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FEATURED REVIEW.

Let G be the group of rational points of a connected reductive group defined over a p-adic field F (not assumed to be of characteristic 0). If P is a parahoric subgroup of G with pro-unipotent radical U the quotient M is the group of rational points of a reductive group defined over the residue field $\mathbf{F}_q$ of F. If $(\pi, \mathcal{V})$ is an irreducible smooth representation of G we say that it is of level 0 if, when restricted to some P as above, the finite-dimensional subspace $\mathcal{V}^U$ of U-fixed vectors is nonzero and contains a cuspidal representation of M. We remark that $\mathcal{V}^U$ is typically zero; if it is not, then, after changing P if necessary, $\mathcal{V}^U$ must contain a cuspidal representation. We say that $(\pi, \mathcal{V})$ is unipotent if it is of level 0 and if, in addition, it contains a unipotent cuspidal representation of M. The best known and most common example of a unipotent representation is the case when $(\pi, \mathcal{V})$ contains a nontrivial Iwahori fixed vector: in other words, the parahoric in question is an Iwahori subgroup I so that M is a torus and the unipotent cuspidal representation is the trivial character.

Here are two facts concerning level-0 representations: (i) If σ is an irreducible cuspidal representation of M as above one can consider the intertwining algebra $\mathcal{H}(\sigma)$ of the smooth representation c-$\mathrm{Ind}_P^G(\check{\sigma})$, where "$^\vee$" denotes the contragredient. The structure of this algebra was investigated by the reviewer [Invent. Math. **114** (1993), no. 1, 1–54; MR1235019 (94g:22035)]; roughly speaking, it is the tensor product of a twisted group algebra by an affine Iwahori-Hecke algebra with not necessarily equal parameters; this is quite analogous to the situation for parabolic induction in groups over finite fields, and indeed the proof uses the theory of R. B. Howlett and G. I. Lehrer [Invent. Math. **58** (1980), no. 1, 37–64; MR0570873 (81j:20017)] in a crucial way. The group algebra may exhaust the intertwining algebra; however, if σ is unipotent the classification of such representations implies that the group algebra piece is very small and comes from automorphisms of P, and the 2-cocycle is trivial. (ii) The category of smooth representations of G which are generated by their σ-isotypic vectors is equivalent to the category of nondegenerate $\mathcal{H}(\sigma)$-modules. This was proved by the reviewer [Compositio Math., to appear], and by A. Moy and G. Prasad [Comment. Math. Helv. **71** (1996), no. 1, 98–121; MR1371680 (97c:22021)]; although these proofs are different, each relies to some extent on the theory of types due to C. J. Bushnell and P. Kutzko [Proc. London Math. Soc. (3), to appear]. This generalises a well-known result of A. Borel [Invent. Math. **35** (1976), 233–259; MR0444849 (56 #3196)] for the Iwahori case. It is worth pointing out, however, that under quite general hypotheses one can show that there are bijections between the isomorphism classes of irreducible objects on each side here. This does not depend on knowing a categorical equivalence, and proofs can be found in the paper by the reviewer cited above, as well as in Bushnell and Kutzko [*The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Ann. of Math. Stud., 129, Princeton Univ. Press, Princeton, NJ, 1993; MR1204652 (94h:22007)].

Now assume G is split simple adjoint, and write ${}^LG$ for the connected component of a dual group for G—we are deliberately vague here—so that ${}^LG$ is simply connected. In the present paper the author establishes a bijection between the set of isomorphism classes of irreducible unipotent representations of G and the set of triples (s, N, ρ), where $s \in {}^LG$ is semisimple, $N \in \mathrm{Lie}{}^LG$ is nilpotent, with $\mathrm{Ad}(s)N = qN$, where q is the order of the residue field of F, and ρ is an irreducible representation of the component group $Z_{{}^LG}(s, N)/Z^0_{{}^LG}(s, N)$ which is trivial on the image of the centre $Z_{{}^LG}$. He obtains variants for the inner forms of G, by placing no restriction on ρ. This completely proves an earlier conjecture of the author's [Trans. Amer. Math. Soc. **277** (1983), no. 2, 623–653; MR0694380 (84j:22023)]. The Iwahori part of the conjecture was established earlier by D. Kazhdan and the author [Invent. Math. **87** (1987), no. 1, 153–215; MR0862716 (88d:11121)] using equivariant K-homology: they showed that the equivariant K-homology tensored by $\mathbf{C}$ of the variety of triples $\{(u, B, B')|\ B, B'$ Borel groups in ${}^LG$, u unipotent $\in B \cap B'\}$ provides a model for the regular representation of the Iwahori-Hecke algebra $\mathcal{H}$. They were then able to construct a "standard" $\mathcal{H}$-module for each triple (s, N, ρ); this module has a unique irreducible quotient, and this provides the desired correspondence, by Borel's version of (ii) above. Unfortunately, the equivariant K-homology method runs into serious difficulties if one tries to extend it to the general conjecture. In the paper under review, following a strategy outlined in his address at the 1990 International Mathematics Congress in Kyoto [in *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, 155–174, Math. Soc. Japan, Tokyo, 1991; MR1159211 (93e:20059)], the author proves his conjecture essentially by replacing affine Hecke algebras by graded versions and using his parametrisation of the irreducible modules of these latter algebras. Indeed, in a series of previous papers he has used (ordinary) equivariant homology and perverse sheaves in the equivariant derived category to construct and parametrise such modules. In particular this method provides a new proof of the Iwahori case.

We now try to provide a rough, and sometimes heuristic, sketch of some of the steps in the bijection itself, beginning with the geometric (complex) side.

(a) First, given a nilpotent orbit $\mathcal{C}$ of some Levi component L of a connected reductive complex algebraic group ${}^LG$ and an irreducible equivariant cuspidal local system $\mathcal{F}$ of this orbit the author has constructed a graded Hecke algebra $\overline{H}$ over $\mathbf{C}[r]$ (r an indeterminate) with typically unequal parameters [see Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 145–202; MR0972345 (90e:22029); in *Representations of groups (Banff, AB, 1994)*, 217–275, Amer. Math. Soc., Providence, RI, 1995; MR1357201 (96m:22038)(with errata for Part I)]. For example, if the Levi component is a maximal torus with the trivial nilpotent orbit this algebra is a graded algebra associated to a filtration of the standard affine Hecke algebra over $\mathbf{C}(v, v^{-1})$. In any case the graded Hecke algebra can be usefully identified with the full endomorphism ring of a certain equivariant perverse sheaf associated to $\mathcal{F}$ in the equivariant derived category $\mathcal{D}_{{}^LG\times\mathbf{C}^*}\mathrm{Lie}{}^LG$, or again, as a graded vector space, with the full equivariant homology of a generalised variety of triples with values in an associated local system.

(b) Now fix ${}^LG$ and consider the equivalence relation on the triples $(L, \mathcal{C}, \mathcal{F})$ in (a) obtained via the action of ${}^LG$. This provides a finite collection of graded Hecke algebras $\overline{H}_1, \cdots, \overline{H}_k$. Let $\mathrm{Irr}_{r_0}\overline{H}_i$ denote the set of isomorphism classes of irreducible modules of $\overline{H}_i$ on which the indeterminate r acts via the real number r_0. The author has shown that there is a bijection from

$\coprod_i \mathrm{Irr}_{r_0}\overline{H}_i$ to the set whose elements consist of triples (s, N, ρ) as above except that s is now in $\mathrm{Lie}^L G$ and q is replaced by $2r_0$. We refer the reader to the author's two-part paper [op. cit.; MR0972345 (90e:22029); op. cit.; MR1357201 (96m:22038)], especially Part II, Section 8.

(c) In particular, if ${}^L G$ is simply connected and almost simple with maximal torus ${}^L T = {}^L T_c \times {}^L T_h$ decomposed into its compact and hyperbolic parts, and $t' \in {}^L T_c$, we may apply (a) and (b) above to the connected reductive group $Z_{{}^L G}(t')$.

(d) Fix $({}^L G, {}^L T)$ as in (c) and let I be an index set in one-to-one correspondence with a set of simple affine roots (or the extended Dynkin diagram) for this pair. To $J \subsetneq I$ one can associate a connected reductive group ${}^L G_J \subsetneq {}^L G$. Fix a nilpotent orbit $\mathcal{C}$ on $\mathrm{Lie}^L G_J$ and an irreducible ${}^L G_J$-equivariant cuspidal local system $\mathcal{F}$ on $\mathcal{C}$. In Section 5 of the paper under review the author associates an affine Hecke algebra $H({}^L G, {}^L G_J, \mathcal{C}, \mathcal{F})$ with unequal parameters starting from a certain root datum $(\mathcal{R}, \mathcal{L}, \mathcal{R}', \mathcal{L}')$ arising from the triple $({}^L G_J, \mathcal{C}, \mathcal{F})$. Let $\mathcal{W}$ denote the Weyl group associated to this root datum; let $\mathcal{T}$ denote the complex algebraic torus whose cocharacter group is $\mathcal{L}'$. The centre $\mathcal{Z}$ of $H({}^L G, {}^L G_J, \mathcal{C}, \mathcal{F})$ can be identified with the algebra of regular functions of the quotient $\mathcal{W}\backslash\mathcal{T} \times \mathbf{C}^*$; thus a character of $\mathcal{Z}$ can be identified with a pair $(\mathcal{W}t, v_0)$, $t \in \mathcal{T}$. Let $\mathrm{Irr}_{\mathcal{W}t,v_0} H({}^L G, {}^L G_J, \mathcal{C}, \mathcal{F})$ denote the set of isomorphism classes of irreducible $H({}^L G, {}^L G_J, \mathcal{C}, \mathcal{F})$ modules which transform on $\mathcal{Z}$ by the character $(\mathcal{W}t, v_0)$.

