Nilpotent Orbits in

Semisimple Lie

Algebras

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Preface

Over the last decade, a circle of ideas has emerged relating three very different kinds of objects associated to a complex semisimple Lie algebra: nilpotent orbits, representations of a Weyl group, and primitive ideals in an enveloping algebra (see [11],[9]). Any attempt to understand or exploit these connections ultimately presupposes a good understanding of each of these objects. Unfortunately, many of the fundamental results are scattered throughout the literature or shrouded in folklore, severely limiting accessibility by the nonexpert. The principal aim of this book is to collect together the important results concerning the classification and properties of nilpotent orbits, beginning from the common ground of basic structure theory. The techniques we use are elementary and in the toolkit of any graduate student hoping to enter the field of representation theory. In fact, one could argue that much of what follows is an elegant illustration of the power of $\mathfrak{sl}_2$ theory. After reviewing the basic prerequisites, we will develop the Dynkin-Kostant and Bala-Carter classifications of complex nilpotent orbits, derive the Lusztig-Spaltenstein theory of induction of nilpotent orbits, discuss some basic topological questions, and classify real nilpotent orbits. We will emphasize the special case of classical Lie algebras throughout; here the theory can be simplified by using the combinatorics of partitions and tableaux. We conclude with a survey of advanced topics related to the above circle of ideas.

The general linear group $GL_n\mathbb{C}$ acts on its Lie algebra $\mathfrak{gl}_n$ of all complex $n \times n$ matrices by conjugation; the orbits are of course similarity classes of matrices. The theory of the Jordan form gives a satisfactory parametrization of these classes and allows us to regard two kinds of classes as distinguished: those represented by diagonal matrices, and those represented by strictly upper triangular matrices. In particular, there are only finitely many similarity classes of nilpotent matrices; more precisely, the set of such classes is parametrized by partitions of n. Also, there is a very similar parametrization of nilpotent orbits in any classical semisimple Lie algebra.

The book is divided into three parts. In the first two, we give a more or less self-contained exposition with complete proofs; in the third, we give a survey of various results without proof. In the first part, consisting of Chapters 1–8, we work with complex semisimple Lie algebras. There are two prerequisites for understanding this material: the basic structure theory of such algebras, as given in the first four chapters of Humphreys's classic text [37]; and the elementary Lie group theory contained in F. Warner's book [89]. In the second part (Chapter 9), we work with real semisimple Lie algebras. We assume familiarity with their classification and basic structure theory, as given in Helgason's book [33, §VI,X]. Finally, in the last part (Chapter 10), we discuss various advanced topics. It is here that we describe the connections between nilpotent orbits and representation theory.

Here is a brief outline of the content of each chapter; for more detailed overviews see the chapter introductions.

In Chapter 1, we give the basic definitions and review the basic facts that we need. In Chapter 2, we classify the semisimple orbits in any complex semisimple Lie algebra; this turns out to be much simpler than classifying the nilpotent orbits. We also mention a few of the elementary topological properties of semisimple orbits and give references for their proofs.

Chapter 3 is the heart of the book; in it we derive the classical Dynkin-Kostant classification of complex nilpotent orbits. While this classification does not actually yield a parametrization (a parametrization is obtained in Chapter 8), it does prove one of our fundamental results: there are only finitely many nilpotent orbits. It also gives a uniform method for labeling nilpotent orbits.

In Chapter 4 we construct three canonical nilpotent orbits in any (complex) simple algebra. They may be specified by their positions in the Hasse diagram of nilpotent orbits relative to a certain partial order which we define in the chapter introduction.

All of the remaining theory in the book turns out to be much more concrete and explicit for the classical algebras than in general (and even simpler for $\mathfrak{sl}_n$ than for the other classical algebras). In Chapter 5 we refine the Dynkin-Kostant classification for classical algebras to an explicit parametrization in terms of partitions. We also give a formula for the labels attached in Chapter 3 to the orbit corresponding to a given partition.

Chapter 6 serves as an introduction to the topology of nilpotent orbits. We compute the fundamental groups of classical nilpotent orbits and give an explicit formula for the partial order relation of Chapter 4 in the classical cases. We also define an order-reversing involution on the set of so-called *special* orbits.

Chapter 7 continues the theme of Chapter 4 by constructing certain canonical nilpotent orbits. Instead of starting from nothing, however, we start from nilpotent orbits in smaller semisimple algebras. Once again we are able to give explicit formulas for the behavior of our construction in the classical cases.

As mentioned above, we refine the Dynkin-Kostant classification to a parametrization (this time for general semisimple algebras) in Chapter 8, following the approach of Bala and Carter. We give tables of the exceptional nilpotent orbits there.

Finally, in the last two chapters, we redo the theory of the preceding eight chapters (mostly the material in Chapters 3, 4, and 5) for real nilpotent orbits and then give a survey of some related representation theory. Topics in the last chapter include the Springer correspondence, associated varieties, and the classification of primitive ideals in enveloping algebras.

This book is the by-product of a two-quarter graduate course taught by the authors at the University of Washington in 1991. The material in Chapters 1–7, plus portions of Chapter 10, comprised a one-quarter course attended by graduate students with a background in basic structure theory. Chapters 8 and 9, together with a development of the classification of real forms (which is only recalled here) accounted for the majority of the second course. With careful planning, the entire book could reasonably be covered in a one-semester course. Alternatively, one could cover Chapters 1–5, 8, and 9 in a one-quarter course, giving the classification in the real and complex cases without touching upon the topological issues.

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Notation and Conventions

$\mathbb{Z}, \mathbb{N}, \mathbb{N}^+, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}$ denote the integers, nonnegative integers, strictly positive integers, rational numbers, real numbers, complex numbers, and quaternions, respectively. The notations $[\cdot,\cdot], \dim, |\mathcal{X}|$ denote the bracket operation in a Lie algebra, *complex* dimension (unless otherwise specified), and the cardinality of the set $\mathcal{X}$, respectively. Lowercase Gothic letters (such as $\mathfrak{g}$) without subscripts denote finite-dimensional complex Lie algebras. Lowercase Gothic letters with subscript $\mathbb{R}$ denote finite-dimensional real Lie algebras. Except in Chapter 9, all Lie algebras are assumed to be complex (and finite-dimensional).

Contents

1 Preliminaries

We begin by reviewing certain fundamental concepts from linear algebra. As innocent as this may seem, all that follows can be traced back to the elementary linear algebraic ideas surrounding the study of similarity between matrices. This review material is followed by a primer on symplectic geometry and its relevance to the study of adjoint orbits.

The material in this chapter should be viewed as a quick review (with references) of important elementary concepts from linear algebra, Lie theory, and differential geometry. We need to know that every element of a semisimple Lie algebra can be expressed as the sum of a nilpotent element and a semisimple element. In turn, this leads to the notions of nilpotent and semisimple orbits in a Lie algebra (or its dual) and allows us to formulate the classification problems central to this book. In §1.4, we include a brief tour of the symplectic geometry relevant to our needs. We ultimately want to know that the previously mentioned orbits are symplectic manifolds and to understand the basic implications of having such a geometric structure.

1.1 Nilpotent and Semisimple Elements

Consider a linear transformation X of a finite-dimensional complex vector space V to itself; i.e., an element of $\mathrm{End}(V)$. Recall that, by definition, we say that X is a *semisimple* operator if every X-invariant subspace has an X-invariant complement. We say that X is a *nilpotent* operator if $X^r = X \circ \cdots \circ X = 0$ for some $r > 0$. Since the base field is $\mathbb{C}$, it follows that X is semisimple if and only if X is *diagonalizable* (i.e. V admits a basis of eigenvectors for X) [36]. We emphasize that when working over $\mathbb{R}$ (or any non-algebraically-closed field), the notions of semisimplicity and diagonalizability differ.

Our first order of business is the definition of nilpotent and semisimple elements in a complex semisimple Lie algebra $\mathfrak{g}$. Such elements are easily defined, if we first recall the *adjoint representation*

$$\mathrm{ad} : \mathfrak{g} \mapsto \mathrm{End}(\mathfrak{g});\ \mathrm{ad}_X Y = [X, Y].$$

We say that an element $X \in \mathfrak{g}$ is *nilpotent* if ad_X is a nilpotent endomorphism of the vector space $\mathfrak{g}$. In a similar spirit, $X \in \mathfrak{g}$ is said to be *semisimple* if ad_X is a semisimple operator on $\mathfrak{g}$.

Our definitions of nilpotent and semisimple elements in a Lie algebra give initial preference to the adjoint representation. You might change these definitions by picking a favorite finite dimensional representation

$$\rho : \mathfrak{g} \mapsto \mathrm{End}(V)$$

and defining nilpotence or semisimplicity of $X \in \mathfrak{g}$ in terms of the nilpotence or semisimplicity of the operator $\rho(X)$ on V. This approach would pose an obvious problem if $X \in \ker(\rho)$. (Recall that the kernel of the adjoint representation on *any* Lie algebra is just the algebra's center, which must be trivial in the semisimple case; consequently, the adjoint representation is faithful.) But, assuming ρ is faithful, it is perhaps hard to argue at the outset why our "adjoint definition" is to be preferred to the "ρ definition". In fact, since the classical Lie algebras $\mathfrak{sl}_n, \mathfrak{sp}_n$, and $\mathfrak{so}_n$ are defined directly as subalgebras of some $\mathrm{End}(V)$, that V, rather than $\mathfrak{g}$, might seem the most natural one to use; see [37, §1.2]. As you would expect, our original definition and these proposed variants will ultimately be equivalent. To prove this, begin with an elementary result from linear algebra [37, §4.2].

Theorem 1.1.1 (Jordan Decomposition). *Let X be an endomorphism of a finite dimensional complex vector space V.*

(i) *There exist unique operators $X_s, X_n \in \mathrm{End}(V)$ satisifying the conditions: $X = X_s + X_n$, X_s is semisimple, X_n is nilpotent, X_s and X_n commute.*

(ii) *Each of the operators X_s and X_n are polynomials in X with zero constant term. In particular, X_s and X_n commute with any operator commuting with X and stabilize any subspace of V that X stabilizes.*

Recall that $\mathrm{End}(V)$ is a Lie algebra under the usual commutator bracket operation; we will denote this Lie algebra by $\mathfrak{gl}(V)$. Of course, the careful reader will notice that $\mathfrak{gl}(V)$ is *not* semisimple! Viewing $X \in \mathfrak{gl}(V)$ as an element of $\mathrm{End}(V)$, we can directly apply the Jordan decomposition theorem to find $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent and the two operators commute. Alternatively, we can apply the Jordan decomposition theorem to the operator ad_X viewed as an element in $\mathrm{End}(\mathfrak{gl}(V))$ and obtain

$$\mathrm{ad}_X = (\mathrm{ad}_X)_s + (\mathrm{ad}_X)_n,$$

where $(\mathrm{ad}_X)_s$ is semisimple, $(\mathrm{ad}_X)_n$ is nilpotent, and the two operators commute. The connection between these two observations is straightforward [37, §4.2].

Lemma 1.1.2. *Let $X \in \mathfrak{gl}(V)$. If $X = X_s + X_n$ is the Jordan decomposition of X in $\mathrm{End}(V)$, then $\mathrm{ad}_X = \mathrm{ad}_{X_s} + \mathrm{ad}_{X_n}$ is the Jordan decomposition of ad_X in $\mathrm{End}(\mathfrak{gl}(V))$.*

The previous lemma is used to prove our proposed alternate definitions of semisimplicity and nilpotence coincide, but there is one subtle point needing attention. Since a semisimple Lie algebra $\mathfrak{g}$ can be realized as a subalgebra of $\mathrm{End}(\mathfrak{g})$ (via the adjoint representation), we can use (1.1.2) to decompose each $X \in \mathfrak{g}$ as $X = X_s + X_n$, where $[X_s, X_n] = 0$; we provisionally refer to this as the *abstract Jordan decomposition* of X and define X_s (resp. X_n) to be the *semisimple* (resp. *nilpotent*) part of X. However, *a priori*, we do not know that the elements X_s and X_n lie in the subalgebra $\mathfrak{g}$! The fact that this is true justifies our terminology and insures that the abstract and usual Jordan decompositions of X coincide [37, §6.4]. With all these ingredients in hand, we are led to the following result [14, I.6.3], which ties together the various approaches to defining nilpotent and semisimple elements.

Proposition 1.1.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If $X \in \mathfrak{g}$, then the following are equivalent:*

(i) *X is a semisimple (resp. nilpotent) element of $\mathfrak{g}$.*

(ii) *For every finite dimensional representation (ρ, V) of $\mathfrak{g}$, $\rho(X)$ is a semisimpl (resp. nilpotent) element of $\mathrm{End}(V)$.*

(iii) *If $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{End}(V)$, then X is semisimple (resp. nilpotent as an element of $\mathrm{End}(V)$.*

In the sequel, we will frequently make use of the *spectral decompositio* (or *generalized eigenspace decomposition*) of an operator $X \in \mathrm{End}(V)$. Con sequently, we review the relevant linear algebra; details can be found in [36 Recall that the *annihilator* of X in the polynomial ring $\mathbb{C}[t]$ is defined to be

$$\mathrm{Ann}_{\mathbb{C}[t]} X = \{\, f \in \mathbb{C}[t] \mid f(X) = 0 \,\}.$$

Since the annihilator of X is a nonzero principal ideal, it is generated by a moni polynomial f_X, referred to as the *minimal polynomial* of X. By the Fundament Theorem of Algebra, we can factor

$$f_X(t) = (t - \lambda_1)^{r_1} \dots (t - \lambda_k)^{r_k},$$

where $\lambda_1, \dots, \lambda_k \in \mathbb{C}, r_1, \dots, r_k \in \mathbb{N}^+$. In this context, it is useful to recall tha an endomorphism X is semisimple if and only if the minimal polynomial f_X ha distinct roots. Let

$$W_i = \{\, v \in V \mid (X - \lambda_i \cdot I)^{r_i} v = 0 \,\},$$

which is just the kernel of the operator $(X - \lambda_i \cdot I)^{r_i}$.

Lemma 1.1.4 (Spectral Decomposition). *We have $V = W_1 \oplus \ldots \oplus W_k$. For each $1 \le i \le k$, W_i is invariant under X and the minimal polynomial of $X|_{W_i}$ is $(t-\lambda_i)^{r_i}$.*

We refer to the decomposition in (1.1.4) as the *spectral* or *generalized eigenspace decomposition* of V determined by X.

Example 1.1.5. Recall that a key example of the above ideas is the weight space decomposition of a finite dimensional representation. Namely, let (ρ, V) be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ an abelian subalgebra consisting of semisimple elements. For each $X \in \mathfrak{h}$, by (1.1.3)(ii), $\rho(X)$ is a semisimple operator on V, leading to a spectral decomposition:

$$V = \sum_{\{\lambda_1(X),\ldots,\lambda_t(X)\}\in\mathbb{C}} V_{\lambda_i(X)},$$

where $V_{\lambda_i(X)} = \{\, v \in V \mid \rho(X)v = \lambda_i(X)v \,\}$ is the $\lambda_i(X)$-eigenspace of $\rho(X)$. As we vary X over the abelian algebra $\mathfrak{h}$, we can simultaneously diagonalize the commuting family $\{\, \rho(X) \mid X \in \mathfrak{h} \,\}$. In so doing, the eigenvalues become *eigenfunctionals* on $\mathfrak{h}$ and we have

$$V = \sum_{\{\lambda_1,\ldots,\lambda_t\}\in\mathfrak{h}^*} V_{\lambda_i},$$

where $V_{\lambda_i} = \{\, v \in V \mid \rho(X)v = \lambda_i(X)v,\ X \in \mathfrak{h} \}$. This is the familiar $\mathfrak{h}$-*weight space decomposition of* V.

In the sequel, it will be important to work in a slightly more general setting than that of a semisimple Lie algebra. Roughly speaking, this generality will be forced upon us so that certain inductive constructions will be functorial.

A complex finite-dimensional Lie algebra $\mathfrak{g}$ is said to be *reductive* if the adjoint representation ad: $\mathfrak{g} \mapsto \mathrm{End}(\mathfrak{g})$ is completely reducible. An alternate characterization can be given once we recall some terminology. Denote by $\mathfrak{c}$ the *center* of $\mathfrak{g}$; i.e.,

$$\mathfrak{c} = \{ X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0 \}.$$

Obviously $\mathfrak{c}$ is an ideal of $\mathfrak{g}$, as is the *derived algebra* $[\mathfrak{g}, \mathfrak{g}]$. A proof of the following lemma can be found in [14, I.6.4].

Lemma 1.1.6. *The following conditions on a complex Lie algebra $\mathfrak{g}$ are equivalent:*

(i) *$\mathfrak{g}$ is reductive.*

(ii) *$\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$ with $[\mathfrak{g}, \mathfrak{g}]$ a semisimple ideal of $\mathfrak{g}$.*

(iii) *$\mathfrak{g}$ has a finite-dimensional representation whose associated trace form (see [14]) is nondegenerate.*

The most basic example of a reductive Lie algebra is the set of $n \times n$ matrices over $\mathbb{C}$, denoted $\mathfrak{gl}_n$. Every semisimple Lie algebra is trivially reductive. The upper triangular matrices in $\mathfrak{gl}_n$ offer an example of a Lie algebra which is not reductive.

If $\mathfrak{g}$ is reductive, we define $X \in \mathfrak{g}$ to be *semisimple* if ad_X is a semisimple endomorphism of $\mathfrak{g}$. Since the kernel of ad is $\mathfrak{c}$, this definition requires elements of $\mathfrak{c}$ to be semisimple. Although ad_X is (trivially) a nilpotent endomorphism for $X \in \mathfrak{c}$, such an X will not be considered nilpotent unless it is zero. Rather, we will say $X \in \mathfrak{g}$ is *nilpotent* if ad_X is a nilpotent endomorphism of $\mathfrak{g}$ and $X \in [\mathfrak{g}, \mathfrak{g}]$. Of course, when $\mathfrak{g}$ is semisimple, our definitions collapse to the original ones.

Example 1.1.7. Suppose $\mathfrak{g} = \mathfrak{sl}_2 = \{\, X \in \mathfrak{gl}_2 \mid \mathrm{trace}(X) = 0 \,\}$ and $X \in \mathfrak{sl}_2$. Then the characteristic polynomial for

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $(t-\lambda_1)(t-\lambda_2)$, where λ_1 and λ_2 are complex numbers.

Case 1: Suppose $\lambda_1 \neq \lambda_2$. Then the minimal polynomial of X is $f_X(t) = (t-\lambda_1)(t-\lambda_2)$. Since the roots of the minimal polynomial are distinct, X is diagonalizable with eigenvalues λ_1 and λ_2. Since elements of $\mathfrak{sl}_2$ have trace zero, we see that $\lambda_1 = -\lambda_2$.

Case 2: Suppose $\lambda_1 = \lambda_2 = \lambda$. Since X has trace zero, we get $\lambda = 0$. Then the minimal polynomial for X is either $f_X(t) = t$ or $f_X(t) = t^2$. If t is the minimal polynomial, X is the zero matrix. If t^2 is the minimal polynomial, then X is similar to its Jordan form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In summary, we have verified the following lemma.

Lemma 1.1.8. *Any $X \in \mathfrak{sl}_2$ is similar to either $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, for some $\lambda \in \mathbb{C}$. In the former case, X is nilpotent; in the latter, it is semisimple. In addition, X is both semisimple and nilpotent if and only if it is zero.*

1.2 Adjoint Orbits

One of our ultimate goals is to devise some sort of classification scheme for the nilpotent and semisimple elements in a complex semisimple Lie algebra $\mathfrak{g}$. With this in mind, let

$$\mathcal{S} = \{\, \text{semisimple elements of } \mathfrak{g} \,\},\ \mathcal{N} = \{\, \text{nilpotent elements of } \mathfrak{g} \,\}.$$

In view of the previous section, it is clear that

$$\mathcal{S} \cap \mathcal{N} = \{0\}.$$

We are now led to a fundamental problem in the structure theory of reductive Lie algebras, which will be refined momentarily:

oblem. *Classify the elements in $\mathcal{S}$ and $\mathcal{N}$.*

Rather than focusing our attention on the individual elements in $\mathcal{S}$ and $\mathcal{N}$, we will take a cue from (1.1.8) and study conjugacy classes of elements using the natural action of a Lie group attached to our Lie algebra. The power of this viewpoint will become clearer in the sequel, once the correct group is isolated. Roughly speaking, the geometry of a conjugacy class is much richer than the geometry of a single element. In fact, we will later see that these conjugacy classes carry the structure of symplectic manifolds. Given these remarks and accepting the existence of a natural Lie group action on the sets of semisimple and nilpotent elements (spelled out carefully below), we refine the above problem:

fined Problem. *Classify the conjugacy classes in $\mathcal{S}$ and $\mathcal{N}$.*

Example 1.2.1. Before proceeding in the general case, let's consider the case $\mathfrak{g} = \mathfrak{sl}_2$. We introduce the parameter sets:

$$\Lambda_s = \{\, \mathbb{C}/\{\lambda \sim -\lambda\} \,\},\ \Lambda_n = \{\, 0, 1 \,\}. \tag{1.2.2}$$

Denote by GL_2 the set of invertible complex 2×2-matrices. For each $X \in \mathfrak{sl}_2$, form the *conjugacy class*

$$\mathcal{O}_X = GL_2 \cdot X = \{\, A \cdot X \cdot A^{-1} \mid A \in GL_2 \,\}.$$

Note that since trace(AB)=trace(BA), the GL_2-conjugate of an element in $\mathfrak{sl}_2$ is again in $\mathfrak{sl}_2$. If $\lambda \in \Lambda_s$ (resp. $\lambda \in \Lambda_n$), we put

$$X(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} (\text{resp.} Y(\lambda) = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}).$$

Then, by (1.1.8), we have a partition of $\mathfrak{sl}_2$ into GL_2-conjugacy classes

$$\mathfrak{sl}_2 = \left(\bigcup_{\lambda \in \Lambda_s \setminus \{0\}} \mathcal{O}_{X(\lambda)} \right) \cup (\mathcal{O}_{Y(0)} \cup \mathcal{O}_{Y(1)}) \tag{1.2.3}$$

with the properties:

(i) A conjugacy class $\mathcal{O}_X$ is semisimple if and only if $\mathcal{O}_X = \mathcal{O}_{X(\lambda)}$, some $\lambda \in \Lambda_s$.

(ii) A conjugacy class $\mathcal{O}_X$ is nilpotent if and only if $\mathcal{O}_X = \mathcal{O}_{Y(\lambda)}$, some $\lambda \in \Lambda_n$.

(iii) A conjugacy class is simultaneously nilpotent and semisimple if and only if $\mathcal{O}_X = \mathcal{O}_{X(0)} = \mathcal{O}_{Y(0)} = \mathcal{O}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

From these remarks, we see that there are an infinite number of distinct semisimple GL_2-conjugacy classes, but just two nilpotent GL_2-conjugacy classes. In this example, $\mathcal{S} \setminus \{0\}$ (resp. $\mathcal{N}$) coincides with the first (resp. second) term in (1.2.3). Next, let SL_2 be the subgroup of GL_2 consisting of determinant one matrices and $PSL_2 = SL_2/Z$, where $Z = \{I, -I\}$ is the center of SL_2. Observe that GL_2, SL_2, and PSL_2-conjugacy classes coincide, since

$$A \cdot X \cdot A^{-1} = \left(\frac{A}{\sqrt{\det(A)}} \right) \cdot X \cdot \left(\frac{A}{\sqrt{\det(A)}} \right)^{-1} \tag{1.2.4}$$

for all $X \in \mathfrak{sl}_2, A \in GL_2$. Thus we have described the semisimple and nilpotent elements of $\mathfrak{sl}_2$ up to GL_2, SL_2, and PSL_2-conjugacy.

These arguments can be generalized to show the following: Let $X \in \mathfrak{sl}_n$, where

$$\mathfrak{sl}_n = \{\, Y \in \mathfrak{gl}_n \mid \mathrm{trace}(Y) = 0 \,\},$$

then the GL_n, SL_n and PSL_n-conjugacy classes of X all coincide; here, GL_n denotes the $n \times n$ invertible complex matrices, SL_n the subgroup of elements having determinant one and PSL_n the group SL_n modulo its center. The key point is to replace the term $\sqrt{\det(A)}$ by the n^{th} root of $\det(A)$ in (1.2.4).

The careful reader will recognize that (1.2.1) is a special case of a more general theme developed in elementary linear algebra: the GL_n-conjugacy class of a matrix X is just its similarity class. Thus, the "refined classification problem" posed above is a natural generalization to the setting of a semisimple Lie algebra of the similarity problem in linear algebra.

In our classification of nilpotent and semisimple elements, we want to form conjugacy classes using some natural Lie group. Recall that to each complex Lie algebra $\mathfrak{g}$ we can canonically associate its *adjoint group* G_{ad}, which is a connected complex Lie group. It is defined to be the connected subgroup of $GL(\mathfrak{g})$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}$.

If $\mathfrak{g}$ is semisimple, we can realize this group more explicitly. We begin with the automorphism group

$$\mathrm{Aut}(\mathfrak{g}) = \{\phi \in GL(\mathfrak{g}) \mid [\phi(X), \phi(Y)] = \phi[X, Y], \text{ for all } X, Y \in \mathfrak{g}\,\}. \tag{1.2.5}$$

of $\mathfrak{g}$. Notice that for $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $X \in \mathfrak{g}$ we have

$$\phi \cdot \mathrm{ad}_X \cdot \phi^{-1} = \mathrm{ad}_{\phi(X)}. \tag{1.2.6}$$

For this reason, the semisimplicity (or nilpotence) of an element X is equivalent to the semisimplicity (or nilpotence) of all the elements $\phi(X), \phi \in \mathrm{Aut}(\mathfrak{g})$. Consequently, when classifying the semisimple or nilpotent elements in a Lie algebra, we could approach the problem by classifying such elements "up to the action of $\mathrm{Aut}(\mathfrak{g})$"; i.e., by classifying the sets

$$\mathrm{Aut}(\mathfrak{g}) \cdot X = \{\, \phi(X) \mid \phi \in \mathrm{Aut}(\mathfrak{g}) \,\}.$$

This is one of our ultimate goals, at least when X is semisimple or nilpotent, but that is getting ahead of the game, for $\mathrm{Aut}(\mathfrak{g})$ is not in general connected. More precisely, we have [39, IX,exercises]

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})^o \rtimes \Gamma(\mathfrak{g})$$

where $\mathrm{Aut}(\mathfrak{g})^o$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$ and $\Gamma(\mathfrak{g})$ is a finite group (the *graph automorphism group*); whenever $\mathfrak{g}$ is simple, this finite group is either trivial, $\mathbb{Z}/2\mathbb{Z}$, or S_3. We can then also define the adjoint group G_{ad} to be

$$G_{ad} = \mathrm{Aut}(\mathfrak{g})^o. \tag{1.2.7}$$

An alternate approach to the adjoint group is as follows. Let G be *any* connected semisimple complex Lie group with Lie algebra $\mathfrak{g}$; then we can use the differentials of the automorphisms of conjugation to define the *adjoint representation* of G [89]

$$\mathrm{Ad} : G \mapsto \mathrm{Aut}(\mathfrak{g}). \tag{1.2.8}$$

One can now define Ad_G=Image(Ad) to be the adjoint group of $\mathfrak{g}$. This is consistent, in view of the following fundamental result [33].

Lemma 1.2.9. *Let G be a connected semisimple complex Lie group with Lie algebra $\mathfrak{g}$. Then $\mathrm{Ad}_G = \mathrm{Aut}(\mathfrak{g})^o = G_{ad}$. The center Z of G is a finite subgroup and Z=ker(Ad).*

If G is reductive, then its derived group $[G, G]$ is semisimple and we have $G_{ad} \cong [G, G]_{ad}$. In particular, notice that (1.2.7) may fail in the reductive case (e.g., if $\mathfrak{g}$ is abelian, then $G_{ad} = \{1\}$ and $\mathrm{Aut}^o(\mathfrak{g}) = GL(\mathfrak{g})$).

A final fact about the adjoint group will prove important in Chapter 6. Ultimately, we need to understand the collection of connected semisimple complex Lie groups having the same Lie algebra $\mathfrak{g}$. The theory of covering groups tells us that there exists a simply connected complex Lie group G_{sc} with Lie algebra $\mathfrak{g}$,

and every other connected complex Lie group G with Lie algebra $\mathfrak{g}$ is a quotient of G_{sc} by a (finite) central subgroup. In particular, there are only finitely many such G; the adjoint group is the smallest group with Lie algebra $\mathfrak{g}$, while G_{sc} is the largest.

Example 1.2.10. Consider the case of $\mathfrak{g} = \mathfrak{sl}_2$. Then the group SL_2 is diffeomorphic to KAN, where

$$K = SU(2) = \{x \in SL_2 | x^{-1} = \bar{x}^t\};$$
$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \mid t > 0 \right\};$$
$$N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

All three of the groups K, A, N are simply connected, so SL_2 is simply connected. In other words, $G_{sc} = SL_2$ in this example. Moreover, one can check that $\{\pm I\}$ is the center of SL_2, so that $G_{ad} = PSL_2$.

If $\mathfrak{g}$ is a semisimple Lie algebra and $X \in \mathfrak{g}$, we consider the action of the adjoint group on the vector space $\mathfrak{g}$ and define the *orbit through* X by

$$\mathcal{O}_X := G_{ad} \cdot X = \{\phi(X) \mid \phi \in G_{ad}\}. \tag{1.2.11}$$

For emphasis in the sequel, we may sometimes refer to these as *adjoint orbits* or *conjugacy classes* in $\mathfrak{g}$. An adjoint orbit $\mathcal{O}_X$ is called *semisimple* (resp. *nilpotent*) if X is semisimple (resp. nilpotent) in $\mathfrak{g}$. It follows from (1.2.6) that an orbit is semisimple (resp. nilpotent) if and only if every element in the orbit is semisimple (resp. nilpotent).

Caution 1.2.12. The G_{ad}-orbits and the $\mathrm{Aut}(\mathfrak{g})$-orbits need not coincide, in general. Of course, the problem is that two different G_{ad}-conjugacy classes could coalesce under the action of the full automorphism group, leading to a smaller number of orbits. In the case of SL_2, $PSL_2 = \mathrm{Aut}(\mathfrak{sl}_2)$. However, consider the case of $\mathfrak{sl}_3$. The map φ defined by $\varphi(A) = -A^t$, where $(\cdot)^t$ represents the transpose operation, is the composition of two anti-automorphisms of $\mathfrak{sl}_3$, hence lies in $\mathrm{Aut}(\mathfrak{sl}_3)$. Now,

$$\varphi \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

but these matrices have different eigenvalues. Consequently, the matrices

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix} \text{ and } \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

can't be conjugate under G_{ad}, since such conjugation preserves eigenvalues. In fact, in this example, one can show that

$$\mathrm{Aut}(\mathfrak{sl}_3) = \mathrm{Aut}(\mathfrak{sl}_3)^o \rtimes \{I, \varphi\}.$$

It is important to notice that an adjoint orbit $\mathcal{O}_X$ is a homogeneous complex manifold of the form

$$\mathcal{O}_X \cong G_{ad}/G_{ad}^X,$$

where

$$G_{ad}^X = \{\, x \in G_{ad} \,|\, x \cdot X = X \,\} \tag{1.2.13}$$

is the *centralizer of* X in G_{ad}; here (and elsewhere in the sequel) we use the notation $x \cdot X$ to denote $\mathrm{Ad}_x X$. For a quick review of the relevant theory of homogeneous spaces, see [89, §3], [83, §2]. For now, we emphasize that the group G_{ad}^X need not be connected; understanding the consequences of disconnectedness will be important in Chapter 6. The Lie algebra of G_{ad}^X is given by

$$\mathfrak{g}^X = \{\, Y \in \mathfrak{g} \,|\, [X, Y] = 0 \,\} \tag{1.2.14}$$

and is referred to as the *centralizer of* X in $\mathfrak{g}$. We have the following elementary lemma, which is the first of many results illustrating the important role played by centralizers.

Lemma 1.2.15. *Let $\mathcal{O}_X$ be the adjoint orbit containing the element X. Then $\mathfrak{g}^X$ is a subalgebra of $\mathfrak{g}$ and $\dim(\mathcal{O}_X) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^X)$.*

We close this section with a review of the classical groups. The most fundamental example of a complex reductive Lie group is the *complex general linear group*

$$GL_n = \{\, \text{invertible } n \times n \text{ matrices } A = (a_{ij}), a_{ij} \in \mathbb{C} \,\}.$$

The Lie algebra of GL_n is just the set of $n \times n$ complex matrices; i.e., $\mathfrak{gl}(n)$. Recall the operations $(\cdot)^t, \overline{(\cdot)}$ of transpose and conjugate of a complex matrix. Define I_p to be the $p \times p$ identity matrix and put

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We now recall the definitions of the *classical complex groups*:

$$\begin{aligned} SL_n &= \{\, A \in GL_n \,|\, \det(A) = 1 \,\}; \\ SO_{2n+1} &= \{\, A \in SL_{2n+1} \,|\, A^t \cdot A = I_{2n+1} \,\}; \\ Sp_{2n} &= \{\, A \in GL_{2n} \,|\, A^t \cdot J_n \cdot A = J_n \,\}; \\ SO_{2n} &= \{\, A \in SL_{2n} \,|\, A^t \cdot A = I_{2n} \,\}. \end{aligned} \tag{1.2.16}$$

These are all examples of connected complex semisimple Lie groups; their corresponding Lie algebras are denoted $\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, and $\mathfrak{so}_{2n}$ [37, §1.2].

1.3 Coadjoint Orbits

We need to extend the notions of semisimplicity and nilpotence to orbits and elements of $\mathfrak{g}^*$, the algebraic dual of a reductive Lie algebra $\mathfrak{g}$. Unfortunately, there is a sticky point at the outset, since $\mathfrak{g}^*$ has no Lie algebra structure. However, just as the action of the adjoint group leads to an orbit partitioning of $\mathfrak{g}$, the coadjoint representation leads to an analogous partitioning of $\mathfrak{g}^*$. In detail, let G be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$ and recall the *coadjoint representation* of G on $\mathfrak{g}^*$.

$$\mathrm{Ad}^* : G \mapsto \mathrm{Aut}(\mathfrak{g}^*); \mathrm{Ad}_x^*(f)(Y) = f(\mathrm{Ad}_{x^{-1}}(Y)). \tag{1.3.1}$$

The differential of Ad^* yields the *coadjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}^*$

$$\mathrm{ad}^* : \mathfrak{g} \mapsto \mathrm{End}(\mathfrak{g}^*); \mathrm{ad}_X^* f(Y) = -f(\mathrm{ad}_X Y). \tag{1.3.2}$$

The *coadjoint orbit* of $f \in \mathfrak{g}^*$, denoted $\mathcal{O}_f$, is precisely

$$\mathcal{O}_f = \{\, \mathrm{Ad}_x^*(f) \mid x \in G \,\} = \{\, \mathrm{Ad}_x^*(f) \mid x \in G_{ad} \,\}. \tag{1.3.3}$$

The vector space $\mathfrak{g}^*$ now breaks up into a disjoint union of coadjoint orbits and we can pose the problem of classifying such orbits. However, addressing the issue of defining nilpotent or semisimple elements (or coadjoint orbits) does not fit into the context of §1.1; it makes no sense to realize $\mathfrak{g}^*$ as a Lie subalgebra of some $\mathrm{End}(V)$! There are two ways to deal with this dilemma, each of which deserves discussion.

Lazy approach. If $\mathfrak{g}$ is semisimple, the *Killing form* $\kappa(\cdot,\cdot)$ is a nondegenerate invariant symmetric bilinear form; recall that a bilinear form ψ is *invariant* if $\psi(\mathrm{Ad}_x Y, Z) = \psi(Y, \mathrm{Ad}_{x^{-1}} Z)$ for all $x \in G_{ad}, Y, Z \in \mathfrak{g}$, or equivalently if $\psi([X,Y],Z) = \psi(X,[Y,Z])$ for all $X, Y, Z \in \mathfrak{g}$. In the reductive case, we have the following lemma [14, I.6.4].

Lemma 1.3.4. *If $\mathfrak{g}$ is reductive, there exists an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. Moreover, if ψ is such a form and $\mathfrak{s}$ is a simple summand of $[\mathfrak{g},\mathfrak{g}]$, then $\psi(\cdot,\cdot)|_{\mathfrak{s}}$ is a nonzero constant multiple of the Killing form on $\mathfrak{s}$.*

Beginning with a reductive Lie algebra $\mathfrak{g}$, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via an invariant nondegenerate symmetric bilinear form $\psi(\cdot,\cdot)$. Given such a form $\psi : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{C}$, we induce a vector space map

$$i_\psi : \mathfrak{g} \mapsto \mathfrak{g}^*; i_\psi(X) = \psi(X,\cdot). \tag{1.3.5}$$

The nondegeneracy of ψ forces i_ψ to be a bijection (since $\mathfrak{g}$ is finite-dimensional). Furthermore, by the invariance of the form,

$$i_\psi(\mathrm{Ad}_x X) = \psi(\mathrm{Ad}_x X, \cdot) = \psi(X, \mathrm{Ad}_{x^{-1}}(\cdot)) = \mathrm{Ad}_x^*(i_{\psi(X)}).$$

This implies that

$$i_\psi(\mathcal{O}_X) = \mathcal{O}_{i_\psi(X)}, \tag{1.3.6}$$

whence we get a one-to-one correspondence between adjoint orbits in $\mathfrak{g}$ and coadjoint orbits in $\mathfrak{g}^*$. The correspondence in (1.3.6) can now be used to transfer the notion of semisimple or nilpotent adjoint orbits over to coadjoint orbits. In turn, we say an element $f \in \mathfrak{g}^*$ is semisimple (resp. nilpotent) if and only if f lies in $\mathcal{O}_{i_\psi(X)}$, with X a semisimple (resp. nilpotent) element of $\mathfrak{g}$. We will provisionally refer to these definitions as the *lazy definitions* of semisimple and nilpotent elements (or orbits) in $\mathfrak{g}^*$.

