Embeddings of Discrete Series into Principal Series

Toshihiko Matsuki and Toshio Oshima

§1. Introduction.

Let G be a connected real semisimple Lie group, σ an involution of G, and H an open subgroup of the group G^σ of fixed points for σ. For simplicity we assume that G has a complexification G_c. We fix a Cartan involution θ of G with $\sigma\theta = \theta\sigma$. The involutions of the Lie algebra $\mathfrak{g}$ of G induced by σ and θ are denoted by the same letters, respectively. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decompositions of $\mathfrak{g}$ into $+1$ and -1 eigenspaces for σ and θ, respectively. Let $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$ be the subalgebras of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ defined by

$$\mathfrak{g}^d = \mathfrak{k}\cap\mathfrak{h} + \sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}) + \sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}) + (\mathfrak{p}\cap\mathfrak{q}),$$
$$\mathfrak{k}^d = \mathfrak{k}\cap\mathfrak{h} + \sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}),\quad \mathfrak{h}^d = \mathfrak{k}\cap\mathfrak{h} + \sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}),$$

and let K, G^d, K^d and H^d be the analytic subgroups of G_c with the Lie algebras $\mathfrak{k}$, $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$, respectively. Then the homogeneous space $X^d = G^d/K^d$ is a Riemannian symmetric space of the non-compact type and called the non-compact Riemannian form of the semisimple symmetric space $X = G/H$. The ring $D(X)$ of the invariant differential operators on X is naturally isomorphic to the ring $D(X^d)$ of invariant differential operators on X^d.

Let P^d be a minimal parabolic subgroup of G^d. In **[O1]** a K-finite eigenfunction ψ of $D(X)$ is called a spherical function on X and defines an H^d-invariant closed subset $FBI(\psi)$ of G^d/P^d. Namely, by the Flensted-Jensen isomorphism, ψ corresponds to a simultaneous eigenfunction $\tilde\psi$ of $D(X^d)$ and then $FBI(\psi)$ is the support of the image of $\tilde\psi$ under the boundary value isomorphism defined by **[KKMOOT]**. The main result in **[O1]** shows that $FBI(\psi)$ and the eigenvalue determine the leading terms in a convergent series expansion of ψ at every boundary point of X in $\tilde X$. Here $\tilde X$ is the compact G-manifold constructed in **[O2]** which contains X as an open G-orbit.

Suppose the spherical function ψ generates an irreducible Harish–Chandra module $U(\psi)$. Then by the leading terms we have embeddings of $U(\psi)$ into principal series for X, which is studied in **[O1**, Theorem 5.1]. The key lemma in **[O1]** is **[O1**, Lemma 3.2] which studies a local property of intertwining operators between class 1 principal series for G^d.

In §2 we will give another lemma for the intertwining operators. These lemmas give embeddings of $U(\psi)$ into principal series for X which are not obtained in [O1]. Namely, by the lemmas and the same argument as in [O1], we get the embeddings which do not correspond to any leading term.

In §3 we consider the case where X is a semisimple Lie group. In this case ψ corresponds to a matrix coefficient of an irreducible Harish–Chandra module for the group, $FBI(\psi)$ coincides with the support of the $\mathcal{D}$-module realized in a complex flag manifold through Beilinson–Bernstein's correspondence ([BB1], [V]) and we will give a simple theorem to find embeddings of any irreducible Harish–Chandra module into principal series for the group. The embeddings corresponding to $S(E)_0$ (i.e. the leading terms) in Theorem 3.2 are also studied by [KW] and [BB2].

In §4 we consider the case where X is a classical simple Lie group and give an algorithm to express the H^d-orbit structure on G^d/P^d, which is sufficient to apply the theorem in §3. Thus we can obtain a simple combinatorial algorithm to obtain the embeddings.

The precise argument for the proof of the lemma and its application will be given elsewhere.

§2. Local properties of intertwining operators.

Retain the notation in §1. Let $G = KA_\mathfrak{p}N$ be an Iwasawa decomposition of G and $\mathfrak{a}_\mathfrak{p}$ the Lie algebra of $A_\mathfrak{p}$. Let Σ be the restricted root system for the pair $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$, Σ^+ the positive system corresponding to N and Ψ the fundamental system of Σ. The Weyl group W of Σ is identified with the normalizer $N_K(\mathfrak{a}_\mathfrak{p})$ of $\mathfrak{a}_\mathfrak{p}$ in K modulo the centralizer M of $\mathfrak{a}_\mathfrak{p}$ in K and the group $P = MA_\mathfrak{p}N$ is a minimal parabolic subgroup of G. For any $\alpha \in \Sigma$, we denote by $w_\alpha \in W$ the reflection with respect to α.

For an open subset U of G the space $\mathcal{B}(U)$ of hyperfunctions on U is naturally a left $\mathfrak{g}$-module. Then for an element λ of the complexification $(\mathfrak{a}_\mathfrak{p})^*_c$ of the dual $\mathfrak{a}^*_\mathfrak{p}$ of $\mathfrak{a}_\mathfrak{p}$, the space of hyperfunction sections of class 1 principal series is defined:

$$\begin{aligned}\mathcal{B}(G/P; L_\lambda) = \{ & f \in \mathcal{B}(G);\, f(gman) = f(g)a^{\lambda-\rho} \\ & \text{for } (g,m,a,n) \in G \times M \times A_\mathfrak{p} \times N\}.\end{aligned}$$

For any $\alpha \in \Psi$ there exists a function $T^\lambda_\alpha \in \mathcal{B}(G/P; L_{w\lambda})$ with the

meromorphic parameter $\lambda \in (\mathfrak{a}_\mathfrak{p})^*_c$ so that the linear map

$$\begin{aligned}& T^\lambda_\alpha : \mathcal{B}(G/P; L_\lambda) \to \mathcal{B}(G/P; L_{w\lambda}) \\ & f(x) \mapsto (T^\lambda_\alpha f)(x) = \int_K f(k) T^\lambda_\alpha(k^{-1}x)\,dk\end{aligned} \tag{2.1}$$

is a G-homomorphism which satisfies

$$(T^\lambda_\alpha f)(x) = \int_{\bar{N}_\alpha} f(x\bar{w}_\alpha \bar{n}_\alpha)\,d\bar{n}_\alpha$$

if f is continuous and $\mathrm{Re}\langle\lambda, \beta\rangle < 0$ for any $\beta \in \Psi$. Here $\bar{w}_\alpha$ is a representative of w_α, $\bar{N}_\alpha = \theta(N) \cap \bar{w}_\alpha^{-1} N \bar{w}_\alpha$ and the measures dk and $d\bar{n}_\alpha$ are Haar measures on K and $\bar{N}_\alpha$, respectively.

For a subset S of G/P we define a subset $w[S]$ of G/P by $\overline{SPw^{-1}P}$. For an open subset U of G/P, which is identified with a right P invariant subset of G, we put

$$\begin{aligned}\mathcal{B}(U; L_\lambda) = \{ & f \in \mathcal{B}(U);\;\; f(gman) = f(g)a^{\lambda-\rho} \\ & \text{for } (g,m,a,n) \in U \times M \times A_\mathfrak{p} \times N\}\end{aligned}$$

and define

$$\mathcal{B}(S; L_\lambda) = \varinjlim_{U:\text{open} \supset S} \mathcal{B}(U; L_\lambda).$$

Then the key lemma in [O1] is

Lemma 2.1 [O1, Lemma 3.2]. *Fix an element α of Ψ and a point p of G/P and put $V = w_\alpha[\{p\}]$. Denoting*

$$\mathcal{B}(V, \{p\}; L_\lambda) = \{f \in \mathcal{B}(V; L_\lambda);\;\; p \notin \operatorname{supp} f\},$$

the map (2.1) induces the $\mathfrak{g}$-homomorphism

$$T^\lambda_\alpha : \mathcal{B}(V, \{p\}; L_\lambda) \to \mathcal{B}(\{p\}; L_{w_\alpha\lambda}) \tag{2.2}$$

for any $\lambda \in (\mathfrak{a}_\mathfrak{p})^*_c$ by analytic continuation. Moreover if

$$e_\alpha(\lambda) \neq 0 \quad \textit{and} \quad -\frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle} \notin \{1, 2, 3, \dots\}, \tag{2.3}$$

then (2.2) is injective.

In the above lemma, $\langle\;,\;\rangle$ is the non-degenerate bilinear form on $(\mathfrak{a}_\mathfrak{p})^*_c$ induced from the Killing form of $\mathfrak{g}$,

$$e_\alpha(\lambda) = \Gamma\left(\frac{1}{2}\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}\right)^{-1} \Gamma\left(\frac{1}{2}\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} + \frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha}\right)^{-1}$$

and m_β denotes the multiplicity of the root space for a root $\beta \in \Sigma$.

