Graded Lie algebras

Lusztig II

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G reductive $\mathcal{N} \subset LG$ $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ Springer

$\iota : \mathbb{C}^* \to G$ $LG = \bigoplus_{t \in \mathbb{Z}} L_t G$

Fix $\Delta = \{n, -n\}$ $n \neq 0$

G^ι = centralizer of $\iota(\mathbb{C}^*)$

$(\mathcal{C}, \mathcal{L}')$ is of Springer type if $\mathcal{L}'$ appears in $\mathcal{H}^i \pi_! \mathbb{C}|_{\mathcal{C}}$ for some i

$\Leftrightarrow$ $IC(\bar{\mathcal{C}}, \mathcal{L}')$ [shift] = direct summand of $\pi_! \mathbb{C}$.

$\mathcal{O}$, G^ι orbit in $L_n G$

$\mathcal{L}$ irr. G^ι-eq. local sys on $\mathcal{O}$

$(\mathcal{O}, \mathcal{L})$ Springer type if $\exists$ $(\mathcal{C}, \mathcal{L}')$ of Springer type s.t. $\mathcal{O} \subset \mathcal{C}$

$\mathcal{L}$ appears in $\mathcal{L}'|_{\mathcal{O}}$.

$\mathcal{J}_G = \{ B ; B \in \mathcal{B}^\iota \}$, $\underline{\mathcal{J}}_G = G^\iota \backslash \mathcal{J}_G$ (Borel sys on which G^ι acts by conj, transitively)

Vector space with basis $\underline{\mathcal{J}}_G$

$K_G = \mathbb{Q}(v)$ vs with basis $(I_\mathcal{J})_{\mathcal{J} \in \underline{\mathcal{J}}_G}$

induction Let $Q \in \mathcal{P}^\iota$ $B' \in \mathcal{J}_{\bar{Q}} \to B \in \mathcal{J}_G$

inv image of B' under $Q \to \bar{Q}$

$\mathrm{ind}_Q^G : K_{\bar{Q}} \to K_G$ linear

$I_{(\bar{Q}^\iota\text{-orbit of } B')} \longrightarrow I_{(G^\iota\text{-orbit of } B)}$

Bar $^- : \mathbb{Q}(v) \longrightarrow \mathbb{Q}(v)$ $\overline{v^m} = v^{-m}$

$\beta : K_G \to K_G$ almost linear $\beta(p I_\mathcal{J}) = \bar{p} I_\mathcal{J}$

K_G has a bilinear form:

$$(:) : K_G \times K_G \to \mathbf{Q}(v)$$

$$(I_{\mathcal{J}} : I_{\mathcal{J}'}) = \sum_{\substack{\Omega : G^\iota\text{-orbit on } \mathcal{B}^\iota \times \mathcal{B}^\iota \\ pr_1 \Omega = \mathcal{J},\ pr_2 \Omega = \mathcal{J}'}} (-v)^{\tau(\Omega)}$$

$(B, B') \in \Omega$ $\quad U, U'$ unipotent rad.

$$\tau(\Omega) = -\dim \frac{L_0 U' + L_0 U}{L_0 U' \cap L_0 U} + \dim \frac{L_n U' + L_n U}{L_n U' \cap L_n U}$$ n particular fixed.

$L_0 U = LU \cap L_0 G$

$(:)$ degenerate $\quad \mathcal{R}$ = radical of $(:)$. It is preserved by $\bar{\ }$

$$D_G^{\#} = \{ s \in LG \text{ semisimple } \exists\ \mathfrak{sl}_2 \mathbf{C} \to LG \text{ Lie alg. hom. } \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \to \frac{2}{n} s \}.$$

For $s \in D_G^{\#} \cap L_0 G$. $\quad LG = \bigoplus_{r \in \frac{n}{2}\mathbb{Z}} L^r G$ $\quad$ r-eigenspace of $\mathrm{ad}(s)$

$$= \bigoplus_{\substack{r \in \frac{n}{2}\mathbb{Z} \\ t \in \mathbb{Z}}} L^r_t G \qquad L^r G \cap L_t G =: L^r_t$$

Vinberg Kawanaka

$$LG_s = \bigoplus_{r=t} L^r_t G \qquad LP_{sr} = \bigoplus_{\frac{2t}{n} \le \frac{2r}{n}} L^r_t G$$

$$P_{sr} \cap P_{s,-r} = G_s, \quad d_s = \sum_{\frac{2r}{n} \le -1} \dim L^r_0 G + \sum_{\frac{2r}{n} \geq 2} \dim L^r_n G$$

the dimension of the corr. G^ι-orbits on $L_n G$

$$E_G^{\#} = \{ s \in D_G^{\#} \cap L_0 G ;\ s \in D_{G_s}^{\#} \}.$$

$$\underline{E}_G = G^\iota \backslash E_G^{\#} \leftarrow \text{param. orbits of } G^\iota \text{ on } L_n G$$