Sophisticated approach. An alternative more natural approach to working with coadjoint orbits is as follows. Given $f \in \mathfrak{g}^*$, define

$$\mathfrak{g}^f = \{\, Y \in \mathfrak{g} \,|\, \mathrm{ad}_Y^* f \equiv 0 \,\}, \tag{1.3.7}$$

called the *stabilizer* of f in $\mathfrak{g}$. Since f maps $\mathfrak{g}$ into $\mathbb{C}$, it makes sense to restrict f to $\mathfrak{g}^f$. We say f is *nilpotent* if $f|\mathfrak{g}^f \equiv 0$. We say f is *semisimple* if $\mathfrak{g}^f$ is reductive in $\mathfrak{g}$ (i.e., the adjoint representation of $\mathfrak{g}$ restricted to the subalgebra $\mathfrak{g}^f$ is completely reducible). We refer to these as the *sophisticated definitions* of nilpotent and semisimple elements in $\mathfrak{g}^*$. We then define a coadjoint orbit $\mathcal{O}_f$ to be semisimple (resp. nilpotent) if f is semisimple (resp. nilpotent) in the sophisticated sense. One should note that a coadjoint orbit is semisimple (resp. nilpotent) in the sophisticated sense if and only if every element in the orbit is semisimple (resp. nilpotent) in the sophisticated sense.

We would like to claim that there is no difference between the lazy and the sophisticated definitions described above. We need an elementary lemma.

Lemma 1.3.8. *Let $\mathfrak{g}$ be reductive, ψ a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ and $X \in \mathfrak{g}$. Then*

(i) $\mathfrak{g}^{i_\psi(X)} = \mathfrak{g}^X$.

(ii) *If $Z \in \mathfrak{g}^X$, then* $\mathrm{ad}_X \mathrm{ad}_Z = \mathrm{ad}_Z \mathrm{ad}_X$.

(iii) *Set* $[\mathfrak{g}, X] = \{[W,X] \,|\, W \in \mathfrak{g}\}$. *Then* $\dim(\mathfrak{g}^X) + \dim([\mathfrak{g},X]) = \dim(\mathfrak{g})$.

Proof. Let i_ψ be the vector space isomorphism in (1.3.5). Now

$$\begin{aligned} Y \in \mathfrak{g}^X \quad &\Longleftrightarrow \quad [X,Y] = 0;\\ &\Longleftrightarrow \quad \psi([X,Y],W) = 0, \text{ for all } W \in \mathfrak{g} \text{ (nondegeneracy)};\\ &\Longleftrightarrow \quad \psi(X,[Y,W]) = 0, \text{ for all } W \in \mathfrak{g} \text{ (invariance)};\\ &\Longleftrightarrow \quad i_\psi(X)([Y,W]) = 0, \text{ for all } W \in \mathfrak{g};\\ &\Longleftrightarrow \quad -i_\psi(X)([Y,W]) = (\mathrm{ad}_Y^* i_\psi(X)(W) = 0, \text{ for all } W \in \mathfrak{g};\\ &\Longleftrightarrow \quad Y \in \mathfrak{g}^{i_\psi(X)}. \end{aligned}$$

This proves (i), while (ii) follows from the Jacobi identity. For part (iii), consider the linear map $\mathrm{ad}_X : \mathfrak{g} \mapsto \mathfrak{g}$; observe that $\mathfrak{g}^X$ is the nullspace, and $[\mathfrak{g}, X]$ is the range. □

Remark. It is worth noting that (iii) does not hold on the level of a vector space decomposition; see the proof of the Jacobson-Morozov Theorem in §3.3.

We can now establish the equivalence of our two approaches to defining nilpotent elements in $\mathfrak{g}^*$. We point out that part of the argument given will be used in the proof of the Jacobson-Morozov Theorem, which is a central result in all that follows.

Lemma 1.3.9. *Let $\mathfrak{g}$ be a reductive Lie algebra. The lazy and the sophisticated definitions of nilpotent elements in $\mathfrak{g}^*$ coincide.*

Proof. Construct a vector space isomorphism i_ψ as in (1.3.5), with respect to some fixed nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$; such exists by (1.3.4). Let $f \in \mathfrak{g}^*$, then there exists a unique $X \in \mathfrak{g}$ such that $f = i_\psi(X)$.

Begin by assuming $f = i_\psi(X)$ is nilpotent using the lazy definition. This means that X is a nilpotent element in $\mathfrak{g}$; i.e., $X \in [\mathfrak{g},\mathfrak{g}]$ and ad_X is a nilpotent operator. We know that $\psi(X,Y) = f(Y)$, for all $Y \in \mathfrak{g}$. After possibly passing to the components of X in the simple summands of $[\mathfrak{g},\mathfrak{g}]$, we can appeal to (1.3.4) and assume

$$f(Z) = \psi(X,Z) = \kappa(X,Z) = \mathrm{trace}(\mathrm{ad}_X \mathrm{ad}_Z), \ \ Z \in \mathfrak{g}^f. \tag{1.3.10}$$

By (1.3.8), $(\mathrm{ad}_X \mathrm{ad}_Z)^k = \mathrm{ad}_X^k \mathrm{ad}_Z^k$, for all $k \in \mathbb{N}, Z \in \mathfrak{g}^X$. Since ad_X^k is zero for large k, it follows that $\mathrm{ad}_X \mathrm{ad}_Z$ is nilpotent; so its trace is zero. This implies (1.3.10) is zero so that $f|_{\mathfrak{g}^f} \equiv 0$. This shows f is nilpotent in the sophisticated sense.

Conversely, suppose $f = i_\psi(X)$ is nilpotent in the sophisticated sense, so that

$$f|_{\mathfrak{g}^f} = \psi(X,\cdot)|_{\mathfrak{g}^X} = 0. \tag{1.3.11}$$

By invariance, $\psi([\mathfrak{g}, X], W) = \psi(\mathfrak{g}, [X, W]) = \psi(\mathfrak{g}, \mathrm{ad}_X W) = 0$, for all $W \in \mathfrak{g}^X$. This shows that $[\mathfrak{g}, X]$ is a subspace of $\mathfrak{g}$ which is ψ-orthogonal to $\mathfrak{g}^X$. By the nondegeneracy of the form and (1.3.8)(iii), $\dim(\mathfrak{g}^X)^\perp = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^X) = \dim([\mathfrak{g}, X])$. This shows that $[\mathfrak{g}, X] = (\mathfrak{g}^X)^\perp$. By (1.3.11), $X \in (\mathfrak{g}^X)^\perp$, so $X \in [\mathfrak{g}, X] \subset [\mathfrak{g}, \mathfrak{g}]$. It remains to show that the operator ad_X is nilpotent. Since $X \in [\mathfrak{g}, X]$, we know there exists $H \in \mathfrak{g}$, so that $X = [H, X]$. Consider the operator $\mathrm{ad}_H : \mathfrak{g} \mapsto \mathfrak{g}$ and decompose $\mathfrak{g}$ into generalized ad_H-eigenspaces. In other words, for each $\alpha \in \mathbb{C}$, define

$$\mathfrak{g}_\alpha = \{\, W \in \mathfrak{g} \mid (\mathrm{ad}_H - \alpha \cdot I)^s W = 0, \ \text{some}\, s = s_\alpha \in \mathbb{N} \,\},$$

then

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{C}} \mathfrak{g}_\alpha. \tag{1.3.12}$$

Let $W \in \mathfrak{g}_\alpha$; then by definition

$$(\mathrm{ad}_H - \alpha \cdot I)^s W = 0, \ \text{for some}\, s \in \mathbb{N}^+. \tag{1.3.13}$$

Using the Jacobi identity, we observe that for all $Z \in \mathfrak{g}$

$$\mathrm{ad}_H[X, Z] = [H, [X, Z]] = -[X, [Z, H]] - [Z, [H, X]] = [X, \mathrm{ad}_H Z] + [X, Z].$$

Using the binomial theorem and repeated applications of this identity we find

$$\begin{aligned}
(\mathrm{ad}_H - (\alpha + 1) \cdot I)^s [X, W] &= \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} (\alpha + 1)^{s-k} \mathrm{ad}_H^k [X, W] \\
&= \sum_{k=0}^{s} \binom{s}{k} [X, \sum_{i=0}^{k} \binom{k}{i} \mathrm{ad}_H^i W] (-1)^{s-k} (\alpha + 1)^{s-k} \\
&= [X, \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \alpha^{s-k} \mathrm{ad}_H^k W] \\
&= [X, (\mathrm{ad}_H - \alpha \cdot I)^s W].
\end{aligned}$$

By (1.3.13), this last term is zero, showing $[X, W] \in \mathfrak{g}_{\alpha+1}$; hence $\mathrm{ad}_X \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha+1}$. Since the sum in (1.3.12) only involves a finite number of terms, we see that $(\mathrm{ad}_X)^t = 0$, for some t; i.e., ad_X is a nilpotent operator. This proves that f is nilpotent in the lazy sense. □

For emphasis, we state (without proof) the equivalence of our two definitions of semisimplicity in $\mathfrak{g}^*$. This result will not be needed in the sequel, but you can provide a proof after reviewing the structure theory discussed in Chapter 2.

Lemma 1.3.14. *Let $\mathfrak{g}$ be a reductive Lie algebra. The lazy and the sophisticated definitions of semisimple elements in $\mathfrak{g}^*$ coincide.*

1.4 Orbits as Symplectic Manifolds

Certain arguments can be simplified if we exploit the differential geometry of a coadjoint orbit. In this section, we review the relevant concepts, ultimately describing a complex symplectic structure on any coadjoint orbit. Among other useful properties, the adjoint and coadjoint orbits in the sequel always have even (complex) dimension. All that follows can be developed over the base field $\mathbb{R}$, but we will stick to the complex setting.

To begin with, a *symplectic vector space* is a pair (V, ω) consisting of a complex (finite-dimensional) vector space V and a nondegenerate skew-symmetric bilinear form ω on V. One typically refers to ω as a *symplectic structure* on V.

Example 1.4.1. Let $V = \mathbb{C}^{2n}$ and view vectors in V as column matrices. Define the 2×2 matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and construct the $2n \times 2n$ matrix

$$B_\omega = \begin{pmatrix} S & 0 & 0 & \cdots & 0 \\ 0 & S & 0 & \cdots & 0 \\ 0 & 0 & S & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S \end{pmatrix},$$

which is the block diagonal sum of n copies of S. It is easily checked that

$$\omega(v, w) = v^t \cdot B_\omega \cdot w$$

defines a symplectic structure on V.

Symplectic vector spaces are set apart from standard Euclidean space due to the presence of self-orthogonal subspaces. If W is a subspace of a symplectic vector space (V, ω), we write

$$W^\perp = \{\, v \in V \mid \omega(v, w) = 0, \ \text{for all}\, w \in W \,\};$$

this is the *orthogonal subspace of* W, relative to ω. We say that W is *isotropic* if $W \subset W^\perp$. For example, the skew-symmetry of the form implies that the one-dimensional subspace generated by a nonzero vector is isotropic. For more details on the basic properties of symplectic vector spaces, including the next lemma, we refer the reader to [38].

Lemma 1.4.2. *Let (V,ω) be a symplectic vector space. Then $\dim(V) = 2n$ for some $n \in \mathbb{N}$. If W is an isotropic subspace of V, then $\dim(W) \leq n$.*

We now shift to the setting of a complex manifold M. We will denote by $T_x^*(M)$ (resp. $T^*(M)$) the complex cotangent space at x (resp. complex cotangent bundle over M). In order to make sense of the following definition, we remind you that a smooth *complex 2-form* ω on M is a smooth section of the exterior bundle $\bigwedge^2(T^*(M))$. For each $x \in M$, we have a canonical vector space isomorphism between $\bigwedge^2(T_x^*(M))$ and the vector space of skew-symmetric bilinear forms on $T_x(M)$. Thus, given a section ω of $\bigwedge^2(T^*(M))$, we obtain a family $\{\omega_x\}_{x\in M}$ of skew-symmetric bilinear forms. A *symplectic manifold* is a pair (M,ω), consisting of a complex manifold M and a smooth closed complex valued 2-form ω on M, such that for all $x \in M$, ω_x is nondegenerate. We refer to ω as a *symplectic structure* on M. Otherwise put, for each $x \in M$, we are provided with a nondegenerate skew-symmetric bilinear form ω_x on $T_x(M)$ and these fit together to form a smooth global complex valued 2-form on M. A submanifold N of M is called an *isotropic submanifold* if $T_x(N) \subset T_x(N)^\perp$, for every $x \in N$; here, $\perp$ is relative to ω_x. The following result is immediate from (1.4.2).

Lemma 1.4.3. *If (M,ω) is a symplectic manifold, then $\dim(M)$ is even. If N is an isotropic submanifold of M, then $\dim(N) \leq \frac{1}{2}\dim(M)$.*

Homogeneous complex manifolds $M = G/H$ play a fundamental role in modern Lie theory. Quite often, these manifolds additionally carry a symplectic structure. A fundamental result ensures that locally, all such homogeneous symplectic manifolds of the same finite dimension "look the same". We will now isolate this local structure by exhibiting an important class of symplectic manifolds.

We consider a connected complex Lie group G with Lie algebra $\mathfrak{g}$ and consider the coadjoint orbits of G in $\mathfrak{g}^*$; although (1.3.3) was couched in the reductive setting, the notion of such orbits makes sense in general. Let $\mathcal{O}_f$ be the coadjoint orbit in $\mathfrak{g}^*$ through f. The element f determines the subgroup

$$G^f = \{\, x \in G \mid Ad_x^* f = f \,\} \tag{1.4.4}$$

having Lie algebra (1.3.7). Define

$$\omega_f : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{C};\ \omega_f(X,Y) = f([X,Y]). \tag{1.4.5}$$

Using the Jacobi identity, we see that ω_f is a skew-symmetric bilinear form on $\mathfrak{g}$. Observe that this form is degenerate and has radical $\omega_f = \mathfrak{g}^f$. We conclude that

ω_f defines a nondegenerate symplectic form on $\mathfrak{g}/\mathfrak{g}^f$. We make the coadjoint orbit into a complex homogeneous manifold using the structure it inherits under the identification

$$\mathcal{O}_f \cong G/G^f. \tag{1.4.6}$$

Consequently, we see that ω_f provides a symplectic vector space structure on

$$T_f(\mathcal{O}_f) \cong T_1(G/G^f) \cong \mathfrak{g}/\mathfrak{g}^f.$$

The main result is

Theorem 1.4.7. *Let $\mathcal{O}_f$ be a coadjoint orbit in $\mathfrak{g}^*$. Then the symplectic bilinear form ω_f on $\mathfrak{g}/\mathfrak{g}^f$ extends to a complex symplectic structure on $\mathcal{O}_f$.*

For a proof, we refer you to [88] or [32]. The following useful corollary is an immediate consequence.

Corollary 1.4.8. *The dimension of a coadjoint orbit is even. The same holds for adjoint orbits if $\mathfrak{g}$ is reductive.*

As a final point of interest, we now refine the motivational remark preceeding our discussion of coadjoint orbits, highlighting their geometric importance. Since this result is not needed in the sequel, we refer you to [32, IV.7].

Theorem 1.4.9. *Each homogeneous complex symplectic manifold (relative to a semisimple Lie group) is a covering of some coadjoint orbit.*

2 Semisimple Orbits

For the purpose of contrast and comparison with the nilpotent case, we devote this chapter to a look at the classification of semisimple orbits in $\mathfrak{g}$. The basic facts are easily understood but go to the heart of the structure theory for semisimple Lie algebras and require hard work to establish. This material is reviewed in §2.1, with emphasis placed on understanding the centralizer structure for semisimple elements. These ideas are central in §2.2, where we classify the semisimple orbits in a reductive Lie algebra. The main result says that semisimple orbits are parametrized by points in a fundamental domain for the action of the Weyl group on a Cartan subalgebra. In particular, there are infinitely many semisimple orbits. We conclude with a brief discussion of the topology of semisimple orbits. It should be emphasized that many of the ideas discussed in this chapter are needed in our study of nilpotent orbits.

2.1 Some Structure Theory

It will be important to know that the centralizer $\mathfrak{g}^X$ of a semisimple element X is reductive and that its structure is easily described in terms of a root space decomposition of $\mathfrak{g}$. The desired result is contained in [37] as an exercise. However, because of its fundamental role in the sequel, we include the necessary details. This is followed by a quick review of some key facts concerning Cartan subalgebras.

Let $\mathfrak{g}$ be a reductive Lie algebra. By definition, a *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra consisting of semisimple elements; these algebras are referred to as "maximal toral subalgebras" in [37], where it is also shown that this definition of Cartan subalgebra agrees with the more usual one. You can easily check ([37, §8.1]) that Cartan subalgebras exist, and that in fact any semisimple element is contained in one. Having fixed such a subalgebra $\mathfrak{h}$, we are led to the familiar *root space decomposition* of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{2.1.1}$$

where Φ is the system of roots for $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{g}_\alpha$ is the α-root space. (Of course, this is just the spectral decomposition of (1.1.5), viewing $\mathfrak{g}$ as a module over $\mathfrak{h}$ via the adjoint representation.) You may wish to brush up on the relevant facts about (2.1.1) in [37, §8]. Recall the set S of semisimple elements in §1.2.

emma 2.1.2. *Let $\mathfrak{g}$ be reductive and $X \in S$. Then $\mathfrak{g}^X$ is reductive and there exists a Cartan subalgebra $\mathfrak{h}$ containing X. If Φ denotes the roots for the pair $(\mathfrak{g}, \mathfrak{h})$, then $\mathfrak{g}^X = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_X} \mathfrak{g}_\alpha$, where $\Phi_X = \{\, \alpha \in \Phi \mid \alpha(X) = 0 \,\}$.*

Proof. We noted above that X lies in a Cartan subalgebra $\mathfrak{h}$; this leads to the root space decomposition (2.1.1), relative to the pair $(\mathfrak{g}, \mathfrak{h})$. On the one hand, if $W \in \mathfrak{g}^X$ lies in the root space $\mathfrak{g}_\alpha$, then by definition, $[H, W] = \alpha(H)W$, for all $H \in \mathfrak{h}$. But the element $H = X \in \mathfrak{h}$ must centralize W and so we get $0 = \alpha(X)W$. Let $\Phi_X = \{\alpha \in \Phi \mid \alpha(X) = 0\,\}$; then these remarks show that

$$\mathfrak{g}^X = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_X} \mathfrak{g}_\alpha.$$

It is an easy exercise to verify that Φ_X satisfies the axioms of a root system, leading to a base of simple roots Δ_X for Φ_X. (We caution you that even if Φ is an irreducible root system, Φ_X may be reducible; see (2.2.8) below.)

To show $\mathfrak{g}^X$ is reductive, we will exhibit the decomposition (1.1.6)(ii). Begin by defining

$$\mathfrak{h}_1 = \bigcap_{\alpha \in \Phi_X} \ker \alpha. \tag{2.1.3}$$

Since the span of the roots in Φ_X has dimension equal to the rank of the root system Φ_X, we get that $\dim(\mathfrak{h}_1) = \dim(\mathfrak{h}) - \operatorname{rank}(\Phi_X)$. For each $\alpha \in \Phi$, pick elements $H_\alpha \in \mathfrak{h}$ as in [37, §8.3]. Now define

$$\mathfrak{h}_2 = \sum_{\alpha \in \Delta_X} \mathbb{C} \cdot H_\alpha \tag{2.1.4}$$

Clearly, $\dim(\mathfrak{h}_2) = \operatorname{rank}(\Phi_X)$. We now have the following refined decomposition of $\mathfrak{g}^X$:

$$\mathfrak{g}^X = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \sum_{\alpha \in \Phi_X} \mathfrak{g}_\alpha. \tag{2.1.5}$$

Finally, using the standard properties of root systems in ([37, §8]), we find that $\mathfrak{h}_1 = \mathfrak{c} =$ center of $\mathfrak{g}^X$, and $[\mathfrak{g}^X, \mathfrak{g}^X] = \mathfrak{h}_2 \oplus \sum_{\alpha \in \Phi_X} \mathfrak{g}_\alpha$. We can easily check that the Killing form of $[\mathfrak{g}^X, \mathfrak{g}^X]$ is nondegenerate, whence $[\mathfrak{g}^X, \mathfrak{g}^X]$ is semisimple. We have exhibited the decomposition of (1.1.6)(ii); hence, $\mathfrak{g}^X$ is reductive. ∎

Remark 2.1.6. It is worth emphasizing that the proof of (2.1.2) depends on the fact that X is semisimple. For example, in §3.7 we will describe the centralizer of a nonzero nilpotent element; such a centralizer is not reductive.

For the remainder of this section, we review some basic facts concerning Cartan subalgebras. This material is crucial to the classification of semisimple orbits in §2.2. We begin with a construction which ultimately produces all Cartan subalgebras. Let t be an indeterminate and $X \in \mathfrak{g}$. Define

$$\Omega_X(t) = \text{characteristic polynomial of } \mathrm{ad}_X = \det(t \cdot I - \mathrm{ad}_X).$$

A proof of the next lemma can be found in [83, §3.9].

Lemma 2.1.7. *Let $m = \dim \mathfrak{g}$; then*

$$\Omega_X(t) = \sum_{0 \le i \le m} (-1)^{m-i} p_i(X) t^i,$$

where $p_i(X)$ is a homogeneous polynomial on $\mathfrak{g}$ of degree $m - i$, $0 \le i \le m$. If $\phi \in Aut(\mathfrak{g})$, then $\Omega_{\phi(X)}(t) = \Omega_X(t)$ and each polynomial $p_i(X)$ is $Aut(\mathfrak{g})$-invariant.

Since $\mathrm{ad}_X X{=}0$, the endomorphism ad_X is singular; consequently, $p_0 = \det(\mathrm{ad}_X) = 0$. This implies that there exists a smallest positive integer ℓ in the interval $[1, m]$ such that

$$p_\ell \not\equiv 0.$$

We call ℓ the *rank* of $\mathfrak{g}$. The distinguished polynomial $p_\ell(X)$ is used to define the set

$$\mathfrak{g}' = \{\, X \in \mathfrak{g} \mid p_\ell(X) \neq 0 \,\}, \tag{2.1.8}$$

which is called the set of *regular semisimple elements* (or sometimes *generic elements*) in $\mathfrak{g}$. The next lemma justifies this terminology. At the same time, it provides a general technique to construct Cartan subalgebras and illustrates the important role played by centralizers; see [83, §4.1].

Lemma 2.1.9. *If $X \in \mathfrak{g}'$, then X is a semisimple element and $\mathfrak{g}^X$ is a Cartan subalgebra.*

We can gain some insight into the structure of $\mathfrak{g}'$ (and, in turn, appreciate the abundance of Cartan subalgebras) by studying the polynomial p_ℓ. This is easily done on a fixed Cartan subalgebra; see [83, §4.1].

Lemma 2.1.10. *Let $\mathfrak{g}$ be reductive, $\mathfrak{h}$ a Cartan subalgebra, and Φ the associated system of roots. Choose a system of positive roots Φ^+ and let $r = |\Phi^+|$. Then $p_\ell(X) = (-1)^r \left\{\prod_{\alpha\in\Phi^+} \alpha(X)\right\}^2$ for $X \in \mathfrak{h}$.*

Using this lemma, we can begin to visualize $\mathfrak{g}'$. Relative to the fixed choices in (2.1.10), we have

$$\begin{aligned}\mathfrak{h}' &:= \mathfrak{g}' \cap \mathfrak{h}\\ &= \{X \in \mathfrak{h} \,|\, p_\ell(X) \neq 0\}\\ &= \{X \in \mathfrak{h} | \prod_{\alpha\in\Phi^+} [\alpha(X)]^2 \neq 0\}\\ &= \{X \in \mathfrak{h} \,|\, \alpha(X) \neq 0, \text{ for all } \alpha \in \Phi^+\}.\end{aligned}$$

It is now possible to describe $\mathfrak{g}'$ as the union of semisimple adjoint orbits through points in $\mathfrak{h}'$. To prove this we need a fundamental result in the structure theory of reductive Lie algebras. A first corollary will ensure that every Cartan subalgebra is of the form constructed in (2.1.9), while the second corollary gives the desired description of $\mathfrak{g}'$. For details, refer to [37, §16] and [83, §4.1].

Theorem 2.1.11. *If $\mathfrak{g}$ is reductive and $\mathfrak{h}_1, \mathfrak{h}_2$ are Cartan subalgebras, then there exists $x \in G_{ad}$ so that $x\cdot\mathfrak{h}_1 = \mathfrak{h}_2$. In other words, all Cartan subalgebras are conjugate. Moreover, the dimension of any Cartan subalgebra of $\mathfrak{g}$ equals the rank of $\mathfrak{g}$.*

Corollary 2.1.12. *If $\mathfrak{g}$ is reductive and $\mathfrak{h}$ is a Cartan subalgebra, then there exists a regular semisimple element X in $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}^X$.*

Corollary 2.1.13. *The regular semisimple set $\mathfrak{g}'$ is connected, Zariski-open, and dense. Furthermore, $\mathfrak{g}'$ is G_{ad}-invariant and may be written as*

$$\mathfrak{g}' = \bigcup_{x\in G_{ad}} x\cdot\mathfrak{h}' = \bigcup_{X\in\mathfrak{h}'} \mathcal{O}_X.$$

Proof. From the definition of $\mathfrak{g}'$ as the complement of the zero set of the polynomial p_ℓ on $\mathfrak{g}$, it is clear that the regular semisimple set is Zariski-open and dense (see [37, §23, appendix] for the basic facts about the Zariski topology). The connectedness follows from a general result: if V is a finite dimensional complex vector space and f is a polynomial on V, then $\{v \,|\, f(v) \neq 0\}$ is path connected. (To see this let v, w belong to this set. Then $\lambda v + (1-\lambda)w$ also does for all but finitely many $\lambda \in \mathbb{C}$. Now observe that $\mathbb{C}$ minus finitely many points is still path connected.)

The invariance of the polynomials $p_i(X)$ in (2.1.7) under the action of the full automorphism group $\mathrm{Aut}(\mathfrak{g})$ tells us that the complement of the zero set of each of these polynomials is $\mathrm{Aut}(\mathfrak{g})$-invariant; in particular, $\mathfrak{g}'$ is G_{ad}-invariant. This tells us that $\mathfrak{g}'$ is a union of adjoint orbits. Clearly, if $X \in \mathfrak{h}'$, then any conjugate of X is also regular semisimple, hence the entire orbit $G_{ad}\cdot X = \mathcal{O}_X \subset \mathfrak{g}'$. On the other hand, if $X \in \mathfrak{g}'$, then (2.1.9) tells us that X lies in the Cartan subalgebra $\mathfrak{g}^X$. By (2.1.11), there exists $x \in G_{ad}$, so that $x\cdot\mathfrak{h} = \mathfrak{g}^X$, which implies $X \in x\cdot\mathfrak{h}$. Since the set of regular semisimple elements is G_{ad}-invariant, $X \in x\cdot\mathfrak{h}'$. □

Example 2.1.14. In the case of $\mathfrak{sl}_2$, introduce the usual basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $Z = tH + cX + dY \in \mathfrak{sl}_2$. We compute that the Casimir polynomial $\omega(Z) := \mathrm{trace}(\mathrm{ad}_Z)^2 = 8(t^2 + cd)$. Next, for any semisimple Lie algebra $\mathfrak{g}$, the polynomial p_{m-2} in (2.1.7) has the form

$$\begin{aligned} p_{m-2}(Z) &= \frac{1}{2}\{[\mathrm{trace}(\mathrm{ad}_Z)]^2 - [\mathrm{trace}(\mathrm{ad}_Z)^2]\}\\ &= -\frac{1}{2}[\mathrm{trace}(\mathrm{ad}_Z)^2],\end{aligned}$$

since the adjoint representation maps $\mathfrak{g}$ into $\mathfrak{sl}(\mathfrak{g})$. In the setting of this example, $m = 3$ and $\ell = 1$, so we find that $p_1(Z) = -\frac{1}{2}\omega(Z)$. This implies that $Z \in \mathfrak{sl}_2$ is regular semisimple if and only if $\omega(Z) \neq 0$. In summary, the regular semisimple elements for $\mathfrak{sl}_2$ can be viewed as the complement of the zero set of the polynomial $f(\alpha, \beta, \gamma) = \alpha^2 + \beta\gamma$ in $\mathbb{C}^3$.

For the remainder of this section, we return to (2.1.2) and (2.1.11). It follows from (2.1.11) that

$$\dim(\mathfrak{g}^X) \geq \ell \text{ for all } X \in \mathcal{S};$$

i.e., ℓ is the minimum possible centralizer dimension of a semisimple element. In fact, a more general statement is true.

Lemma 2.1.15. *If $\mathfrak{g}$ is reductive, then $\dim(\mathfrak{g}^X) \geq \ell$ for all $X \in \mathfrak{g}$.*

Proof. Consider the set $\mathcal{M}$ of elements X in $\mathfrak{g}$ such that $\dim(\mathfrak{g}^X)$ is minimal; equivalently, those X such that the rank of the endomorphism ad_X (in $\mathrm{End}(\mathfrak{g})$) is maximal. The set $\mathcal{M}$ is Zariski-open, whence it must meet the dense open set $\mathfrak{g}'$ by (2.1.13); now (2.1.11) implies the lemma. □

This leads to a definition. We will define an element X to be *regular* if the dimension of the centralizer of X in $\mathfrak{g}$ is minimal, i.e., if its dimension is ℓ (by the previous lemma). Denote by $\mathcal{R}$ the set of all regular elements. Note that by

(1.2.15) an element X is regular if and only if the dimension of the adjoint orbit $\mathcal{O}_X$ is maximal. It now follows from the above discussion that

$$\mathfrak{g}' = \mathcal{S} \cap \mathcal{R} \subset \mathcal{R},$$

but we do *not* have $\mathcal{R} \subset \mathcal{S}$.

In fact, as you will see in Chapter 4, there exist regular nilpotent elements, and $\mathcal{R} \cap \mathcal{N}$ is a single adjoint orbit.

Caution. In some references, our regular semisimple elements are simply called regular; for example, see [90], [83].

Classification of Semisimple Orbits

One consequence of our discussion in the previous section is a first approximation to the classification of semisimple orbits in $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ and define a map

$$\tilde{\mu} : \mathfrak{h} \to \{\text{semisimple orbits}\};\ \tilde{\mu}(X) = \mathcal{O}_X. \tag{2.2.1}$$

Then the next corollary tells us that $\tilde{\mu}$ is surjective.

Corollary 2.2.2. *Let $\mathfrak{h}$ be a Cartan subalgebra of a reductive Lie algebra $\mathfrak{g}$ and assume $\mathcal{O}$ is a semisimple adjoint orbit. Then $\mathcal{O} = \mathcal{O}_X$, for some $X \in \mathfrak{h}$.*

Proof. By definition $\mathcal{O} = \mathcal{O}_{X'}$ for some semisimple element $X' \in \mathcal{S}$, and X' lies in a Cartan subalgebra $\mathfrak{h}'$, which is conjugate to $\mathfrak{h}$ by (2.1.11). Thus X' is conjugate to some $X \in \mathfrak{h}$. □

In this section, we will identify the fibers of $\tilde{\mu}$ and obtain the desired classification of semisimple orbits. This requires that we make an observation about the Weyl group. To see the idea, recall that our discussion in §1.2 shows that the semisimple orbits in $\mathfrak{sl}_2$ are parametrized by the set $\Lambda_s = \mathbb{C}/\{\lambda \sim -\lambda\}$. This parameter space can be realized using the Weyl group. To see this, recall that one Cartan subalgebra of $\mathfrak{sl}_2$ is $\mathbb{C}H$, where H is as in (2.1.14). The Weyl group $W = \{1, s_\alpha\}$ acts on $\mathfrak{h}$ and $\mathfrak{h}^*$ by reflecting through the origin: $s_\alpha(X) = -X$. In other words, we may identify Λ_s with $\mathfrak{h}/W$. This suggests our general result, which depends on the following lemma; see [83, §4].

Lemma 2.2.3. *Let $\mathfrak{g}$ be semisimple and $\mathfrak{h}$ a Cartan subalgebra. If W is the Weyl group of the root system attached to the pair $(\mathfrak{g}, \mathfrak{h})$, then there is a natural isomorphism*

$$W \cong N_{G_{ad}}(\mathfrak{h})/C_{G_{ad}}(\mathfrak{h}),$$

where

$$\begin{aligned} N_{G_{ad}}(\mathfrak{h}) &= \{x \in G_{ad} \mid x \cdot \mathfrak{h} = \mathfrak{h}\}, \quad \text{and} \\ C_{G_{ad}}(\mathfrak{h}) &= \{x \in G_{ad} \mid x \cdot Y = Y \text{ for every } Y \in \mathfrak{h}\} \end{aligned}$$

denote the normalizer and centralizer of $\mathfrak{h}$ in G_{ad}, respectively. This isomorphism is realized via the natural W-action on $\mathfrak{h}^*$, transferred to $\mathfrak{h}$ via the contragredient representation. In particular, every $w \in W$, regarded as an automorphism of $\mathfrak{h}$, extends to an automorphism of $\mathfrak{g}$.

Theorem 2.2.4 (Classification). *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and W the associated Weyl group. Then the set of semisimple orbits is in bijective correspondence with $\mathfrak{h}/W$.*

Proof. Consider the map

$$\mu : \mathfrak{h}/W \longmapsto \{\mathcal{O}_X \mid X \in \mathcal{S}\};\ \mu([Y]) = \mathcal{O}_Y, \tag{2.2.5}$$

where Y is any representative of the equivalence class $[Y]$. Then $\mathcal{O}_Y$ is clearly a semisimple orbit. Any two representatives of the class $[Y]$, say Y and Y_1, are G_{ad}-conjugate by (2.2.3), so $\mathcal{O}_Y = \mathcal{O}_{Y_1}$. This shows the map μ is well defined.

That μ is onto follows since $\tilde{\mu}$ (defined in (2.2.1)) is onto.

Finally, we must show that μ is injective. Suppose that $\mu([X_1]) = \mu([X_2])$, then $\mathcal{O}_{X_1} = \mathcal{O}_{X_2}$. So there exists $x \in G_{ad}$ such that $x \cdot X_1 = X_2$. Notice that $\mathfrak{h}$ and $x \cdot \mathfrak{h}$ are two Cartan subalgebras of $\mathfrak{g}$ containing X_2. Since Cartan subalgebras are abelian, this says that both $\mathfrak{h}$ and $x \cdot \mathfrak{h}$ centralize X_2; i.e.,

$$\mathfrak{h},\ x \cdot \mathfrak{h} \subset \mathfrak{g}^{X_2}. \tag{2.2.6}$$

These are both Cartan subalgebras of $\mathfrak{g}^{X_2}$, which is reductive by (2.1.2). We may apply (2.1.11) to conclude the two Cartan subalgebras in (2.2.6) are conjugate *inside* $\mathfrak{g}^{X_2}$. This tells us there exists an element y of the adjoint group of $\mathfrak{g}^{X_2}$ so that $y \cdot x \cdot \mathfrak{h} = \mathfrak{h}$ and $y \cdot x \cdot X_1 = y \cdot X_2 = X_2$, since y centralizes X_2. This tells us that X_1 is conjugate to X_2 by an element of $N_{G_{ad}}(\mathfrak{h})$. By (2.2.3), X_1 is conjugate to X_2 by an element of W. In other words, $[X_1] = [X_2]$, so that μ is one to one. □

Remarks. (i) This result extends easily to reductive $\mathfrak{g}$, with essentially the same proof.

(ii) It follows that semisimple orbits may be parametrized by any fundamental domain for the action of W on $\mathfrak{h}$. We give an example of such a domain. Choose a set Δ of simple roots for the root system Φ of $\mathfrak{g}$ relative to $\mathfrak{h}$, and then take the domain to be

$$D_\Delta = \left\{ x \in \mathfrak{h} \;\middle|\; \begin{array}{l} \mathrm{Re}(\alpha(x)) \geq 0 \text{ and whenever } \mathrm{Re}(\alpha(x)) = 0, \\ \text{then } \mathrm{Im}(\alpha(x)) \geq 0, \text{ for all } \alpha \in \Delta \end{array} \right\}.$$

This D_Δ may be regarded as the complexification of the (closed) fundamental dominant Weyl chamber of [37, §10].

Corollary 2.2.7. *There exist infinitely many distinct semisimple orbits.*

We emphasize this corollary, which was already proved for $\mathfrak{g} = \mathfrak{sl}_2$ in §1.2. This stands in contrast to one of the main results concerning nilpotent orbits; namely, that there are only finitely many of them.

Example 2.2.8. We conclude this section with a discussion of the content of (2.2.4) in the case of $\mathfrak{sl}_4$; you can easily extrapolate to the $\mathfrak{sl}_n$ case. The diagonal matrices in $\mathfrak{sl}_4$, denoted $\mathfrak{h}$, form a 3-dimensional Cartan subalgebra. For each $1 \leq i \leq 4$, define a linear functional e_i in the dual space $\mathfrak{h}^*$ by

$$e_i \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{pmatrix} = h_i.$$

Then the root space decomposition relative to this choice of $\mathfrak{h}$ leads to the standard root system Φ of type A_3. The standard choices of positive and simple roots are given by

$$\Phi^+ = \{ e_i - e_j \mid 1 \leq i < j \leq 4 \} \text{ and } \Delta = \{ e_i - e_{i+1} \mid 1 \leq i \leq 3 \}.$$

Consider the following four matrices:

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Each of these elements is semisimple and lies in $\mathfrak{h}$. We now describe the adjoint orbits $\mathcal{O}_{X_k}$ for $1 \leq k \leq 4$.

Case of X_1: In this case, notice that $\alpha(X_1) \neq 0$ for all $\alpha \in \Delta$. By (2.1.10), this implies that $X_1 \in \mathfrak{h}'$; consequently, by (2.1.2), $\mathfrak{g}^{X_1} = \mathfrak{h}$ is a Cartan subalgebra. By (1.2.15),

$$\dim(\mathcal{O}_{X_1}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^{X_1}) = 15 - 3 = 12.$$

Note that (2.1.13) says that if we take the union of all these 12-dimensional orbits over the 3-dimensional parameter set $\mathfrak{h}'$ we get the 15-dimensional set $\mathfrak{g}'$.

