UNITARY DUAL OF THE HERMITIAN QUATERNIONIC GROUP OF THE SPLIT RANK 2

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We classify the irreducible non-cuspidal representations of the hermitian quaternionic group of the split rank 2. We also find the complete non-cuspidal unitary dual of this non-quasi split group.

1. Introduction and preliminaries

We are interested in studying the composition series and unitary dual of the p-adic hermitian quaternionic group of the semi-simple rank 2. (We denote it by $G_2(D)$.) This group is not quasi-split, so the calculations of the unitary dual can not be obtained by the direct applications of Shahidi methods. Also, this group has an interesting feature: it has an isolated unitary representation, the phenomenon which occurs, for example, in the case of the exceptional group G_2. We also calculate the unitary dual supported on the non–Siegel maximal parabolic subgroup using the global methods similar to those used in [**19**] for the Siegel case, but resolving some obstacles related to the Langlands correspondence between the hermitian quaternionic group of the semi-simple rank 1 and it's split form. Thus, we follow the similar objectives for the classical groups, done by Sally and Tadić ([**20**]) for the p-adic $GSp(2,F)$ and $Sp(2,F)$, and for the quasi split unitary group by Konno ([**13**]). In the case of the exceptional groups, the classification for the group G_2 was done by Muić ([**18**]). In the classification of the subquotients of the principal series of the hermitian quaternionic group we use the structure of the ψ–Hopf module on the Grothendieck group of the representations of the finite length. This structure in the case of the split connected groups with the root system of types C_n and B_n was observed by Tadić ([**30**]) and then, in the case of $O(2n,F)$, the similar result was obtained by Ban ([**3**]).
In the preliminaries, we recall the structure of the hermitian quaternionic groups. In the next subsections, we state a result about the aforementioned structure of the ψ–Hopf module on the Grothendieck group. Also, we state the Langlands' classification and the criterion for the square integrability. We resolve the questions of the reducibility of the induced representations for the hermitian quaternionic group of the semi-simple rank 1 and make some useful observations about its structure. In the second section we deal with the reducibility and composition series for the principal series for the group

$G_2(D)$. The third and the fourth section are devoted to the determination of reducibility in the case of the induction from the maximal parabolic subgroups. The fifth section is devoted to the classification of the non-cuspidal part of the unitary dual of the group $G_2(D)$.

For the admissible representation σ of any group we consider, we denote by ω_σ its central character (if it exists). We will denote the Steinberg representation of the group G by St_G. If H is a subgroup of the group G, and $g \in G$ normalizes H, for the representation σ of the group H, we denote by ${}^g\sigma$ the representation of the group H defined by ${}^g\sigma(h) = \sigma(g^{-1}hg)$. We denote by $\{\alpha, \beta\}$ the basis of the root system corresponding to the maximal F–split torus in $G_2(D)$. The choice of the maximal F–split torus will be given in the next subsection. Also $\{\alpha, \beta\}$ will denote the basis of the root system with respect to the diagonal subgroup in $SO(4, F)$.

1.1. Hermitian quaternionic groups. Let F be a non-archimedean local field of characteristic zero, with the residual field with q elements. We choose the uniformizer of the field and denote it by $\overline{\omega}$. Let D be a quaternionic algebra, central over F and let τ be an involution, fixing the center of D (involution of the first kind). By the [**17**], the division algebra D defines a reductive group G over F as follows. Let

$$V_n = e_1D \oplus \cdots \oplus e_nD \oplus e_{n+1}D \oplus \cdots \oplus e_{2n}D$$

be a right vector space over D. Relations $(e_i, e_{2n-j+1}) = \delta_{ij}$ for $i = 1, 2, \dots, n$ define the hermitian form on V_n:

$$(v, v') = \varepsilon\tau((v', v)), v, v' \in V_n,\ \varepsilon \in \{-1, 1\}$$
$$(vx, v'x') = \tau(x)(v, v')x', x, x' \in D.$$

We extend the involution τ on $M(k, D)$ in the following way (and denote it by *):

$$g^* = (g_{ij})^* = (\tau(g_{ij}))^t.$$

For the smooth representation τ of the group $GL(n, D)$, we define the representation

$$\tau^*(g) = \tau(g^{-*}).$$

By the observation in [**19**], for the irreducible smooth representation τ of the group $GL(\cdot, D)$, the relation $\tau^* \cong \widetilde{\tau}$ holds. Let $G_n(D, \varepsilon)$ be the group of the isometries of the form $(\cdot, \cdot)$. It is easy to see that $G_n(D, \epsilon)$ is the group of F-rational points of the reductive algebraic group, an inner form of the split group $SO(4n)$ or $Sp(4n)$, if $\epsilon = -1$ or $\epsilon = 1$, respectively. In this paper, we will do explicit calculations for the case $\varepsilon = -1$, so we drop ε from the notation and, unless otherwise specified, assume $\varepsilon = -1$.

We can describe the group $G_n(D)$ also as

$$G_n(D) = \left\{ g \in GL(2n, D) : g^* \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\},$$

where

$$J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}.$$

We fix the maximal split torus A_0 in $G_n(D)(\overline{F})$ which has the following set of the F–rational points

$$A_0(F) = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \lambda_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2^{-1} & \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} : \lambda_i \in F^* \right\}.$$

We also denote elements from $A_0(F)$ by $diag\{\lambda_1, \ldots, \lambda_n\}$. By $X(A_0)$ we denote the group of F–rational characters on A_0, which can be identified with characters on $A_0(F)$. The root system corresponding to the maximal F-split torus is of the type C_n, with the set of the simple roots $\{\alpha_i,\ i = 1, \ldots, n\}$ where

$$\alpha_i(diag\{\lambda_1, \ldots, \lambda_n\}) = \lambda_i \lambda_{i+1}^{-1},\ i = 1, \ldots, n-1,$$
$$\alpha_n(diag\{\lambda_1, \ldots, \lambda_n\}) = \lambda_n^2.$$

Standard Levi F–subgroups correspond to the subsets θ of the set of simple roots $\Delta = \Delta(G_n(D), A_0)$ in the root system and we denote them by M_θ. Just for the purpose of the description of the Levi subgroups $M_\theta(F)$, for the quadratic matrices A_l of the order l we denote by $B_l = J_l(A_l^{-1})^* J_l$.
(i) If $\alpha_n \notin \theta$ there exist positive integers $n_1, n_2, \ldots, n_k$ such that $\Sigma n_i = n$ and

$$M_\theta(F) = \left\{ \begin{pmatrix} A_{n_1} & & & & & \\ & \ddots & & & & \\ & & A_{n_k} & & & \\ & & & B_{n_k} & & \\ & & & & \ddots & \\ & & & & & B_{n_1} \end{pmatrix} : A_{n_i} \in GL(n_i, D) \right\},$$

(ii) If $\alpha_n \in \theta$ there exist positive integers $n_1, n_2, \ldots, n_k, r$ such that $\Sigma n_i + r = n$ and

$$M_\theta(F) = \left\{ \begin{pmatrix} A_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & A_{n_k} & & & & \\ & & & G_r(D) & & & \\ & & & & B_{n_k} & & \\ & & & & & \ddots & \\ & & & & & & B_{n_1} \end{pmatrix} : A_{n_i} \in GL(n_i, D) \right\}.$$

Now, we describe the Langlands classification following [**4**]. Let us denote $\nu(x) = |x|_F$ if $x \in F$ and $\nu(x) = |det(x)|_F$ if $x \in D$ and det is the norm homomorphism. For any essentially square integrable representation (mod center) δ of the group $GL(m, D)$, there exists a unique real number $e(\delta)$ and an unique square integrable representation δ^u such that $\delta = \nu^{e(\delta)} \delta^u$. For a (ordered) multi-set $(\delta_1, \delta_2, \ldots, \delta_k)$ of the essentially square integrable representations of the $GL(\cdot, D)$-groups we say that it is in the standard order if $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_k)$.

For the representations δ_i of the $GL(\cdot, D)$ groups and the representation τ of the group $G_r(D)$, we write

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau = \mathrm{Ind}_P^{G_n(D)}(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau),$$

where P is a standard parabolic subgroup of the group $G_n(D)$. For the multiset of the irreducible essentially square integrable representations $(\delta_1, \delta_2, \ldots, \delta_k)$ which is in the standard order, and such that $e(\delta_k) > 0$ and an irreducible tempered representation τ of $G_r(D)$, we consider the representation $\delta = \delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau$ of the corresponding standard Levi subgroup, and we denote

$$e(\delta) = (e(\delta_1), e(\delta_1), \ldots, e(\delta_k), e(\delta_k), 0, \ldots, 0) \in X(A_0) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n.$$

In the above relation the number $e(\delta_i)$ appears in $e(\delta)$ exactly n_i times, if δ_i is a representation of the group $GL(n_i, D)$ and 0 appears r times. We introduce a partial order on $X(A_0) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ related to the root system of the type $\mathrm{C_n}$. We say that $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ if and only if we have the following inequalities

$$\begin{gathered} x_1 \leq y_1, \\ x_1 + x_2 \leq y_1 + y_2, \\ \vdots \\ x_1 + x_2 + \cdots + x_n \leq y_1 + y_2 + \cdots + y_n. \end{gathered}$$

The order just described is the one obtained in general as follows. Let $(\cdot, \cdot)$ be the Weyl group–invariant scalar product on $X(A_0) \otimes \mathbb{R}$ and let

$(\beta_1, \beta_2, \ldots, \beta_n)$ be the basis bidual to $(\alpha_1, \ldots, \alpha_n)$. Then for $\nu_1, \nu_2 \in X(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$ we say $\nu_1 \leq \nu_2$ if and only if $(\nu_1, \beta_i) \leq (\nu_2, \beta_i)$ for all $i = 1, \ldots, n$.

Lemma 1.1 (Langlands' classification [**4**]). *The induced (standard) representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where the irreducible essentially square integrable representations $(\delta_1, \delta_2, \ldots, \delta_k)$ are in the standard order with $e(\delta_k) > 0$, and where τ is the irreducible tempered representation of the group $G_r(D)$, has a unique irreducible quotient, denoted $L(\delta_1, \delta_2, \ldots, \delta_k; \tau)$, which is of the multiplicity one in the induced representation. In this way, we obtain every irreducible representation of the group $G_n(D)$. If the standard representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ has an irreducible subquotient σ, other then its Langlands quotient, where $\sigma = L(\delta'_1, \ldots, \delta_{k'}; \tau')$ then $e(\delta'_1 \otimes \delta'_2 \otimes \cdots \otimes \delta'_{k'} \otimes \tau') < e(\delta_1 \otimes \delta_2 \otimes \cdots \delta_k \otimes \tau)$.*

For an admissible irreducible representation π of the group $G_n(D)$ and $\alpha = (n_1, n_2, \ldots, n_k)$, an ordered partition of the $n - r$, $s_{(\alpha)}(\pi)$ denotes the normalized Jacquet module of that representation with respect to the standard parabolic subgroup P_α with the Levi subgroup isomorphic to $GL(n_1, D) \times GL(n_2, D) \times \cdots \times GL(n_k, D) \times G_r(D)$. For the irreducible representation π, let P_α denote (one of the associated) standard parabolic subgroups minimal with the property that $s_{(\alpha)}(\pi) \neq 0$. Each irreducible subquotient of $s_{(\alpha)}(\pi)$ is necessarily cuspidal. The criterion for the square integrability for the general p-adic reductive groups [**5**] readily applies, and we obtain the following criterion.

Lemma 1.2 (The square integrability criterion). *The irreducible admissible representation π is square integrable if and only if for each irreducible subquotient $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau$ of $s_{(\alpha)}(\pi)$ as above the following holds:*

$$(e(\sigma), \beta_{n_1 + \cdots + n_i}) > 0, \ \forall i = 1, \ldots, k.$$

For the admissible representation σ of the standard Levi subgroup M_θ and the element w of the Weyl group such that $w(\theta) = \theta'$ is subset of the set of the simple roots, we set $N_w = N_0 \cap w\overline{N_\theta}w^{-1}$, where $\overline{N_\theta}$ is the unipotent radical of the parabolic subgroup opposite to P_θ. For $m \in M_{\theta'}$, we define the representation of $M_{\theta'}$ by ${}^w\sigma(m) = \sigma(w^{-1}mw)$. We define (only formally), for $f \in \mathrm{Ind}_{M_\theta}^{G_n(D)}(\sigma)$

$$A_w(\sigma)f(g) = \int_{N_w} f(w^{-1}ng)\mathrm{d}n.$$

If this integral converges for every f, it defines the intertwining operator

$$A_w(\sigma) : \mathrm{Ind}_{M_\theta}^{G_n(D)}(\sigma) \to \mathrm{Ind}_{M_{\theta'}}^{G_n(D)}({}^w\sigma).$$

Often, the operator A_w will have some additional arguments (the complex numbers) which usually will denote the action of the family of intertwining operators on the family of the representations, which in the obvious will depend on these complex numbers. If the element w is the longest one in the relative Weyl group, we call the operator A_w the long intertwining operator. Sometimes, we will use a different definition of the long intertwining operator: we denote by $\delta_1 \underline{\times} \delta_2 \underline{\times} \cdots \underline{\times} \delta_k \underline{\rtimes} \tau$ the representation of M_θ induced from the opposite (lower-triangular) parabolic subgroup. The long-intertwining operator from the representation space of the standard representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ to the representation space of the representation $\delta_1 \underline{\times} \delta_2 \underline{\times} \cdots \underline{\times} \delta_k \underline{\rtimes} \tau$, is denoted by R and defined (formally) by the following relation

$$R(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau, N_\theta, \overline{N_\theta}) f(g) = \int_{\overline{N_\theta}} f(\overline{n} g) \mathrm{d}\overline{n}.$$

If this operator is injective or surjective, the representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ is irreducible.

We now briefly recollect some results about the reducibilities of the induced representations of the group $GL(n, D)$ ([**1**],[**28**]). To the irreducible cuspidal representation σ of the group $GL(n, D)$ by Jacquet-Langlands correspondence an irreducible essentially square integrable representation σ' of the group $GL(2n, F)$ is attached. If σ' is a cuspidal representation, we set $s(\sigma) = 1$, and if σ' is a subquotient of the induced representation $\tau \times \tau\nu$ for some irreducible cuspidal representation τ of the group $GL(n, F)$ (these are the only possibilities), we set $s(\sigma) = 2$. We then set $\nu_\sigma = \nu^{s(\sigma)}$. Then, for the irreducible cuspidal representations σ_i of the groups $GL(n_i, D)$, $i = 1, 2$, the representation $\sigma_1 \times \sigma_2$ is reducible if and only if $n_1 = n_2$, $s(\sigma_1) = s(\sigma_2)$ and $\sigma_1 = \nu_{\sigma_2}^{\pm 1} \sigma_2$.