Here we give another lemma.

LEMMA 2.2. *Use the notation as in Lemma 2.1.*

1) *Suppose λ satisfies*

$$\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \{0,1,2,\dots\}. \tag{2.4}$$

Then the function T_α^μ has a pole of order 1 at $\mu = \lambda$ and the residue defines the $\mathfrak{g}$-homomorphism

$$\operatorname{Res} \mathcal{T}_\alpha^\lambda : \mathcal{B}(V;L_\lambda) \to \mathcal{B}(V;L_{w_\alpha\lambda}) \tag{2.5}$$

and if the support of f in $\mathcal{B}(V;L_\lambda)$ is not equal to V, then

$$\operatorname{supp} f = \operatorname{supp}(\operatorname{Res} \mathcal{T}_\alpha^\lambda)f. \tag{2.6}$$

2) *If $m_\alpha = 1$ and*

$$2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \{0,1,2,\dots\}, \tag{2.7}$$

then there exists a $\mathfrak{g}$-homomorphism

$$\mathcal{S}_\alpha^\lambda : \mathcal{B}(V;L_\lambda) \to \mathcal{B}(V;L_{w_\alpha\lambda}) \tag{2.8}$$

such that if the support of f in $\mathcal{B}(V;L_\lambda)$ is not equal to V, then

$$\operatorname{supp} f = \operatorname{supp} \mathcal{S}_\alpha^\lambda f. \tag{2.9}$$

3) *If $e_\alpha(\lambda)e_\alpha(-\lambda) \neq 0$, then the analytic continuation of $\Gamma\left(\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right)^{-1} \mathcal{T}_\alpha^\lambda$ defines a bijective $\mathfrak{g}$-homomorphism*

$$\tilde{\mathcal{T}}_\alpha^\lambda : \mathcal{B}(V;L_\lambda) \to \mathcal{B}(V;L_{w_\alpha\lambda}). \tag{2.10}$$

§3. Group cases.

Let G be a connected real semisimple Lie group with a simply connected complexification G_c and $G = KA_\mathfrak{p}N$ an Iwasawa decomposition of G. Let K_c be an analytic subgroup of G_c with the Lie algebra $\mathfrak{k}_c$ which is the complexification of the Lie algebra $\mathfrak{k}$ of K, B a Borel subgroup of G_c which contains $A_\mathfrak{p}N$ and $\mathfrak{j}_c$ a Cartan subalgebra of $\mathfrak{g}_c$ which satisfies $A_\mathfrak{p} \subset \exp(\mathfrak{j}_c) \subset B$. Let $\Sigma(\mathfrak{j})$ be the root system for the pair $(\mathfrak{g}_c,\mathfrak{j}_c)$ by denoting $\mathfrak{j} = \mathfrak{g} \cap \mathfrak{j}_c$, $\Sigma(\mathfrak{j})^+$ the positive system corresponding

to B, $\Psi(\mathfrak{j}) = \{\alpha_1,\dots,\alpha_\ell\}$ the fundamental system and ρ half the sum of the positive roots. The Weyl group W of $\Sigma(\mathfrak{j})$ is generated by the reflections s_j with respect to simple roots α_j $(j = 1,\dots,\ell)$.

Let E be an irreducible Harish–Chandra module with an integral infinitesimal character $-\lambda$. Here we choose the element λ of the complex dual $\mathfrak{j}_c^*$ of $\mathfrak{j}_c$ with

$$\langle\lambda,\alpha\rangle \geq 0 \quad \text{for any} \quad \alpha \in \Sigma(\mathfrak{j})^+. \tag{3.1}$$

Let L_λ be the holomorphic line bundle over the flag manifold $Y = G_c/B$ induced from the holomorphic character τ_λ of B which satisfies $\tau_\lambda(\exp(Z)) = \exp\langle\rho-\lambda,Z\rangle$ for $Z \in \mathfrak{j}_c$. The twisted sheaf of differential operators $\mathcal{D}_\lambda$ on Y is defined by

$$\mathcal{D}_\lambda = L_\lambda \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} L_\lambda^{-1}. \tag{3.2}$$

Here $\mathcal{O}_Y$ (resp. $\mathcal{D}_Y$) are the sheaf of holomorphic functions (resp. that of differential operators) on Y in the Zariski topology. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_c$. For a matrix coefficient ψ of E we put $FBI(E) = FBI(\psi)$. Then $FBI(E)$ is well-defined and a closure of a single K_c-orbit on Y and satisfies

$$FBI(E) = \operatorname{supp}(\mathcal{D}_\lambda \underset{U(\mathfrak{g})}{\otimes} E). \tag{3.3}$$

If $\operatorname{rank}(G) = \operatorname{rank}(K)$ and E is the Harish–Chandra module belonging to the discrete series of G, then E is isomorphic to $H_V^n(Y,L_\lambda)$ with a compact K_c-orbit V on Y. Here n is the codimension of V in Y.

Let L be the centralizer of $A_\mathfrak{p}$ in G and L_0 its identity component. Then $P = LN$ is a minimal parabolic subgroup of G and $P_0 = L_0N$ is its identity component. Let π_λ be the irreducible representation of P_0 whose restriction on L_0 has the lowest weight $\rho-\lambda$ and U_λ the Harish–Chandra module of the representation of G induced from π_λ. Then U_λ is a finite direct sum of principal series of G in the category of Harish–Chandra modules.

By denoting $B_j = \overline{Bs_jB}$ for any $\alpha_j \in \Psi(\mathfrak{j})$, we have

DEFINITION 3.1. *For any closed subset V of Y, we put*

$S(V) = \{w \in W;$ *there exists a reduced expression*

$$w = s_{\nu(k)} \cdots s_{\nu(1)}$$

with the length k of w and a map

$$\varepsilon : \{1, \dots, k\} \to \{0, 1\}$$

such that

$V_k = Y$ *and* $V_{i-1} \neq V_{i-1}B_{\nu(i)}$ *for* $i = 1, \dots, k$

by inductively denoting

$$V_i = \begin{cases} V_{i-1}B_{\nu(i)} & \text{if } \varepsilon(i) = 1, \\ V_{i-1} & \text{if } \varepsilon(i) = 0, \end{cases} \quad \text{for } i = 1, \dots, k.\}$$

For the irreducible Harish–Chandra module E we put $S(E) = S(FBI(E))$. Since $FBI(E) = \bar{D}$ with a K_c-orbit D of Y, each V_i in the above definition is a closure of a single K_c-orbit D_i and $\dim D_i = \dim D_{i-1} + \varepsilon(i)$. Then for a non-negative integer j, we put

$$S(E)_j = \{w \in S(E);\ \text{the length of } w \text{ equals } j + \operatorname{codim} D\}.$$

By a similar argument as in [O1, §4, 5] with the lemmas in §2, we have

THEOREM 3.2. *Retain the above notation.*

1) *For any $w \in S(E)$, there exists an embedding of E into $U_{w\lambda}$.*
2) ([O1, Theorem 4.1]) *Let ψ be a matrix coefficient of E. Suppose λ is regular for simplicity. Then there exists a positive number ε, non-zero real analytic functions $a_w(g, g')$ of $G \times G$ for $w \in S(E)_0$ such that*
$$\begin{aligned} &\psi(g \cdot \exp Z \cdot g') \\ &= \sum_{w \in S(E)_0} a_w(g, g') e^{\langle w\lambda - \rho, Z\rangle} \\ &+ 0\Big(\sum_{w \in S(E)_0} \left| e^{\langle w\lambda - (1+\varepsilon)\rho, Z\rangle} \right|\Big) \end{aligned}$$
for $Z \in \mathfrak{a}_\mathfrak{p}$ and $(g, g') \in G \times G$ when $\alpha(Z) \to \infty$ for all $\alpha \in \Psi(\mathrm{j})$ with $\alpha|_{\mathfrak{a}_\mathfrak{p}} \neq 0$. Here the estimate is locally uniform on $G \times G$.
3) *If E is embedded in U_μ with an element $\mu \in \mathrm{j}_c^*$, then there exist $v \in W$ and $w \in S(E)_0$ satisfying $\mu = v\lambda$ and $v \geq w$ with respect to Bruhat ordering.*

For an element w of $S(E)_j$ we put $\partial w = \{v \in S(E)_{j-1}; v < w\}$. Then we have

CONJECTURE 3.3. *$(\partial, S(E))$ is isomorphic to a regular contractible CW-complex.*

On the other hand, we have

PROPOSITION 3.4. $\sum(-1)^j \# S(E)_j = 1$.

