corresponding simple root on H (extending our fixed Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ to a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$). Then the labels are not necessarily nonnegative; but they are integers bounded in absolute value by $\dim \mathfrak{g}_{\mathbb{C}}$. (By the representation theory of $\mathfrak{sl}_2$, none of the irreducible constituents of $\mathfrak{g}_{\mathbb{C}}$ under the action of the normal triple can have dimension greater than that of $\mathfrak{g}_{\mathbb{C}}$ itself.) This new labeled diagram is *not* an invariant of the orbit $K_{\mathbb{C}} \cdot X$, but it does completely determine this orbit. Thus, there are only finitely many nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$. This argument is taken from [54]. We summarize it in eorem 9.5.3. *There are only finitely many nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$.*

6 Tables of the Real Exceptional Orbits

We conclude this chapter by giving Djoković's tables in [24, 25] of the weighted diagrams of the real exceptional orbits. For each orbit $\mathcal{O}_{\mathbb{R}}$, we give its weighted diagram $\Delta_K(\mathcal{O}_{\mathbb{R}})$ together with the diagram of its complexification, which is denoted $\Delta(\mathcal{O}_{\mathbb{C}})$. Whenever the orbit lives in a Hermitian symmetric real form, we extend the weighted diagram, as in the last section, so that it is a complete invariant; the additional label lies to the right of the others. Thus, for every exceptional real form $\mathfrak{g}_{\mathbb{R}}$, we can read off which orbits in $\mathfrak{g}_{\mathbb{C}}$ meet $\mathfrak{g}_{\mathbb{R}}$. (We showed how to do this for classical forms $\mathfrak{g}_{\mathbb{R}}$ in §9.3.) By combining these tables with the ones in the last chapter, we can also compute the dimension of any real exceptional nilpotent orbit. (Djoković also computes the reductive part of the centralizer of any real nilpotent element; we omit this information.)

To make sense out of the tables, we make a few additional notational remarks. Fix a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}$, which can be extended to a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$; in fact, if $\mathfrak{g}_{\mathbb{C}}$ is a simple real exceptional form, then $E_{6(6)}$ and $E_{6(-26)}$ are the only cases where $\mathfrak{t}_{\mathbb{C}}$ must be extended. Given the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$, we can form the *extended Dynkin diagram* that arises when we append the negative of the highest root of $\Phi^+(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ to its base $\Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ of simple roots; this leads to the extended base of simple roots, denoted $\Delta_{\text{ext}}(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$. The key fact to recall is that we can find a base of simple roots $\Delta(\mathfrak{t}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ for $\Phi^+(\mathfrak{t}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ so that $\Delta(\mathfrak{t}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}) \subset \Delta_{\text{ext}}(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$. If ℓ (resp. ℓ_K) denotes the rank of $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{k}_{\mathbb{C}}$), then denote by $\alpha_1, \ldots, \alpha_\ell$ (resp. $\beta_1, \ldots, \beta_{\ell_K}$) the simple roots. In the tables below, we indicate the location of each α_i (resp. β_i) in the Dynkin diagram (resp. extended Dynkin diagram).

Before giving the tables, we mention that Djoković's method for classifying normal triples in $\mathfrak{g}_{\mathbb{C}}$ is essentially the same as Dynkin's method for classifying standard triples in $\mathfrak{g}_{\mathbb{C}}$, described in §8.1. It would be interesting to classify real nilpotent orbits using Bala and Carter's ideas.

Nilpotent Orbits in $G_{2(2)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$		$\Delta_K(\mathcal{O}_{\mathbb{R}})$	
	α_1	α_2	β_1	β_2
0	0	0	0	0
1	0	1	1	1
2	1	0	1	3
3	0	2	2	2
4	0	2	0	4
5	2	2	4	8

Nilpotent Orbits in $F_{4(4)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$				$\Delta_K(\mathcal{O}_{\mathbb{R}})$			
	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	1	1
2	0	0	0	1	1	0	0	2
3	0	0	0	1	0	1	0	0
4	0	1	0	0	0	0	1	3
5	0	1	0	0	1	0	1	1
6	2	0	0	0	0	0	0	4
7	2	0	0	0	2	0	0	0
8	2	0	0	0	0	0	2	2
9	0	0	0	2	0	2	0	0
10	0	0	1	0	1	1	0	2
11	2	0	0	1	1	0	2	4
12	2	0	0	1	0	1	2	2
13	0	1	0	1	1	1	1	1
14	1	0	1	0	1	0	3	1
15	1	0	1	0	1	1	1	3
16	0	2	0	0	0	0	4	0
17	0	2	0	0	0	2	0	4
18	0	2	0	0	2	0	2	2
19	2	2	0	0	0	0	4	8
20	2	2	0	0	2	0	4	4
21	1	0	1	2	1	3	1	3
22	0	2	0	2	0	4	0	4
23	0	2	0	2	2	2	2	2
24	2	2	0	2	2	2	4	4
25	2	2	0	2	4	0	4	8
26	2	2	2	2	4	4	4	8

Nilpotent Orbits in $F_{4(-20)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$				$\Delta_K(\mathcal{O}_{\mathbb{R}})$			
	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1
2	0	0	0	2	4	0	0	0

Nilpotent Orbits in $E_{6(-14)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$ α_2; α_1 α_3 α_4 α_5 α_6 (listed:$\alpha_1(H),\ldots,\alpha_6(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$ β_5; β_1 β_2 β_3 β_4 β_6 (listed:$\beta_1(H_K),\ldots\beta_5(H_K)$, $\beta_6(H_K)$)	
		(recall Hermitian symmetric case conventions)	
0	0 0 0 0 0 0	0 0 0 0 0	0
1	0 1 0 0 0 0	0 0 0 0 1	0
2	0 1 0 0 0 0	0 0 0 1 0	-2
3	1 0 0 0 0 1	1 0 0 0 0	1
4	1 0 0 0 0 1	1 0 0 0 0	-2
5	1 0 0 0 0 1	0 0 0 1 1	-2
6	0 2 0 0 0 0	0 2 0 0 0	-2
7	1 1 0 0 0 1	1 1 0 1 0	-2
8	1 1 0 0 0 1	1 1 0 0 1	-3
9	2 0 0 0 0 2	4 0 0 0 0	-2
10	1 2 0 0 0 1	0 0 0 1 3	-2
11	1 2 0 0 0 1	0 0 0 3 1	-6
12	2 2 0 0 0 2	0 2 0 2 2	-6

Nilpotent Orbits in $E_{6(-26)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$ α_2; α_1 α_3 α_4 α_5 α_6 (listed:$\alpha_1(H),\ldots,\alpha_6(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$ β_1 β_2 β_3 β_4			
0	0 0 0 0 0 0	0	0	0	0
1	1 0 0 0 0 1	0	0	0	1
2	2 0 0 0 0 2	0	0	0	2

Nilpotent Orbits in $E_{6(2)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$ α_2; α_1 α_3 α_4 α_5 α_6 (listed:$\alpha_1(H),\ldots,\alpha_6(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$ β_6; β_1 β_2 β_3 β_4 β_5 (listed:$\beta_1(H_K),\ldots\beta_5(H_K)$, $\beta_6(H_K)$)	
0	0 0 0 0 0 0	0 0 0 0 0	0
1	0 1 0 0 0 0	0 0 1 0 0	1
2	1 0 0 0 0 1	1 0 0 0 1	2
3	1 0 0 0 0 1	0 1 0 1 0	0
4	0 0 0 1 0 0	0 0 1 0 0	3
5	0 0 0 1 0 0	1 0 1 0 1	1
6	0 2 0 0 0 0	0 0 0 0 0	4
7	0 2 0 0 0 0	2 0 0 0 2	0
8	0 2 0 0 0 0	0 0 2 0 0	2
9	1 1 0 0 0 1	2 1 0 0 1	1
10	1 1 0 0 0 1	1 0 0 1 2	1
11	2 0 0 0 0 2	0 2 0 2 0	0
12	0 0 1 0 1 0	3 0 1 0 0	0
13	0 0 1 0 1 0	0 0 1 0 3	0
14	0 0 1 0 1 0	1 1 0 1 1	2
15	1 2 0 0 0 1	1 0 2 0 1	4
16	1 2 0 0 0 1	0 1 2 1 0	2
17	1 0 0 1 0 1	1 1 1 1 1	1
18	0 1 1 0 1 0	1 0 3 0 1	1
19	0 1 1 0 1 0	1 1 1 1 1	3
20	0 0 0 2 0 0	0 0 4 0 0	0
21	0 0 0 2 0 0	0 2 0 2 0	4
22	0 0 0 2 0 0	2 0 2 0 2	2
23	0 2 0 2 0 0	0 0 4 0 0	8
24	0 2 0 2 0 0	2 0 4 0 2	4
25	2 2 0 0 0 2	4 0 0 0 4	4
26	2 2 0 0 0 2	2 2 0 2 2	0
27	1 1 1 0 1 1	1 2 1 1 3	1
28	1 1 1 0 1 1	3 1 1 2 1	1
29	1 2 1 0 1 1	3 1 3 1 0	4
30	1 2 1 0 1 1	0 1 3 1 3	4
31	2 1 1 0 1 2	1 3 1 3 1	3
32	2 0 0 2 0 2	2 2 2 2 2	2
33	2 0 0 2 0 2	0 4 0 4 0	4
34	2 2 0 2 0 2	2 2 4 2 2	4
35	2 2 0 2 0 2	4 0 4 0 4	8
36	2 2 2 0 2 2	4 4 0 4 4	4
37	2 2 2 2 2 2	4 4 4 4 4	8

Nilpotent Orbits in $E_{6(6)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$ α_2; α_1 α_3 α_4 α_5 α_6 (listed:$\alpha_1(H),\ldots,\alpha_6(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$ β_1	β_2	β_3	β_4
0	0 0 0 0 0 0	0	0	0	0
1	0 1 0 0 0 0	0	0	0	1
2	1 0 0 0 0 1	0	1	0	0
3	0 0 0 1 0 0	1	0	0	1
4	0 2 0 0 0 0	0	0	0	2
5	0 2 0 0 0 0	2	0	0	0
6	2 0 0 0 0 2	0	2	0	0
7	1 2 0 0 0 1	0	1	0	2
8	1 1 0 0 0 1	0	1	0	1
9	2 2 0 0 0 2	0	2	0	2
10	0 0 1 0 1 0	1	0	1	0
11	1 0 0 1 0 1	1	1	0	1
12	0 0 0 2 0 0	2	0	0	2
13	0 2 0 2 0 0	2	0	0	4
14	2 1 1 0 1 2	1	2	1	1
15	0 1 1 0 1 0	1	0	1	1
16	1 1 1 0 1 1	1	1	1	1
17	1 2 1 0 1 1	1	1	1	2
18	2 2 2 0 2 2	2	2	2	2
19	2 0 0 2 0 2	2	2	0	2
20	2 2 2 2 2 2	4	2	2	4
21	2 2 0 2 0 2	2	2	0	4
22	2 0 0 2 0 2	0	2	2	0
23	0 0 0 2 0 0	0	0	2	0