(e) Take a pair $(\mathcal{W}t, v_0)$ as in (d) above and decompose $t = t_c t_h$ into its compact and hyperbolic parts. We assume v_0 is a positive real number. To t_c the author associates an element $t_c^\psi \in T_c$, and ${}^L G_J$ is a Levi component of $Z_{{}^L G}(t_c^\psi)$. One then has the graded Hecke algebra $\overline{H}(Z_{{}^L G}(t_c^\psi), {}^L G_J, \mathcal{C}, \mathcal{F})$ as in (a). The element $\log t_h$ lies in $\mathrm{Lie}\,\mathcal{T}$ and this can be identified with $\mathrm{Lie}\, Z_{{}^L G_J} \subset \mathrm{Lie}^L T$. Just as in (d) one has the set $\mathrm{Irr}_{\mathcal{W}_{t_c}\log t_h, r_0}\overline{H}(Z_{{}^L G}(t_c^\psi), {}^L G_J, \mathcal{C}, \mathcal{F})$, where $r_0 = \log v_0$ and the Weyl group $\mathcal{W}_{t_c}$ may be identified with $N_{Z_{{}^L G}(t_c^\psi)}{}^L(G_J)/{}^L G_J$. The author then applies the results of his paper [J. Amer. Math. Soc. **2** (1989), no. 3, 599–635; MR0991016 (90e:16049)] to establish a bijection $\mathrm{Irr}_{\mathcal{W}t,v_0} H({}^L G, {}^L G_J, \mathcal{C}, \mathcal{F}) \to \mathrm{Irr}_{\mathcal{W}_{t_c}\log t_h, r_0}\overline{H}(Z_{{}^L G}(t_c^\psi), {}^L G_J, \mathcal{C}, \mathcal{F})$.

(f) Fix a positive number v_0, and let $\overline{\mathfrak{G}}$ denote the collection of triples $(J, \mathcal{C}, \mathcal{F})$. By varying the orbits of $\mathcal{T}$ under $\mathcal{W}$, and combining the results in (c)–(e), the author obtains a bijection of $\coprod_{(J,\mathcal{C},\mathcal{F})\in\overline{\mathfrak{G}}} \mathrm{Irr}\, H_{v_0}({}^L G, {}^L G_J, \mathcal{C}, \mathcal{F})$ with the set $\mathfrak{G}(v_0)$ of triples in the introduction, except that q is replaced by v_0^2. (See Theorem 5.21 of the paper.) Here $H_{v_0}(\cdots)$ denotes the algebra $H(\cdots)$ specialised at v_0. Note that if one takes $v_0 = 1$, the left side of the bijection becomes the collection of irreducible representations of a certain finite set of generalised affine Weyl groups, while the right side becomes the collection of pairs (g, ρ): $g \in {}^L G$, ρ an irreducible representation of the component group of $Z_{{}^L G}(g)$.

(g) Let $S'(I)$ denote the (finite) group of characters of $Z_{{}^L G}$. If $\chi \in S'(I)$ then in (f) above we can consider the sets $\overline{\mathfrak{G}}_\chi, \mathfrak{G}_\chi(v_0)$: the first set denotes triples $(J, \mathcal{C}, \mathcal{F})$ where $Z_{{}^L G}$ acts on each stalk of $\mathcal{F}$ by χ; the second denotes triples (s, N, ρ) where $\rho|\mathrm{image}\, Z_{{}^L G}$ acts by χ. The bijection in (f) respects this decomposition. The elements $\overline{\mathfrak{G}}_\chi$ are called by the author geometric diagrams of type χ.

We remark that much of §§ 2–5 of the paper is concerned with carefully establishing the bijections in (d)–(f) above. We now pass to the p-adic (arithmetic) side.

(h) Let $\tilde{F}$ denote the unramified closure of the field F. We take a split adjoint simple F-group $\mathbf{G}_0$ and fix an inner form $\mathbf{G}$ of $\mathbf{G}_0$; the Bruhat-Tits classification implies that $\mathbf{G}$ splits over $\tilde{F}$ and we identify $\mathbf{G}$ with its $\tilde{F}$-points. Take the complex group ${}^L G$ in (a)–(g) above to be the connected component of the dual group of $\mathbf{G}$. (We are being careless here.) The Galois action on $\mathbf{G}$ is determined by a Frobenius element Φ which induces a permutation u on the vertices $\mathbf{I}$ of the ($\tilde{F}$-) local Dynkin diagram of $\mathbf{G}$. This permutation can be identified with an element χ of the group $S'(I)$ in (g) by a result of Kneser-Kottwitz.

(i) Let P be a parahoric subgroup of $G = \mathbf{G}^\Phi$; we write $\widehat{P}$ for the full centraliser of the facet in the affine enlarged building of G fixed by P. Up to G-conjugacy P corresponds to a u-stable subset $\mathbf{J}$ of $\mathbf{I}$, and we suppose that its reductive quotient M admits a unipotent cuspidal representation $\mathbf{E}$, with contragredient $\check{\mathbf{E}}$. This forces strong restrictions on both M and P. Let ψ be a character of the "component" group $\widehat{P}/P$. The triple $(\mathbf{J}, \mathbf{E}, \psi)$ is called by Lusztig an arithmetic diagram of type u; let $\overline{\mathfrak{A}}_u$ denote the collection of such diagrams. Write $\mathcal{H}(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$ for the intertwining algebra End_G (c-$\mathrm{Ind}_P^G(\check{\mathbf{E}})$) (compact induction); the irreducible unital modules of this algebra are in one-to-one correspondence with the irreducible admissible representations of G which contain $\mathbf{E}$ by (ii) above. Now, this algebra can be decomposed into a direct sum of two-sided ideals $\mathcal{H}^\psi$ corresponding to central idempotents parametrised by the ψ above, and $\mathcal{H}^\psi$ is isomorphic with the algebra End_G (c-$\mathrm{Ind}_{\widehat{P}}^G(\check{\mathbf{E}}_\psi)$), where $\check{\mathbf{E}}_\psi$ is an extension of $\check{\mathbf{E}}$ to $\widehat{P}$, parametrised by ψ. This last assertion depends on the fact that $\mathbf{E}$ is unipotent. Each of the algebras $\mathcal{H}^\psi$ is noncanonically isomorphic to a fixed algebra $\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$ which has the same presentation as End_G (c-$\mathrm{Ind}_P^G(\check{\mathbf{E}})$) except that the group algebra part can be identified with the group algebra of $N_G(P)/\widehat{P}$. Let $\mathcal{U}(G)$ denote the set of isomorphism classes of irreducible unipotent representations of G, and let $\mathrm{Irr}\,\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$ denote isomorphism classes of irreducible unital modules of the algebra $\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$. Combining these facts one obtains a bijection $\mathcal{U}(G) \to \coprod_{(\mathbf{J}, \mathbf{E}, \psi) \in \overline{\mathfrak{A}}_u} \mathrm{Irr}\,\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$.

(j) Fix $\chi \in S'(I)$ corresponding to u. The desired bijection $\coprod_{(\mathbf{J}, \mathbf{E}, \psi) \in \overline{\mathfrak{A}}_u} \mathrm{Irr}\,\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E}) \to \coprod_{(J, \mathcal{C}, \mathcal{F}) \in \overline{\mathfrak{G}}_\chi} \mathrm{Irr}\, H_{v_0}({}^L G, {}^L G_J, \mathcal{C}, \mathcal{F})$ is realised in §7 of the paper under review by matching families of arithmetic diagrams with families of geometric diagrams case by case, and specialising the bijection in (g) to $v_0 = \sqrt{q}$, where q is the order of the residue field of F. For this the author is guided by earlier bijections he has established in the classification of character sheaves. (The interested reader may wish to look at the reviewer's paper [Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 5, 639–667; MR1399618 (97g:22013)] for a special case.)

This is an important paper which will repay close reading.

Reviewed by *Lawrence Morris*

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Classification of Unipotent Representations of Simple p-adic Groups

George Lusztig

0 Introduction

0.1

Let us first consider the group $\mathcal{G}_0$ of $\mathbf{F}_q$-rational points of a reductive connected algebraic group defined over a finite field $\mathbf{F}_q$. Among the irreducible (complex) representations of $\mathcal{G}_0$, there is a remarkable class of representations, called *unipotent*, which were defined in [DL] by a cohomological method.

The unipotent representations of $\mathcal{G}_0$ were classified in [L6]; their classification is independent of q. At one extreme, one has the irreducible representations which have nonzero vectors fixed by a Borel subgroup. (They are in bijection with the irreducible representations of the Weyl group, assuming that we are in the split case.) At the other extreme, one has the unipotent cuspidal representations. These are quite rare: for example, the group $SO_{2n+1}(\mathbf{F}_q)$ (and also $Sp_{2n}(\mathbf{F}_q)$) has a unique unipotent cuspidal representation if n is *twice a triangular number*, and none, otherwise.