§4. Orbit structures on complex flag manifolds of classical type.

Let G_c be a connected complex reductive Lie group with a connected real form G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g} = \operatorname{Lie} G$ with respect to a Cartan involution θ and K_c the analytic subgroup of G for $\mathfrak{k}_c$.

Let B be a Borel subgroup of G_c, $\mathfrak{b}$ its Lie algebra and $Y = G_c/B$ the flag manifold for G_c. Since Y is identified with the set of all Borel subalgebras in $\mathfrak{g}_c$ on which G_c acts by the adjoint action, the K_c-orbit structure on Y depends only on $\mathfrak{g}^s = [\mathfrak{g}, \mathfrak{g}]$.

Let $\tilde{K}_c$ be a subgroup of G_c such that $K_c \subset \tilde{K}_c \subset N_{G_c}(K_c)$, where $N_{G_c}(K_c)$ is the normalizer of K_c in G_c. Then all the K_c-orbits contained in a $\tilde{K}_c$-orbit are diffeomorphic to each other. Let D_1 and D_2 be two K_c-orbits on Y with $\tilde{K}_c D_1 = \tilde{K}_c D_2$. Then we can easily obtain $S(\bar{D}_1) = S(\bar{D}_2)$. Hence in order to get $S(\bar{D})$ for a K_c-orbit D in Y, we have only to study the $\tilde{K}_c$-orbit structure on Y for some $\tilde{K}_c$.

For any $gB \in Y$, the Borel subalgebra $\operatorname{Ad}(g)\mathfrak{b}$ has a split component $\mathfrak{a}$ such that $\theta\mathfrak{a} = \mathfrak{a}$ ([M1], [R]). Note that $\mathfrak{a}_c$ is a θ-stable Cartan subalgebra of $\mathfrak{g}_c$ contained in $\operatorname{Ad}(g)\mathfrak{b}$. Let Σ be the root system for the pair $(\mathfrak{g}_c, \mathfrak{a}_c)$, Σ^+ the positive system for $\operatorname{Ad}(g)\mathfrak{b}$, Ψ the set of simple roots in Σ^+ and $\mathfrak{g}(\mathfrak{a}; \alpha)$ the root space for a root $\alpha \in \Sigma$.

In this section we parametrize the $\tilde{K}_c$-orbit structure on Y when $\mathfrak{g}^s$ is a simple Lie algebra of classical type.

Suppose that $G_c = GL(n, \mathbf{C})$, $SO(2n+1, \mathbf{C})$, $\mathrm{Sp}(n, \mathbf{C})$ or $SO(2n, \mathbf{C})$. (Later we will consider the case when $\mathfrak{g}^s$ is complex simple.) Take the orthogonal basis $\{e_1, \dots, e_n\}$ of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ such that

$$\Psi = \begin{cases} \{\alpha_1, \dots, \alpha_{n-1}\} & \text{if } G = GL(n, \mathbf{C}), \\ \{\alpha_1, \dots, \alpha_n\} & \text{otherwise,} \end{cases}$$

where $\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_n, 2e_n$ or $e_{n-1} + e_n$ if $G = SO(2n+1, \mathbf{C})$, $\mathrm{Sp}(n, \mathbf{C})$ or $SO(2n, \mathbf{C})$, respectively.

Since θ induces an involution of Σ, we have a permutation φ of $\{1, 2, \dots, n\}$ such that $\varphi^2 = \mathrm{id}$ and that

$$\theta e_i = \pm e_{\varphi(i)}$$

for every $i = 1, \dots, n$. We can assign to the pair $(\mathfrak{a}, \Psi)$ an ordered set $\{\varepsilon_1, \dots, \varepsilon_n\}$, which we call "a clan", of n "persons" with the following structure:

Each person ε_i is an element of the set $\{+, -, \circ\}$ of three elements, the signs $+$ and $-$ and the circle $\circ$, which we call "a boy", "a girl" and "an adult", respectively. Some of the adults in a clan form pairs and no adult belongs to two different pairs. Each pair is "a young couple" or "an old couple".

A young couple and an old couple are expressed by joining the corresponding two circles with a line and an arrow, respectively. Here we ignore the direction of the arrow.

The clan has the following property:

($\pm$) If $\theta e_i = e_i$, then $\varepsilon_i = +$ or $-$. Moreover ε_i and ε_j are the same sign if and only if $\mathfrak{g}(\mathfrak{a}; e_i - e_j) \subset \mathfrak{k}_c$, that is, the root $e_i - e_j$ is a compact root.

(a) If $\theta e_i = e_j$ with $i \neq j$, then ε_i and ε_j are adults and form a young couple.

(A) If $\theta e_i = -e_j$ with $i \neq j$, then ε_i and ε_j are adults and form an old couple.

($\circ$) If $\theta e_i = -e_i$, then ε_i is an adult which does not belong to any pair, which we call "the aged".

THEOREM 4.1. *The $\tilde{K}_c$-orbits on Y and the clans with the conditions in Table 1 are in one-to-one correspondance.*

REMARK 4.2 (i) In Table 1, for example, the condition $(A, \circ)_n$ means that the clan consists of n persons and there exists no boy, no girl or no young couple.

(ii) N_+, N_- and N_A are the members of boys, girls and old couples, respectively.

(iii) For BI, $\mathfrak{g}(\mathfrak{a}; \alpha_n) \subset \mathfrak{k}_c \Leftrightarrow (N_+ - N_- = p - q \quad \text{and} \quad \varepsilon_n = +)$ or $(N_+ - N_- = p - q + 1 \quad \text{and} \quad \varepsilon_n = -)$.
For CI, $\mathfrak{g}(\mathfrak{a}; 2e_i) \not\subset \mathfrak{g}_c$ if $\theta e_i = e_i$.
For CII, $\mathfrak{g}(\mathfrak{a}; 2e_i) \subset \mathfrak{k}_c$ if $\theta e_i = e_i$.
For DI, $\mathfrak{g}(\mathfrak{a}; \alpha_{n-1}) \subset \mathfrak{k}_c \Leftrightarrow \mathfrak{g}(\mathfrak{a}; \alpha_n) \subset \mathfrak{k}_c$.
For $DIII$, $\mathfrak{g}(\mathfrak{a}; \alpha_{n-1}) \subset \mathfrak{k}_c \Leftrightarrow \mathfrak{g}(\mathfrak{a}; \alpha_n) \not\subset \mathfrak{k}_c$.

(iv) For the compact orbits and open orbits, see Table 1'.

Table 1 ($p + q = n$)

Type	$\mathfrak{g}^s$	G_c	$\tilde{K}_c$	Condition for the clans
AI	$\mathfrak{sl}(n, \mathbb{R})$	$GL(n, \mathbb{C})$	$O(n, \mathbb{C})$	$(A, \circ)_n$
AII	$\mathfrak{su}^*(n)$	$GL(n, \mathbb{C})$	$Sp(n/2, \mathbb{C})$	$(A)_n$ ($n =$ even)
AIII	$\mathfrak{su}(p, q)$	$GL(n, \mathbb{C})$	$GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$	$(\pm, a)_n$ $N_+ - N_- = p - q$
BI	$\mathfrak{so}(2p + 1, 2q)$	$SO(2n + 1, \mathbb{C})$	$S(O(2p + 1, \mathbb{C}) \times O(2p, \mathbb{C}))$	$(\pm, a, A, \circ)_n$ $N_+ - N_- = p - q$ or $p - q + 1$
CI	$\mathfrak{sp}(n, \mathbb{R})$	$Sp(n, \mathbb{C})$	$GL(n, \mathbb{C})$	$(\pm, a, A, \circ)_n$
CII	$\mathfrak{sp}(p, q)$	$Sp(n, \mathbb{C})$	$Sp(p, \mathbb{C}) \times Sp(q, \mathbb{C})$	$(\pm, a, A)_n$ $N_+ - N_- = p - q$
DI	$\mathfrak{so}(2p, 2q)$	$SO(2n, \mathbb{C})$	$S(O(2p, \mathbb{C}) \times O(2q, \mathbb{C}))$	$(\pm, a, a, \circ)_n$ $N_+ - N_- = p - q$
DI'	$\mathfrak{so}(2p + 1, 2q - 1)$	$SO(2n, \mathbb{C})$	$S(O(2p + 1, \mathbb{C}) \times O(2q - 1, \mathbb{C}))$	$(\pm, a, A, \circ)_n$ $N_+ - N_- = p - q + 1$
DIII	$\mathfrak{so}^*(2n)$	$SO(2n, \mathbb{C})$	$\mathbb{C}^\times \times PSL(n, \mathbb{C})$	$(\pm, a, A)_n$ $N_+ - N_- + 2N_A \equiv n \pmod 4$