Nilpotent Orbits in $E_{7(7)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$ α_2; α_1 α_3 α_4 α_5 α_6 α_7 (listed:$\alpha_1(H),\ldots,\alpha_7(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$ β_1	β_2	β_3	β_4	β_5	β_6	β_7
0	0 0 0 0 0 0 0	0	0	0	0	0	0	0
1	1 0 0 0 0 0 0	0	0	0	1	0	0	0
2	0 0 0 0 0 1 0	0	1	0	0	0	1	0
3	0 0 0 0 0 0 2	0	2	0	0	0	0	0
4	0 0 0 0 0 0 2	0	0	0	0	0	2	0
5	0 0 1 0 0 0 0	1	0	0	1	0	0	1
6	2 0 0 0 0 0 0	2	0	0	0	0	0	2
7	2 0 0 0 0 0 0	0	0	0	2	0	0	0
8	0 1 0 0 0 0 1	1	1	0	0	1	0	0
9	0 1 0 0 0 0 1	0	0	1	0	0	1	1
10	1 0 0 0 0 1 0	2	0	1	0	0	0	1
11	1 0 0 0 0 1 0	1	0	0	0	1	0	2
12	1 0 0 0 0 1 0	0	1	0	1	0	1	0
13	0 0 0 1 0 0 0	3	0	0	0	1	0	0
14	0 0 0 1 0 0 0	0	0	1	0	0	0	3
15	0 0 0 1 0 0 0	1	0	1	0	1	0	1
16	0 2 0 0 0 0 0	4	0	0	0	0	0	0
17	0 2 0 0 0 0 0	0	0	0	0	0	0	4
18	0 2 0 0 0 0 0	2	0	0	0	2	0	0
19	0 2 0 0 0 0 0	0	0	2	0	0	0	2
20	2 0 0 0 0 1 0	0	1	0	2	0	1	0
21	0 0 0 0 0 2 0	0	2	0	0	0	2	0
22	2 0 0 0 0 0 2	0	2	0	2	0	0	0
23	2 0 0 0 0 0 2	0	0	0	2	0	2	0
24	0 0 1 0 0 1 0	1	1	0	1	0	1	1
25	1 0 0 1 0 0 0	1	0	1	1	1	0	1
26	0 0 2 0 0 0 0	2	0	0	2	0	0	2
27	0 0 2 0 0 0 0	0	0	2	0	2	0	0
28	1 0 0 0 1 0 1	1	1	1	1	0	1	0
29	1 0 0 0 1 0 1	0	1	0	1	1	1	1
30	2 0 2 0 0 0 0	2	0	0	4	0	0	2

Nilpotent Orbits in $E_{7(7)}$ (continued)								
bel	$\Delta(\mathcal{O}_{\mathbb{C}})$	$\Delta_K(\mathcal{O}_{\mathbb{R}})$						
31	0 1 1 0 0 0 1	2	1	0	1	1	0	1
32	0 1 1 0 0 0 1	1	0	1	1	0	1	2
33	0 1 1 0 0 0 1	0	1	2	0	1	0	1
34	0 1 1 0 0 0 1	1	0	1	0	2	1	0
35	0 0 0 1 0 1 0	1	0	3	0	0	1	0
36	0 0 0 1 0 1 0	0	1	0	0	3	0	1
37	0 0 0 1 0 1 0	1	1	1	0	1	1	1
38	2 0 0 0 0 2 0	2	2	0	0	0	2	2
39	0 0 0 0 2 0 0	0	0	4	0	0	0	0
40	0 0 0 0 2 0 0	0	0	0	0	4	0	0
41	0 0 0 0 2 0 0	2	0	2	0	0	2	0
42	0 0 0 0 2 0 0	0	2	0	0	2	0	2
43	2 0 0 0 0 2 0	0	2	0	2	0	2	0
44	2 0 0 0 0 2 2	0	4	0	2	0	2	0
45	2 0 0 0 0 2 2	0	2	0	2	0	4	0
46	2 1 1 0 0 0 1	2	1	0	3	1	0	1
47	2 1 1 0 0 0 1	1	0	1	3	0	1	2
48	1 0 0 1 0 1 0	3	1	0	1	0	2	1
49	1 0 0 1 0 1 0	1	2	0	1	0	1	3
50	1 0 0 1 0 1 0	1	1	1	1	1	1	1
51	2 0 0 1 0 1 0	3	0	1	3	0	1	0
52	2 0 0 1 0 1 0	0	1	0	3	1	0	3
53	2 0 0 1 0 1 0	1	1	1	2	1	1	1
54	0 0 0 2 0 0 0	2	0	2	0	2	0	2
55	2 0 0 0 2 0 0	4	0	0	4	0	0	0
56	2 0 0 0 2 0 0	0	0	0	4	0	0	4
57	2 0 0 0 2 0 0	2	0	2	2	0	2	0
58	2 0 0 0 2 0 0	0	2	0	2	2	0	2
59	1 0 0 1 0 2 0	1	2	1	1	1	2	1
60	1 0 0 1 0 1 2	1	3	1	1	1	1	1
61	1 0 0 1 0 1 2	1	1	1	1	1	3	1
62	0 0 2 0 0 2 0	2	2	0	2	0	2	2
63	0 0 2 0 0 2 0	0	2	2	0	2	2	0
64	0 1 1 0 1 0 2	1	3	1	0	3	0	1
65	0 1 1 0 1 0 2	1	0	3	0	1	3	1
66	2 0 2 0 0 2 0	2	2	0	4	0	2	2
67	0 0 0 2 0 0 2	2	2	2	0	2	0	2
68	0 0 0 2 0 0 2	2	0	2	0	2	2	2
69	0 0 0 2 0 0 2	0	4	0	0	4	0	0
70	0 0 0 2 0 0 2	0	0	4	0	0	4	0
71	0 0 0 2 0 2 0	2	2	2	0	2	2	2
72	2 1 1 0 1 0 2	3	0	1	3	1	3	1
73	2 1 1 0 1 0 2	1	3	1	3	1	0	3
74	2 1 1 0 1 1 0	3	1	1	3	1	2	1

Nilpotent Orbits in $E_{7(7)}$ (continued)								
label	$\Delta(\mathcal{O}_{\mathbb{C}})$	$\Delta_K(\mathcal{O}_{\mathbb{R}})$						
75	2 1 1 0 1 1 0	1	2	1	3	1	1	3
76	2 0 0 2 0 0 2	2	2	2	2	2	0	2
77	2 0 0 2 0 0 2	2	0	2	2	2	2	2
78	2 0 0 2 0 0 2	4	0	0	4	0	4	0
79	2 0 0 2 0 0 2	0	4	0	4	0	0	4
80	2 0 0 2 0 2 0	4	2	2	0	2	2	4
81	2 0 0 2 0 2 0	2	2	2	2	2	2	2
82	2 1 1 0 1 2 2	3	4	1	3	1	3	1
83	2 1 1 0 1 2 2	1	3	1	3	1	4	3
84	2 0 2 2 0 2 0	4	2	2	4	2	2	4
85	2 0 0 2 0 2 2	2	4	2	2	2	2	2
86	2 0 0 2 0 2 2	2	2	2	2	2	4	2
87	2 0 0 2 0 2 2	4	4	0	4	0	4	0
88	2 0 0 2 0 2 2	0	4	0	4	0	4	4
89	2 2 2 0 2 0 2	4	4	0	4	4	0	4
90	2 2 2 0 2 0 2	4	0	4	4	0	4	4
91	2 2 2 0 2 2 2	4	4	4	4	0	4	4
92	2 2 2 0 2 2 2	4	4	0	4	4	4	4
93	2 2 2 2 2 2 2	8	4	4	4	4	4	4
94	2 2 2 2 2 2 2	4	4	4	4	4	4	8

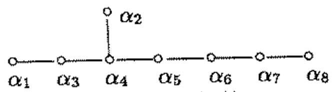
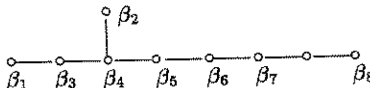
Nilpotent Orbits in $E_{7(-5)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$ α_2 $\alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7$ (listed:$\alpha_1(H),\ldots,\alpha_7(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$ β_6 $\beta_1\ \beta_2\ \beta_3\ \beta_4\ \beta_5\quad \beta_7$ (listed:$\beta_1(H_K),\ldots\beta_6(H_K)$, $\beta_7(H_K)$)
0	0 0 0 0 0 0 0	0 0 0 0 0 0 0
1	1 0 0 0 0 0 0	0 0 0 0 1 0 1
2	0 0 0 0 0 1 0	0 1 0 0 0 0 2
3	0 0 0 0 0 1 0	0 0 0 1 0 0 0
4	0 0 1 0 0 0 0	0 0 0 0 1 0 3
5	0 0 1 0 0 0 0	0 1 0 0 1 0 1
6	2 0 0 0 0 0 0	0 0 0 0 0 0 4
7	2 0 0 0 0 0 0	0 0 0 0 2 0 2
8	2 0 0 0 0 0 0	0 2 0 0 0 0 0
9	1 0 0 0 0 1 0	1 1 0 0 0 1 1
10	0 0 0 1 0 0 0	2 0 0 1 0 0 0
11	0 0 0 1 0 0 0	0 1 0 1 0 0 2
12	2 0 0 0 0 1 0	0 1 0 0 2 0 4
13	2 0 0 0 0 1 0	0 0 0 1 2 0 2
14	0 0 0 0 0 2 0	4 0 0 0 0 0 0
15	0 0 0 0 0 2 0	0 0 0 2 0 0 0
16	0 0 1 0 0 1 0	0 1 0 1 1 0 1
17	1 0 0 1 0 0 0	0 1 0 0 3 0 1
18	1 0 0 1 0 0 0	0 1 0 1 1 0 3
19	0 0 2 0 0 0 0	0 0 0 0 4 0 0
20	0 0 2 0 0 0 0	0 0 0 2 0 0 4
21	0 0 2 0 0 0 0	0 2 0 0 2 0 2
22	2 0 2 0 0 0 0	0 0 0 0 4 0 8
23	2 0 2 0 0 0 0	0 2 0 0 4 0 4
24	0 0 0 1 0 1 0	2 0 1 0 1 1 2
25	2 0 0 0 0 2 0	0 4 0 0 0 0 4
26	2 0 0 0 0 2 0	0 2 0 2 0 0 0
27	1 0 0 1 0 1 0	1 1 1 1 1 0 1
28	2 0 0 1 0 1 0	2 0 1 0 3 1 4
29	0 0 0 2 0 0 0	0 0 4 0 0 0 0
30	1 0 0 1 0 2 0	0 1 0 3 1 0 3
31	0 0 2 0 0 2 0	0 2 0 2 2 0 0
32	0 0 2 0 0 2 0	0 0 0 4 0 0 4
33	2 0 2 0 0 2 0	0 2 0 2 4 0 4
34	2 0 2 0 0 2 0	0 4 0 0 4 0 8
35	0 0 0 2 0 2 0	4 0 0 4 0 0 0
36	2 0 0 2 0 2 0	0 4 0 4 0 0 4
37	2 0 2 2 0 2 0	0 4 0 4 4 0 8