0.2

Now let G be a simply connected almost simple algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. Consider the set of all pairs $(\mathcal{C}, \mathcal{F})$ where $\mathcal{C}$ is a nilpotent G-orbit on $\mathfrak{g}$ and $\mathcal{F}$ is an irreducible, G-equivariant local system on $\mathcal{C}$ (up to isomorphism). The set of all such pairs is in a certain sense similar to the set of unipotent representations of $\mathcal{G}_0$. At one extreme, one has the pairs $(\mathcal{C}, \mathcal{F})$, which enter in the Springer correspondence [S]. (They are in bijection with the irreducible representations of the Weyl group.) At the other extreme,

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one has the cuspidal pairs (see [L5]). These are quite rare: for example, the group $Sp_{2n}(\mathbf{C})$ has a unique cuspidal pair if n is a *triangular number*, and none, otherwise. Note that the picture here, although reminiscent of that in 0.1, is different from that in 0.1.

0.3

The main theme of this paper is that the pictures in 0.1 and 0.2 come together in the representation theory of $\mathcal{G}$, the group of rational points of a split adjoint simple algebraic group over a nonarchimedean local field.

We shall be interested in a particular class of irreducible admissible representations of $\mathcal{G}$ that we call *unipotent*, since they are analogous to the unipotent representations of a reductive group over a finite field. Namely, an irreducible admissible representation of $\mathcal{G}$ is said to be unipotent if its restriction to some parahoric subgroup contains a subspace on which the parahoric subgroup acts through a unipotent cuspidal representation of the "reductive quotient" of that parahoric subgroup (a reductive group over a finite field).

(If, in this definition, we replace $\mathcal{G}$ by $\mathcal{G}_0$ of 0.1 and "parahoric" by "parabolic," we recover the class of unipotent representations of $\mathcal{G}_0$.)

According to the Langlands philosophy, it should be possible to parametrize the set of isomorphism classes of unipotent representations of our group in terms of the geometry of G (as in 0.2) assumed to be of type dual to that of $\mathcal{G}$. Indeed, in [L4] it was conjectured that the set of isomorphism classes of unipotent representations of $\mathcal{G}$ is naturally in one-to-one correspondence with the set of triples $(s, y, \mathcal{V})$ (modulo the natural action of G) where

s is a semisimple element of G,

y is a nilpotent element of the Lie algebra of G such that $\mathrm{Ad}(s)y = qy$ (here q is the number of elements in the residue field of our local field), and

$\mathcal{V}$ is an irreducible representation (up to isomorphism) of the group of components of the simultaneous centralizer of s and y in G, on which the centre of G acts trivially.

In the case when we restrict ourselves to representations with nonzero vectors fixed by an Iwahori subgroup, our conjecture specializes to a conjecture of Deligne-Langlands (in a refined form, due to the presence of $\mathcal{V}$). In [L7], the idea that equivariant K-theory should play a role in this problem was formulated, and in [KL], the proof of the (refined form of the) Deligne-Langlands conjecture was given using equivariant K-theory. (In [G], a partial proof of that conjecture was given; see also [CG].)

In this paper we prove the general conjecture stated above. The strategy of the proof was sketched in [L12], and several steps of that strategy were carried out in [L10], [L11], [L13], [L14]. First, one observes that the endomorphism algebra of a representation induced by a unipotent cuspidal representation of (the reductive quotient of a) parahoric subgroup is an affine Hecke algebra with an explicit presentation similar to that given by Iwahori and Matsumoto, but the parameters can be unequal and almost arbitrary. Then, one would like to know that the unipotent representations under consideration can be parametrized by the simple modules of the endomorphism algebras above. This was recently established by Moy and Prasad [MP2] and Morris [M2], generalizing an earlier result of Borel [B] and Matsumoto [Ma]. Thus, we are reduced to the problem of classifying the simple modules of certain affine Hecke algebras with unequal parameters. One would like to see these affine Hecke algebras in a purely geometric context, in terms of the complex group G. In this paper we show that, given a subgroup of G which is the centralizer of some semisimple element and a unipotent class of the subgroup which carries a cuspidal local system, one can construct explicitly an affine Hecke algebra (usually with unequal parameters). Remarkably, the affine Hecke algebras arising from the geometry of G are the same as the affine Hecke algebras arising from the representation theory of $\mathcal{G}$. We are therefore reduced to the problem of classifying the simple modules of the affine Hecke algebras attached to cuspidal local systems.

Our approach is as follows: the affine Hecke algebra in question can be "linearized" with respect to various points in the spectrum of its centre. The linearized algebras are the "graded Hecke algebras" (see [L10], [L11]). These algebras are simpler than the affine Hecke algebra; however, from their representation theory, one can recover the representation theory of the affine Hecke algebra, as shown in [L11]. (This is in some sense similar to the way in which much of the representation theory of a Lie group can be recovered from that of its Lie algebra.) We are therefore reduced to the problem of classifying the simple modules of the linearized algebras. It turns out that all the linearized algebras arising in our case are of the type studied in [L10], using equivariant homology and perverse sheaves. In particular, their representation theory is understood from [L14].

Thus, the desired classification can be accomplished. (We note that our proof does not use K-theory at all.)

The classification of the unipotent representations of inner forms of $\mathcal{G}$ is also obtained in this paper (see Corollary 6.5). The ideas of this paper are expected to apply, more generally, to any form of $\mathcal{G}$ which becomes split after an unramified extension of the ground field; there is some discussion of this in §8.

Contents

1 Endomorphism algebras of certain induced representations

1.1

Let $\mathbf{K}$ be a nonarchimedean local field. Let A be the ring of integers of $\mathbf{K}$, let $\mathfrak{m}$ be the maximal ideal of A, and let q be the cardinal of the finite field $A/\mathfrak{m}$; we write $\mathbf{F}_q$ instead of $A/\mathfrak{m}$. Let $\tilde{\mathbf{K}}$ be a maximal unramified extension of $\mathbf{K}$. Let $\tilde{A}$ be the ring of integers of $\tilde{\mathbf{K}}$, and let $\tilde{\mathfrak{m}}$ be the maximal ideal of $\tilde{A}$. Then $\mathbf{k} = \tilde{A}/\tilde{\mathfrak{m}}$ is an algebraic closure of $A/\mathfrak{m} = \mathbf{F}_q$. Let $F : \tilde{\mathbf{K}} \to \tilde{\mathbf{K}}$ be the unique field automorphism such that $F(x) = x$ for all $x \in \mathbf{K}$ and $F(x) = x^q \bmod \tilde{\mathfrak{m}}$ for all $x \in \tilde{A}$.

Let $\mathbf{G}$ be a connected, adjoint simple algebraic group, defined over $\mathbf{K}$. (For the classification of the possible such $\mathbf{G}$, see Kneser [K] for the case where $\mathbf{K}$ has characteristic zero, and Bruhat and Tits [BT] for general $\mathbf{K}$.)

Unless otherwise stated, it is assumed that $\mathbf{G}$ is split over $\tilde{\mathbf{K}}$. Then $\mathbf{G}(\mathbf{K})$ is the fixed-point set of the natural action of the Galois group $\mathrm{Gal}(\tilde{\mathbf{K}}/\mathbf{K})$ on $\mathbf{G}(\tilde{\mathbf{K}})$. Since F is a topological generator of $\mathrm{Gal}(\tilde{\mathbf{K}}/\mathbf{K})$, the action above gives us a bijection $F : \mathbf{G}(\tilde{\mathbf{K}}) \to \mathbf{G}(\tilde{\mathbf{K}})$ whose fixed-point set is $\mathbf{G}(\mathbf{K})$. We shall write $\mathbf{G}$ instead of $\mathbf{G}(\tilde{\mathbf{K}})$; then we have $F : \mathbf{G} \to \mathbf{G}$ (*"Frobenius map"*) and $\mathbf{G}(\mathbf{K}) = \mathbf{G}^F$ (fixed-point set). (We shall generally denote the fixed-point set of a map $f : Z \to Z$ by Z^f.)

1.2 Parahoric subgroups

Let $\mathcal{B}$ be the set of all subgroups of $\mathbf{G}$ that are conjugate to the subgroup of $\mathbf{G}$ denoted by B in [IM, 2.1] (whose definition uses a fixed *épinglage* of G). The subgroups in $\mathcal{B}$ are called *Iwahori subgroups* of $\mathbf{G}$. Note that any Iwahori subgroup of $\mathbf{G}$ is contained in $\mathbf{G}'$, the derived group of $\mathbf{G}$ (a subgroup of finite index of $\mathbf{G}$).

A subgroup of $\mathbf{G}$ is said to be *parahoric* (cf. [BT]) if it contains some Iwahori subgroup and is contained strictly in $\mathbf{G}'$.

Any parahoric subgroup $\mathbf{P}$ can be naturally regarded as a proalgebraic group over $\mathbf{k}$. It has a canonical normal subgroup $U_{\mathbf{P}}$ which is prounipotent, so that the quotient $\bar{\mathbf{P}} = \mathbf{P}/U_{\mathbf{P}}$ is naturally a connected, reductive algebraic group over $\mathbf{k}$.

The set $\mathcal{B}^F = \{B \in \mathcal{B} \mid FB = B\}$ is nonempty (see [BT]).