Table 1'

Type	$\mathfrak{g}^s$	$\tilde{N}_c$	N_c	compact orbits	open orbits	codimension of compact orbits
AI	$\mathfrak{sl}(n,\mathbf{R})$	1	$\begin{cases} n: \text{even } 2 \\ n: \text{odd } 1 \end{cases}$	$\begin{cases} AB\cdots BA \\ AB\cdot\cdot\circ\cdot\cdot BA \end{cases}$	$(\circ)_n$	$\begin{cases} n^2/4 \\ (n^2-1)/4 \end{cases}$
AII	$\mathfrak{su}^*(n)$	1	1	$AB\cdots BA$	$AABB\cdots$	$n(n-2)/4$
AIII	$\mathfrak{su}(p,q)$	$\binom{n}{p}$	$\binom{n}{p}$	$(\pm)_n$	$ab\cdot\cdot(+)_{p-q}\cdot\cdot ba$	pq
BI	$\mathfrak{so}(2p+1,2q)$	$\binom{n}{p}$	$2\binom{n}{p}$	$(\pm)_n$	$\begin{cases} p\geq q\ \circ\cdot\cdot\circ(+)_{p-q} \\ p<q\ \circ\cdot\cdot\circ(-)_{q-p-1} \end{cases}$	$(2p+1)q$
CI	$\mathfrak{sp}(n,\mathbf{R})$	2^n	2^n	$(\pm)_n$	$(\circ)_n$	$n(n+1)/2$
CII	$\mathfrak{sp}(p,q)$	$\binom{n}{p}$	$\binom{n}{p}$	$(\pm)_n$	$AABB\cdot\cdot(+)_{p-q}$	$2pq$
DI	$\mathfrak{so}(2p,2q)$	$\binom{n}{p}$	$2\binom{n}{p}$	$(\pm)_n$	$\circ\cdot\cdot\circ(+)_{p-q}$	$2pq$
DI'	$\mathfrak{so}(2p+1,2q-1)$	$\binom{n-1}{p}$	$\binom{n-1}{p}$	$(\pm)_{n-1}\circ$	$\circ\cdot\cdot\circ(+)_{p-q+1}$	$2pq-p+q-1$
DIII	$\mathfrak{so}^*(2n)$	2^{n-1}	2^{n-1}	$(\pm)_n$	$AABB\cdots$	$n(n-1)/2$

Here $\tilde{N}_c$ is the number of the compact $\tilde{K}_c$-orbits, N_c is the number of compact K_c-orbits and $(+)_k$ denotes the row of $+$'s of length k if $k \geq 0$ and that of $-$'s of length $-k$ otherwise.

Each $\tilde{K}_c$-orbit in Table 1 can be expressed by a symbol $\varepsilon_1\varepsilon_2\ldots\varepsilon_n$ with lines and arrows, where $\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\}$ is the clan corresponding to the orbit. The following are examples of $\tilde{K}_c$-orbits of Type CI with $n = 5$ in Table 1.

$$+-+\circ- \qquad +\circ+\circ- \qquad +\circ\circ\circ\circ$$

To express the orbit more easily we give "a family name" for each pair and then we can write the above example as follows, respectively:

$$+-+\circ- \qquad +a+a- \qquad +abaB.$$

Here each couple in a clan has a family name consisting of letter to distinguish couples in a clan. A young couple is expressed by the same small letters and an old couple is expressed by the small letter and the capital letter corresponding to the family name. In some cases, we express an old couple by the same capital letters. We remark that the following expressions also correspond the last orbit in the above example:

$$+abaB \qquad +babA \qquad +aBaB \qquad +bAbA$$

Let B_i be the parabolic subgroup of G_c for $\{-\alpha_i\}\cup\Sigma^+$. Let $Y_i = G/B_i$ and $\pi_i : Y \to Y_i$ be the canonical projection. Let D_1 and D_2 be two $\tilde{K}_c$-orbits on Y. Then we write

$$D_1 \xrightarrow{i} D_2$$

if and only if $\pi_i(D_1) = \pi_i(D_2)$ and $\dim D_1 < \dim D_2$, which implies $\dim D_2 = \dim D_1 + 1$.

Proposition 4.3. ([V, §5], [M2]). *Choose a pair $(\mathfrak{a},\Psi)$ corresponding to an element of D_1. Then $D_1 \xrightarrow{i} D_2$ for some $\tilde{K}_c$-orbit D_2 if and only if one of the following conditions holds.*

(I) *$\theta\alpha_i = \alpha_i$ and $\mathfrak{g}(\mathfrak{a};\alpha_i) \not\subset \mathfrak{k}_c$, that is, α_i is a non-compact simple root.*

(II)
$$\theta\alpha_i \in \Sigma^+ - \{\alpha_i\}.$$

We will give the necessary and sufficient condition for $D_1 \xrightarrow{i} D_2$ in Table 2 and examples of the $\tilde{K}_c$-orbit structure on Y of Typ^ AI,..., DIII in Fig. 1~ Fig. 20.

Table 2

We express the orbits by rows consisting of $+, -$ and letters. Let $\delta_1 \dots \delta_n$ and $\delta'_1 \dots \delta'_n$ be the expressions corresponding to D_1 and D_2, respectively.

(i) Here $i = 1, \dots, n-1$ and the old couple (resp. young couple) is expressed by the capital letters (resp. small letters) corresponding to the family name. Then $D_1 \xrightarrow{i} D_2$ if and only if $\delta_j = \delta'_j$ for $j = 1, \dots, i-1, i+2, \dots, n$ and $\delta_i, \delta_{i+1}, \delta'_i$ and δ'_{i+1} equal to one of the following lists, where the letters p, P, q, Q correspond to suitable family names.

$\delta_i\delta_{i+1}$	$\delta'_i\delta'_{i+1}$	Condition	$\theta\alpha_i$
$+-$	pp		α_i
$-+$	pp		α_i
pp	$\circ\circ$		α_i
$p\pm$	$\pm p$	$\varphi(i) < i+1$	$e_{\varphi(i)} - e_{i+1}$
$\pm p$	$p\pm$	$i < \varphi(i+1)$	$e_i - e_{\varphi(i+1)}$
pq	qp	$\varphi(i) < \varphi(i+1)$	$e_{\varphi(i)} - e_{\varphi(i+1)}$
$\pm P$	$P\pm$		$e_i + e_{\varphi(i+1)}$
PQ	QP	$\varphi(i+1) < \varphi(i)$	$e_{\varphi(i+1)} - e_{\varphi(i)}$
pQ	Qp		$e_{\varphi(i)} + e_{\varphi(i+1)}$
$\pm\circ$	$\circ\pm$		$e_i + e_{i+1}$
$p\circ$	$\circ p$		$e_{\varphi(i)} + e_{i+1}$
$P\circ$	$\circ P$	$i+1 < \varphi(i)$	$e_{i+1} - e_{\varphi(i)}$
$\circ P$	$P\circ$	$\varphi(i+1) < i$	$e_{\varphi(i+1)} - e_i$

(ii) Here $i = n$ and the old couple (resp. young couple) is expressed by the small letter and capital letter (resp. small letter) corresponding to the family name. If δ_j and $\delta_{j'}$ are an old couple with $j < j'$, then we express δ_j by a small letter and $\delta_{j'}$ by the corresponding capital letter.

(1) In the cases BI, CI and CII, $D_1 \xrightarrow{n} D_2$ if and only if $\delta_j = \delta'_j$ for $j = 1, \dots, n-1$ and δ_n and δ'_n are one of the following:

δ_n	δ'_n	$\theta\alpha_n$
$\pm$	$\circ$	α_n
p	P	$e_{\varphi(n)}$

(2) In the case DI, $D_1 \xrightarrow{n} D_2$ if and only if $\delta_j = \delta'_j$ for $j = 1, \dots, n-2$

and δ_{n-1}, δ_n, δ'_{n-1} and δ'_n are one of the following:

$\delta_{n-1}\delta_n$	$\delta'_{n-1}\delta'_n$	Condition	$\theta\alpha_n$
$+-$	PP		α_n
$-+$	PP		α_n
pp	$\circ\circ$		α_n
$\pm p$	$P\pm$		$e_{\varphi(n)} + e_{n-1}$
$p\pm$	$\pm P$		$e_{\varphi(n-1)} + e_n$
pq	QP		$e_{\varphi(n-1)} + e_{\varphi(n)}$
pQ	qP	$\varphi(n-1) < \varphi(n)$	$e_{\varphi(n-1)} - e_{\varphi(n)}$
Qp	Pq	$\varphi(n) < \varphi(n-1)$	$e_{\varphi(n)} - e_{\varphi(n-1)}$
$\pm\circ$	$\circ\pm$		$e_{n-1} - e_n$
$p\circ$	$\circ P$		$e_{\varphi(n-1)} - e_n$
$\circ p$	$P\circ$		$e_{\varphi(n)} - e_{n-1}$