Case of X_2: In this case, if $\alpha \in \Phi$, notice that $\alpha(X_2) = 0$ if and only if $\alpha = \pm(e_1 - e_2)$. This implies that $X_2 \notin \mathfrak{h}'$ and $\Phi_{X_2} = \{ \pm(e_1 - e_2) \}$. By (2.1.2), $\mathfrak{g}^{X_2} = \mathfrak{h} \oplus \mathfrak{g}_{e_1 - e_2} \oplus \mathfrak{g}_{e_2 - e_1}$ is reductive with derived algebra $[\mathfrak{g}^{X_2}, \mathfrak{g}^{X_2}] = \mathfrak{sl}_2$ and (1.2.15) implies

$$\dim(\mathcal{O}_{X_2}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^{X_2}) = 15 - 5 = 10.$$

Case of X_3: In this case, $\Phi_{X_3} = \{ \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3) \}$. By (2.1.2), $\mathfrak{g}^{X_3}$ is reductive with derived algebra $[\mathfrak{g}^{X_3}, \mathfrak{g}^{X_3}] = \mathfrak{sl}_3$ and (1.2.15) implies

$$\dim(\mathcal{O}_{X_3}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^{X_3}) = 15 - 9 = 6.$$

Case of X_4: In this case, $\Phi_{X_4} = \{ \pm(e_1 - e_2), \pm(e_3 - e_4) \}$. By (2.1.2), $\mathfrak{g}^{X_4}$ is reductive with derived algebra $[\mathfrak{g}^{X_4}, \mathfrak{g}^{X_4}] = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$; this provides an example where Φ_X is reducible and Φ is irreducible. By (1.2.15),

$$\dim(\mathcal{O}_{X_4}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^{X_4}) = 15 - 7 = 8.$$

Finally, notice that the dimension of the adjoint orbit $\mathcal{O}_{X_k}$ is even for all k, as it must be by (1.4.8).

2.3 Basic Topology of Semisimple Orbits

Aside from the fact that semisimple orbits are rather easily parametrized, they also may be topologically characterized among all adjoint orbits.

Theorem 2.3.1 (Borel, Harish-Chandra). *Assume $\mathfrak{g}$ is a reductive Lie algebra. An element $X \in \mathfrak{g}$ is semisimple if and only if $\mathcal{O}_X$ is closed.*

We will not need this result in the sequel; see [90, 1.3.5.5] for a proof. Actually, semisimple adjoint orbits of any connected algebraic group are closed; see [79, 4.4.5]. The theorem implies that the closure of any semisimple orbit is smooth. This is *not* the case for nilpotent orbits, and understanding the geometry of nilpotent orbit closures is a topic of current research.

A second important fact related to semisimple orbits concerns the structure of the centralizer subgroups in G_{ad}. Namely, for X in $\mathfrak{g}$, we define

$$G_{ad}^X = \{\, x \in G_{ad} \,|\, x \cdot X = X \,\}. \tag{2.3.2}$$

Because the adjoint orbit $\mathcal{O}_X$ may be identified with the homogeneous space G_{ad}/G_{ad}^X, its topology will be affected by the connectedness of G_{ad}^X; this is discussed fully in §6.1. In the case of semisimple orbits, the following result is of central importance. The proof of this nontrivial fact can be found in [79, 7.3.5].

Theorem 2.3.3. *Assume $\mathfrak{g}$ is a reductive Lie algebra. If X is a semisimple element of $\mathfrak{g}$, then G_{sc}^X is connected.*

Actually, the proof shows that the same result holds for any algebraic Lie algebra, not just a reductive one. We will make no use of this, however. In Chapter 6, we will use this result to show that the fundamental group of *any* adjoint orbit is isomorphic to the equivariant fundamental group of some nilpotent orbit (in a possibly different reductive Lie algebra). As a consequence, we get

Corollary 2.3.4. *If X is a semisimple element of $\mathfrak{g}$, then $\mathcal{O}_X$ is simply connected.*

The topology of nilpotent orbits is much more complicated and interesting. In §6.1, we will compute the fundamental group of each nilpotent orbit, which is often nontrivial. In fact, a slight variant of the fundamental group–called the G_{ad}-equivariant fundamental group–is important in relating nilpotent orbits to Weyl group representations via the Springer correspondence. We will discuss this in more detail in Chapter 10.

3 The Dynkin-Kostant Classification

In this chapter, we derive the classical Dynkin-Kostant classification of complex nilpotent orbits. In the end, we establish a one-to-one correspondence between nilpotent orbits and a certain finite collection of semisimple orbits, which we call "distinguished":

$$\{\text{nilpotent orbits in } \mathfrak{g}\} \Longleftrightarrow \{\text{distinguished semisimple orbits}\}.$$

Whereas the set of all semisimple orbits is infinite (recall (2.2.7)), we will see that the set of distinguished semisimple orbits is finite. These distinguished orbits can then be catalogued using their so-called weighted Dynkin diagrams. Each of these consists of the Dynkin diagram together with certain nonnegative integer labels attached to the nodes. Describing the possible labels is tantamount to parametrizing the nilpotent orbits in $\mathfrak{g}$.

Carrying out the above scheme involves several steps. First, we establish a one-to-one correspondence between nilpotent orbits and conjugacy classes of certain small semisimple subalgebras of $\mathfrak{g}$. The proof that this correspondence is surjective depends on the famous Jacobson-Morozov Theorem and a conjugacy theorem of Kostant proves it is injective. The second step is to show that conjugacy classes of these small subalgebras are in one-to-one correspondence with certain distinguished semisimple orbits; this uses a second conjugacy theorem, this time due to Mal'cev. Once we understand the procedure for attaching a distinguished semisimple orbit to a nilpotent orbit, we obtain the weighted Dynkin diagrams.

For the classical cases, the weighted Dynkin diagram classification can be replaced by a "partition classification"; this naturally generalizes the situation for $\mathfrak{sl}_n$, which is discussed in §3.1. We will devote Chapter 5 to a discussion of nilpotent orbits in the classical Lie algebras and Chapter 8 will deal with the exceptional Lie algebras.

3.1 Type A_n

Before beginning the general classification of nilpotent orbits, it is both illuminating and suggestive to study the special case of $\mathfrak{g} = \mathfrak{sl}_n$. In this setting, we will describe how the nilpotent orbits stand in one-to-one correspondence with partitions of the integer n. In particular, there are only finitely many nilpotent orbits; this should be contrasted with (2.2.7), which tells us that there are infinitely many semisimple orbits.

We first produce a distinguished class of nilpotent orbits in $\mathfrak{sl}_n$. Recall that a *partition* of n is a tuple $[d_1, d_2, \ldots, d_k]$ of positive integers having the two properties

$$d_1 \geq d_2 \geq \ldots \geq d_k > 0 \text{ and } d_1 + d_2 + \ldots + d_k = n. \tag{3.1.1}$$

It is useful to relax this definition and use the notation $[d_1, \ldots, d_n]$ to represent a partition of n, with the understanding that there exists an index $k, 1 \leq k \leq n$, such that

$$d_1 \geq d_2 \geq \ldots \geq d_k > 0 \text{ and } d_{k+1} = \ldots = d_n = 0.$$

We agree to identify two partitions $[d_1, \ldots, d_n]$ and $[p_1, \ldots, p_n]$ when their nonzero parts agree. On occasion we may represent a partition by a boldface letter $\mathbf{d} = [d_1, \ldots, d_n]$. In addition, it is also useful to introduce *exponential notation* for partitions. We write

$$[t_1^{i_1}, \ldots, t_r^{i_r}]$$

to denote the partition $[d_1, \ldots, d_n]$, where

$$d_j = \begin{cases} t_1 & 1 \leq j \leq i_1 \\ t_2 & i_1 + 1 \leq j \leq i_1 + i_2 \\ t_3 & i_1 + i_2 + 1 \leq j \leq i_1 + i_2 + i_3 \\ \vdots & \vdots \end{cases}$$

For example, $[4, 3^2, 2^3, 1, 0^{10}] = [4, 3, 3, 2, 2, 2, 1, 0, \ldots, 0]$ is a partition of 17. Denote by $\mathcal{P}(n)$ the set of all partitions of n. So $\mathcal{P}(2) = \{[2], [1^2]\}$ and

$$\mathcal{P}(6) = \{[6], [5, 1], [4, 2], [4, 1^2], [3^2], [3, 2, 1], [3, 1^3], [2^3], [2^2, 1^2], [2, 1^4], [1^6]\}.$$

Given a positive integer i, we construct the $i \times i$ matrix

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Recall that J_i is called an elementary *Jordan block of type i*. Clearly, J_i is a nilpotent endomorphism of $\mathbb{C}^i$. Given a partition $[d_1, d_2, \ldots, d_n] \in \mathcal{P}(n)$, let k be the largest index such that $d_k > 0$ and form the diagonal sum of elementary Jordan blocks $J_{d_1}, \ldots, J_{d_k}$, denoted

$$X_{[d_1, d_2, \ldots, d_n]} = \begin{pmatrix} J_{d_1} & 0 & 0 & \cdots & 0 \\ 0 & J_{d_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{d_k} \end{pmatrix}. \tag{3.1.2}$$

This is a nilpotent endomorphism of $\mathbb{C}^{d_1 + d_2 + \ldots + d_k} = \mathbb{C}^n$. By (1.1.3), we conclude that $X_{[d_1, d_2, \ldots, d_n]}$ is a nilpotent element of $\mathfrak{sl}_n$. For $\mathfrak{sl}_n$, the adjoint group is given by the group PSL_n defined in Chapter 1. This allows us to form the nilpotent orbit

$$\mathcal{O}_{X_{[d_1, d_2, \ldots, d_n]}} = PSL_n \cdot X_{[d_1, d_2, \ldots, d_n]}. \tag{3.1.3}$$

To make notation simpler, we will often write

$$\mathcal{O}_{X_{[d_1, d_2, \ldots, d_n]}} = \mathcal{O}_{[d_1, d_2, \ldots, d_n]}. \tag{3.1.4}$$

Given two different partitions $[d_1, d_2, \ldots, d_n]$ and $[p_1, p_2, \ldots, p_n]$ of n, we observe that the orbits $\mathcal{O}_{[d_1, d_2, \ldots, d_n]}$ and $\mathcal{O}_{[p_1, p_2, \ldots, p_n]}$ are disjoint by the uniqueness of the Jordan normal form. We have now made

Observation 3.1.5. *The number of nilpotent orbits in $\mathfrak{sl}_n$ is at least $|\mathcal{P}(n)|$.*

This discussion suggests that we use the set $\mathcal{P}(n)$ of partitions of n as our parameter set for nilpotent orbits. If $X \in \mathfrak{sl}_n$ is nilpotent, then (1.1.3) tells us that X is a nilpotent endomorphism of $\mathbb{C}^n$. By elementary linear algebra, X has Jordan normal form $X_{[d_1, d_2, \ldots, d_n]}$ for some partition $[d_1, d_2, \ldots, d_n] \in \mathcal{P}(n)$. We observed before (1.2.5) that

$$\text{the } GL_n, SL_n, \text{ and } PSL_n\text{-conjugacy classes in } \mathfrak{sl}_n \text{ coincide.} \tag{3.1.6}$$

In other words, X is PSL_n-conjugate to $X_{[d_1, d_2, \ldots, d_n]}$. Thus, we get

oposition 3.1.7. *In the case of $\mathfrak{g} = \mathfrak{sl}_n$, there is a one-to-one correspondence between the set of nilpotent orbits and the set $\mathcal{P}(n)$ of partitions of n. The correspondence sends a nilpotent element X to the partition determined by the block sizes in its Jordan normal form. The zero orbit corresponds to the partition $[1^n]$. In particular, the set of nilpotent orbits is finite.*

In summary, the classification of nilpotent orbits in $\mathfrak{sl}_n$ is a straightforward linear algebra exercise. You might be tempted to use the same idea for the other classical algebras $\mathfrak{so}_{2n}, \mathfrak{sp}_{2n}$, and $\mathfrak{so}_{2n+1}$. After all, each of these algebras is a subalgebra of an appropriate $\mathfrak{sl}_N$, and we could then try to use (3.1.7). Of course, the tricky point here is that we do not have an analogue of (3.1.6); in fact, it fails for $\mathfrak{so}_{4n}$. Thus, it is not yet clear that there are only finitely many nilpotent orbits in the other classical Lie algebras. We will sort out the appropriate "partition type" classification scheme for the classical cases in Chapter 5. For now we will develop the theory of weighted Dynkin diagrams for arbitrary semisimple $\mathfrak{g}$.

.2 Strategy

In this section, we outline our strategy for classifying nilpotent orbits in a complex semisimple Lie algebra. We will use three main tools.

- A correspondence between nilpotent orbits in $\mathfrak{g}$ and small semisimple subalgebras of $\mathfrak{g}$.
- A correspondence between small semisimple subalgebras of $\mathfrak{g}$ and distinguished semisimple orbits in $\mathfrak{g}$.
- A correspondence between distinguished semisimple orbits in $\mathfrak{g}$ and certain weighted Dynkin diagrams.

Before presenting the details, we make an elementary observation about nilpotent elements. For this we need to recall some additional structure theory of semisimple Lie algebras. A *Borel subalgebra* of $\mathfrak{g}$ is defined to be a maximal solvable subalgebra. The next lemma summarizes its basic properties; see [37, §16] or [21, §1.10].

emma 3.2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. If $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, then there exists a direct sum decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$; it consists exactly of the nilpotent elements in $\mathfrak{b}$. Moreover, relative to the root space decomposition (2.1.1), there is a choice of positive roots Φ^+ such that $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.*

Remember that a *nil ideal* of a Lie algebra $\mathfrak{g}$ is an ideal $\mathfrak{n}$ such that ad_X is nilpotent for all $X \in \mathfrak{n}$. Every Lie algebra contains a unique maximal nil ideal $\mathfrak{m}$. We call $\mathfrak{m}$ the *nilradical* of $\mathfrak{g}$; see [83, §3.8]. Observe that the nilradical is always contained in the radical. We are now led to the following characterization of nilpotent elements.

Lemma 3.2.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then X is nilpotent if and only if X lies in the nilradical of some Borel subalgebra of $\mathfrak{g}$.*

Proof. Note first that the elements of $\mathfrak{n}$ act nilpotently on all of $\mathfrak{g}$ (not just $\mathfrak{b}$), since they map any root space of $\mathfrak{g}$ to a sum of root spaces belonging to strictly higher roots in the standard partial order on Φ relative to Φ^+. (Here we regard the Cartan subalgebra $\mathfrak{h}$ as the 0 root space $\mathfrak{g}_0$ of $\mathfrak{g}$.) Conversely, suppose X is nilpotent in $\mathfrak{g}$; then the subalgebra $\mathfrak{a} = \mathbb{C}X$ is nilpotent and hence solvable. We can now find a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{a}$. Then X lies in the nilradical of $\mathfrak{b}$ by (3.2.1). □

On the one hand, while interesting, this description is far from illuminating. In effect, we have transformed the problem of parametrizing nilpotent elements into the problem of keeping track of the nilradicals of all Borel subalgebras in $\mathfrak{g}$. This latter problem is involved, since there are a lot of Borel subalgebras. (In fact, the set $\mathcal{B}$ of all Borel subalgebras of $\mathfrak{g}$ forms a smooth variety of dimension $|\Phi^+|$ on which the adjoint group acts transitively.) On the other hand, this lemma suggests a strategy one might use to classify nilpotent elements or orbits; namely, trying to organize them using special subalgebras of $\mathfrak{g}$.

Among the semisimple subalgebras of $\mathfrak{g}$, those of minimal dimension play a fundamental role. The smallest possible dimension of a semisimple Lie algebra is 3 and any 3-dimensional semisimple Lie algebra is isomorphic to $\mathfrak{sl}_2$. Any subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ is spanned by a *standard triple* $\{H, X, Y\}$ of nonzero elements in $\mathfrak{g}$ satisfying the bracket relations:

$$[H, X] = 2X, \ [H, Y] = -2Y \text{ and } [X, Y] = H.$$

We call the element H (resp. X, Y) the *neutral* (resp. *nilpositive*, *nilnegative element*) of the triple $\{H, X, Y\}$. Note that the neutral element H of a standard triple $\{H, X, Y\}$ is semisimple in the subalgebra $\mathfrak{a} = \mathbb{C}\langle H, X, Y\rangle$, since ad_H has distinct eigenvalues 0, −2, and 2 on $\mathfrak{a}$. Using (1.1.3), it follows that H is a semisimple element of the ambient Lie algebra $\mathfrak{g}$. Similarly, we see that the nilpositive and nilnegative elements of a standard triple are nilpotent in the Lie algebra $\mathfrak{g}$.

Let's indicate why standard triples play a central role by showing that each of the nonzero nilpotent orbits in $\mathfrak{sl}_n$ (which we classified in (3.1.7)) has a representative that is the nilpositive element of a standard triple. We will need to use the finite-dimensional representation theory of $\mathfrak{sl}_2$. Recall that $\mathfrak{sl}_2$ is spanned by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.2.3}$$

and that these satisfy the bracket relations

$$[H,X]=2X,\ [H,Y]=-2Y\,,\ \text{and}\ [X,Y]=H. \tag{3.2.4}$$

Fix an integer $r \geq 0$ and define a linear map

$$\rho_r : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_{r+1}, \tag{3.2.5}$$

via

$$\rho_r(H) = \begin{pmatrix} r & 0 & 0 & \cdots & 0 & 0 \\ 0 & r-2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -r+2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -r \end{pmatrix},$$

$$\rho_r(X) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho_r(Y) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \mu_1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \mu_r & 0 \end{pmatrix},$$

where $\mu_i = i(r+1-i)$ for $1 \leq i \leq r$.

Lemma 3.2.6. *The map ρ_r is an irreducible representation of $\mathfrak{sl}_2$ of dimension $r+1$. If $\{e_1, e_2, \ldots, e_{r+1}\}$ represents the standard basis of $\mathbb{C}^{r+1}$, then e_1 is a vector of highest weight r. Moreover, if π is an irreducible finite-dimensional representation of $\mathfrak{sl}_2$, then there exists an r such that $\pi \cong \rho_r$.*

Proof. This is well known (see [21, §1.8] or [37, §7]). You can check directly that ρ_r is a Lie algebra homomorphism and that it defines an irreducible representation of $\mathfrak{sl}_2$, but it takes a bit of work to see that every irreducible finite-dimensional representation of $\mathfrak{sl}_2$ arises in this way. □

Now let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sl}_n$. By (3.1.7), there exists a partition $[d_1, d_2, \ldots, d_n] \in \mathcal{P}(n)$ such that $\mathcal{O} = \mathcal{O}_{[d_1,d_2,\ldots,d_n]}$. Let k be the largest index such that $d_k > 0$ and define a homomorphism

$$\phi_{\mathcal{O}} : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_n \text{ via } \phi_{\mathcal{O}} = \bigoplus_{1\leq i\leq k} \rho_{d_i-1}.$$

Then $\phi_{\mathcal{O}}$ has the property that $\phi_{\mathcal{O}}(X) = X_{[d_1,d_2,\ldots,d_n]}$, where X is defined in (3.2.4). Indeed, the image of $\phi_{\mathcal{O}}$ is a subalgebra of $\mathfrak{sl}_n$ containing $X_{[d_1,d_2,\ldots,d_n]}$, which is either zero or isomorphic to $\mathfrak{sl}_2$. If the image is not zero we get a standard triple

$$\{H_{[d_1,\ldots,d_n]}, X_{[d_1,\ldots,d_n]}, Y_{[d_1,\ldots,d_n]}\} := \{\phi_{\mathcal{O}}(H), \phi_{\mathcal{O}}(X), \phi_{\mathcal{O}}(Y)\}.$$

Hence we obtain

Corollary 3.2.7. *To each nonzero nilpotent orbit $\mathcal{O} = \mathcal{O}_X$ in $\mathfrak{sl}_n$, we can attach a standard triple $\{H, X, Y\}$ whose nilpositive element is X.*

The above digression suggests a close connection between standard triples in $\mathfrak{g}$ and nilpotent orbits in $\mathfrak{g}$. Note that any nonzero homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ yields a standard triple $\{\phi(H), \phi(X), \phi(Y)\}$. If $\mathrm{Hom}^\times(\mathfrak{sl}_2, \mathfrak{g})$ denotes the set of nonzero homomorphisms from $\mathfrak{sl}_2$ to $\mathfrak{g}$, we get a bijection

$$\mathrm{Hom}^\times(\mathfrak{sl}_2, \mathfrak{g}) \Leftrightarrow \{\text{standard triples in } \mathfrak{g}\},$$

which will be used throughout the sequel. Furthermore, notice that each of these sets is invariant under the obvious action of G_{ad}. With these conventions, we can now define

$$\begin{aligned} \mathcal{A}_{hom} &= \{\, G_{ad} - \text{conjugacy classes in } \mathrm{Hom}^\times(\mathfrak{sl}_2, \mathfrak{g})\,\} \\ \mathcal{A}_{triple} &= \{\, G_{ad} - \text{conjugacy classes of standard triples in } \mathfrak{g}\,\}. \end{aligned} \tag{3.2.8}$$

These two sets are isomorphic as G_{ad}-sets. In addition, there is a map Ω defined by

$$\Omega : \mathcal{A}_{triple} \longrightarrow \{\,\text{nonzero nilpotent orbits in } \mathfrak{g}\,\};\ \Omega(\{H, X, Y\}) = \mathcal{O}_X. \tag{3.2.9}$$

We can now precisely state the first step in the classification strategy.

Theorem 3.2.10 (Classification Step 1). *The map Ω is a one-to-one correspondence between the set $\mathcal{A}_{triple}$ and the set of nonzero nilpotent orbits in $\mathfrak{g}$.*

Two facts we have seen suggest that Ω should be surjective. First, Corollary (3.2.7) shows this is indeed the case for $\mathfrak{g} = \mathfrak{sl}_n$. Second, any nilpotent element X in any $\mathfrak{g}$ lies in the nilradical $\mathfrak{n}$ of some Borel subalgebra $\mathfrak{b}$, by (3.2.2). Recall that $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ for some choice Φ^+ of positive roots, by (3.2.1). If $X = X_\alpha$ is actually a root vector, then the structure theory in [37, §8.3] yields elements $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{h}$ satisfying the bracket relations of (3.2.4):

$$[H_\alpha, X_\alpha] = 2X_\alpha,\ [H_\alpha, Y_\alpha] = -2Y_\alpha\ ,\ [X_\alpha, Y_\alpha] = H_\alpha. \tag{3.2.11}$$

In other words, every root vector X_α is the nilpositive element of a standard triple $\{H_\alpha, X_\alpha, Y_\alpha\}$. Thus the image of Ω contains any nilpotent orbit through a root vector. If X is not a root vector, then it is not obvious that it belongs to a standard triple, but fortunately it does.

rem (Jacobson-Morozov). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If X is a nonzero nilpotent element of $\mathfrak{g}$, then it is the nilpositive element of a standard triple. Equivalently, for any nilpotent element X, there exists a homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ such that $\phi\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) = X$.*

We prove this in the next section. To complete the proof of (3.2.10), it suffices to show that Ω is injective. If $\mathfrak{g} = \mathfrak{sl}_n$, this follows from the theory of the Jordan normal form. In general, we will need the following conjugacy result of Kostant; a slightly strengthened version of this is proved in §3.4.

rem (Kostant). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Any two standard triples $\{H, X, Y\}$ and $\{H', X, Y'\}$ with the same nilpositive element are conjugate by an element of G_{ad}. In particular, the map Ω of (3.2.10) is injective.*

Once (3.2.10) is known, the next step is to relate conjugacy classes of standard triples in $\mathfrak{g}$ to certain semisimple orbits in $\mathfrak{g}$. There is an obvious way to do this. Define a map

$$\Upsilon : \mathcal{A}_{triple} \to \{\mathcal{O}_Z \mid Z \in \mathcal{S}\} \text{ via } \Upsilon(\{H, X, Y\}) = \mathcal{O}_H. \tag{3.2.12}$$

We have seen that the image of Υ lies in the specified set. We define

$$\mathcal{S}_{dist} = \operatorname{Image} \Upsilon, \tag{3.2.13}$$

and call $\mathcal{S}_{dist}$ the set of *distinguished semisimple orbits* in $\mathfrak{g}$. Our next result is

rem 3.2.14 (Classification Step 2). *The map Υ establishes a one-to-one correspondence between the set $\mathcal{A}_{triple}$ and the set $\mathcal{S}_{dist}$ of distinguished semisimple orbits in $\mathfrak{g}$.*

To prove this, it suffices to show that Υ is injective and this follows from

rem (Mal'cev). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Any two standard triples $\{H, X, Y\}$ and $\{H, X', Y'\}$ with the same neutral element are conjugate by an element of G_{ad}.*

We will prove this in §3.4. This completes the second step of the classification scheme. We now know that the nilpotent orbits of $\mathfrak{g}$ are in one-to-one correspondence with a certain subset of the set of semisimple orbits; this latter set was parametrized in (2.2.4). However, even after all this work, it is not obvious that the set of distinguished semisimple orbits is a finite set! If $\mathcal{O}_H$ is a distinguished semisimple orbit, we will need to observe that certain integrality conditions hold for the element H. Let $\mathfrak{h}$ be a Cartan subalgebra containing H

and choose a base of simple roots Δ such that H lies in the fundamental domain D_Δ defined before (2.2.7). In §3.5, we show that $\alpha(H) \in \{0, 1, 2\}$ for $\alpha \in \Delta$. Thus, we get an obvious labeling of the Dynkin diagram known as the *weighted Dynkin diagram* (of H or $\mathcal{O}_H$). Since there are only $3^{\operatorname{rank}(\mathfrak{g})}$ possible labelings, we obtain the following result.

Corollary 3.2.15 (Classification Step 3). *There is a one-to-one correspondence between the set of nilpotent orbits in $\mathfrak{g}$ and weighted Dynkin diagrams of distinguished semisimple orbits. In particular, there are at most $3^{\operatorname{rank}(\mathfrak{g})}$ weighted Dynkin diagrams, and thus at most that number of nilpotent orbits in $\mathfrak{g}$.*

Of course, in order to give a parametrization, we need to understand which labelings of the Dynkin diagram of $\mathfrak{g}$ are weighted Dynkin diagrams. This is tricky, as the following example suggests.

Example 3.2.16. Let $\mathfrak{g} = \mathfrak{sl}_6$, then (3.1.7) tells us there are exactly $|\mathcal{P}(6)| = 11$ nilpotent orbits. On the other hand, (3.2.15) tells us only that there are at most $3^5 = 243$ orbits!

In the classical cases, rather than computing the weighted Dynkin diagrams directly, we will carry out a "partition type" classification in the spirit of §3.1. We will then be able to read off the weighted Dynkin diagrams from the partition classification. We do this for $\mathfrak{sl}_n$ in §3.6 and in Chapter 5 for the remaining classical cases. In the exceptional cases, we will go back to first principles to determine the weighted Dynkin diagrams, using the work of Bala and Carter. This is done in Chapter 8.

3.3 The Jacobson-Morozov Theorem

This section is devoted to the proof of the following fundamental result.

Theorem 3.3.1 (Jacobson-Morozov). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If X is a nonzero nilpotent element of $\mathfrak{g}$, then there exists a standard triple for $\mathfrak{g}$ whose nilpositive element is X. Hence, the map Ω in (3.2.10) is onto.*

Proof. We will argue by induction on the dimension of $\mathfrak{g}$. If this is 3, then $\mathfrak{g}$ must be isomorphic to $\mathfrak{sl}_2$ and the result follows from (3.2.7). This gets the induction started. (Actually, the argument for the inductive step below also handles the base case.) Assume $\dim(\mathfrak{g}) > 3$. If X lies in a proper semisimple subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, then by induction we can already find a standard triple $\{H, X, Y\}$ in $\mathfrak{a}$ with X as its nilpositive element; clearly, this will then be a standard triple in $\mathfrak{g}$. Thus we may assume for the remainder of the proof that

$$X \text{ does not lie in any proper semisimple subalgebra of } \mathfrak{g}. \tag{3.3.2}$$

Let κ be the Killing form of $\mathfrak{g}$. We saw in the proof of (1.3.9) that

$$\kappa(X, \mathfrak{g}^X) = 0 \tag{3.3.3}$$

and thus

$$X \in (\mathfrak{g}^X)^\perp = [\mathfrak{g}, X] \tag{3.3.4}$$

where the orthogonal complement is taken relative to the Killing form. Thus,

$$[H', X] = 2X \text{ for some } H' \in \mathfrak{g}. \tag{3.3.5}$$

Claim 3.3.6. There exists a semisimple element H such that $[H, X] = 2X$.

To see this, let $H'_s + H'_n$ be the Jordan decomposition of H' in $\mathfrak{g}$. By (1.1.1), H'_s acts semisimply and H'_n acts nilpotently on the subspace $\mathbb{C}X$, whence $[H'_s, X] = 2X, [H'_n, X] = 0$. Thus we may take $H = H'_s$.

Claim 3.3.7. If H is as in (3.3.6), then $H \in [\mathfrak{g}, X]$.

Accepting this claim for the moment, we show how to finish the proof of the theorem. We have $Y \in \mathfrak{g}$ with $H = [X, Y]$. Since ad_H acts semisimply on $\mathfrak{g}$, we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_{\lambda_1} \oplus \ldots \oplus \mathfrak{g}_{\lambda_k} \tag{3.3.8}$$

of $\mathfrak{g}$ into ad_H-eigenspaces. Write $Y = Y_1 + \ldots + Y_k$ with $Y_i \in \mathfrak{g}_{\lambda_i}$. An easy calculation shows that

$$[X, \mathfrak{g}_{\lambda_i}] \subset \mathfrak{g}_{\lambda_i+2}, \; 1 \le i \le k. \tag{3.3.9}$$

We have $H \in \mathfrak{g}_0$ since $\mathrm{ad}_H H = 0$. On the other hand, we also have

$$H = [X, Y] = \sum_{i=1}^{k} [X, Y_i],$$

whence by (3.3.9) we get

$$H = [X, Y_{-2}]. \tag{3.3.10}$$

Thus, if we replace Y by its component Y_{-2} in the -2-eigenspace of ad_H, then $[H, Y] = -2Y$ and $\{H, X, Y\}$ is a standard triple containing X as its nilpositive element, as desired. (Note: If we had not first chosen H to be semisimple, we would not have Y with honest eigenvalue -2; we would have known only that the generalized eigenvalue is -2.)

It remains to prove (3.3.7). We argue by contradiction, assuming that $H \notin [\mathfrak{g}, X]$. We noted in (3.3.4) that $[\mathfrak{g}, X] = (\mathfrak{g}^X)^\perp$, so we must have

$$\kappa(H, \mathfrak{g}^X) \neq 0. \tag{3.3.11}$$

Also, a calculation with the Jacobi identity shows that ad_H leaves $\mathfrak{g}^X$ invariant (see (3.4.4) below) and must act semisimply, so we may decompose $\mathfrak{g}^X$ into ad_H eigenspaces:

$$\mathfrak{g}^X = \mathfrak{g}^X_{\tau_1} \oplus \ldots \oplus \mathfrak{g}^X_{\tau_t}, \tau_i \in \mathbb{C}.$$

Note that $\mathfrak{g}^X_0 = \{ Z \in \mathfrak{g}^X \mid [H, Z] = 0 \}$ is just the centralizer of H in $\mathfrak{g}^X$, so that we have

$$\mathfrak{g}^X = (\mathfrak{g}^X)^H \oplus \sum_{\tau_i \neq 0} \mathfrak{g}^X_{\tau_i}. \tag{3.3.12}$$

By the invariance of the Killing form,

$$\kappa(H, [H, \mathfrak{g}^X]) = \kappa([H, H], \mathfrak{g}^X) = 0. \tag{3.3.13}$$

If Z is a nonzero element of $\mathfrak{g}^X_{\tau_i}$ with $\tau_i \neq 0$, then by (3.3.13),

$$0 = \kappa(H, [H, Z]) = \kappa(H, \tau_i Z) = \tau_i \kappa(H, Z).$$

This shows that

$$H \in (\mathfrak{g}^X_{\tau_i})^\perp \text{ for all } \tau_i \neq 0. \tag{3.3.14}$$

Combining (3.3.11), (3.3.12), and (3.3.14) we arrive at:

$$\text{There exists } Z \in (\mathfrak{g}^X)^H \text{ such that } \kappa(H, Z) \neq 0. \tag{3.3.15}$$

If Z is nilpotent, then we argue as in (1.3.9) to get $\kappa(H, Z) = \mathrm{trace}(\mathrm{ad}_H \mathrm{ad}_Z) = 0$, a contradiction. Hence, $Z_s \neq 0$. By (1.1.1)(ii),

$$Z_s \text{ is a nonzero semisimple element in } (\mathfrak{g}^X)^H. \tag{3.3.16}$$

By (2.1.2), $\mathfrak{g}^{Z_s}$ is reductive, whence $[\mathfrak{g}^{Z_s}, \mathfrak{g}^{Z_s}]$ is a semisimple subalgebra of $\mathfrak{g}$. It is a proper subalgebra, since $\mathfrak{g}^{Z_s} = \mathfrak{g}$ only if $Z_s = 0$. We have now shown that $H \in \mathfrak{g}^{Z_s}$ and $X \in \mathfrak{g}^{Z_s}$ by (3.3.16), so $2X = [H, X] \in [\mathfrak{g}^{Z_s}, \mathfrak{g}^{Z_s}]$. This places our nilpotent element X in a proper semisimple subalgebra of $\mathfrak{g}$, contrary to (3.3.2). This contradiction establishes our claim in (3.3.7). □

Remark. In Chapter 9 we will generalize this result to real semisimple Lie algebras. In fact it holds over arbitrary basefields of characteristic 0, and under some restrictions in characteristic p as well.

Theorems of Kostant and Mal'cev

In this section, we prove two conjugacy theorems that lead to a bijective correspondence between nilpotent orbits and distinguished semisimple orbits. This requires a crude understanding of the centralizer structure for nilpotent elements. While the centralizer of a nonzero nilpotent element is not a reductive subalgebra, we will ultimately show that it is the direct sum of a nilpotent ideal and a reductive subalgebra. The proof of this fact is postponed until §3.7, but a preliminary picture will emerge below.

Let X be a nonzero nilpotent element of a semisimple Lie algebra $\mathfrak{g}$. By the Jacobson-Morozov Theorem, X embeds into a standard triple $\{H, X, Y\}$. Since ad_H acts semisimply on $\mathfrak{g}$, we can decompose $\mathfrak{g}$ into its ad_H eigenspaces:

$$\mathfrak{g} = \bigoplus_{\lambda\in\mathbb{C}} \mathfrak{g}_\lambda, \ \mathfrak{g}_\lambda = \{\, Z \in \mathfrak{g} \,|\, \mathrm{ad}_H Z = [H, Z] = \lambda Z \,\}.$$

Note that our standard triple $\{H, X, Y\}$ spans a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. Under the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$, we know that $\mathfrak{g}$ decomposes as a direct sum of irreducible finite-dimensional modules. By the representation theory of $\mathfrak{sl}_2$ (cf. (3.2.6)), ad_H has integral eigenvalues on the irreducible summands, whence ad_H has integral eigenvalues on $\mathfrak{g}$. We can now write

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i. \tag{3.4.1}$$

The Jacobi identity shows that $\mathfrak{g}^X$ is ad_H-stable. Thus we have

$$\mathfrak{g}^X = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i^X, \ \mathfrak{g}_i^X = \mathfrak{g}^X \cap \mathfrak{g}_i. \tag{3.4.2}$$

By the representation theory of $\mathfrak{sl}_2$, $\mathfrak{g}^X$ is the sum of the highest weight spaces for the action of $\mathfrak{a}$ on $\mathfrak{g}$ and every such weight space has a nonnegative weight. Thus, we have

Lemma 3.4.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra and X be a nonzero nilpotent element of $\mathfrak{g}$. Let $\{H, X, Y\}$ be a Jacobson-Morozov standard triple for $\mathfrak{g}$ containing X as its nilpositive element.*

(i) *The Killing form restricted to $\mathfrak{g}^H$ is nondegenerate.*

(ii) *We have $\mathfrak{g}^X = \oplus_{i\geq 0}\mathfrak{g}_i^X$.*

Proof. We established (ii) above; (i) follows from the proof of (2.1.2). □

We now prove the first of three results relating two standard triples with elements in common.

Lemma 3.4.4. *Let $\{H, X, Y\}$ and $\{H, X, Y'\}$ be standard triples for $\mathfrak{g}$ having the same neutral and nilpositive elements. Then $Y = Y'$.*

Proof. We have $[X, Y - Y'] = [X, Y] - [X, Y'] = H - H = 0$, so $Y - Y' \in \mathfrak{g}^X$. On the other hand, $Y - Y'$ lies in the -2-eigenspace of ad_H, since Y and Y' do. Lemma 3.4.3 implies that this eigenspace meets $\mathfrak{g}^X$ in $\{0\}$, so $Y = Y'$. □

The subalgebra $\mathfrak{u}^X$ introduced in the next lemma will play a crucial role in the proof of Kostant's conjugacy theorem and the structural results of §3.7.

Lemma 3.4.5. *Let X be a nonzero nilpotent element of a semisimple Lie algebra $\mathfrak{g}$. Then $\mathfrak{u}^X := \mathfrak{g}^X \cap [\mathfrak{g}, X]$ is an ad_H-invariant nilpotent ideal of $\mathfrak{g}^X$. Moreover, in the notation of (3.4.3), $\mathfrak{u}^X = \oplus_{i>0}\mathfrak{g}_i^X$.*

Proof. The first assertion follows at once from (3.4.3) and the second assertion, since $[\mathfrak{g}_i^X, \mathfrak{g}_j^X] \subset \mathfrak{g}_{i+j}^X$. The last assertion follows from the representation theory of $\mathfrak{sl}_2$. □

Let U^X be the connected subgroup of G_{ad}^X with Lie algebra $\mathfrak{u}^X$. (We will justify the notations $U^X, \mathfrak{u}^X$ in §3.7 by defining a group U and its Lie algebra $\mathfrak{u}$ and showing that $U^X, \mathfrak{u}^X$ are the centralizers of X in $U, \mathfrak{u}$, respectively.) The group U^X is simply connected, and the exponential map Exp: $\mathfrak{u}^X \to U^X$ is a diffeomorphism, since $\mathfrak{u}^X$ is nilpotent. See [33]. Recall that for any $Z \in \mathfrak{u}^X$, we have

$$\mathrm{Exp}\ Z \cdot H = H + [Z, H] + \frac{1}{2}[Z, [Z, H]] + \ldots = \sum_{i=0}^{\infty} \frac{(\mathrm{ad}_Z)^i H}{i!}. \tag{3.4.6}$$

This is a finite sum, since each ad_Z is nilpotent. Moreover, since $\mathfrak{u}^X$ is ad_H-invariant, every term in (3.4.6) except H lies in $\mathfrak{u}^X$. Thus, $\mathrm{Exp}\ Z \cdot H \in H + \mathfrak{u}^X$, whence we get $U^X \cdot H \subset H + \mathfrak{u}^X$. In fact, more is true.