1.3

A number of properties of $\mathbf{G}$ or of its subgroups are analogues of familiar properties of reductive groups over a finite field. For example,

(a) *if* $B \in \mathcal{B}^F$, *the Lang map* $b \to b^{-1}F(b)$ *of* B *into itself is surjective.*

In other words, "Lang's theorem" holds for B. This can be deduced from the usual Lang's theorem for connected algebraic groups over $\mathbf{k}$, since B is connected and proalgebraic over $\mathbf{k}$. We will often use this extension of Lang's theorem to proalgebraic groups like $U_{\mathbf{P}}, \mathbf{P}, \ldots$.

From (a), it follows, in the standard way, that the action of $\mathbf{G}^F$ on $\mathcal{B}^F$ (by conjugation) is transitive.

1.4

Let $\mathbf{P}$ be a parahoric subgroup of $\mathbf{G}$ such that $F\mathbf{P} = \mathbf{P}$. Then $F : \mathbf{P} \to \mathbf{P}$ induces a bijection $F : \bar{\mathbf{P}} \to \bar{\mathbf{P}}$ which may be regarded as a Frobenius map of an $\mathbf{F}_q$-rational structure of the reductive group $\bar{\mathbf{P}}$. Taking fixed points of F, we obtain subgroups $\mathbf{P}^F, U_{\mathbf{P}}^F, \bar{\mathbf{P}}^F$. (The last group is finite.) Note that $U_{\mathbf{P}}^F$ is a normal subgroup of $\mathbf{P}^F$ and $\bar{\mathbf{P}}^F = \mathbf{P}^F/U_{\mathbf{P}}^F$. (To see this, we use "Lang's theorem" for $U_{\mathbf{P}}$.) Let $\pi_{\mathbf{P}} : \mathbf{P}^F \to \bar{\mathbf{P}}^F$ be the natural projection.

1.5

We now fix $\mathbf{P}$ as in 1.4, and we assume that there is given a unipotent cuspidal (irreducible) representation (over $\mathbf{C}$) of the finite group $\bar{\mathbf{P}}^F$; let $\mathbf{E}$ be the vector space (over $\mathbf{C}$) of this representation. We regard $\mathbf{E}$ as a $\mathbf{P}^F$-module via the surjective homomorphism $\pi_{\mathbf{P}} : \mathbf{P}^F \to \bar{\mathbf{P}}^F$. Let $\mathcal{X}$ be the $\mathbf{C}$-vector space consisting of all functions $f : \mathbf{G}^F \to \mathbf{E}$ such that

(a) $f(gp) = p^{-1}f(g)$ for all $g \in \mathbf{G}^F, p \in \mathbf{P}^F$ and

(b) $\operatorname{supp} f = \{g \in \mathbf{G}^F \mid f(g) \neq 0\}$ is contained in a union of finitely many $\mathbf{P}^F$-cosets in $\mathbf{G}^F/\mathbf{P}^F$.

For any $g' \in \mathbf{G}^F$ and $f \in \mathcal{X}$, the function $g'f : \mathbf{G}^F \to \mathbf{E}$ given by $(g'f)(g) = f(g'^{-1}g)$ belongs to $\mathcal{X}$. This defines a representation of $\mathbf{G}^F$ on the vector space $\mathcal{X}$. We have a direct sum decomposition

$$\mathcal{X} = \oplus_{g\mathbf{P}^F} \mathcal{X}_{g\mathbf{P}^F}$$

where $g\mathbf{P}^F$ runs over the cosets $\mathbf{G}^F/\mathbf{P}^F$ and $\mathcal{X}_{g\mathbf{P}^F} = \{f \in \mathcal{X} \mid \operatorname{supp} f \in g\mathbf{P}^F\}$.

Let $\mathcal{H}(\mathbf{G}^F; \mathbf{P}, \mathbf{E}) = \mathcal{H}$ be the endomorphism algebra of the $\mathbf{G}^F$-module $\mathcal{X}$. Let $\operatorname{Irr}\mathcal{H}$ be the set of isomorphism classes of (finite-dimensional) simple $\mathcal{H}$-modules.

1.6

Let $\operatorname{Irr}(\mathbf{G}^F; \mathbf{P}, \mathbf{E})$ be the set of isomorphism classes of irreducible admissible representations of $\mathbf{G}^F$ with the following property: The subspace of vectors fixed by $U_{\mathbf{P}}^F$, regarded as a $\bar{\mathbf{P}}^F$-module in the natural way, contains the $\bar{\mathbf{P}}^F$-module $\mathbf{E}$. We shall need the following result.

(a) *There is a natural one-to-one correspondence between the sets* $\operatorname{Irr}(\mathbf{G}^F; \mathbf{P}, \mathbf{E})$ *and* $\operatorname{Irr}\mathcal{H}$.

This result was proved by Morris [M2], under the assumption that $\mathbf{G}$ is split over $\mathbf{K}$, and by Moy and Prasad [MP2], without any assumption. Both of these proofs rely on the paper [BK] of Bushnell and Kutzko. In these references, the bijection (a) is obtained from a stronger statement, namely an equivalence of two categories of representations. (The assumption that $\mathbf{E}$ is unipotent is not needed for these results.) The case where $\mathbf{P}$ is an Iwahori subgroup and $\mathbf{E}$ is the unit representation was considered earlier by Borel [B] and Matsumoto [Ma].

Let us now consider another F-stable parahoric subgroup $\mathbf{P}'$ of $\mathbf{G}$ and a unipotent cuspidal (irreducible) representation (over $\mathbf{C}$) of the finite group $\bar{\mathbf{P}}'^F$ on the vector space $\mathbf{E}'$.

(b) *The sets* $\operatorname{Irr}(\mathbf{G}^F; \mathbf{P}, \mathbf{E}), \operatorname{Irr}(\mathbf{G}^F; \mathbf{P}', \mathbf{E}')$ *are either equal or disjoint. The first alternative occurs if and only if there exists* $g \in \mathbf{G}^F$ *which conjugates* $\mathbf{P}$ *to* $\mathbf{P}'$ *and* $\mathbf{E}$ *to a representation isomorphic to* $\mathbf{E}'$.

From [MP1, 5.2] it follows that $\mathbf{P}, \mathbf{P}'$ are associate; the fact that $\mathbf{E}, \mathbf{E}'$ are unipotent cuspidal then forces $\mathbf{P}, \mathbf{P}'$ to be conjugate under $\mathbf{G}^F$.

1.7

Let us fix $(\mathbf{P}, \mathbf{E})$ as in 1.5. From 1.6 (a), we see that the problem of parametrizing the set $\operatorname{Irr}(\mathbf{G}^F; \mathbf{P}, \mathbf{E})$ is the same as the problem of parametrizing the set $\operatorname{Irr}\mathcal{H}$. It is therefore desirable to describe the algebra $\mathcal{H}$ explicitly.

It is a remarkable fact that the algebra $\mathcal{H}$ is an affine Hecke algebra with not necessarily equal parameters. (In the case where $\mathbf{G}$ is split over $\mathbf{K}$, $\mathbf{P}$ is an Iwahori subgroup, and E is the unit representation, this goes back to the fundamental paper of Iwahori and Matsumoto [IM]; the parameters are all equal to q in this case.) In the general case, this fact was stated very briefly in the introduction to [L10]; more details were given in [L12] where the types of the Hecke algebras appearing and the values of the parameters were described without proof. (However, the proof is almost the same as that of the analogous statement in the representation theory of reductive groups over a finite field [L3, 3.26]; the proof of this last statement is given in [L1, §5] for exceptional groups and in [L2, §5] for classical groups.) The proof (which will be sketched in the remainder of this section) is not only analogous to that of [L3, 3.26], but some of its key steps actually rely on [L3, 3.26]. (See also [M1].)

1.8

For any $(\mathbf{P},\mathbf{P})$-double coset Θ in $\mathbf{G}$, we denote by $\mathcal{H}_\Theta$ the set consisting of those $\phi \in \mathcal{H}$ such that

$$\phi(\mathcal{X}_{g\mathbf{P}^F}) \subset \oplus_{g'\mathbf{P}^F \in \mathbf{G}^F/\mathbf{P}^F; g'^{-1}g \in \Theta} \mathcal{X}_{g'\mathbf{P}^F}$$

for some $g\mathbf{P}^F \in \mathbf{G}^F/\mathbf{P}^F$, or equivalently, for any $g\mathbf{P}^F \in \mathbf{G}^F/\mathbf{P}^F$. It is immediate that

$$\mathcal{H} = \oplus_\Theta \mathcal{H}_\Theta$$

where Θ runs over the $(\mathbf{P},\mathbf{P})$-double cosets in $\mathbf{G}$; we may in fact restrict ourselves to Θ such that $F\Theta = \Theta$, since for other Θ we have $\mathcal{H}_\Theta = 0$.

The condition that $F\Theta = \Theta$ is equivalent to the condition that $\Theta \cap \mathbf{G}^F \neq \emptyset$. The second condition implies the first condition trivially; the first condition implies the second condition since Θ is a homogeneous space for $\mathbf{P} \times \mathbf{P}$ and "Lang's theorem" holds for $\mathbf{P} \times \mathbf{P}$. Moreover, if these conditions are satisfied, then $\Theta \cap \mathbf{G}^F$ is a single $(\mathbf{P}^F,\mathbf{P}^F)$-double coset in $\mathbf{G}^F$. (Here we use the fact that "Lang's theorem" holds for the isotropy group in $\mathbf{P} \times \mathbf{P}$ of a point of $\Theta \cap \mathbf{G}^F$, which is an intersection of $\mathbf{P}$ and a $\mathbf{G}^F$-conjugate of $\mathbf{P}$.)

