04/12/06

Graded Lie algebras and inters. cohom.

G. Lusztig

G connected alg. gp acts on alg. variety X with finitely many orbits. $\mathcal{I}(X) = \{ (\mathcal{O}, \mathcal{F}) \mid \mathcal{O} = G\text{-orbit in } X,\ \mathcal{F} = \text{irr. } G\text{-equiv. local system on } \mathcal{O} \}$

unitriangular matrix IC-matrix $= (m_{\mathcal{O}', \mathcal{F}' ; \mathcal{O}, \mathcal{F}})$

$$m_{\mathcal{O}', \mathcal{F}' ; \mathcal{O}, \mathcal{F}} = \begin{cases} \sum_i \left(\mathcal{F}' : \mathcal{H}^i \, IC(\bar{\mathcal{O}}, \mathcal{F})|_{\mathcal{O}'} \right) v^{-i} & \text{if } \mathcal{O}' \subset \bar{\mathcal{O}} \\ 0 & \text{otherwise} \end{cases}$$

Elements are polynomials in $\mathbb{Z}[v^{-1}]$.

IC is the trans. matrix when $v = 1$
$\mathcal{I}(X) \leftrightarrow$ std module
$\mathcal{I}(X) \leftrightarrow$ simple module

Examples 1) G reductive $X = (\text{flag manifold})^2$ $\exists$ algorithm K-L

2) G reductive, $X =$ nilpotent elements in $\operatorname{Lie}(G)$

3) $\iota : \mathbb{C}^\times \longrightarrow G$

$$G^\iota = \{ g \in G \mid g\, \iota(t) = \iota(t)\, g \quad \forall t \in \mathbb{C}^\times \}.$$

$$\operatorname{Lie} G = \bigoplus_{n \in \mathbb{Z}} \operatorname{Lie}_n G \quad \text{eigenspaces of } \iota.$$

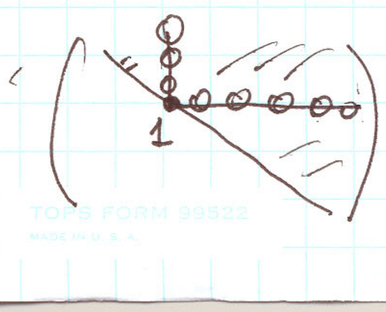
$X = L_n G$ if $n \neq 0$

corresp. to the similar problem for the graded Hecke alg?

(previous work 1995)

In ex 2, $(\mathcal{O}, \mathcal{F})$ is cuspidal $\iff m_{\mathcal{O}', \mathcal{F}' ; \mathcal{O}, \mathcal{F}} = 0 \ \forall \mathcal{O}' \subset \bar{\mathcal{O}} - \mathcal{O}$

and $m_{\mathcal{O}, \mathcal{F} ; \mathcal{O}'', \mathcal{F}''} = 0 \ \forall \mathcal{O}'', \mathcal{O} \subset \bar{\mathcal{O}''} \subset \ldots$



$(\mathcal{O}, \mathcal{F})$ is semi-cuspidal if $m_{\mathcal{O}', \mathcal{F}'; \mathcal{O}, \mathcal{F}} = 0 \ \forall\, \mathcal{O}' \subset \bar{\mathcal{O}} - \mathcal{O}$.

classification exists.

Ex $G = SL_2(\mathbb{C})$ $\mathcal{O}$ = regular nilpotent orbit.

$\mathcal{F}$ = nontrivial irr. local system.

This is the only cuspidal.

But $(\{0\}, \underset{\text{triv.}}{\mathbb{C}})$ is semi cuspidal.

Problem 3) (use analogy with Quantum gps)

Ex $G = GL_n(\mathbb{C})$. $V = \bigoplus_{n \in \mathbb{Z}} V_n$ $L_2 G = \{T : V \to V \mid TV_n \subset V_{n+2} \ \forall n$

$G' = \prod GL(V_n)$.

U^+ = generated by $e_i \ i \in I$ $U^+ = \bigoplus_\nu U^+_\nu$

G^i orbits in $L_2 G \longleftrightarrow$ PBW basis of U^+_ν

canonical basis of U^+_ν (of monomials).

monomials in $U^+_\nu \longleftrightarrow$ Borel subgp of $G \supset i(\mathbb{C}^\times)$ up to G'-action

Generalization

(i, G) rigid $\iff \varphi : sl_2(\mathbb{C}) \to G$, $\varphi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = i(t) \ \forall t \mod Z_G$