Let us, just for the purpose of the next lemma, denote $\widetilde{\pi}$ by π^{-1}. Using the factorization of the long intertwining operator [**26**] we obtain the following lemma (see also [**29**]).

Lemma 1.3 (The reducibility of the principal series)**.** *For the irreducible admissible representations τ_i of the D^* the principal series representation $\tau_1 \times \cdots \times \tau_n \rtimes 1$ of the group $G_n(D)$ reduces if and only if one of the following holds:*

(i) *There exists i such that $\tau_i^{\epsilon} \rtimes 1$ reduces in $G_1(D)$*
(ii) *There exist i and j, $i \neq j$ such that $\tau_i^{\epsilon_1} \times \tau_j^{\epsilon_2}$ reduces in $GL(2, D)$, for some $\epsilon, \epsilon_1, \epsilon_2 \in \{1, -1\}$.*

We will describe reducibility in $G_1(D)$ shortly.

There is a well-known structure of Hopf algebra on the Grothendieck group of the smooth representations of finite length of the group $GL(n, D)$

([**34**],[**28**]). In more detail, we denote by R_n the Grothendieck group of the smooth representations of finite length of the group $GL(n, D)$. In what follows, $R(*)$ will denote the Grothendieck group related to the corresponding reductive group. Let $R = \oplus_{n\geq 0} R_n$. The multiplication $m : R \otimes R \to R$ is defined by the induction, and comultiplication $m^* : R \to R \otimes R$ by Jacquet modules:

$$m^*(\pi) = \sum_{k=0}^{n} s.s(r_{(k,n-k),(n)}(\pi)) \in R \otimes R.$$

Here π is the smooth representation of the finite length of the $GL(n, D)$, and $r_{(k,n-k),(n)}(\pi) \in R_k \otimes R_{n-k}$ denotes the normalized Jacquet module with respect to the maximal standard parabolic subgroup with Levi subgroup isomorphic to $GL(k, D) \times GL(n-k, D)$. By the linearity, we extend the definition of m and m^* to R. The tensor product $R \otimes R$ has an algebra structure in an usual way. The comultiplication m^* is a ring homomorphism; the proof of that fact can be found in [**34**] for the case of the general linear group over the field F.

Let us denote $R(G) = \oplus R(G_n(D))$. $R(G)$ is obviously R-module, and comodule structure is defined analogously to one in the GL -case: for smooth, finite length representation π of the group $G_n(D)$ we put

$$\mu^*(\sigma) = \sum_{k=0}^{n} s.s(s_{(k)}(\sigma)).$$

Let us denote by $s : R \otimes R \to R \otimes R$ the linear map such that $s(\pi_1 \otimes \pi_2) = \pi_2 \otimes \pi_1$ for representations π_1 and π_2. Also, we define the ring homomorphism $\Psi : R \to R \otimes R$ by

$$\Psi = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*.$$

We have the following proposition

Proposition 1.4 (The structure of Ψ–Hopf module on $R(G)$)**.** *For the smooth, finite length representation π of the group $GL(m, D)$, and smooth, finite length representation of the group $G_k(D)$ we have*

$$\mu^*(\pi \rtimes \sigma) = \Psi^*(\pi) \rtimes \mu^*(\sigma).$$

Proof. As in the split case done in [**30**]. □

Proposition 1.5 (R-groups)**.** *Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the discrete series representations of the general linear groups over the division algebra D, and τ the discrete series representation of $G_r(D)$. Then the representation*

$$\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k \rtimes \tau$$

is multiplicity free, and its length is 2^d where d denotes the number of mutually nonequivalent σ_i such that $\sigma_i \rtimes \tau$ reduces.

Proof. This is proved in [**12**]. □

1.2. The structure and the reducibility results for the group $G_1(D)$. It is easy to see that we can describe the group $G_1(D)$ also as

$$(1)\quad G_1(D) = \left\{ \begin{bmatrix} ad_1 & bd_1 \\ cd_1 & dd_1 \end{bmatrix}; a, b, c, d \in F, d_1 \in D, (ad - bc)d_1\tau(d_1) = 1 \right\}.$$

So, there is an epimorphism

$$\phi : G_1(D) \to F^*/(F^*)^2,\ \phi(g) = (ad - bc)(F^*)^2$$

with the kernel isomorphic to $SL(2, F)D_1$, where D_1 denotes the subgroup of the elements of norm 1 in D^*, and is realized as a subgroup of diagonal matrices in $G_1(D)$. Also $SL(2, F) \cap D_1 = \{\pm I\}$.

From now on, for the irreducible smooth representations of D^* (which are always finite-dimensional) we will use χ to denote (unitary) characters, and τ to denote (unitary) higher-dimensional representations. The distinction is important, because, by the Jacquet-Langlands correspondence, characters correspond to twists of the Steinberg representation of $GL(2, F)$, and the higher dimensional representations of D^* correspond to the cuspidal representations of $GL(2, F)$ ([**1**]). Each (continuous) character χ of D^* is of the form $\chi = \chi' \circ det$, for some character χ' of the field F.

Proposition 1.6. *Assume that τ and χ are irreducible admissible representations of D^*. Then we have the following:*

(i) *If $\tau \ncong \widetilde{\tau}$, then $\tau\nu^s \rtimes 1$ is irreducible for all $s \in \mathbb{R}$.*
(ii) *If $\chi^2 \neq 1$, then $\chi\nu^s \rtimes 1$ is irreducible for all $s \in \mathbb{R}$*
(iii) *Assume $\tau \cong \widetilde{\tau}$. Then $\tau\nu^s \rtimes 1$ reduces if and only if $s = \pm\frac{1}{2}$ and $\omega_\tau = 1$ or $s = 0$ and $\omega_\tau \neq 1$.*
(iii) *If $\chi^2 = 1$, $\chi^s \rtimes 1$ reduces if and only if $s = \pm\frac{1}{2}$.*

In the both cases, when we have reducibility at $s = \pm\frac{1}{2}$, the induced representation has length 2, and one of the subquotients is a square integrable representation, which is denoted $\delta[\chi\nu^{\frac{1}{2}}; 1]$ (or $\delta[\tau\nu^{\frac{1}{2}}; 1]$). When the representation $\tau \rtimes 1$ reduces, it is direct sum of the two nonequivalent tempered representations. The square integrable representations obtained this way are mutually inequivalent.

Proof. Let w_0 be the unique nontrivial element of the Weyl group of the group $G_1(D)$. Applying the standard result of Harish–Chandra ([**3**]), and taking into account the action of the Weyl group, the representation $\tau\nu^s \rtimes 1$ of $G_1(D)$ reduces for some $s \in \mathbb{R}$ only if $\tau^* \cong \widetilde{\tau} \cong \tau$. So we assume that $\tau \cong \widetilde{\tau}$. Let

$$A_{w_0}(\tau, s)f_s(e) = \int_U f_s(w_0^{-1}u)\mathrm{d}u$$

be the action of the standard intertwining operator, where f denotes function in the "compact" picture of the representation $\tau \rtimes 1$, and f_s its analytic section. We can identify $U \cong F$. We have the explicit calculation:

$$A_{w_0}(\tau,s)f_s(e) = \int_{|n|\leq 1} f_s(w_o^{-1}n)\mathrm{d}n + \sum_{k=1}^{\infty} q^{-2ks}\omega_\tau(\overline{\omega}^k)\int_{O^*} f\left(\begin{bmatrix} u^{-1} & 0 \\ \overline{\omega}^k & u\end{bmatrix}\right)\mathrm{d}u.$$

We denote by K_m the m-th congruence subgroup in $GL(2,D)$. If we denote the first integral above, which always converges, by I_1, and if f is $K_m \cap G_1(D)$–right–invariant, we get

$$A_{w_0}(\tau,s)f_s(e) = I_1 + \sum_{k=1}^{m-1} q^{-2ks}\omega_\tau(\overline{\omega}^k)\int_{O^*}\omega_\tau(u) f\left(\begin{bmatrix} 1 & 0 \\ u^{-1}\overline{\omega}^k & 1\end{bmatrix}\right)\mathrm{d}u$$
$$+ \sum_{k=m}^{+\infty} q^{-2ks}\omega_\tau(\overline{\omega}^k)f(e)\int_{O^*}\omega_\tau(u)\mathrm{d}u.$$

We conclude that the operator $A_{w_0}(\tau,s)$ is holomorphic at $s=0$, if and only if, the central character ω_τ of the representation τ is nontrivial. So, by the results of Harish–Chandra, in this situation the induced representation is reducible at $s=0$, and this is the only point of the reducibility ([**25**]). Let us consider now the selfcontragredient representations τ (or χ) with trivial central character so the induced representation is irreducible at $s=0$.

We now determine the poles of Plancherel measure by computing the composition of the intertwining operators $A_{w_0}(\tau,s)A_{w_0}(\tau,-s)$. Let us denote $\widetilde{f_s} = A_{w_0}(\tau,-s)f_{-s}$. Then

$$A_{w_0}(\tau,s)\widetilde{f_s}(e) = \int_F |\xi_0|^{-2s-1}\widetilde{f_s}\left(\begin{bmatrix} 1 & 0 \\ \xi_0^{-1} & 1\end{bmatrix}\right)\mathrm{d}\xi_0 =$$
$$\int_F |\xi_0|^{-2s-1}\int_F |\xi|^{2s-1} f_{-s}\left(\begin{bmatrix} 1 & 0 \\ \xi^{-1}+\xi_0^{-1} & 1\end{bmatrix}\right)\mathrm{d}\xi\mathrm{d}\xi_0.$$

In order to detect the poles of Plancherel measure, it is enough to consider such f that $\mathrm{supp} f_{-s} \subset P_0\overline{U_0}$, and such that

$$f_{-s|\overline{U_0}}\left(\begin{bmatrix} 1 & 0 \\ \xi & 1\end{bmatrix}\right) = \begin{cases} 0, & |\xi| > 1 \\ v_0, & |\xi| \leq 1.\end{cases}$$

The vector v_0 belongs to the representation space of τ. After some straightforward calculation, we get that the composition of those intertwining operators is trivial only for $s = \pm\frac{1}{2}$. Let us remark that the characters of D^* which have order less or equal to 2, are necessarily trivial on F^*. □

2. The principal series representations

In this section we write down all the composition sequences for the principal series representations. So, we identify the occurrence of the square integrable and tempered irreducible subquotients.

Let us recall that to each square integrable representation of $GL(n,D)$ is attached a segment of cuspidal representations ([**28**]). So, the unique (essentially) square integrable subquotient of the representation $\rho\nu_\rho^k \times \rho\nu_\rho^{k-1} \times \cdots \times \rho$ is denoted $\delta(\rho\nu_\rho^k,\rho)$. Here, ρ denotes the cuspidal representation of some $GL(m,D)$. In our case, for segments of characters of D^* $\nu_\chi=\nu^2$ and for segments of higher dimensional irreducible cuspidal representations of D^* $\nu_\tau=\nu$. Now, in the following two propositions we describe the composition series of all principal series induced from the characters. Lemma 1.3 and Proposition 1.6 give the reducibilities.

Proposition 2.1. *Assume that χ_1 is a unitary character of D^*, and $\alpha\in\mathbb{R}$. Then in the Grothendieck group $R(G_2(D))$ we have the following*

$$\chi_1\nu^\alpha \times \chi_1\nu^{\alpha+2} \rtimes 1 =$$
$$\begin{cases} L(\chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1});1)+\pi_1+L(\chi_1\nu^{\frac{5}{2}};\delta[\chi_1\nu^{\frac{1}{2}};1])+ \\ \qquad +L(\chi_1\nu^{\frac{5}{2}},\chi_1\nu^{\frac{1}{2}};1) & \text{if } \chi_1^2=1,\ \alpha=\frac{1}{2}, \\ L(\chi_1\nu^{\frac{1}{2}}\delta(\nu,\nu^{-1});1)+\pi_2+L(\chi_1\nu^{\frac{3}{2}};\delta[\chi_1\nu^{\frac{1}{2}};1])+ \\ \qquad +L(\chi_1\nu^{\frac{3}{2}},\chi_1\nu^{\frac{1}{2}};1) & \text{if } \chi_1^2=1,\ \alpha=-\frac{1}{2}, \\ L(\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1});1)+L(\chi_1\nu^{\alpha+2},\chi_1\nu^{\alpha};1) & \text{if } \alpha\in\mathbb{R}^+\setminus\{\frac{1}{2}\}, \\ L(\chi_1\nu^{1}\delta(\nu,\nu^{-1});1)+L(\chi_1\nu^{2};\chi_1\rtimes 1) & \text{if } \alpha=0, \\ L(\chi_1\nu^{1}\delta(\nu,\nu^{-1});1)+L(\chi_1\nu^{\alpha+2},\chi_1^{-1}\nu^{-\alpha};1) & \text{if } \alpha\in(-1,0)\setminus\{-\frac{1}{2}\}, \\ L(\chi_1\delta(\nu,\nu^{-1})\rtimes 1)+L(\chi_1\nu,\chi_1^{-1}\nu;1) & \text{if } \alpha=-1, \\ L(\chi_1^{-1}\nu^{-1-\alpha}\delta(\nu,\nu^{-1});1)+L(\chi_1^{-1}\nu^{-\alpha},\chi_1\nu^{\alpha+2};1) & \text{if } \alpha\in(-2,-1), \\ L(\chi_1^{-1}\nu\delta(\nu,\nu^{-1});1)+L(\chi_1^{-1}\nu^{2};\chi_1\rtimes 1) & \text{if } \alpha=-2, \\ L(\chi_1^{-1}\nu^{-1-\alpha}\delta(\nu,\nu^{-1});1)+L(\chi_1^{-1}\nu^{-\alpha},\chi_1^{-1}\nu^{-\alpha-2};1) & \text{if } \alpha<-2. \end{cases}$$

The representations π_i, $i=1,2$ are mutually inequivalent, square-integrable representations.