1.9

If $\mathbf{P}_1, \mathbf{P}_2$ are parahoric subgroups of $\mathbf{G}$ in the $\mathbf{G}$-conjugacy class of $\mathbf{P}$, then $\mathbf{P}'_1 = U_{\mathbf{P}_1}(\mathbf{P}_1 \cap \mathbf{P}_2)$ and $\mathbf{P}'_2 = U_{\mathbf{P}_2}(\mathbf{P}_1 \cap \mathbf{P}_2)$ are parahoric subgroups. We say that the pair $(\mathbf{P}_1,\mathbf{P}_2)$ is *good* if $\mathbf{P}'_1 = \mathbf{P}_1$ or, equivalently, if $\mathbf{P}'_2 = \mathbf{P}_2$. We say that the pair $(\mathbf{P}_1,\mathbf{P}_2)$ is *bad* if $\mathbf{P}'_1 \neq \mathbf{P}_1$ or, equivalently, if $\mathbf{P}'_2 \neq \mathbf{P}_2$.

A $(\mathbf{P},\mathbf{P})$-double coset Θ in $\mathbf{G}$ is said to be *good* (resp. *bad*) if the pair $(\mathbf{P}, g\mathbf{P}g^{-1})$ is good (resp. bad) for some, or equivalently any, $g \in \Theta$.

An argument similar to one used by Harish Chandra for reductive groups over a finite field shows that, if Θ is a bad double coset, then $\mathcal{H}_\Theta = 0$. (The argument is based on the fact that, in this case, for $g \in \Theta$, the image of $\mathbf{P} \cap g\mathbf{P}g^{-1}$ in $\bar{\mathbf{P}}$ is a proper parabolic subgroup of $\bar{\mathbf{P}}$. Then the hypothesis that $\mathbf{E}$ is cuspidal is used.) Moreover, if Θ is a good double coset (stable under F), then $\mathcal{H}_\Theta$ has dimension 0 or 1. More precisely, let $g \in \Theta \cap \mathbf{G}^F$. Then we can define two isomorphisms ι_1, ι_2 from $\bar{\mathbf{P}}$ to $\overline{g\mathbf{P}g^{-1}}$. Namely, ι_1 is induced by the homomorphism $\mathbf{P} \to g\mathbf{P}g^{-1}$ given by conjugation by g and ι_2 is defined by the condition $\iota_2(\pi_P(x)) = \pi_{gPg^{-1}}(x)$ for any $x \in \mathbf{P} \cap g\mathbf{P}g^{-1}$. Then $\iota = \iota_2^{-1}\iota_1$ is an automorphism of $\bar{\mathbf{P}}$ defined over $\mathbf{F}_q$, so it restricts to an automorphism of the finite group $\bar{\mathbf{P}}^F$. Composing this automorphism with the homomorphism $\bar{\mathbf{P}}^F \to GL(\mathbf{E})$ defining the representation $\mathbf{E}$ gives us a new $\bar{\mathbf{P}}^F$-module ${}^g\mathbf{E}$; if the $\bar{\mathbf{P}}^F$-module $\mathbf{E}$, ${}^g\mathbf{E}$ are isomorphic, then $\dim \mathcal{H}_\Theta = 1$; otherwise, we have $\dim \mathcal{H}_\Theta = 0$. (The condition that $\mathbf{E}$, ${}^g\mathbf{E}$ are isomorphic is independent of the choice of g in $\Theta \cap \mathbf{G}^F$.) So far, the assumption that $\mathbf{E}$ is unipotent has not been used. We now use this assumption. Then $\mathbf{E}$ is isomorphic to the $\bar{\mathbf{P}}^F$-module obtained by composing $\mathbf{E}$ with any automorphism of $\bar{\mathbf{P}}^F$ which is the restriction of an automorphism of the reductive group $\bar{\mathbf{P}}$, defined over $\mathbf{F}_q$ (a known property of unipotent cuspidal representations). Hence in our case, we have $\dim \mathcal{H}_\Theta = 1$ for any good, F-stable double coset Θ.

1.10

The minimal elements (under inclusion) of the set of parahoric subgroups that are not Iwahori subgroups fall into finitely many $\mathbf{G}'$-orbits $(\mathcal{P}_i)_{i \in \mathbf{I}}$ (under conjugation). Here $\mathbf{I}$ is a (finite) indexing set.

We choose an Iwahori subgroup B such that $B \subset \mathbf{P}$ and $F(B) = B$. Let $\mathbf{W}$ be an indexing set for the set of (B, B)-double cosets in $\mathbf{G}$. The set $\mathbf{W}$ has been described in [IM]. We denote by O_w the double coset corresponding to $w \in \mathbf{W}$. Let $\mathbf{W}' = \{w \in \mathbf{W} \mid O_w \subset \mathbf{G}'\}$, and let $\Omega = \{w \in \mathbf{W} \mid O_w \subset NB\}$. (Here NB is the normalizer of B in $\mathbf{G}$.)

If $i \in \mathbf{I}$, and $\mathbf{P}_i$ denotes the unique element of $\mathcal{P}_i$ containing B, then $\mathbf{P}_i - B = O_w$ for a well-defined $w \in \mathbf{W}'$; we set $w = s_i$. Two (B, B)-double cosets $O_w, O_{w'}$ are said to be *composable* if there exists $w'' \in \mathbf{W}$ such that multiplication defines a bijection $O_w \times_B O_{w'} \xrightarrow{\sim} O_{w''}$. (Here B acts on $O_w \times O_{w'}$ by $b : (g, g') \mapsto (gb^{-1}, bg')$.) We then set $w \cdot w' = w''$.

There is a unique group structure on $\mathbf{W}$ such that the product of w, w' is $w \cdot w'$ (as above) whenever $O_w, O_{w'}$ are composable, and $s_i^2 = 1$ for all $i \in \mathbf{I}$. (The unit element 1 is the (B, B)-double coset B.) Then $\mathbf{W}'$ is the subgroup of $\mathbf{W}$ generated by $\{s_i \mid i \in \mathbf{I}\}$. This is a normal subgroup of $\mathbf{W}$, and Ω is an abelian subgroup of $\mathbf{W}$ which maps isomorphically onto $\mathbf{W}/\mathbf{W}'$. The group $\mathbf{W}'$ together with $\{s_i \mid i \in \mathbf{I}\}$ is a Coxeter group (in fact, an irreducible affine Weyl group).

1.11

Let $\tilde{S}(\mathbf{I})$ be the group of all permutations $u' : \mathbf{I} \xrightarrow{\sim} \mathbf{I}$ such that $s_i \mapsto s_{u'(i)}$ (for $i \in \mathbf{I}$) extends to an automorphism of the group $\mathbf{W}'$.

The elements of $\mathbf{W}'$ which have centralizer of finite index in $\mathbf{W}'$ form a normal subgroup $\mathcal{T}$ of finite index in $\mathbf{W}'$.

A permutation $u' \in \tilde{S}(\mathbf{I})$ is said to be *special* if the corresponding automorphism of $\mathbf{W}'$ restricts to an automorphism of $\mathcal{T}$ given by conjugation by some element of $\mathbf{W}'$.

The special permutations of $\mathbf{I}$ form a normal subgroup $S(\mathbf{I})$ of $\tilde{S}(\mathbf{I})$. If $w \in \Omega$, then there is a unique special permutation $u' : \mathbf{I} \xrightarrow{\sim} \mathbf{I}$ such that $ws_iw^{-1} = s_{u'(i)}$ for all $i \in \mathbf{I}$. This gives an isomorphism $\Omega \xrightarrow{\sim} S(\mathbf{I})$.

1.12

Let $l : \mathbf{W}' \to \mathbf{N}$ be the length function of the Coxeter group $\mathbf{W}'$. We extend this to a function $l : \mathbf{W} \to \mathbf{N}$ by $l(w_1w_2) = l(w_2w_1) = l(w_1)$ for $w_1 \in \mathbf{W}', w_2 \in \Omega$. Then $O_w, O_{w'}$ are composable precisely when $l(ww') = l(w) + l(w')$.

The orbits of $\mathbf{G}'$ on the set of parahoric subgroups (under conjugation) are naturally indexed by the subsets $\mathbf{J}$ of $\mathbf{I}$, distinct from $\mathbf{I}$; the orbit $\mathcal{P}_\mathbf{J}$ corresponding to $\mathbf{J}$ consists of those parahoric subgroups $\mathbf{P}'$ such that the following holds: There exists an Iwahori subgroup B' such that, if $\mathbf{P}'_i$ denotes the unique element of $\mathcal{P}_i$ containing B', then we have $\mathbf{P}'_i \subset \mathbf{P}'$ if and only if $i \in \mathbf{J}$.

For any subset $\mathbf{J}$ of $\mathbf{I}$ distinct from $\mathbf{I}$, we denote by $\mathbf{W}_\mathbf{J}$ the subgroup of $\mathbf{W}'$ generated by $\{s_i \mid i \in \mathbf{J}\}$.

