LEFT CELLS IN WEYL GROUPS

G. Lusztig*
Department of Mathematics, M.I.T.
Cambridge, Massachussetts 02139

In $[KL_1]$, Kazhdan and myself have defined a partition of an arbitrary Coxeter group into subsets called left cells. These subsets enter in an essential way in the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra. In this paper we shall generalize the definition of $[KL_1]$ to include the case where the simple reflections are given different weights. We shall give an application of this to Schubert varieties. We shall also give some examples concerning a Weyl group of type B_n.

1. Let (W,S) be a Coxeter group and let $\varphi: W \to \Gamma$ be a map of W into an abelian group Γ such that $\varphi(s_1 s_2 \dots s_p) = \varphi(s_1)\varphi(s_2)\dots \varphi(s_p)$ for any reduced expression $s_1 s_2 \dots s_p$ in W. We shall set $\varphi(w) = q_w^{1/2}$, $(w \in W)$. Let H_φ be the Hecke algebra of W with respect to φ; this is an algebra over the group ring $\mathbb{Z}[\Gamma]$. As a $\mathbb{Z}[\Gamma]$-module, it is free with basis T_w, $(w \in W)$. The multiplication is defined by

$$(T_s + 1)(T_s - q_s) = 0, \qquad (s \in S)$$

$$T_{s_1 s_2 \dots s_p} = T_{s_1} T_{s_2} \dots T_{s_p}, \quad \text{if } s_1 s_2 \dots s_p \text{ is a reduced expression in } W.$$

The unit element is T_e. It will be convenient to introduce a new basis $\tilde{T}_w = q_w^{-1/2} T_w$, $(w \in W)$. We then have

$$(\tilde{T}_s + q_s^{-\frac{1}{2}})(\tilde{T}_s - q_s^{\frac{1}{2}}) = 0, \qquad (s \in S)$$

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$$\tilde{T}_{s_1 s_2 \dots s_p} = \tilde{T}_{s_1}\tilde{T}_{s_2}\dots\tilde{T}_{s_p}, \qquad \text{if } s_1 s_2 \dots s_p \text{ is a reduced expression in } W.$$

Let $a \to \bar{a}$ be the involution of the ring $\mathbb{Z}[\Gamma]$ which takes γ to γ^{-1} for any $\gamma \in \Gamma$. We extend it to an involution $h \to \bar{h}$ of the ring H_φ by the formula

$$\overline{\sum_w a_w \tilde{T}_w} = \sum_w \bar{a}_w \tilde{T}^{-1}_{w^{-1}}, \qquad (a_w \in \mathbb{Z}[\Gamma]).$$

(Note that $\tilde{T}_s^{-1} = \tilde{T}_s + (q_s^{-\frac{1}{2}} - q_s^{\frac{1}{2}})$, $s \in S$, hence $\tilde{T}_w$ is ivertible for all $w \in W$.) Let us define elements $R^*_{x,y} \in \mathbb{Z}[\Gamma]$, $(x,y \in W)$, by

$$\tilde{T}^{-1}_{y^{-1}} = \sum_x \bar{R}^*_{x,y}\tilde{T}_x.$$

It is easy to see that $R^*_{x,y} = 0$ unless $x \le y$ in the standard partial order of W. Using the fact that $h \to \bar{h}$ is an involution, we see that

$$\sum_{x \le y \le z} \bar{R}^*_{x,y} R^*_{y,z} = \delta_{x,z} \tag{1.1}$$

for all $x \le z$ in W. Note also that $q_x^{-1/2} q_y^{1/2} R^*_{x,y} \in \mathbb{Z}[\Gamma^2]$.

For example, $R^*_{x,x} = 1$ for all $x \in W$. Let ℓ be the length function on W.

(1.2) If $x < y$, $\ell(y) = \ell(x) + 1$, then x is obtained by dropping $s \in S$ in a reduced expression of y, and we have $R^*_{x,y} = q_s^{1/2} - q_s^{-1/2}$.

(1.3) If $x < y$, $\ell(y) = \ell(x) + 2$, then x is obtained by dropping $s \in S$ and $t \in T$ in a reduced expression of y, and we have $R^*_{x,y} = (q_s^{1/2} - q_s^{-1/2})(q_t^{1/2} - q_t^{-1/2})$.