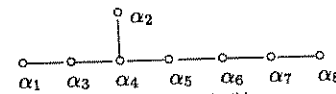
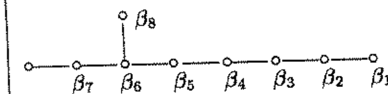
Nilpotent Orbits in $E_{7(-25)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$ α_2 $\alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7$ (listed:$\alpha_1(H),\ldots,\alpha_7(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$ β_2 $\beta_1\ \beta_3\ \beta_4\ \beta_5\ \beta_6\ \beta_7$ (listed:$\beta_1(H_K),\ldots\beta_6(H_K)$, $\beta_7(H_K)$)
		(Recall Hermitian symmetric case conventions)
0	0 0 0 0 0 0 0	0 0 0 0 0 0 0
1	1 0 0 0 0 0 0	1 0 0 0 0 0 0
2	1 0 0 0 0 0 0	0 0 0 0 0 1 -2
3	0 0 0 0 0 1 0	0 0 0 0 0 1 0
4	0 0 0 0 0 1 0	1 0 0 0 0 0 -2
5	0 0 0 0 0 1 0	1 0 0 0 0 1 -2
6	0 0 0 0 0 0 2	0 0 0 0 0 0 2
7	0 0 0 0 0 0 2	0 0 0 0 0 0 -2
8	0 0 0 0 0 0 2	0 0 0 0 0 2 -2
9	0 0 0 0 0 0 2	2 0 0 0 0 0 -2
10	2 0 0 0 0 0 0	0 2 0 0 0 0 -2
11	1 0 0 0 0 1 0	0 1 0 0 1 0 -2
12	1 0 0 0 0 1 0	0 1 1 0 0 0 -3
13	2 0 0 0 0 1 0	3 0 0 0 0 1 -2
14	2 0 0 0 0 1 0	1 0 0 0 0 3 -6
15	0 0 0 0 0 2 0	2 0 0 0 0 2 -4
16	2 0 0 0 0 0 2	2 0 0 0 0 2 -2
17	2 0 0 0 0 0 2	4 0 0 0 0 0 -2
18	2 0 0 0 0 0 2	0 0 0 0 0 4 -6
19	2 0 0 0 0 0 2	2 0 0 0 0 2 -6
20	2 0 0 0 0 2 0	2 2 0 0 0 2 -6
21	2 0 0 0 0 2 2	4 0 0 0 0 4 -6
22	2 0 0 0 0 2 2	4 0 0 0 0 4 -10

Nilpotent Orbits in $E_{8(-24)}$

ıbel	$\Delta(\mathcal{O}_{\mathbb{C}})$  (listed:$\alpha_1(H),\ldots,\alpha_8(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$  (listed:$\beta_1(H_K),\ldots\beta_7(H_K)$,	$\beta_8(H_K)$)
0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0	0
1	0 0 0 0 0 0 0 1	0 0 0 0 0 0 1	1
2	1 0 0 0 0 0 0 0	1 0 0 0 0 0 0	2
3	1 0 0 0 0 0 0 0	0 0 0 0 0 1 0	0
4	0 0 0 0 0 0 1 0	0 0 0 0 0 0 1	3
5	0 0 0 0 0 0 1 0	1 0 0 0 0 0 1	1
6	0 0 0 0 0 0 0 2	0 0 0 0 0 0 0	4
7	0 0 0 0 0 0 0 2	0 0 0 0 0 0 2	2
8	0 0 0 0 0 0 0 2	2 0 0 0 0 0 0	0
9	1 0 0 0 0 0 0 1	1 1 0 0 0 0 0	1
10	0 0 0 0 0 1 0 0	1 0 0 0 0 1 0	2
11	0 0 0 0 0 1 0 0	0 0 0 1 0 0 0	0
12	1 0 0 0 0 0 0 2	1 0 0 0 0 0 2	4
13	1 0 0 0 0 0 0 2	0 0 0 0 0 1 2	2
14	2 0 0 0 0 0 0 0	0 0 0 0 0 2 0	0
15	1 0 0 0 0 0 1 0	1 0 0 0 0 1 1	1
16	0 0 0 0 0 1 0 1	1 0 0 0 0 1 1	3
17	0 0 0 0 0 1 0 1	1 0 0 0 0 0 3	1
18	0 0 0 0 0 0 2 0	0 0 0 0 0 2 0	4
19	0 0 0 0 0 0 2 0	0 0 0 0 0 0 4	0
20	0 0 0 0 0 0 2 0	2 0 0 0 0 0 2	2
21	0 0 0 0 0 0 2 2	0 0 0 0 0 0 4	8
22	0 0 0 0 0 0 2 2	2 0 0 0 0 0 4	4
23	1 0 0 0 0 1 0 0	0 1 1 0 0 0 1	2
24	2 0 0 0 0 0 0 2	4 0 0 0 0 0 0	4
25	2 0 0 0 0 0 0 2	2 0 0 0 0 2 0	0
26	1 0 0 0 0 1 0 1	1 0 1 0 0 1 1	1
27	1 0 0 0 0 1 0 2	0 1 1 0 0 0 3	4
28	0 0 0 0 0 2 0 0	0 0 0 2 0 0 0	0
29	2 0 0 0 0 1 0 1	1 0 0 0 0 3 1	3
30	2 0 0 0 0 0 2 0	2 0 0 0 0 2 2	2
31	2 0 0 0 0 0 2 0	0 0 0 0 0 4 0	4
32	2 0 0 0 0 0 2 2	2 0 0 0 0 2 4	4
33	2 0 0 0 0 0 2 2	4 0 0 0 0 0 4	8
34	2 0 0 0 0 2 0 0	0 0 0 2 0 2 0	0
35	2 0 0 0 0 2 0 2	4 0 0 0 0 4 0	4
36	2 0 0 0 0 2 2 2	4 0 0 0 0 4 4	8

Nilpotent Orbits in $E_{8(8)}$

label	$\Delta(\mathcal{O}_{\mathbb{C}})$  (listed:$\alpha_1(H),\ldots,\alpha_8(H)$)	$\Delta_K(\mathcal{O}_{\mathbb{R}})$  (listed:$\beta_1(H_K),\ldots\beta_8(H_K)$)
0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 1	0 0 0 0 0 0 1 0
2	1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0
3	0 0 0 0 0 0 1 0	0 1 0 0 0 0 1 0
4	0 0 0 0 0 0 0 2	0 2 0 0 0 0 0 0
5	0 0 0 0 0 0 0 2	0 0 0 0 0 0 2 0
6	0 1 0 0 0 0 0 0	1 0 0 0 1 0 0 0
7	1 0 0 0 0 0 0 1	1 1 0 0 0 0 0 1
8	1 0 0 0 0 0 0 1	0 0 0 1 0 0 1 0
9	0 0 0 0 0 1 0 0	2 0 0 1 0 0 0 0
10	0 0 0 0 0 1 0 0	0 1 0 0 0 1 0 0
11	1 0 0 0 0 0 0 2	0 0 0 1 0 0 2 0
12	0 0 1 0 0 0 0 0	3 0 0 0 0 0 0 1
13	0 0 1 0 0 0 0 0	1 0 0 1 0 0 0 1
14	2 0 0 0 0 0 0 0	4 0 0 0 0 0 0 0
15	2 0 0 0 0 0 0 0	2 0 0 0 0 0 0 2
16	2 0 0 0 0 0 0 0	0 0 0 2 0 0 0 0
17	1 0 0 0 0 0 1 0	0 1 0 1 0 0 1 0
18	0 0 0 0 0 1 0 1	0 1 0 0 0 1 1 0
19	0 0 0 0 0 0 2 0	0 2 0 0 0 0 2 0
20	0 0 0 0 0 0 2 0	0 0 0 0 0 2 0 0
21	0 0 0 0 0 0 2 2	0 2 0 0 0 0 4 0
22	0 0 0 0 1 0 0 0	1 0 1 0 0 1 0 0
23	0 0 1 0 0 0 0 1	1 0 0 1 0 0 1 1
24	0 1 0 0 0 0 1 0	1 1 0 0 1 0 1 0
25	0 1 0 0 0 0 1 0	0 0 1 0 0 1 0 1
26	1 0 0 0 0 1 0 0	2 0 1 0 0 0 1 1
27	1 0 0 0 0 1 0 0	1 0 0 0 1 0 0 2
28	1 0 0 0 0 1 0 0	0 1 0 1 0 1 0 0
29	2 0 0 0 0 0 0 2	0 2 0 2 0 0 0 0
30	2 0 0 0 0 0 0 2	0 0 0 2 0 0 2 0
31	0 0 0 1 0 0 0 0	0 0 1 0 0 0 0 3
32	0 0 0 1 0 0 0 0	1 0 1 0 1 0 0 1
33	0 1 0 0 0 0 1 2	1 1 0 0 1 0 3 0
34	0 2 0 0 0 0 0 0	0 0 0 0 0 0 0 4
35	0 2 0 0 0 0 0 0	2 0 0 0 2 0 0 0

Nilpotent Orbits in $E_{8(8)}$ (continued)

label	$\Delta(\mathcal{O}_{\mathbb{C}})$	$\Delta_K(\mathcal{O}_{\mathbb{R}})$
36	0 2 0 0 0 0 0 0	0 0 2 0 0 0 0 2
37	1 0 0 0 0 1 0 1	1 1 1 1 0 0 1 0
38	1 0 0 0 0 1 0 1	0 1 0 1 0 1 1 0
39	1 0 0 0 1 0 0 0	1 0 1 1 0 1 0 0
40	1 0 0 0 0 1 0 2	2 0 1 0 0 0 3 1
41	1 0 0 0 0 1 0 2	0 1 0 1 0 1 2 0
42	0 0 0 1 0 0 0 1	2 1 0 1 0 1 0 0
43	0 0 0 1 0 0 0 1	0 1 2 0 0 1 0 0
44	0 0 0 1 0 0 0 1	1 0 1 0 1 0 1 1
45	0 0 0 0 0 2 0 0	0 0 4 0 0 0 0 0
46	0 0 0 0 0 2 0 0	0 2 0 0 0 2 0 0
47	2 0 0 0 0 1 0 1	0 1 0 2 0 1 1 0
48	0 0 0 1 0 0 0 2	3 0 0 0 1 0 3 0
49	0 0 0 1 0 0 0 2	1 0 1 0 1 0 2 1
50	0 0 1 0 0 1 0 0	1 1 0 1 0 1 0 1
51	0 2 0 0 0 0 0 2	4 0 0 0 0 0 4 0
52	0 2 0 0 0 0 0 2	0 0 2 0 0 0 2 2
53	0 2 0 0 0 0 0 2	2 0 0 0 2 0 2 0
54	2 0 0 0 0 0 2 0	0 2 0 2 0 0 2 0
55	2 0 0 0 0 0 2 0	0 0 0 2 0 2 0 0
56	2 0 0 0 0 0 2 2	0 2 0 2 0 0 4 0
57	0 0 0 1 0 0 1 0	1 1 1 0 1 0 1 1
58	1 0 0 1 0 0 0 1	1 0 1 1 1 0 1 1
59	0 0 1 0 0 1 0 1	1 1 0 1 0 1 1 1
60	0 1 1 0 0 0 1 0	2 1 0 1 1 0 1 1
61	0 1 1 0 0 0 1 0	1 0 1 0 2 1 0 0
62	1 0 0 0 1 0 1 0	1 1 1 1 0 1 1 0
63	1 0 0 0 1 0 1 0	0 1 0 1 1 1 0 1
64	0 0 0 1 0 1 0 0	0 1 0 0 3 0 0 1
65	0 0 0 1 0 1 0 0	1 1 1 0 1 1 0 1
66	1 0 0 0 1 0 1 2	1 1 1 1 0 1 3 0
67	0 0 0 0 2 0 0 0	2 0 2 0 0 2 0 0
68	0 0 0 0 2 0 0 0	0 0 0 0 4 0 0 0
69	0 0 0 0 2 0 0 0	0 2 0 0 2 0 0 2
70	2 0 0 0 0 2 0 0	4 0 0 4 0 0 0 0
71	2 0 0 0 0 2 0 0	0 2 0 2 0 2 0 0
72	0 1 1 0 0 0 1 2	2 1 0 1 1 0 3 1
73	0 1 1 0 0 0 1 2	0 1 2 0 1 0 3 1
74	1 0 0 1 0 1 0 0	1 1 1 1 1 1 0 1
75	0 0 0 1 0 1 0 2	1 1 1 0 1 1 2 1
76	0 0 0 1 0 1 0 2	1 0 3 0 0 1 3 0
77	2 0 0 0 0 2 0 2	0 4 0 2 0 2 0 0
78	2 0 0 0 0 2 0 2	0 2 0 2 0 2 2 0
79	0 0 0 0 2 0 0 2	0 2 0 0 2 0 2 2
80	0 0 0 0 2 0 0 2	0 0 4 0 0 0 4 0