: For G^L-orbit η in $E_G^\#$ and $n \in \Delta$ set $P_{\eta n} = P_{s n}$ $\quad s \in \eta$

$\bar{s}$ = image of s under $P_{\eta n} \to \bar{P}_{\eta n}$

$E^\#_{\bar{P}_{\eta n}} \ni \underline{\eta}_n = \bar{P}^i_{\eta n}$-orbit of $\underline{s}$ $\qquad d_\eta = d_s \quad s \in \eta$

To any $\eta \in \underline{E}^\#_G$, $n \in \Delta$ we associate

$$\begin{tikzcd} Z_n^\eta \arrow[r, phantom, "\subset"] \arrow[dr, leftrightarrow] & K_G^\# \\ & U_n^\eta \arrow[u, phantom, "\cup"] \end{tikzcd}$$

desired properties: $\qquad$ 4 bases should be constr. simultaneously

$Z_n = \bigcup_\eta Z_n^\eta$ disjoint union $\qquad$ both form basis mod Rad

$U_n = \bigcup_\eta U_n^\eta$. —"— $\qquad$ $U_n = U_{-n}$ in $K_G^\# / \mathrm{Rad}$.

Z_n, U_n related by upper Δ matrix with entries in $\mathbb{Z}[v]$

This is the multiplicity matrix.

When G torus, $L_n G = 0$, $\eta = \{0\}$.

Can assume the subsets Z_n^η, U_n^η are defined when G is replaced by $\bar{P}$ where $P \in \mathcal{P}^i$, $P \neq G$.

Let $E_G^! = \{\eta \in \underline{E}_G \,;\, P_{\eta n} \neq G\}$

If $\eta \in E_G^!$, $Z_n^{\overline{\eta_n}} \subset K_{\bar{P}_{\eta, n}}$ is defn by induction

Set $Z_n^\eta = \mathrm{Ind}_{P_{\eta n}}^G (Z_n^{\overline{\eta_n}})$

$Z'_u = \bigcup_{\eta \in E'_G} Z^\eta_u$ claim: • this is disjoint union

• $(\xi : \xi')_{\xi, \xi' \in Z'_u}$ nonsingular

If ι not rigid done ✓

$V_u = \mathbb{Q}(v)$-sub sp of $K_G^{\#}$ spanned by Z'_u

Z^0_u basis of V

For $x \in K_G^{\#}$, $x = \underset{\substack{\uparrow \\ V_u}}{Y_u(x)} + \underset{\substack{\uparrow \\ V_u^\perp}}{Y_u^\perp(x)}$ uniquely

$\beta' : V_u \to V_u$

$x \longmapsto Y_u(\beta(x))$

Claim: $x \in V_u \to Y_u^\perp(x) \in \text{Rad}$. It follows that $\beta'\beta' = 1$.

Claim: $\xi \in Z'_u \to \beta'(\xi) = \xi + \sum_{\xi' < \xi} * \xi'$ ← in $\mathbb{Z}[v, v^{-1}]$

$\xi' < \xi$: $\xi' \in Z^{\eta'}_u$. $d_{\eta'} < d_\eta$

$\xi \in Z^\eta_u$

Define: $w^\xi_u \in V_u$ $\beta'(w^\xi_u) = w^\xi_u$

$$\boxed{w^\xi_u = \xi + \sum_{\xi' < \xi} * \xi'}$$ in $v\mathbb{Z}[v]$

$U'^{\#}_u = \{ w^\xi_u : \xi \in Z'_u \}$

If (G, ι) is not rigid we set $Z_u = Z'^{\#}_u$ $U_u = U'^{\#}$

Now assume (G, ι) rigid

$J_{-u} = \{ \xi_0 \in Z'_{-u} ; Y^\perp(w^{\xi_0}_{-u}) \notin \text{Rad}.$

$J_{-u} \to K_G$ $\xi_0 \mapsto Y^\perp_u(w^{\xi_0}_{-u})$ image $= Z^{\eta_0}_u$

η_0 : unique element of $E_G - E'_G$

$\shortparallel$

single $s \in E_G$ s.t. $s \in LG_{der}$ ad $s(x) = mx$ for $x \in L_m G$

Claim: $\mathcal{J}_{-m} \xrightarrow[h_m]{\approx} Z_m^{\eta_0}$

For $\xi \in Z_m^{\eta_0}$ set $W_m^{\xi} = W_{-m}^{h_m^{-1}(\xi)}$

$$\mathcal{U}_m^{\eta_0} = \{ W_m^{\xi}, \ \xi \in Z_m^{\eta_0} \}.$$