We now assume given a total order on Γ compatible with the group structure on Γ. Let Γ_+ be the set of elements of Γ which are strictly positive for this total order and let $\Gamma_- = (\Gamma_+)^{-1}$. We shall assume that $q_s^{1/2} \in \Gamma_+$ for all $s \in S$. We have

2. Proposition. Given $w \in W$, there is a unique element $C'_w \in H_\varphi$ such that

$$\bar{C}'_w = C'_w$$

$$C'_w = \sum_{y \le w} P^*_{y,w} \tilde{T}_y$$

where $P^*_{w,w} = 1$ and, for any $y < w$, $P^*_{y,w}$ is a $\mathbb{Z}$-linear combination of elements in Γ_-. Moreover, $q_y^{-1/2} q_w^{1/2} P^*_{y,w} \in \mathbb{Z}[\Gamma^2]$.

(When f is constant on S, this is the same as (1.1.c) of $[KL_1]$.)

We must show that the system of equations

$$(2.1) \qquad P^*_{w,w} = 1$$

$$(2.2) \qquad \bar{P}^*_{x,w} - P^*_{x,w} = \sum_{x<y\le w} R^*_{x,y} P^*_{y,w} \qquad ,(\forall x < w),$$

with unknowns $P^*_{x,w}$,has a unique solution such that $P^*_{x,w}$ is a $\mathbb{Z}$-linear combination of elements in Γ_-, for $x < w$. This is shown by induction on $\ell(w) - \ell(x)$. The uniqueness is clear. To show existence, we shall use a suggestion of O. Gabber, which simplifies somewhat the original proof in $[KL_1]$. We fix $x < w$ and assume that for all y, $x < y \le w$, the $P^*_{y,w}$ have been already constructed and have the required property. It is then enough to show that $\overline{\sum_{x<y\le w} R^*_{x,y} P^*_{y,w}} = - \sum_{x<y\le w} R^*_{x,y} P^*_{y,w}$. But we have

$$\overline{\sum_{x<y\le w} R^*_{x,y} P^*_{y,w}} = \sum_{x<y\le z\le w} \overline{R^*_{x,y}} R^*_{y,z} P^*_{z,w}$$

$$= \sum_{x<z\le w} \Big(\sum_{x\le y\le z} \bar{R}^*_{x,y} R^*_{y,z} P^*_{z,w} - R^*_{x,z} P^*_{z,w} \Big)$$

and, using (1.1), this equals $- \sum_{x<z\le w} R^*_{x,z} P^*_{z,w}$, as required. The last assertion follows from (2.2). This completes the proof of the proposition.

3. Now let $s \in S$, $w \in W$ be such that $w < sw$. For each y such that $sy < y < w$, we define an element

$$M^s_{y,w} \in \mathbb{Z}[\Gamma]$$

by the inductive condition

(3.1) $\sum_{\substack{y \le z < w \\ sz < z}} P^*_{y,z} M^s_{z,w} - q_s^{1/2} P^*_{y,w}$ is a combination of elements in Γ_-

and by the symmetry condition

$$\bar{M}^s_{y,w} = M^s_{y,w}. \tag{3.2}$$

The condition (3.1) determines uniquely the coefficient of γ in $M^s_{y,w}$ for all $\gamma \in \Gamma - \Gamma_-$; the condition (3.2) determines the remaining coefficients. We have $q_s^{-1/2} q_y^{-1/2} q_w^{1/2} M^s_{y,w} \in \mathbb{Z}[\Gamma^2]$.

4. Proposition. Let $s \in S$ and let $w \in W$. Then:

(4.1) $(\tilde{T}_s + q_s^{-1/2}) C'_w = C'_{sw} + \sum_{\substack{z < w \\ sz < z}} M^s_{z,w} C'_z$, if $w < sw$

(4.2) $(\tilde{T}_s - q_s^{1/2}) C'_w = 0$, if $w > sw$

(compare with (2.3.a), (2.3.c) in [KL_1]).

Proof. If $w = e$, then (4.1) is clearly true. Now assume that $w \neq e$ and that the proposition is already proved for all $w' < w$. Using (4.2), we see that

(4.3) $P^*_{u,z} = q_s^{-1/2} P^*_{su,z}$ if $u < su \le z$, $sz < z < w$.