Nilpotent Orbits in $E_{8(8)}$ (continued)

label	$\Delta(\mathcal{O}_{\mathbb{C}})$	$\Delta_K(\mathcal{O}_{\mathbb{R}})$
81	0 0 0 0 2 0 0 2	2 0 2 0 0 2 2 0
82	2 0 0 0 0 2 2 2	0 4 0 2 0 2 4 0
83	2 1 1 0 0 0 1 2	2 1 0 3 1 0 3 1
84	1 0 0 1 0 1 0 1	3 1 0 1 0 2 1 1
85	1 0 0 1 0 1 0 1	1 1 1 1 1 1 1 1
86	1 0 0 1 0 1 1 0	1 2 1 1 1 1 1 1
87	1 0 0 1 0 1 0 2	1 3 1 1 1 1 0 1
88	1 0 0 1 0 1 0 2	1 1 1 1 1 1 2 1
89	2 0 0 1 0 1 0 2	1 1 1 2 1 1 2 1
90	2 0 0 1 0 1 0 2	3 0 1 3 0 1 3 0
91	0 0 0 2 0 0 0 2	2 0 2 0 2 0 2 2
92	0 0 0 2 0 0 0 2	0 4 0 0 4 0 0 0
93	2 0 0 0 2 0 0 2	0 2 0 2 2 0 2 2
94	2 0 0 0 2 0 0 2	4 0 0 4 0 0 4 0
95	2 0 0 0 2 0 0 2	2 0 2 2 0 2 2 0
96	1 0 0 1 0 1 2 2	1 3 1 1 1 1 4 1
97	0 1 1 0 1 0 2 2	1 3 1 0 3 0 4 1
98	0 0 0 2 0 0 2 0	0 0 4 0 0 4 0 0
99	0 0 0 2 0 0 2 0	2 2 2 0 2 0 2 2
100	2 1 1 0 1 1 0 1	3 1 1 3 1 2 1 1
101	0 0 0 2 0 0 2 2	2 2 2 0 2 0 4 2
102	0 0 0 2 0 0 2 2	0 4 0 0 4 0 4 0
103	2 1 1 0 1 0 2 2	1 3 1 3 1 0 4 3
104	2 0 0 2 0 0 2 0	2 2 2 2 2 0 2 2
105	2 0 0 2 0 0 2 0	4 0 0 4 0 4 0 0
106	2 0 0 2 0 0 2 2	2 2 2 2 2 0 4 2
107	2 0 0 2 0 0 2 2	0 4 0 4 0 0 4 4
108	2 1 1 0 1 2 2 2	3 4 1 3 1 3 4 1
109	2 0 0 2 0 2 0 2	2 2 2 2 2 2 2 2
110	2 0 0 2 0 2 0 2	4 4 0 4 0 4 0 0
111	2 0 0 2 0 2 2 2	2 4 2 2 2 2 4 2
112	2 0 0 2 0 2 2 2	4 4 0 4 0 4 4 0
113	2 2 2 0 2 0 2 2	4 4 0 4 4 0 4 4
114	2 2 2 0 2 2 2 2	4 4 4 4 0 4 4 4
115	2 2 2 2 2 2 2 2	8 4 4 4 4 4 4 4

10 Advanced Topics

So far we have concentrated exclusively on the structure theory of semisimple Lie algebras $\mathfrak{g}$. In this last chapter, we give a survey (without proofs) of some of the many connections between nilpotent orbits and representation theory. For simplicity we will assume that $\mathfrak{g}$ is complex. We begin by describing a fundamental connection discovered by Springer between nilpotent orbits and representations of a Weyl group. This leads naturally into our next topic, which is the classification of primitive ideals in $U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$. (This classification is most naturally stated in terms of Weyl group representations. To do this, we need to introduce the notion of the associated variety of a primitive ideal; associated varieties turn out to be closures of nilpotent orbits.) In the last section we will refine the relationship between nilpotent orbits and primitive ideals slightly. We also discuss some of the geometry of nilpotent orbits and its ideal-theoretic consequences. Here we will allow ourselves to use some definitions and theorems from commutative algebra; a basic reference for these is [63].

10.1 The Springer Correspondence

Since $\mathfrak{g}$ has only finitely many nilpotent orbits but infinitely many inequivalent irreducible representations, it is not at all obvious that nilpotent orbits have anything to do with representations of $\mathfrak{g}$. What one needs is a finite group closely related to $\mathfrak{g}$ whose representation theory can act as an intermediary between that of $\mathfrak{g}$ and the theory of nilpotent orbits. It should perhaps come as no surprise that the Weyl group W of $\mathfrak{g}$ fills the bill. In this section, we describe Springer's method for producing a representation π of W from a nilpotent orbit $\mathcal{O}$; in §10.3 we show how to pass from π to a representation of $\mathfrak{g}$ (or rather to several such representations).

As usual, the situation is clearest when $\mathfrak{g} = \mathfrak{sl}_n$. Then we have $W \cong S_n$, the symmetric group on n letters. If you are familiar with the standard parametrization of the set $\widehat{W}$ of irreducible representations of W, you will recognize an obvious parallel between it and (3.1.7). The main facts are summarized next. We remind you that the sign representation sgn of any Weyl group W' is the one on which any $w' \in W'$ acts by a scalar equal to the determinant of its action on the dual of a Cartan subalgebra. If $\mathcal{Y}_{\mathbf{d}}$ is the Young diagram corresponding to a partition $\mathbf{d}$, then a labeling of its boxes with the integers from 1 to n so that labels increase across rows and down columns is called a *standard Young tableau* (of shape $\mathbf{d}$).

orem 10.1.1 (Young). *Elements of $\widehat{S}_n$ are parametrized by partitions of n. If $\mathbf{d} = [d_1, d_2, \dots, d_n] \in \mathcal{P}(n)$ and $\mathbf{f} = \mathbf{d^t} = [f_1, f_2, \dots, f_n]$, then the representation $\pi_{\mathbf{d}}$ corresponding to $\mathbf{d}$ is uniquely characterized by two properties: its restriction to the subgroup $\prod_{i=1}^{n} S_{d_i}$ of S_n contains a copy of the trivial representation, while its restriction to $\prod_{j=1}^{n} S_{f_j}$ contains a copy of the sign representation sgn. One has $\pi_{\mathbf{f}} \cong \pi_{\mathbf{d}} \otimes sgn$. The dimension of $\pi_{\mathbf{d}}$ equals the number of standard Young tableaux of shape $\mathbf{d}$.*

For a proof and an elegant "hook formula" for the number of standard Young tableaux of a given shape, see [40, §§11,20]. Combining (3.1.7) and (10.1.1), we get a bijective correspondence between the set of nilpotent orbits and $\widehat{S}_n$; this correspondence is the Springer correspondence described below. The principal orbit corresponds to the trivial representation while the zero orbit corresponds to the sign representation. (It turns out that these last two facts hold for any semisimple algebra $\mathfrak{g}$.)

Before giving the general definition of the Springer correspondence, we describe how it works for the other classical algebras $\mathfrak{g}$, as this description is generally more useful than the precise definition. If $\mathfrak{g}$ is of type B_n or C_n, then W is the semidirect product of S_n and $(\mathbb{Z}/2\mathbb{Z})^n$; it acts on $\mathbb{C}^n$ by permuting and changing the signs of the coordinates (as noted in §5.2). Since $(\mathbb{Z}/2\mathbb{Z})^n$ is abelian, Wigner and Mackey's method of "little groups" (described in [75, §8.2]) yields the following result (cf. [64]). Given a partition $\mathbf{d}$, we use the notation $|d|$ to denote the sum of its parts.

orem 10.1.2. *If $\mathfrak{g}$ is of type B_n or C_n, then elements of $\widehat{W}$ are parametrized by ordered pairs $(\mathbf{d}, \mathbf{f})$ of partitions such that $|d| + |f| = n$. The representation $\pi_{(\mathbf{d},\mathbf{f})}$ is characterized by the following property. Let σ be the subspace of $\pi_{(\mathbf{d},\mathbf{f})}$ consisting of all vectors on which the first $|d|$ copies of $\mathbb{Z}/2$ act trivially while the remaining $|f|$ copies act by -1. Then $S_{|d|} \times S_{|f|}$ acts on σ according to the representation $\pi_{\mathbf{d}} \times \pi_{\mathbf{f}}$ defined in (10.1.1). We have $\pi_{(\mathbf{f^t},\mathbf{d^t})} \cong \pi_{(\mathbf{d},\mathbf{f})} \otimes sgn$ and $\dim \pi_{(\mathbf{d},\mathbf{f})} = \binom{n}{|d|}(\dim \pi_{\mathbf{d}})(\dim \pi_{\mathbf{f}})$, where $\binom{n}{|d|}$ as usual denotes the binomial coefficient $\frac{n!}{|d|!(n-|d|)!}$.*

This time the correspondence between nilpotent orbits and representations of W is far from obvious. To define it, we need to use Lusztig's notion of the symbol of a representation [57], which we now introduce. Suppose first that $\mathfrak{g}$ is of type B_n and let $\mathbf{d} = [d_1, d_2, \dots, d_k] \in \mathcal{P}_1(2n+1)$. Choose the notation so that all parts of $\mathbf{d}$ are nonzero; then k is odd. Define a new *increasing* sequence $e_1 < e_2 < \dots < e_k$ of integers via $e_i = d_{k+1-i} + i - 1$. Enumerate the odd e_i as $2f_1 + 1 < 2f_2 + 1 < \dots < 2f_a + 1$ and the even e_i as $2g_1 < 2g_2 < \dots < 2g_b$.

Then it turns out that $a = (k+1)/2 = b + 1$. Form the alternating sequence $(f_1, g_1, f_2, g_2, \dots, g_b, f_a)$ and write it down so that all the f_i are on one line while the g_j are on a lower line. The resulting arrangement is called the *symbol* of $\mathbf{d}$. For example, if $\mathbf{d} = [3^2, 1^3]$, then $(e_1, e_2, \dots) = (1, 2, 3, 6, 7), (f_1, f_2, f_3) = (0, 1, 3)$, and $(g_1, g_2) = (1, 3)$. The symbol of $\mathbf{d}$ is usually written

$$\begin{pmatrix} 0 & & 1 & & 3 \\ & 1 & & 3 & \end{pmatrix}$$

If $\mathbf{d} = [2^4, 1]$, then the symbol of $\mathbf{d}$ is

$$\begin{pmatrix} 0 & & 1 & & 2 \\ & 2 & & 3 & \end{pmatrix}$$

Given a symbol, we can produce a pair of partitions as in (10.1.2) as follows: subtract $(i-1)$ from the i^{th} element of the top row (counting from the left) and deal similarly with the elements in the bottom row. Then the new top and bottom rows (when rearranged in nonincreasing order) correspond to a pair $(\mathbf{p}, \mathbf{q})$ of partitions satisfying the conditions of (10.1.2). For example, if $\mathbf{d} = [3^2, 1^3]$, then $(\mathbf{p}, \mathbf{q}) = ([1], [2, 1])$; if $\mathbf{d} = [2^4, 1]$, then $(\mathbf{p}, \mathbf{q}) = (\emptyset, [2^2])$.