Proof. In the course of the proof we will make an extensive use of the Remark 3.2 and Lemma 3.7 from [**31**]. We have

$$\chi_1\nu^\alpha \times \chi_1\nu^{\alpha+2} \rtimes 1 = \chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1 + L(\chi_1\nu^{\alpha+2},\chi_1\nu^\alpha)\rtimes 1.$$

Using the structure of Ψ-Hopf module (Proposition 1.4) we obtain that

$$s_{(2)}(\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1})\rtimes 1) =$$
$$\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1})\otimes 1+\chi_1^{-1}\nu^{-(\alpha+1)}\widetilde{\delta(\nu,\nu^{-1})}\otimes 1+\chi_1^{-1}\nu^{-\alpha}\times\chi_1\nu^{\alpha+2}\otimes 1$$

and

$$s_{(1)}(\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1) = \chi_1\nu^{\alpha+2} \otimes \chi\nu^{\alpha} \rtimes 1 + \chi_1^{-1}\nu^{-\alpha} \otimes \chi\nu^{\alpha+2} \rtimes 1.$$

First, assume that all three expressions $\chi_1^{-1}\nu^{-\alpha} \times \chi_1\nu^{\alpha+2} \otimes 1$, $\chi\nu^{\alpha} \rtimes 1$ and $\chi\nu^{\alpha+2} \rtimes 1$ are irreducible. Then, applying Lemma 3.7 from [**31**] on Jacquet subquotients, we see that, in that case, the representation $\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1$ is irreducible. In general, (without assumptions on reducibility of those three expressions) with the aid of the Aubert involution ([**2**]), we obtain that $\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1$ and $L(\chi_1\nu^{\alpha+2},\chi_1\nu^{\alpha}) \rtimes 1$ have the same length. So, with previous assumption, the representation $\chi_1\nu^{\alpha} \times \chi_1\nu^{\alpha+2} \rtimes 1$ has length 2.

Second, assume reducibility of $\chi_1^{-1}\nu^{-\alpha} \times \chi_1\nu^{\alpha+2}$. This implies $\chi_1^2 = 1$ and $\alpha \in \{0,-2\}$. For each choice of α from this set, we get the representations which are the same in the Grothendieck group. We have

$$\chi_1\nu^2 \times \chi_1 \rtimes 1 = \chi_1\nu\delta(\nu,\nu^{-1}) \rtimes 1 + \chi_1\nu 1_{GL(2,D)} \rtimes 1.$$

Lemma 2.2. *The representation $\chi_1\nu\delta(\nu,\nu^{-1}) \rtimes 1$ is irreducible.*

Proof. We can apply ideas from [**31**], Section 6. □

Third, assume that $\chi\nu^{\alpha} \rtimes 1$ is reducible. Follows that $\chi_1^2 = 1$, and $\alpha \in \{\pm\frac{1}{2}\}$. We have (in the Grothendieck group)

$$\begin{aligned}(2)\qquad \chi_1\nu^{\frac{5}{2}} \times \chi_1\nu^{\frac{1}{2}} \rtimes 1 &= \chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1}) \rtimes 1 + \chi_1\nu^{\frac{3}{2}}1_{GL(2,D)} \rtimes 1\\ &= \chi_1\nu^{\frac{5}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1] + \chi_1\nu^{\frac{5}{2}} \rtimes L(\chi_1\nu^{\frac{1}{2}};1).\end{aligned}$$

We have

$$\begin{aligned}(3)\quad & s_{(2)}(\chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1}) \rtimes 1) =\\ & \chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1}) \otimes 1 + \chi_1\nu^{-\frac{3}{2}}\delta(\nu,\nu^{-1}) \otimes 1 + \chi_1\nu^{\frac{5}{2}} \times \chi_1\nu^{-\frac{1}{2}} \otimes 1,\end{aligned}$$

$$(4)\quad s_{(2)}(\chi_1\nu^{\frac{5}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]) = \chi_1\nu^{\frac{1}{2}} \times \chi_1\nu^{\frac{5}{2}} \otimes 1 + \chi_1\nu^{\frac{-5}{2}} \times \chi_1\nu^{\frac{1}{2}} \otimes 1.$$

From this, applying Remark 3.2 from [**31**], follows that both of the representations

$$\chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1}) \rtimes 1 \text{ and } \chi_1\nu^{\frac{5}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]$$

are reducible and that they have an irreducible subquotient in common. Examining Jacquet modules in equations (3) and (4), we conclude that there is only one such subquotient, denoted π_1, and it is a square-integrable representation. Analogously we conclude that $\chi_1\nu^{\frac{3}{2}}1_{GL(2,D)} \rtimes 1$ and $\chi_1\nu^{\frac{5}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]$ have a common irreducible subquotient. We also conclude that each of the representations which appear on the right-hand side of the equation (2) has length at most 3. If we explore Jacquet modules of the representation $\chi_1\nu^{\frac{5}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]$ with respect to the minimal parabolic subgroup, we see

that this is impossible. So we obtain the decomposition of the principal series into 4 irreducible subquotients.
In the case $\alpha = -\frac{1}{2}$, the discussion is quite similar, but here we find a common square integrable subquotient π_2 in $\chi_1\nu^{\frac{1}{2}}\delta(\nu,\nu^{-1}) \rtimes 1$ and $\chi_1\nu^{\frac{3}{2}} \rtimes L(\chi_1\nu^{\frac{1}{2}};1)$. Also examining the Jacquet modules with respect to the minimal parabolic subgroup, we find that the principal series has length 4.
Fourth, reducibility of the $\chi_1\nu^{\alpha+2} \rtimes 1$ leads to the representations already seen above. □

Proposition 2.3. *We assume $\chi_1^2 = 1$. Then, we have*

$$\chi_2\nu^{\alpha} \times \chi_1\nu^{\frac{1}{2}} \rtimes 1 =$$
$$\begin{cases} L(\chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1});1) + \pi_1 + L(\chi_1\nu^{\frac{5}{2}};\delta[\chi_1\nu^{\frac{1}{2}};1]) + \\ \qquad + L(\chi_1\nu^{\frac{5}{2}},\chi_1\nu^{\frac{1}{2}};1) & \text{if } \chi_2=\chi_1,\ \alpha=\frac{5}{2} \\ L(\chi_1\nu^{\frac{1}{2}}\delta(\nu,\nu^{-1});1) + \pi_2 + L(\chi_1\nu^{\frac{3}{2}};\delta[\chi_1\nu^{\frac{1}{2}};1]) + \\ \qquad + L(\chi_1\nu^{\frac{3}{2}},\chi_1\nu^{\frac{1}{2}};1) & \text{if } \chi_2=\chi_1,\ \alpha=\frac{3}{2} \\ L(\chi_2\nu^{\frac{1}{2}};\delta[\chi_1\nu^{\frac{1}{2}};1]) + \pi_3 + +L(\chi_1\nu^{\frac{1}{2}},\chi_2\nu^{\frac{1}{2}};1) + \\ \qquad + L(\chi_1\nu^{\frac{1}{2}};\delta[\chi_2\nu^{\frac{1}{2}};1]) & \text{if } \chi_2^2=1,\ \chi_2\neq\chi_1,\ \alpha=\frac{1}{2} \\ \chi_2 \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1] + L(\chi_1\nu^{\frac{1}{2}};\chi_2\rtimes 1),\ (\alpha=0) \\ L(\chi_2\nu^{\alpha};\delta[\chi_1\nu^{\frac{1}{2}};1]) + L(\chi_2\nu^{\alpha},\chi_1\nu^{\frac{1}{2}};1),\ (\alpha>0, \text{ other cases}) \\ L(\chi_2^{-1}\nu^{-\alpha};\delta[\chi_1\nu^{\frac{1}{2};1}]) + L(\chi_2^{-1}\nu^{-\alpha},\chi_1\nu^{\frac{1}{2}};1), (\alpha<0, \text{ other cases}) \end{cases}$$

The representations π_i, $i = 1,2,3$ are mutually inequivalent, square integrable representations.

Proof. We have

$$\chi_2\nu^{\alpha} \times \chi_1\nu^{\frac{1}{2}} \rtimes 1 = \chi_2\nu^{\alpha} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1] + \chi_2\nu^{\alpha} \rtimes L(\chi_1\nu^{\frac{1}{2}};1).$$

Also

$$s_{(2)}(\chi_2\nu^{\alpha} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]) = \chi_2\nu^{\alpha} \times \chi_1\nu^{\frac{1}{2}} + \chi_2^{-1}\nu^{-\alpha} \times \chi_1\nu^{\frac{1}{2}},$$

and

$$s_{(1)}(\chi_2\nu^{\alpha} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]) =$$
$$\chi_1\nu^{\frac{1}{2}} \otimes +\chi_2\nu^{\alpha} \rtimes 1 + \chi_2\nu^{\alpha} \otimes \delta[\chi_1\nu^{\frac{1}{2}};1] + \chi_2^{-1}\nu^{-\alpha} \otimes \delta[\chi_1\nu^{\frac{1}{2}};1].$$

The Assumptions of irreducibility of $\chi_2\nu^{\alpha} \times \chi_1\nu^{\frac{1}{2}}$, $\chi_2^{-1}\nu^{-\alpha} \times \chi_1\nu^{\frac{1}{2}}$ and of $\chi_2\nu^{\alpha} \rtimes 1$ with the use of Lemma 3.7 of [**31**] leads to the conclusion of the irreducibility of the representations $\chi_2\nu^{\alpha} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]$ and $\chi_2\nu^{\alpha} \rtimes L(\chi_1\nu^{\frac{1}{2}};1)$. By dropping these assumptions, only new cases to consider are

$$\chi_2\nu^{\frac{1}{2}} \times \chi_1\nu^{\frac{1}{2}} \rtimes 1,$$

with $\chi_2^2 = 1$.

First, suppose that $\chi_2 = \chi_1$. The representations $\chi_1\nu^{\frac{1}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]$ and $\chi_1\nu^{\frac{1}{2}} \rtimes L(\chi_1\nu^{\frac{1}{2}};1)$ are irreducible. Namely, representation $\chi_1\nu^{\frac{1}{2}} \times \chi_1\nu^{-\frac{1}{2}}$ is an irreducible unitarizable representation of $GL(2,D)$, so representation $\chi_1\nu^{\frac{1}{2}} \times \chi_1\nu^{-\frac{1}{2}} \rtimes 1$ is also unitarizable, and the representation $\chi_1\nu^{\frac{1}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]$ is its' quotient. But it is also a standard representation, so it is irreducible.

Now, suppose that $\chi_2 \neq \chi_1$. By examining $s_{(2)}(\chi_2\nu^{\frac{1}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1])$ we see that $\chi_2\nu^{\frac{1}{2}} \rtimes \delta[\chi_1\nu^{\frac{1}{2}};1]$ has length at most 2. Also we see from Remark 3.2 of [**31**] that this representation and $\chi_1\nu^{\frac{1}{2}} \rtimes L(\chi_2\nu^{\frac{1}{2}};1)$ have one common subquotient, a square integrable representation denoted π_3 . □

Now, we describe the composition series of all principal series induced from higher–dimensional representations. The principal series representation of the form $\tau_1\nu^{\alpha} \times \tau_2\nu^{\beta} \rtimes 1$ where $\dim \tau_i > 1$, $i = 1,2$ is reducible only in the situations covered by the next four propositions.

Proposition 2.4. *Let τ_1 denote an irreducible, admissible, unitary representation of D^* of dimension greater than 1. If τ_1 is not a selfdual representation, we have the following*

$$\tau_1\nu^{\alpha+1} \times \tau_1\nu^{\alpha} \rtimes 1 =$$

$$\begin{cases} L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1) + L(\tau_1\nu^{\alpha+1},\tau_1\nu^{\alpha};1) & \text{if } \alpha > 0,\\ L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1) + L(\tau_1\nu;\tau_1 \rtimes 1) & \text{if } \alpha = 0,\\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1) + L(\tau_1\nu^{\alpha+1},\widetilde{{}_1}\nu^{-\alpha};1) & \text{if } \alpha \in (-\frac{1}{2},0),\\ L(\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}}) \rtimes 1) + L(\tau_1\nu^{\frac{1}{2}},\widetilde{\tau_1}\nu^{-\frac{1}{2}};1) & \text{if } \alpha = -\frac{1}{2},\\ L(\nu^{-\alpha-\frac{1}{2}}\delta(\widetilde{\tau_1}\nu^{\frac{1}{2}},\widetilde{\tau_1}\nu^{-\frac{1}{2}});1) + L(\widetilde{\tau_1}\nu^{-\alpha},\tau_1\nu^{\alpha+1};1) & \text{if } \alpha \in (-1,-\frac{1}{2}),\\ L(\nu^{-\alpha-\frac{1}{2}}\delta(\widetilde{\tau_1}\nu^{\frac{1}{2}},\widetilde{\tau_1}\nu^{-\frac{1}{2}});1) + L(\widetilde{\tau_1}\nu^{-\alpha},\widetilde{\tau_1}^{-\alpha-1};1) & \text{if } \alpha < -1. \end{cases}$$

Proof. We have

$$\tau_1\nu^{\alpha+1} \times \tau_1\nu \rtimes 1 = \nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})) \rtimes 1 + \nu^{\frac{1}{2}}L(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}}) \rtimes 1.$$

Analogously to the proof of the previous proposition, we examine

$$s_{(2)}(\nu^{\frac{1}{2}+\alpha}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})) \rtimes 1) =$$

$$\nu^{\frac{1}{2}+\alpha}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}}) \otimes 1 + \nu^{-\frac{1}{2}-\alpha}\delta(\widetilde{\tau_1\nu^{\frac{1}{2}},\tau_1}\nu^{-\frac{1}{2}}) \otimes 1 + \widetilde{\tau_1}\nu^{-\alpha} \times \tau_1\nu^{\alpha+1} \otimes 1.$$

Also we have

$$s_{(1)}(\nu^{\frac{1}{2}+\alpha}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})) = \tau_1\nu^{\alpha+1} \otimes \tau_1\nu^{\alpha} \rtimes 1 + \widetilde{\tau_1}\nu^{-\alpha} \otimes \tau_1\nu^{\alpha+1} \rtimes 1.$$

With the assumptions that $\widetilde{\tau_1}\nu^{-\alpha} \times \tau_1\nu^{\alpha+1}$, $\tau_1\nu^{\alpha} \rtimes 1$, and $\tau_1\nu^{\alpha+1} \rtimes 1$ are irreducible, and applying Lemma 3.7 of [**31**] we obtain that the representation

$(\nu^{\frac{1}{2}+\alpha}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ is irreducible, and, by the property of the Aubert involution, also that the representation $\nu^{\frac{1}{2}}L(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ is irreducible. These assumptions are met when $\tau_1\not\cong\widetilde{\tau_1}$. □

Proposition 2.5. *Let τ_1 denote an irreducible, unitary, selfdual representation of D^* of dimension greater then 1.*
(i) If $\chi_{\tau_1}=1$, then we have the following

$$\tau_1\nu^{\alpha+1}\times\tau_1\nu^{\alpha}\rtimes 1=$$
$$\begin{cases} L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)+L(\tau_1\nu;\tau_1\rtimes 1) & \textit{if } \alpha=0,\\ L(\nu\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)+L(\tau_1\nu^{\frac{3}{2}};\delta[\tau\nu^{\frac{1}{2}};1])+\pi_4+L(\tau_1\nu^{\frac{3}{2}},\tau_1\nu^{\frac{1}{2}};1) & \textit{if } \alpha=\frac{1}{2},\\ L(\tau_1\nu^{\frac{1}{2}};\delta[\tau_1\nu^{\frac{1}{2}};1])+L(\tau\nu_1^{\frac{1}{2}},\tau\nu_1^{\frac{1}{2}};1)+T_1+T_2 & \textit{if } \alpha=-\frac{1}{2},\\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)+L(\tau_1\nu^{\alpha+1},\tau_1\nu^{\alpha};1) & \textit{if } \alpha\in\mathbb{R}^+\setminus\{\frac{1}{2}\}.\end{cases}$$

(ii) If $\chi_{\tau_1}\neq 1$ and $\tau_1\rtimes 1=T_3'+T_4'$, then we have the following

$$\tau_1\nu^{\alpha+1}\times\tau_1\nu^{\alpha}\rtimes 1=$$
$$\begin{cases} L(\tau_1\nu;T_3')+L(\tau_1\nu;T_4')+2L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)+\pi_5+\pi_6 & \textit{if } \alpha=0,\\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)+L(\tau_1\nu^{\alpha+1},\tau_1\nu^{\alpha};1) & \textit{if } \alpha>0.\end{cases}$$

The representations $\pi_i,\ i=4,5,6$ are mutually inequivalent square–integrable representations, and $T_i,\ i=1,2$ are mutually inequivalent tempered(non square-integrable) representations.