$\mathcal{I}(L_{\pm 2} G) = \{(P, \xi) \mid P$ parabolic in G, $P \supset i(\mathbb{C}^\times)$ $(i, \bar{P})$ rigid; ξ cuspidal local sys on the nilp. orbit $L\bar{P}$ defined by i, $\bar{P}$-equiv. $\}$

$\bar{P}$ = reductive part

G^i acts by conjugation on $\mathcal{I}(L_{\pm 2} G)$ and has finitely many orbits

Define $K(L_{\pm 2} G) = \mathbb{Q}(v)$ -vect. space spaned by $\mathcal{I}(L_{\pm 2} G) / G^i$

$\mu : J(L_{\pm 2}G) \longrightarrow \{$ semicuspidal pairs $(\mathcal{O}, \mathcal{F})$ for $G \}$

(P, ξ)

$\cup \quad \downarrow$

L local syst.

$\| \quad$ on C_L some

$\bar{P}$ nilpot. orbit in Lie(L) $\subset \mathcal{O}$ nilpot. orbit in LG

$\mathcal{F}$ local syst restr. to ξ

$(\mathcal{F}|_{C_L} = \xi)$

$$J(L_{\pm 2}G)_{\mathcal{F}} = \mu^{-1}(\mathcal{F}) \qquad Y_{\mathcal{F}} = \{((P,\xi),(P',\xi')) : P, P' \text{ have same Levi}\}$$

$$(P,\xi) \in J(L_{\pm 2}G)_{\mathcal{F}}, \quad (P',\xi') \in J(L_{\pm 2}G)_{\mathcal{F}^*}$$

G^i acts with finitely many orbits on $Y_{\mathcal{F}}$.

$$\tau : G^i \backslash Y_{\mathcal{F}} \longrightarrow \mathbb{Z} \qquad \tau((P,\xi),(P',\xi')) = -\dim \frac{L_0 U_{P'} + L_0 U_P}{L_0 U_{P'} \cap L_0 U_P} + \dim \frac{L_2 U_{P'} + L_2 U_P}{L_2 U_{P'} \cap L_2 U_P}$$

$(,)$ bilinear form on $K(L_{\pm 2}G)$ values on $\mathbb{Q}(v)$

$$\left(\underset{\mathcal{F}}{\overset{\mu\downarrow}{I_y}}, \underset{\mathcal{F}'}{\overset{\downarrow\mu}{I_{y'}}}\right) = \begin{cases} 0, & \mathcal{F}' = \mathcal{F}^* \\ \sum\limits_{\substack{\Omega \in G^i \backslash Y_{\mathcal{F}} \\ pr_1 \Omega = \varphi \\ pr_2 \Omega = \varphi'}} (v)^{\tau(\Omega)} & \end{cases}$$

Claim $\dim K(L_{\pm 2}G)/\mathrm{rad}(,) = |J(L_2 \xi G)| = |J(L_{-2} \xi G)|$

Claim PBW-"basis"$_2$ in $K(L_{\pm 2}G)$ (still different become bases after quotienting by rad (,))

The defn is inductive (cannot be separated) PBW-"basis"$_{-2}$ — " —

canonical "basis"$_2$

canonical "basis"$_{-2}$ > same after quotient

The algorithm works because of the following

Geom property

$(\mathcal{O}, \mathcal{F})$ $L_2 G$

either 1) $\overline{\mathcal{O}} \neq L_2 G$. or 2) $\overline{\mathcal{O}} = L_2 G$ but $\mathcal{F}(IC(\overline{\mathcal{O}}, \mathcal{F})) = IC(\overline{\mathcal{O}'}, \mathcal{F}')$ (Fourier transf.)

$(\mathcal{O}', \mathcal{F}')$ related to $L_{-2}G$

and $\overline{\mathcal{O}'} \neq L_{-2}G$

or

3) (i, G) must be rigid. $\exists$ semicuspidal pair (C, ξ) for LG

C = nilpotent attached to i

$\mathcal{O}$ = open orbit in $L_2 G$ $IC(\overline{\mathcal{O}}, \xi) = \xi$ extended by 0.

$\mathcal{F} = \xi|_{\mathcal{O}}$

G^i-orbits on $L_2 G \longleftrightarrow G^i$-orbits on $\mathcal{E}$

where $\mathcal{E} = \left\{ \underset{\text{semis}}{\tau \in LG} \;\middle|\; \begin{array}{l} \tau \in L_0 G \\ \tau \text{ is of Dynkin type w.r.to } LG \\ \tau \text{ — " — } \bigoplus_{n} {}_nL_n G \\ \tau \text{ — " — } \bigoplus_{n \geq n'} {}_n L_{n'} G \end{array} \right\}$

One other application: the dim of A-weight spaces of the standard modules.