Let $\mathbf{J}$ be the subset of $\mathbf{I}$ such that $\mathbf{P} \in \mathcal{P}_\mathbf{J}$. If Θ is a $(\mathbf{P},\mathbf{P})$-double coset in $\mathbf{G}$, then Θ is a union $\cup O_w$ where w runs over a $(\mathbf{W}_\mathbf{J},\mathbf{W}_\mathbf{J})$-double coset in $\mathbf{W}$. This gives a one-to-one correspondence between the set of $(\mathbf{P},\mathbf{P})$-double cosets in $\mathbf{G}$ and the set of $(\mathbf{W}_\mathbf{J},\mathbf{W}_\mathbf{J})$-double cosets in $\mathbf{W}$. A $(\mathbf{P},\mathbf{P})$-double coset Θ in $\mathbf{G}$ is good if and only if the corresponding $(\mathbf{W}_\mathbf{J},\mathbf{W}_\mathbf{J})$-double coset in $\mathbf{W}$ is contained in $N\mathbf{W}_\mathbf{J}$ (the normalizer of $\mathbf{W}_\mathbf{J}$ in $\mathbf{W}$). In each $(\mathbf{W}_\mathbf{J},\mathbf{W}_\mathbf{J})$-double coset in $\mathbf{W}$ contained in $N\mathbf{W}_\mathbf{J}$, there is a unique element of minimal length. Hence the set $\mathcal{W}$, consisting of the elements of minimal length of the various $(\mathbf{W}_\mathbf{J},\mathbf{W}_\mathbf{J})$-double cosets in $\mathbf{W}$ contained in $N\mathbf{W}_\mathbf{J}$, is an indexing set for the good $(\mathbf{P},\mathbf{P})$-double cosets in $\mathbf{G}$. We shall denote the good $(\mathbf{P},\mathbf{P})$-double coset corresponding to an element $w \in \mathcal{W}$ by Θ_w. Actually, $\mathcal{W}$ is a subgroup of $\mathbf{W}$ (and of $N\mathbf{W}_\mathbf{J}$).

Now F permutes among themselves the (B, B)-double cosets in $\mathbf{G}$; hence it defines a permutation of $\mathbf{W}$, denoted by u. This is a group automorphism which maps $\mathbf{W}'$ into itself, maps Ω into itself, and satisfies $l(u(w)) = l(w)$ for all $w \in \mathbf{W}$. In particular, u restricts to a permutation of $\{s_i \mid i \in \mathbf{I}\}$. We denote again by u the permutation of $\mathbf{I}$ such that $u(s_i) = s_{u(i)}$ for all i. This permutation leaves stable $\mathbf{J}$. Hence u maps $\mathbf{W_J}$ into itself, so that u maps $\mathcal{W}$ into itself. Clearly $w \in \mathcal{W}$ is fixed by u if and only if $F\Theta_w = \Theta_w$.

1.13

From 1.8 and 1.12, we see that $\mathcal{H} = \oplus_{w \in \mathcal{W}^u} \mathcal{H}_{\Theta_w}$; from 1.9, we see that $\dim \mathcal{H}_{\Theta_w} = 1$ for all $w \in \mathcal{W}^u$.

1.14

Let $w, w' \in \mathcal{W}$. We say that $\Theta_w, \Theta_{w'}$ are *composable* if there exists $w'' \in \mathcal{W}$ such that multiplication defines a bijection $\Theta_w \times_{\mathbf{P}} \Theta_{w'} \xrightarrow{\sim} \Theta_{w''}$. (Here $\mathbf{P}$ acts on $\Theta_w \times \Theta_{w'}$ by $b : (g, g') \mapsto (gb^{-1}, bg')$.)

If $\Theta_w, \Theta_{w'}$ are composable and $u(w) = w$, $u(w') = w'$, then w'' above satisfies $u(w'') = w''$ and, using "Lang's theorem" for $\mathbf{P}$, it follows that multiplication defines a bijection $(\Theta_w^F) \times_{\mathbf{P}^F} (\Theta_{w'}^F) \xrightarrow{\sim} \Theta_{w''}^F$. (Here $\mathbf{P}^F$ acts on $(\Theta_w^F) \times (\Theta_{w'}^F)$ by $b : (g, g') \mapsto (gb^{-1}, bg')$.)

For example, if $w, w' \in \mathcal{W}^u$ are such that $l(ww') = l(w) + l(w')$ (length in $\mathbf{W}$), then $\Theta_w, \Theta_{w'}$ are composable and w'' above is equal to ww'. (Compare to [L1, 5.4].) Hence, multiplication defines a bijection $(\Theta_w^F) \times_{\mathbf{P}^F} (\Theta_{w'}^F) \xrightarrow{\sim} \Theta_{ww'}^F$. From this, one can deduce that, for w, w' such that $l(ww') = l(w) + l(w')$, the following holds.

(a) *If* t_w *(resp.* $t_{w'}$*) is a basis element of the one-dimensional vector space* $\mathcal{H}_{\Theta_w}$ *(resp.* $\mathcal{H}_{\Theta_{w'}}$*), then* $t_w t_{w'}$ *is a basis element of the one-dimensional vector space* $\mathcal{H}_{\Theta_{ww'}}$.

(Here the product $t_w t_{w'}$ is given by composition of endomorphisms of $\mathcal{X}$.)

1.15

Let $\mathcal{W}'$ (resp. $\bar{\Omega}$) be the set consisting of the elements of minimal length of the various $(\mathbf{W_J}, \mathbf{W_J})$-double cosets in $\mathbf{W}'$ (resp. Ω) contained in $N\mathbf{W_J}$. Thus, $\mathcal{W}' = \mathcal{W} \cap \mathbf{W}'$, $\bar{\Omega} = \mathcal{W} \cap \Omega = N\mathbf{W_J} \cap \Omega$. Then $\mathcal{W}'^u$ is a normal subgroup of $\mathcal{W}^u$, and the subgroup $\bar{\Omega}^u$ of $\mathcal{W}^u$ maps isomorphically onto $\mathcal{W}^u/\mathcal{W}'^u$ under the canonical map $\mathcal{W}^u \to \mathcal{W}^u/\mathcal{W}'^u$. We shall need the following results.

(a) *If* u *has exactly one orbit on* $\mathbf{I} - \mathbf{J}$ *(that is, if* $\mathbf{P}$ *is maximal among* F*-stable parahoric subgroups), then* $\mathcal{W}'^u = \{1\}$.

(b) *If* u *has at least two orbits on* $\mathbf{I}-\mathbf{J}$ *(that is, if* $\mathbf{P}$ *is not maximal among* F*-stable parahoric subgroups), then* $\mathcal{W}'^u$ *is a Coxeter group on certain generators* $s_{\underline{k}}$ *where* $\underline{k}$ *runs over the set of* u*-orbits on* $\mathbf{I}-\mathbf{J}$.

These statements follow from 2.30 and 2.28. (From the classification of unipotent cuspidal representations [L6], we see that the subset $\mathbf{J}$ of $\mathbf{I}$ is u-excellent in the sense of 2.28, and hence the results in 2.28 and 2.30 are applicable.) More precisely, $s_{\underline{k}}$ can be characterized as the unique element of $\mathcal{W}'^u \cap \mathbf{W}_{\mathbf{J}\cup\underline{k}}$ distinct from 1. It follows that the conjugation action of an element of $\bar{\Omega}^u$ on $\mathcal{W}'^u$ maps the set of generators $(s_{\underline{k}})$ into itself.

1.16

Let $N\mathbf{P}$ be the normalizer of $\mathbf{P}$ in $\mathbf{G}$, and let $\overline{N\mathbf{P}} = N\mathbf{P}/U_{\mathbf{P}}$ (a possibly disconnected reductive group over $\mathbf{k}$, with identity component $\bar{\mathbf{P}}$, on which F acts naturally as a Frobenius map for an $\mathbf{F}_q$-rational structure). We have $\overline{N\mathbf{P}}^F = N\mathbf{P}^F/U_{\mathbf{P}}^F$ (by "Lang's theorem" for $U_{\mathbf{P}}^F$), hence $\bar{\mathbf{P}}^F \subset N\mathbf{P}^F/U_{\mathbf{P}}^F$. We denote the natural map $N\mathbf{P}^F \to N\mathbf{P}^F/U_{\mathbf{P}}^F$ by $\pi_{N\mathbf{P}}$. Using the classification of unipotent cuspidal representations [L6], it is easy to see that the $\bar{\mathbf{P}}^F$-module $\mathbf{E}$ can be extended to a $N\mathbf{P}^F/U_{\mathbf{P}}^F$-module. Assume that this extension has been chosen.

1.17

From the definitions, we see that an element $w \in \mathcal{W}^u$ belongs to $\bar{\Omega}^u$ if and only if $\Theta_w \subset N\mathbf{P}$. We shall define, for each $w \in \bar{\Omega}^u$ a basis element T_w of $\mathcal{H}_{\Theta_w}$ as follows. For $f \in \mathcal{X}$ we define $T_w f : \mathbf{G}^F \to \mathbf{E}$ by

$$(T_w f)(g) = \pi_{N\mathbf{P}}(y) f(gy)$$

where $y \in \Theta_w \cap \mathbf{G}^F \subset N\mathbf{P}^F$. (Here $\pi_{N\mathbf{P}}(y)$ acts on $f(gy) \in \mathbf{E}$ by the chosen extension of $\mathbf{E}$ to a $N\mathbf{P}^F/U_{\mathbf{P}}^F$-module.) It is clear that $T_w f$ is independent of the choice of y, that it belongs to $\mathcal{X}$, and that $f \mapsto T_w f$ is a well-defined element in $\mathcal{H}_{\Theta_w}$ (in fact, a basis element of it). Using the definitions, we see that

(a) $T_w T_{w'} = T_{ww'}$ for any $w, w' \in \mathcal{W}^u$.