Proof. Dropping the assumptions of the irreducibility of the representations $\widetilde{\tau_1}\nu^{-\alpha}\times\tau_1\nu^{\alpha+1},\tau_1\nu^{\alpha}\rtimes 1,$ and $\tau_1\nu^{\alpha+1}\rtimes 1$ mentioned in the proof of the previous proposition, we are left to deal with the following representations

$$\tau_1\nu\times\tau_1\rtimes 1,\ \text{when } \chi_{\tau_1}=1. \tag{5}$$

Analogously to the Lemma 2.2 we conclude the irreducibility of $\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$. Another representation to consider is

$$\tau_1\nu\times\tau_1\rtimes 1,\ \text{when } \chi_{\tau_1}\neq 1 \tag{6}$$

Here we obtain the single case where the multiplicity one fails, this is also the only induced representation of the length equal to 6. Examining the Jacquet modules we learn that the representation $\tau_1\rtimes T_3'$ has length 3, at most, and that it is reducible (because it has the same length as $\tau_1\rtimes T_4'$). If we assume that it has length equal to 2, then also

$$\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1=L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)+\pi,(\text{ in } R(G_2(D)))$$

where π is some subrepresentation. We see that $L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)$ has to be then the subrepresentation of $\tau_1\nu\times\tau_1\rtimes 1$, but $\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ is also the subrepresentation of $\tau_1\nu\times\tau_1\rtimes 1$. But this leads to conclusion that either $L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)$ is a subrepresentation of $\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$, or the multiplicity of $L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)$ in $\tau_1\nu\times\tau_1\rtimes 1$ is greater then 1; both of them false (in this situation). So, we conclude that $\tau_1\rtimes T_3'$ and $\tau_1\rtimes T_4'$ both have length 3, and both have the unique subrepresentations which are square integrable (denoted π_5 and π_6). By the careful examining of the composition sequences of the Jacquet modules, we conclude that representations $\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ and $\nu^{\frac{1}{2}}L(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ have one irreducible quotient in common. In the Grothendieck group we have:

$$\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1 = L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)+\pi_5+\pi_6$$
$$\tau_1\nu\rtimes T_3' = L(\tau_1\nu;T_3')+\pi_5+L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1).$$

The next case is

$$\tau_1\nu^{\frac{3}{2}}\times\tau_1\nu^{\frac{1}{2}}\rtimes 1;\ \text{if } \chi_{\tau_1}=1 \tag{7}$$

Examining the length of Jacquet modules, similarly to the case of inducing from the characters, we see that the length of $\nu\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ can't be 3, because of inconsistency with associativity of Jacquet modules. The rest is straightforward. Similarly with the last representation to consider

$$\tau_1\nu^{\frac{1}{2}}\times\tau_1\nu^{\frac{1}{2}}\rtimes 1;\ \text{if } \chi_{\tau_1}=1 \tag{8}$$

□

Proposition 2.6. *Let τ_2 be an unitary, irreducible selfdual representation of D^* of dimension greater than 1, with trivial central character, and let τ_1 denote a unitary irreducible representation of D^* of dimension greater then 1.*
(a) If $\tau_1\ncong\widetilde{\tau_1}$, then we have the following

$$\tau_1\nu^{\alpha}\times\tau_2\nu^{\frac{1}{2}}\rtimes 1=\begin{cases} L(\tau_1\nu^{\alpha};\delta[\tau_2\nu^{\frac{1}{2}};1])+L(\tau_1\nu^{\alpha},\tau_2\nu^{\frac{1}{2}};1) & \text{if } \alpha>0,\\ T_3+L(\tau_2\nu^{\frac{1}{2}};\tau_1\rtimes 1) & \text{if } \alpha=0,\end{cases}$$

where $T_3=\tau_1\rtimes\delta[\tau_2\nu^{\frac{1}{2}};1])$ is an irreducible tempered representation.
(b) If $\tau_1\cong\widetilde{\tau_1}$, then we have the following two cases: (i) and (ii)
(i) If $\chi_{\tau_1}=1$, then we have the following

$$\tau_1\nu^{\alpha}\times\tau_2\nu^{\frac{1}{2}}\rtimes 1 = L(\tau_1\nu^{\alpha};\delta[\tau_2\nu^{\frac{1}{2}};1])+L(\tau_1\nu^{\alpha},\tau_2\nu^{\frac{1}{2}};1)$$

if $|\alpha|\notin\mathbb{R}_0^+\setminus\{0,\pm\frac{1}{2},\pm\frac{3}{2}\}$;

$$\tau_1\nu^{\alpha}\times\tau_2\nu^{\frac{1}{2}}\rtimes 1 =$$
$$L(\tau_1\nu^{\frac{1}{2}};\delta[\tau_2\nu^{\frac{1}{2}};1])+L(\tau_2\nu^{\frac{1}{2}};\delta[\tau_1\nu^{\frac{1}{2}};1])+L(\tau_1\nu^{\frac{1}{2}},\tau_2\nu^{\frac{1}{2}};1)+\pi_7$$

if $|\alpha \models \frac{1}{2}$ *and* $\tau_1 \ncong \tau_2$;

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = L(\tau_1\nu^{\frac{1}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau\nu_1^{\frac{1}{2}}, \tau\nu_1^{\frac{1}{2}}; 1) + T_1 + T_2$$

if $|\alpha| = \frac{1}{2}$ *and* $\tau_1 \cong \tau_2$; *and finally*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 =$$
$$L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + \pi_4 + L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1)$$

if $|\alpha| = \frac{3}{2}$ *and* $\tau_1 \cong \tau_2$.

The representation π_7 *is a square–integrable representation.*

(ii) If $\chi_{\tau_1} \neq 1$ *and* $\tau_1 \rtimes 1 = T_3' + T_4'$, *then we have the following*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = \begin{cases} L(\tau_1\nu^{\alpha}; \delta[\tau_2\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{\alpha}, \tau_2\nu^{\frac{1}{2}}; 1) & \text{if } \alpha > 0, \\ L(\tau_2\nu^{\frac{1}{2}}; T_3') + L(\tau_2\nu^{\frac{1}{2}}; T_4') + T_5 + T_4 & \text{if } \alpha = 0. \end{cases}$$

The representations $T_i,\ i = 4, 5$ *are the irreducible tempered representations.*

Proof. Only new case left to check, after dealing with ones which are solved by Lemma 3.7 of [**31**] is

$$\tau_1\nu^{\frac{1}{2}} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1;\ \text{if } \tau_1 \ncong \tau_2,\ \chi_{\tau_1} = \chi_{\tau_2} = 1 \tag{9}$$

This case is resolved in the same way as for the characters. □

Proposition 2.7. *Let* τ_2 *be unitary irreducible, self-dual representation of* D^* *such that* $\chi_{\tau_2} \neq 1$, *so that* $\tau_2 \rtimes 1 = T_3' \oplus T_4'$, *and let* τ_1 *be an irreducible unitary representation of* D^*. *Then, we have the following*
(a) If $\alpha > 0$, *then we have the following*

$$\tau_1\nu^{\alpha} \times \tau_2 \rtimes 1 =$$
$$\begin{cases} L(\tau_1\nu^{\frac{1}{2}}; T_3') + L({}_1\nu^{\frac{1}{2}}; T_4') + T_4 + T_5 \ \text{if } \alpha = \frac{1}{2},\ \tau_1 \cong \widetilde{\tau}_1,\ \omega_{\tau_1} = 1, \\ L(\tau_2\nu; T_3') + L(\tau_2\nu; T_4') + 2L(\nu^{\frac{1}{2}}\delta(\tau_2\nu^{\frac{1}{2}}, \tau_2\nu^{-\frac{1}{2}}); 1) + \pi_5 + \pi_6 \ \text{if } \tau_1 \cong \tau_2,\ \alpha = 1, \\ L(\tau_1\nu^{\alpha}; T_3') + L(\tau_1\nu^{\alpha}; T_4') \ \text{in other cases.} \end{cases}$$

(b) If $\alpha = 0$ *then we have the following*

$$\tau_1 \times \tau_2 \rtimes 1 = \begin{cases} T_6 + T_7 + T_8 + T_9 & \text{if } \tau_1 \cong \widetilde{\tau}_1,\ \omega_{\tau_1} \neq 1,\ \tau_1 \ncong \tau_2, \\ T_{10} + T_{11} & \text{in other cases.} \end{cases}$$

The representations $T_i,\ i = 6, \ldots, 11$ *are mutually inequivalent tempered (non square-integrable) representations.*

Proof. The cases in the first part of this Proposition were already covered, and the statements of the second part follow from [**12**]. □

We now settle the mixed case of the principal series representations.

Proposition 2.8. *Let χ be a unitary character of D^*, and let τ be irreducible, admissible unitary representation of D^* of dimension greater than 1. Then the principal series representation $\chi\nu^\alpha \times \tau\nu^\beta \rtimes 1$ (for $\alpha, \beta \in \mathbb{R}$) reduces only in the following cases:*
(a) If $\chi^2 = 1$, then we have the following

$$\tau\nu^\beta \times \chi^{\frac{1}{2}} \rtimes 1 =$$
$$\begin{cases} L(\tau\nu^\beta; \delta[\chi\nu^{\frac{1}{2}}; 1]) + L(\tau\nu^\beta, \chi\nu^{\frac{1}{2}}; 1) \text{ if } \beta > 0 \text{ and } \tau\nu^\beta \rtimes 1 \text{ is irreducible,} \\ \tau \rtimes \delta[\chi\nu^{\frac{1}{2}}; 1] + L(\chi\nu^{\frac{1}{2}}; \tau \rtimes 1) \text{ if } \beta = 0 \text{ and } \tau \rtimes 1 \text{ is irreducible,} \\ T_7 + T_8 + L(\chi\nu^{\frac{1}{2}}, T_3') + L(\chi\nu^{\frac{1}{2}}, T_4') \text{ if } \tau \cong \widetilde{\tau},\ \omega_\tau \neq 1,\ \beta = 0, \\ \begin{array}{c} L(\tau\nu^{\frac{1}{2}}; \delta[\chi\nu^{\frac{1}{2}}; 1]) + \pi_8 + L(\chi\nu^{\frac{1}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + \\ + L(\tau\nu^{\frac{1}{2}}, \chi\nu^{\frac{1}{2}}; 1) \end{array} \text{ if } \tau \cong \widetilde{\tau},\ \omega_\tau = 1,\ |\beta| = \frac{1}{2}, \end{cases}$$

where $\tau \rtimes 1 = T_3' + T_4'$ in the third case above. The tempered representation $\tau \rtimes \delta[\chi\nu^{\frac{1}{2}}; 1]$ from the second case is irreducible, and the representation π_8 is a square integrable representation.
(b) If $\tau\nu^{\frac{1}{2}} \rtimes 1$ reduces, then we have the following

$$\chi\nu^\alpha \times \tau\nu^{\frac{1}{2}} \rtimes 1 =$$
$$\begin{cases} L(\chi\nu^\alpha; \delta[\tau\nu^{\frac{1}{2}}; 1]) + L(\chi\nu^\alpha, \tau\nu^{\frac{1}{2}}; 1) \text{ if } \alpha > 0 \text{ and } \chi\nu^\alpha \rtimes 1 \text{ is irreducible} \\ L(\tau\nu^{\frac{1}{2}}; \delta[\chi\nu^{\frac{1}{2}}; 1]) + L(\chi\nu^{\frac{1}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + L(\tau\nu^{\frac{1}{2}}, \chi\nu^{\frac{1}{2}}; 1) + \pi_8 \text{ if } \chi^2 = 1,\ \alpha = \frac{1}{2} \\ \chi \rtimes \delta[\tau\nu^{\frac{1}{2}}; 1] + L(\tau\nu^{\frac{1}{2}}; \chi \rtimes 1) \text{ if } \alpha = 0 \end{cases}$$

The tempered representation $\chi \rtimes \delta[\tau\nu^{\frac{1}{2}}; 1]$ in the third case is irreducible.
(c) If $\tau \rtimes 1 = T_3' \oplus T_4'$, then the following holds

$$\chi\nu^\alpha \times \tau \rtimes 1 =$$
$$\begin{cases} L(\chi\nu^\alpha; T_3') + L(\chi\nu^\alpha; T_4') & \text{if } \alpha > 0,\ \chi\nu^\alpha \rtimes 1 \text{ is irreducible,} \\ \chi \rtimes T_3' + \chi \rtimes T_4' & \text{if } \alpha = 0, \\ T_7 + T_8 + L(\chi\nu^{\frac{1}{2}}, T_3') + L(\chi\nu^{\frac{1}{2}}, T_4') & \text{if } \chi^2 = 1,\ |\alpha| = \frac{1}{2}. \end{cases}$$

Proof. Using Lemma 3.7 of [**31**]. □

3. Induced representations of the group $G_2(D)$; the Siegel case

In this section, we consider the reducibility of the representations of the form

$$\sigma\nu^s \rtimes 1,$$

where σ is an irreducible admissible cuspidal representation of $GL(2, D)$, s is a real number. By the result of Harish–Chandra ([**3**]) if this induced representation is reducible for some s, σ must be a self-contragredient. So, from now on, we assume that $\sigma \cong \widetilde{\sigma}$. Let σ' be the square integrable representation of $GL(4,F)$ corresponding to σ by Jacquet-Langlands correspondence. It is actually a cuspidal representation also ([**1**]).