Case 1: $w < sw$. Consider the left hand side minus the right hand side of (4.1). The coefficient of $\tilde{T}_y$ in that expression is

$$f_y = q_s^{\varepsilon/2} P^*_{y,w} + P^*_{sy,w} - P^*_{y,sw} - \sum_{\substack{y \le z < w \\ sz < z}} P^*_{y,z} M^s_{z,w}$$

where $\varepsilon = \begin{cases} 1 & \text{if} \quad sy < y \\ -1 & \text{if} \quad sy > y \end{cases}$ and $P^*_{x,x'}$ is defined to be zero whenever $x \nleq x'$. If $sy < y$, then (3.1) shows that f_y is a $\mathbb{Z}$-linear combination of elements in Γ_-. If $sy > y$, then applying (4.3) we see that

$$f_y = q_s^{-1/2} P^*_{y,w} + P^*_{sy,w} - P^*_{y,sw} - q_s^{-1/2} \sum_{\substack{sy \leq z < w \\ sz < z}} P^*_{sy,z} M^s_{z,w}.$$

It follows that

$$f_y = q_s^{-1/2} f_{sy} + q_s^{-1/2} P^*_{sy,sw} - P^*_{y,sw}$$

hence, again, f_y is a $\mathbb{Z}$-linear combination of elements in Γ_-. But $(\tilde{T}_s + q_s^{-1/2}) = C'_s$, C'_w, $M^s_{z,w}$ are each fixed by the involution $h \to \bar{h}$. Hence $\overline{\sum_y f_y \tilde{T}_y} = \sum_y f_y \tilde{T}_y$. Assume that some f_{y_0} is non-zero. We can take y_0 to have maximal possible length subject to the property $f_{y_0} \neq 0$. Then the coefficient of $\tilde{T}_{y_0}$ in $\overline{\sum_y f_y \tilde{T}_y}$ is equal to $\overline{f_{y_0}}$. Thus $f_{y_0} = \bar{f}_{y_0}$. This contradicts the fact that f_{y_0} is a non-zero combination of elements in Γ_-. Thus we have $f_y = 0$ for all $y \in W$ and (4.1) is proved for w.

<u>Case 2</u>: $w > sw$. Applying (4.1) to sw, we see that

$$C'_w = (\tilde{T}_s + q_s^{-1/2}) C'_{sw} - \sum_{\substack{z < sw \\ sz < z}} M^s_{z,sw} C'_z.$$

Clearly, $(\tilde{T}_s - q_s^{1/2})(\tilde{T}_s + q_s^{-1/2}) = 0$ and, by the induction hypothesis, $(\tilde{T}_s - q_s^{1/2}) C'_z = 0$ for all z, $(z < sw,\ sz < z)$. Hence $(\tilde{T}_s - q_s^{1/2}) C'_w = 0$, as required.

5. <u>Proposition</u>. <u>Let</u> $y < w$ <u>be such that</u> $\ell(w) = \ell(y) + 1$. <u>Then</u> y <u>is obtained by dropping a simple reflection</u> s <u>in a reduced expression of</u> w.

(a) <u>We have</u> $P^*_{y,w} = q_s^{-1/2}$

(b) <u>Let</u> t <u>be a simple reflection such that</u> $ty < y < w < tw$. <u>Then</u>

$$M^t_{y,w} = \begin{cases} 0, & \text{if } q_t^{1/2} < q_s^{1/2} \\ 1, & \text{if } q_t^{1/2} = q_s^{1/2} \\ q_s^{1/2}q_t^{-1/2} + q_s^{-1/2}q_t^{1/2}, & \text{if } q_t^{1/2} > q_s^{1/2} \end{cases}$$

<u>Proof</u>. From (1.2) and (2.2) we see that

$$\bar{P}^*_{y,w} - P^*_{y,w} = R^*_{y,w} = q_s^{1/2} - q_s^{-1/2}$$

and (a) follows. If t is as in (b), then by (3.1) and (a), $M^t_{y,w} - q_t^{1/2}q_s^{-1/2}$ must be a $\mathbb{Z}$-linear combination of elements in Γ_-. From this and from (3.2), the desired formula for $M^t_{y,w}$ follows.