Combining the above recipes with (5.1.2), we get a map from nilpotent orbits to representations of W, as desired. Note, however, that it is now only an injection and not a bijection; we compute directly that the representation corresponding to $([1^2], \emptyset)$ does not (yet) correspond to any nilpotent orbit in type B_2. We will see below how to refine and extend the Springer correspondence into a bijection between $\widehat{W}$ and a certain set of ordered pairs $(\mathcal{O}, \mu)$, where $\mathcal{O}$ is a nilpotent orbit and μ is an irreducible representation of the group $A(\mathcal{O})$ defined in Chapter 6. (Then a representation π of W corresponding to an orbit $\mathcal{O}$ under the above map corresponds to $(\mathcal{O}, \mathbf{1})$ under its extension, where $\mathbf{1}$ denotes the trivial representation. Recall from Chapter 6 that all groups $A(\mathcal{O})$ are trivial in type A, so it is not necessary to extend the Springer correspondence in that case.)

Next let $\mathfrak{g}$ be of type C_n. Of course, the Weyl group W is the same as in type B_n. Given a partition $\mathbf{d} = [d_1, d_2, \dots, d_k] \in \mathcal{P}_{-1}(2n)$, we compute its symbol by slightly modifying the recipe for type B_n, as follows. If k is even, replace it by $k+1$ and add a 0 as the last part of $\mathbf{d}$; otherwise, leave $\mathbf{d}$ and k unchanged. Define a sequence $e_1 < e_2 < \dots < e_k$ exactly as above. Define sequences $f_1 < f_2 < \dots < f_a, g_1 < g_2 < \dots < g_b$ as above, except that the roles of even and odd e_i are reversed. Just as in type B_n, it turns out that $a = b + 1$. Construct the symbol of $\mathbf{d}$ from the sequences $(f_1, f_2, \dots)$ and $(g_1, g_2, \dots)$ and the pair of partitions from the symbol as in type B_n. Once again, the resulting map from nilpotent orbits to representations is injective but not bijective. (Its range is not the same as in type B_n.) It too will be extended to a bijection between $\widehat{W}$ and a set of ordered pairs $(\mathcal{O}, \mu)$ below. For example, in type C_3, if $\mathbf{d} = [2^2, 1^2]$, then we ultimately obtain the pair of partitions $(\mathbf{p}, \mathbf{q}) = ([1], [1^2])$.

Finally, let $\mathfrak{g}$ be of type D_n. This time W is a semidirect product of S_n and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$; it acts on $\mathbb{C}^n$ by permuting and changing an even number of the signs of the coordinates. Embed it in the Weyl group W' of type B_n via this action. The method of little groups yields the next result; see [65].

eorem 10.1.3. *If $\mathfrak{g}$ is of type D_n, then elements of $\widehat{W}$ are parametrized by unordered pairs $\{\mathbf{d},\mathbf{f}\}$ of partitions such that $|d|+|f|=n$, except that if $n=2m$ is even and $\mathbf{d}=\mathbf{f}$, then the unordered pair $\{\mathbf{d},\mathbf{d}\}$ corresponds to two representations, denoted $\pi^I_{\mathbf{d},\mathbf{d}}$ and $\pi^{II}_{\mathbf{d},\mathbf{d}}$. The representation $\pi_{\mathbf{d},\mathbf{f}}$ corresponding to the unordered pair $\{\mathbf{d},\mathbf{f}\}$ is just the restriction to W of the representation $\pi_{(\mathbf{d},\mathbf{f})}$ of W' corresponding to the ordered pair $(\mathbf{d},\mathbf{f})$, provided that $\mathbf{d}\neq\mathbf{f}$. If $\mathbf{d}=\mathbf{f}$, then $\pi_{(\mathbf{d},\mathbf{d})}$ decomposes over W as the direct sum of $\pi^I_{\mathbf{d},\mathbf{d}}$ and $\pi^{II}_{\mathbf{d},\mathbf{d}}$. Thus, $\pi_{\mathbf{d}^t,\mathbf{f}^t}=\pi_{\mathbf{f}^t,\mathbf{d}^t}\cong\pi_{(\mathbf{d},\mathbf{f})}\otimes sgn$ and $\dim\pi_{\mathbf{d},\mathbf{f}}=\dim\pi_{(\mathbf{d},\mathbf{f})}$ if $\mathbf{d}\neq\mathbf{f}$. If $\mathbf{d}=\mathbf{f}$, then $\dim\pi^I_{\mathbf{d},\mathbf{d}}=\dim\pi^{II}_{\mathbf{d},\mathbf{d}}=\frac{1}{2}\dim\pi_{(\mathbf{d},\mathbf{d})}$. If a representation $\pi^I_{\mathbf{d},\mathbf{d}}$ or $\pi^{II}_{\mathbf{d},\mathbf{d}}$ is tensored with sgn, then the partition $\mathbf{d}$ is replaced by its transpose. The labels I or II stay the same if $m=n/2$ is even, but interchange if m is odd.*

For the definitions of the labels I and II, see [65]; the notations 1 and 2 are used there. Now let $\mathbf{d}=[d_1,d_2,\ldots,d_k]\in\mathcal{P}_1(2n)$. As above, the key tool for passing from $\mathbf{d}$ to a representation of W is the symbol of $\mathbf{d}$, which we now define. As in type B, choose the largest index k so that $d_k>0$; then k is even. Define the sequence $e_1<e_2<\ldots<e_k$ as in types B and C. Enumerate and label the odd and even e_i exactly as in type B. This time it turns out that $a=b$. Write down the f_i on one line and the g_j on another line directly below the first one. The resulting arrangement is called the symbol of $\mathbf{d}$. For example, the symbol of $\mathbf{d}=[5,3,2^2]$ is

$$\begin{pmatrix}1 & 2\\ 1 & 4\end{pmatrix}$$

and that of $\mathbf{f}=[3,2^2,1]$ is

$$\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}$$

We construct an (unordered) pair of partitions from a symbol exactly as above. Moreover, we handle Roman numerals in the "obvious" way: it turns out that the elements in the unordered pair coincide if and only if the original partition of $2n$ is very even, in which case we give the numeral of the partition to the pair. As in types B and C, the resulting map from orbits to representations is injective but not surjective. It too will be extended to a bijection below. In the two examples just given, the nilpotent orbit $\mathcal{O}_{[5,3,2^2]}$ (resp. $\mathcal{O}_{[3,2^2,1]}$) corresponds to the W representation $\pi_{[1^2],[1,3]}$ (resp. $\pi_{\emptyset,[2^2]}$).

Before we show how to extend the Springer correspondence in the classical cases to a bijection and define it in the exceptional cases, we mention that the special orbits of §6.3 admit a particularly simple characterization in terms of their symbols. In type A, every orbit is special (as observed in Chapter 6; note that no such orbit has a symbol). In types B and C, an orbit with symbol

$$\begin{pmatrix}p_1 & & p_2 & & \ldots & & p_n & & p_{n+1}\\ & q_1 & & & & q_n & & \end{pmatrix}$$

is special if and only if $p_1\le q_1\le p_2\le q_2\le\ldots\le q_n\le p_{n+1}$. Similarly, in type D_n, an orbit with symbol

$$\begin{pmatrix}p_1 & p_2 & \ldots & p_n\\ q_1 & q_2 & & q_n\end{pmatrix}$$

is special if and only if $p_1\le q_1\le p_2\le q_2\le\ldots\le p_n\le q_n$. There are order-preserving isomorphisms from special orbits in type B_n to special orbits in type C_n and vice versa; they both correspond to the identity map on symbols. We call a representation of W *special* if it corresponds to a special orbit via the Springer map. (Actually, Lusztig originally defined special representations of W first; he then defined special orbits as their inverse images under the Springer map.)

Now we are finally ready to give the general definition of the (extended) Springer correspondence. Let $\mathfrak{g}$ be an arbitrary complex semisimple Lie algebra. Let $\mathcal{B}$ denote the *flag variety* of $\mathfrak{g}$; that is, the set of its Borel subalgebras. Then $\mathcal{B}$ is a homogeneous space for G_{ad} (since all Borel subalgebras are conjugate); the isotropy subgroup of any of its elements is a Borel subgroup. Let $X\in\mathfrak{g}$ be nilpotent. Denote by $\mathcal{B}_X$ the *Dynkin variety* of all Borel subalgebras containing X. Obviously G^X_{ad} acts on $\mathcal{B}_X$. It is not difficult to show that the induced action of $(G^X_{ad})^o$ on the cohomology of $\mathcal{B}_X$ (with complex coefficients) is trivial; so $A(\mathcal{O}_X)$ acts on the cohomology of $\mathcal{B}_X$.

Theorem 10.1.4 (Springer [78]).

(i) *There is a natural action of W on $H^*(\mathcal{B}_X,\mathbb{C})$ commuting with the action of $A:=A(\mathcal{O}_X)$.*

(ii) *The natural map $H^*(\mathcal{B},\mathbb{C})\mapsto H^*(\mathcal{B}_X,\mathbb{C})$ is W-equivariant (here W acts on the domain via the so-called "Borel picture"; see [7]).*

(iii) *The top degree cohomology $H^{\dim\mathcal{B}_X}(\mathcal{B}_X,\mathbb{C})$ decomposes as a direct sum $\bigoplus_{\mu\in\widehat{A}}(\pi_\mu\otimes V_\mu)$, where π_μ is either 0 or an irreducible representation of W on which A acts trivially, while V_μ is a module on which A acts by μ and W acts trivially. Here "dim" denotes real dimension, contrary to our usual convention.*

(iv) *We have $\pi_{\mathbf{1}}\neq 0$, where $\mathbf{1}$ as above denotes the trivial representation.*

(v) *Any irreducible W-module is isomorphic to π_μ for a unique nilpotent orbit $\mathcal{O}_X$ and a unique $\mu\in\widehat{A}=\widehat{A}(\mathcal{O}_X)$.*

The (extended) Springer correspondence is the map sending $(\mathcal{O}_X, \mu)$ to π_μ. We note that this correspondence is now universally defined slightly differently than it was in [78]; Springer's π_μ differs from ours by tensoring by the sign representation sgn. In particular, as mentioned above, this correspondence always sends $(\mathcal{O}_{prin}, 1)$ to the trivial representation and $(\mathcal{O}_0, 1)$ to the sign representation. Actually, Lusztig has extended the correspondence of (10.1.4) even further, namely, to all representations of $\pi_1(\mathcal{O}_X)$; see [60].

We gave formulas above for all representations $\pi_{\mathbf{1}}$ if $\mathfrak{g}$ is classical. The general formula for π_μ in this case is somewhat complicated to state (see [18, §13]), but it is not difficult to compute the orbit corresponding to a given representation of W via the Springer map. We have already done this for type A above. In types B_n and C_n, let $(\mathbf{d}, \mathbf{f})$ be the ordered pair of partitions corresponding to a representation π of W. Run the above recipes for constructing pairs of partitions from symbols in reverse to produce a symbol s. Now s is not in general the symbol of any partition in $\mathcal{P}_1(2n+1)$ or $\mathcal{P}_{-1}(2n)$. Nevertheless, we can run the recipes for producing symbols from partitions in types B and C in reverse to produce partitions p_1, p_2 of $2n+1, 2n$, respectively. Then the partition of the SO_{2n+1}-orbit corresponding to π is the B-collapse $(p_1)_B$ of p_1 (defined in §6.3). Similarly, the Sp_{2n}-orbit corresponding to π is the C-collapse $(p_2)_C$ of p_2. In type D_n, the procedure for computing the orbit attached to a typical π is similar but slightly more complicated. Start with the unordered pair $\{\mathbf{d}, \mathbf{f}\}$ of partitions corresponding to π and assume first that $\mathbf{d} \neq \mathbf{f}$. Construct a symbol s from $\{\mathbf{d}, \mathbf{f}\}$ as above. Then the top row of s is different from its bottom row, so that at least one integer i appears only once in s. Let i_0 be the smallest such integer and interchange the rows of s if necessary to put i_0 into the top row. Then compute the partition $\mathbf{p}$ with symbol s as above. Once again, it is not necessarily true that $\mathbf{p} \in \mathcal{P}_1(2n)$. Nevertheless, $\mathbf{p}$ has a D-collapse p_D. The SO_{2n}-orbit with this partition corresponds to π. If $\mathbf{d} = \mathbf{f}$, we can construct this orbit more quickly. Now the symbol s *is* the symbol of a partition $\mathbf{q}$ in $\mathcal{P}_1(2n)$, which is necessarily very even. The orbit corresponding to π has partition $\mathbf{q}$ and the same Roman numeral as $\{\mathbf{d}, \mathbf{d}\}$.