Proposition 3.1. *Let us denote by σ irreducible, admissible, selfdual cuspidal representation of the group $GL(2,D)$. The representation $\sigma \rtimes 1$ is irreducible if and only if $L(s,\sigma',\Lambda^2\rho_4)$ has a pole at $s=0$. If this is so, the representation $\sigma\nu^s \rtimes 1,\ s\in\mathbb{R}$ reduces only for $s=\pm\frac{1}{2}$.*

Proof. Let us remind that the Plancherel measure is defined as

$$R(s,\sigma,\overline{N}(F),N(F))R(s,\sigma,N(F),\overline{N}(F)) = \mu^{-1}(s,\sigma).$$

Notation is as in [**19**]. By the results from [**19**], we have

$$\mu(s,\sigma)=\mu(s,\sigma')$$

where on the left hand side we have the Plancherel measure in the group $G_2(D)$, and on the right hand side, the Plancherel measure corresponding to the representation induced from σ', in $SO(8,F)$. Because of the fact that σ' is cuspidal, the reducibility of $\sigma'\nu^s \rtimes 1$ can be directly learned from the Plancherel measure: there exists the unique $s_0 \geq 0$ such that $\sigma'\nu^{s_0} \rtimes 1$ reduces ([**25**]) and

$$s_0 = 0 \text{ if and only if } \mu(s_0,\sigma') \neq 0,$$

$$s_0 > 0 \text{ if and only if } \mu(s,\sigma') \text{ has a pole at } s=s_0.$$

So, $\sigma\nu^s \rtimes 1$ is reducible if and only if $\sigma'\nu^s \rtimes 1$ is reducible, and by the work of Shahidi ([**23**]) $s_0 \in \{\frac{1}{2},0\}$. Because σ' is generic, Plancherel measure is expressible in terms of L–functions. To conclude, $\sigma' \rtimes 1$ is irreducible, if and only if $L(s,\sigma',\Lambda^2\rho_4)$ has a pole at $s=0$ ([**24**]). □

4. Induced representations of the group $G_2(D)$; the non–Siegel case

We are now considering the representations of the form

$$\tau\nu^s \rtimes \delta,$$

where τ is irreducible admissible unitary representation of D^*, and δ is irreducible cuspidal representation of $G_1(D)$. Analogously to the results of the previous section, for examining the reducibility, it is enough to assume that $\tau \cong \widetilde{\tau}$. Throughout this section we keep this assumption. For a algebraic number field k, we denote its ring of adeles by A_k. We restrict

$$\delta_{|SL(2,F)D_1} = \sum_{i=1}^{k} \tau_i \otimes \delta_i,$$

according to the observation about the structure of $G_1(D)$ (the equation(1)). The procedure we use is as follows: we choose a summand in the above restriction, for example, $\tau_1 \otimes \delta_1$, and lift it to the discrete series representation $\tau_1 \otimes \delta_1'$ of the group $SL(2,F)SL(2,F)$. Then we will find a representations δ' and δ'' of $SO(4,F)$ such that the representation $\tau_1 \otimes \delta_1'$ is a component in the the restrictions of the representation δ' and δ'' to $SL(2,F)SL(2,F)$. Then, using global methods, we will prove that $\mu(s, \tau \otimes \delta)^2 = \mu(s, \tau' \otimes \delta')\mu(s, \tau' \otimes \delta'')$. The difficulty in applying the global methods lies in the fact that there are global L–packets for the group $SL(2, A_k)$ such that some global representations in that L–packets are automorphic, and others are not ([**14**]), so we have to make some adjustments. Also, in order to ensure that the representations δ' and δ'' differ only in quadratic character, i.e. they have the same restriction on $SL(2,F)SL(2,F)$, we have to be carefull when varying representations in the local L–packets of τ_1 and δ_1.

We can choose a number field k such that there exist two places of k, v_1 and v_2 such that $k_{v_1} \cong k_{v_2} \cong F$, and a division algebra $\mathbf{D}$ of rank 4 over k, such that it splits only at v_1 and v_2, and $\mathbf{D}_{v_i} \cong D,\ i = 1, 2$. Then we can define the reductive group $\mathbf{G}_1$ over k, such that $\mathbf{G}_1(k_v) \cong SO(4, k_v)$, for all $v \notin \{v_1, v_2\}$, and $\mathbf{G}_1(k_{v_i}) \cong G_1(D)$ for $i = 1, 2$. Analogously, we define $\mathbf{G}_2$ over k. Also, we can define $\mathbf{D}_1$–the subgroup of elements of norm 1 in $\mathbf{D}^*$ such that $\mathbf{D}_1(k_v) \cong SL(2, k_v)$ for $v \notin \{v_1, v_2\}$ and $\mathbf{D}_1(k_{v_i}) \cong D_1$, for $i = 1, 2$. We choose any of the summands from the restriction of δ. E.g. $\tau_1 \otimes \delta_1$.
First, assume that $\dim \delta_1 > 1$. Consider set of ideles $((\pm I)_v)$ which can be observed as a subgroup of $\mathbf{D}_1$. Then we can form the character $\omega = \prod \omega_v$ on that set such that $\omega_{v_i} = \omega_{\delta_1},\ i = 1, 2$ and ω_v are almost everywhere trivial. Then we can introduce the space $L(\mathbf{D}_1(A_k))$ as in [**7**], and study the representations of the functions from $C(\mathbf{D}_1(A_k))$ on the space $L(\mathbf{D}_1(A_k))$ (all notations as in [**7**]). We usually denote f from $C(\mathbf{D}_1(A_k))$ as $f = \otimes f_v$. We can choose f in such way that $f_{v_i},\ i = 1, 2$ are coefficients of the representation δ_1, and on all other non-archimedean places f_v are spherical. Then we can adjust the support of f_v on the archimedean places in such way then we can reason analogously to [**7**], Proposition §3.3 and Theorem §4.3. We obtain the existence of the automorphic cuspidal representation $\pi_1' = \otimes_v \pi_{1,v}'$ of the group $\mathbf{D}_1(A_k)$ with the central character ω such that $\pi_{1,v_i}' \cong \delta_1,\ i = 1, 2$. Then, there exists the Grossencharacter $\omega' = \otimes_v \omega_v'$ such that $\omega'_{|((\pm I_v))} = \omega$. Also, we can find automorphic cuspidal representation $\pi' = \otimes_v \pi_v'$ of the group $\mathbf{D}^*(A_k)$ with the central character ω' such that π_1' embeds in π' (the proof is analogous to one in [**8**]). Note that π_{v_i}' are cuspidal representations of D^* of dimension greater then 1, so by Jacquet-Langlands correspondence, they correspond to the cuspidal representations of $GL(2,F)$. This enables us to use results of [**9**] about lifts of representations of $\mathbf{D}^*(A_k)$ to the representations of $GL(2, A_k)$ with one fixed cuspidal place. So, there exists a automorphic cuspidal representation $\pi = \otimes_v \pi_v$ of

$GL(2, A_k)$ such that $\pi_v \cong \pi'_v$, $\forall v \notin \{v_1, v_2\}$, and π_{v_i} and π'_{v_i} correspond. Let $\pi_1 = \otimes_v \pi_{1,v}$ denote some automorphic cuspidal representation of $SL(2, A_k)$ embedded in the representation π. We can accomplish that $\pi_{1,v} \cong \pi'_{1,v}$ for every place v different from v_i, $i = 1, 2$. Indeed, let $\{\phi\}$ be an admissible homomorphism $\{\phi\} : W_{K/k} \to PGL(2) \times W_{K/k}$ defined by representation $\mathrm{Ind}(W_{K/k}, W_{K/E}, \theta)$, where K is some large, but finite Galois extension of k, E a quadratic extension of k contained in K, and θ a Grossencharacter of E that doesn't factor through $Nm_{E/k}$. Let π_2 be some automorphic cuspidal representation of $SL(2, A_k)$ embedded in π. If π_2 does not belong to the L–packet parameterized by $\{\phi\}$, we define the representation π_1 of $SL(2, A_k)$ in the following way:

$$\pi_{1,v} = \pi'_{1,v},\ \forall v \notin \{v_1, v_2\}, \pi_{1,v_i} = \pi_{2,v_i},\ i = 1, 2.$$

The representation π_1 is in the same L–packet as π_2 and it is automorphic by [**14**]. If π_2 corresponds to $\{\phi\}$ as above, we can form π_1 as above, on split places, but on v_1 and v_2 we must adjust representations to obtain the representation which is in the same L–packet as π_2 but is also automorphic. We can do that, because the multiplicity with which π_1 occurs in the space of cusp forms is

$$\frac{1}{[S_\phi^\circ \setminus S_\phi]} \sum_{s \in S_\phi^\circ \setminus S_\phi} < s, \pi_1 > .$$

Notation is as in [**14**]. So, we want to make $< s, \pi_1 >= \prod < s, \pi_{1,v} >$ a trivial character. But we can easily do that fixing on the place v_1 the representation which defines the trivial character on the local group $S_{\phi_{v_1}}^\circ \setminus S_{\phi_{v_1}}$, and on the place v_2 adjust accordingly.
Second, if dim $\delta_1 = 1$, i.e. $\delta_1 = 1$, we fix a non-archimedean place u outside $\{v_1, v_2\}$ and fix some cuspidal representation π_u of $SL(2, k_u)$ on that place. As before, we can choose an automorphic cuspidal representation π'_1 of $\mathbf{D}_1$ which has that component on the place u, and which is unramified on the places v_1 and v_2, i.e. equal to $\delta_1 = 1$. Now there exists a lift from the automorphic cuspidal representations of $\mathbf{D}^*$ to such representations of $GL(2, A_k)$, with fixed place u with cuspidal component, and, as before, we obtain the representation $SL(2, A_k)$ which has properties as in the previous case.
Note that if the finite set $\{g_i = \left[\begin{smallmatrix} 1 & 0 \\ 0 & x_i \end{smallmatrix}\right]\}$ is the set of the representatives of $GL(2, F)/SL(2, F)F^*$, then the set $g'_i = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x_i & 0 & 0 \\ 0 & 0 & x_i^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$ is the set of the representatives (in our realization) of $SO(4, F)/SL(2, F)SL(2, F)$. Now, with the same character ω as before, we can find an automorphic cuspidal representation $\epsilon = \otimes_v \epsilon_v$ of $SL(2, A_k)$ with the central character ω such that

(i) If $\pi_{1,v_1} \cong \pi_{1,v_2}$ then $\epsilon_{v_1} \cong \epsilon_{v_2} \cong \tau_1$, and

(ii) If $\pi_{1,v_2} \cong {}^{g_i}\pi_{1,v_1}$ then $\epsilon_{v_1} \cong \tau_1$ and $\epsilon_{v_2} \cong {}^{g_i}\tau_1$

There exists an automorphic cuspidal representation σ' of the group $\mathbf{G}_1(A_k)$ such that $\epsilon \otimes \pi_1'$ embeds in it as a representation of the group $SL(2, A_k)\mathbf{D}_1(A_k)$. Analogously, there exists an automorphic cuspidal representation σ of the group $SO(4, A_k)$ such that $\epsilon \otimes \pi_1$ embeds in it as a representation of $SL(2, A_k)SL(2, A_k)$. We can accomplish

(i) $\sigma'_{v_1} \cong \sigma'_{v_2} \cong \delta$
(ii) $\sigma'_v \cong \sigma_v$, $\forall v \notin \{v_1, v_2\}$

Let $\gamma' = \otimes_v \gamma'_v$ be an automorphic cuspidal representation of the $\mathbf{D}^*(A_k)$ such that $\gamma'_{v_1} \cong \gamma'_{v_2} \cong \tau$ and γ its lift to $GL(2; A_k)$ such that $\gamma'_v \cong \gamma_v$, $\forall v \notin \{v_1, v_2\}$ and $\gamma_{v_1} \cong \gamma_{v_2} \cong \tau'$, where τ' corresponds to τ by Jacquet-Langlands correspondence (This can be arranged by [**9**]).
We have to normalize measures on the unipotent radicals of the groups considered in order to get the global functional equation right. We can decompose $D = F \oplus D^-$, where we can look at the center F of the algebra D as a τ–hermitian part of D, and D^- as τ–anti–hermitian part. Now, the unipotent radical of the non-Siegel parabolic subgroup in the group $G_2(D)$ in the case $\epsilon = -1$ (the case we are now considering) is $N(F) \cong D \oplus D \oplus F$, and in the case $\epsilon = 1$ $N'(F) \cong D \oplus D \oplus D^-$. Let ψ_F denotes a nontrivial additive character of F. Introduce self-dual measures on $N(F)$ and $N'(F)$ by the use of the F-form $< x, y >= \sum_{i=1}^5 x_i y_i + \tau(x_i y_i)$ on the D^5, and the character ψ_F, so that the self-dual measure on D^5 is product of a self-dual measure α_F on $N(F)$ and α'_F on $N'(F)$. Fix a nontrivial character $\psi = \otimes_v \psi_v$ of A_k trivial on k, such that $\psi_{v_1} = \psi_{v_2} = \psi_F$. Analogously as above, on the each split place we can get a self-dual measure α_v on $N(k_v)$ with respect to ψ_v, and a self dual measure α on $N(A_k)$. In this way, we get a coherent family of measures $\{\alpha_v\}$, such that $\alpha = \prod \alpha_v$, and α is actually the Tamagawa measure [**33**, §$VII.2$, Corollary 1.] meaning that $\alpha(N(A_k)/N(k)) = 1$. The Plancherel measure is defined analogously as in the Siegel case.

Proposition 4.1. $\mu(s, \tau \otimes \delta)^2 = \mu(s, \tau' \otimes \sigma_{v_1})\mu(s, \tau' \otimes \sigma_{v_2})$.