6. Let $j\colon H_\varphi \to H_\varphi$ be the ring involution defined by $j(\sum_w a_w\tilde{T}_w) = \sum_w \bar{a}_w\varepsilon_w\tilde{T}_w$, where $\varepsilon_w = (-1)^{\ell(w)}$. It commutes with the involution $h \to \bar{h}$. Let $C_w = \varepsilon_w j(C'_w)$. Then

$$\bar{C}_w = C_w \quad \text{and} \quad C_w = \sum_{y\le w} \varepsilon_y\varepsilon_w \bar{P}^*_{y,w}\tilde{T}_y. \qquad \text{(Compare [KL}_1\text{, 1.1].)}$$

Applying j to (4.1) and (4.2) we get:

$$(6.1) \qquad (\tilde{T}_s - q_s^{1/2})C_w = C_{sw} - \sum_{\substack{z<w \\ sz<z}} \varepsilon_z\varepsilon_w M^s_{z,w}C_z, \qquad \text{if } w < sw$$

$$(6.2) \qquad (\tilde{T}_s + q_s^{-1/2})C_w = 0, \qquad \text{if } w > sw.$$

Let $j'\colon H_\varphi \to H_\varphi$ be the anti-automorphism of the ring H_φ defined by $j'(\tilde{T}_w) = \tilde{T}_{w^{-1}}$ and $j'(a) = a$ for $a \in \mathbb{Z}[\Gamma]$. It is easy to see that $j'(C_w) = C_{w^{-1}}$. Therefore, from (6.1) and (6.2) we can deduce

$$(6.3)\quad C_w(\tilde{T}_s - q_s^{1/2}) = C_{ws} - \sum_{\substack{z<w\\ zs<z}} \varepsilon_z\varepsilon_w M^s_{z^{-1},w^{-1}} C_z, \qquad \text{if } w < ws$$

$$(6.4)\quad C_w(\tilde{T}_s + q_s^{-1/2}) = 0, \qquad \text{if } w > ws.$$

Let $\underset{L,\varphi}{\leq}$ be the preorder relation on W generated by the relation "$x' \underset{L,\varphi}{\leq} x$ if there exists $s \in S$ such that $C_{x'}$ appears with non-zero coefficient in $\tilde{T}_s \cdot C_x$ (expressed in the C_w-basis)." We call it the left preorder. The equivalence relation associated to $\underset{L,\varphi}{\leq}$ is denoted $\underset{L,\varphi}{\sim}$ and the corresponding equivalence classes in W are called the left cells of W (with respect to φ). Given $x,y \in W$, we say that $x \underset{LR,\varphi}{\leq} y$ if there exists a sequence $x = x_0, x_1, \dots, x_n = y$ of elements in W such that for $i = 0,1,\dots,n-1$, we have either $x_i \underset{L,\varphi}{\leq} x_{i+1}$ or $x_i^{-1} \underset{L,\varphi}{\leq} x_{i+1}^{-1}$. The equivalence relation on W corresponding to the preorder $\underset{LR,\varphi}{\leq}$ is denoted $\underset{LR,\varphi}{\sim}$ and the correspondint equivalence classes on W are called the two sided cells of W. (These notions were introduced in [KL_1] in the case where φ is constant on S.)

For any $x \in W$, we denote I_x^L (resp. $\hat{I}_x^L$) the $\mathbb{Z}[\Gamma]$-submodule of H_φ spanned by the elements C_y, $y \underset{L,\varphi}{\leq} x$, (resp. by the elements C_y, $y \underset{L,\varphi}{\leq} x$, $y \underset{L,\varphi}{\not\sim} x$). We define similarly I_x^{LR} and $\hat{I}_x^{LR}$, by replacing $\underset{L,\varphi}{\leq}$, $\underset{L,\varphi}{\sim}$ by $\underset{LR,\varphi}{\leq}$, $\underset{LR,\varphi}{\sim}$ in the previous definition. It is clear from (6.1)-(6.4) that I_x^L, $\hat{I}_x^L$ are left ideals of H_φ and that I_x^{LR}, $\hat{I}_x^{LR}$ are two-sided ideals of H_φ. Hence $I_x^L/\hat{I}_x^L$ is a left H_φ-module with a natural basis given by the images of C_y for y in the left cell of x; $I_x^{LR}/\hat{I}_x^{LR}$ is a two sided H_φ-module with a natural basis given by the images of C_y, for y in the two sided cell of x. With respect to this basis, the action of $\tilde{T}_s$ $(s \in S)$ is given by a matrix which is completely determined by the elements $M^s_{y,w}$.