Lemma 3.4.7 (Kostant). *For every $V \in \mathfrak{u}^X$ there exists a unique $Z \in \mathfrak{u}^X$ so that $\mathrm{Exp}\ Z \cdot H = H + V$. In particular, $H + \mathfrak{u}^X = U^X \cdot H$.*

Proof. We have

$$\mathfrak{u}^X = \bigoplus_{i=1}^{m} \mathfrak{g}_i^X$$

for some m. We produce the claimed Z inductively, by constructing successive approximations to it on $\oplus_{1\leq i\leq j}\mathfrak{g}_i^X$ for $1 \leq j \leq m$. Begin by setting $Z_1 = -V_1$, where V_1 is the component of V in $\mathfrak{g}_1^X$. Then $Z_1 \in \mathfrak{g}_1^X \subset \mathfrak{u}^X$ and $\mathrm{ad}_{Z_1} H = -[H, Z_1] = -Z_1 = V_1$. Thus,

$$\begin{aligned} &\operatorname{Exp} Z_1 \cdot H - (H+V) = \\ &= H + \mathrm{ad}_{Z_1} H + \tfrac{1}{2}(\mathrm{ad}_{Z_1})^2 H + \ldots + \tfrac{1}{i!}(\mathrm{ad}_{Z_1})^i H + \ldots - (H+V) \\ &= \tfrac{1}{2}(\mathrm{ad}_{Z_1})^2 H + \ldots + \tfrac{1}{i!}(\mathrm{ad}_{Z_1})^i H + \ldots + V_1 - V + H - H \\ &\in \mathfrak{g}_2^X + \mathfrak{g}_3^X + \ldots = \textstyle\bigoplus_{2\le i\le m} \mathfrak{g}_i^X . \end{aligned} \tag{3.4.8}$$

Now assume inductively that we have constructed Z_j with

$$\begin{aligned} &\text{(i) } Z_j \in \textstyle\bigoplus_{1\le i\le j} \mathfrak{g}_i^X, \text{ and} \\ &\text{(ii) } \operatorname{Exp} Z_j \cdot H - (H+V) \in \textstyle\bigoplus_{j+1\le i\le m} \mathfrak{g}_i^X . \end{aligned} \tag{3.4.9}$$

Set

$$Z'_{j+1} = \text{the component of } \{\operatorname{Exp} Z_j \cdot H - (H+V)\} \text{ in } \mathfrak{g}_{j+1}^X .$$

Let

$$Z_{j+1} = Z_j + \frac{1}{j+1} Z'_{j+1} \in \bigoplus_{1\le i\le j+1} \mathfrak{g}_i^X .$$

Then one checks immediately from (3.4.6) that properties (i) and (ii) carry over to Z_{j+1}. Thus, if we put $Z = Z_m$, we get $\operatorname{Exp} Z \cdot H = H + V$, as desired. Moreover, if $\operatorname{Exp} W$ also sends H to $H+V$, then we can retrace the above argument to deduce that the projection of W to each $\oplus_{1\le i\le j}\mathfrak{g}_i^X$ agrees with Z_j, whence $W = Z$. □

We can now prove our first conjugacy theorem. Combined with the Jacobson-Morozov Theorem, we obtain a proof of Theorem 3.2.10.

rem 3.4.10 (Kostant). *Let $\{H, X, Y\}$ and $\{H', X, Y'\}$ be two standard triples in $\mathfrak{g}$ with the same nilpositive element X. Then there exists $x \in U^X$ such that $x \cdot H = H'$, $x \cdot X = X$ and $x \cdot Y = Y'$. In particular, the map Ω in (3.2.10) is injective.*

Proof. We have $[H', X] = 2X = [H, X]$, whence $H' - H \in \mathfrak{g}^X$. In addition, $[X, Y' - Y] = H' - H$, which implies that $H' - H \in [\mathfrak{g}, X]$. Hence,

$$H' - H \in \mathfrak{u}^X . \tag{3.4.11}$$

It follows from (3.4.7) that there is $x \in U^X$ with $x \cdot H = H', x \cdot X = X$. By (3.4.4), we also have $x \cdot Y = Y'$, as desired. □

One bonus of the proof of Kostant's conjugacy theorem is an explicitly constructed element in the adjoint group that conjugates one standard triple to another. By contrast, the proof of Mal'cev's conjugacy theorem (our next result) proceeds on a more abstract level and requires transcendental methods.

To state the theorem, we first need a piece of notation. Given a semisimple element $H \in \mathfrak{g}$, put

$$(G_{ad}^H)^o = \text{connected component of the identity of } G_{ad}^H .$$

Remark. We could invoke (2.3.3), which says that G_{ad}^H is already connected.

Theorem 3.4.12 (Mal'cev). *Any two standard triples with the same neutral element are conjugate by some element of $(G_{ad}^H)^o$. In particular, the map Υ in (3.2.14) is injective.*

This result completes the proof of Theorem 3.2.14 (Classification Step 2).

Proof. Let $\{H, X, Y\}$ and $\{H, X', Y'\}$ be two standard triples with the same neutral element. Decompose $\mathfrak{g}$ into ad_H eigenspaces

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$$

as in (3.4.1). Define

$$\mathcal{P} = \{ Z \in \mathfrak{g}_2 \mid \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0 \}. \tag{3.4.13}$$

By (3.4.3) and the definition of a standard triple we have

$$X, X' \in \mathcal{P}. \tag{3.4.14}$$

Next, since

$$[\mathfrak{g}_0, \mathfrak{g}_2] \subset \mathfrak{g}_2 , \tag{3.4.15}$$

we see that $\mathfrak{g}_2$ is invariant under the action of the group $(G_{ad}^H)^o$ defined above. Moreover, if $x \in (G_{ad}^H)^o$,

$$\begin{aligned} Z \in \mathcal{P} &\iff \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0 \\ &\iff x \cdot (\mathfrak{g}^Z \cap \mathfrak{g}_{-2}) = 0 \\ &\iff \mathfrak{g}^{x\cdot Z} \cap x \cdot (\mathfrak{g}_{-2}) = 0 \\ &\iff \mathfrak{g}^{x\cdot Z} \cap \mathfrak{g}_{-2} = 0 \text{ (by (3.4.15))} \\ &\iff x \cdot Z \in \mathcal{P}. \end{aligned}$$

Thus we are reduced to proving that $(G_{ad}^H)^o$ acts transitively on $\mathcal{P}$, for then any $x \in (G_{ad}^H)^o$ sending X to X' fixes H, whence it must send Y to Y' by (3.4.4). This transitivity follows from the next two claims:

Claim 3.4.16. $\mathcal{P}$ is a path connected open subset of $\mathfrak{g}_2$.

Claim 3.4.17. Each $(G_{ad}^H)^o$-orbit in $\mathcal{P}$ is open and closed.

To prove (3.4.16), define

$$T : \mathfrak{g}_2 \to \mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_0);\ T(Z) = \mathrm{ad}_Z.$$

Then for each $Z \in \mathfrak{g}_2$, $T(Z)$ is represented by a $\dim(\mathfrak{g}_0) \times \dim(\mathfrak{g}_{-2})$ matrix with entries depending linearly on Z. Note that $\mathrm{Image}(T(Z)) = [Z, \mathfrak{g}_{-2}]$ and $\mathrm{kernel}(T(Z)) = \mathfrak{g}_{-2} \cap \mathfrak{g}^Z$. We deduce that

$$\begin{aligned} Z \in \mathcal{P} &\iff \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0 \\ &\iff \mathrm{kernel}(T(Z)) = 0 \\ &\iff \dim(\mathrm{Image}T(Z)) = \dim([Z, \mathfrak{g}_{-2}]) = \dim(\mathfrak{g}_{-2}) = \dim(\mathfrak{g}_2), \end{aligned} \tag{3.4.18}$$

where the last equality follows from $\mathfrak{sl}_2$ theory. Hence, we see that

$$Z \in \mathcal{P} \iff T(Z) \text{ has full rank.}$$

But now we have identified $\mathcal{P}$ with a Zariski-open subset of $\mathfrak{g}_2$. Any such set is path connected: if $A, B \in \mathcal{P}$, then there are only finitely many $\lambda \in \mathbb{C}$ such that $\lambda A + (1 - \lambda)B$ is not contained in $\mathcal{P}$, and $\mathbb{C}$ minus a finite number of points is path connected (cf. the proof of (2.1.13)). This proves (3.4.16).

Next we compute the dimension of any $(G_{ad}^H)^o$-orbit inside $\mathcal{P}$. If $Z \in \mathfrak{g}_2$, then the invariance of the Killing form and $[\mathfrak{g}^Z, Z] = 0$ imply that

$$0 = \kappa([\mathfrak{g}^Z \cap \mathfrak{g}_0, Z], \mathfrak{g}_{-2}) = \kappa(\mathfrak{g}^Z \cap \mathfrak{g}_0, [Z, \mathfrak{g}_{-2}]).$$

Hence,

$$[Z, \mathfrak{g}_{-2}] \subset (\mathfrak{g}^Z \cap \mathfrak{g}_0)^\perp \cap \mathfrak{g}_0. \tag{3.4.19}$$

By (3.4.18) and the nondegeneracy of κ on $\mathfrak{g}_0$ (3.4.3), this implies that

$$\dim(\mathfrak{g}_{-2}) = \dim([Z, \mathfrak{g}_{-2}]) \leq \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_0 \cap \mathfrak{g}^Z).$$

Rewriting, this becomes

$$\dim(\mathfrak{g}_0 \cap \mathfrak{g}^Z) \leq \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_{-2}) = \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_2).$$

If $Z \in \mathcal{P}$, we can now compute

$$\begin{aligned} \dim((G_{ad}^H)^o \cdot Z) &= \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_0 \cap \mathfrak{g}^Z) \\ &\geq \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_0) + \dim(\mathfrak{g}_2) = \dim(\mathfrak{g}_2). \end{aligned}$$

This shows that every $(G_{ad}^H)^o$-orbit in $\mathcal{P}$ has full dimension, hence is open. Since the complement of an orbit is a union of orbits, the orbits are also closed. By (3.4.16), $\mathcal{P}$ must be a single orbit, whence (3.4.17) follows. □

Remarks. (i) We can generalize this result to semisimple Lie algebras over any algebraically closed basefield of characteristic 0, by using the Zariski rather than the Euclidean topology on $\mathfrak{g}_2$ in the proof. Indeed, any $(G_{ad}^H)^o$-orbit in $\mathcal{P}$ is Zariski-open in its closure [79, 4.3.1], which must be all of $\mathfrak{g}_2$ by dimension counting. Since the vector space $\mathfrak{g}_2$ is irreducible as a variety, there can be only one orbit in $\mathcal{P}$.

(ii) In Chapter 9, we will establish versions of (3.4.10) and (3.4.12) over $\mathbb{R}$.

In Chapter 4, a closer analysis of the set $\mathcal{P}$ in a special case will imply the existence of a certain canonically defined nilpotent orbit in $\mathfrak{g}$ called the subregular orbit. We close this section by looking at an example.

Remark. One can show that, in fact, the set $\mathcal{P}$ is principal open; that is, it consists of the nonzeros of a single polynomial.

3.5 Weighted Dynkin Diagrams

At this point, we know that nilpotent orbits are in bijective correspondence with distinguished semisimple orbits, via

$$\mathcal{O}_X \rightsquigarrow \{H, X, Y\} \rightsquigarrow \mathcal{O}_H,$$

where $\{H, X, Y\}$ is a Jacobson-Morozov standard triple containing X as its nilpositive element. We now show that very few semisimple orbits $\mathcal{O}_H$ arise in this way. This will ultimately lead to a proof that the number of nilpotent orbits is finite and allow us to attach a weighted Dynkin diagram to each nilpotent orbit. Moreover, this weighted Dynkin diagram will be a complete invariant of the nilpotent orbit: Any two nilpotent orbits with the same weighted diagram coincide.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$, a system of positive roots Φ^+, and a base of simple roots Δ. Write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, as in (3.2.1). Let $\bar{\mathfrak{b}} = \mathfrak{h} \oplus \bar{\mathfrak{n}}$ be the opposite Borel subalgebra, so that $\bar{\mathfrak{n}} = \sum_{\alpha \in -\Phi^+} \mathfrak{g}_\alpha$. We say that an element $Z \in \mathfrak{h}$ is Δ-*dominant* if $\alpha(Z)$ is real and nonnegative for all $\alpha \in \Delta$.

Let $\mathcal{O}$ be a nonzero nilpotent orbit. By the Jacobson-Morozov Theorem, any representative X of $\mathcal{O}$ embeds into a standard triple $\{H, X, Y\}$. Since H is semisimple, there exists a Cartan subalgebra $\mathfrak{h}_X$ containing H and (2.1.11) ensures that $\mathfrak{h}_X$ is conjugate to $\mathfrak{h}$ by an element of G_{ad}. Replacing $\{H, X, Y\}$ by a conjugate, we may therefore assume that $H \in \mathfrak{h}$; this does not affect the orbit $\mathcal{O}_X$. By the remark after (2.2.4), we may further conjugate $\{H, X, Y\}$ so that H lies in the fundamental domain D_Δ defined there.

We now want to show that H is very special. Begin by noting that H is completely determined by the values $\alpha(H), \alpha \in \Delta$, since Δ is a basis of $\mathfrak{h}^*$. Now (3.4.1) implies that H is Δ-dominant and integral:

$$\alpha(H) \in \mathbb{N} = \{0, 1, 2, \ldots\} \text{ for all } \alpha \in \Delta. \tag{3.5.1}$$

We also have $Y \in \bar{\mathfrak{n}}$, since $Y \in \mathfrak{g}_{-2}$. Since any negative root is a sum of negatives of simple roots, we get

$$[X_\alpha, Y] \in \bar{\mathfrak{b}} \text{ for } \alpha \in \Delta. \tag{3.5.2}$$

If $[X_\alpha, Y] = 0$, then $X_\alpha \in \mathfrak{g}^Y$. The proof of (3.4.3) shows that $\mathfrak{g}^Y = \oplus_{i \leq 0}(\mathfrak{g}^Y \cap \mathfrak{g}_i)$. This implies that $\alpha(H) \in -\mathbb{N} \cap \mathbb{N} = \{0\}$. If $[X_\alpha, Y] \neq 0$, then the ad_H eigenvalue of $[X_\alpha, Y]$ is $\alpha(H) - 2$. For (3.5.2) to hold, we must have

$$\alpha(H) - 2 \in -\mathbb{N}.$$

Combining this with (3.5.1), we deduce that $\alpha(H) \in \{0, 1, 2\}$ for any $\alpha \in \Delta$.

Hence, if we label every node of the Dynkin diagram of $\mathfrak{g}$ with the eigenvalue $\alpha(H)$ of H on the corresponding simple root space $\mathfrak{g}_\alpha$, then all labels are 0,1, or 2. We call such a labeled Dynkin diagram the *weighted Dynkin diagram* of $\mathcal{O}_X$ (or $\mathcal{O}_H$ or H) and denote it by $\Delta(\mathcal{O}_X)$.

Remark 3.5.3. By convention, the zero nilpotent orbit has a weighted Dynkin diagram with every node labeled 0, even though we do not call $\{0, 0, 0\}$ a standard triple. We will see in the next chapter that one Dynkin diagram has every node labeled 2 and belongs to the so-called principal nilpotent orbit.

Theorem 3.5.4 (Kostant). *There are only finitely many, and in fact at most $3^{\mathrm{rank}\,\mathfrak{g}}$, nilpotent orbits in $\mathfrak{g}$. The weighted Dynkin diagram of a nilpotent orbit is a complete invariant; i.e., $\Delta(\mathcal{O}) = \Delta(\mathcal{O}')$ if and only if $\mathcal{O} = \mathcal{O}'$.*

Proof. If $\Delta(\mathcal{O}) = \Delta(\mathcal{O}')$ and at least one label is nonzero, then the above discussion shows that there exist two standard triples, $\{H, X, Y\}$ and $\{H', X', Y'\}$, with nilpositive elements in $\mathcal{O}$, $\mathcal{O}'$, respectively, and with the same neutral element. By Mal'cev's Theorem (3.4.13), these two standard triples are conjugate. Thus $\mathcal{O} = \mathcal{O}'$. This proves the last assertion; the first one is immediate. □

It should be noted that Dynkin [27] classified the conjugacy classes of standard triples in a semisimple Lie algebra; he was studying conjugacy classes of semisimple subalgebras of $\mathfrak{g}$. Kostant's work [52] is required to tell us that these conjugacy classes of triples are in bijective correspondence with nilpotent orbits.

Caution. We have already seen in (3.2.16) that not every labeling of the nodes of a Dynkin diagram with the integers 0,1,2 is a weighted Dynkin diagram. In fact the number of weighted Dynkin diagrams in type A_{n-1} equals the number of partitions of n, which is asymptotically equivalent to

$$\frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}},$$

which is far less than the potential 3^n.

These remarks suggest that we develop more direct classification schemes. In Chapter 5, we do this in the classical cases, using "partition type" parametrizations. When we show how to attach partitions to nilpotent orbits, we will also show how to read off the weighted Dynkin diagram of an orbit from its partition. We will do this for $\mathfrak{sl}_n$ in the next section and for the other classical algebras in Chapter 5. In the exceptional cases, partition schemes are not available, and we will need to determine explicitly the weighted Dynkin diagrams. Fortunately, in the case of E_8, we will see there are just 70 nilpotent orbits, which is again far less than the potential 3^8. The exceptional cases will be discussed in Chapter 8.

3.6 Weighted Dynkin Diagrams for Type A_n

Let $\mathbf{d} = [d_1, \ldots, d_n]$ be a partition of n. Recall the explicit triple in $\mathfrak{sl}_n$ attached to $\mathbf{d}$ that was constructed before (3.2.7). We will describe an algorithm that computes the weighted Dynkin diagram of the corresponding orbit $\mathcal{O}_{[d_1,\ldots,d_n]}$. Denote the associated triple by

$$\{H_{[d_1,\ldots,d_n]}, X_{[d_1,\ldots,d_n]}, Y_{[d_1,\ldots,d_n]}\}; \tag{3.6.1}$$

this is a standard triple unless $\mathbf{d} = [1^n]$, in which case $\mathcal{O}_{[1^n]}$ is the zero orbit. Let k be the largest index such that the entry d_k of the partition $[d_1, \ldots, d_n]$ is positive. Then the neutral element of this triple takes the following form:

$$H_{[d_1,\ldots,d_n]} = \phi_{\mathcal{O}_{[d_1,\ldots,d_n]}}(H) = \begin{pmatrix} D(d_1) & 0 & \ldots & 0 \\ 0 & D(d_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D(d_k) \end{pmatrix}, \tag{3.6.2}$$

where

$$D(d_i) = \begin{pmatrix} d_i-1 & 0 & 0 & \dots & 0 & 0 \\ 0 & d_i-3 & 0 & \dots & 0 & 0 \\ 0 & 0 & d_i-5 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -d_i+3 & 0 \\ 0 & 0 & 0 & \dots & 0 & -d_i+1 \end{pmatrix}$$

for $1 \le i \le k$. The diagonal matrices of trace 0 form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n$. We may choose a set of positive roots in such a way that the corresponding Borel subalgebra consists of the upper triangular matrices of trace 0. With this choice of positive roots, an element $\operatorname{diag}(h_1, \dots, h_n) \in \mathfrak{h}$ is Δ-dominant if and only if

$$h_1 \ge h_2 \ge \dots \ge h_n.$$

The Weyl group is just $W = S_n$, the symmetric group on n letters; by (2.2.3) it acts on a diagonal matrix by permuting its diagonal elements. In order to compute $\Delta(\mathcal{O}_{[d_1,\dots,d_n]})$, it suffices to conjugate $H_{[d_1,\dots,d_n]}$ to a Δ-dominant matrix via W. This is done as follows. Form the set

$$S_{[d_1,\dots,d_n]} = \bigcup_{1\le i\le k} \{d_i-1, d_i-3, \dots, -d_i+1\},$$

possibly with multiplicities. Reorder the elements of this set as $h_1 \ge h_2 \ge \dots \ge h_n$, and form the diagonal matrix

$$\tilde{H}_{[d_1,\dots,d_n]} = \begin{pmatrix} h_1 & 0 & 0 & \dots & 0 \\ 0 & h_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h_n \end{pmatrix}. \tag{3.6.3}$$

This is the desired Δ-dominant matrix. We have

mma 3.6.4. *If $\tilde{H}_{[d_1,\dots,d_n]}$ is chosen as in (3.6.3), then the weighted Dynkin diagram $\Delta(\mathcal{O}_{[d_1,\dots,d_n]})$ is given by*

$$\overset{h_1-h_2}{\circ}\!\!-\!\!-\!\!\overset{h_2-h_3}{\circ}\!\!-\!\!-\; \cdots \;-\!\!-\!\!\overset{h_{n-2}-h_{n-1}}{\circ}\!\!-\!\!-\!\!\overset{h_{n-1}-h_n}{\circ}.$$

We also obtain a general property of weighted Dynkin diagrams for $\mathfrak{sl}_n$.

mma 3.6.5. *In the case of $\mathfrak{sl}_n$, the weights of $\Delta(\mathcal{O}_{[d_1,\dots,d_n]})$ attached to a nilpotent orbit are symmetric about the center of the diagram; i.e., the weighted Dynkin diagrams are invariant under the nontrivial graph automorphism.*

Example 3.6.6. Consider the case of $\mathfrak{sl}_4$. The table below gives the nilpotent orbits, the elements $H_{[d_1,\dots,d_n]}$ and $\tilde{H}_{[d_1,\dots,d_n]}$, and the associated weighted Dynkin diagrams.

Nilpotent orbits in $\mathfrak{sl}_4$

Orbit	$H_{[d_1,\dots,d_n]}$	$\tilde{H}_{[d_1,\dots,d_n]}$	$\Delta(\mathcal{O}_{[d_1,\dots,d_n]})$
$\mathcal{O}_{[4]}$	diag(3,1,-1,-3)	diag(3,1,-1,-3)	$\overset{2}{\circ}-\overset{2}{\circ}-\overset{2}{\circ}$
$\mathcal{O}_{[3,1]}$	diag(2,0,-2,0)	diag(2,0,0,-2)	$\overset{2}{\circ}-\overset{0}{\circ}-\overset{2}{\circ}$
$\mathcal{O}_{[2^2]}$	diag(1,-1,1,-1)	diag(1,1,-1,-1)	$\overset{0}{\circ}-\overset{2}{\circ}-\overset{0}{\circ}$
$\mathcal{O}_{[2,1^2]}$	diag(1,-1,0,0)	diag(1,0,0,-1)	$\overset{1}{\circ}-\overset{0}{\circ}-\overset{1}{\circ}$
$\mathcal{O}_{[1^4]}$	diag(0,0,0,0)	diag(0,0,0,0)	$\overset{0}{\circ}-\overset{0}{\circ}-\overset{0}{\circ}$

3.7 Centralizer Structure

Let X be the nilpositive element of a standard triple $\{H, X, Y\}$. Recall that introduced a group U^X and its Lie algebra $\mathfrak{u}^X$ in §3.4. As promised there, now define a group U and its Lie algebra $\mathfrak{u}$ such that $U^X, \mathfrak{u}^X$ are the centraliz of X in $U, \mathfrak{u}$. We also define another important subgroup of G_{ad}. Put $\mathfrak{u} = \oplus_{i>}$ (notation (3.4.1)), and let U be the connected subgroup of G_{ad} with Lie algel $\mathfrak{u}$. By the result in [33] quoted in §3.4, we have $U = \operatorname{Exp} \mathfrak{u}$. It follows that U is the centralizer of X in U. Now set

$$\mathfrak{g}^\phi = \{\, Z \in \mathfrak{g} \mid [Z, V] = 0, \text{ for all } V \in \mathbb{C}\langle H, X, Y\rangle \,\}, \qquad (3.7$$

the centralizer of the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ spanned by $\{H, X, Y\}$. The letter denotes the homomorphism of $\mathfrak{sl}_2$ into $\mathfrak{g}$ determined by the standard trip Similarly let G^ϕ_{ad} denote the centralizer of $\mathfrak{a}$ in G_{ad}. This subgroup need *not* connected; let $(G^\phi_{ad})^o$ denote its identity component and

$$A(\mathcal{O}_X) := G^\phi_{ad}/(G^\phi_{ad})^o \qquad (3.7$$

denote its component group. Despite the notation, the definition of $A(\mathcal{O}$ depends on the choice of ϕ; two different maps ϕ attached to the same orbit yield isomorphic groups $A(\mathcal{O})$, but the isomorphism is uniquely defined only to an inner automorphism. Thus, we have

Lemma 3.7.3 (Barbasch-Vogan, Kostant). *Retain the above notations. There is a direct sum decomposition $\mathfrak{g}^X = \mathfrak{u}^X \oplus \mathfrak{g}^\phi$ in which $\mathfrak{u}^X$ is a nil ideal of $\mathfrak{g}^X$ and $\mathfrak{g}^\phi$ is reductive. This lifts to a semidirect product decomposition $G_{ad}^X = U^X \cdot G_{ad}^\phi$, and the map $G_{ad}^\phi \hookrightarrow G_{ad}^X$ induces an isomorphism from $A(\mathcal{O}_X)$ to the component group of G_{ad}^X. (This justifies our notation.)*

Proof. We have already observed that $\mathfrak{u}^X$ is a nilpotent ideal of $\mathfrak{g}^X$, whence U^X is a normal subgroup of G_{ad}^X. From $\mathfrak{sl}_2$ theory we get $\mathfrak{g}^\phi = \mathfrak{g}_0^X$ (notation (3.4.3)). The direct sum decomposition follows at once; we also get $U^X \cap G_{ad}^\phi = 1$, since $U^X = \operatorname{Exp} \mathfrak{u}^X$. To see that

$$G_{ad}^X = U^X \cdot G_{ad}^\phi,$$

let $x \in G_{ad}^X$ and note that

$$x \cdot \phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x \cdot X = X.$$

Then $\{H, X, Y\}$ and $x \cdot \{H, X, Y\}$ are two standard triples with the same nilpositive element. By Kostant's conjugacy theorem (3.4.10), there exists $y \in U^X$ such that

$$yx \cdot \phi = \phi.$$

This implies that $yx \in G_{ad}^\phi$, whence the semidirect product decomposition follows. The last assertion follows from the connectedness of U^X.

It remains to prove that $\mathfrak{g}^\phi$ is a reductive Lie algebra. By (1.1.6)(iii), it is enough to show that the Killing form κ is nondegenerate when restricted to $\mathfrak{g}^\phi$ (this restriction is the trace form of the adjoint representation of $\mathfrak{g}^\phi$ on $\mathfrak{g}$). For the remainder of the proof, we will use the notation $\perp$ to denote the κ-orthogonal. By the proof of the Jacobson-Morozov theorem (see (3.3.4)), we know that $(\mathfrak{g}^X)^\perp = [\mathfrak{g}, X]$. Then (3.4.5) implies that

$$\mathfrak{u}^X = \mathfrak{g}^X \cap [\mathfrak{g}, X] = \mathfrak{g}^X \cap (\mathfrak{g}^X)^\perp. \tag{3.7.4}$$

Thus κ restricts to a nondegenerate form on $\mathfrak{g}^X/(\mathfrak{g}^X \cap (\mathfrak{g}^X)^\perp) = \mathfrak{g}^X/\mathfrak{u}^X \cong \mathfrak{g}_0^X = \mathfrak{g}^\phi$, as desired. □

Remarks 3.7.5 (i) It follows from $\mathfrak{sl}_2$ theory that

$$\mathfrak{g}^Y = \bigoplus_{i \le 0} (\mathfrak{g}^Y \cap \mathfrak{g}_i)$$

and, thus, that

$$\mathfrak{g}^X \cap \mathfrak{g}^Y = \mathfrak{g}^\phi.$$

(ii) It should be noted that the assertion $G_{ad}^X/(G_{ad}^X)^o = G_{ad}^\phi/(G_{ad}^\phi)^o$ holds when G_{ad} is replaced by any complex connected semisimple Lie group G with Lie algebra $\mathfrak{g}$. This will be important in §6.1 where we compute not only the group $A(\mathcal{O}_X)$ but also the fundamental group of a nilpotent orbit.

3.8 Parabolic Subalgebras

In this section, we want to see how centralizers of nilpotent elements fit into the set of subalgebras of $\mathfrak{g}$. We define a *parabolic subalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ to be any subalgebra of $\mathfrak{g}$ containing a Borel subalgebra. Ultimately we want to relate the centralizer of a nilpotent element to a particular parabolic subalgebra. Consequently, we begin with a few general structure-theoretic remarks.

Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra, $\mathfrak{n} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and Δ denotes the set of simple roots determined by our choice of positive roots Φ^+. There is a standard construction of all possible parabolic subalgebras containing $\mathfrak{b}$. Let $\Theta \subset \Delta$ and define $\mathfrak{p}_\Theta$ to be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and all of the root spaces $\mathfrak{g}_\alpha$ such that $\alpha \in \Delta$ or $-\alpha \in \Theta$. Let $\langle\Theta\rangle$ denote the subroot system of Φ generated by Θ and put $\langle\Theta\rangle^+ = \langle\Theta\rangle \cap \Phi^+$. The following lemma is fundamental in the structure theory of semisimple Lie algebras; see [37, p.87,ex.6].

Lemma 3.8.1. *Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{b}$ a Borel subalgebra, and Δ the system of simple roots determined by $\mathfrak{b}$.*

(i) *If $\Theta \subset \Delta$, then $\mathfrak{p}_\Theta$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$.*

(ii) *Every parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ is of the form $\mathfrak{p}_\Theta$, for some $\Theta \subset \Delta$.*

(iii) *Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to one of the form $\mathfrak{p}_\Theta$, for some $\Theta \subset \Delta$. Moreover, $\mathfrak{p}_\Theta$ is G_{ad}-conjugate to $\mathfrak{p}_\Xi$ if and only if $\Theta = \Xi$. In particular, there are $2^{\operatorname{rank}[\mathfrak{g},\mathfrak{g}]}$ conjugacy classes of parabolic subalgebras in $\mathfrak{g}$.*

(iv) *We have a direct sum decomposition $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$, where $\mathfrak{l}_\Theta$, called a Levi subalgebra of $\mathfrak{g}$, is reductive and $\mathfrak{n}_\Theta$ is the nilradical of $\mathfrak{p}_\Theta$. More precisely, we have $\mathfrak{l}_\Theta = \mathfrak{h} \oplus \sum_{\alpha \in \langle\Theta\rangle} \mathfrak{g}_\alpha$ and $\mathfrak{n}_\Theta = \sum_{\alpha \in \Phi^+ \setminus \langle\Theta\rangle^+} \mathfrak{g}_\alpha$, where $\setminus$ denotes set subtraction.*

(v) *Two Levi subalgebras $\mathfrak{l}_\Theta$ and $\mathfrak{l}_{\Theta'}$ are G_{ad}-conjugate if and only if the root systems $\langle\Theta\rangle$ and $\langle\Theta'\rangle$ are W-conjugate.*

Example 3.8.2. Consider the case of $\mathfrak{sl}_4$ and choose Δ in (2.2.8). By (3.8.1), there are $2^3 = 8$ conjugacy classes of parabolic subalgebras in $\mathfrak{sl}_4$. In the table below (after 3.8.8), we catalogue these eight parabolic subalgebras by starring the simple roots in the Dynkin diagram that lie in the set Θ. We use the following (standard) enumeration of the simple roots in the Dynkin diagram of type A_3:

$$\begin{array}{ccccc} e_1 - e_2 & & e_2 - e_3 & & e_3 - e_4 \\ 0 & \text{---} & 0 & \text{---} & 0\,. \end{array}$$

The table also indicates the form of the matrices in the Levi subalgebra and nilradical of each $\mathfrak{p}_\Theta$. This is accomplished via a figure of the following sort:

$$\mathfrak{p}_\Theta = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

which means that $\mathfrak{p}_\Theta$ consists of all matrices in $\mathfrak{sl}_4$ with zeroes in the indicated places; the other entries are arbitrary.

The parabolic subalgebras $\mathfrak{p}_\Theta$ are often called the *standard parabolic subalgebras* of $\mathfrak{g}$ (relative to $\mathfrak{b}$). The decomposition in part (iv) is called a *Levi decomposition* of $\mathfrak{p}_\Theta$. The two extreme cases $\mathfrak{p} = \mathfrak{b}$ and $\mathfrak{p} = \mathfrak{g}$ correspond to $\Theta = \emptyset$ and $\Theta = \Delta$, respectively. If $\Theta = \{\beta\}$ contains a single simple root, then $\mathfrak{l}_\Theta = \mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$ and $\mathfrak{n}_\Theta = \sum_{\alpha\in\Phi^+\setminus\{\beta\}} \mathfrak{g}_\alpha$; in particular, the semisimple part of the Levi subalgebra is isomorphic to $\mathfrak{sl}_2$. Since any two Cartan subalgebras of $\mathfrak{g}$ contained in the same Borel subalgebra $\mathfrak{b}$ are conjugate under the adjoint group of $\mathfrak{b}$ [37, §16.2], it follows that any two Levi subalgebras of the same parabolic subalgebra $\mathfrak{p}$ are conjugate.

With notation as in (3.4.1), define

$$\begin{aligned} \mathfrak{q} &= \textstyle\bigoplus_{i\geq 0} \mathfrak{g}_i, \\ \mathfrak{l} &= \mathfrak{g}_0, \\ \mathfrak{u} &= \textstyle\bigoplus_{i>0} \mathfrak{g}_i; \end{aligned} \tag{3.8.3}$$

note that $\mathfrak{u}$ was already defined in the last section.

Lemma 3.8.4. *Let $\{H,X,Y\}$ be a standard triple in $\mathfrak{g}$ and define $\mathfrak{q}$ as in (3.8.3). Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Moreover, $\mathfrak{g}^X = \mathfrak{u}^X \oplus \mathfrak{g}^\phi$ is a subalgebra of $\mathfrak{q}$ and $\mathfrak{g}^\phi$ is a subalgebra of $\mathfrak{l}$.*

Proof. We observed in §3.5 that we can choose a Cartan subalgebra $\mathfrak{h}$ containing H and a set of positive roots of $\mathfrak{h}$ in $\mathfrak{g}$ such that $\alpha(H)$ is a nonnegative integer for all positive roots α. It follows at once that $\mathfrak{q}$ contains a Borel subalgebra. The other assertions are left to the reader. □

Remarks 3.8.5. (i) The parabolic subalgebra $\mathfrak{q}$ attached to the standard triple $\{H,X,Y\}$ in (3.8.4) is also denoted $\mathfrak{q}_X$ and called the *Jacobson-Morozov parabolic subalgebra* of X. Although its definition seems to depend on the choice of standard triple $\{H,X,Y\}$, it is actually uniquely determined by X alone (by (3.4.10)), since any two such parabolic subalgebras are G_{ad}^X-conjugate and $\mathfrak{g}^X \subset \mathfrak{q}$. The importance of this link between nilpotent elements and parabolic subalgebras will become more apparent in Chapter 8, where we discuss the Bala-Carter classification of nilpotent orbits.

(ii) By contrast, recall from (2.1.2) that centralizers of semisimple elements are reductive.

We can form a number of auxiliary subalgebras:

$$\begin{aligned} \bar{\mathfrak{q}} &= \mathfrak{l} \oplus \bar{\mathfrak{u}} = \mathfrak{l} \oplus \textstyle\sum_{i<0} \mathfrak{g}_i \ \text{ the opposite parabolic subalgebra to } \mathfrak{q}; \\ \mathfrak{q}_2 &= \textstyle\sum_{i\geq 2} \mathfrak{g}_i \subset \mathfrak{q}; \\ \bar{\mathfrak{q}}_2 &= \textstyle\sum_{i\leq -2} \mathfrak{g}_i \subset \bar{\mathfrak{q}}; \\ \mathfrak{q}_3 &= \textstyle\sum_{i\geq 3} \mathfrak{g}_i \subset \mathfrak{q}; \\ \bar{\mathfrak{q}}_3 &= \textstyle\sum_{i\leq -3} \mathfrak{g}_i \subset \bar{\mathfrak{q}}. \end{aligned} \tag{3.8.6}$$

If $\mathfrak{g}_1 = 0$, then we call $X, \mathcal{O}_X, \{H,X,Y\}$ an *even nilpotent element*, *even nilpotent orbit*, and *even standard triple*, respectively. The zero nilpotent orbit $\{0\}$ is also called even. Using $\mathfrak{sl}_2$ theory, it is easy to verify the next lemma and its corollary.