(3) In the case DIII, $D_1 \xrightarrow{n} D_2$ if and only if $\delta_j = \delta'_j$ for $j = 1, \dots, n-2$ and δ_{n-1}, δ_n, δ'_{n-1} and δ'_n are one of the following:

$\delta_{n-1}\delta_n$	$\delta'_{n-1}\delta'_n$	Condition	$\theta\alpha_n$
$++$	PP		α_n
$--$	PP		α_n
$\pm p$	$P\mp$		$e_{\varphi(n)} + e_{n-1}$
$p\pm$	$\mp P$		$e_{\varphi(n-1)} + e_n$
pq	QP		$e_{\varphi(n-1)} + e_{\varphi(n)}$
pQ	qP	$\varphi(n-1) < \varphi(n)$	$e_{\varphi(n-1)} - e_{\varphi(n)}$
Qp	Pq	$\varphi(n) < \varphi(n-1)$	$e_{\varphi(n)} - e_{\varphi(n-1)}$

Next consider the case where $\mathfrak{g}^s$ is a classical complex simple Lie algebra. Then we may suppose $G_c = G'_c \times G'_c$ with $G'_c = GL(n, \mathbb{C})$, $SO(2n+1, \mathbb{C})$, $\mathrm{Sp}(n, \mathbb{C})$ or $SO(2n, \mathbb{C})$, $\theta(g, g') = (g', g)$ for $(g, g') \in G_c$, $K_c = \tilde{K}_c = \{(g, g); g \in G'_c\}$ and $B = B' \times B'$ with a Borel subgroup B' of G'_c. Then

$$K_c \setminus G_c/B \xrightarrow{\sim} B' \setminus G'_c/B'$$

by the map $(g, g') \mapsto g^{-1}g'$ of G_c onto G'_c. Thus the K_c-orbit structure on $Y = G_c/B$ is reduced to the structure of the Bruhat decomposition of G'_c. Then by taking the orthogonal bases $\{e_1, \dots, e_n, e'_1, \dots, e'_n\}$ of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ in a natural way as before, we can express the K_c-orbit structure as in Fig. 21 ~ Fig. 23.

Now we give some examples of $S(E)$ in Theorem 3.2, which are easily obtained from the diagram of the $\tilde{K}_c$-orbit structure. Suppose $G = SU(2,1)$. Then the corresponding diagram is Fig. 5. Since the closed subset V_i is a closure of one K_c-orbit on Y, we express it by the corresponding clan. Suppose E is the Harish-Chandra module belonging to the discrete series of $SU(2,1)$. Then $FBI(E) = -++$ or $+-+$ or $++-$ and we have $S(-++) = \{s_2s_1 = \binom{123}{231}\}$, $s(++-) = \{s_1s_2 = \binom{123}{312}\}$ and $S(+-+) = \{s_2s_1, s_1s_2, s_1s_2s_1 = \binom{123}{321}\}$. Here we identify the Weyl group W with the permutation group of 3 numbers 1, 2 and 3 and the elements of $S(+-+)$ are obtained as in the following table:

	V_0	$\nu(1)$	$\varepsilon(1)$	V_1	$\nu(2)$	$\varepsilon(2)$	V_2	$\nu(3)$	$\varepsilon(3)$	V_3
s_2s_1	$+-+$	1	1	$aa+$	2	1	$a+a$			
s_1s_2	$+-+$	2	1	$+aa$	1	1	$a+a$			
$s_1s_2s_1$	$+-+$	1	0	$+-+$	2	1	$+aa$	1	1	$a+a$

Suppose $G = SU(p,q)$ with $p \geq q$. For a Harish-Chandra module E belonging to a discrete series of G, we can obtain $S(E)_0$ in the following way:

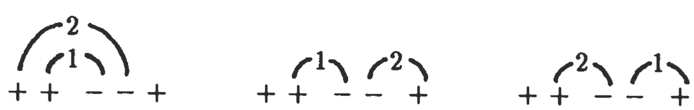
Consider the ordered q pairs in the clan corresponding to the compact K_c-orbit $FBI(E)$ which satisfies the following conditions.

Let $(\varepsilon_{I(i)}, \varepsilon_{J(i)})$ denote the i-th pair with $I(i) < J(i)$.

(1) Each pair consists of a boy and a girl.
(2) There exist $p-q$ boys who do not belong to any pair.
(3) If there exist i and j with $I(i) < j < J(i)$, then there exists i' with $i' < i$ such that $I(i) < I(i') < J(i') < J(i)$ and $j \in \{I(i'), J(i')\}$.

For the ordered q pairs, we attach an element $\begin{pmatrix} 1 & \cdots & n \\ j_1 & \cdots & j_n \end{pmatrix}$ of the permutation group of n numbers $1, \ldots, n$ satisfying $j_k = J(i_k)$ and $j_{n+1-k} = I(i_k)$ for $k = 1, \ldots, q$ and $j_{q+1} < j_{q+2} < \cdots < j_{n-q}$. Identifying the permutation group with the Weyl group of $SL(n, \mathbb{C})$ and considering all the definitions of the above q pairs, we obtain all elements of $S(E)_0$.

For example, if $FBI(E)$ corresponds to $++--+$, then there exist 3 types of the ordered q pairs



from which we have the following elements, respectively:

$$\begin{pmatrix} 12345 \\ 34512 \end{pmatrix} \quad \begin{pmatrix} 12345 \\ 35142 \end{pmatrix} \quad \begin{pmatrix} 12345 \\ 53124 \end{pmatrix}.$$

In the case where $\mathfrak{g}_c$ is classical and simple, a computer programme to calculate $S(\bar{D})$ for any K_c-orbit D on Y was written by the second

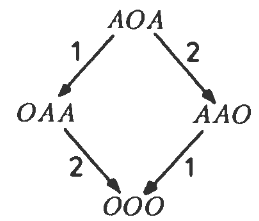
named author. By the program, for example, we have

$$\#S(+--+-) = 7, \#S(+-+-+) = 35, \#S(+-+-+-) = 135, \ldots$$

in the case of type $SU(p,q)$. It takes about 2 minutes by a microcomputer (CPU:80286, Clock:10MHz) to get all $S(+-+-+-)$. The programme with source code in C is available from the second named author.

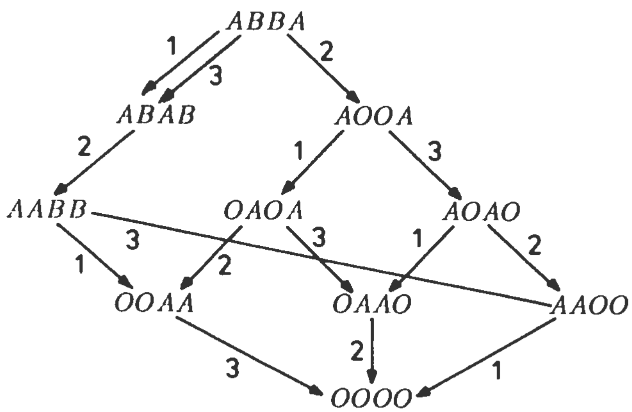
REMARK 4.4. Suppose G is a classical simple Lie group whose real rank equals 1. Suppose E is a Harish-Chandra module belonging to the discrete series of G. Then comparing a result in [C], we can obtain all the embeddings of E into principal series by Theorem 3.2 except the following cases:

$FBI(E) = ++\cdots+-$ and $\mathfrak{g} = \mathfrak{sp}(p,1)$ with $p \geq 2$. In these cases there are two embeddings of E into principal series of G but Theorem 3.2 gives only one embedding corresponding to the leading term of the matrix coefficient of E.