From now on, we assume that Γ is the infinite cyclic group with generator $q^{1/2}$ with the order relation $q^{i/2} \leq q^{j/2} \Leftrightarrow i \leq j$. We then have $q_w^{1/2} = q^{m(w)/2}$ where $m\colon W \to \{1,2,3,\dots\}$. In this case, we have

$P^*_{y,w} \in \mathbb{Z}[q^{-1/2}]$ and $q^{\frac{m(w)-m(y)}{2}} P^*_{y,w} \in \mathbb{Z}[q]$ for any $y \leq w$. Moreover $P^*_{y,w}$ has no constant term if $y < w$. From (3.1) we see that, when defined, $M^s_{y,w}\, q^{\frac{m(s)-1}{2}} \in \mathbb{Z}[q^{1/2}]$. In particular, $M^s_{y,w}$ is a constant whenever $m(s) = 1$. (This is the case considered in [KL_1].)

Consider, for example, the case where (W,S) is a Weyl group of type B_2 with $S = \{s_1, s_2\}$, $(s_1 s_2)^4 = 1$, and let $m(s_1) = 1$, $m(s_2) = c \geq 2$. We have:

$$P^*_{s_2, s_2 s_1 s_2} = q^{-\frac{c+1}{2}} - q^{-\frac{c-1}{2}}, \qquad P^*_{e, s_2 s_1 s_2} = q^{-\frac{2c+1}{2}} - q^{-\frac{2c-1}{2}}$$

$$P^*_{s_1, s_1 s_2 s_1} = q^{-\frac{c+1}{2}} + q^{-\frac{c-1}{2}}, \qquad P^*_{e, s_1 s_2 s_1} = q^{-\frac{c+2}{2}} + q^{-\frac{c}{2}}$$

and $P^*_{y,w} = q^{\frac{m(y)}{2} - \frac{m(w)}{2}}$ for all other pairs $y \leq w$. (In particular, $P^*_{y,w}$ may have negative coefficients.) We have

$$M^{s_2}_{s_2 s_1, s_1 s_2 s_1} = M^{s_2}_{s_2, s_1 s_2} = q^{\frac{c-1}{2}} + q^{\frac{-c+1}{2}}, \quad M^{s_1}_{s_1 s_2, s_2 s_1 s_2} = M^{s_1}_{s_1, s_2 s_1} = 0.$$

The left cells are:

$$\{e\},\ \{s_1\},\ \{s_2, s_1 s_2\},\ \{s_2 s_1 s_2\},\ \{s_2 s_1, s_1 s_2 s_1\},\ \{s_1 s_2 s_1 s_2\}.$$

The corresponding H_φ-modules $I^L_x/\hat{I}^L_x$ (with scalars extended to an algebraic closure of $\mathbb{Q}(q^{1/2})$) are all irreducible. (This is in contrast with the situation when $m(s_1) = m(s_2) = 1$ in which case there are only four left cells.) The two-sided cells are $\{e\}$, $\{s_1\}$, $\{s_2 s_1 s_2\}$, $\{s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$, $\{s_1 s_2 s_1 s_2\}$.

7. If we specialize $q^{1/2}$ to 1, and take coefficients in $\mathbb{Q}$, the H_φ-modules $I^L_x/\hat{I}^L_x$ become left W-modules; they give a direct sum decomposition of the left regular representation of W; they are said to be the W-modules carried by the left cells (with respecto to φ). Simi-

larly, $I_x^{LR}/\hat{I}_x^{LR}$ become two sided W-modules; they give a direct sum decomposition of the two sided regular representation of W. Hence the two sided cells give rise to an equivalence relation on the set of irreducible representations of W: two representations are equivalent if they can be connected by a chain such that any two consecutive ones appear in the same $I_x^{LR}/\hat{I}_x^{LR}$. The equivalence relation on the representations is known in the case where $\varphi(s) = q$ $(\forall s \in S)$, by the work of Barbasch-Vogan [BV]. It coincides with the equivalence relation described in [L_1]. It is likely that, in general, the equivalence relation should still be that in [L_1], except that instead of the a-function used there one should use an a-function which depends on φ: for any irreducible W-module E, we define $a_\varphi(E)$ to be the order at 0 of the rational function in q giving the formal degree of the Hecke algebra H_φ corresponding to E. In particular, it should be true that the a_φ-function should be constant on each equivalence class.