We conclude this section by mentioning that the Springer correspondence has also been computed for exceptional Weyl groups W; see [18, §13]. It turns out to be quite efficient in the sense that very few pairs $(\mathcal{O}, \mu)$ get mapped to the zero representation. Furthermore, Kazhdan and Lusztig have defined a map from nilpotent orbits to conjugacy classes in W [48] that has been computed explicitly by Spaltenstein [77] in the classical cases.

0.2 Associated Varieties of Primitive Ideals

It is much more difficult to produce representations of $\mathfrak{g}$ than representations of W from nilpotent orbits. It turns out to be much easier to proceed in the reverse direction and attach nilpotent orbits $\mathcal{O}$ to representations π. Unfortunately,

the orbit $\mathcal{O}$ can tell only so much about π, for we will see that the first step in constructing $\mathcal{O}$ is to replace π by its annihilator $\mathrm{Ann}(\pi)$ in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Thus, any two representations π, π' with the same annihilator get the same attached orbit, even though π and π' can certainly look very different. Nevertheless, the ideals $\mathrm{Ann}(\pi)$, called *primitive*, are of considerable interest in their own right, and we will see in the next section that a careful analysis of the fibers of the map sending $\mathrm{Ann}(\pi)$ to $\mathcal{O}$ leads to a complete classification of primitive ideals.

So let π be an irreducible left $U(\mathfrak{g})$-module with annihilator I in $U(\mathfrak{g})$. Such an ideal I is called *primitive*, by definition. For the properties of $U(\mathfrak{g})$ and its general (two-sided) ideals see [21, chs.2,3]. The basic philosophy is that $U(\mathfrak{g})$ is close to being a polynomial algebra on $\dim \mathfrak{g}$ generators. (This is made precise in the famous Poincaré-Birkhoff-Witt Theorem, to be recalled below.) Hence, I is close to being a prime ideal in a polynomial ring. Thus, it should have an irreducible associated variety $\mathcal{V}$ of common zeros in $\mathfrak{g}^*$. Since I turns out to meet the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ in an ideal of codimension one, it is not difficult to show (using (4.4.14)) that $\mathcal{V}$ sits inside the nilpotent cone of $\mathfrak{g}^*$ (defined in §1.3). Since it is irreducible, it turns out to be, not quite a nilpotent orbit, but the *closure* of such an orbit. But we saw in the introduction to Chapter 4 that any adjoint or coadjoint orbit is uniquely determined by its closure. Thus, we get a well-defined map sending $I = \mathrm{Ann}(\pi)$ to a nilpotent orbit, as desired.

More precisely, we begin with the following result. Let $\{U_n(\mathfrak{g})\}$ be the standard filtration of $U(\mathfrak{g})$, so that $U_0(\mathfrak{g})$ identifies with the scalar field $\mathbb{C}$, $U_1(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g}$, and in general $U_n(\mathfrak{g})$ is spanned by products of at most n elements of $\mathfrak{g}$. Then we can form the associated graded algebra

$$\mathrm{gr}\, U(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$$

where we define $U_{-1}(\mathfrak{g})$ to be 0.

Theorem 10.2.1 (Poincaré-Birkhoff-Witt). *There is a natural G_{ad}-equivariant isomorphism of graded algebras $gr\, U(\mathfrak{g}) \cong S(\mathfrak{g})$, where $S(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$, is defined as in Chapter 4 to be the algebra of polynomial functions on $\mathfrak{g}^*$. It is a polynomial algebra on* $\dim \mathfrak{g}$ *generators, graded by assigning a degree of 1 to each generator.*

For a proof see [37, §17.4] or [21, 2.3.6]. Now we bring our ideal I into the picture. It inherits a filtration $\{I_n\}$ from the standard filtration of $U(\mathfrak{g})$. The associated graded object $\mathrm{gr}\, I$ is clearly a graded ideal of $\mathrm{gr}\, U(\mathfrak{g}) \cong S(\mathfrak{g})$. As such, it has an associated variety $\mathcal{V}(\mathrm{gr}\, I)$ of common zeros in $\mathfrak{g}^*$. Since I is graded, this variety is a cone in $\mathfrak{g}^*$. It is usually denoted simply $\mathcal{V}(I)$ and called the *associated variety* of I. Since I is stable under the (locally finite) action of G_{ad} on $U(\mathfrak{g})$, $\mathcal{V}(I)$ is also G_{ad}-stable. Clearly, it is also Zariski-closed. By a lemma of Dixmier ([21, 2.6.9]), the ideal I meets the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ in an ideal of codimension one. Since the associated graded algebra of $Z(\mathfrak{g})$ identifies with the space of G_{ad}-invariant polynomials $S(\mathfrak{g})^{G_{ad}}$ in $S(\mathfrak{g})$, it follows that gr I meets gr $Z(\mathfrak{g})$ in its augmentation ideal (consisting of all G_{ad}-invariant polynomials with zero constant term). Then (4.4.12) implies that $\mathcal{V}(I)$ sits inside the nilcone $\mathcal{N}^*$ of $\mathfrak{g}^*$ (defined in §1.3, where we also observed that it is isomorphic to the nilcone of $\mathfrak{g}$ as an algebraic variety). The upshot of all these observations is a theorem of Borho and Kraft [12]: $\mathcal{V}(I)$ is a closed finite union of nilpotent orbits. But actually we have a much stronger result, whose statement should come as no surprise by now.

eorem 10.2.2 (Borho, Brylinski, Joseph). *Let I be a primitive ideal of $U(\mathfrak{g})$. Then $\mathcal{V}(I)$ is the closure $\overline{\mathcal{O}}$ of one nilpotent orbit $\mathcal{O}$.*

This result, often referred to as the Irreducibility Theorem, was proved by Borho and Brylinski in the special case where $\mathfrak{g}$ is classical and I has integral infinitesimal character [10]. (We will explain this condition in the next section.) Joseph then proved it in general [47]. As a consequence, we obtain the desired map from representations π to nilpotent (coadjoint) orbits $\mathcal{O}$; it sends π to the unique open dense orbit in $\mathcal{V}(\mathrm{Ann}(\pi))$. We can also define the associated variety $\mathcal{V}(\pi)$ of π itself, but this turns out to be much more difficult to understand than $\mathcal{V}(\mathrm{Ann}(\pi))$ (unless π happens to be a so-called Harish-Chandra module for a complex group; we define these in the next section). For example, it need not be irreducible even if π is. We will discuss the fibers of our map in some detail in the next section. For now, we note that Barbasch and Vogan have shown how to give an elegant general definition of the order-reversing surjection d of §6.3 from nilpotent orbits to special nilpotent orbits in terms of associated varieties of certain maximal ideals in $U(\mathfrak{g})$. They also show that the restriction of d to the special orbits (which is an involution) admits a very simple description (in most cases) on the level of the Springer correspondence. More precisely, for every special nilpotent orbit $\mathcal{O}$, with one exception in type E_7 and two in E_8, the Springer representation attached to $(d(\mathcal{O}), 1)$ is obtained from that attached to $(\mathcal{O}, 1)$ by tensoring with the sign representation. See [6, §3].

0.3 Classification of Primitive Ideals

We now investigate the precise extent to which a primitive ideal is determined by its associated variety. The starting point in the modern theory of primitive ideals is the following celebrated result, which was first proved analytically by Duflo [26] and then algebraically by Joseph [44]. We assume you are familiar with the definition and basic properties of highest weight modules; see [37, ch.VI] or [21, ch.7].

eorem 10.3.1. *Any primitive ideal I in $U(\mathfrak{g})$ is the annihilator of some simple highest weight module L (relative to some choice $\mathfrak{b}$ of Borel subalgebra containing a Cartan subalgebra $\mathfrak{h}$).*

The proof uses the fact noted in the last section that I meets the center $Z(\mathfrak{g})$ in an ideal of codimension one. An immediate consequence of this fact is that $Z(\mathfrak{g})$ acts by a character on a simple highest weight module L, called the *infinitesimal character* of L or of I. A basic fact, first observed by Harish-Chandra, is that there are only finitely many simple highest weight modules (relative to $\mathfrak{b}$ and $\mathfrak{h}$) with a fixed infinitesimal character χ; more precisely, the infinitesimal character of any simple highest weight module is completely determined by the W-orbit of its highest weight plus ρ, the half sum of the positive roots (again relative to $\mathfrak{b}$ and $\mathfrak{h}$). Assume now for simplicity that the infinitesimal character χ is *regular integral*; that is, it is the infinitesimal character of some finite-dimensional simple module. Then there are exactly $|W|$ nonisomorphic simple highest weight modules of infinitesimal character χ. We can completely classify primitive ideals of this infinitesimal character by determining which of these highest weight modules have the same annihilator. Algorithms for this purpose are available if $\mathfrak{g}$ is of classical type ([45], [28]), but we content ourselves here with an abstract parametrization of the set of primitive ideals with infinitesimal character χ and a given associated variety.

Theorem 10.3.2 (Barbasch, Vogan, Joseph). *Assume that χ is regular and integral. The associated variety $\mathcal{V}(I)$ of any primitive ideal I of infinitesimal character χ is the closure of a special nilpotent orbit $\mathcal{O}$. The set of such ideals with associated variety equal to a fixed $\overline{\mathcal{O}}$ is parametrized by a basis of the special W-representation π corresponding via the Springer map to $(\mathcal{O}, 1)$. Thus, the number of primitive ideals of infinitesimal character χ equals the sum of the dimensions of the special representations of W.*

This is an extremely difficult result; it was the culmination of six years of intensive work in the subject following Duflo's announcement of Theorem 10.3.1. See [4, 5, 46]. Roughly speaking, the parametrization is realized as follows. Fix a special orbit $\mathcal{O}$ and consider the set $C(\mathcal{O})$ of simple highest weight modules $L(\lambda)$ of highest weight λ and infinitesimal character χ such that $\mathcal{V}(\mathrm{Ann}\ L(\lambda)) = \overline{\mathcal{O}}$. For each such highest weight λ, consider the set of simple highest weight modules $L(\lambda')$ where λ' ranges over integral weights in the same Weyl chamber as λ. The Goldie rank of the quotients $U(\mathfrak{g})/\mathrm{Ann}\ L(\lambda)$ turns out to depend polynomially on λ, and the polynomial p expressing it as a function of λ extends in a natural way to $\mathfrak{h}^*$. Thus, p identifies with an element of $S(\mathfrak{h})$, and as such it transforms in a certain way under the natural action of W on $S(\mathfrak{h})$. The set of all polynomials p obtained as $L(\lambda)$ ranges over $C(\mathcal{O})$ spans a W-submodule of $S(\mathfrak{h})$ isomorphic to the Springer representation attached to $(\mathcal{O}, 1)$. The proof relies much more on the formal properties of the Springer correspondence than on its precise definition. For a nice exposition of most of the theory involved, see [41, Kap. 14,16].

If the infinitesimal character χ is not regular integral, then the set of primitive ideals of infinitesimal character χ and a fixed associated variety $\mathcal{V}$ is parametrized by a basis of the subspace of W''-fixed vectors in a suitable special representation of W', where W' and W'' are the so-called integral and singular Weyl subgroups of W relative to χ. For their precise definitions see [41, Kap. 2]. This parametrization follows easily from the ideas in the proof of (10.3.2). We note that $\mathcal{V}$ need *not* be the closure of a special orbit in this situation.