AI $\mathfrak{g}^s = \mathfrak{sl}(3,\mathbf{R})$ $O(3)\backslash GL(3,\mathbf{C})/B$ 1 2

Fig. 1

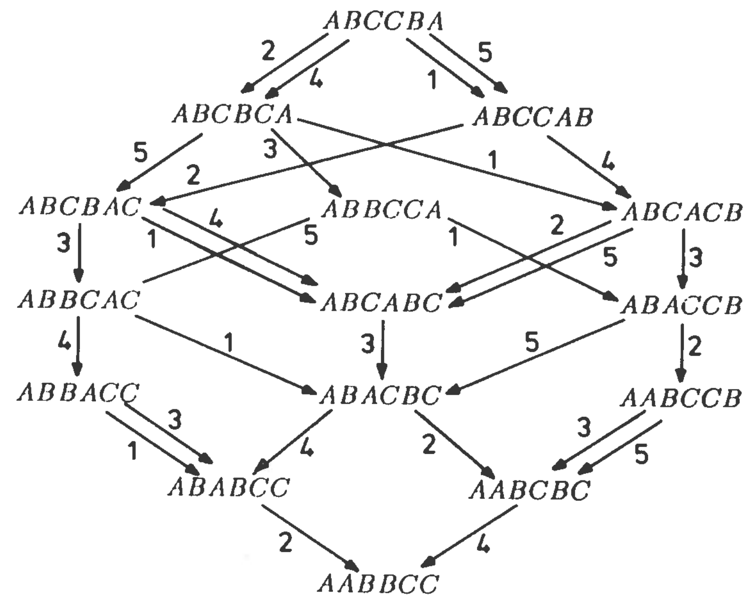


AI $\mathfrak{g}^s = \mathfrak{sl}(4,\mathbf{R})$ $O(4)\backslash GL(4,\mathbf{C})/B$ 1 2 3

Fig. 2

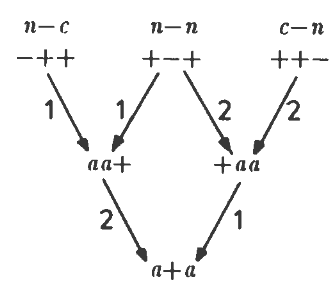
AII $\mathfrak{g}^s = \mathfrak{su}^*(4)$ $Sp(2,\mathbf{C})\backslash GL(4,\mathbf{C})/B$ 1 2 3

Fig. 3



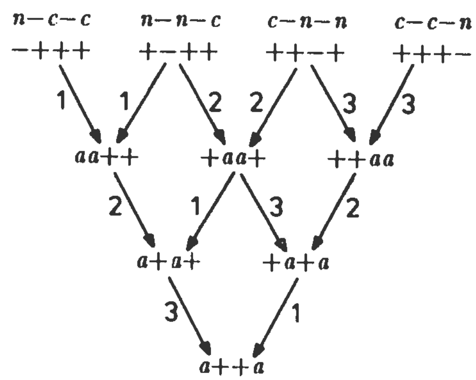
AII $\mathfrak{g}^s = \mathfrak{su}^*(6)$ $Sp(3,\mathbf{C})\backslash GL(6,\mathbf{C})/B$ 1 2 3 4 5

Fig. 4



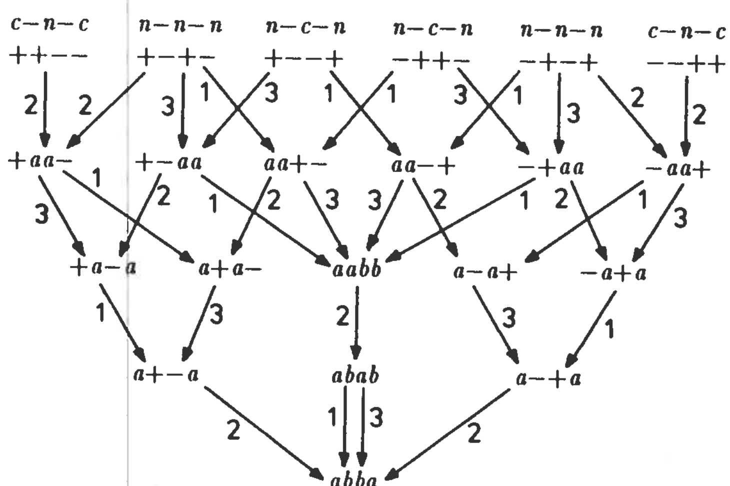
AIII $\mathfrak{g}^s = \mathfrak{su}(2,1)$ $GL(2,\mathbf{C}) \times GL(1,\mathbf{C})\backslash GL(3,\mathbf{C})/B$ 1 2

Fig. 5



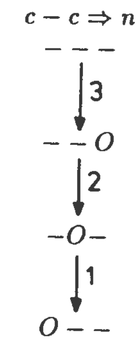
AIII $\mathfrak{g}^s = \mathfrak{su}(3,1)$ $GL(3,\mathbb{C}) \times GL(1,\mathbb{C}) \backslash GL(4,\mathbb{C})/B$ 1 2 3

Fig. 6



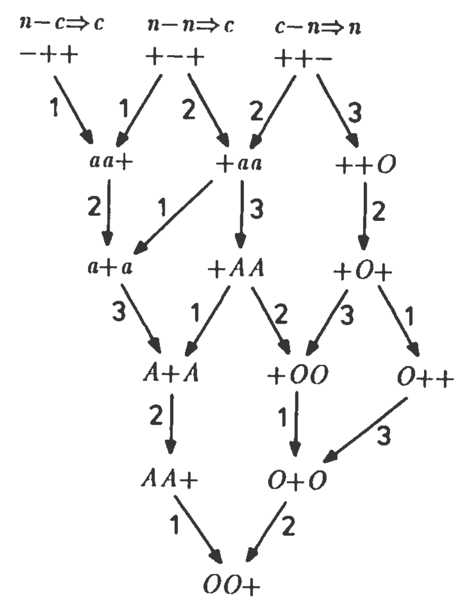
AIII $\mathfrak{g}^s = \mathfrak{su}(2,2)$ $GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \backslash GL(4,\mathbb{C})/B$ 1 2 3

Fig. 7



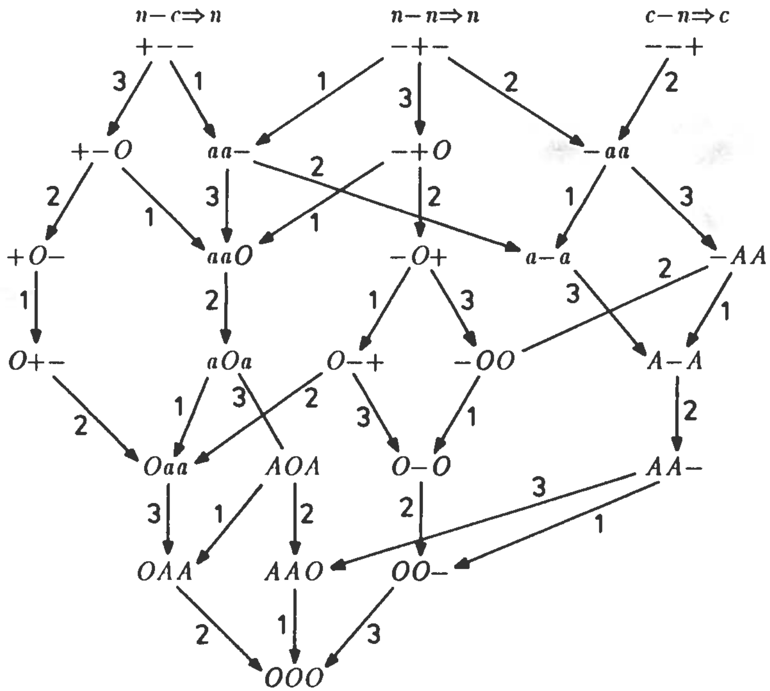
BI $\mathfrak{g}^s = \mathfrak{so}(1,6)$ $S(O(1,\mathbb{C}) \times O(6,\mathbb{C})) \backslash SO(7,\mathbb{C})/B$ 1 2 3

Fig. 8



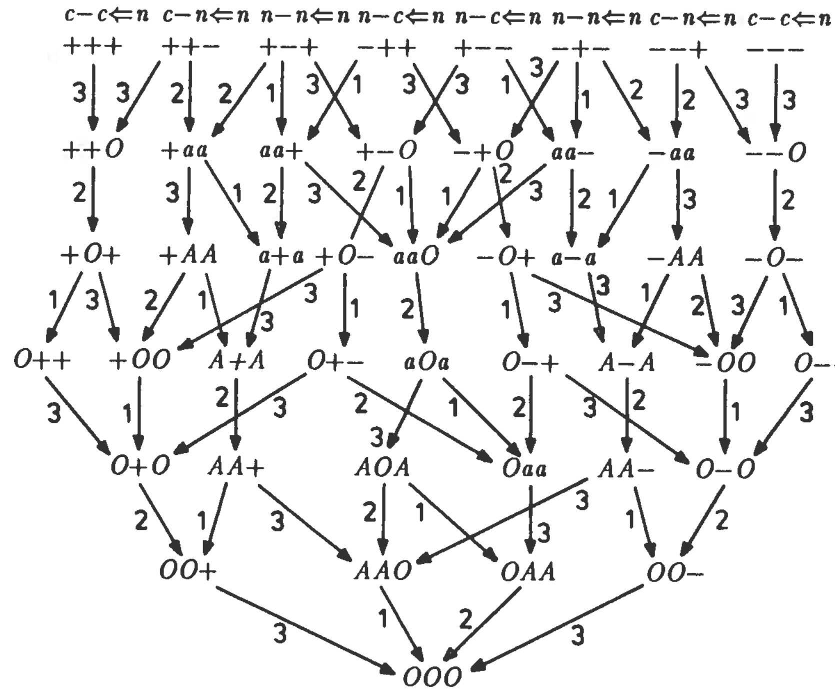
BI $\mathfrak{g}^s = \mathfrak{so}(5,2)$ $S(O(5,\mathbb{C}) \times O(2,\mathbb{C})) \backslash SO(7,\mathbb{C})/B$ 1 2 3