8. Let G be a simple adjoint group defined over $\mathbb{C}$ and let $\alpha: G \to G$ be an outer automorphism which leaves stable a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We assume that the corresponding map α: $W \to W$ (W = Weyl group of G) is non-trivial. Let W_1 be the fixed point set of α on W. It is well known that W_1 is a Coxeter group with a set of generators S_1 corresponding to the orbits of α on S (= the simple reflections of W); to an orbit $\mathcal{O}$, there corresponds the longest element in the subgroup generated by $\mathcal{O}$. Let $\varphi: W_1 \to \{q^{\mathbb{Z}}\}$ be the function defined by $\varphi(w_1) = q^{\ell(w_1)}$ where $\ell(w_1)$ is the length of w_1 with respect to (W,S). Let y, w be two elements of W_1 such that $y \leq w$. We shall give an interpretation of $P^*_{y,w}$ (defined with respect to φ) in terms of Schubert varieties, analogous to [KL_2]. Let $\bar{\mathcal{B}}_w \subset G/B$ be the Schubert variety corresponding to w and let $\mathcal{B}_y$ be the Bruhat cell corresponding to y. Then α acts naturally on $\bar{\mathcal{B}}_w$ and on its subvariety $\mathcal{B}_y$. Let $\mathcal{H}^{2i}_{\mathcal{B}_y}(\bar{\mathcal{B}}_w)$ be the stalk of the 2i-th

intersection cohomology sheaf of $\bar{\mathcal{B}}_w$ at a point of $\mathcal{B}_y$ which is fixed by α. Then α acts naturally on $\mathcal{H}^{2i}_{\mathcal{B}_y}(\bar{\mathcal{B}}_w)$ and we have

$$\sum_{i\geq 0} \mathrm{Tr}(\alpha, \mathcal{H}^{2i}_{\mathcal{B}_y}(\bar{\mathcal{B}}_w))q^i = q^{\frac{\ell(w)-\ell(y)}{2}} P^*_{y,w}. \tag{8.1}$$

(Note that ℓ is here the length function on W, not on W_1.) The proof is similar to that of [KL_2]. The formula (8.1) explains why the coefficients of $P^*_{y,w}$ may be negative.

9. The multiplication in the Hecke algebra can be interpreted in terms of a multiplication of complexes in a derived category of constructible sheaves over the flag manifold. (See [LV], [S].) This interpretation together with (8.1) allows us to deduce the following.

Let $y,w \in W_1$ and let us write $C_y \cdot C_w = \sum_{z_j \in W_1} N_{y,w,z_1} C_{z_1}$, $N_{y,w,z_1} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ (an identity in the Hecke algebra of W_1, with respect to φ). We can also consider y, w as elements in W and attach to them elements $\tilde{C}_y$, $\tilde{C}_w$ in the Hecke algebra of W, with respect to $\varphi(w) = q^{\ell(w)}$. ($\tilde{C}_w$ is just C_w with respect to W.) We then have $\tilde{C}_y \cdot \tilde{C}_w = \sum_{z \in W} \tilde{N}_{y,w,z} \tilde{C}_z$, $\tilde{N}_{y,w,z} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$. The coefficients of $\varepsilon_y \varepsilon_w \varepsilon_z \tilde{N}_{y,w,z}$, ($\varepsilon_w = (-1)^{\ell(w)}$), are ≥ 0 and can be interpreted as dimensions of certain vector spaces on which α acts whenever $z \in W_1$. Moreover the trace of α on that vector space is the corresponding coefficient of $N_{y,w,z}$. It follows that

(9.1) If $z \in W_1$ and $N_{y,w,z} \neq 0$ then $\tilde{N}_{y,w,z} \neq 0$.

From the definition of the left preorder $\underset{L,\varphi}{\leq}$ it now follows easily that

(9.2) If $y,w \in W_1$ satisfy $y \underset{L,\varphi}{\leq} w$ with respect to the left preorder of W_1,φ then they satisfy the similar inequality with respect to the left preorder of W,φ. Hence any left cell of W_1 (with respect to φ) is contained in a left cell of W (with respect to φ).