Lemma 3.8.7. *Let X be a nonzero nilpotent element in a semisimple Lie algebra $\mathfrak{g}$. Then X is even if and only if $\mathfrak{g}_{2k+1} = 0$, for all $k \in \mathbb{Z}$. In other words, if X is the nilpositive element of a standard triple and we decompose $\mathfrak{g}$ as the direct sum of irreducible representations of the subalgebra spanned by this triple, then all of the summands have even highest weight.*

Corollary 3.8.8. *$\mathcal{O}_X$ is an even nilpotent orbit if and only if its weighted Dynkin diagram involves only the labels 0 and 2.*

Parabolic Subalgebras in $\mathfrak{sl}_4$

Θ	$\mathfrak{p}_\Theta$	$\mathfrak{l}_\Theta$	$\mathfrak{n}_\Theta$
○—○—○	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—○—○	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
○—⊛—○	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
○—○—⊛	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—⊛—○	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
○—⊛—⊛	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—○—⊛	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—⊛—⊛	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4 Principal, Subregular, and Minimal Nilpotent Orbits

With the Finiteness Theorem behind us, the time has come to develop basic methods for constructing nilpotent orbits. In this chapter, we use general principles to establish the existence of three canonical nilpotent orbits: the *principal (or regular) orbit*, the *subregular orbit*, and the *minimal orbit*. To describe the sense in which these orbits are canonical, we need to introduce a partial order on the set of nilpotent orbits, which depends on the Zariski closure operation. We say that $\mathcal{O} \leq \mathcal{O}'$ if $\overline{\mathcal{O}} \subseteq \overline{\mathcal{O}}'$. To see that $\leq$ really defines a partial order, we need a basic fact from the theory of algebraic groups that was already mentioned above: every orbit is Zariski-open in its closure [79, 4.3.1]. It follows at once that the boundary of an orbit (defined to be its closure minus the orbit itself) must have a smaller dimension than the orbit, since it is a Zariski-closed proper subvariety of the closure. (This should not be too surprising. We already noted above that orbits are smooth manifolds, whence orbit closures are analogous to (though not quite) manifolds with boundary.) Thus, it is in particular impossible for distinct orbits to have the same closure. Now the three orbits mentioned above are completely determined by their positions in the Hasse diagram ([81, p.98]) of this partial order. More precisely, if $\mathfrak{g}$ is simple, one has the following rudimentary picture of this Hasse diagram.

orbit	dimension
$\mathcal{O}_{prin}$	$\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g})$
$\mathcal{O}_{subreg}$	$\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g}) - 2$
fuzzy structure	
$\mathcal{O}_{min}$	See (4.3.5)
$\{0\}$	0

In §6.2, after we have general partition type classification schemes for the classical cases, we will describe the "fuzzy structure" in this Hasse diagram more precisely. In fact, in the classical cases, we will give a simple partition criterion for deciding when one orbit is less than another. The partial order has also been worked out completely in the exceptional cases; see [76].

One can also construct nilpotent orbits in $\mathfrak{g}$ from nilpotent orbits in Levi factors of parabolic subalgebras. The technique for doing this is called *induction* and will be described in Chapter 7.

We will conclude with a survey of the theory of exponents of a semisimple Lie group. We will exhibit a beautiful connection between the principal nilpotent orbit, the cohomology of a complex Lie group, invariant theory, and finite Chevalley groups. This material is not used in the sequel and can be omitted on first reading.

4.1 The Principal Nilpotent Orbit

At this point, although we have defined the set $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$ and shown that it consists of finitely many orbits, we don't yet know much else about it; for example, we have not computed its dimension as an algebraic variety. We will do this now and simultaneously refine our discussion of regular elements in §2.1.

We begin this section with two easy lemmas.

Lemma 4.1.1. *Let $\mathfrak{n}$ be the nilradical of a Borel subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Then $G_{ad} \cdot \mathfrak{n} = \mathcal{N}$. In particular, if $\mathcal{X}$ is a dense subset of $\mathfrak{n}$, then $G_{ad} \cdot \mathcal{X}$ is dense in $\mathcal{N}$.*

Proof. This follows from (3.2.1) and (3.2.2). □

Lemma 4.1.2. *Let $\mathfrak{g}$ be a reductive Lie algebra. Then the trivial orbit $\{0\}$ is the unique nilpotent orbit of minimal dimension.*

Proof. By definition, the only nilpotent element in the center $\mathfrak{z}$ of $\mathfrak{g}$ is 0. Now the assertion follows at once from (1.2.15). □

The formula (1.2.15) for computing orbit dimensions is often difficult to apply in practice. The next lemma gives a handy alternative formula.

Lemma 4.1.3. *Let $\{H, X, Y\}$ be a standard triple in a semisimple Lie algebra $\mathfrak{g}$ and form the ad_H-eigenspace decomposition $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ in (3.4.1). Then $\dim(\mathcal{O}_X) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_1)$. In particular, if $\mathcal{O}_X$ is an even nilpotent orbit, then $\dim(\mathcal{O}_X) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_0)$.*

Proof. Let $\mathfrak{a}$ be the $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}$ spanned by the standard triple. Decompose $\mathfrak{g}$ as a direct sum of irreducible $\mathfrak{a}$-modules. By $\mathfrak{sl}_2$ theory, each such module contributes a 1-dimensional subspace to $\mathfrak{g}^X$, and $\mathfrak{g}^X$ is the direct sum of these subspaces. On the other hand, each such module also contributes a 1-dimensional subspace to one of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ (accordingly as its highest weight is even or odd), and $\mathfrak{g}_0 + \mathfrak{g}_1$ is the direct sum of these subspaces. Now the result follows from (1.2.15) (cf. [37, §7.2]). □

Remarks. (i) This lemma does *not* say that $\mathfrak{g}^X = \mathfrak{g}_0 + \mathfrak{g}_1$; in fact, we saw in §3.7 that this is far from the case.

(ii) It follows that the dimension of a nilpotent orbit may be easily read off from its weighted Dynkin diagram.

The next result plays a fundamental rôle in all our subsequent work on orbit closures.

Lemma 4.1.4 (Kostant [52]). *Let X be the nilpositive element of a standard triple $\{H, X, Y\}$. Write $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ as above and recall the subalgebras $\mathfrak{q}_2 = \oplus_{i\geq 2}\mathfrak{g}_i$, $\mathfrak{q}_3 = \oplus_{i\geq 3}\mathfrak{g}_i$, $\mathfrak{q} = \mathfrak{q}_X = \oplus_{i\geq 0}\mathfrak{g}_i$, defined in §3.8. Let Q be the (closed) connected Lie subgroup of G_{ad} having Lie algebra $\mathfrak{q}$. Set $\mathcal{P} = (G_{ad}^H)^o \cdot X$, as in the proof of Mal'cev's Theorem (3.4.12). Then $\mathcal{O}_X \cap \mathfrak{q}_2 = Q\cdot X = \mathcal{P} + \mathfrak{q}_3 := \{W + Z \mid W \in \mathcal{P}, Z \in \mathfrak{q}_3\}$. In particular, $\mathcal{O}_X \cap \mathfrak{q}_2$ is open dense in $\mathfrak{q}_2$ (by (3.4.12)).*

Proof. From $\mathfrak{sl}_2$ theory we get $\mathfrak{q}_3 \subset [X, \mathfrak{u}]$, whence it follows as in the proof of (3.4.7) that

$$X + \mathfrak{q}_3 \subset Q \cdot X.$$

But we saw in the proof of (3.4.12) that *any* element of $\mathcal{P}$ embeds into a standard triple with neutral element H. Hence,

$$\mathcal{P} + \mathfrak{q}_3 \subset Q \cdot X.$$

The definitions of $\mathcal{P}, Q$ show that the reverse inclusion also holds. Thus $Q\cdot X = \mathcal{P} + \mathfrak{q}_3$ and $Q \cdot X$ is open dense in $\mathfrak{q}_2$. It only remains to show that

$$Q \cdot X = (G_{ad} \cdot X) \cap \mathfrak{q}_2. \tag{4.1.5}$$

Let $Z \in (G_{ad}\cdot X)\cap\mathfrak{q}_2$; then $\dim\mathfrak{g}^Z = \dim\mathfrak{g}^X$. Since $\mathfrak{g}^X \subset \mathfrak{q}$ by (3.4.3), it follows that $\dim\mathfrak{q}^Z = \dim(\mathfrak{g}^Z \cap \mathfrak{q}) \leq \dim\mathfrak{q}^X$ and, hence, that $\dim Q\cdot Z \geq \dim Q \cdot X$. Thus, the orbit $Q \cdot Z$ has full dimension in $\mathfrak{q}_2$ and so must be an open subset of the latter. Since $Q \cdot X$ is dense in $\mathfrak{q}_2$, $Q\cdot Z$ must meet $Q \cdot X$. Thus, $Q\cdot Z = Q\cdot X$, as desired. □

Remark. Like Mal'cev's Theorem, this result generalizes immediately to semisimple Lie algebras over any algebraically closed basefield of characteristic zero. One simply replaces the Euclidean by the Zariski topology and invokes the Zariski-openness of orbits in their closures, which we also observed at the beginning of the chapter. A similar remark applies to the next theorem.

heorem 4.1.6 (de Siebenthal, Dynkin, Kostant). *Let $\mathfrak{g}$ be a semisimple Lie algebra. There exists a unique nilpotent orbit in $\mathfrak{g}$ of dimension equal to* $\dim(\mathfrak{g})-\operatorname{rank}(\mathfrak{g})$, *denoted $\mathcal{O}_{prin}$. This is an even nilpotent orbit that is open and dense in $\mathcal{N}$. The weighted Dynkin diagram of this orbit has every node labeled 2.*

We call this orbit *principal* or *regular*. It follows from (2.1.15) that a nilpotent element is regular if and only if it belongs to this orbit; this justifies our terminology. We also see that the neutral element of any standard triple with a regular nilpositive element is also regular.

Proof. We use an argument of Kostant. Fix a Cartan subalgebra $\mathfrak{h}$, a system of roots Φ, and a basis of simple roots Δ. Let $\mathfrak{b} = \mathfrak{h}\oplus\mathfrak{n}$ be the corresponding Borel subalgebra, as in (3.2.1). For each $\alpha \in \Delta$, form a standard triple $\{H_\alpha, X_\alpha, Y_\alpha\}$, where $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$. Recall that these standard triples exist just by the basic structure theory of semisimple Lie algebras, so that one need not invoke the Jacobson-Morozov Theorem. Choose $H \in \mathfrak{h}$ so that $\alpha(H) = 2$ for all $\alpha \in \Delta$. Because $\{H_\alpha \,|\, \alpha \in \Delta\}$ spans $\mathfrak{h}$, we can find $a_\alpha \in \mathbb{C}$ so that

$$H = \sum_{\alpha\in\Delta} a_\alpha H_\alpha.$$

Set

$$X = \sum_{\alpha\in\Delta} X_\alpha, \quad Y = \sum_{\alpha\in\Delta} a_\alpha Y_\alpha.$$

Then

$$\begin{aligned}
[H,X] &= \sum_{\alpha\in\Delta} \alpha(H) X_\alpha = 2X,\\
[H,Y] &= \sum_{\alpha\in\Delta} a_\alpha(-\alpha(H))Y_\alpha = -2Y, \text{ and}\\
[X,Y] &= \sum_{\alpha,\beta\in\Delta} a_\beta [X_\alpha, Y_\beta] = \sum_{\alpha\in\Delta} a_\alpha [X_\alpha, Y_\alpha] = \sum_{\alpha\in\Delta} a_\alpha H_\alpha = H,
\end{aligned}$$

since the difference of two simple roots is never a root.

Thus, $\{H, X, Y\}$ is a standard triple. We claim that the nilpotent orbit $\mathcal{O}_X$ has the desired properties. Clearly it is even, and in fact every node in its weighted Dynkin diagram is labeled 2. We also check immediately that

$$\mathfrak{q}_2 = \sum_{i\geq 2} \mathfrak{g}_i = \mathfrak{n} \text{ and } \mathfrak{g}_2 = \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha. \tag{4.1.7}$$

Then $\overline{\mathcal{O}}_X = G_{ad}\cdot\mathfrak{n} = \mathcal{N}$ by (4.1.1) and (4.1.4). We also see that $\mathcal{O}_X$ is open dense in $\mathcal{N}$ (since $\mathcal{O}_X \cap \mathfrak{n}$ is open dense in $\mathfrak{n}$). It follows that $\mathcal{O}_X$ is the unique open and the unique dense orbit in $\mathcal{N}$. It only remains to compute its dimension. We clearly have $\mathfrak{g}_0 = \mathfrak{h}, \mathfrak{g}_1 = 0$. Now (4.1.3) completes the proof. □

Corollary 4.1.8. *We have* $\dim(\mathcal{N}) = \dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g})$.

Remark. The set $\mathcal{N}$ is an irreducible affine variety. At the end of §4.4 we will give a set of defining equations for it.

In §5.4, we will identify the principal nilpotent orbit in the classical cases. For now, we content ourselves with a look at the situation for $\mathfrak{sl}_n$.

Example 4.1.9. In the case of $\mathfrak{sl}_n$, we compute as in §3.6 that the nilpotent orbit $\mathcal{O}_{[n]}$ attached to the partition $[n]$ has weighted Dynkin diagram with every node labeled 2. Thus, $\mathcal{O}_{[n]}$ is the principal nilpotent orbit in $\mathfrak{sl}_n$.

4.2 The Subregular Nilpotent Orbit

Recall the notion of a regular element, defined after (2.1.15). We say that an element Z in $\mathfrak{g}$ is *subregular* if $\dim(\mathfrak{g}^Z) = \operatorname{rank}(\mathfrak{g}) + 2$. By (1.2.15) and (1.4.8) the subregular elements are precisely those whose orbits have *second* largest dimension. By (2.1.2), the subregular semisimple elements are precisely those whose centralizers are reductive with the semisimple part isomorphic to $\mathfrak{sl}_2$. We would now like to address the existence, uniqueness, and properties of nilpotent orbits consisting of subregular elements. We will postpone one step of the proof of the next result to §7.1, since it is more convenient to obtain it as a special case of results proved there than to perform a lengthy ad hoc construction here.

Theorem 4.2.1 (Steinberg [82]). *Assume $\mathfrak{g}$ is a simple Lie algebra and let $\mathcal{O}_{prin}$ be the principal nilpotent orbit. There exists a unique nilpotent orbit that is open and dense in $\overline{\mathcal{O}}_{prin} \setminus \mathcal{O}_{prin} = \mathcal{N} \setminus \mathcal{O}_{prin}$, denoted $\mathcal{O}_{subreg}$ (where $\setminus$ denotes set subtraction, as in (3.8.1)). The dimension of this orbit is* $\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g}) - 2$.

It is clear that the orbit $\mathcal{O}_{subreg}$ consists of subregular elements, so we call it the *subregular nilpotent orbit*.

Proof. Let $\{H, X, Y\}$ be the standard triple constructed in the proof of (4.1.6), so that $\mathcal{O}_X = \mathcal{O}_{prin}$. Define the subalgebras $\mathfrak{g}_2, \mathfrak{q}, \mathfrak{q}_2, \mathfrak{q}_3$ as in (4.1.4) (relative to H). The first step is to identify the *complement* of the set $\mathcal{P}$ occurring there precisely. Recall that $\mathcal{P}$ consists exactly of the elements X' of $\mathfrak{g}_2$, such $\{H, X', Y'\}$ is a standard triple for some Y' (actually unique) in $\mathfrak{g}_{-2}$. We saw in the proof of (4.1.6) that we may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, and the corresponding set Δ of simple roots so that $H \in \mathfrak{h}$ and $\mathfrak{g}_2$ is the sum of the simple root spaces of $\mathfrak{h}$. Since the difference of two simple roots is never a root, the proof of (4.1.6) shows that any linear combination $X' = \sum_{\alpha\in\Delta} c_\alpha X_\alpha$ of simple root vectors $X_\alpha \in \mathfrak{g}_\alpha$ belongs to $\mathcal{P}$ whenever all c_α are nonzero. Conversely, suppose that at least one c_α is zero. Then any element of $[X', \mathfrak{g}_{-2}]$ is a linear combination of various H_β for $\beta \in \Delta, \beta \neq \alpha$. Since our neutral element H *cannot* be realized as such a combination, it follows that $X' \notin \mathcal{P}$. Hence, the complement $\mathcal{P}'$ of $\mathcal{P}$ in $\mathfrak{g}_2$ is a union of hyperplanes

$$\mathcal{D}_\alpha = \{\, Z \in \mathfrak{g}_2 \,|\, Z = \sum_{\beta\in\Delta} c_\beta X_\beta, c_\alpha = 0 \,\},$$

one for each $\alpha \in \Delta$. Put $\mathcal{C}_\alpha = \mathcal{D}_\alpha + \mathfrak{q}_3$; then the complement of $Q \cdot X$ in $\mathfrak{q}_2$ is the union of the $\mathcal{C}_\alpha$.

Fix $\alpha \in \Delta$. The next step is to locate an orbit $\mathcal{O}_\alpha$ in $\mathcal{N}$ whose intersection with $\mathcal{C}_\alpha$ is open dense in the latter. It is here that we invoke the theory of §7.1 (which does not depend on this theorem). The hyperplane $\mathcal{C}_\alpha$ coincides with $\mathfrak{n}_{\{\alpha\}}$, the nilradical of the parabolic subalgebra $\mathfrak{p}_{\{\alpha\}}$ containing $\mathfrak{b}$ (notation of (3.8.1) with $\Theta = \{\alpha\}$). Now (7.1.1) and (7.1.4) guarantee the existence and uniqueness of $\mathcal{O}_\alpha$ and furthermore show that its dimension is twice that of $\mathcal{C}_\alpha$. We obtain

$$\dim(\mathcal{O}_\alpha) = \dim(\mathcal{O}_\beta) = \dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g}) - 2 \text{ for all } \alpha, \beta \in \Delta. \tag{4.2.2}$$

To complete the proof of the theorem, it is enough to show that $\mathcal{O}_\alpha = \mathcal{O}_\beta = \mathcal{O}$ for all $\alpha, \beta \in \Delta$, for then it is clear that $\mathcal{O}$ is dense and thus open in $\mathcal{N} \setminus \mathcal{O}_{prin}$, whence it is the unique orbit with these properties (and the unique orbit of its dimension in $\mathcal{N}$). Since $\mathfrak{g}$ is simple, it is enough to show that $\mathcal{O}_\alpha = \mathcal{O}_\beta$ if α and β are adjacent simple roots. Now, in the language of §7.1, $\mathcal{O}_\alpha$ is induced from the zero orbit in $\mathfrak{l}_{\{\alpha\}}$; similarly $\mathcal{O}_\beta$ is induced from the zero orbit in $\mathfrak{l}_{\{\beta\}}$. By (7.1.4), we could also have obtained $\mathcal{O}_\alpha$ by first inducing the zero orbit in $\mathfrak{l}_{\{\alpha\}}$ to the Levi subalgebra $\mathfrak{l}_{\{\alpha,\beta\}}$ and then inducing to $\mathfrak{g}$; a similar remark applies to $\mathcal{O}_\beta$. So it suffices to show that the zero orbits in $\mathfrak{l}_{\{\alpha\}}, \mathfrak{l}_{\{\beta\}}$ induce to the same orbit in $\mathfrak{l}_{\{\alpha,\beta\}}$. Equivalently, it suffices to assume that $\mathfrak{g}$ has rank 2 with simple roots α and β, and then prove that $\mathcal{O}_\alpha = \mathcal{O}_\beta$ in this situation. This is done by a case-by-case computation: for every possible $\mathfrak{g}$, we produce an orbit that simultaneously satisfies the defining properties of $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$. Indeed, suppose first that $\mathfrak{g} \cong \mathfrak{sl}_3$, and let $\mathcal{O} = \mathcal{O}_V$ be the orbit of the highest root vector V (with respect to the simple roots α and β). One computes that $[\mathfrak{p}_{\{\alpha\}}, V] = \mathfrak{n}_{\{\alpha\}}, [\mathfrak{p}_{\{\beta\}}, V] = \mathfrak{n}_{\{\beta\}}$, whence $\mathcal{O}_V$ meets both $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ in an open set. Next let $\mathfrak{g} \cong \mathfrak{sp}_4$, and consider the orbit $\mathcal{O} = \mathcal{O}_V$ of the highest *short* root vector V. One checks as for $\mathfrak{sl}_3$ that $\mathcal{O}_V$ meets both $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ in an open set. Finally, let $\mathfrak{g}$ be of type G_2. This time the orbit $\mathcal{O} = \mathcal{O}_V$ through a sum V of two root vectors belonging to orthogonal roots does the trick. This completes the proof. □

The proof of this theorem depends strongly on special properties of the principal orbit. Given a general nilpotent orbit $\mathcal{O}$, it can happen that $\overline{\mathcal{O}} - \mathcal{O}$ does not contain an open dense orbit; we will see numerous examples of this in the sequel. Indeed, under the closure ordering, it can happen that $\overline{\mathcal{O}} - \mathcal{O}$ contains two maximal elements of different dimensions; otherwise put, the ordered poset of nilpotent orbits is not ranked by dimension. Second, even if $\overline{\mathcal{O}} - \mathcal{O}$ does contain a dense suborbit, it need not have dimension $\dim(\mathcal{O}) - 2$.

In §5.4, we will identify the partition of the subregular nilpotent orbit in a classical algebra $\mathfrak{g}$ and discuss when this orbit is even. For now, we content ourselves with observing that the partition of the subregular orbit in $\mathfrak{sl}_n$ is $[n-1, 1]$. This follows by computing the dimension of $\mathcal{O}_{[n-1,1]}$, using (3.6.4) and (4.1.3).

4.3 The Minimal Nilpotent Orbit

Whereas the principal orbit is the largest orbit in $\mathcal{N}$, we can look at the other end of the spectrum and establish the existence of small nilpotent orbits. The trivial orbit $\{0\}$ is the unique nilpotent orbit of minimal dimension, by (4.1.2). We actually have

Lemma 4.3.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. If $\mathcal{O}$ is a nilpotent orbit, then $\{0\} \subset \overline{\mathcal{O}}$.*

Proof. Let $\mathcal{O} = \mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}$. The key here is to make the following useful observation:

$$\text{If } X \text{ is nilpotent and } 0 \neq k \in \mathbb{C}, \text{ then } kX \in \mathcal{O}_X. \tag{4.3.2}$$

We will reduce this to an $\mathfrak{sl}_2$ calculation. If $X = 0$, this is obvious. Otherwise X embeds into a standard triple $\{H, X, Y\}$. Clearly any nonzero multiple of X lies in the principal orbit of $\mathbb{C}\langle H, X, Y\rangle \cong \mathfrak{sl}_2$ as the only other orbit is 0. The desired result follows at once. (Actually, we can show more precisely that any kX is conjugate to X via Exp cH for some $c \in \mathbb{C}$.) □

Remark. The proof shows that the closure of any orbit in $\mathcal{N}$ is a cone; i.e., a union of lines through the origin.

We saw in the last section that $\mathcal{N}$ has a unique "second largest" orbit if $\mathfrak{g}$ is simple; we might ask if $\mathcal{N}$ has a second smallest orbit as well. The answer is yes.

Theorem 4.3.3. *Let $\mathfrak{g}$ be a simple Lie algebra. Then there exists a nonzero nilpotent orbit of minimal dimension, denoted $\mathcal{O}_{min}$, which is contained in the closure of any nonzero nilpotent orbit. We have $\mathcal{O}_{min} = \mathcal{O}_{X_{\alpha^\#}}$, where $X_{\alpha^\#}$ is a nonzero vector in the highest root space (with respect to some Cartan subalgebra $\mathfrak{h}$ and choice of positive roots).*

Proof. Let $\mathcal{O} = \mathcal{O}_X$ be a nonzero nilpotent orbit in $\mathfrak{g}$. It is enough to prove that $\mathcal{O}_X$ contains elements arbitrarily close to $X_{\alpha^\#}$. Write

$$X = X_0 + \sum_{\alpha\in\Phi} X_\alpha, \quad X_0 \in \mathfrak{h}, X_\alpha \in \mathfrak{g}_\alpha$$

where Φ, of course, denotes the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. We claim that we may replace X by a conjugate to ensure that its component in the $\alpha^\#$-root space is not zero. If this is not already the case, choose β maximal in the standard partial order on $\Phi \cup \{0\}$ (relative to Φ^+) so that $X_\beta \neq 0$. There is a simple root α such that $\beta+\alpha$ is a root. Let Z_α be an α-root vector. The formula for $\mathrm{Exp}\, Z_\alpha \cdot X$ shows that we can find $c \in \mathbb{C}$ so that the component of $\mathrm{Exp}\, cZ_\alpha \cdot X$ in the $\beta+\alpha$-root space is nonzero. Continuing in this fashion, we arrive at a conjugate of X whose component in the $\alpha^\#$-root space is nonzero, as desired. Now choose $H \in \mathfrak{h}$ so that $\alpha(H) = 2$ for all simple roots α (as in the proof of (4.1.6)). Choose a norm on the vector space $\mathfrak{g}$ compatible with its topology. Conjugating X by $\mathrm{Exp}\, rH$ where $r \in \mathbb{R}$ is large, we can make the norm of the $\alpha^\#$-root space component of X arbitrarily large compared to the sum of the norms of its components in the other root spaces and $\mathfrak{h}$. Conjugating X to a multiple of itself by the proof of (4.3.1), we may assume that the $\alpha^\#$-root space component of X is exactly $X_{\alpha^\#}$, while $X - X_{\alpha^\#}$ has an arbitrarily small norm. Now the desired result follows. □

Remark 4.3.4. Since the boundary of any orbit is a union of orbits of smaller dimension, it follows from (4.3.1) that $\overline{\mathcal{O}}_{min} = \mathcal{O}_{min} \cup \{0\}$. We also see that $\mathcal{O}_{min}$ is the unique nilpotent orbit of its dimension. Actually, one can show that if $\mathfrak{g}$ is not of type A_n, then $\mathcal{O}_{min}$ is the unique adjoint orbit of its dimension. If $\mathfrak{g}$ is of type A_n, then it admits a 1-parameter family of semisimple orbits of the same dimension as $\mathcal{O}_{min}$. If $\mathfrak{g}$ is simple and has two root lengths, then it has another small nilpotent orbit that is sometimes useful, namely the one through a highest *short* root vector.

There is a slightly messy formula for the dimension of $\mathcal{O}_{min}$.

Lemma 4.3.5. *Let $\mathfrak{g}$ be a simple Lie algebra. Then $\dim(\mathcal{O}_{min})$ equals one plus the number of positive roots not orthogonal to the highest root $\alpha^\#$.*

Proof. Let $X_{\alpha^\#}$ be a highest root vector relative to a Cartan subalgebra $\mathfrak{h}$ and a choice Φ^+ of positive roots. Since the span of $X_{\alpha^\#}$ is ad $\mathfrak{h}$-stable, so is $\mathfrak{g}^{X_{\alpha^\#}}$. But now any positive root vector annihilates $X_{\alpha^\#}$, as does an appropriate hyperplane of $\mathfrak{h}$. A negative root vector, lying say in the β-root space, annihilates $X_{\alpha^\#}$ if and only if $(\alpha^\#, \beta) = 0$, since $\alpha^\# - \beta$ is not a root [37, §§8.4,9.4]. Now the formula for $\dim \mathcal{O}_{min}$ follows from (1.2.15). □

In §5.4, we will identify the minimal nilpotent orbit in the classical cases and give its weighted Dynkin diagram. We close by looking at the case of $\mathfrak{sl}_n$.

Example 4.3.6. Suppose $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices. Choose the positive and simple roots in the usual way:

$$\Phi^+ = \{e_i - e_j \mid 1 \leq i,j \leq n\,,\, i<j\}; \Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}.$$

Then $\alpha^\# = e_1 - e_n$ is the highest root and we can choose

$$X_{\alpha^\#} = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then by (4.3.3), we have $\mathcal{O}_{X_{\alpha^\#}} = \mathcal{O}_{min}$. On the other hand, we have already shown in §3.1 that $\mathcal{O}_{min} = \mathcal{O}_{[d_1,\dots,d_k]}$ for some partition $[d_1,\dots,d_k] \in \mathcal{P}(n)$. A short calculation shows that $X_{\alpha^\#}$ is conjugate to the matrix

$$X_{[2,1^{n-2}]} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus the partition of $\mathcal{O}_{min}$ is $[2,1^{n-2}]$. By (3.6.4), its weighted Dynkin diagram is

$$\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\cdots-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}.$$

Example 4.3.7. Consider the case $\mathfrak{g} = \mathfrak{sl}_4$. We know there are $\#|\mathcal{P}(4)| = 5$ nilpotent orbits; these were tabulated in (3.6.6). The work above tells us that the respective partitions and dimensions of $\mathcal{O}_{prin}, \mathcal{O}_{subreg}, \mathcal{O}_{min}$ are $[4], [3,1], [2,1^2]$ and 12,10,6. We also have $\mathcal{O}_{[1^4]} = 0$. There is one more orbit, namely $\mathcal{O}_{[2^2]}$. One computes from (3.6.4) and (4.1.3) that its dimension is 8. By (4.1.6) and (4.2.1), $\mathcal{O}_{subreg}$ lies below $\mathcal{O}_{prin}$ in the closure order, while every other orbit lies below $\mathcal{O}_{subreg}$. We obtain the following Hasse diagram for the closure order, in view of (4.3.3).

Hasse Diagram for $\mathfrak{sl}_4$

Orbit	dimension
$\mathcal{O}_{[4]}$	12
$\mathcal{O}_{[3,1]}$	10
$\mathcal{O}_{[2^2]}$	8
$\mathcal{O}_{[2,1^2]}$	6
$\mathcal{O}_{[1^4]}$	0

.4 The Exponents of a Semisimple Group

In this section, we describe a beautiful connection between the cohomology of a complex Lie group, the principal nilpotent orbit, invariant theory, and the formula for the order of a finite Chevalley group. We omit proofs but give ample references to the literature.

We begin by fixing a connected complex simple Lie group G. Since the group G is a manifold, we can form the *de Rham cohomology groups* $H^i(G,\mathbb{C})$, which are complex vector spaces for each $i \in \mathbb{N}$; recall [89, §5]. A fundamental problem in the cohomology theory of Lie groups is to compute the dimensions of these cohomology groups. To this end, we define a polynomial

$$p_G(t) = \sum_{i\geq 0} \dim(H^i(G,\mathbb{C}))t^i, \tag{4.4.1}$$

called the *Poincaré polynomial of* G; computing the dimensions of the de Rham cohomology groups amounts to giving a formula for this polynomial.

We begin by recalling the theory of compact real forms as given in [83, §4] or [33]. Let $\mathfrak{g}$ be the complex Lie algebra of G; then there exists a connected compact Lie group U whose real Lie algebra $\mathfrak{u}_{\mathbb{R}}$ is a real Lie subalgebra of $\mathfrak{g}$ with complexification $\mathfrak{g}$. We will need the following structure-theoretic result [33, VI.6.3].

Lemma 4.4.2. *Let G be a connected complex semisimple Lie group and U a compact real form. Then U is a deformation retract of G. In particular, $H^i(G,\mathbb{C}) = H^i(U,\mathbb{C})$, for all i.*

Thus $p_G(t) = p_U(t)$. Now we can apply the following qualitative theorem of Hopf [31, IV.4].

Theorem 4.4.3 (Hopf). *Let U be a compact connected Lie group. Then the cohomology algebra $H^*(U,\mathbb{C})$ is the cohomology algebra of a finite product of $\ell = rank(\mathfrak{g})$ spheres of odd dimension.*

The de Rham cohomology of an odd dimensional sphere S^{2m+1} is given by

$$\dim(H^i(S^{2m+1},\mathbb{C})) = \begin{cases} 1 & \text{if } i = 0, 2m+1 \\ 0 & \text{otherwise} \end{cases},$$

which implies that

$$p_G(t) = p_U(t) = \prod_{i=1}^{\ell}(1+t^{2m_i+1}) \tag{4.4.4}$$

for some nonnegative integers $m_1, m_2, \ldots, m_\ell$. We call the collection of integers $\{m_1, m_2, \ldots, m_\ell\}$ the *exponents* of G or U.

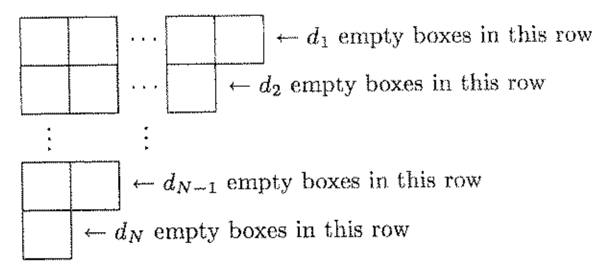
If we examine the chain complex $(\mathcal{E}^*(G), d^*)$ used to compute the de Rham cohomology groups of a complex Lie group G, we can form a subcomplex $\mathcal{E}^*(G)_{inv}$ consisting of the left invariant complex valued differential forms on G. This subcomplex is preserved by d^*, whence one obtains the left invariant de Rham cohomology groups $H^i(G,\mathbb{C})_{inv}$. An elementary exercise (see [51, §IV]) shows that $H^i(G,\mathbb{C})_{inv}$ is isomorphic to $H^i(\mathfrak{g},\mathbb{C})$, the i^{th} Lie algebra cohomology group of $\mathfrak{g}$ with coefficients in $\mathbb{C}$. Finally, it is known (see [31]) that $H^i(U,\mathbb{C}) \cong H^i(U,\mathbb{C})_{inv}$ whenever U is a connected compact Lie group. Thus we may also call the m_i the exponents of $\mathfrak{g}$.

It was Bott [13] who first succeeded in extracting the exponents from the root space decomposition of $\mathfrak{g}$ in a uniform manner using Morse theory; the exponents had earlier been calculated case by case by Chevalley and Yen. A. Shapiro developed a simpler (but empirical) approach to computing the exponents. His procedure involves a canonical involution on the set of partitions, which will be used frequently in the sequel.

Given a partition $\mathbf{d} = [d_1, \ldots, d_N] \in \mathcal{P}(N)$, define a new partition $\mathbf{d^t} = [d_1^t, \ldots, d_N^t]$, where

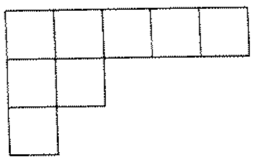
$$d_j^t = |\{ i \mid d_i \geq j \}|. \tag{4.4.5}$$

We call $\mathbf{d^t}$ the *transpose partition* of $\mathbf{d}$. Note that the map $\mathbf{d} \to \mathbf{d^t}$ defines an involution on the set $\mathcal{P}(N)$; i.e., $(\mathbf{d^t})^{\mathbf{t}} = \mathbf{d}$ for all $\mathbf{d} \in \mathcal{P}(N)$. To picture this duality operation, we use Young diagrams, which are defined as follows. Given a partition $[d_1, \ldots, d_N]$ of N, form a left-justified array of N rows of empty boxes such that the ith row has d_i boxes. We call this array the *Young diagram* of $\mathbf{d}$. We can picture the Young diagram of $\mathbf{d}$ as follows:

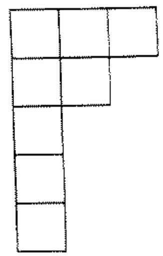


Thus we have a bijection between $\mathcal{P}(N)$ and a certain set $\mathcal{Y}(N)$ of Young diagrams. There is an obvious involution on $\mathcal{Y}(N)$ defined as follows: given a Young diagram, form a new one whose rows are the columns of the old one from left to right, where we add empty rows as necessary to bring the total number of rows to N. A moment's thought shows that transferring this involution to $\mathcal{P}(N)$ realizes the duality operation. For example, if $N = 8$ and $\mathbf{d} = [5, 2, 1]$, then $\mathbf{d^t} = [3, 2, 1^3]$ and the corresponding Young diagrams are

$[5, 2, 1]$



$[3, 2, 1^3]$



Given a simple Lie algebra $\mathfrak{g}$, let N be the number of its positive roots. Define the *height* of a positive root β, denoted $ht(\beta)$, to be the sum of the coefficients occurring when β is written as a linear combination of simple roots. Define a partition

$$\mathbf{d} = [t_1, t_2, \ldots, t_{ht(\alpha^\#)}],$$

by letting t_i be the number of positive roots of height i; here $\alpha^\#$ is the highest root. (It turns out that the t_i are nonincreasing, so they do form a partition according to our conventions.) Now put

$$\mathbf{d}_\Phi = \mathbf{d}^{\mathbf{t}} = [m'_1, m'_2, \ldots]. \tag{4.4.6}$$

Clearly $t_1 = \ell = \text{rank}(\mathfrak{g})$, so that the partition $\mathbf{d}_\Phi$ has exactly ℓ parts. We will call the integers $\{m'_1, m'_2, \ldots, m'_\ell\}$ the *exponents of* Φ. Then we have

Conjecture 4.4.7 (Shapiro). *Let $\mathfrak{g}$ be simple. Then the set of exponents of $\mathfrak{g}$ coincides with the set of exponents of Φ.*

In a very different direction, Coleman was studying the properties of Weyl group elements of the form

$$w^\sharp = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell}, \tag{4.4.8}$$

where $\{\alpha_1, \ldots, \alpha_\ell\}$ is an arbitrary ordering of any set Δ of simple roots for $\mathfrak{g}$. An element of the form (4.4.8) is called a *Coxeter-Killing transformation.* The set of all such transformations in W forms a single conjugacy class ([19, 10.3.1]); thus all such transformations have the same order. The next result links the eigenvalues of Coxeter-Killing transformations to the conjecture above; see [20].

Theorem 4.4.9 (Coleman). *Assume that the order $o(w^\sharp)$ of a Coxeter-Killing element $w^\sharp$ satisfies $o(w^\sharp)\ell = 2|\Phi^+|$. Then the eigenvalues of $w^\sharp$ have the form ω^{m_i}, where ω is a primitive $o(w^\sharp)$-th root of unity and $\{m_1, m_2, \ldots, m_\ell\}$ are the exponents of $\mathfrak{g}$.*

One can now verify the conjecture by proving that $o(w^\sharp) = \frac{2|\Phi^+|}{\ell}$ and that the set of exponents of Φ coincides with the set

$$\{u < o(w^\sharp) \mid \omega^u \text{ is an eigenvalue of } w^\sharp\}.$$

An elementary proof of these two facts can be found in [19, §10]. Given the validity of (4.4.7), it is now an easy matter to compute the exponents of a simple Lie algebra.