So far in this section we have discussed only annihilators of irreducible representations. We now give a brief account of what can be said about representations π with a fixed annihilator I. The first point is that it is hopeless to say anything sensible about such representations π in general: there are far too many of them, and they can be too badly behaved. We must restrict attention to a manageable subclass of simple $U(\mathfrak{g})$-modules. The most obvious one in view of (10.3.1) is that of simple highest weight modules. For simplicity, we assume once again that all our modules have regular integral infinitesimal character.

eorem 10.3.3 (Barbasch, Joseph, Lusztig, Vogan). *The set of all simple highest weight modules having a fixed primitive annihilator I of regular integral infinitesimal character is parametrized by a basis of a certain W-representation π' (called the left cell representation of I). If $\mathfrak{g}$ is of type A, then π' is irreducible and, in fact, isomorphic to the representation π corresponding to I via Theorem (10.3.2). In general, there is a unique copy of π inside π'.*

The ideas and techniques of the proof are similar to those of the proof of (10.3.2), but we must work even harder. The parametrization turns out to be rather less canonical than that of (10.3.2). The possibilities for π' have been worked out completely by Lusztig [58, 61], who gives an interesting description of them in terms of representations of various finite groups. There is a generalization of (10.3.3) to arbitrary infinitesimal characters similar to the one we gave of (10.3.2), but we omit it; see [41, Kap. 16].

We conclude this section by considering one further class of simple $U(\mathfrak{g})$-modules, namely, Harish-Chandra modules (for G_{ad}). A *Harish-Chandra module* M is a $(U(\mathfrak{g}), U(\mathfrak{g}))$ *bimodule* such that the "adjoint $\mathfrak{g}$-action" given by

$$X \cdot m = Xm - mX$$

is locally finite. Thus, modules of this sort certainly look very different from highest weight modules; for example, the latter are only left modules. Nevertheless, there is a remarkable equivalence between a certain large subcategory of the category of Harish-Chandra modules and a certain natural category containing the highest weight modules [41, 6.27]. Combining this equivalence with a fundamental calculation of Joseph [43] and a deep result of Lusztig [59, 12.15], we obtain

heorem 10.3.4. *The set of simple Harish-Chandra modules with fixed left and right primitive annihilators I_L, I_R of regular integral infinitesimal character is either empty or parametrized by a basis of the space of homomorphisms between a suitable pair of left cell representations of W. The latter alternative holds if and only if I_L and I_R have the same associated variety; in this case, the left cell representations are those attached to I_L and I_R by Theorem 10.3.3.*

A simple example of a Harish-Chandra module is a quotient $U(\mathfrak{g})/I$ of $U(\mathfrak{g})$ by any (two-sided) ideal I (even the zero ideal). If I is primitive, then $U(\mathfrak{g})/I$ belongs to the "large subcategory" of the category of Harish-Chandra modules alluded to above and has some particularly nice properties. It was by studying these properties that Duflo was led to discover Theorem 10.3.1.

10.4 Primitive Ideals and the Geometry of Orbits

In this last section, we describe some more recent work on primitive ideals of $U(\mathfrak{g})$, focusing on the connections between them and nilpotent orbits. We are led to expect such connections by the results of §10.2 together with the well-known identification of any prime quotient $S(\mathfrak{g})/I$ of $S(\mathfrak{g})$ with the coordinate ring of its associated irreducible variety. Since we now know a precise sense in which $U(\mathfrak{g})$ is not far removed from $S(\mathfrak{g})$, it is natural to investigate the relationship between a primitive quotient $U(\mathfrak{g})/I$ and the coordinate ring $R_I := S(\mathfrak{g})/\sqrt{\mathrm{gr}\, I}$ of *its* associated variety. By (10.2.2), the radical $\sqrt{\mathrm{gr}\, I}$ of gr I is prime, though gr I itself need not be. If gr I does happen to be prime, then it coincides with its radical. Then the quotient $U(\mathfrak{g})/I$ admits a natural filtration with associated graded algebra isomorphic to $S(\mathfrak{g})/\mathrm{gr}\, I$. Thus, $U(\mathfrak{g})/I$ looks just like R_I in some sense and in fact does share many properties of the latter. For example, it is easy to show that $U(\mathfrak{g})/I$ is completely prime (i.e., has no zero divisors) and is isomorphic to R_I as a G_{ad}-module in this situation.

In general, it turns out that there is a precise measure of the difference between $S(\mathfrak{g})/\mathrm{gr}\, I$ and $S(\mathfrak{g})/\sqrt{\mathrm{gr}\, I}$, and hence also of the difference between $U(\mathfrak{g})/I$ and R_I. Indeed, both $S(\mathfrak{g})/\mathrm{gr}\, I$ and $S(\mathfrak{g})/\sqrt{\mathrm{gr}\, I}$ are finitely generated $S(\mathfrak{g})$-modules whose Hilbert polynomials have leading terms at^d, bt^d involving the same power of the indeterminate t (the exponent d is the dimension of the associated variety $\mathcal{V}(I)$). The ratio a/b of the leading coefficients turns out to be a positive integer n. The formal scalar multiple $n\mathcal{V}(I)$ of $\mathcal{V}(I)$ by n is called the *characteristic cycle* of I or of $U(\mathfrak{g})/I$. Vogan has shown how to replace it by an n-dimensional representation of a certain compact subgroup of G_{ad}, which can then be used to deduce much useful information about $U(\mathfrak{g})/I$ [86]; actually, he works in a much more general context.

Thus, one would like to understand the coordinate rings R_I not only as interesting objects in their own right, but also for the insights they reveal into the quotients $U(\mathfrak{g})/I$. Unfortunately, most of the known results pertain not to R_I directly but rather to its normalization (or integral closure). This normalization may be identified with the coordinate ring $R(\mathcal{O})$ of the open dense orbit $\mathcal{O}$ in the associated variety of I [12]. For example, $R(\mathcal{O})$ is known to be Gorenstein and have rational singularities [35]; thus, it is also Cohen-Macaulay. A great deal of work has been done on determining which orbits $\mathcal{O}$ have normal closures, so that the corresponding coordinate rings R_I are already normal. It turns out that they all do in type A and most of them do in the other classical types. The known methods in principle settle the question of whether the closure of a classical nilpotent orbit is normal, with the exception of certain very even orbits in type D [56]. For the exceptional cases, on the other hand, we have only a fairly long list of orbits whose closures are known to be *nonnormal*, but so far no corresponding list of orbits with provably normal closures [73]. There are, however, three general results, due to Kostant, Broer, and Kempf, respectively: the principal, subregular, and minimal nilpotent orbits all have normal closures [53, 16, 49].

The most useful result so far for computations seems to be the following formula for the G_{ad}-module structure of $R(\mathcal{O})$. To state it, we need some notation. Given a nonzero nilpotent orbit $\mathcal{O}$, let $\{H, X, Y\}$ be a standard triple in $\mathfrak{g}$ with $X \in \mathcal{O}$. Decompose $\mathfrak{g}$ as a direct sum of eigenspaces $\mathfrak{g}_i$ of ad_H, as in (3.4.1). Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing H and a choice of positive roots of $\mathfrak{t}$ in $\mathfrak{g}$ such that $\alpha(H) \geq 0$ for every positive root α. Let T denote the connected Lie subgroup of G_{ad} with Lie algebra $\mathfrak{t}$; then T is a maximal torus in G_{ad}. For each weight λ in the root lattice of $\mathfrak{t}$, let e^λ denote the corresponding (exponentiated) character of T. The set Λ of all such characters has an obvious structure of abelian group (written multiplicatively); moreover, we have the familiar law of exponents $e^\lambda e^\mu = e^{\lambda+\mu}$. If $\lambda \in \Lambda$, then we define the formal sum $\mathrm{Ind}_T^G(e^\lambda)$ of G_{ad}-modules by Frobenius reciprocity: a typical finite-dimensional simple module π appears with coefficient n_λ in $\mathrm{Ind}_T^G(e^\lambda)$, where n_λ denotes the multiplicity of the λ-weight space in π. We extend $\mathrm{Ind}_T^G(\cdot)$ to elements of the group ring $\mathbb{Z}[\Lambda]$ by linearity; then its image lies in the Grothendieck group K_0 generated by the finite-dimensional G_{ad}-modules.

eorem 10.4.1 (McGovern [66]). *With notation as above, we have*

$$R(\mathcal{O}) = \mathrm{Ind}_T^G\,\Big(\prod_\alpha (1 - e^\alpha)\Big)$$

in K_0, where α runs over the positive roots of $\mathfrak{t}$ in $\mathfrak{g}$ such that the corresponding root space $\mathfrak{g}_\alpha$ lies in $\mathfrak{g}_0 + \mathfrak{g}_1$.

This result also holds for the zero orbit if we make the obvious convention that $\mathfrak{g}_0 = \mathfrak{g}$ in that case.

We mentioned in the introduction to this chapter that the geometry of nilpotent orbits has ideal-theoretic consequences. We conclude by amplifying this remark slightly. It is natural to ask whether the well-known bijection between prime ideals in $S(\mathfrak{g})$ and irreducible subvarieties of $\mathfrak{g}^*$ extends to $U(\mathfrak{g})$. The most recent work suggests that it does not, but there should still be an injective map between some presently undefined class $\mathcal{C}$ of geometric objects (including all coadjoint orbits, nilpotent or not) and another class $\mathcal{C}'$ of completely prime algebra extensions of finite type over primitive quotients of $U(\mathfrak{g})$ (including all the completely prime primitive quotients themselves). We call algebras of this sort *Dixmier algebras*. Whenever a coadjoint orbit admits a nontrivial cover, that cover should also belong to $\mathcal{C}$, so that one may attach a Dixmier algebra to it. Similarly, whenever an orbit $\mathcal{O}$ has a nonnormal closure $\overline{\mathcal{O}}$, both $\mathcal{O}$ and $\tilde{\overline{\mathcal{O}}}$ should belong to $\mathcal{C}$, so that both get Dixmier algebras attached to them. We refer you to [84, 85, 67, 68].

Bibliography

[1] A. Alexeevsky. Component groups of centralizer for unipotent elements in semisimple algebraic groups [in Russian], Trudy Tbilissi. Mat. Inst. Razmazde Akad. Nauk. Gruzin SSR 62, (1979), 5-27.

[2] P. Bala and R. Carter. Classes of unipotent elements in simple algebraic groups I, Math. Proc. Camb. Phil. Soc. **79** (1976), 401–425.

[3] P. Bala and R. Carter. Classes of unipotent elements in simple algebraic groups II, Math. Proc. Camb. Phil. Soc. **80** (1976), 1–18.

[4] D. Barbasch and D. Vogan. Primitive ideals and orbital integrals in complex classical groups, Math. Ann. **259** (1982), 153–199.

[5] D. Barbasch and D. Vogan. Primitive ideals and orbital integrals in complex exceptional groups, J. Alg. **80** (1983), 350–382.

[6] D. Barbasch and D. Vogan. Unipotent representations of complex semisimple groups, Ann. of Math. **121** (1985), 41–110.

[7] A. Borel. *Topics in the homology theory of fiber bundles.* Lecture Notes in Math. **36**, Springer-Verlag, New York, 1967.

[8] W. Borho. Über Schichten halbeinfacher Lie-Algebren, Inv. Math. **65** (1981), 283–317.

[9] W. Borho. A survey of enveloping algebras of semisimple Lie algebras, I, in *Lie Algebras and Related Topics*, Canad. Math. Soc. Conf. Proc. **5**, Amer. Math. Soc. for CMS, Providence, 1986, 19–50.