(Here $T_w T_{w'}$ is a composition of endomorphisms of $\mathcal{X}$.) Note also that $T_1 = 1$.

1.18

Let $(\mathbf{I}-\mathbf{J})/u$ be the set of orbits of u on $\mathbf{I}-\mathbf{J}$. Assume that $(\mathbf{I}-\mathbf{J})/u$ has at least two elements. If $\underline{k}$ is one of these elements, there is a unique basis element $T_{s_{\underline{k}}}$ of $\mathcal{H}_{s_{\underline{k}}}$ such that

(a) $(T_{s_{\underline{k}}} + 1)(T_{s_{\underline{k}}} - q^{L(\underline{k})}) = 0$

for some integer $L(\underline{k}) \geq 1$. The uniqueness is obvious. The existence follows from the study of induction of unipotent cuspidal representations in reductive groups over a finite field [L1, §5], [L2, §5].

If $\underline{k}$ is as above and $w \in \bar{\Omega}^u$, then $ws_{\underline{k}}w^{-1} = s_{\underline{k}'}$ for some $\underline{k}' \in (\mathbf{I}-\mathbf{J})/u$. We have

(b) $T_w T_{s_{\underline{k}}} T_w^{-1} = T_{s_{\underline{k}'}}$.

Indeed, using 1.14 (a), we see that $T_w T_{s_{\underline{k}}} T_w^{-1} \in \mathcal{H}_{\Theta_{s_{\underline{k}'}}}$. On the other hand, from $(T_{s_{\underline{k}}}+1)(T_{s_{\underline{k}}} - q^{L(\underline{k})}) = 0$, it follows that $(T_w T_{s_{\underline{k}}} T_w^{-1}+1)(T_w T_{s_{\underline{k}}} T_w^{-1} - q^{L(\underline{k})}) = 0$. From the definition of $T_{s_{\underline{k}'}}$, we now see that (b) holds.

Let $\underline{k}', \underline{k}''$ be two distinct elements of $(\mathbf{I}-\mathbf{J})/u$ such that $s_{\underline{k}'}s_{\underline{k}''}$ has finite order m in $\mathcal{W}'^u$. We have

(c) $T_{s_{\underline{k}'}} T_{s_{\underline{k}''}} T_{s_{\underline{k}'}} = \cdots = T_{s_{\underline{k}''}} T_{s_{\underline{k}'}} T_{s_{\underline{k}''}} \cdots$

where both products have m factors. This equality follows from the study of induction of unipotent cuspidal representations in reductive groups over a finite field [L1, §5], [L2, §5].

For $w' \in \mathcal{W}'^u$, we define

(d) $T_{w'} = T_{s_{\underline{k}_1}} T_{s_{\underline{k}_2}} \cdots T_{s_{\underline{k}_p}}$

where $w' = s_{\underline{k}_1} s_{\underline{k}_2} \cdots s_{\underline{k}_p}$ is a reduced expression in the Coxeter group $\mathcal{W}'^u$. This is well defined (independent of the choice of reduced expression) by (b). From 2.29 (b), it follows that we necessarily have $l(w') = l(s_{\underline{k}_1}) + l(s_{\underline{k}_2}) + \cdots + l(s_{\underline{k}_p})$ (where l is the length function of $\mathbf{W}'$), and from 1.14 (a) it follows that $T_{w'} \in \mathcal{H}_{\Theta_{w'}}$. Note that $T_{w'} \neq 0$ (it is a product of invertible elements of $\mathcal{H}$) and hence $T_{w'}$ is a basis element of $\mathcal{H}_{\Theta_{w'}}$.

For $w \in \mathcal{W}^u$, we define

(e) $T_w = T_{w'} T_{w''}$

where $w' \in \mathcal{W}'^u$, $w'' \in \bar{\Omega}^u$, and $w = w'w''$; here, $T_{w'}$ is given by (d) and $T_{w''}$ is as in 1.17. From 1.14 (a) it follows that $T_w \in \mathcal{H}_{\Theta_w}$. Moreover, T_w is an invertible element of $\mathcal{H}$ since both $T_{w'}, T_{w''}$ are so; hence T_w is a basis element of $\mathcal{H}_{\Theta_w}$. Using 1.13, we see that

(f) $\{T_w \mid w \in \mathcal{W}^u\}$ *is a* $\mathbf{C}$*-basis of* $\mathcal{H}$.

Now (a)–(f) and 1.17 (a) show that $\mathcal{H}$ is naturally the affine Hecke algebra attached (in the Iwahori-Matsumoto–presentation (IM-presentation), as in 5.12, and specialized at $v = \sqrt{q}$) to the Coxeter group $\mathcal{W}'^u$ with Coxeter generators $s_{\underline{k}}$ indexed by $(\mathbf{I}-\mathbf{J})/u$, to the function $\underline{k} \mapsto L(\underline{k})$, and to the finite abelian group $\bar{\Omega}^u$, which acts by conjugation on $\mathcal{W}'^u$ respecting the set of generators and the function L.

From the definitions, we see that the parameters $L(\underline{k})$ are determined as follows. Let $\mathbf{P}'$ be the parahoric subgroup in $\mathcal{P}_{\mathbf{J}\cup\underline{k}}$ containing $\mathbf{P}$ (see 1.12). Then $\mathbf{P}'$ is F-stable, and $\bar{\mathbf{P}}^F$ may be regarded as a Levi quotient of a maximal F-stable parabolic subgroup Q of $\bar{\mathbf{P}}'^F$. We can induce $\mathbf{E}$ from Q^F to $\bar{\mathbf{P}}'^F$; the resulting representation splits into the direct sum of two irreducible representations of dimensions $d > d'$. Then $q^{L(\underline{k})} = d/d'$; the explicit value of $L(\underline{k})$ can be extracted from the table in [L3, p. 35]. On the other hand, the integers m in (c) are determined as in 2.28 (a) (with $(I = \mathbf{I}, J = \mathbf{J})$.

1.19

Now assume that $(\mathbf{I} - \mathbf{J})/u$ has exactly one element. Then $\mathcal{W}^u = \bar{\Omega}^u$ (see 1.15(a)), and hence $\{T_w \mid w \in \bar{\Omega}^u\}$ (defined as in 1.17) form a basis of $\mathcal{H}$. From 1.17 (a), we see that $\mathcal{H}$ is naturally the group algebra of $\bar{\Omega}^u$.

1.20

In the setup of 1.18 or 1.19, we note that the permutation action of $\bar{\Omega}^u$ on $\mathbf{I}$ induces a homomorphism of $\bar{\Omega}^u$ into the group of permutations of the set $(\mathbf{I} - \mathbf{J})/u$. Let $\bar{\Omega}^u_1$ (resp. $\bar{\Omega}^u_2$) be the kernel (resp. image) of this homomorphism.

Note that the subspace of $\mathcal{H}$ spanned by $\{T_w \mid w \in \bar{\Omega}^u_1\}$ is a subalgebra of $\mathcal{H}$ contained in the centre of $\mathcal{H}$. This subalgebra is the group algebra of $\bar{\Omega}^u_1$; it is naturally a direct sum of one-dimensional subalgebras corresponding to the various group homomorphisms $\psi : \bar{\Omega}^u_1 \to \mathbf{C}^*$. The one-dimensional subspace corresponding to ψ generates a two-sided ideal $\mathcal{H}^\psi$ of $\mathcal{H}$. We have a direct sum decomposition $\mathcal{H} = \oplus_\psi \mathcal{H}^\psi$ as algebras. Let $\mathcal{H}'$ be the algebra defined by the same IM-presentation as $\mathcal{H}$ except that $\bar{\Omega}^u$ is replaced by $\bar{\Omega}^u_2$. For any ψ, we define an algebra isomorphism $\mathcal{H}' \xrightarrow{\sim} \mathcal{H}^\psi$ as follows. We choose a cross section $j : \bar{\Omega}^u_2 \to \bar{\Omega}^u$ of $\bar{\Omega}^u \to \bar{\Omega}^u_2$. We can find a function $\zeta : \bar{\Omega}^u_2 \to \mathbf{C}^*$ (depending on j and on ψ) such that the assignment

$$T_{w'_2} T_{w''} \mapsto \zeta(w'_2) T_{j(w'_2)} T_{w''}$$

is an algebra homomorphism for any $w'_2 \in \bar{\Omega}^u_2, w'' \in \mathcal{W}'^u$. (The last group is defined to be $\{1\}$ in the setup of 1.19.) This is then automatically an algebra isomorphism.

Thus, we may identify $\mathcal{H}$ with a direct sum of copies of the algebra $\mathcal{H}'$, the various copies being indexed by $\mathrm{Hom}(\bar{\Omega}^u_1, \mathbf{C}^*)$.

We shall denote $\mathcal{H}'$ (with its IM-presentation) by $\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$.

1.21

An irreducible admissible representation of $\mathbf{G}^F$ is said to be *unipotent* if its isomorphism class belongs to $\mathrm{Irr}(\mathbf{G}^F; \mathbf{P}, \mathbf{E})$ for some $(\mathbf{P}, \mathbf{E})$ as in 1.5. Let $\mathcal{U}(\mathbf{G}^F)$ be the set of isomorphism classes of unipotent representations of $\mathbf{G}^F$.