Exponents of $\mathfrak{g}$

$\mathfrak{g}$	Exponents
$\mathfrak{sl}_n$	$1,2,\ldots,n-1$
$\mathfrak{so}_{2n+1}$	$1,3,5,\ldots,2n-1$
$\mathfrak{sp}_{2n}$	$1,3,5,\ldots,2n-1$
$\mathfrak{so}_{2n}$	$1,3,5,\ldots,2n-3,n-1$
E_6	1,4,5,7,8,11
E_7	1,5,7,9,11,13,17
E_8	1,7,11,13,17,19,23,29
F_4	1,5,7,11
G_2	1,5

It is interesting to note that the exponents of $\mathfrak{g}$ also arise when computing the order of a finite Chevalley group $G(q)$ over the field $\mathbb{F}_q$ of q elements. For example, let $PSL_n(q)$ denote the projective special linear group of type A_{n-1} over $\mathbb{F}_q$. Then

$$|PSL_n(q)| = \frac{1}{gcd(n, q-1)} q^{n(n-1)/2}(q^2-1)(q^3-1)\ldots(q^n-1).$$

See [18, §9,10] for further details and similar formulas for other finite groups of Lie type.

Kostant [52] discovered a connection between the exponents of $\mathfrak{g}$ and the principal nilpotent orbit. Let $\mathcal{O}_{prin}$ be the principal nilpotent orbit and $\{H, X, Y\}$ the standard triple constructed in the proof of (4.1.6). Recall that $\dim(\mathfrak{g}^X) = \ell$. Let $k_1 \leq k_2 \leq \ldots \leq k_\ell$ enumerate the highest weights of $\mathfrak{g}$ as a module over the span of $\{H, X, Y\}$.

emma 4.4.10 (Kostant). *If $\mathfrak{g}$ is simple, then each of the highest weights $k_i = 2m_i^*$ is even. We have $m_1^* = 1, m_\ell^* = ht(\alpha^\#)$, and $m_{\ell-1}^* < m_\ell^*$, where $\alpha^\sharp$, as above, is the highest root.*

This is not difficult to verify, but one actually has

heorem 4.4.11 (Kostant). *Retain the above notation. Then the m_i^* are the exponents of $\mathfrak{g}$, even if $\mathfrak{g}$ is only semisimple.*

We close this section by giving yet another context in which the exponents arise. Recall that the G_{ad}-action on $\mathfrak{g}^*$ extends in a natural way to the symmetric algebra $S(\mathfrak{g}^*)$; this algebra may be identified with the polynomial functions on $\mathfrak{g}$. Thus one can form the subalgebra $S(\mathfrak{g}^*)^{G_{ad}}$ of G_{ad}-invariant polynomials on $\mathfrak{g}$. Then one has ([21, 8.1.1,8.1.3])

heorem 4.4.12. *$S(\mathfrak{g}^*)^{G_{ad}}$ is a polynomial algebra. Let $f_1, f_2, \ldots, f_\ell$ be algebraically independent homogeneous generators of degrees $\nu_1, \nu_2, \ldots, \nu_\ell$. Then $\nu_i = 1 + m_i$, where $m_1, m_2, \ldots, m_\ell$ are the exponents of $\mathfrak{g}$. Moreover, the ideal in $S(\mathfrak{g})$ generated by the f_i, or equivalently by the G_{ad}-invariant polynomials with zero constant term, is prime and has the nilpotent cone $\mathcal{N}$ as its variety of common zeros.*

5 Nilpotent Orbits in the Classical Algebras

We saw in Chapter 3 that the theory developed there is much more concrete and explicit in the special case $\mathfrak{g} = \mathfrak{sl}_n$ than in general. Thanks to the Jordan normal form, we were able to prove the Jacobson-Morozov Theorem by a direct calculation, parametrize the nilpotent orbits in (3.1.7), and construct an explicit standard triple for each orbit in (3.2.7). This chapter is devoted to accomplishing the same tasks for $\mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}$, and $\mathfrak{so}_{2n}$. This is done in the first three sections; in the last one we identify the partitions and weighted diagrams of the principal, subregular, and minimal nilpotent orbits. The results in the first two sections are due to Gerstenhaber [30], with one exception, while the results in the last two sections are due to Springer and Steinberg. Our exposition follows [80].

For now we mention that our viewpoint will be rather different from what it was in the last two chapters. Given a copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$, we will consider its action on the natural, not the adjoint, representation of $\mathfrak{g}$; of course, we need to assume that $\mathfrak{g}$ is classical to speak of its natural representation. It follows from (3.2.6) and Weyl's Theorem on complete reducibility that any finite-dimensional representation of $\mathfrak{sl}_2$ is determined up to isomorphism by a partition. Thus, we will parametrize nilpotent orbits by partitions, just as we did for $\mathfrak{sl}_n$. We will classify nilpotent orbits in the exceptional algebras in Chapter 8 by an entirely different method (which actually works in general).

5.1 Partition Type Classifications

We begin by stating the main results on the parametrization of nilpotent orbits and then indicate a cleaner statement that can be made at the expense of introducing the full orthogonal groups. We begin by restating (3.1.7).

Theorem 5.1.1 (Type A_n). *Nilpotent orbits in $\mathfrak{sl}_n$ are in one-to-one correspondence with the set $\mathcal{P}(n)$ of partitions of n.*

Theorem 5.1.2 (Type B_n). *Nilpotent orbits in $\mathfrak{so}_{2n+1}$ are in one-to-one correspondence with the set of partitions of $2n+1$ in which even parts occur with even multiplicity.*

We use the same notation $\mathcal{O}_{[d_1,\ldots,d_r]}$ for the orbit attached to the partition $[d_1,\ldots,d_r]$ that we did in §3.1. For example, in $\mathfrak{so}_7$, there are seven nilpotent orbits: $\mathcal{O}_{[7]}, \mathcal{O}_{[5,1^2]}, \mathcal{O}_{[3,1^4]}, \mathcal{O}_{[3,2^2]}, \mathcal{O}_{[3^2,1]}, \mathcal{O}_{[2^2,1^3]}, \mathcal{O}_{[1^7]}$.

Theorem 5.1.3 (Type C_n). *Nilpotent orbits in $\mathfrak{sp}_{2n}$ are in one-to-one correspondence with the set of partitions of $2n$ in which odd parts occur with even multiplicity.*

For example, in $\mathfrak{sp}_6$, there are eight nilpotent orbits: $\mathcal{O}_{[6]}, \mathcal{O}_{[4,2]}, \mathcal{O}_{[4,1^2]}, \mathcal{O}_{[3^2]}, \mathcal{O}_{[2^3]}, \mathcal{O}_{[2^2,1^2]}, \mathcal{O}_{[2,1^4]}, \mathcal{O}_{[1^6]}$.

Theorem 5.1.4 (Type D_n, Springer-Steinberg). *Nilpotent orbits in $\mathfrak{so}_{2n}$ are parametrized by partitions of $2n$ in which even parts occur with even multiplicity, except that "very even" partitions $[d_1,\ldots,d_{2n}]$ (those with only even parts, each having even multiplicity) correspond to two orbits, denoted $\mathcal{O}^I_{[d_1,\ldots,d_{2n}]}$ and $\mathcal{O}^{II}_{[d_1,\ldots,d_{2n}]}$.*

For example, in $\mathfrak{so}_8$, there are ten partitions of eight in which even parts occur with even multiplicity, two of which are very even ($[4^2]$ and $[2^4]$). Thus, we get exactly twelve nilpotent orbits:

$$\begin{gathered}\mathcal{O}_{[7,1]},\ \mathcal{O}_{[5,3]},\ \mathcal{O}^I_{[4^2]},\ \mathcal{O}^{II}_{[4^2]},\ \mathcal{O}_{[5,1^3]},\ \mathcal{O}_{[3^2,1^2]},\\ \mathcal{O}_{[3,2^2,1]},\ \mathcal{O}^I_{[2^4]},\ \mathcal{O}^{II}_{[2^4]},\ \mathcal{O}_{[3,1^5]},\ \mathcal{O}_{[2^2,1^4]},\ \mathcal{O}_{[1^8]}.\end{gathered}$$

We will define the labels I and II precisely in §5.3.

By considering orbits under a slightly larger group than G_{ad}, we can give a cleaner and more uniform statement of the last three results. Let $\epsilon = \pm 1$ and consider a nondegenerate bilinear form $\langle\cdot,\cdot\rangle_\epsilon$ on $\mathbb{C}^m$ such that

$$\langle A,B\rangle_\epsilon = \epsilon\langle B,A\rangle_\epsilon \ \text{ for all } A,B\in\mathbb{C}^m.$$

If $\epsilon = -1$ (resp. $\epsilon = 1$), then the form $\langle\cdot,\cdot\rangle_\epsilon$ is symplectic (resp. symmetric). We then define

$$\begin{aligned} I(\langle\cdot,\cdot\rangle_\epsilon) &= \{x\in GL_m(\mathbb{C}) \mid \langle xA,xB\rangle_\epsilon = \langle A,B\rangle_\epsilon \text{ for all } A,B\in\mathbb{C}^m\},\\ \mathfrak{g}_\epsilon &= \{X\in\mathfrak{sl}_m \mid \langle XA,B\rangle_\epsilon = -\langle A,XB\rangle_\epsilon \text{ for all } A,B\in\mathbb{C}^m\}.\end{aligned}$$

Thus $I(\langle\cdot,\cdot\rangle_\epsilon)$ is just the isometry group of the form $\langle\cdot,\cdot\rangle_\epsilon$ on $\mathbb{C}^m$, and $\mathfrak{g}_\epsilon$ is its Lie algebra. If $\epsilon = -1$, then $m = 2n$ must be even and $I(\langle\cdot,\cdot\rangle_{-1}) \cong Sp_{2n}$. If $\epsilon = 1$, then m need not be even. We have $I(\langle\cdot,\cdot\rangle_1) \cong O_m$ and $\mathfrak{g}_1 \cong \mathfrak{so}_m$. Now set

$$\mathcal{P}_\epsilon(m) = \{[d_1,\ldots,d_m]\in\mathcal{P}(m) \mid \#\{j \mid d_j = i\} \text{ is even for all } i \text{ with } (-1)^i = \epsilon\}. \tag{5.1.5}$$

One readily checks that a partition belongs to $\mathcal{P}_\epsilon(m)$ if and only if it satisfies the appropriate conditions in (5.1.2)–(5.1.4). Then we have

Theorem 5.1.6 (Gerstenhaber). *Nilpotent $I(\langle\cdot,\cdot\rangle_\epsilon)$-orbits in $\mathfrak{g}_\epsilon$ are in one-to-one correspondence with the set of partitions in $\mathcal{P}_\epsilon(m)$.*

If $\epsilon = -1$, then the adjoint group of $\mathfrak{g}_\epsilon$ is $PSp_{2n} := Sp_{2n}/\{\pm I\}$, and it is clear that its orbits are the same as those of Sp_{2n}. If $\epsilon = 1$ and m is odd then the isometry group O_m is the direct product of its center $\{\pm I\}$ and the adjoint group SO_m of $\mathfrak{g}_\epsilon$, whence the orbits under the isometry group again coincide with those under the adjoint group. But if $\epsilon = 1$ and m is even, then the adjoint group of $\mathfrak{g}_\epsilon$ becomes $PSO_m := SO_m/\{\pm I\}$, and its orbits do not coincide with those of O_m. In fact, (5.1.6) implies that there can be only one O_m-orbit attached to a very even partition $\mathbf{d}$ of m. This orbit turns out to be the union $\mathcal{O}^I_{\mathbf{d}} \cup \mathcal{O}^{II}_{\mathbf{d}}$ of the two orbits attached to $\mathbf{d}$ by (5.1.4).

We turn now to the proofs of the above theorems. Let $\mathfrak{g}$ be a classical algebra with standard representation on $\mathbb{C}^N$. Let X be a nilpotent element of $\mathfrak{g}$. We saw in Chapter 1 that X is also nilpotent when regarded as an element of $\mathfrak{sl}_N$. Thus it already has a partition $\mathbf{d} = [d_1,\ldots,d_N]$ attached to it via (3.1.7) and belongs to a standard triple in $\mathfrak{sl}_N$ constructed explicitly after (3.2.6). But now X is also the nilpositive element of a standard triple $\{H,X,Y\}$ in $\mathfrak{g}$, which is conjugate under GL_N to the first triple. Letting $\mathfrak{a}$ denote the span of $\{H,X,Y\}$ we get

Observation 5.1.7. *The nonzero d_i are exactly the dimensions of the irreducible summands of the standard representation $\mathbb{C}^N$ of $\mathfrak{g}$, regarded as an $\mathfrak{a}$-module.*

Suppose now that $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $\langle\cdot,\cdot\rangle$ denote the nondegenerate symplectic form on $\mathbb{C}^{2n}$ preserved by G_{ad}. We have an $\mathfrak{a}$-module decomposition

$$\mathbb{C}^{2n} \cong \bigoplus_{r\geq 0} M(r) \tag{5.1.8}$$

where $M(r)$ is a finite (possibly empty) direct sum of irreducible representations of $\mathfrak{sl}_2$ of highest weight r (and dimension $r+1$). We know by (5.1.7) that we can read off the dimensions of the summands occurring in (5.1.8) from the partition $\mathbf{d}$ of X regarded as a matrix in $\mathfrak{sl}_{2n}$; this partition is an invariant of the G_{ad}-orbit of X. Now we need to see what requirements the form $\langle\cdot,\cdot\rangle$ places on the parts of $\mathbf{d}$. For $r \geq 0$, denote by $H(r)$ the highest weight space in $M(r)$. From $\mathfrak{sl}_2$ theory we get

$$\dim H(r) = \operatorname{mult}(\rho_r, M(r)) \tag{5.1.9}$$

where ρ_r denotes the irreducible representation of $\mathfrak{sl}_2$ of highest weight r. Now we want to use $\langle\cdot,\cdot\rangle$ to produce a new nondegenerate bilinear form $(\cdot,\cdot)_r$ on $H(r)$. If $v,w \in H(r)$, put

$$(v, w)_r = (v, Y^r \cdot w)$$

where $Y^r \cdot w$ denotes Y acting r times on w. A simple calculation using the $\mathfrak{g}$-invariance of $\langle \cdot, \cdot \rangle$ yields

mma 5.1.10. *The form $(\cdot, \cdot)_r$ is symplectic or symmetric according as r is even or odd.*

A key result follows.

mma 5.1.11. *The form $(\cdot, \cdot)_r$ is nondegenerate for all r.*

Proof. Note first that the r-weight space of $\mathbb{C}^{2n}$ is $\langle \cdot, \cdot \rangle$-orthogonal to its s-weight space whenever $s \neq -r$, by the invariance of ad_H relative to $\langle \cdot, \cdot \rangle$. Now suppose that $r \geq 0$. The subspace $H(r)$ has a canonical complement in the full r-weight space; it is spanned by all vectors in this weight space lying in the range of Y. Since Y^{r+1} kills $H(r)$, we see that $H(r)$ is $(\cdot, \cdot)_r$-orthogonal to this complement. It follows from $\mathfrak{sl}_2$ theory that $Y^r \cdot H(r)$ is the lowest weight space in $M(r)$, and it pairs nondegenerately with $H(r)$ via $\langle \cdot, \cdot \rangle$. Thus, $(\cdot, \cdot)_r$ is nondegenerate. □

Since the irreducible representation of highest weight r has dimension $r+1$ and nondegenerate symplectic forms exist only in even dimension (by §1.4), we deduce

orollary 5.1.12. *The partition* **d** *of X lies in $\mathcal{P}_{-1}(2n)$; that is, its odd parts occur with even multiplicity.*

Hence we get a map

$$\Pi_{-1} : \{\text{nilpotent } I(\langle \cdot, \cdot \rangle) - \text{orbits in } \mathfrak{sp}_{2n}\} \to \mathcal{P}_{-1}(2n), \tag{5.1.13}$$

which is defined as above for nonzero orbits and which sends the zero orbit to the partition $[1^{2n}]$. Now we construct the analogue Π_1 of this map for orthogonal orbits; we will show below that both Π_1 and Π_{-1} are bijections.

So let $\mathfrak{g} = \mathfrak{so}_m$, where m may be even or odd. Once again let $\langle \cdot, \cdot \rangle$ be the nondegenerate symmetric form on $\mathbb{C}^m$ preserved by G_{ad}. Let $\mathfrak{a}$ be the span of a standard triple $\{H, X, Y\}$. For every $r \geq 0$, define the subspaces $M(r), H(r)$, and the induced form $(\cdot, \cdot)_r$ exactly as above. Then (5.1.11) extends easily to $(\cdot, \cdot)_r$. A calculation as above yields

emma 5.1.14. *The form $(\cdot, \cdot)_r$ is symmetric or symplectic according as r is even or odd.*

Thus we get

orollary 5.1.15. *The partition* **d** *of X lies in $\mathcal{P}_1(m)$; that is, its even parts occur with even multiplicity.*

We also get a map

$$\Pi_1 : \{\text{nilpotent } I(\langle \cdot, \cdot \rangle) - \text{orbits in } \mathfrak{so}_m\} \to \mathcal{P}_1(m), \tag{5.1.16}$$

sending the zero orbit to the partition $[1^m]$.

The main lemma is

Lemma 5.1.17 (Wall [87]). *The maps $\Pi_{\pm 1}$ are bijections.*

Proof. The basic idea is simply to show how to recover the ambient form $\langle \cdot, \cdot \rangle$ from its induced forms $(\cdot, \cdot)_r$, but the details are a bit complicated. We give them in case $\mathfrak{g} = \mathfrak{sp}_{2m}$; the orthogonal case is similar. We first show that Π_{-1} is surjective. Given a partition $\mathbf{d} = [d_1^{i_1}, \ldots, d_r^{i_r}]$ in $\mathcal{P}_{-1}(2n)$, let V be a direct sum of r vector spaces V_j, each having dimension i_j. Define a form $(\cdot, \cdot)$ on V as follows. We declare that $(V_i, V_j) = 0$ whenever $i \neq j$. If d_j is odd, then we require $(\cdot, \cdot)$ to be nondegenerate and symplectic on V_j; if d_j is even, we require $(\cdot, \cdot)$ to be nondegenerate and symmetric on V_j. Such a form $(\cdot, \cdot)$ exists on V and is unique up to equivalence. We now replace V by a larger space V' (which will be isomorphic to the standard representation $\mathbb{C}^{2n}$). The larger space V' will admit an $\mathfrak{sl}_2$ action invariant with respect to an ambient symplectic form $\langle \cdot, \cdot \rangle$; its subspace V will be the sum of the subspaces $H(r)$ defined above. We begin by replacing each summand V_j of V with $d_j \neq 1$ by a direct sum $W_j \oplus W_j'$ of two isomorphic copies of itself. We replace $(\cdot, \cdot)$ on V_j by a symplectic form $\langle \cdot, \cdot \rangle_j$ on $W_j \oplus W_j'$ in such a way that W_j is paired nondegenerately with W_j' and each of W_j, W_j' is self-orthogonal. Again, there is a unique way to do this up to equivalence. Obviously, the direct sum of all the W_j and W_j' inherits a symplectic form $\langle \cdot, \cdot \rangle'$, which is just the orthogonal sum of the $\langle \cdot, \cdot \rangle_j$. Now, using the formulas in [37, §7.2] for the action of the standard basis vectors of $\mathfrak{sl}_2$ on a finite-dimensional irreducible module, we enlarge each $W_j \oplus W_j'$ to a $d_j i_j$-dimensional $\mathfrak{sl}_2$ module whose highest weight space is W_j and whose lowest weight space is W_j'. This module is the direct sum of i_j irreducible submodules, each of highest weight $d_j - 1$. It admits a nondegenerate symplectic form extending $\langle \cdot, \cdot \rangle_j$ and invariant under the $\mathfrak{sl}_2$ action. The form is unique up to $\mathfrak{sl}_2$-equivariant equivalence, by Schur's Lemma. If $d_k = 1$, then V_k itself may already be regarded as a (trivial) $\mathfrak{sl}_2$ module with a nondegenerate symplectic form. Finally, we let V' be the direct sum of all of these $\mathfrak{sl}_2$ modules with the inherited symplectic form $\langle \cdot, \cdot \rangle$. Then V' is isomorphic to the standard representation. Clearly $\mathfrak{sp}(V')$ has a nilpotent element with partition **d**. Hence Π_{-1} is surjective. Its injectivity follows from the canonical nature of the above construction: any two images of $\mathfrak{sl}_2$ in $\mathfrak{sp}_{2n}$ giving rise to the same partition of $2n$ via (3.1.7) must be conjugate under an isometry of the symplectic form. This concludes the proof. □

Now Theorems 5.1.2, 5.1.3, and 5.1.6 follow from (5.1.17) and the remarks before Theorem 5.1.6. To prove Theorem 5.1.4, we must work a little harder. As observed above, the problem is that G_{ad} is isomorphic to PSO_m, and while the orbits of this group are the same as those of SO_m, they do not coincide with those of $I(\langle\cdot,\cdot\rangle_1) \cong O_m$ if m is even. Given two actions of $\mathfrak{sl}_2$ on $\mathbb{C}^m$ invariant under $\langle\cdot,\cdot\rangle_1$, suppose they are conjugate under an element g of $I(\langle\cdot,\cdot\rangle_1)$. Suppose that the determinant of the matrix g is -1; then we must decide when we can replace g by a matrix of determinant 1. Assume first that at least one part of the partition $\mathbf{d}$ corresponding to either action of $\mathfrak{sl}_2$ is odd. Then the proof of (5.1.17) shows that we can find an irreducible odd-dimensional summand of $\mathbb{C}^m$ under the first action that pairs nondegenerately with itself under $\langle\cdot,\cdot\rangle_1$. Multiplying g by -1 on this summand S and leaving it unchanged on the orthogonal complement of S, we obtain a new g that also conjugates the first action to the second but has determinant 1. Hence, the two actions are already conjugate under SO_m or PSO_m. Now assume that all parts of $\mathbf{d}$ are even, so they all occur with even multiplicity. Then the proof of (5.1.17) shows that the commutant in O_m of either $\mathfrak{sl}_2$ action is a direct product of symplectic groups, one for each distinct part of $\mathbf{d}$. Since a symplectic transformation automatically has determinant 1, it is impossible to replace any g of determinant -1 by one of determinant 1. Hence, very even partitions of m (which exist only if m is a multiple of 4) correspond to two orbits: given a representative of one of them, one obtains a representative of the other by conjugating by an orthogonal matrix of determinant -1. Other partitions of m correspond to one orbit, as asserted in (5.1.4).

Remark 5.1.18. We have just observed that one bonus of our proof of (5.1.17) is a formula for the commutant in $I(\langle\cdot,\cdot\rangle)_\epsilon$ of any $\langle\cdot,\cdot\rangle_\epsilon$-invariant $\mathfrak{sl}_2$ action on the standard representation, or equivalently the centralizer in $I(\langle\cdot,\cdot\rangle)_\epsilon$ of any image of $\mathfrak{sl}_2$ in $\mathfrak{g}_\epsilon$. This centralizer is a direct product of orthogonal and symplectic groups. If $\epsilon = 1$, one gets one orthogonal group O_{r_i} for every distinct odd part i of the corresponding partition $\mathbf{d}$ of the action; here r_i is the multiplicity of i in $\mathbf{d}$. Similarly, one gets one symplectic group Sp_{r_j} in the centralizer for every distinct even part j of $\mathbf{d}$; again r_j denotes the multiplicity of j in $\mathbf{d}$, which must now be even. If $\epsilon = -1$, the situation is the same except that the roles of even and odd parts are reversed. We will use this formula for the centralizer in §6.1. For now, we note that the main disadvantage of our proof of (5.1.17) is that the recipe for constructing an $\mathfrak{sl}_2$ action corresponding to a given partition is rather abstract. We will remedy this situation in the next section, where we construct explicit standard triples in $\mathfrak{g}$ attached to partitions $\mathbf{d} \neq [1^m]$.

5.2 Explicit Standard Triples

We now construct (more or less) canonical representatives of conjugacy classes of standard triples in the classical algebras. These will provide us with a normal form representative of each nilpotent orbit and enable us to compute its weighted Dynkin diagram.

Given a classical $\mathfrak{g}$, we begin by fixing a choice of Cartan subalgebra $\mathfrak{h}$ together with a (standard) coordinate system on $\mathfrak{h}$. We then write down all

roots and root spaces of $\mathfrak{h}$ in $\mathfrak{g}$; we also fix a choice of positive roots. Given a partition $\mathbf{d}$ corresponding via §§3.1,5.1 to a nonzero nilpotent orbit in $\mathfrak{g}$, we construct a standard triple $\{H, X, Y\}$ such that $H \in \mathfrak{h}$, X is a sum of vectors in certain positive root spaces, and Y is a sum of vectors in the corresponding negative root spaces. In the next section, we conjugate H to an element of the dominant Weyl chamber relative to our choice of positive roots and then read off the weighted Dynkin diagram of $\{H, X, Y\}$.

Suppose first that $\mathfrak{g} = \mathfrak{sl}_n$. As in §3.6, we let $\mathfrak{h}$ be the set of traceless diagonal matrices and coordinatize it in the obvious fashion. Define linear functionals $e_i \in \mathfrak{h}^*$ by $e_i(H) = i^{th}$ diagonal entry of H; here $1 \le i \le n$. Then the root system of $\mathfrak{g}$ is $\{e_i - e_j \mid 1 \le i, j \le n, i \neq j\}$. As in §3.6, we choose $\{e_i - e_j \mid i < j\}$ as our set of positive roots. The $(e_i - e_j)$-root space is spanned by the elementary matrix $E_{i,j}$ having a 1 as its ij-entry and zeros elsewhere.

Next let $\mathfrak{g} = \mathfrak{sp}_{2n}$. We may realize $\mathfrak{g}$ as the following set of matrices:

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \mid Z_i \in M_n(\mathbb{C}), Z_2, Z_3 \text{ symmetric} \right\}.$$

Let $\mathfrak{h}$ be the Cartan subalgebra of all diagonal matrices in $\mathfrak{g}$. These take the form

$$H = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$$

where

$$D = \begin{pmatrix} h_1 & 0 & 0 & \dots & 0 \\ 0 & h_2 & 0 & \dots & 0 \\ 0 & 0 & h_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h_n \end{pmatrix}.$$

Define linear functionals $e_i \in \mathfrak{h}^*$ by $e_i(H) = h_i$, for $1 \le i \le n$. The root system of $\mathfrak{g}$ is then $\{\pm e_i \pm e_j, \pm 2e_i \mid 1 \le i, j \le n, i \neq j\}$. We make the standard choice $\{e_i \pm e_j, 2e_k \mid 1 \le i < j \le n, 1 \le k \le n\}$ of positive roots. The α-root space of $\mathfrak{g}$ is spanned by the root vector X_α defined below.

$$\begin{aligned} X_{e_i - e_j} &= E_{i,j} - E_{j+n,i+n} \\ X_{e_i + e_j} &= E_{i,j+n} + E_{j,i+n} \\ X_{-e_i - e_j} &= E_{i+n,j} + E_{j+n,i} \\ X_{2e_i} &= E_{i,i+n} \\ X_{-2e_i} &= E_{i+n,i} \end{aligned}$$

Now let $\mathfrak{g} = \mathfrak{so}_{2n+1}$. We may realize $\mathfrak{g}$ as the following set of matrices:

$$\left\{ \begin{pmatrix} 0 & u & v \\ -v^t & Z_1 & Z_2 \\ -u^t & Z_3 & -Z_1^t \end{pmatrix} \mid u, v \in \mathbb{C}^n, Z_i \in M_n(\mathbb{C}), Z_2, Z_3 \,\text{skew-symmetric} \right\}$$

Note that this is *not* the set of skew-symmetric matrices but rather the set of skew-adjoint matrices relative to the quadratic form $z_1^2 + 2(z_2 z_{n+2} + z_3 z_{n+3} + \dots + z_{n+1} z_{2n+1})$ on $\mathbb{C}^{2n+1}$. Once again we take the set of diagonal matrices in $\mathfrak{g}$ as our Cartan subalgebra $\mathfrak{h}$. Then the matrices in $\mathfrak{h}$ take exactly the same form as they did for $\mathfrak{sp}_{2n}$ except that they have an extra row and column of zeros at the top and left. Define linear functionals $e_i \in \mathfrak{h}^*$ as above for $\mathfrak{sp}_{2n}$, where again i ranges from 1 to n. Thus, e_i picks out the $(i+1)^{st}$ diagonal entry of a matrix in $\mathfrak{h}$. The root system of $\mathfrak{g}$ is now $\{\pm e_i \pm e_j, \pm e_j \mid 1 \le i, j \le n, i \ne j\}$. We take $\{e_i \pm e_j, e_k \mid 1 \le i < j \le n, 1 \le k \le n\}$ as our set of positive roots. The α-root space of $\mathfrak{g}$ is spanned by the vector X_α given below.

$$\begin{aligned} X_{e_i - e_j} &= E_{i+1,j+1} - E_{j+n+1,i+n+1} \\ X_{e_i + e_j} &= E_{i+1,j+n+1} - E_{j+1,i+n+1} \qquad (i < j) \\ X_{-e_i - e_j} &= E_{i+n+1,j+1} - E_{j+n+1,i+1} \qquad (i < j) \\ X_{e_i} &= E_{1,i+n+1} - E_{i+1,1} \\ X_{-e_i} &= E_{1,i+1} - E_{i+n+1,1} \end{aligned}$$

Finally let $\mathfrak{g} = \mathfrak{so}_{2n}$. We realize $\mathfrak{g}$ as

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \mid Z_i \in M_n(\mathbb{C}), Z_2, Z_3 \,\text{skew-symmetric} \right\}.$$

This is the set of skew-adjoint matrices relative to the quadratic form $z_1 z_{n+1} + \dots + z_n z_{2n}$ on $\mathbb{C}^{2n}$. Again, we take the diagonal matrices in $\mathfrak{g}$ as our Cartan subalgebra $\mathfrak{h}$. The matrices in $\mathfrak{h}$ take exactly the same form as they did for $\mathfrak{sp}_{2n}$; we define linear functionals $e_i \in \mathfrak{h}^*$ as in that case. Now the root system for $\mathfrak{g}$ is $\{\pm e_i \pm e_j \mid 1 \le i, j \le n, i \ne j\}$. We take $\{e_i \pm e_j \mid 1 \le i < j \le n\}$ as a set of positive roots. The α-root space is spanned by the vector X_α given below.

$$\begin{aligned} X_{e_i - e_j} &= E_{i,j} - E_{n+j,n+i} \\ X_{e_i + e_j} &= E_{i,n+j} - E_{j,n+i} \qquad (i < j) \\ X_{-e_i - e_j} &= E_{n+i,j} - E_{n+j,i} \qquad (i < j) \end{aligned}$$

Now we are ready to give our recipes for attaching standard triples to partitions. Given $\mathbf{d}$, we will break up its parts into chunks each consisting of one or two parts. We will attach a set of positive roots to each chunk in such a way that positive roots attached to distinct chunks are orthogonal. Our nilpotent element X with partition $\mathbf{d}$ will be a sum of positive root vectors, one for each root attached to each chunk of $\mathbf{d}$. (In most cases, the choice of root vector within the root space is arbitrary, but there is one exception noted below.) As promised above, we will also construct a standard triple $\{H, X, Y\}$ such that $H \in \mathfrak{h}$ and Y is a sum of negative root vectors, one for every positive root vector appearing in X.

Recipe 5.2.1 (Type A_n). The chunks of $\mathbf{d}$ are just its parts d_i (each repeated as often as its multiplicity). For each chunk $\{d_i\}$, choose a block of consecutive indices $\{N_i + 1, \dots, N_i + d_i\}$ in such a way that disjoint blocks are attached to distinct chunks. To every chunk $\{d_i\}$ attach the set of simple roots

$$\mathcal{C}^+ = \mathcal{C}^+(d_i) = \{e_{N_i+1} - e_{N_i+2}, \dots, e_{N_i+d_i-1} - e_{N_i+d_i}\}.$$

Note that the set $\mathcal{C}^+$ is empty whenever $d_i = 1$. For every simple root α in $\mathcal{C} := \cup_i \mathcal{C}^+(d_i)$, let X_α be an α-root vector. Let X be the sum of the X_α. By the discussion before (3.2.6), there is a sum Y of root vectors $X_{-\alpha}$, one for every $\alpha \in \mathcal{C}$, and an $H \in \mathfrak{h}$ such that $\{H, X, Y\}$ is a standard triple. We have $H = \sum_i H_{\mathcal{C}(d_i)}$ where

$$H_{\mathcal{C}(d_i)} = \sum_{1 \le \ell \le d_i} (d_i - 2\ell + 1) E_{N_i+\ell, N_i+\ell}.$$

Recipe 5.2.2 (Type C_n). Given $\mathbf{d} \in \mathcal{P}_{-1}(2n)$, break it up into chunks of the following two types: pairs $\{2r+1, 2r+1\}$ of equal odd parts, and single even parts $\{2q\}$. Now attach sets of positive (but not necessarily simple) roots to each chunk $\mathcal{C}$ as follows. If $\mathcal{C} = \{2q\}$, choose a block $\{j+1, \dots, j+q\}$ of consecutive indices and let $\mathcal{C}^+ = \mathcal{C}^+(2q) = \{e_{j+1} - e_{j+2}, \dots, e_{j+q-1} - e_{j+q}, 2e_{j+q}\}$. If $\mathcal{C} = \{2r+1, 2r+1\}$, choose a block $\{\ell+1, \dots, \ell+2r+1\}$ of consecutive indices and let $\mathcal{C}^+ = \mathcal{C}^+(2r+1, 2r+1) = \{e_{\ell+1} - e_{\ell+2}, \dots, e_{\ell+2r} - e_{\ell+2r+1}\}$. (Note that $\mathcal{C}^+$ is empty if $\mathcal{C} = \{1, 1\}$.) As always, we require that the blocks of indices attached to distinct chunks be disjoint; this can always be arranged. For example, if $\mathfrak{g} = \mathfrak{sp}_{20}$ and $\mathbf{d} = [6, 5^2, 2^2]$, then its chunks are $\{6\}, \{5, 5\}, \{2\}$, and $\{2\}$, and we may take

$$\begin{aligned} \mathcal{C}^+(6) &= \{e_1 - e_2, e_2 - e_3, 2e_3\}, \\ \mathcal{C}^+(5,5) &= \{e_4 - e_5, e_5 - e_6, e_6 - e_7, e_7 - e_8\}, \\ \mathcal{C}^+(2) &= \{2e_9\} \\ \mathcal{C}^+(2) &= \{2e_{10}\}. \end{aligned}$$

Now we define X exactly as we did for $\mathfrak{sl}_n$. Then there is a sum Y of negative root vectors and $H \in \mathfrak{h}$ such that $\{H, X, Y\}$ is a standard triple. (The key point to observe in verifying this is that if α, β are distinct positive roots such that corresponding root vectors appear in X, then $\alpha - \beta$ is not a root; cf. the proof of (4.1.6).) We have

$$H = \sum_{\mathcal{C}} H_{\mathcal{C}}$$

where $H_{\mathcal{C}}$ is defined to be

$$\sum_{\ell=1}^{q}(2q-2\ell+1)(E_{j+\ell,j+\ell}-E_{n+j+\ell,n+j+\ell})$$

if $C^+ = \{e_{j+1}-e_{j+2},\ldots,e_{j+q-1}-e_{j+q},2e_{j+q}\}$ and

$$\sum_{m=0}^{2r}(2r-2m)(E_{\ell+1+m,\ell+1+m}-E_{n+\ell+1+m,n+\ell+1+m})$$

if $C^+ = \{e_{\ell+1}-e_{\ell+2},\ldots,e_{\ell+2r}-e_{\ell+2r+1}\}$.

position 5.2.3. *If we view $\mathfrak{sp}_{2n}$ as a subalgebra of $\mathfrak{sl}_{2n}$, then the partition attached to the standard triple $\{H,X,Y\}$ via (3.1.7) is* $\mathbf{d}$.

Proof. This follows from the definition of H and the formula in §3.1 for the neutral element (up to conjugacy) of any standard triple with partition $\mathbf{d}$, for the eigenvalues of the neutral element determine the partition uniquely. □

It follows that (contrary to the situation in $\mathfrak{sl}_n$) not every nilpotent orbit in $\mathfrak{sp}_{2n}$ is represented by a sum of simple root vectors. For example, the nilpotent orbit $\mathcal{O}_{[4,2]}$ in $\mathfrak{sp}_6$ is not represented by any sum of simple root vectors (as one verifies by considering all the possibilities).

The last two recipes are unfortunately a bit more complicated.