[10] W. Borho and J. Brylinski. Differential operators on homogeneous spaces I: irreducibility of the associated variety, Inv. Math. **69** (1982), 437–476.

[11] W. Borho, J. Brylinski, and R. MacPherson. *Nilpotent Orbits, Primitive Ideals, and Characteristic Classes: A Geometric Perspective in Ring Theory.* Progress in Math. **78**, Birkhäuser, Boston, 1989.

[12] W. Borho and H. Kraft. Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen, Comm. Math. Helv. **54** (1979), 61–104.

[13] R. Bott. An application of Morse theory to the topology of Lie groups, Bull. Soc. Math. Fr. **84** (1956), 251–281.

[14] N. Bourbaki. *Elements of Mathematics: Lie Groups and Lie Algebras, Chapters 1-3.* Springer-Verlag, New York, 1989.

[15] N. Bourbaki. *Éléments de mathématique: Groupes et Algèbres de Lie; Chapitres 4-6, 7-8.* Hermann, Paris, 1968.

[16] A. Broer. On the subregular nilpotent variety, preprint, Amsterdam, 1991.

[17] N. Burgoyne and R. Cushman. Conjugacy classes in the linear groups, J. Alg. **44** (1977), 339–362.

[18] R. Carter. *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters.* John Wiley & Sons, London, 1985.

[19] R. Carter. *Simple Groups of Lie Type.* Wiley Classics Library, John Wiley & Sons, London, 1989.

[20] A. Coleman. The Betti numbers of the simple Lie groups, Canad. J. Math. **10** (1958), 349–356.

[21] J. Dixmier. *Enveloping Algebras.* North-Holland, Amsterdam, 1977.

[22] D. Djoković. Closures of conjugacy classes in classical real linear Lie groups, in *Algebra, Carbondale, 1980*, Lecture Notes in Math. **848**, Springer-Verlag, New York, 1980, 63–83.

[23] D. Djoković. Proof of a conjecture of Kostant, Trans. Amer. Math. Soc. **302** (1987), 577–585.

[24] D. Djoković. Classification of nilpotent elements in simple exceptional real Lie algebras of inner type and description of their centralizers, J. Alg. **112** (1988), 503–524.

[25] D. Djoković. Classification of nilpotent elements in the simple real Lie algebras $E_{6(6)}$ and $E_{6(-26)}$ and description of their centralizers, J. Alg. **116** (1988), 196–207.

[26] M. Duflo. Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple, Ann. of Math. **105** (1977), 107–120.

[27] E. Dynkin. Semisimple subalgebras of semisimple Lie algebras, Amer. Math. Soc. Transl. Ser. 2, **6** (1957), 111–245.

[28] D. Garfinkle. On the classification of primitive ideals for complex classical Lie algebras, I, Comp. Math. **75** (1990), 135–169.

[29] M. Gerstenhaber. On dominance and varieties of commuting matrices, Ann. of Math. **73** (1961), 324–348.

[30] M. Gerstenhaber. Dominance over the classical groups, Ann. of Math. **74** (1961), 532–569.

[31] W. Greub, S. Halperin, and R. Vanstone. *Connections, Curvature and Cohomology II.* Academic Press, New York, 1973.

[32] V. Guillemin and S. Sternberg. *Geometric Asymptotics.* Mathematical Surveys 14, American Mathematical Society, Providence, 1977.

[33] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, New York, 1978.

[34] W. Hesselink. Singularities in the nilpotent scheme of a classical group, Trans. Amer. Math. Soc. **222** (1979), 1–32.

[35] V. Hinich. On the singularities of nilpotent orbits, Israel J. Math. **73** (1991), 297–308.

[36] K. Hoffman and R. Kunze. *Linear Algebra*, 2^d ed., Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

[37] J. Humphreys. *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Math. **9**, Springer-Verlag, New York, 1972.

[38] N. Jacobson. *Basic Algebra I.* W. H. Freeman and Company, New York, 1985.

[39] N. Jacobson. *Lie Algebras.* Dover, New York, 1962.

[40] G. D. James. *Representation Theory of the Symmetric Groups*, Lecture Notes in Math. **682**, Springer-Verlag, New York, 1978.

[41] J. C. Jantzen. *Einhüllende Algebren halbeinfacher Lie-Algebren.* Ergebnisse der Mathematik **3**, Springer-Verlag, New York, 1983.

[42] D. S. Johnston and R. W. Richardson. Conjugacy classes in parabolic subgroups of semisimple algebraic groups, II, Bull. Lond. Math. Soc. **9** (1977), 245–250.

[43] A. Joseph. On the annihilators of the simple subquotients of the principal series, Ann. Sci. Ec. Norm. Sup. (4) **10** (1977), 419–440.

[44] A. Joseph. Dixmier's problem for Verma and principal series submodules, J. London Math. Soc. **20** (1979), 193–204.

[45] A. Joseph. Towards the Jantzen conjecture II, Comp. Math. **40** (1980), 69–78.

[46] A. Joseph. Goldie rank in the enveloping algebra of a semisimple Lie algebra, I,II, J. Alg. **65** (1980), 269–306.

[47] A. Joseph. On the associated variety of a primitive ideal, J. Alg. **93** (1985), 509–523.

[48] D. Kazhdan and G. Lusztig. Fixed point varieties on affine flag manifolds, Israel J. Math. **62** (1988), 129–168.

[49] G. Kempf. On the collapsing of homogeneous bundles, Inv. Math. **37** (1976), 229–239.

[50] G. Kempken. Induced conjugacy classes in classical Lie-algebras, Abh. Math. Sem. Univ. Hamb. **53** (1983), 53–83.

[51] A. W. Knapp. Lie groups, Lie algebras, and cohomology, Math. Lecture Notes **34**, Princeton University Press, Princeton, 1988.

[52] B. Kostant. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. **81** (1959), 973–1032.

[53] B. Kostant. Lie group representations on polynomial rings, Amer. J. Math. **85** (1963), 327–404.

[54] B. Kostant and S. Rallis. Orbits and representations associated with symmetric spaces, Amer. J. Math. **93** (1971), 753–809.

[55] H. Kraft. Parametrisierung von Konjugationsklassen in $\mathfrak{sl}_n$, Math. Ann. **234** (1978), 209–220.

[56] H. Kraft and C. Procesi. On the geometry of conjugacy classes in classical groups, Comm. Math. Helv. **57** (1982), 539–602.

[57] G. Lusztig. A class of irreducible representations of a Weyl group, Indag. Math. **41** (1979), 323–335.

[58] G. Lusztig. A class of irreducible representations of a Weyl group II, Indag. Math. **44** (1982), 219–226.

[59] G. Lusztig. *Characters of Reductive Groups over a Finite Field*. Annals of Math. Studies **107**, Princeton University Press, Princeton, 1984.

[60] G. Lusztig. Intersection cohomology complexes on a reductive group, Inv. Math. **75** (1984), 205–272.

[61] G. Lusztig. Sur les cellules gauches des groupes de Weyl, C. R. Acad. Sci. Paris (A) **302** (1986), 5–8.

[62] G. Lusztig and N. Spaltenstein. Induced unipotent classes, J. London Math. Soc. **19** (1979), 41–52.

[63] H. Matsumura. *Commutative Ring Theory*, tr. by M. Reid. Cambridge University Press, Cambridge, 1986.

[64] S. J. Mayer. On the characters of the Weyl group of type C, J. Alg. **33** (1975), 59–67.

[65] S. J. Mayer. On the characters of the Weyl group of type D, Math. Proc. Camb. Phil. Soc. **77** (1975), 259–264.

[66] W. McGovern. Rings of regular functions on nilpotent orbits and their covers, Inv. Math. **97** (1989), 209–217.

[67] W. McGovern. Unipotent representations and Dixmier algebras, Comp. Math. **69** (1989), 241–276.

[68] W. McGovern. Dixmier algebras and the orbit method, in *Actes du colloque en l'honneur de Jacques Dixmier*. Progress in Math. **92**, Birkhäuser, Boston, 1990, 397–416.

[69] G. D. Mostow. Some new decomposition theorems for semi-simple Lie groups, Mem. Amer. Math. Soc. **14** (1955), 31–54.

[70] T. Ohta. Closures of nilpotent orbits in classical symmetric pairs and their singularities, Tohoku Math. J. (2) **43** (1991), 161–213

[71] H. Ozeki and M. Wakimoto. On polarizations of certain homogeneous spaces, Hiroshima Math. J. **2** (1972), 445–482.

[72] R. W. Richardson. Conjugacy classes in parabolic subgroups of semisimple algebraic groups, Bull. London Math. Soc. **6** (1974), 21–24.

[73] R. W. Richardson. Derivatives of invariant polynomials on a semisimple Lie algebra, in *Proceedings, Miniconference on Harmonic Analysis* **15** (1987), Australian National University, Canberra, 1987, 228–242.

[74] J. Sekiguchi. Remarks on nilpotent orbits of a symmetric pair, J. Math. Soc. Japan **39** (1987), 127–138.

[75] J. P. Serre. *Linear Representations of Finite Groups*, tr. by Leonard Scott, Graduate Texts in Math. **42**, Springer-Verlag, New York, 1977.

[76] N. Spaltenstein. *Classes Unipotentes et Sous-Groupes de Borel*, Lecture Notes in Math. **946**, Springer-Verlag, New York, 1982.

[77] N. Spaltenstein. Nilpotent orbits and conjugacy classes in Weyl groups, Astérisque **168** (1988), 191–217.

[78] T. A. Springer. A construction of representations of Weyl groups, Inv. Math. **44** (1978), 279–293.

[79] T. Springer. *Linear Algebraic Groups*. Progress in Mathematics **9**, Birkhauser, Boston 1981.

[80] T. Springer and R. Steinberg. Conjugacy Classes, in *Seminar on Algebraic Groups and Related Finite Groups*, Lecture Notes in Math. **131**, Springer-Verlag, New York, 1970, 167–266.

[81] R. Stanley. *Enumerative Combinatorics*. Advanced Mathematics Series, Wadsworth and Brooks/Cole, Pacific Grove, Calif. 1986.

[82] R. Steinberg. *Conjugacy Classes in Algebraic Groups*. Lecture Notes in Math. **366**, Springer-Verlag, New York, 1974.

[83] V. Varadarajan. *Lie Groups, Lie Algebras, and their Representations.*, 2d ed., Springer-Verlag, New York, 1984.

[84] D. Vogan. The orbit method and primitive ideals for semisimple Lie algebras, in *Lie Algebras and Related Topics*, Canad. Math. Soc. Conf. Proc. **5**, Amer. Math. Soc. for CMS, Providence, 1986, 281–319.

[85] D. Vogan. Dixmier algebras, sheets, and representation theory, in *Actes du colloque en l'honneur de Jacques Dixmier*. Progress in Math. **92**, Birkhäuser, Boston, 1990, 333–397.

[86] D. Vogan. Associated varieties and unipotent representations, in *Proceedings, Harmonic Analysis on Reductive Groups, Bowdoin College*. Progress in Math. **101**, Birkhäuser, Boston, 1991.

[87] G. Wall. On the conjugacy classes in the unitary, symplectic and orthogonal groups, J. Austr. Math. Soc. **3** (1963), 1–62.

[88] N. Wallach. *Symplectic Geometry and Fourier Analysis*. Math Sci Press, Brookline, Massachusetts, 1977.

[89] F. Warner. *Foundations of Differential Manifolds and Lie Groups.*, 2d ed., Springer-Verlag, New York, 1983.

[90] G. Warner. *Harmonic Analysis on Semi-simple Lie groups I,II*. Springer-Verlag, New York, 1972.

Index

Index