Proof. Denote by S a finite set of places containing v_1 and v_2, all the places of the residual characteristic 2, and all the places where γ_v, σ_v, and ψ_v are ramified. For every $v \notin S$, let $f_{v,s}$ denote the unique unramified vector in $\gamma'_v \nu^s \rtimes \sigma'_v$, normalized to be equal to 1 on the maximal compact subgroup K_v. Analogously $\overline{f}_{v,s}$ in $\gamma'_v \nu^s \underline{\rtimes} \sigma'_v$. In the L–group $SO(8, \mathbb{C})$ of $SO(8, k_v)$, the action (representation) of the $GL(2, \mathbb{C}) \times SO(4, \mathbb{C})$ on the unipotent radical is equal to $\Lambda^2(\mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^4)$. So, for $v \notin S$ we can explicitly calculate constants $c(v, s, \gamma'_v \otimes \sigma'_v)$ in terms of L–functions, where

$$R(s, \gamma'_v \otimes \sigma'_v, N(k_v), \overline{N}(k_v))f_{v,s} = c(v, s, \gamma'_v \otimes \sigma'_v)\overline{f}_{v,s}.$$

We then have

$$c(v,s,\gamma'_v\otimes\sigma'_v)=\frac{L(s,\gamma'_v\otimes\sigma'_v,\rho_2\otimes\rho_4)(1-\chi_{\gamma'_v}(\overline{\omega})q_v^{-s})^{-1}}{L(1+s,\gamma'_v\otimes\sigma'_v,\rho_2\otimes\rho_4))(1-\chi_{\gamma'_v}(\overline{\omega})q_v^{-s-1})^{-1}}.$$

It is easily seen that the product

$$c_S(s,\gamma'\otimes\sigma')=\prod_{v\notin S}c(v,s,\gamma'_v\otimes\sigma'_v)$$

converges in some right half-plane, and it continues to the meromorphic function on $\mathbb{C}$. Analogously we have

$$R(s,\gamma'_v\otimes\sigma'_v,\overline{N}(k_v),N(k_v))\overline{f}_{v,s}=c(v,-s,\widetilde{\gamma'_v}\otimes\sigma'_v)f_{v,s}.$$

We now take $f_s=\otimes_v f_{v,s}\in\gamma'\nu^s\rtimes\sigma'$, where, for each $v\notin S$ we have chosen spherical $f_{v,s}$ as above. Because we have chosen the Tamagawa measure on the (global) unipotent radical we have the global functional equation (Theorem $\S IV.1.10$ from [**16**])

$$R(s,\gamma'\otimes\sigma',\overline{N}(A_k),N(A_k))R(s,\gamma'\otimes\sigma',N(A_k),\overline{N}(A_k))f_s=f_s.$$

When the right-hand side of the above global equation is written as product of local intertwining operators, the local Plancherel measures appear. So we have

$$\prod_{v\in S}\mu(s,\gamma'_v\otimes\sigma'_v)c_S(s,\gamma'\otimes\sigma')c_S(-s,\widetilde{\gamma'}\otimes\sigma')=1.$$

Analogously to the previous equation for $\mathbf{G}_2(A_k)$, we have equation

$$\prod_{v\in S}\mu(s,\gamma_v\otimes\sigma_v)c_S(s,\gamma\otimes\sigma)c_S(-s,\widetilde{\gamma}\otimes\sigma)=1$$

in $SO(8,A_k)$. But on each split place we have an isomorphism $\mathbf{G}_2(k_v)\cong SO(8,k_v)$ which preserves unipotent radicals, and we have isomorphic representations, so we have equality of the Plancherel measures. From this follows

$$\mu(s,\tau\otimes\delta)^2=\mu(s,\tau'\otimes\sigma_{v_1})\mu(s,\tau'\otimes\sigma_{v_2}).$$

□

Remark. Because of our adjustment of the representation ϵ, we conclude that the representations σ_{v_1} and σ_{v_2} differ in the quadratic character, trivial on the $SL(2,F)SL(2,F)$.

Let us now compute the above Plancherel measure. Because τ_1 is generic representation for some nontrivial character of F, and π_{v_1} is generic representation for some nontrivial character, we can conclude that σ_{v_1} is generic representation of the group $SO(4,F)$. We can now use the results of Shahidi

[**23**] and express the Plancherel measure in terms of γ–factors. We fix a non-trivial additive character ψ of F and obtain, up to the exponential factor,

$$\mu(s,\tau'\otimes\sigma_{v_1})\approx\frac{\gamma(2s,\tau',\Lambda^2\rho_2,\psi)\gamma(s,\tau'\times\sigma_{v_1})}{\gamma(1+2s,\tau',\Lambda^2\rho_2,\psi)\gamma(1+s,\tau'\times\sigma_{v_1})}.$$

Only difficulty appears in the calculation of the Rankin-Selberg γ–factor of the groups $GL(2,F)\times SO(4,F)$. If σ_{v_1} is non–cuspidal, the computation is straightforward, using the multiplicativity of γ–factors.([**22**]). So, assume that the representations σ_{v_i} appearing in the previous proposition are square integrable, non-cuspidal representations. This is the case when, with the previous notations, $\delta_1=1$. Let π denote the cuspidal unitary representation of $GL(2,F)$ with the trivial central character such that $\tau_1\hookrightarrow\pi|_{SL(2,F)}$. Then π is a self-contragredient representation. Such π's differ mutually by a quadratic character. We denote the basis of the root system for $SO(4,F)$ by $\{\alpha,\beta\}$. Now, the standard Levi subgroup M_α is diagonally embedded in $SO(4,F)$ and contains the diagonal version of $SL(2,F)$, and M_β isn't diagonal and also contains the other copy of $SL(2,F)$. Let us consider the representation $\mathrm{Ind}_{M_\beta}^{SO(4,F)}\pi\nu^{\frac12}$. It is easy to see that this representation restricted on $SL(2,F)SL(2,F)$ decomposes as

$$\mathrm{Ind}_{M_\beta}^{SO(4,F)}\pi\nu^{\frac12}|_{SL(2,F)SL(2,F)}=\sum\nu\rtimes 1\otimes\tau_i,$$

where τ_i's are components of the restriction of π on $SL(2,F)$. Let δ' denote the unique square integrable subquotient of $\mathrm{Ind}_{M_\beta}^{SO(4,F)}\pi\nu^{\frac12}$. Then $\mathrm{St}_{SL(2,F)}\otimes\tau_1\hookrightarrow\delta'$, so σ_{v_i} differ from δ' by a quadratic character. We can conclude

Corollary 4.2. *Assume that the representations σ_{v_i} are not cuspidal, and $\sigma_{v_1}\hookrightarrow\mathrm{Ind}_{P_\beta}^{SO(4,F)}\pi\nu^{\frac12}$.*

(i) *If $\dim\tau>1$ then*
   (a) *If $\chi_{\tau'}(\overline{\omega})\neq 1$ the representation $\tau\nu^s\rtimes\delta$ reduces only for $s=0$.*
   (b) *If $\chi_{\tau'}(\overline{\omega})=1$ then $\tau\rtimes\delta$ is irreducible, and reduces at $s=\frac12$ or at $s=\frac32$. It reduces at $s=\frac12$ if $\tau'\ncong\pi$ and at $s=\frac32$ if $\tau'\cong\pi$.*

(ii) *If $\tau=\chi$ is a quadratic character, the representation $\chi\nu^s\rtimes\delta$ reduces only at $s=\frac12$.*

Proof. In the formula for Plancherel measure we include the expression for the Rankin-Selberg factor

$$\gamma(s,\tau'\times\sigma_{v_1},\psi)=\gamma(s,\tau'\times\pi\nu^{\frac12},\psi)\gamma(s,\tau'\times\pi\nu^{-\frac12},\psi).$$

If τ' is cuspidal, we obtain the claim, and if $\tau'=\chi\mathrm{St}_{GL(2,F)}$ we have

$$\gamma(s,\chi\mathrm{St}_{GL(2,F)}\times\pi\nu^{\frac12},\psi)=\gamma(s,\chi\nu^{\frac12}\times\pi\nu^{\frac12},\psi)\gamma(s,\chi\nu^{-\frac12}\times\pi\nu^{\frac12},\psi)=1.$$

$\square$

Let us denote by $^\epsilon$ conjugation in $SO(4,F)$ by an element

$$\epsilon = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

of $O(4,F)\backslash SO(4,F)$, and, accordingly, if π is the representation of $SO(4,F)$ we denote the representation of $SO(4,F)$ obtained using involution $^\epsilon$ by $^\epsilon\pi$. Keeping the assumptions from the proposition, in the case of the cuspidal σ_{v_i}'s we have

Corollary 4.3. (i) *If $\tau = \chi$ is a quadratic character of D^* we have*

(a) *If $^\epsilon\sigma_{v_1} \ncong \sigma_{v_1}$ the representation $\chi\nu^s \rtimes \delta$ reduces only at $s = \frac{1}{2}$.*

(b) *If $^\epsilon\sigma_{v_1} \cong \sigma_{v_1}$ (then also $^\epsilon\sigma_{v_2} \cong \sigma_{v_2}$ holds) then if at least one of the representations $\chi \rtimes \widetilde{\sigma_{v_i}}$, $i = 1,2$ of the group $SO(6,F)$ is irreducible, $\chi\nu^s \rtimes \delta$ reduces (only) for $s = \frac{3}{2}$. On the other hand, if both of the representations $\chi \rtimes \widetilde{\sigma_{v_i}}$, $i = 1,2$ reduce, $\chi\nu^s \rtimes \delta$ reduces for $s = \frac{1}{2}$.*

(ii) *If $\dim\tau > 1$ then*

(a) *If $\chi_\tau(\overline{\omega}) = 1$ the representation $\tau\nu^{\frac{1}{2}} \rtimes \delta$ reduces*

(b) *If $\chi_\tau(\overline{\omega}) = -1$ then the points of reducibility are at $s = 0$ or $s = 1$. We have reducibility at $s = 1$ if at least one of the representations $\tau' \rtimes \sigma_{v_i}$ of the group $SO(8,F)$ is irreducible.*

Proof. Now we prove (i). Using the multiplicativity of γ–factors, we come to γ–factors $\gamma(s, \chi\nu^{\frac{1}{2}} \times \sigma_{v_i})$, which leads us to consider the reducibilities of the representations $\chi\nu^s \rtimes \sigma_{v_i}$ of the group $SO(6,F)$. It is well known that in order to have a reducibility for some real number s, the non-trivial element of the Weyl group has to fix the representation $\chi \otimes \sigma_{v_i}$, which amounts to the statement that $^\epsilon\sigma_{v_1} \cong \sigma_{v_1}$. If it isn't so, the mentioned L–functions and γ–factors appearing in the case of $SO(6,F)$ are trivial, so the only pole of the Plancherel measure comes from Hecke L–function for the τ'. If $^\epsilon\sigma_{v_1} \cong \sigma_{v_1}$, the irreducibility of $\chi \rtimes \widetilde{\sigma_{v_1}}$ implies that the $L(s, \chi \times \sigma_{v_1})$ has a pole at $s = 0$. This implies that $\mu(s, \tau \otimes \delta)$ has a pole at $s = \frac{3}{2}$, no matter the situation with $\chi \rtimes \widetilde{\sigma_{v_2}}$. On the other hand, if both of the representations $\chi \rtimes \widetilde{\sigma_{v_i}}$ reduce, the Rankin-Selberg L–functions are holomorphic for real s and the only poles of $\mu(s, \tau \otimes \delta)$ come from Hecke L–function for $\chi\mathrm{St}_{GL(2,F)}$.
The proof of (ii) follows from the results from [**10**, Theorem 4.8.]. $\square$

5. Unitary dual of the group $G_2(D)$

We are interested in finding the hermitian, and especially irreducible unitarizable representations of $G_2(D)$. We are list them by grouping together the ones with the same cuspidal support.

5.1. Unitary subquotients of the principal series. Let χ_1 and χ_2 denote unitary characters of D^*. Denote $\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1$. So, we can assume that $s_1 \geq s_2 \geq 0$.

Proposition 5.1. *Assume that we have unitary characters χ_1 and χ_2 such that $\chi_1^2 \neq 1$, and $\chi_2^2 \neq 1$.*

1) *If $\chi_1 \neq \chi_2^{\pm 1}$ then the representation π has a hermitian subquotient if and only if $s_1 = s_2 = 0$ and then it is an irreducible tempered representation.*
2) *If $\chi_1 = \chi_2$ then the representationπ has a hermitian subquotient if and only if $s_1 = s_2 = 0$ and then it is an irreducible tempered representation.*
3) *If $\chi_1 = \chi_2^{-1}$ then if $s_2 = 0$ the representation π has a hermitian subquotient only if $s_1 = 0$ and we obtain an irreducible tempered representation (isomorphic to one obtained in the previous case for the same χ_1).*
   If $s_2 > 0$ the representation π has a hermitian subquotient only if $s_1 = s_2$; then for all $s_1 > 0$ all the subquotients of the representation π are hermitian. For $s_1 \in (0,1)$ we have $\pi = \chi_1\nu^{s_1} \times \chi_1^{-1}\nu^{s_1} \rtimes 1 = L(\chi_1\nu^{s_1}, \chi_1^{-1}\nu^{s_1}; 1)$ and it is an unitary representation. For $s_1 > 1$ it is not an unitary representation. For $s_1 = 1$ we have

   $$\pi = \chi_1\delta(\nu, \nu^{-1}) \rtimes 1 + L(\chi_1\nu, \chi_1^{-1}\nu; 1),$$

   where the first subquotient is a tempered representation, and the other is unitary (non-tempered).