Fig. 9



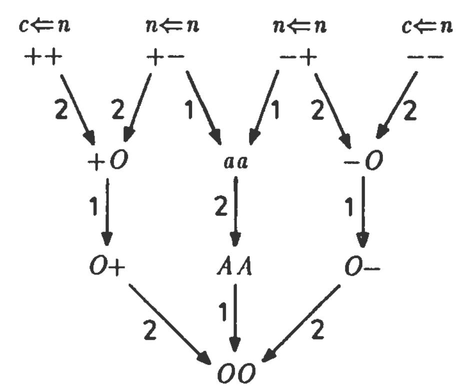
BI $\mathfrak{g}^s = \mathfrak{so}(3,4)$ $S(O(3,\mathbf{C}) \times O(4,\mathbf{C}))\backslash SO(7,\mathbf{C})/B$ 1 2 3

Fig. 10



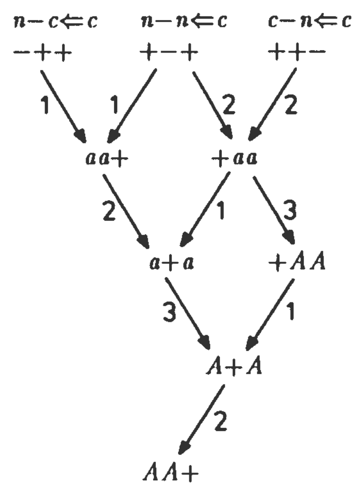
CI $\mathfrak{g}^s = \mathfrak{sp}(3,\mathbf{R})$ $GL(3,\mathbf{C})\backslash Sp(3,\mathbf{C})/B$ 1 2 3

Fig. 11



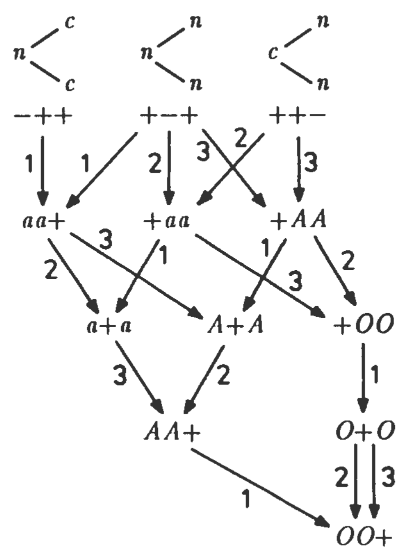
CI $\mathfrak{g}^s = \mathfrak{sp}(2,\mathbf{R})$ $GL(2,\mathbf{C})\backslash Sp(2,\mathbf{C})/B$ 1 2

Fig. 12



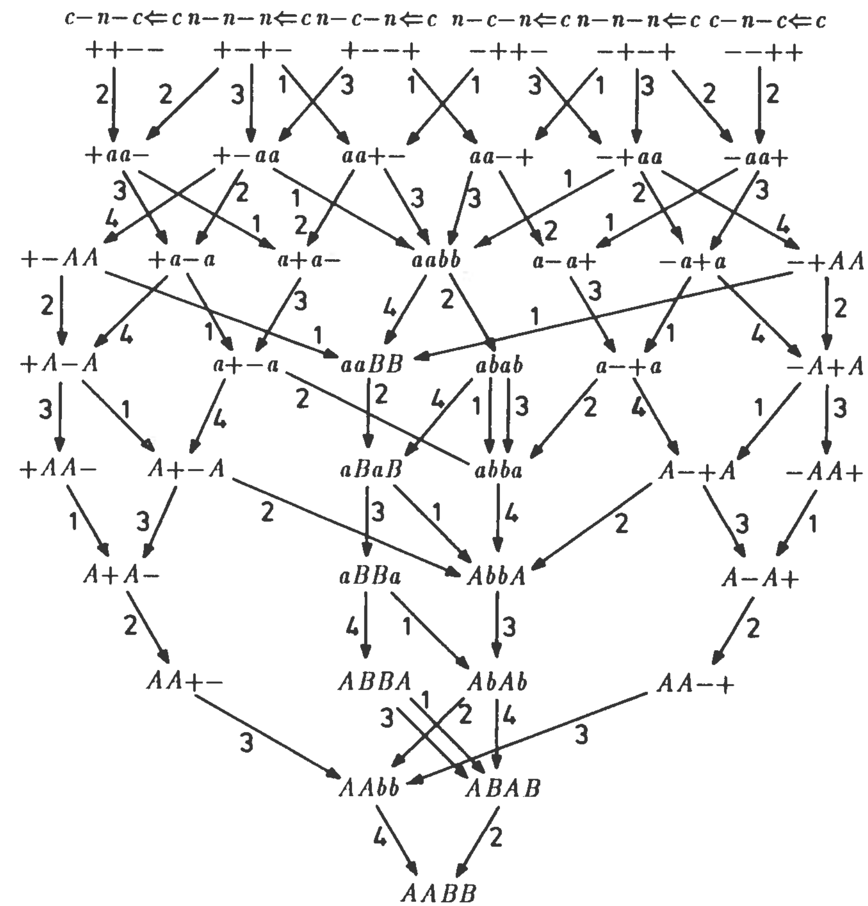
CII $\mathfrak{g}^s = \mathfrak{sp}(2,1)$ $Sp(2,\mathbb{C}) \times Sp(1,\mathbb{C}) \backslash Sp(3,\mathbb{C})/B$ 1 2 3

Fig. 13



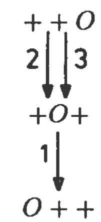
DI $\mathfrak{g}^s = \mathfrak{so}(4,2)$ $S(O(4,\mathbb{C}) \times O(2,\mathbb{C})) \backslash SO(6,\mathbb{C})/B$ 1 2 3

Fig. 14



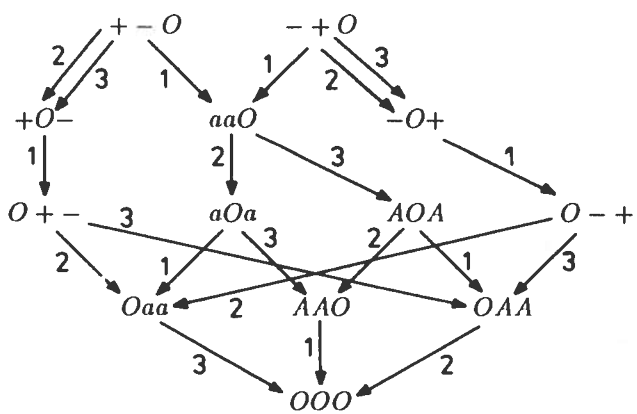
CII $\mathfrak{g}^s = \mathfrak{sp}(2,2)$ $Sp(2,\mathbb{C}) \times Sp(2,\mathbb{C}) \backslash Sp(4,\mathbb{C})/B$ 1 2 3 4