10. Assume, for example that (W,S) is a Weyl group of type A_n $(n \geq 3)$ and that α is the unique automorphism of order 2 of (W,S). Then (W_1,S_1) is a Weyl group of type $B_{n/2}$ if n is even and $B_{(n+1)/2}$ if n is odd. The restriction of $\varphi(w) = q^{\ell(w)}$ to S_1 has values $q^2,q^2,\ldots,q^2,q^3$ if n is even and $q^2,q^2,\ldots,q^2,q$ if n is odd. It is known that each left cell of W contains a unique involution and it carries an irreducible representation of W. It is clear that $\alpha\colon W \to W$ permutes among themselves the left cells of W, and it maps each two sided cell of W into itself (since α is an inner automorphism of W). Let $n(W_1)$ be the number of left cells of W_1 (with respect to φ) and let $n(W)$ be the number of α-stable left cells of W. We have

$$n(W_1) \geq n(W). \tag{10.1}$$

Indeed, each α-stable left cell of W contains some element of W_1 (for example, it contains a unique involution which is necessarily fixed by α), hence by (9.2) it contains a left cell of W_1 (with respect to φ). Let $n'(W_1)$ be the sum of the dimensions of the irreducible representations of W_1. Since the representations carried by the left cells of W_1 (with respect to φ) give a direct sum decomposition of the left regular representation of W_1, we see that

$$n'(W_1) \geq n(W_1) \tag{10.2}$$

with equality if and only if each left of cell of W_1,φ carries an irreducible representation of W_1.

Now let E be an irreducible representation of W. According to [KL_1], E admits a natural basis (e_i) in 1-1 correspondence with the set of left cells contained in the two-sided cell Ω of W corresponding to E. This basis has the following property: the permutation defined by α on the set of left cells in Ω corresponds to the permutation of the basis (e_i) defined by the action of $\pm w_0$ on E, where w_0 is the longest element in W. Thus, the number of α-stable left cells in Ω is equal to $|\mathrm{Tr}(w_0,E)|$, hence

$$(10.3) \qquad n(W) = \sum_E |\mathrm{Tr}(w_0,E)|,$$

with the sum over all irreducible representations E of W.

Next, we note that there exists an imbedding $W_1^\vee \subset W^\vee$ of the set of irreducible representations of W_1 into the analogous set for W with the following property: if $E_1 \in W_1^\vee$ corresponds to $E \in W^\vee$, then $\mathrm{Tr}(w_0,E) = \pm\dim E_1$, and if $E \in W^\vee$ is not in the image of our imbedding, then $\mathrm{Tr}(w_0,E) = 0$. This can be seen by direct computation, using Murnaghan's rule, or one can argue as follows. We can regard W as the Weyl group of a unitary group over a finite field F_{p^r} and W_1 as the relative Weyl group. The unipotent representations of the unitary group are parametrized by the elements of $W^\vee$ and the unipotent representation corresponding to $E \in W^\vee$ has degree given by a polynomial in p^r which for $r \to 0$ becomes $\pm\mathrm{Tr}(w_0,E)$. This polynomial is divisible by $(p^r - 1)$ unless the unipotent representation is in the principal series. The unipotent representations in the principal series are parametrized by the irreducible representations of W_1 and the representation corresponding to $E_1 \in W_1^\vee$ has degree given by a polynomial in p^r which for $r \to 0$ becomes $\dim E_1$. Hence our assertion follows. We seen then from (10.3) that

$$n(W) = \sum \dim(E_1),$$

with the sum over all irreducible representations E_1 of $W_1^\vee$, hence

$$n(W) = n'(W_1).$$

Comparing with (10.1) and (10.2) it follows that

$$n(W_1) = n(W) = n'(W_1)$$

hence we have proved the following.

11. Theorem. *Each left cell of W_1 (with respect to φ) carries an irreducible representation of W_1. It contains a unique involution and is the intersection of W_1 with a left cell of W (= symmetric group $\mathfrak{S}_{n+1}$).*

12. In the case where (W,S) is a Weyl group and $\varphi(s) = q$ for all $s \in S$, one can use the character formulas $[L_2]$ for the unipotent representations of a semisimple group over a finite field, to get some new information on the structure of left cells in W. For example, when (W,S) is of type B_n or D_n one can prove $[L_2]$ that

(12.1) *Any left cell in W carries a representation of W which is multiplicity free and has a number of irreducible components equal to a power of 2. Moreover the set of irreducible components can be organized in a natural way as a vector space over the field with 2 elements.*

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