Proof. The first two cases follow from the criterion of the hermiticity of the Langlands quotient. For the third case we observe that $\chi_1\nu^{s_1} \times \chi_1\nu^{-s_1}$ is the complementary series of the group $GL(2, D)$ for $\alpha \in (0,1)$. From this follows that $\chi_1\nu^{s_1} \times \chi_1\nu^{-s_1} \rtimes 1$ has all the unitarizable subquotients for $s_1 \in (0,1]$. □

Proposition 5.2. *Let χ_1 and χ_2 be unitary characters, such that $\chi_1^2 = 1$ and $\chi_2^2 \neq 1$. Again denote $\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1$. The representation π has a hermitian subquotient only if $s_2 = 0$; then all of its subquotients are hermitian representations. In this case, for $s_1 = 0$ π is irreducible tempered representation, for $s_1 \in (0, \frac{1}{2}]$ $L(\chi_1\nu^{s_1}; \chi_2 \rtimes 1)$ is unitarizable, and for $s_1 > \frac{1}{2}$ the representation π is irreducible and non-unitarizable.*

Proof. Let us just comment the case $s_2 = 0$. We have the standard intertwining operators $A_{w_{2\alpha+\beta}}(s_1) : \chi_1\nu^{s_1} \times \chi_2 \rtimes 1 \to \chi_1\nu^{-s_1} \times \chi_2 \rtimes 1$ which converge for $s_1 > 0$. These operators define, for $s_1 > \frac{1}{2}$, a continuous family, indexed by s_1, of the nondegenerate hermitian forms on the compact picture

X of the representation $\chi_1 \times \chi_2 \rtimes 1$ by the means of

$$(f_1, f_2)_{s_1} = \int_K < f_{1,s_1}(k), A_{w_{2\alpha+\beta}}(s_1) f_{2,s_1}(k) > \mathrm{d}k.$$

Here, f_1 and f_2 belong to X, and f_{i,s_1} denote theirs holomorphic sections that are identified with elements from $\chi_1\nu^{s_1} \times \chi_2 \rtimes 1$. If one of this forms would be unitarizable, meaning that the irreducible representation $\chi_1\nu^{s_1} \times \chi_2 \rtimes 1$ is unitarizable, all the other forms would have to be unitarizable, too, because of the connectedness of the indexing set. But for $s_1 > \frac{5}{2}$ the representation $\chi_1\nu^{s_1} \times \chi_2 \rtimes 1$ has unbounded matrix coefficients, which implies non–unitarizability. The operator $A_{w_{2\alpha+\beta}}(s_1)$ has a pole at $s_1 = 0$, but we can normalize it by multiplying it by an appropriate real polynomial; we obtain, for $s_1 \in [0, \frac{1}{2})$ a family of (normalized) intertwining operators $A'_{w_{2\alpha+\beta}}(s_1)$ which also define a continuous family of non-degenerate hermitian forms on X. By the same argumentation, we obtain the unitarizability of the representations considered for $s_1 \in [0, \frac{1}{2})$. In this way, we obtained the complementary series representations, and, by the results of [**15**] the subquotients of the representation on the edge of the complementary series ($s_1 = \frac{1}{2}$) are unitarizable.

□

Proposition 5.3. *With the notations as before assume now that $\chi_1^2 \neq 1$ and $\chi_2^2 = 1$. Then the representation π has a hermitian subquotient only if $s_1 = s_2 = 0$, and π is then irreducible and tempered. This representation is already described in the previous proposition.*

Proof. The proof is left to the reader. □

Proposition 5.4. *Let us assume now that we have unitary characters χ_1 and χ_2 such that $\chi_1^2 = \chi_2^2 = 1$ and $\chi_1 \neq \chi_2$. Let us denote $\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1$. With the Figure 1 bellow we have*

(i) *The representation π for (s_1, s_2) from the closed region I has, for all its subquotients, unitarizable representations. The composition series are given in the Proposition* 2.3.
(ii) *For (s_1, s_2) from each of the open regions II, III, IV we obtain representations with the subquotients which are all hermitian, but none unitarizable.*

Proof. For (s_1, s_2) with $s_1 \geq s_2 > 0$ from the open region I we have the family of the standard long intertwining operators $A(s_1, s_2) : \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1 \to \chi_1\nu^{-s_1} \times \chi_2\nu^{-s_2} \rtimes 1$ which defines a continuous family of nondegenerate hermitian forms on the compact picture. So, we can fix such (s_1, s_2) and form one-parameter family $t \to (ts_1, ts_2)$ for $t \geq 0$. Because $\chi_1 \times \chi_2 \rtimes 1$ is irreducible, we can assume that for $t \geq 0$ (until $\chi_1\nu^{ts_1} \times \chi_2\nu^{ts_2} \rtimes 1$ reduces) we have unitarizable representations. Analogous reasoning and unboundedness

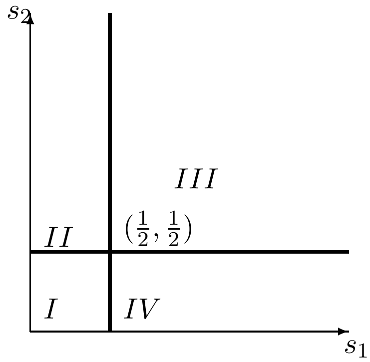

Figure 1

of matrix coefficients insures the non-unitarizability of the representations on the unbounded regions. We, of course, could apply immediately [**27**] to conclude that. □

Proposition 5.5. *Assume that we have an quadratic character χ_1 of D^*. We denote $\pi = \chi_1\nu^{s_1} \times \chi_1\nu^{s_2} \rtimes 1$. All of its subquotients are hermitian representations. With the notations as in the Figure 2 below, we have*

(i) *Considering the open regions from the figure, only for (s_1, s_2) which belongs to (open) regions I and III we have that π is unitarizable (irreducible) non-tempered representation. Now we consider the boundaries*
(ii) *Besides the square integrable and tempered subquotients, other unitarizable subquotients which appear for (s_1, s_2) which lay on the boundaries of the regions are all subquotients of π for (s_1, s_2) from the boundary of the regions I and III and $L(\chi_1\nu^{\frac{5}{2}}, \chi_1\nu^{\frac{1}{2}}; 1)$.*

Proof. As in the previous proposition, we obtain the unitarizability of the region I on the Figure 2. Consider $\chi_1\nu^{s_1} \times \chi_1\nu^{-s_1}$ in $GL(2, D)$ for $s_1 \in (0, 1)$. These (irreducible) representations belong to the complementary series of $GL(2, D)$ so all the subquotients of the representation $\chi_1\nu^{s_1} \times \chi_1\nu^{s_1} \rtimes 1$ are unitarizable. In the open region III this gives unitarizability of some representations, but then follows the unitarizability on all the region. This gives, by [**15**] the unitarizability of all subquotients for π from the boundaries of I and III. Consider now the representation $\chi_1\nu^{\frac{5}{2}} \times \chi_1\nu^{\frac{1}{2}} \rtimes 1$. If we prove

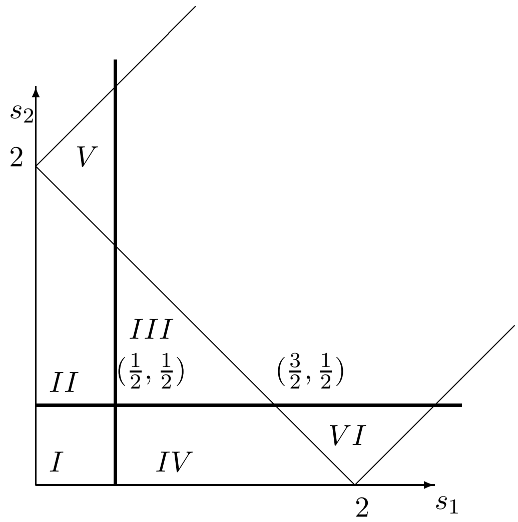

Figure 2

that it has some non-unitarizable subquotients, this would imply the non-unitarizability on the open region VI (and, simetrically, on V). We have

$$\chi_1\nu^{\frac{5}{2}} \times \chi_1\nu^{\frac{1}{2}} \rtimes 1 = L(\chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1}); 1) + \pi_1 \\ + L(\chi_1\nu^{\frac{5}{2}}; \delta[\chi_1\nu^{\frac{1}{2}}; 1]) + L(\chi_1\nu^{\frac{5}{2}}, \chi_1\nu^{\frac{1}{2}}; 1).$$

The unitarizability of $L(\chi_1\nu^{\frac{5}{2}}, \chi_1\nu^{\frac{1}{2}}; 1)$ is proved using global methods ([**11**]). In the case $\chi_1 = 1$, $L(\chi_1\nu^{\frac{5}{2}}, \chi_1\nu^{\frac{1}{2}}; 1)$ is a trivial character, so we know that the only unitarizable subquotients of $\nu^{\frac{5}{2}} \times \nu^{\frac{1}{2}} \rtimes 1$ are the Steinberg and the trivial representation([**6**]). In general, the non-unitarizability of $L(\chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1}); 1)$ and $L(\chi_1\nu^{\frac{5}{2}}; \delta[\chi_1\nu^{\frac{1}{2}}; 1])$ can be proved using Howe-Moore theorem ([**4**]). Denote $L(\chi_1\nu^{\frac{3}{2}}\delta(\nu,\nu^{-1}); 1)$ by π. Consider the unbounded set
$S = \{a_0 \in A_0 : |\alpha_1(a_0)|_F \leq 1,\ |\alpha_2(a_0)| = 1\}$. We have

$$s_{(1,1)}(\pi) = \chi_1\nu^{\frac{1}{2}} \otimes \chi_1\nu^{\frac{5}{2}} + \chi_1\nu^{\frac{1}{2}} \otimes \chi_1\nu^{-\frac{5}{2}} + \chi_1\nu^{-\frac{5}{2}} \otimes \chi_1\nu^{\frac{1}{2}}.$$

Let v and $\widetilde{v}$ be the canonical lifts of the vectors in Jacquet modules corresponding to the last summand in the above sum. Then, there exists $\epsilon > 0$ such that for every a_0 from $A_0(\epsilon)$ we have

$$< \pi(a_0)v, \widetilde{v} >= \delta_{P_0}^{\frac{1}{2}}(a_0)(\chi_1\nu^{-\frac{5}{2}} \otimes \chi_1\nu^{\frac{1}{2}})(a_0) < j_{P_0}(v), \widetilde{j}_{P_0}(\widetilde{v}) > .$$

Here, $A_0(\epsilon) = \{a_0 \in A_0 : |\alpha_1(a_0)|_F \leq \epsilon,\ |\alpha_2(a_0)|_F \leq \epsilon\}$. The vectors $j_{P_0}(v)$ and $\widetilde{j}_{P_0}(\widetilde{v})$ denote the projection on the corresponding Jacquet module. We fix an element a_0 from $A_0(\epsilon)$. Then $a_0 S$ is the subset of $A_0(\epsilon)$. So, we can find un unbounded sequence of elements in $a_0 S$ which defines a sequence of the matrix coefficients of π that does not vanish at the infinity. By the Howe-Moore theorem π is not uniformly bounded representation, hence is not unitarizable. Quite analogously we prove non-unitarizability of $L(\chi_1\nu^{\frac{5}{2}}; \delta[\chi_1\nu^{\frac{1}{2}}; 1])$. Consider now the representation

$$\chi_1\nu^2 \times \chi_1 \rtimes 1 = L(\chi_1\nu\delta(\nu, \nu^{-1}); 1) + L(\chi_1\nu^2, \chi_1 \rtimes 1).$$

We prove that both subquotients on the right-hand side are non-unitarizable. We have a holomorphic family of the irreducible hermitian representations $\chi_1\nu^s\delta(\nu, \nu^{-1}) \rtimes 1$ for $s \in [1, \frac{3}{2})$, because we have a holomorphic family of non-degenerate hermitian forms obtained by standard intertwining operators on the compact picture of the representation $\chi_1\delta(\nu, \nu^{-1}) \rtimes 1$. If we assume unitarizability at $s = 1$, this would imply unitarizability on the whole interval, and unitarizability of all the subquotients on the edge of the interval, at $s = \frac{3}{2}$, which is false by above reasoning. Analogously, the unitarizability of $L(\chi_1\nu^2; \chi_1 \rtimes 1) = \chi_1\nu L(\nu, \nu^{-1}) \rtimes 1$ would imply the unitarizability of all the subquotients of $\chi_1\nu^{\frac{3}{2}} L(\nu, \nu^{-1}) \rtimes 1$, but we have proved that this is not the case. By this, we proved non-unitarizability on the region IV and on the remaining boundaries. $\square$

We continue with the examination of the principal series representations induced by the higher–dimensional representations of D^*. Let τ_1 and τ_2 be the unitarizable representations of D^* of the dimension greater then 1. Let us denote $\pi = \tau_1\nu^{s_1} \times \tau_2\nu^{s_2} \rtimes 1$. We can assume that $s_1 \geq s_2 \geq 0$. The following four propositions are completely analogous to the first four propositions in the previous subsection, so we just note them.

Proposition 5.6. *With the notation as above, assume that $\tau_1 \ncong \widetilde{\tau_1}$ and $\tau_2 \ncong \widetilde{\tau_2}$. Then*

1) *If $\tau_1 \ncong \tau_2$ and $\tau_1 \ncong \widetilde{\tau_2}$, then the representation π has a hermitian subquotient if and only if $s_1 = s_2 = 0$, and then π is an irreducible tempered representation.*
2) *If $\tau_1 \cong \tau_2$ then π has a hermitian subquotient if and only if $s_1 = s_2 = 0$, and then π is an irreducible tempered representation.*
3) *If $\tau_1 \cong \widetilde{\tau_2}$ the representation π has a hermitian subquotient if and only if $s_1 = s_2$. In that case all of its subquotients are hermitian. For $s_1 \in \left(0, \frac{1}{2}\right)$ the representation $L(\tau_1\nu^{s_1}, \widetilde{\tau_1}\nu^{s_1}; 1)$ is unitarizable, for $s_1 > \frac{1}{2}$ non-unitarizable. We get also tempered subquotients for $s_1 \in \left\{0, \frac{1}{2}\right\}$.*

Proposition 5.7. *Assume that $\tau_1 \cong \widetilde{\tau_1}$ and $\tau_2 \ncong \widetilde{\tau_2}$. Then the representation π has a hermitian subquotient only if $s_2 = 0$. In that case, if $\omega_{\tau_1} = 1$, π has a tempered subquotient only if $s_1 = \frac{1}{2}$, and if $s_1 = 0$. On the other hand, $L(\tau_1\nu^{s_1}, \tau_2 \rtimes 1)$ is unitarizable for $s_1 \in \left(0, \frac{1}{2}\right)$, and for $s_1 > \frac{1}{2}$ it is a hermitian non-unitarizable representation; if $\omega_{\tau_1} \neq 1$ π has a tempered subquotient only if $s_1 = 0$, in that case π is a sum of two nonequivalent tempered representations. Then $L(\tau_1\nu^{s_1}, \tau_2 \rtimes 1)$ is hermitian non-unitarizable representation for every positive s_1.*

Proposition 5.8. *Assume that $\tau_1 \ncong \widetilde{\tau_1}$ and $\tau_2 \cong \widetilde{\tau_2}$. Then π has a hermitian subquotient only if $s_1 = s_2 = 0$. Then π is an irreducible tempered representation, or sum of two nonequivalent tempered representations, depending on the central character of τ_1. These tempered representations were described in the previous proposition.*