Fig. 15



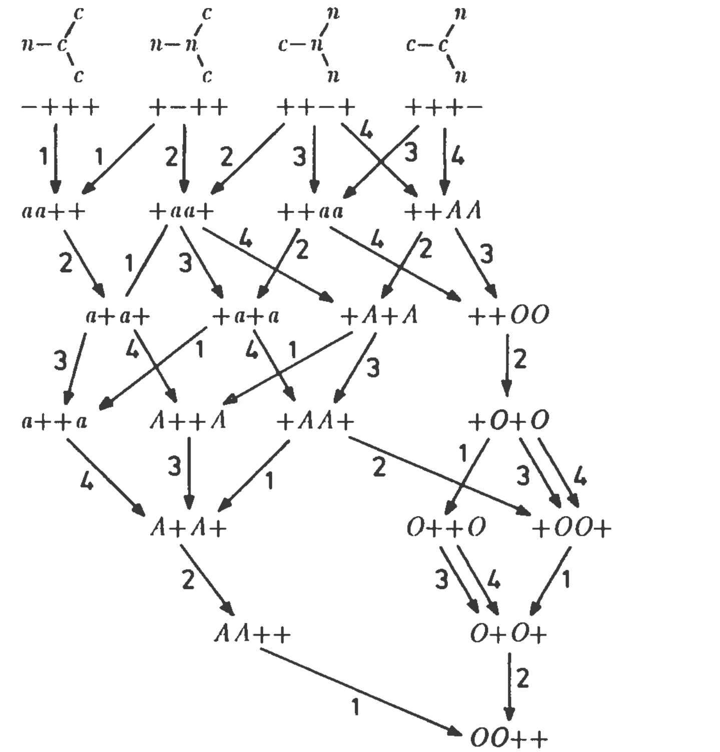
DI′ $\mathfrak{g}' = \mathfrak{so}(5,1)$ $S(O(5,\mathbf{C}) \times O(1,\mathbf{C}))\backslash SO(6,\mathbf{C})/B$ 1 2 3

Fig. 16



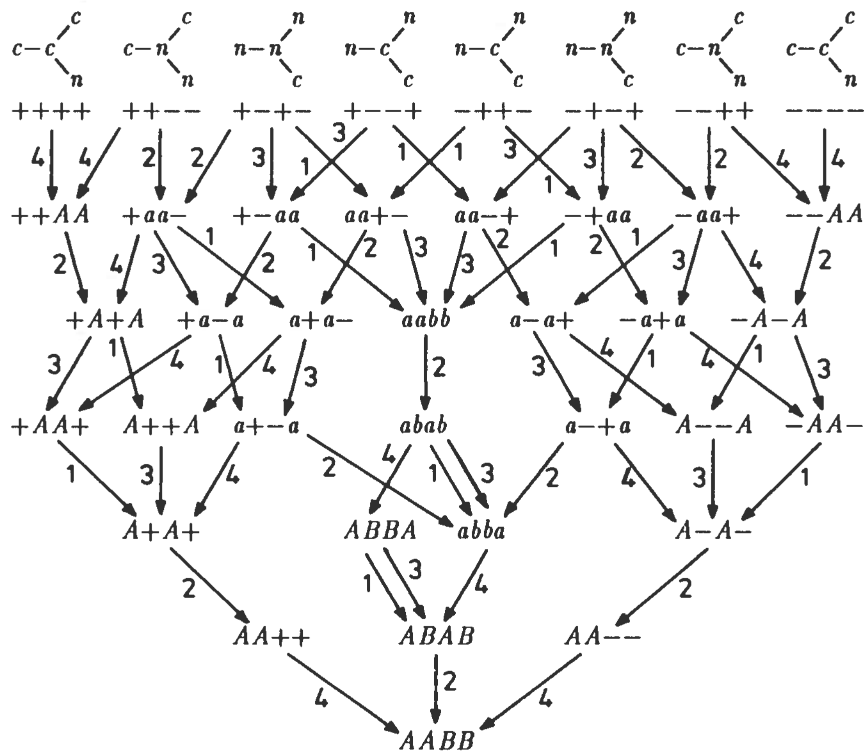
DI′ $\mathfrak{g}' = \mathfrak{so}(3,3)$ $S(O(3,\mathbf{C}) \times O(3,\mathbf{C}))\backslash SO(6,\mathbf{C})/B$ 1 2 3

Fig. 17



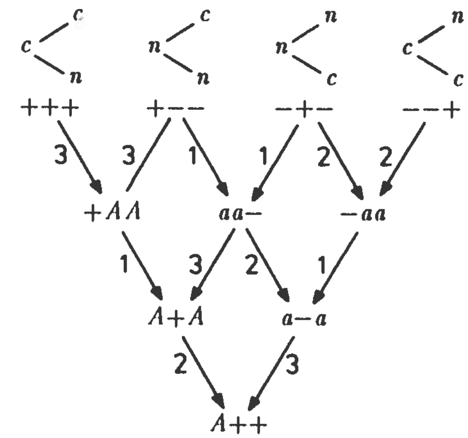
DI $\mathfrak{g}' = \mathfrak{so}(6,2)$ $S(O(6,\mathbf{C}) \times O(2,\mathbf{C}))\backslash SO(8,\mathbf{C})/B$ 1 2 3 4

Fig. 18



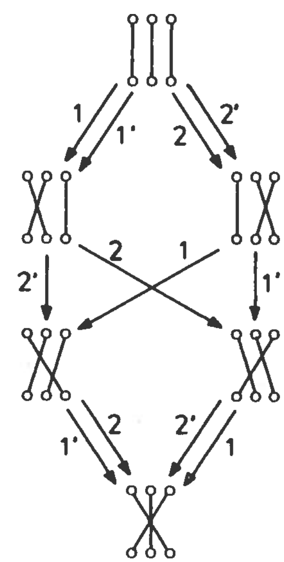
DIII $\mathfrak{g}^s = \mathfrak{so}^*(8)$ $\mathbf{C}^\times \times PSL(4,\mathbf{C})\backslash SO(8,\mathbf{C})/B$

Fig. 19



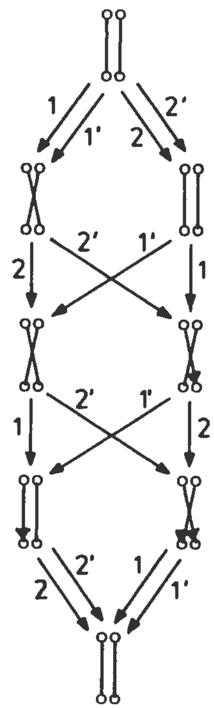
DIII $\mathfrak{g}^s = \mathfrak{so}^*(6)$ $\mathbf{C}^\times \times PSL(3,\mathbf{C})\backslash SO(6,\mathbf{C})/B$

Fig. 20



A_2 $\mathfrak{g}^s = \mathfrak{sl}(3,\mathbf{C})$ $\Delta GL(3,\mathbf{C})\backslash GL(3,\mathbf{C}) \times GL(3,\mathbf{C})/B$

Fig. 21



B_2 $\mathfrak{g}^s = \mathfrak{so}(5,\mathbb{C})$ $\Delta SO(5,\mathbb{C})\backslash SO(5,\mathbb{C}) \times SO(5,\mathbb{C})/B$

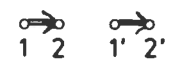
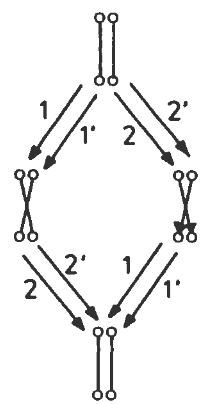


Fig. 22



D_2 $\mathfrak{g}^s = \mathfrak{so}(4,\mathbb{C})$ $\Delta SO(4,\mathbb{C})\backslash SO(4,\mathbb{C}) \times SO(4,\mathbb{C})/B$ ∘1 ∘2 ∘1' ∘2'

Fig. 23

References

[BB1] A.A. Beilinson and J. Bernstein, *Localization de $\mathfrak{g}$-modules*, C.R. Acad. Sci. Paris **292** (1981), 15-18.

[BB2] ——————, *A generalization of Casselman's submodule theorem*, in "Representation Theory of Reductive Groups;" edited by P.C. Trombi, Progress in Mathematics 40, Birkhäuser, Boston, 1983, pp. 35-52.

[C] D.H. Collingwood, *Representations of Rank One Lie Groups*. Pitman Advanced Pub. Program, Boston, 1985.

[KKMOOT] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. **107** (1978), 1-39.

[KW] A.W. Knapp and N.R. Wallach, *Szegö kernels associated with discrete series*. Invent. Math. **34** (1976), 163-200, **62** (1980), 341-346.

[M1] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331-357.

[M2] ————, *Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups*, Advanced Studies in Pure Math. **14** (1988), 541-559.

[O1] T. Oshima, *Asymptotic behavior of spherical functions on semisimple symmetric spaces*, Advances Studies in Pure Math. **14** (1988), 561-601.

[O2] ————, *A realization of semisimple symmetric spaces and construction of boundary value maps*, Advanced Studies in Pure Math. **14** (1988), 603-650.

[R] W. Rossman, *The structure of semisimple symmetric spaces*, Canad. J. Math. **31** (1979), 157-180.

[V] D. Vogan, *Irreducible characters of semisimple Lie groups III*, Invent. Math. **71** (1983), 381-417.

Toshihiko Matsuki
Faculty of General Education
Tottori University
Tottori 680, Japan

Toshio Oshima
Department of Mathematics
Faculty of Science
University of Tokyo
Tokyo 113, Japan