Recipe 5.2.4 (Type B_n). Given $\mathbf{d}\in\mathcal{P}_1(2n+1)$, break it up into chunks of the following three types: pairs $\{r,r\}$ of equal parts, pairs $\{2s+1,2t+1\}$ of unequal odd parts, and a single odd part $\{2u+1\}$. There must be a unique chunk of the last type. Attach blocks of indices (subject to the usual disjointness requirement) and sets C^+ of positive roots to chunks C as follows. If $C=\{r,r\}$, choose a block $\{j+1,\ldots,j+r\}$ of consecutive indices and let $C^+=C^+(r,r)=\{e_{j+1}-e_{j+2},\ldots,e_{j+r-1}-e_{j+r}\}$. If $C=\{2u+1\}$, choose a block $\{\ell+1,\ldots,\ell+u\}$ of consecutive indices and let $C^+=C^+(2u+1)=\{e_{\ell+1}-e_{\ell+2},\ldots,e_{\ell+u-1}-e_{\ell+u},e_{\ell+u}\}$. If $C=\{2s+1,2t+1\}$ with $s>t$, choose a block $B=B(C)=\{m+1,\ldots,m+s+t+1\}$ of $s+t+1$ consecutive indices. Put

$$H_C=\textstyle\sum_{i=1}^{s-t}(2s+2-2i)(E_{m+i+1,m+i+1}-E_{n+m+i+1,n+m+i+1}) + \sum_{j=1}^{t}(2t+2-2j)F_{m+s-t+2j},$$

where F_a denotes $E_{a,a}-E_{a+n,a+n}+E_{a+1,a+1}-E_{a+n+1,a+n+1}$. Now let $C^+=C^+(2s+1,2t+1)$ consist exactly of those positive roots $\alpha=e_j\pm e_k$ such that $j,k\in B$ and $\alpha(H_C)=2$. For example, if $C=\{5,3\}$ and $B=\{1,2,3,4\}$, then $H_C=4E_{2,2}+2E_{3,3}+2E_{4,4}-4E_{n+2,n+2}-2E_{n+3,n+3}-2E_{n+4,n+4}$ and $C^+=\{e_1-e_2,e_1-e_3,e_2\pm e_4,e_3\pm e_4\}$. (Recall that we shifted indices by one in the definition of e_i in type B.) If $C=\{7,5\}$ and $B=\{1,2,3,4,5,6\}$, then

$$C^+=\{e_1-e_2,e_1-e_3,e_2-e_4,e_3-e_4,e_2-e_5,e_3-e_5,e_4\pm e_6,e_5\pm e_6\}.$$

For every chunk of the form $\{r,r\}$ or $\{2u+1\}$ and every α in $C^+(r,r)$ or $C^+(2u+1)$, let X_α be an arbitrary α-root vector. For every chunk of the form $\{2s+1,2t+1\}$ and every β in $C^+(2s+1,2t+1)$, choose a β-root vector X_β and a $-\beta$-root vector $X_{-\beta}$ such that $[\sum_\beta X_\beta,\sum_\beta X_{-\beta}]=H_C$. This is possible by the proof of (3.4.12) applied to the Lie algebra of type D_{s+t} generated by the $(\pm e_j\pm e_k)$-root spaces for $j,k\in B$. (In case $t=0$, the X_β may be chosen arbitrarily.) Let X be the sum of the X_α and X_β defined above. Then (just as for $\mathfrak{sp}_{2n}$) there is a sum Y of negative root vectors, one for every root in the union of the sets C^+, and an element of $\mathfrak{h}$, such that $\{H,X,Y\}$ is a standard triple. We have $H=\sum_C H_C$ where H_C is defined as follows. If C is a chunk of the form $\{2s+1,2t+1\}$, then H_C was already defined above. If C takes the form $\{r,r\}$ and corresponds to the block of indices $\{j+1,\ldots,j+r\}$, then

$$H_C=\sum_{k=1}^{r}(r-2k+1)(E_{j+k+1,j+k+1}-E_{n+j+k+1,n+j+k+1}).$$

If C takes the form $\{2u+1\}$ and corresponds to the block of indices $\{\ell+1,\ldots,\ell+u\}$, then

$$H_C=\sum_{k=1}^{u}(2u+2-2k)(E_{\ell+k+1,\ell+k+1}-E_{n+\ell+k+1,n+\ell+k+1}).$$

Arguing as in (5.2.3), we get

Proposition 5.2.5. *If we view $\mathfrak{so}_{2n+1}$ as a subalgebra of $\mathfrak{sl}_{2n+1}$, then the partition of the standard triple $\{H,X,Y\}$ is* $\mathbf{d}$.

Recipe 5.2.6 (Type D_n). Given $\mathbf{d}\in\mathcal{P}_1(2n)$, break it up into chunks as for $\mathfrak{so}_{2n+1}$, except that there must not be any chunks of the form $\{2u+1\}$. If $\mathbf{d}$ is not very even, then we define a standard triple $\{H,X,Y\}$ exactly as we did for $\mathfrak{so}_{2n+1}$ (except that all indices in the formulas for H_C are shifted down by one, thanks to the differences in the above formulas for root vectors in types B and D). If $\mathbf{d}$ is very even, then recall by (5.1.4) that we must attach a Roman numeral I or II to it to make it correspond to a unique nilpotent orbit. If this numeral is I, then we again follow Recipe 5.2.4 to attach a standard triple $\{H,X,Y\}$ to $(\mathbf{d},\mathrm{I})$. If the numeral is II, then we modify this recipe slightly. Note that *all* chunks of $\mathbf{d}$ take the form $\{2r,2r\}$ in this situation. Choose a chunk C arbitrarily and let $\{j+1,\ldots,j+2r\}$ be its block of indices. Now replace the positive root $e_{j+2r-1}-e_{j+2r}$ in the above recipe for C^+ by $e_{j+2r-1}+e_{j+2r}$. Leave every other root in this C^+ and all the others unchanged. The formula for H_C is almost the same as it was before; the only difference is that the coefficient of $E_{j+2r,j+2r}-E_{n+j+2r,n+j+2r}$ changes from $-(2r-1)$ to $2r-1$.

Example 5.2.7. Let $\mathfrak{g} = \mathfrak{so}_4, \mathbf{d} = [2^2]$. The neutral element of the standard triple attached to ($\mathbf{d}$,I) is $E_{1,1} - E_{3,3} - E_{2,2} + E_{4,4}$. The corresponding element for ($\mathbf{d}$,II) is $E_{1,1} - E_{3,3} + E_{2,2} - E_{4,4}$.

osition 5.2.8. *If we view $\mathfrak{so}_{2n}$ as a subalgebra of $\mathfrak{sl}_{2n}$, then the partition of $\{H, X, Y\}$ is $\mathbf{d}$, whether or not $\mathbf{d}$ is very even.*

Again this is proved in the same way as for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$.

We conclude this section with two remarks. First, if we had been content merely to construct a representative of every nilpotent orbit in a classical algebra $\mathfrak{g}$ (as opposed to a complete standard triple with nilpositive element in the orbit), then we could have simplified the above definition of $\mathcal{C}^+(2s+1, 2t+1)$ considerably. Given the block of indices $\{m+1, \ldots, m+s+t+1\}$, we could have put

$$\mathcal{C}^+(2s+1, 2t+1) = \{e_{m+1} - e_{m+2}, \ldots, e_{m+s} - e_{m+s+1}, \\ e_{m+s} + e_{m+s+1}, \ldots, e_{m+s+t} + e_{m+s+t+1}\}.$$

To define the component C of the representative X coming from $\mathcal{C}^+(2s+1, 2t+1)$, we could then have proceeded as for any other chunk. That is, if we let X_α be an arbitrary α-root vector for every $\alpha \in \mathcal{C}^+(2s+1, 2t+1)$, then we could have taken C to be the sum of the X_α.

Second, given a partition $\mathbf{d} = [d_1, \ldots, d_k]$ in $\mathcal{P}_1(m)$ with only odd parts, there is another way to construct a representative of $\mathcal{O}_{\mathbf{d}}$ in $\mathfrak{so}_m$. It is obvious from the definition of $\mathfrak{so}_m$ that it contains a copy of $\mathfrak{so}_{d_1} \times \cdots \times \mathfrak{so}_{d_k}$ (although it is awkward to write down formulas akin to those before (5.2.1) for root vectors in the various $\mathfrak{so}_{d_i}$ in terms of those in $\mathfrak{so}_m$). Now any sum of principal nilpotent elements of each $\mathfrak{so}_{d_i}$ represents $\mathcal{O}_{\mathbf{d}}$. Similarly, given a standard triple in each $\mathfrak{so}_{d_i}$ with principal nilpositive element, one easily obtains a standard triple in $\mathfrak{so}_m$ with nilpositive element in $\mathcal{O}_{\mathbf{d}}$. These recipes are useful for computing centralizers of nilpotent elements with rather odd partitions in the sense of §6.1, but they do not lead to convenient formulas for sums of *root vectors* representing the orbits $\mathcal{O}_{\mathbf{d}}$.

Weighted Dynkin Diagrams

In §3.6, we showed how to compute the weighted Dynkin diagram of any nilpotent orbit in $\mathfrak{sl}_n$. We now show how to do the same thing for an arbitrary classical algebra $\mathfrak{g}$. Thanks to our work in the last section, we now have explicit standard triples $\{H, X, Y\}$ corresponding to every nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{g}$. We need only conjugate H by the Weyl group so as to make it dominant with respect to our choice of positive roots, and then read off its eigenvalues on the simple roots.

For each $\mathfrak{g}$, we fixed a choice of Cartan subalgebra consisting of diagonal matrices in the last section. A typical matrix in $\mathfrak{h}$ looks like

$$\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$$

where

$$D = \begin{pmatrix} h_1 & 0 & 0 & \ldots & 0 \\ 0 & h_2 & 0 & \ldots & 0 \\ 0 & 0 & h_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & h_n \end{pmatrix}$$

if $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$; if $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the matrices in $\mathfrak{h}$ look the same, except tha each has an extra row and column of zeros at the top and left. Relative to ou choices of positive roots, the dominance condition is

$$h_1 \geq h_2 \geq \ldots \geq h_n \geq 0$$

for $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and

$$h_1 \geq h_2 \geq \ldots \geq h_{n-1} \geq |h_n|$$

for $\mathfrak{g} = \mathfrak{so}_{2n}$; here, of course, all h_i must be real.

Now suppose that $\mathfrak{g} = \mathfrak{sp}_{2n}$. The set of simple roots corresponding t our choice above of positive roots is $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}$ The Weyl group acts on the matrices in $\mathfrak{h}$ by permuting and changing the sign of their first n diagonal entries. In view of the formula for H in (5.2.2), w obtain the following recipe for its conjugate $\tilde{H}$ in the dominant chamber. Le $\mathbf{d} = [d_1, \ldots, d_{2n}] \in \mathcal{P}_{-1}(2n)$. Form a sequence $(s_1, s_2, \ldots, s_{2n})$ of integers b concatenating the sequences $(d_i - 1, d_i - 3, \ldots, -d_i + 1)$ for $1 \leq i \leq 2n$. Rearrang the sequence $(s_1, \ldots, s_{2n})$ so that its nonnegative terms occur in nonincreasin order, followed by their negatives. Then the rearranged sequence takes the forn $(h_1, h_2, \ldots, h_n, -h_1, \ldots, -h_n)$. Its terms are precisely the diagonal entries of $\tilde{H}$ We obtain

Lemma 5.3.1. *If $\mathfrak{g} = \mathfrak{sp}_{2n}$, then the weighted Dynkin diagram of $\mathcal{O}_{\mathbf{d}}$ is*

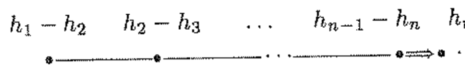
$$\begin{array}{ccccc} h_1 - h_2 & h_2 - h_3 & \ldots & h_{n-1} - h_n & 2h_n \\ \bullet\!\!\longrightarrow & \!\!\bullet\!\!\longrightarrow & \cdots & \!\!\bullet\Leftarrow & \!\!\bullet \end{array}$$

Example 5.3.2. In the following chart we list the matrices $\tilde{H}$ and weighte diagrams for $\mathfrak{g} = \mathfrak{sp}_6$.

Next let $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Now the appropriate set of simple roots is $\Delta =$ $\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}$. The Weyl group acts on the matrices i

$\mathfrak{h}$ by permuting and changing the signs of their second through $(n+1)^{st}$ diagonal entries. The formula for H in (5.2.3) leads to essentially the same recipe for $\tilde{H}$ as in the case $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $\mathbf{d} = [d_1, \ldots, d_{2n+1}] \in \mathcal{P}_1(2n+1)$. Form a sequence of integers $(s_1, \ldots, s_{2n+1})$ as above and rearrange it so that a 0 comes first, followed by the remaining nonnegative terms in nonincreasing order, followed by their negatives. This time the rearranged sequence takes the form $(0, h_1, h_2, \ldots, h_n, -h_1, \ldots, -h_n)$. Its terms are the diagonal entries of $\tilde{H}$. We obtain

emma 5.3.3. *If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then the weighted Dynkin diagram of $\mathcal{O}_{\mathbf{d}}$ is*

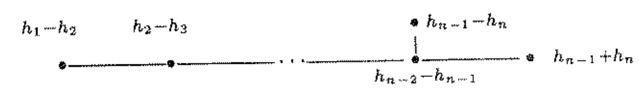


Nilpotent Orbits in $\mathfrak{sp}_6$		
Orbit $\mathcal{O}$	$\tilde{H}$	$\Delta(\mathcal{O})$
$\mathcal{O}_{[6]}$	diag(5,3,1,-5,-3,-1)	$2 - 2 \Leftarrow 2$
$\mathcal{O}_{[4,2]}$	diag(3,1,1,-3,-1,-1)	$2 - 0 \Leftarrow 2$
$\mathcal{O}_{[4,1^2]}$	diag(3,1,0,-3,-1,0)	$2 - 1 \Leftarrow 0$
$\mathcal{O}_{[3^2]}$	diag(2,2,0,-2,-2,0)	$0 - 2 \Leftarrow 0$
$\mathcal{O}_{[2^3]}$	diag(1,1,1,-1,-1,-1)	$0 - 0 \Leftarrow 2$
$\mathcal{O}_{[2^2,1^2]}$	diag(1,1,0,-1,-1,0)	$0 - 1 \Leftarrow 0$
$\mathcal{O}_{[2,1^4]}$	diag(1,0,0,-1,0,0)	$1 - 0 \Leftarrow 0$
$\mathcal{O}_{[1^6]}$	diag(0,0,0,0,0,0)	$0 - 0 \Leftarrow 0$

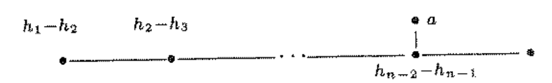
Finally let $\mathfrak{g} = \mathfrak{so}_{2n}$. The appropriate set of simple roots is $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$. The Weyl group acts on the matrices in $\mathfrak{h}$ by permuting and changing an even number of the signs of their first n diagonal entries. Let $\mathbf{d} = [d_1, \ldots, d_{2n}] \in \mathcal{P}_1(2n)$, and assume first for simplicity that $\mathbf{d}$ is not very even. Form a sequence of integers $(s_1, \ldots, s_{2n})$ as usual and rearrange it as above for $\mathfrak{g} = \mathfrak{sp}_{2n}$. The rearranged sequence once again takes the form $(h_1, \ldots, h_n, -h_1, \ldots, -h_n)$. Its terms are the diagonal entries of $\tilde{H}$.

Before we give the weighted diagram of $\mathcal{O}_{\mathbf{d}}$, we must introduce a convention for the correspondence between simple roots and nodes of a Dynkin diagram of type D. We will portray such a diagram as a row of dots connected by single lines, together with one additional dot lying over the *second* rightmost dot in the row and connected to it by a single line. We agree that the row of dots corresponds to the simple roots $e_1 - e_2, e_2 - e_3, \ldots, e_{n-2} - e_{n-1}, e_{n-1} + e_n$, while the single dot corresponds to the root $e_{n-1} - e_n$. Then we have

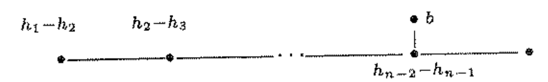
Lemma 5.3.4. *Let $\mathfrak{g} = \mathfrak{so}_{2n}$, and assume that $\mathbf{d}$ is not very even. Then the weighted Dynkin diagram of $\mathcal{O}_{\mathbf{d}}$ is*



Lemma 5.3.5. *Let $\mathfrak{g} = \mathfrak{so}_{2n}$ and assume that $\mathbf{d}$ is very even. Then the weighted diagrams of $\mathcal{O}^{I}_{\mathbf{d}}$ and $\mathcal{O}^{II}_{\mathbf{d}}$ are given by*



and



respectively, where $a = 0$ if n is a multiple of 4, $a = 2$ otherwise, and $b = 2 - a$ in either case.

Remark 5.3.6. In particular, we see that the recipe in (5.3.4) for weighted diagrams actually holds for arbitrary partitions $\mathbf{d} \in \mathcal{P}_1(2n)$ given our formula for H. We also see that if $\mathbf{d}$ is not very even, then the upper and rightmost nodes in its weighted diagram have the same label (since $h_n = 0$ in this situation). If $\mathbf{d}$ is very even, then the diagrams of $\mathcal{O}^{I}_{\mathbf{d}}$ and $\mathcal{O}^{II}_{\mathbf{d}}$ differ only in the labels attached to their upper and rightmost nodes. By specifying these diagrams, we have now given a precise meaning to the labels I and II used in the last two sections. We will use this definition in the next chapter. Although it leads to a nonuniform formula for the labels a and b in (5.3.5), it does lead to a uniform formula for a representative of a very even orbit with a fixed Roman numeral. (It also simplifies the formulas for the partitions of induced orbits in §7.3.)

Example 5.3.7. In the table below we list the first four diagonal entries of our matrices $\tilde{H}$ and weighted diagrams for the orbits in $\mathfrak{g} = \mathfrak{so}_8$.

Nilpotent Orbits in $\mathfrak{so}_8$		
Orbit $\mathcal{O}$	4-tuple	$\Delta(\mathcal{O})$
$\mathcal{O}_{[7,1]}$	(6,4,2,0)	$\begin{smallmatrix} & 2 & \\ 2 & 2 & 2 \end{smallmatrix}$
$\mathcal{O}_{[5,3]}$	(4,2,2,0)	$\begin{smallmatrix} & 2 & \\ 2 & 0 & 2 \end{smallmatrix}$
$\mathcal{O}_{[4^2]^I}$	(3,3,1,1)	$\begin{smallmatrix} & 0 & \\ 0 & 2 & 2 \end{smallmatrix}$
$\mathcal{O}_{[4^2]^{II}}$	(3,3,1,-1)	$\begin{smallmatrix} & 2 & \\ 0 & 2 & 0 \end{smallmatrix}$
$\mathcal{O}_{[5,1^3]}$	(4,2,0,0)	$\begin{smallmatrix} & 0 & \\ 2 & 2 & 0 \end{smallmatrix}$
$\mathcal{O}_{[3,2^2,1]}$	(2,1,1,0)	$\begin{smallmatrix} & 1 & \\ 1 & 0 & 1 \end{smallmatrix}$
$\mathcal{O}_{[2^4]^I}$	(1,1,1,1)	$\begin{smallmatrix} & 0 & \\ 0 & 0 & 2 \end{smallmatrix}$
$\mathcal{O}_{[2^4]^{II}}$	(1,1,1,-1)	$\begin{smallmatrix} & 2 & \\ 0 & 0 & 0 \end{smallmatrix}$
$\mathcal{O}_{[3,1^5]}$	(2,0,0,0)	$\begin{smallmatrix} & 0 & \\ 2 & 0 & 0 \end{smallmatrix}$
$\mathcal{O}_{[2^2,1^4]}$	(1,1,0,0)	$\begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \end{smallmatrix}$
$\mathcal{O}_{[1^8]}$	(0,0,0,0)	$\begin{smallmatrix} & 0 & \\ 0 & 0 & 0 \end{smallmatrix}$

5.4 Partitions Corresponding to $\mathcal{O}_{prin}$, $\mathcal{O}_{subreg}$, and $\mathcal{O}_{min}$

In Chapter 4, we established the existence of three important nilpotent orbits, which we called principal, subregular, and minimal. Now we can write down the partitions attached to them in §5.1 for classical $\mathfrak{g}$.

Proposition 5.4.1. *Let $\mathfrak{g}$ be a classical Lie algebra.*

(i) *If $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathcal{O}_{prin} = \mathcal{O}_{[n]}$, $\mathcal{O}_{subreg} = \mathcal{O}_{[n-1,1]}$, and $\mathcal{O}_{min} = \mathcal{O}_{[2,1^{n-2}]}$.*

(ii) *If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then $\mathcal{O}_{prin} = \mathcal{O}_{[2n+1]}$, $\mathcal{O}_{subreg} = \mathcal{O}_{[2n-1,1^2]}$, and $\mathcal{O}_{min} = \mathcal{O}_{[2^2,1^{2n-3}]}$.*

(iii) *If $\mathfrak{g} = \mathfrak{sp}_{2n}$, then $\mathcal{O}_{prin} = \mathcal{O}_{[2n]}$, $\mathcal{O}_{subreg} = \mathcal{O}_{[2n-2,2]}$, and $\mathcal{O}_{min} = \mathcal{O}_{[2,1^{2n-2}]}$.*

(iv) *If $\mathfrak{g} = \mathfrak{so}_{2n}$, then $\mathcal{O}_{prin} = \mathcal{O}_{[2n-1,1]}$, $\mathcal{O}_{subreg} = \mathcal{O}_{[2n-3,3]}$, and $\mathcal{O}_{min} = \mathcal{O}_{[2^2,1^{2n-4}]}$.*

Proof. The assertions about $\mathcal{O}_{prin}$ follow at once by computing the weighted diagram of the orbit with the given partition as in the last section and then invoking (4.1.6). Those about $\mathcal{O}_{subreg}$ and $\mathcal{O}_{min}$ follow in a similar way: one computes the weighted diagrams of the orbits with the given partitions and their dimensions by (4.1.3), and then invokes (4.2.1) and (4.3.5). The computation is somewhat tedious for $\mathcal{O}_{min}$; as an alternative, one can appeal to the recipes of §5.2 to show that the given orbit $\mathcal{O}_{\mathbf{d}}$ is indeed represented by a highest root vector. □

Remark 5.4.2. We mentioned in (4.3.4) that if $\mathfrak{g}$ is simple and has two root lengths, then another small orbit in $\mathfrak{g}$ is the one through a highest short root vector. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, the partition of this orbit is $[2^2, 1^{2n-4}]$, while for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the partition is $[3, 1^{2n-2}]$.

We can also write down the weighted diagrams of the subregular and minimal orbits. This information is summarized below.

Corollary 5.4.3. *Let $\mathfrak{g}$ be a classical Lie algebra.*

(i) *The subregular nilpotent orbit is even, except in the case of $\mathfrak{sl}_{2k+1}$.*

(ii) *The minimal nilpotent orbit is never even, except in the case of $\mathfrak{sl}_2$.*

We leave it as an exercise to compute the dimensions of $\mathcal{O}_{prin}$, $\mathcal{O}_{subreg}$, and $\mathcal{O}_{min}$ for classical $\mathfrak{g}$.

The Subregular and Minimal Orbits		
$\mathfrak{g}$	$\Delta(\mathcal{O}_{subreg})$	$\Delta(\mathcal{O}_{min})$
$\mathfrak{sl}_n$	$2 - 2 - \cdots - 0 - \cdots - 2 - 2 \quad n = 2k$	$2 \quad n = 2$
	$2 - 2 - \cdots - 1 - 1 - \cdots - 2 - 2 \quad n = 2k+1$	$1 - 0 - \cdots - 0 - 1 \quad n > 2$
$\mathfrak{so}_{2n+1}$	$2 - 2 - \cdots - 2 \Rightarrow 0$	$0 - 1 - 0 - \cdots - 0 \Rightarrow 0$
$\mathfrak{sp}_{2n}$	$2 - 2 - \cdots - 0 \Leftarrow 2$	$1 - 0 - \cdots - 0 \Leftarrow 0$
$\mathfrak{so}_{2n}, n \geq 4$	$2 - 2 - \cdots - 2 - 0 \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$	$0 - 1 - \cdots - 0 - 0 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$

6 Topology of Nilpotent Orbits

We have seen that the topology of a semisimple orbit is quite simple: semisimple orbits are closed simply connected submanifolds of $\mathfrak{g}$. In this chapter, we study the topology of nilpotent orbits. This is much richer, having close connections to subjects ranging from the geometry of Schubert varieties to the classification of primitive ideals in enveloping algebras. We will confine ourselves here to two topics: fundamental groups of nilpotent orbits, and the partial order on orbits by containment of their closures (defined in the introduction to Chapter 4). We give an explicit formula for $\pi_1(\mathcal{O})$ whenever $\mathcal{O}$ is classical and make some remarks about it in the exceptional cases. We then give a simple partition criterion for a classical orbit to be contained in the closure of another. In general, it turns out that there is a remarkable order-reversing map d from the set of nilpotent orbits to itself that restricts to an involution on its range. Orbits in the range of d are called *special*; it also turns out that there is a unique smallest special orbit lying above any given orbit in the partial order. We will define the map d and the set of special orbits for every classical algebra $\mathfrak{g}$. The map d has also been defined and completely computed in the exceptional cases; see [76].

6.1 The Fundamental Group and $A(\mathcal{O})$

Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}$. We want to study the fundamental group $\pi_1(\mathcal{O}_X)$ of $\mathcal{O}_X$ (ignoring basepoints); for this purpose it is convenient to look at the universal cover $\widetilde{\mathcal{O}}_X$ of $\mathcal{O}_X$. Recall from §1.2 that the universal covering space G_{sc} of G_{ad} is itself a complex Lie group, and the canonical projection $G_{sc} \twoheadrightarrow G_{ad}$ is a surjective homomorphism. In fact, we saw in (1.2.9) that the kernel of this map is precisely the center Z of G_{sc}.

Lemma 6.1.1. *The space $\widetilde{\mathcal{O}}_X$ is a homogeneous space for G_{sc} isomorphic to $G_{sc}/(G_{sc}^X)^o$. The group $\pi_1(\mathcal{O}_X)$ is isomorphic to the component group $G_{sc}^X/(G_{sc}^X)^o$ of the centralizer of X in G_{sc}.*

Proof. Since G_{sc} is simply connected it acts transitively on $\widetilde{\mathcal{O}}_X$ (cf. the proof that G_{sc} has a group structure in [89]). Let $X' \in \widetilde{\mathcal{O}}_X$ lie in the fiber F over X of the canonical projection $\widetilde{\mathcal{O}}_X \twoheadrightarrow \mathcal{O}_X$. Then the $(G_{sc}^X)^o$-orbit of X' in $\widetilde{\mathcal{O}}_X$ is a connected subspace of the discrete space F, whence it consists of X' alone. Hence, the isotropy group of $\widetilde{\mathcal{O}}_X$ at X' contains $(G_{sc}^X)^o$. On the other hand, it is clear that $G_{sc}/(G_{sc}^X)^o$ is a covering space of $\mathcal{O}_X$, which must in turn be covered by $\widetilde{\mathcal{O}}_X$. The first assertion follows at once. To prove the second, just note that $\pi_1(\mathcal{O}_X)$ is the group of deck transformations of $\widetilde{\mathcal{O}}_X$. $\square$

Now it is natural to make the following definition. Let G be any complex Lie group with Lie algebra $\mathfrak{g}$. Define the G-equivariant fundamental group $\pi_1^G(\mathcal{O}_X)$ of $\mathcal{O}_X$ via $\pi_1^G(\mathcal{O}_X) := G^X/(G^X)^o$, the component group of G^X. This is just the group of deck transformations of the largest covering space of $\mathcal{O}_X$ with a G-action. Obviously $\pi_1^{G_{sc}}(\mathcal{O}_X) \cong \pi_1(\mathcal{O}_X)$. We write $A(\mathcal{O}_X)$ for $\pi_1^{G_{ad}}(\mathcal{O}_X)$, as in (3.7.2). To compute $\pi_1^G(\mathcal{O}_X)$ for any G, we first observe that

$$G^X/(G^X)^o = G^\phi/(G^\phi)^o \tag{6.1.2}$$

by (3.7.5)(ii) (notation §3.7). Thus, we are reduced to studying the centralizer in G of a copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$. Assume now that $\mathfrak{g}$ is classical. If $\mathfrak{g} = \mathfrak{sl}_n$, then $G_{sc} = SL_n$; similarly, if $\mathfrak{g} = \mathfrak{sp}_{2n}$, then $G_{sc} = Sp_{2n}$. On the other hand, if $\mathfrak{g} = \mathfrak{so}_N$, then G_{sc} is a double cover of SO_N denoted $Spin_N$.

For any group H, let H_Δ^n denote the diagonal copy of H inside the direct product of n copies of H. If $H_1, \ldots, H_m$ are m matrix groups, let $S(\prod_i H_i)$ denote the subgroup of $\prod_i H_i$ consisting of m-tuples of matrices whose determinants have product 1. Although any H_Δ^n is of course isomorphic to H, we note that $S(H \times K \times \ldots)$ need *not* be isomorphic to $S(H_\Delta^m \times K_\Delta^n \times \ldots)$. The simplest example occurs when $H = GL_n$ and $m = 2$. Then $S(H) = SL_n$, but $S(H_\Delta^2)$ is (isomorphic to) a larger group sometimes denoted by $SL_n^\pm$; it consists of all $n \times n$ complex matrices of determinant ± 1.

Theorem 6.1.3 (Springer-Steinberg). *Let $\mathfrak{g}$ be a classical Lie algebra and $\mathcal{O}_X$ a nilpotent orbit in $\mathfrak{g}$. Write $\mathcal{O}_X = \mathcal{O}_{[d_1,d_2,\ldots,d_N]}$, for some $\mathbf{d} = [d_1, d_2, \ldots, d_N]$ in $\mathcal{P}(N)$ (notation §5.1). Put $r_i = |\{ j \mid d_j = i \}|$ and $s_i = |\{ j \mid d_j \geq i \}|$. Then*

$$G_{sc}^\phi \cong \begin{cases} S(\prod_i (GL_{r_i})_\Delta^i) & \mathfrak{g} = \mathfrak{sl}_n \\ \prod_{i\,\mathrm{odd}} (Sp_{r_i})_\Delta^i \times \prod_{i\,\mathrm{even}} (O_{r_i})_\Delta^i & \mathfrak{g} = \mathfrak{sp}_{2n} \\ \text{double cover of } C := (S(\prod_{i\,\mathrm{even}} (Sp_{r_i})_\Delta^i \times \prod_{i\,\mathrm{odd}} (O_{r_i})_\Delta^i) & \mathfrak{g} = \mathfrak{so}_N \end{cases}$$

$$G_{ad}^\phi \cong \begin{cases} S(\prod_i (GL_{r_i})_\Delta^i)/\{\text{scalar matrices in } SL_n\} & \mathfrak{g} = \mathfrak{sl}_n \\ G_{sc}^\phi/\{\pm I\} & \mathfrak{g} = \mathfrak{sp}_{2n} \\ C & \mathfrak{g} = \mathfrak{so}_{2n+1} \\ C/\{\pm I\} & \mathfrak{g} = \mathfrak{so}_{2n}. \end{cases}$$

In addition, the dimension of $\mathfrak{g}^X$ is given by

$$\dim(\mathfrak{g}^X) = \begin{cases} \sum_i s_i^2 - 1 & \mathfrak{g} = \mathfrak{sl}_n \\ \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i\,\mathrm{odd}} r_i & \mathfrak{g} = \mathfrak{sp}_{2n} \\ \frac{1}{2}\sum_i s_i^2 - \frac{1}{2}\sum_{i\,\mathrm{odd}} r_i & \mathfrak{g} = \mathfrak{so}_N. \end{cases}$$

Proof. The formulas for G_{sc}^ϕ and G_{ad}^ϕ follow from (5.1.18). To prove the assertions about dimension, we must work harder. Suppose first that $\mathfrak{g} = \mathfrak{sl}_n$. Given $X \in \mathfrak{g}$ as in the hypothesis of the theorem, we investigate the conditions under which a typical Z in $\mathfrak{gl}_n$ (not $\mathfrak{g}$) commutes with X. Decompose the standard representation V of $\mathfrak{g}$ under the action of a standard triple containing X, as in §5.1, and adopt the notation used there. This time, however, let $L(d)$ denote the *lowest* weight space of $M(d)$. Once again the number of irreducible summands of $M(d)$ equals $\dim L(d)$. Clearly it also equals r_{d+1}. Note first that the action of Z on V is completely determined by what Z does to $L(d)$, for any vector in V can be obtained by applying X repeatedly to some vector in $L(d)$. Conversely, if $v \in L(d)$, then the only constraint on $Z \cdot v$ is that it must be killed by $d+1$ applications of X. Now let $d \leq e$. If we restrict the matrices in $\mathfrak{gl}_n^X$ to $L(d)$ and project their ranges to $M(e)$, we get a subspace $S_{d,e}$ of $r_{d+1} \times [(e+1)(r_{e+1})]$ matrices of dimension $(d+1)r_{d+1}r_{e+1}$. If instead $e < d$, the dimension of $S_{d,e}$ is $(e+1)r_{d+1}r_{e+1}$. Since the set of restrictions of matrices in $\mathfrak{gl}_n^X$ to the sum of the $L(d)$ is the direct sum of its projections to the various $S_{d,e}$, it follows that the dimension of $\mathfrak{gl}_n^X$ is

$$\begin{aligned} & r_1(r_1 + 2r_2 + 2r_3 + \ldots) + r_2(2r_2 + 4r_3 + 4r_4 + \ldots) + \ldots \\ &= r_1(r_1 + 2s_2) + r_2(2r_2 + 4s_3) + r_3(3r_3 + 6s_4) + \ldots \\ &= (s_1 - s_2)(s_1 + s_2) + (s_2 - s_3)(2s_2 + 2s_3) + (s_3 - s_4)(3s_3 + 3s_4) + \ldots \\ &= s_1^2 - s_2^2 + 2s_2^2 - 2s_3^2 + 3s_3^2 - 3s_4^2 + \ldots \\ &= s_1^2 + s_2^2 + \ldots \end{aligned}$$

Passing from $\mathfrak{gl}_n^X$ to $\mathfrak{g}^X$, we lose exactly one dimension (because there are no nonzero scalar matrices in $\mathfrak{g}$). The formula for $\dim \mathfrak{g}^X$ follows.

Next let $\mathfrak{g} = \mathfrak{sp}_{2n}$. The basic idea is of course to imitate the argument of the above paragraph, but now we must keep track of the ambient symplectic form $\langle \cdot, \cdot \rangle$ on V. The matrices of $\mathfrak{g}^X$ are skew-adjoint with respect to this form. Once again, any $Z \in \mathfrak{g}^X$ is completely determined by its restriction to $L(d)$. If Z sends $L(d)$ to $M(e)$ with $d \neq e$, then skew-adjointness forces it to send $L(e)$ to $M(d)$ and moreover completely determines its action on $L(e)$. Conversely, any linear map from $M(d)$ to $M(e)$ commuting with X extends uniquely to a skew-adjoint map from $M(d) \oplus M(e)$ to itself also commuting with X. So it suffices to study skew-adjoint maps from $M(d)$ to itself commuting with X. If such a map sends $L(d)$ to $H(d)$, then it defines a bilinear form $(\cdot,\cdot)_d$ on $L(d)$ via $(v,w)_d = \langle v, Z \cdot w \rangle$. One checks immediately that this form is symmetric or symplectic according as d is even or odd and that the action of Z on $L(d)$ is completely determined by this form. The set of all possible forms $(\cdot,\cdot)_d$ is thus a vector space of dimension $\frac{1}{2}r_{d+1}(r_{d+1}+1)$ if d is even and $\frac{1}{2}r_{d+1}(r_{d+1}-1)$ if d is odd. Now suppose that Z sends $L(d)$ to the $(d-2)$-weight space of $M(d)$. The analogue of $(\cdot,\cdot)_d$ is the bilinear form $(\cdot,\cdot)_{d-2}$ defined by $(v,w)_{d-2} = \langle v, X \cdot Z \cdot w \rangle$. Once again the action of Z on $L(d)$ is completely determined by $(\cdot,\cdot)_{d-2}$, but this time this form is symmetric or symplectic according as d is odd or even. Repeating this argument for all the weight spaces of $M(d)$, and combining it with earlier arguments, we arrive at the following formula for $\dim \mathfrak{g}^X$.

$$r_1(\tfrac{1}{2}r_1 + \tfrac{1}{2} + r_2 + \dots) + r_2(2 \cdot \tfrac{1}{2}r_2 + \dots) + r_3(3 \cdot (\tfrac{1}{2}r_3 + \tfrac{1}{2}) + \dots) + \dots$$

$$\begin{aligned} &= (s_1 - s_2)\tfrac{1}{2}(s_1 + s_2) + \tfrac{1}{2}r_1 + (s_2 - s_3)(s_2 + s_3) + \\ &\qquad + (s_3 - s_4)\tfrac{3}{2}(s_3 + s_4) + \tfrac{1}{2}r_3 + \dots \end{aligned}$$

$$= \tfrac{1}{2}s_1^2 - \tfrac{1}{2}s_2^2 + \tfrac{1}{2}r_1 + s_2^2 - s_3^2 + \tfrac{3}{2}s_3^2 - \tfrac{3}{2}s_4^2 + \tfrac{1}{2}r_3 + \dots$$

$$= \tfrac{1}{2}(s_1^2 + s_2^2 + \dots) + \tfrac{1}{2}(r_1 + r_3 + r_5 + \dots)$$

∎

This is the desired result. The argument for $\mathfrak{g} = \mathfrak{so}_N$ is similar.

Using (1.2.15), it is an easy matter to compute the dimension of any nilpotent orbit in a classical Lie algebra.

Corollary 6.1.4. *Using the notation of (6.1.3), we have*

$$\dim(\mathcal{O}_X) = \begin{cases} n^2 - \sum_i s_i^2, & \text{if } \mathfrak{g} = \mathfrak{sl}_n \\ 2n^2 + n - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i \text{ odd}} r_i, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1} \\ 2n^2 + n - \frac{1}{2}\sum_i s_i^2 - \frac{1}{2}\sum_{i \text{ odd}} r_i, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n} \\ 2n^2 - n - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i \text{ odd}} r_i, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n}. \end{cases}$$

Examples 6.1.5. We now give several examples illustrating (6.1.3) and (6.1.4).

(i) Let $\mathfrak{g} = \mathfrak{sl}_6$ and $\mathcal{O} = \mathcal{O}_{[2^3]}$. Then $s_1 = s_2 = 3$ and $s_3 = 0$. We have $G_{sc}^\phi = S((GL_3)_\Delta^2) \cong SL_3^\pm$; it has two components, but $G_{ad}^\phi \cong SL_3$ is connected. One computes that $\dim(\mathfrak{g}^X) = 17$ if $X \in \mathcal{O}$ and $\dim(\mathcal{O}) = 18$. From (6.1.1) and (6.1.2) we get $\pi_1(\mathcal{O}) = \mathbb{Z}/2\mathbb{Z}$ and $A(\mathcal{O}) = 1$.

(ii) Let $\mathfrak{g} = \mathfrak{sl}_{10}, \mathcal{O} = \mathcal{O}_{[7,3]}$. Then $r_3 = r_7 = 1$ and $r_i = 0, i \neq 3, 7$. Now $G_{sc}^\phi = S(GL_1 \times GL_1) \cong GL_1 \cong G_{ad}^\phi$, which is connected. One computes that $\dim(\mathfrak{g}^{X_{[7,3]}}) = 15$ and $\dim(\mathcal{O}_{[7,3]}) = 84$. By (6.1.1), $\pi_1(\mathcal{O}_{[7,3]}) = A(\mathcal{O}_{[7,3]}) = 1$.