Proposition 5.9. *Assume that τ_1 and τ_2 are mutually non-isomorphic unitary, selfcontragredient representations. We keep our assumption $s_1 \geq s_2 \geq 0$. Then all the subquotients of the representation π are hermitian and*

1) *If $\omega_{\tau_1} = \omega_{\tau_2} = 1$, we have the same situation as on the Figure 1, π has a unitarizable subquotients only inside closed region I, the tempered (non-square integrable) subquotients appear for $(s_1, s_2) \in \left\{(\frac{1}{2}, 0), (0, 0)\right\}$, and the square integrable representation appears for $(s_1, s_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$.*
2) *If $\omega_{\tau_1} = 1$ and $\omega_{\tau_2} \neq 1$ π has unitarizable subquotients only for $s_2 = 0$ and $s_1 \in [0, \frac{1}{2}]$. Tempered (non-square-integrable) representations appear for $(s_1, s_2) \in \left\{(\frac{1}{2}, 0), (0, 0)\right\}$.*
3) *If $\chi_{\tau_1} \neq 1$ and $\chi_{\tau_2} \neq 1$ π is unitarizable only for $s_1 = s_2 = 0$ and π (as we have already seen) is a sum of the 4 non-equivalent tempered representations.*

Proposition 5.10. *Assume that $\tau_1 \cong \tau_2$ and τ_1 is selfcontragredient, such that $\omega_{\tau_1} \neq 1$. Then all the irreducible subquotients of the representation π are hermitian, and the unitarizable subquotients appear only for (s_1, s_2) from the closed region I on the Figure 3 (when all of them are unitarizable). The square integrable subquotients appear for $(s_1, s_2) = (1, 0)$ and the tempered (non square integrable) subquotients appear for $(s_1, s_2) = (0, 0)$.*

Proposition 5.11. *We keep all the assumptions of the previous proposition, except now $\omega_{\tau_1} = 1$. On the Figure 4, considering the open regions, we have unitarizability only on the region I, where we have non tempered representations. On the boundaries, we have a square integrable subquotient for $(s_1, s_2) = \left(\frac{3}{2}, \frac{1}{2}\right)$, and tempered subquotients for $(s_1, s_2) \in \left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0, 0)\right\}$. We have other unitary subquotients on the boundary of the region I, and $L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1)$ is an unitarizable subquotient.*

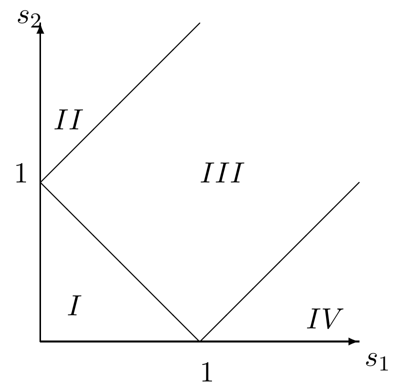

Figure 3

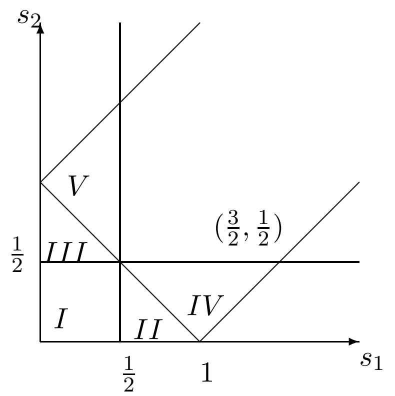

Figure 4

Proof. Let us consider the representation

$$\tau_1\nu^{\frac{3}{2}} \times \tau_1\nu^{\frac{1}{2}} \rtimes 1 =$$
$$L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + \pi_4 + L(\tau\nu^{\frac{3}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1).$$

The unitarizability of the representation $L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1)$ is proved by global methods ([**11**]). This is an isolated unitary representation in the unitary dual. We will prove the non-unitarizability of $L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1)$ and of $L(\tau\nu^{\frac{3}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1])$. We will do that in the following way: we will calculate the Plancherel measure $\mu(s, \delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}))$. Let us again denote by $A(s)$ standard intertwining operator such that

$$A(s) : \delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}})\nu^s \rtimes 1 \to \delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}})\nu^{-s} \rtimes 1.$$

We will prove that the Plancherel measure has a simple pole at $s = 1$ and that $A(s)$ has no pole at $s = -1$. We will apply these observations on the calculation of Jantzen filtrations near $s = 1$. This will give us non-unitarizability. Let us denote $\delta = \delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}})$. First, we use the previously mentioned result which states $\mu(s, \delta) = \mu(s, \delta')$. The representation δ' is generic, so we can apply the results from [**23**] to compute the Plancherel measure in terms of γ–factors. Up to an exponential factor, we have

$$\mu(\frac{s}{2}, \delta') \approx \frac{\gamma(s, \delta', \Lambda^2\rho_4, \psi)}{\gamma(1+s, \delta', \Lambda^2\rho_4, \psi)},$$

where ψ is nontrivial additive character of F, given in advance. By multiplicativity of γ–factors ([**22**]) we have

$$\gamma(s, \delta', \Lambda^2\rho_4, \psi) = \gamma(s, \tau_1\nu^{\frac{1}{2}}, \det, \psi)\gamma(s, \tau_1\nu^{-\frac{1}{2}}, \det, \psi)\gamma(s, \tau_1\nu^{\frac{1}{2}} \times \tau_1\nu^{\frac{1}{2}}, \psi).$$

The last is Rankin-Selberg γ–factor. When γ factor is expressed in terms of L–function, we obtain, up to exponential factor,

$$\mu(\frac{s}{2}, \delta') \approx \frac{(1-q^{-1-s})(1-q^{1-s})(1-q^{1+s})(1-q^{-1+s})(1-q^{-rs})(1-q^{rs})}{(1-q^{s})(1-q^{-s})(1-q^{-2+s})(1-q^{-2-s})(1-q^{-r+rs})(1-q^{-r-rs})}.$$

In the above equality r is a natural number satisfying $L(s, \tau_1 \times \tau_1) = (1-q^{rs})^{-1}$. Indeed $\mu(s, \delta')$ has a simple pole at $s = 1$. Denote by w_ϵ the reflection in the Weyl group with respect to the root ϵ. Then consider the intertwining operator

$$A_{w_{\alpha+\beta}}(s) : \tau_1\nu^{s+\frac{1}{2}} \times \tau_1\nu^{s-\frac{1}{2}} \rtimes 1 \to \tau_1\nu^{-s+\frac{1}{2}} \times \tau_1\nu^{-s-\frac{1}{2}} \rtimes 1.$$

The poles of the operator $A(s)$ are among the poles of the operator $A_{w_{\alpha+\beta}}(s)$ and $A_{w_{\alpha+\beta}}(s)|_{\delta\nu^s \rtimes 1} = A(s)$. But using the factorization of the operator $A_{w_{\alpha+\beta}}(s)$ ([**21**]), we see that it has no poles at $s = -1$. Let X denotes the compact picture of the representation $\delta\nu^s \rtimes 1$. We will consider the Jantzen filtrations of the space X, for $s \in [0, 1]$. For $s \in (0, 1)$ the representations $\delta\nu^s \rtimes 1$ are irreducible, and the mentioned interval parameterizes a nondegenerate family of hermitian forms on the compact picture X. For s=0 $A(s)$ is holomorphic, and, normalized, generates the intertwining algebra of the representation $\delta \rtimes 1 = T_1 + T_2$ (The proposition 2.4). The operator $A(0)$ endows the space of this representation with the hermitian form which is of a different sign on each of T_i's. This gives us the non-unitarizability of $\delta\nu^s \rtimes 1$ for $s \in (0,1)$. By the theory of Jantzen filtrations ([**32**]), at $s=1$ we consider filtrations

$$X = X_1^0 \supset X_1^1 \supset \cdots \supset 0.$$

Because we have a standard representation, we have $X_1^1 = \pi_4$, a square integrable representation. We will prove that $X_1^2 = \{0\}$, i.e. that a hermitian form defined on X_1^1 by

$$< v, v' >_1 = \lim_{s\to 1} \int_K < v(k), \frac{1}{s-1} A(s) v_s'(k) > \mathrm{d}k$$

is nondegenerate, so it's radical, namely X_1^2, is trivial. Because of the simplicity of the pole of the Plancherel measure at $s=1$, we have

$$A(-s)\frac{1}{s-1}A(s) = h(s),$$

where h is holomorphic function in the neighborhood of $s=1$, and $h(1) \neq 0$. From this follows that for no-null $v' \in X$ such that $v_1' \in \pi_4$ stands $\lim_{s\to 1}\frac{1}{s-1}A(s)v_s' \notin L(\delta\nu, 1)$. Now, we can choose $v \in X_1^1$ such that

$$< v, v' >_1 = \lim_{s\to 1} \int_K < v(k), \frac{1}{s-1} A(s) v_s'(k) > \mathrm{d}k \neq 0.$$

If we denote the signature of $L(\delta\nu, 1)$ by (p_0, q_0) (by [**32**]), we can obtain the signature of $\delta\nu^s \rtimes 1$ for $s > 1$ and for $s < 1$ in terms of signatures (p_0, q_0) and (p_1, q_1). But we know that on these segments we have a non-unitarizable representations. We conclude $p_0 \neq 0$ and $q_0 \neq 0$ which is equivalent to non-unitarizability.

The proof of the non-unitarizability of $L(\tau_1\nu^{\frac{3}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1])$ follows the same pattern: We will compute the Plancherel measure of $\mu(s, \tau_1 \otimes \delta[\tau_1\nu^s; 1])$. We can easily extend the results from the fourth section to the case when we consider the square integrable representations instead of the cuspidal ones. So we have

$$\mu(s, \tau_1 \otimes \delta[\tau_1\nu^s; 1])^2 = \mu(s, \tau_1' \otimes \sigma_1)\mu(s, \tau_1' \otimes \sigma_2).$$

The representation $\sigma_i,\ i = 1, 2$ from the above equation are obtained originally by considering the restrictions of the representations to the groups $SL(2,F)D_1$ or $SL(2,F)SL(2,F)$. It is not hard to see that, in this case, $\sigma_i \hookrightarrow \mathrm{Ind}_{M_\alpha}^{SO(4,F)}(\tau_1'\chi_i)$,where χ_i is a quadratic character on F^*.

Here M_α is the standard Levi subgroup which is diagonally positioned in $SO(4,F)$, and τ_1' is a Langlands' lift of the representation τ_1. Because of the genericity of the representations σ_i we can apply the results of Shahidi about multiplicativity of γ–factors. We obtain that the Plancherel measure $\mu(s, \tau_1' \otimes \sigma_i)$ can have, at $s = \frac{3}{2}$, a pole of order one. We obtain a pole there if and only if, $\tau_1' \cong \tau_1'\chi_i$. But $\mu(s, \tau_1 \otimes \delta[\tau_1\nu^{\frac{1}{2}}; 1])$ must have a pole there, so it is a pole of order one. Analogously to the previous case, we conclude that the intertwining operators appearing in the definition of the Plancherel measure $\mu(s,\tau_1\otimes\delta[\tau_1\nu^s;1])$ are holomorphic near $s=\pm\frac{3}{2}$. Now we can conclude, as in the previous discussion, that $L(\tau_1\nu^{\frac{3}{2}};\delta[\tau_1\nu^{\frac{1}{2}};1])$ is a non-unitarizable representation. The only Langlands quotient left to settle is $L(\tau_1\nu;\tau_1\rtimes 1)=\nu^{\frac{1}{2}}L(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$. We obtain the hermiticity of the representations $\pi_s=\nu^s L(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ for $s\in(0,1)$ using the action of the long intertwining operator acting on the space $\tau_1\nu^{s+\frac{1}{2}}\times\tau_1\nu^{s-\frac{1}{2}}\rtimes 1$. But unitarity of the representation π_s at $s=\frac{1}{2}$ would imply unitarizability of all the subquotients at $s=1$, which contradicts what we have just proved. □

Again, let us denote $\pi=\tau_1\nu^{s_1}\times\chi_1\nu^{s_2}\rtimes 1$. We can assume $s_1,s_2\geq 0$. Then

Proposition 5.12. (i) *If τ_1 is not a selfcontragredient representation and $\chi_1^2\neq 1$, π has a hermitian quotient only if $(s_1,s_2)=(0,0)$.*

(ii) *If τ_1 is selfdual, but $\chi_1^2\neq 1$ π has a hermitian quotient only if $s_2=0$ (then all the subquotients are hermitian), and has the unitarizable subquotients for (s_1,s_2) from the segment $[0,\frac{1}{2}]$ if $\omega_{\tau_1}=1$ and only in the origin if $\omega_{\tau_1}\neq 1$.*

(iii) *If $\chi_1^2=1$ and τ_1 is not a selfdual representations, analogously to the previous case, and the unitarizability is obtained on the segment $\{(0,s_2):s_2\in[0,\frac{1}{2}]\}$.*

(iv) *If $\chi_1^2=1$ and $\tau_1\rtimes 1$ reduces, every subquotient of π is hermitian, and π has unitarizable subquotients on the same segment as in the previous case*

(iv) *If $\chi_1^2=1$ and $\tau_1\nu^{\frac{1}{2}}\rtimes 1$ reduces, we have the analogous situation as for the characters; unitarizability of all subquotients of π on the closed region I on the Figure 1.*

Proof. Straightforward. □

5.2. The unitary dual supported on the non-minimal parabolic subgroups. Once we have handled the reducibility questions in this case, the rest is straightforward.

Proposition 5.13. (i) *We consider induction from the Siegel parabolic subgroup: If τ is an irreducible cuspidal representation of $GL(2,D)$ then let us denote $\pi_s=\tau\nu^s\rtimes 1$. If τ is not self-dual, π_s is hermitian only when $s=0$; then it is a tempered representation. If τ is self-dual, π_s reduces for $s\in\{0,\frac{1}{2}\}$, and all of it subquotients are always hermitian. If, in this case, $s=0$ π_0 is the sum of two non-equivalent tempered representations, and otherwise, π_s are non-unitarizable. If $s=\frac{1}{2}$, π_s is non-tempered irreducible unitary representation for $s\in$*

$(0, \frac{1}{2})$, tempered for $s = 0$, and for $s = \frac{1}{2}$ the representation $\pi_{\frac{1}{2}}$ has two irreducible unitary subquotients; one of them is non-tempered and the other is a square integrable representation.

(ii) *We are considering induction from non-Siegel parabolic subgroup: the result is analogous to one in Siegel case, and we saw that the reducibility for the representation $\tau\nu^s \rtimes \delta$ implies $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$.*

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