(iii) Let $\mathfrak{g} = \mathfrak{sp}_{12}, \mathcal{O} = \mathcal{O}_{[4^2,2^2]}$. Now $r_2 = r_4 = 2$, while $r_1 = r_3 = 0$. We have $G_{sc}^\phi \cong (O_2)_\Delta^4 \times (O_2)_\Delta^2$ while $G_{ad}^\phi \cong G_{sc}^\phi/\{\pm I\}$. We also have $\dim \mathfrak{g}^X = 20$ if $X \in \mathcal{O}$ and $\dim \mathcal{O} = 58$. Here $\pi_1(\mathcal{O}) \cong A(\mathcal{O}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. On the other hand, if $\mathcal{O} = \mathcal{O}_{[6,2^2,1^2]}$, then $G_{sc}^\phi \cong (O_1)_\Delta^6 \times (O_2)_\Delta^2 \times Sp_2, G_{ad}^\phi = G_{sc}^\phi/\{\pm I\}$. We again get $\pi_1(\mathcal{O}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, but this time $A(\mathcal{O}) \cong \mathbb{Z}/2\mathbb{Z}$. Once again $\dim \mathcal{O} = 58$.

(iv) Finally, put $\mathfrak{g} = \mathfrak{so}_{12}, \mathcal{O} = \mathcal{O}_{[3^2,2^2,1^2]}$. Then G_{sc}^ϕ is a double cover of $S[(O_2)_\Delta^3 \times (Sp_2)_\Delta^2 \times O_2]$. It may also be realized as a subgroup of index 2 of $Pin_2 \times Sp_2 \times O_2$, where Pin_n for any n denotes a double cover of O_n corresponding to the double cover $Spin_n$ of SO_n. We have $G_{ad}^\phi = G_{sc}^\phi/\{\pm I\}$. Here $\pi_1(\mathcal{O}) = A(\mathcal{O}) = \mathbb{Z}/2\mathbb{Z}$ and $\dim \mathcal{O} = 40$. If $\mathcal{O} = \mathcal{O}_{[3,2^2,1^5]}$, then G_{sc}^ϕ is a double cover of $S[(O_1)_\Delta^3 \times (Sp_2)_\Delta^2 \times O_5]$. We still have $\pi_1(\mathcal{O}) \cong \mathbb{Z}/2\mathbb{Z}$, but now $A(\mathcal{O}) = 1$. Here $\dim \mathcal{O} = 32$.

We can generalize these examples by giving formulas for $\pi_1(\mathcal{O})$ and $A(\mathcal{O})$ for any orbit $\mathcal{O} = \mathcal{O}_{[d_1,\dots,d_N]}$ in a classical algebra $\mathfrak{g}$. To do this we need some notation. Put

$$\begin{aligned} a &= \text{number of distinct odd } d_i \\ b &= \text{number of distinct even nonzero } d_i \\ c &= \gcd(d_1, \dots, d_N). \end{aligned}$$

A partition is called *rather odd* if all of its odd parts have multiplicity one; in particular, a very even partition is vacuously rather odd. A group E is called a *central extension* of a group H by a group K if there exists a short exact sequence $1 \to K \to E \to H \to 1$, where K is a central subgroup of E.

Corollary 6.1.6. *With notation as above, the groups $\pi_1(\mathcal{O})$ and $A(\mathcal{O})$ are given in the following table, for all classical orbits $\mathcal{O}$.*

Classical Equivariant Fundamental Groups		
Algebra	$\pi_1(\mathcal{O}_{\mathbf{d}})$	$A(\mathcal{O}_{\mathbf{d}})$
$\mathfrak{sl}_n$	$\mathbb{Z}/c\mathbb{Z}$	1
$\mathfrak{so}_{2n+1}$	If $\mathbf{d}$ is rather odd, a central extension by $\mathbb{Z}/2\mathbb{Z}$ of $(\mathbb{Z}/2\mathbb{Z})^{a-1}$; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{a-1}$	$(\mathbb{Z}/2\mathbb{Z})^{a-1}$
$\mathfrak{sp}_{2n}$	$(\mathbb{Z}/2\mathbb{Z})^b$	$(\mathbb{Z}/2\mathbb{Z})^b$ if all even parts have even multiplicity; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{b-1}$
$\mathfrak{so}_{2n}$	If $\mathbf{d}$ is rather odd, a central extension by $\mathbb{Z}/2\mathbb{Z}$ of $(\mathbb{Z}/2\mathbb{Z})^{\max(0,a-1)}$; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{\max(0,a-1)}$	$(\mathbb{Z}/2\mathbb{Z})^{\max(0,a-1)}$ if all odd parts have even multiplicity; otherwise $(\mathbb{Z}/2\mathbb{Z})^{\max(0,a-2)}$

Corollary 6.1.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra of classical type and $\mathcal{O}$ a nilpotent orbit. Then $A(\mathcal{O}_X)$ is either trivial or a finite product of copies of $\mathbb{Z}/2\mathbb{Z}$. In particular, it is always abelian.*

Remarks. (i) The precise structure of $\pi_1(\mathcal{O}_{\mathbf{d}})$ for rather odd $\mathbf{d}$ in types B, D is somewhat complicated to describe, but one can show that this group is built up out of copies of Di_8 (the dihedral group of order 8), Q_8 (the quaternion group of order 8), $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/4\mathbb{Z}$ by taking central products. Moreover, $\pi_1(\mathcal{O}_{\mathbf{d}})$ depends only on the congruence classes modulo 4 of the odd parts of $\mathbf{d}$. See [60, §14.3].

(ii) If $\mathfrak{g}$ is exceptional, then $\pi_1(\mathcal{O})$ takes the form $S_r \times \mathbb{Z}/d\mathbb{Z}$ for any nilpotent orbit $\mathcal{O}$, where S_r denotes the symmetric group on r letters. (This result is due to Alexeevski [1] and (independently) Mizuno. The proof is far from conceptual: one first makes a list, then observes that every entry has the claimed form.) We have $1 \le r \le 5$ and $1 \le d \le 3$. More precisely, we have $d = 1$, except in types E_6, E_7, where it can also be 3,2, respectively. (The factor $\mathbb{Z}/d\mathbb{Z}$ comes from the center of G_{sc}, so that $A(\mathcal{O})$ is obtained from $\pi_1(\mathcal{O})$ by omitting this factor.) There is only one orbit for which $r = 5$ (resp. $r = 4$); it lives in type E_8 (resp. F_4). If $r \ge 3$, then $d = 1$, except for one orbit in type E_7. We will give tables of the exceptional orbits and their fundamental groups at the end of Chapter 8.

We observed in the introduction to Chapter 4 that the boundary of any adjoint or coadjoint orbit $\mathcal{O}$ lives in codimension at least two; from this it can be shown that $\pi_1(\mathcal{O}) = \pi_1(\overline{\mathcal{O}})$. We close the section by observing that the fundamental group of any adjoint orbit coincides with an *equivariant* (not necessarily the full) fundamental group of a nilpotent orbit.

Proposition 6.1.8. *Let $\mathcal{O}_X$ be the adjoint orbit through any $X \in \mathfrak{g}$. Let $X = X_s + X_n$ be the Jordan decomposition of X. Then $\pi_1(\mathcal{O}_X)$ is isomorphic to the $G_{sc}^{X_s}$-equivariant fundamental group of the $G_{sc}^{X_s}$-orbit through X_n. In particular, every semisimple orbit in $\mathfrak{g}$ is simply connected.*

Proof. Combining (6.1.1) (which actually applies to any adjoint orbit) with the uniqueness of the Jordan decomposition in $\mathfrak{g}$, we get

$$\pi_1(\mathcal{O}_X) = G_{sc}^X/(G_{sc}^X)^o = (G_{sc}^{X_s})^{X_n}/((G_{sc}^{X_s})^{X_n})^o;$$

from which the first assertion follows immediately. The second follows from (2.3.3), as already noted in (2.3.4). □

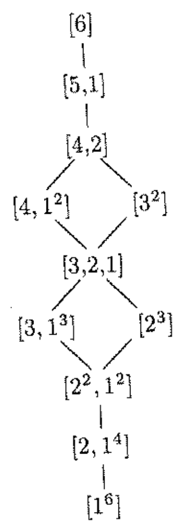
The reason that $\pi_1(\mathcal{O}_X)$ need not be isomorphic to the full fundamental group of a nilpotent orbit is simply that the centralizer $G_{sc}^{X_s}$, although certainly reductive (by (2.1.2)), need not be simply connected.

6.2 The Closure Ordering

In the introduction of Chapter 4 we defined a partial order on the set of nilpotent orbits via $\mathcal{O} \le \mathcal{O}'$ if $\overline{\mathcal{O}} \subset \overline{\mathcal{O}'}$. We now describe this order for classical $\mathfrak{g}$ in terms of partitions. We begin by defining a partial order on the set $\mathcal{P}(N)$ of partitions of N. Given two partitions $\mathbf{d} = [d_1, \ldots, d_N]$ and $\mathbf{f} = [f_1, \ldots, f_N]$, we say that $\mathbf{d}$ *dominates* $\mathbf{f}$ and write $\mathbf{d} \ge \mathbf{f}$, if the following condition holds:

$$\sum_{1 \le j \le k} d_j \ge \sum_{1 \le j \le k} f_j \text{ for } 1 \le k \le N. \qquad (6.2.1)$$

The corresponding Hasse diagram for $\mathcal{P}(6)$ follows.



Before stating and proving our main theorem, we need two combinatorial lemmas.

Lemma 6.2.2. *Let $\mathcal{O}_{\mathbf{d}}, \mathcal{O}_{\mathbf{f}}$ be the nilpotent orbits in $\mathfrak{sl}_n$ corresponding to $\mathbf{d}$ and $\mathbf{f}$, respectively, and let $X \in \mathcal{O}_{\mathbf{d}}, Y \in \mathcal{O}_{\mathbf{f}}$. Then $\mathbf{d} \geq \mathbf{f}$ if and only if rank$(X^k) \geq$ rank(Y^k) for all $k \geq 0$, where X^k of course denotes the k^{th} matrix power of X.*

Proof. One readily computes that

$$\operatorname{rank}(X^k) = \sum_{\{i \mid d_i \geq k\}} (d_i - k). \tag{6.2.3}$$

Suppose that $\mathbf{d} \not\geq \mathbf{f}$. Let j be the smallest integer with $\sum_{i=1}^{j} d_i < \sum_{i=1}^{j} f_i$. Then clearly $d_j < f_j$. No term d_i with $i > j$ contributes to rank(X^{d_j}), so it follows that rank$(X^{d_j}) <$ rank(Y^{d_j}). Conversely, suppose that rank$(X^k) <$ rank(Y^k) for some k. Let m be the largest index with $f_m \geq k$. Then rank$(Y^k) = \sum_{i=1}^{m}(f_i - k)$, while $\sum_{i=1}^{m}(d_i - k) \leq$ rank(X^k). Hence $\sum_{i=1}^{m} d_i < \sum_{i=1}^{m} f_i$, so that $\mathbf{d} \not\geq \mathbf{f}$, as required. □

Lemma 6.2.4 (Gerstenhaber [29]). *Let $\mathbf{d}, \mathbf{f} \in \mathcal{P}(N)$ with $\mathbf{d} = [d_1, \ldots, d_N]$. Then $\mathbf{d}$ covers $\mathbf{f}$ in the order $\leq$ (that is, $\mathbf{d} > \mathbf{f}$ and there is no partition $\mathbf{e}$ with $\mathbf{d} > \mathbf{e} > \mathbf{f}$) if and only if $\mathbf{f}$ can be obtained from $\mathbf{d}$ by the following procedure. Choose an index i and let j be the smallest index greater than i with $0 \leq d_j < d_i - 1$. Assume that either $d_j = d_i - 2$ or $d_k = d_i$ whenever $i < k < j$. Then the parts of $\mathbf{f}$ are obtained from the d_k by replacing d_i, d_j by $d_i - 1, d_j + 1$, respectively (and rearranging).*

Proof. It is clear that any $\mathbf{d}'$ obtained from $\mathbf{d}$ as in the statement must be covered by the latter. Conversely, if $\mathbf{d}$ covers $\mathbf{f}$, then $d_k > f_k$ for at least one k; let i be the least such k. Let $\mathbf{d}'$ be obtained from $\mathbf{d}$ as in the statement of the lemma, using the indices i and j. One readily computes that $\mathbf{d}' \geq \mathbf{f}$, whence we must have equality if $\mathbf{d}$ covers $\mathbf{f}$. □

The main result of this section is

Theorem 6.2.5 (Gerstenhaber, Hesselink). *Let $\mathfrak{g}$ be a classical Lie algebra, and let $\mathbf{d}, \mathbf{f}$ be the partitions of two nilpotent orbits $\mathcal{O}_{\mathbf{d}}, \mathcal{O}_{\mathbf{f}}$ in $\mathfrak{g}$. Then $\mathcal{O}_{\mathbf{d}} > \mathcal{O}_{\mathbf{f}}$ if and only if $\mathbf{d} > \mathbf{f}$.*

Proof. Let $X \in \mathcal{O}_{\mathbf{d}}, Y \in \mathcal{O}_{\mathbf{f}}$. Since the rank of any power of a matrix is invariant under conjugation, and since the condition that the rank of a matrix be strictly less than some fixed integer m is a (Zariski-)closed condition, we deduce that $\mathcal{O}_{\mathbf{d}} > \mathcal{O}_{\mathbf{f}}$ only if rank$(X^k) >$ rank(Y^k) for all k, which holds if and only if $\mathbf{d} > \mathbf{f}$, by (6.2.2). To prove the converse, suppose first that $\mathfrak{g} = \mathfrak{sl}_n$. Then it suffices to treat the case where $\mathbf{d}$ covers $\mathbf{f}$. Choose a standard triple $\{H, X, Y\}$ in $\mathfrak{g}$ with $X \in \mathcal{O}_{\mathbf{d}}$ as in Recipe 5.2.1 and define the subalgebra $\mathfrak{q}_2$ of $\mathfrak{g}$ as in §3.8. Using (5.2.1) again, it is straightforward to check that $\mathcal{O}_{\mathbf{f}}$ is represented by an element of $\mathfrak{q}_2$. Then the desired result follows from (4.1.4). For the other classical cases, one must work harder, replacing (6.2.4) by a more complicated condition for one partition in $\mathcal{P}_1(2n+1), \mathcal{P}_1(2n)$, or $\mathcal{P}_{-1}(2n)$ to cover another in the same set. We refer to [34] for the details. □

We mention that Gerstenhaber proved most of (6.2.5) in [30], but his arguments were incomplete; Hesselink patched them up in [34]. You may wonder why we did not replace the $>$ sign in (6.2.5) by $\geq$. In fact, we could have done so in types A, B, and C, but the difficulty in type D is that the two orbits in type D_{2n} attached to a very even partition $\mathbf{d}$ in $\mathcal{P}_1(4n)$ are incomparable (as can be seen by observing that they have the same dimension). Thus, we could also have replaced the $>$ sign by $\geq$ if we looked at O_{2n}-orbits in type D_{2n}.

Examples 6.2.6.(i) For $\mathfrak{g} = \mathfrak{sl}_6$, the Hasse diagram of nilpotent orbits coincides with that of $\mathcal{P}(6)$; it was given previously.

(ii) For $\mathfrak{g} = \mathfrak{sp}_6$, we get the following Hasse diagram of nilpotent orbits:

$$
\begin{tikzcd}
& \mathcal{O}_{[6]} \arrow[d, dash] & \\
& \mathcal{O}_{[4,2]} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_{[4,1^2]} \arrow[dr, dash] & & \mathcal{O}_{[3^2]} \arrow[dl, dash] \\
& \mathcal{O}_{[2^3]} \arrow[d, dash] & \\
& \mathcal{O}_{[2^2,1^2]} \arrow[d, dash] & \\
& \mathcal{O}_{[2,1^4]} \arrow[d, dash] & \\
& \mathcal{O}_{[1^6]} &
\end{tikzcd}
$$

Note that this Hasse diagram, unlike that of $\mathcal{P}(6)$, does not admit an order-reversing involution. We will partially rectify this situation in the next section.

(iii) For $\mathfrak{g} = \mathfrak{so}_8$ the Hasse diagram is given on the left; if $\mathfrak{g} = \mathfrak{so}_{10}$ the Hasse diagram is on the right (see p. 97). One observes from the diagram that the poset of nilpotent orbits is not ranked in this case; see [81] for a discussion of ranked posets.

(iv) Finally, we remark that the corresponding Hasse diagrams have been completely computed for exceptional $\mathfrak{g}$. As in Chapter 4, we refer you to [76].

$$
\begin{tikzcd}
& \mathcal{O}_{[7,1]} \arrow[d, dash] & \\
& \mathcal{O}_{[5,3]} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathcal{O}^{I}_{[4^2]} \arrow[dr, dash] & \mathcal{O}_{[5,1^3]} \arrow[d, dash] & \mathcal{O}^{II}_{[4^2]} \arrow[dl, dash] \\
& \mathcal{O}_{[3^2,1^2]} \arrow[d, dash] & \\
& \mathcal{O}_{[3,2^2,1]} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathcal{O}^{I}_{[2^4]} \arrow[dr, dash] & \mathcal{O}_{[3,1^5]} \arrow[d, dash] & \mathcal{O}^{II}_{[2^4]} \arrow[dl, dash] \\
& \mathcal{O}_{[2^2,1^4]} \arrow[d, dash] & \\
& \mathcal{O}_{[1^8]} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{O}_{[9,1]} \arrow[d, dash] & \\
& \mathcal{O}_{[7,3]} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_{[7,1^3]} \arrow[dr, dash] & & \mathcal{O}_{[5^2]} \arrow[dl, dash] \\
& \mathcal{O}_{[5,3,1^2]} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_{[4^2,1^2]} \arrow[dr, dash] & & \mathcal{O}_{[5,2^2,1]} \arrow[dl, dash] \arrow[dd, dash] \\
& \mathcal{O}_{[3^3,1]} \arrow[dl, dash] & \\
\mathcal{O}_{[3^2,2^2]} \arrow[dr, dash] & & \mathcal{O}_{[5,1^5]} \arrow[dl, dash] \\
& \mathcal{O}_{[3^2,1^4]} \arrow[d, dash] & \\
& \mathcal{O}_{[3,2^2,1^3]} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_{[2^4,1^2]} \arrow[dr, dash] & & \mathcal{O}_{[3,1^7]} \arrow[dl, dash] \\
& \mathcal{O}_{[2^2,1^6]} \arrow[d, dash] & \\
& \mathcal{O}_{[1^{10}]} &
\end{tikzcd}
$$

6.3 Special Nilpotent Orbits in the Classical Algebras

One of the reasons the partial order on nilpotent orbits is worth studying is that it reveals a symmetry in the structure of the latter that was not obvious before. In this section, we give a precise account of this symmetry. Following [76], we define an order-reversing map d on the set of nilpotent orbits in $\mathfrak{g}$ that becomes an involution when restricted to its range. Orbits in the range of d are called *special*; you will see in Chapter 10 that they play a key role in the classification of primitive ideals in the enveloping algebra of $\mathfrak{g}$. Lusztig originally defined special orbits by means of a sophisticated correspondence between nilpotent orbits and representations of a Weyl group (this is the so-called *Springer correspondence* and will be discussed in Chapter 10). For classical $\mathfrak{g}$, however, there is a straightforward combinatorial definition of specialness for orbits in terms of their attached partitions. We give this definition and then show how to construct the map d. Our results are due to Spaltenstein [76], with a couple of exceptions noted below.

Assume first (as usual) that $\mathfrak{g} = \mathfrak{sl}_n$. In this case, it turns out that every orbit is special, so it suffices to define the map d. To do this, we need to recall a couple of definitions from §4.4. A *Young diagram* is a left-justified array of empty boxes arranged so that no row is shorter than the one below it. There is an obvious bijection between such objects and partitions: given a partition $[p_1, p_2, \ldots, p_n]$, the corresponding diagram has p_1 boxes in its first row, p_2 in its second, and so on. (For some pictures of Young diagrams, see §4.4.) Now let $\mathbf{p} = [p_1, \ldots, p_n]$ be a partition of n. Define $\mathbf{p^t}$, the *transpose partition* of $\mathbf{p}$, via $\mathbf{p^t} = [p_1^t, \ldots, p_n^t]$, where $p_i^t = |\{j \mid p_j \geq i\}|$ (cf. the statement of (6.1.3)). The map $\mathbf{p} \mapsto \mathbf{p^t}$ admits a simple geometric description on the level of Young diagrams: the rows of the diagram of $\mathbf{p^t}$ are just the columns of the diagram of $\mathbf{p}$ (from left to right). Again see §4.4 for an example. A key result is

Lemma 6.3.1 (Gerstenhaber). *The map $\mathbf{p} \mapsto \mathbf{p^t}$ is an order-reversing involution on the set $\mathcal{P}(n)$ of partitions of n.*

Proof. It is clear that this map is an involution on $\mathcal{P}(n)$. Suppose that $\mathbf{p^t} \ngeq \mathbf{q^t}$ and let j be the least index for which $\sum_{i=1}^j p_i^t < \sum_{i=1}^j q_i^t$. Then we must have $p_j^t < q_j^t$. Note that the number r_i of parts of $\mathbf{p}$ equal to i is just $p_i^t - p_{i+1}^t$; similarly the number s_i of parts of $\mathbf{q}$ equal to i is $q_i^t - q_{i+1}^t$. Set $k = p_j^t, \ell = q_j^t$. Then

$$\begin{aligned}\textstyle\sum_{i=1}^k p_i &= n - \textstyle\sum_{m=1}^{j-1} m r_m \\ &= n - (j-1)(p_{j-1}^t - p_j^t) \\ &\quad -(j-2)(p_{j-2}^t - p_{j-1}^t) \\ &\quad - \cdots - (p_1^t - p_2^t) \\ &= n - (p_1^t + \cdots + p_{j-1}^t) + (j-1)k.\end{aligned}$$

On the other hand, we have $q_i \geq j$ for any $i \leq \ell$. Hence

$$\begin{aligned}\textstyle\sum_{i=1}^k q_i &= n - \textstyle\sum_{m=1}^{j-1} m s_m - \sum_{r=k}^{\ell} q_r \\ &\leq n - (q_1^t + \cdots + q_{j-1}^t) + (j-1)\ell - j(\ell - k).\end{aligned}$$

It follows easily that $\sum_{i=1}^k p_i > \sum_{i=1}^k q_i$, so that $\mathbf{p} \nleq \mathbf{q}$. Since the transpose map is an involution, we have $\mathbf{p} \nleq \mathbf{q} \iff \mathbf{p^t} \ngeq \mathbf{q^t}$, or equivalently $\mathbf{p} \leq \mathbf{q} \iff \mathbf{p^t} \geq \mathbf{q^t}$, as desired. □

Combining this lemma with (6.2.5), we deduce

Theorem 6.3.2 (Gerstenhaber). *There is an order-reversing involution d on the set of nilpotent orbits in $\mathfrak{sl}_n$ sending the orbit $\mathcal{O}_{\mathbf{p}}$ with partition $\mathbf{p}$ to $\mathcal{O}_{\mathbf{p^t}}$.*

As soon as we try to generalize this result to other classical algebras $\mathfrak{g}$, we run into a fundamental difficulty: The transpose of a partition in $\mathcal{P}_1(2n)$ need not lie in $\mathcal{P}_1(2n)$! Clearly, what is wanted is a method for passing from an arbitrary partition of $2n+1$ (resp. $2n$) to one in $\mathcal{P}_1(2n+1)$ (resp. $\mathcal{P}_1(2n), \mathcal{P}_{-1}(2n)$). This is provided by

Lemma 6.3.3 (Gerstenhaber). *Let $\mathbf{p} = [p_1, \ldots, p_{2n+1}]$ be a partition in $\mathcal{P}(2n+1)$. Then there is a unique largest partition in $\mathcal{P}_1(2n+1)$ dominated by $\mathbf{p}$. This partition, called the B-collapse of $\mathbf{p}$ and denoted by $\mathbf{p}_B$, may be defined as follows. If $\mathbf{p}$ is not already in $\mathcal{P}_1(2n+1)$, then at least one of its even parts must occur with odd multiplicity; let q be the largest such part. Replace the last occurrence of q in $\mathbf{p}$ by $q-1$ and the first subsequent part r strictly less than $q-1$ by $r+1$; we may have to add a 0 to $\mathbf{p}$ to find such an r. (Recall that we identify any two partitions whenever their nonzero parts agree.) Repeat this process until a partition in $\mathcal{P}_1(2n+1)$ is obtained. Similarly, there are unique largest partitions $\mathbf{q}_C, \mathbf{q}_D$ in $\mathcal{P}_{-1}(2n), \mathcal{P}_1(2n)$ dominated by any given partition $\mathbf{q}$ of $2n$. They are called the C- and D-collapses of $\mathbf{q}$; their definitions are the obvious analogues of that of $\mathbf{p}_B$.*

Proof. The argument is similar to the proof of (6.2.4). We give the details in case $\mathbf{p} \in \mathcal{P}_1(2n+1)$; the other cases are similar. Obviously $\mathbf{p}_B \in \mathcal{P}_1(2n+1)$ and $\mathbf{p} \geq \mathbf{p}_B$. We must show that $\mathbf{p}_B$ is the unique largest partition with these properties. Note that $\mathbf{p}_B$ is obtained from $\mathbf{p}$ in stages; at each stage a partition $\mathbf{e}$ is replaced by another one $\mathbf{e}'$ obtained by a single application of the algorithm above. It suffices to assume that $\mathbf{f} = [f_1, \ldots, f_n] \in \mathcal{P}_1(2n+1)$ and $\mathbf{f} \leq \mathbf{e} = [e_1, \ldots, e_n]$ and prove that $\mathbf{f} \leq \mathbf{e}'$. Now $\mathbf{e}'$ differs from $\mathbf{e}$ in only two parts, say the q^{th} and r^{th} with $q < r$. Then e_q is even and $e_r \leq e_q - 2$. The q^{th} part of $\mathbf{e}'$ is $e_q - 1$ and its r^{th} part is $e_r + 1$. Now let i be the largest index (if any) with $f_i \geq e_q$. If $i \geq r-1$, then a moment's reflection shows that it suffices to prove that $\sum_{j=1}^{r-1} f_j < \sum_{j=1}^{r-1} e_j$. Now $\sum_{j=1}^{r-1} e_j$ is even or odd according as r is even or odd. By considering the parity of $\sum_{j=1}^{r-1} f_j$, we see that this sum could equal $\sum_{j=1}^{r-1} e_j$ only if $f_r = e_q$. But since $e_r \leq e_q - 2$ and $\sum_{j=1}^{r} f_j < \sum_{j=1}^{r} e_j$, we see that $f_r = e_q$ forces $\sum_{j=1}^{r-1} f_j < \sum_{j=1}^{r-1} e_j$, as desired. Similarly, if $q \leq i < r-1$, it suffices to show that $\sum_{j=1}^{i} f_j < \sum_{j=1}^{i} e_j$. But these sums have opposite parity, so they cannot be equal. If $i < q$, then it is easy to see that $\mathbf{f} \leq \mathbf{e}'$. Hence, in all cases, $\mathbf{f} \leq \mathbf{e}'$, as desired. □

Examples 6.3.4. (i) If $\mathbf{d} = [3, 1^3]$, then $\mathbf{d}_C = [2^2, 1^2]$.
(ii) If $\mathbf{d} = [3, 2, 1]$, then $\mathbf{d}_C = [2^3]$ and $\mathbf{d}_D = [3, 1^3]$.
(iii) If $\mathbf{d} = [3, 2, 1^2]$, then $\mathbf{d}_B = [3, 1^4]$.

As an immediate consequence, we see that whenever $\mathbf{p}, \mathbf{q} \in \mathcal{P}(n), \mathbf{p} \leq \mathbf{q}$, and both $\mathbf{p}_X$ and $\mathbf{q}_X$ are defined $(X = B, C, D)$, then $\mathbf{p}_X \leq \mathbf{q}_X$. We deduce

ollary 6.3.5 (Spaltenstein). *The map $d : \mathbf{p} \mapsto \mathbf{p}_B^{\mathbf{t}}$ (resp. $\mathbf{p} \mapsto \mathbf{p}_C^{\mathbf{t}}, \mathbf{p} \mapsto \mathbf{p}_D^{\mathbf{t}}$) is an order-reversing map from $\mathcal{P}_1(2n+1)$ (resp. $\mathcal{P}_{-1}(2n), \mathcal{P}_1(2n)$) to itself with the further property that $d^2(\mathbf{p}) \geq \mathbf{p}$ for any $\mathbf{p}$. It induces order-reversing maps on the corresponding sets of nilpotent orbits if we decree that the numerals I and II attached to a very even partition in $\mathcal{P}_1(2n)$ get interchanged by d if n is odd but preserved if n is even.*

(The strange convention on numerals has no effect on dominance of partitions, by (6.2.5). The reason for it will emerge in Chapter 10.)

We can now prove the remaining properties of d listed in the introduction to this chapter.

rollary 6.3.6. *We have $d(\mathbf{p}) = d^3(\mathbf{p})$ for any $\mathbf{p}$ in $\mathcal{P}_1(2n+1), \mathcal{P}_1(2n)$, or $\mathcal{P}_{-1}(2n)$, so that d becomes an involution when restricted to its range. More generally, $d^2(\mathbf{p})$ is the unique smallest partition dominating $\mathbf{p}$ and in the range of d.*

Proof. Let $\mathbf{p}$ belong to $\mathcal{P}_1(2n+1), \mathcal{P}_1(2n)$, or $\mathcal{P}_{-1}(2n)$. We have $d^3(\mathbf{p}) \geq d(\mathbf{p})$ since $d^2(\mathbf{q}) \geq \mathbf{q}$ for any $\mathbf{q}$. But we also have $d^3(\mathbf{p}) \leq d(\mathbf{p})$, since $d^2(\mathbf{p}) \geq \mathbf{p}$ and d is order-reversing. Hence $d^3(\mathbf{p}) = d(\mathbf{p})$. Clearly $d^2(\mathbf{p})$ is in the range of d and dominates $\mathbf{p}$. If $d(\mathbf{q}) \geq \mathbf{p}$, then $d^3(\mathbf{q}) = d(\mathbf{q}) \geq d^2(\mathbf{p})$, since d^2 is order-preserving. The corollary follows. □

We now give a fairly simple criterion for a partition to lie in the range of d. Call a partition $\mathbf{p} \in \mathcal{P}_1(2n+1)$ *special* if it has an even number of odd parts between any two consecutive even parts and an odd number of odd parts greater than the largest even part. (Here and throughout, all parts are counted with multiplicities.) We define special partitions in $\mathcal{P}_1(2n)$ in the same way, except that we require that there be an even number of odd parts greater than the largest even part. Finally, we call a partition in $\mathcal{P}_{-1}(2n)$ special (relative of course to $\mathcal{P}_{-1}(2n)$, not $\mathcal{P}_1(2n)$) if it has an even number of even parts between any two consecutive odd ones and an even number of even parts greater than the largest odd part. A nilpotent orbit in type B, C, or D is called *special* if its partition is special; of course, as noted above, every nilpotent orbit in type A is called special. (Note also that any orbit in type D whose partition is very even is special).

We have the following criterion for specialness whose straightforward proof is left to you. Note the curious situation that arises for $\mathcal{P}_1(2n)$.

Proposition 6.3.7.

(i) *A partition $\mathbf{p} \in \mathcal{P}_1(2n+1)$ is special if and only if $\mathbf{p}^{\mathbf{t}} \in \mathcal{P}_1(2n+1)$.*

(ii) *A partition $\mathbf{p} \in \mathcal{P}_{-1}(2n)$ is special if and only if $\mathbf{p}^{\mathbf{t}} \in \mathcal{P}_{-1}(2n)$.*

(iii) *A partition $\mathbf{p} \in \mathcal{P}_1(2n)$ is special if and only if $\mathbf{p}^{\mathbf{t}} \in \mathcal{P}_{-1}(2n)$.*

Before we can prove that the special partitions are precisely those in th range of d, we need to give an explicit construction of the smallest special par tition dominating a given one. First, we reformulate the recipe of (6.3.3) fo passing from a partition to its B-,C-, or D-collapse.

Lemma 6.3.8. *Given $\mathbf{p} \in \mathcal{P}(2n+1)$, add a 0 to its parts if necessary to make it hav an even number of even parts. Enumerate its even parts (with multiplicities as $p_1^* \geq \cdots \geq p_{2n}^*$. Enumerate the indices i with $p_{2i-1}^* > p_{2i}^*$ as $i_1 < \cdots < i_m$ Then the B-collapse $\mathbf{p}_B$ of $\mathbf{p}$ may be obtained by replacing each pair of part $p_{2i_j-1}^*, p_{2i_j}^*$ by $p_{2i_j-1}^* - 1, p_{2i_j}^* + 1$, respectively, and leaving the other parts alone If $\mathbf{p} \in \mathcal{P}(2n)$, then its D-collapse $\mathbf{p}_D$ may be constructed in the same way (not that $\mathbf{p}$ automatically has an even number of even parts in this case). One ca also construct the C-collapse $\mathbf{p}_C$ of $\mathbf{p}$ in the same way, except that the p_i^* shoul enumerate the odd parts of $\mathbf{p}$.*

We again omit the straightforward proof. Intuitively, the lemma says that w need only "pass through" the parts of $\mathbf{p}$ once to compute its collapses. Now ther is a completely parallel procedure for computing the smallest special partitio dominating a given one in $\mathcal{P}_1(2n+1), \mathcal{P}_1(2n)$, or $\mathcal{P}_{-1}(2n)$. Assuming for moment that such a partition exists, call it the *X-expansion* of $\mathbf{p}$ and denote i by $\mathbf{p}^X$ $(X = B, C, D)$.

Lemma 6.3.9. *Let $\mathbf{p} = [p_1, \ldots, p_{2m+1}] \in \mathcal{P}_1(2n+1)$ and assume that all of its parts ar nonzero. Enumerate the indices $i \geq 1$ such that $p_{2i-1} = p_{2i}$ is even and p_{2i-2} i either undefined or different from p_{2i-1} as $i_1 < \cdots < i_k$. Then $\mathbf{p}^B$ exists and ma be computed by replacing each pair of parts p_{2i_j-1}, p_{2i_j} by $p_{2i_j-1}+1, p_{2i_j}-1$ respectively, and leaving the other parts alone. If instead $\mathbf{p} = [p_1, \ldots, p_{2m}]$ $\mathcal{P}_1(2n)$ with $p_{2m} \neq 0$, then enumerate the indices i such $p_{2i} = p_{2i+1}$ is even an $p_{2i-1} \neq p_{2i}$ as $i_1 < \cdots < i_k$. Then $\mathbf{p}^D$ exists and may be obtained from $\mathbf{p}$ b replacing each pair of parts p_{2i_j}, p_{2i_j+1} by $p_{2i_j}+1, p_{2i_j+1}-1$, respectively, an leaving the other parts alone. Finally, if $\mathbf{p} \in \mathcal{P}_{-1}(2n)$, then we construct $\mathbf{p}^C$ b the same recipe as for $\mathbf{p}^D$, replacing the condition that $p_{2i} = p_{2i+1}$ be even i the definition of the i_j with the condition that it be odd.*

The proof is left to you; it is similar to those of (6.3.3) and (6.3.8).

Examples 6.3.10. (i) If $\mathbf{p} = [6, 4^2, 3^2, 1^2] \in \mathcal{P}_{-1}(22)$, then $\mathbf{p}^C = [6, 4^3, 2^2]$.
(ii) If $\mathbf{p} = [7, 4^2, 2^4, 1] \in \mathcal{P}_1(24)$, then $\mathbf{p}^D = [7, 5, 3^2, 2^2, 1^2]$.
(iii) If $\mathbf{p} = [4^2, 2^4, 1] \in \mathcal{P}_1(17)$, then $\mathbf{p}^B = [5, 3^2, 2^2, 1^2]$.

The main result is

orem 6.3.11. *If* $\mathbf{p}$ *belongs to* $\mathcal{P}_1(2n+1), \mathcal{P}_1(2n)$, *or* $\mathcal{P}_{-1}(2n)$, *then* $d^2(\mathbf{p}) = \mathbf{p}^X$, *where* X *is the appropriate letter in* $\{B, C, D\}$. *In particular, the special partitions or orbits are precisely those in the range of* d.

Proof. In types B and C, the proof amounts to showing that $(\mathbf{p}^X)^{\mathbf{t}} = (\mathbf{p}^{\mathbf{t}})_X$, where X is either B or C; we again leave the proof of this to you. In type D, the details are somewhat complicated (though not really difficult); we must study the relationship between the C- and D-collapses of a partition of $2n$. We refer to [76, ch. III]. □

rollary 6.3.12. *The map* d *coincides with the transpose map when restricted to special partitions in types* A, B, *and* C *(but not in type* D*).*

We now give some examples of Hasse diagrams of special orbits. First let $\mathfrak{g} = \mathfrak{sp}_6$. All partitions in $\mathcal{P}_{-1}(6)$ are special except for $[4, 1^2]$ and $[2, 1^4]$. The Hasse diagram $\mathcal{P}_{-1}(6)_{special}$ of special orbits follows. The map d is obvious in this case. Next let $\mathfrak{g} = \mathfrak{so}_8$, then all partitions in $\mathcal{P}_1(8)$ are special except $[3, 2^2, 1]$. The Hasse diagram $\mathcal{P}_1(8)_{special}$ of special orbits follows, and the map d is again obvious, given that it preserves Roman numerals in this case (by (6.3.5)). Finally, let $\mathfrak{g} = \mathfrak{so}_{10}$. The only non-special partitions in $\mathcal{P}_1(10)$ are $[3, 2^2, 1^3]$ and $[5, 2^2, 1]$. The Hasse diagram $\mathcal{P}_1(10)_{special}$ of special orbits follows, and the map d is obvious once we specify that $d([7, 1^3]) = [4, 1^6]_D = [3, 1^7]$.

We refer you to [76] for the definition of the map d for exceptional algebras $\mathfrak{g}$. We give a list of the special orbits in the exceptional algebras in Chapter 8. The picture on the cover of this book is the Hasse diagram of special orbits in E_8. The blackened node in this picture represents the unique nilpotent orbit with fundamental group S_5, mentioned in the remarks after (6.1.7); the labels in the diagram will be described in Chapter 8.