For any u-stable proper subset $\mathbf{J}$ of $\mathbf{I}$, we denote by $\mathcal{U}^0(\mathbf{J}, u)$ the set of isomorphism classes of unipotent cuspidal representations of an adjoint algebraic group over $\mathbf{F}_q$, with Dynkin diagram given by the full subgraph $\mathbf{J}$ of $\mathbf{I}$ and with twisting described by $u : \mathbf{J} \to \mathbf{J}$.

1.22

For any $u \in \tilde{S}(\mathbf{I})$ (see 1.11), we define $\bar{\mathfrak{A}}_u$ to be the set of all triples $(\mathbf{J}, \mathbf{E}, \psi)$ where $\mathbf{J}$ runs through a set of representatives for the Ω^u-orbits of proper u-stable subsets of $\mathbf{I}$, $\mathbf{E}$ runs through $\mathcal{U}^0(\mathbf{J}, u)$, and ψ runs through $\mathrm{Hom}(\bar{\Omega}^u_{\mathbf{I}}, \mathbf{C}^*)$. Here $\bar{\Omega}^u_{\mathbf{I}}$ is defined in terms of $\mathbf{J}, u$ as in 1.15 and 1.20. The triples $(\mathbf{J}, \mathbf{E}, \psi)$ are called *arithmetic diagrams of type* u.

The results in this section provide a bijection

(a) $\mathcal{U}(\mathbf{G}^F) \leftrightarrow \sqcup_{(\mathbf{J},\mathbf{E},\psi)\in\bar{\mathfrak{A}}_u} \mathrm{Irr}\,\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$

where $F : \mathbf{G} \to \mathbf{G}$ is as in 1.1 and $u : \mathbf{I} \to \mathbf{I}$ is induced by F as in 1.12. (The affine Hecke algebra $\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$ is independent of ψ.)

2 Relative affine Cartan matrices

2.1

Let (a_{ij}) be an affine Cartan matrix indexed by $I \times I$. (I is a finite set with at least two elements.) We include irreducibility in the definition of an affine Cartan matrix. There are uniquely defined strictly positive integers d_i, r_i, n_i ($i \in I$) such that

(a) $d_i a_{ij} = d_j a_{ji}$ for all i, j and $d_i = 1$ for some i,

(b) $\sum_i r_i a_{ij} = 0$ for all j and $r_i = 1$ for some i.

(c) $\sum_j a_{ij} n_j = 0$ for all i and $n_i = 1$ for some i.

We say that (a_{ij}) is *untwisted* if for any i such that $n_i = 1$, we have $d_i = \max_{j\in I} d_j$. An equivalent condition is that n_i/r_i is an integer for all i. From now until the end of §4, we assume that (a_{ij}) is untwisted.

2.2

Let V be an $\mathbf{R}$-vector space with basis $(\alpha_i)_{i\in I}$, and let V' be the dual vector space; we denote by $\langle\, ,\, \rangle : V' \times V \to \mathbf{R}$ the obvious bilinear pairing. For any $i \in I$, let $\varpi_i \in V'$ and $h_i \in V'$ be defined by

$$\langle \varpi_i, \alpha_j \rangle = \delta_{ij}, \qquad \langle h_i, \alpha_j \rangle = a_{ij}.$$

The symmetric bilinear form $(,) : V \times V \to \mathbf{R}$ defined by $(\alpha_i, \alpha_j) = d_i a_{ij}$ is positive semi-definite with radical spanned by the vector

$$D = \sum_{j\in I} n_j \alpha_j;$$

hence it induces a positive definite form on $V/\mathbf{R}D$. The hyperplane

$$H^0 = \{x' \in V' \mid \langle x', D\rangle = 0\}$$

in V' is exactly the subspace of V' generated by the h_i. We have $\sum_i r_i h_i = 0$. Clearly, $\langle\, ,\, \rangle$ defines a perfect pairing $H^0 \times (V/\mathbf{R}D) \to \mathbf{R}$. Hence there is a unique linear map $\iota : V \to H^0$ such that $\langle \iota(x), y\rangle = (x, y)$ for all $x, y \in V$; this map takes α_i to $d_i h_i$ for all i and induces an isomorphism $V/\mathbf{R}D \xrightarrow{\sim} H^0$. We use this to carry the form $(,)$ from $V/\mathbf{R}D$ to H^0, and we obtain a positive definite form on H^0 denoted again by $(,)$; it satisfies $(h_i, h_j) = a_{ij}/d_j$ for all i, j.

For $i \in I$, let $s_i : V' \to V'$ be the reflection given by $s_i(x') = x' - \langle x', \alpha_i\rangle h_i$, and let $s_i : V \to V$ be the reflection given by $s_i(x) = x - \langle h_i, x\rangle \alpha_i$. Let W be the subgroup of $GL(V)$ generated by the s_i; taking contragredients, we identify W with the subgroup of $GL(V')$ generated by the s_i (in this identification, $s_i : V \to V$ corresponds to $s_i : V' \to V'$). The action of W on V preserves the form $(\, ,\,)$ on V and hence also the form $(\, ,\,)$ on H^0. (W keeps D fixed and leaves H^0 stable.)

The affine hyperplane

$$H = \{x' \in V' \mid \langle x', D\rangle = 1\}$$

in V', parallel to H^0, is W-stable. Let $\mathfrak{H}(H)$ be the smallest W-stable collection of affine hyperplanes in H that contains $\{x' \in H \mid \langle x', \alpha_i\rangle = 0\}$ for any $i \in I$. The chambers and facets of H can be defined in terms of this collection of hyperplanes as in [NB, Ch. V, §1]. Any facet is the image under some $w \in W$ of a facet of the form

$$C_K = \{x' \in V' \mid x' = \sum_{i\in K} c_i \varpi_i; c_i \in \mathbf{R}_{>0}, \sum_i n_i c_i = 1\} \subset H$$

for a unique nonempty subset $K \subset I$. We then say that the facet has type K. A chamber is a facet of type I.

2.3

Let K be a nonempty subset of I and let $J = I - K$. The affine subspace

$$P_K = \{x' \in H \mid \langle x', \alpha_j\rangle = 0 \quad \forall j \in J\} = \Big\{x' \in V' \mid x' = \sum_{i\in K} c_i \varpi_i; c_i \in \mathbf{R}, \sum_i n_i c_i = 1\Big\}$$

of H is parallel to the following linear subspace of H^0:

$$P_K^0 = \{x' \in H^0 \mid \langle x', \alpha_j\rangle = 0 \quad \forall j \in J\} = \{x' \in H^0 \mid x' + P_K = P_K\}.$$

Let $\mathfrak{H}(P_K)$ be the collection of all codimension-one affine subspaces of P_K that are of the form $P_K \cap X$ for some $X \in \mathfrak{H}(H)$. Then $\mathfrak{H}(P_K)$ is a locally finite collection of affine hyperplanes in P_K, and hence one can define in terms of it the notion of facet, chamber, and wall (as in [NB, Ch. V, §1]); here we shall call them P_K-facet, P_K-chamber, and wall. A P_K-facet is just a facet of H that is contained in P_K, while a P_K-facet is a facet of H that is contained as an open set in P_K. Note that C_K is a P_K-chamber, but that in general a P_K-chamber may have type other than K, when regarded as a facet of H. It is known [NB, Ch. V, §1, pp. 62–63] that

(a) *any P_K-facet of codimension one in P_K is contained in the closure of exactly two P_K-chambers.*

From this we shall deduce that

(b) *given two P_K-chambers $C \neq C'$, there exists a sequence of P_K-chambers $C = C_0 \neq C_1 \neq \cdots \neq C_s = C'$ and a sequence of P_K-facets $D_0, D_1, \ldots, D_{s-1}$ of codimension one in P_K such that D_i is in the closure of C_i and of C_{i+1} for $i = 0, 1, \ldots, s-1$.*

To verify (b), we fix $p \in C$ and choose $q \in C'$ general enough so that the segment $[p, q]$ does not meet any facet in P_K of codimension greater than one in P_K and meets any codimension-one facet in P_K in at most one point. This segment will cross successively the facets $C = C_0, D_0, C_1, D_1, \ldots, C_s = C'$ with the required properties.

2.4

For any $J \subset I$, let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$. If $J \neq I$, then W_J is finite and has a unique longest element w_0^J (for the standard length function $l : W \to \mathbf{N}$ on the Coxeter group W); we then have a well-defined involution $\tau_J : J \to J$ such that $w_0^J(\alpha_j) = -\alpha_{\tau_J(j)}$ for all $j \in J$.

We say that J is an *excellent* subset of I if $J \neq I$ and if, for any $J' \subset I$ such that $J \subset J', J' \neq I$, we have $\tau_{J'}(J) = J$.

We assume, until the end of 2.27, that we are given an excellent subset J *of* I *and that the set* $K = I - J$ *has at least two elements.*

We shall write P, P^0 instead of P_K, P_K^0, and we will use the terms P-chamber and P-facet, instead of P_K-chamber and P_K-facet. Note that $\dim P \geq 1$. Let V_J be the subspace of V spanned by $\{\alpha_j \mid j \in J\}$.

For any $k \in K$, conjugation by $w_0^{J \cup k}$ defines a Coxeter group automorphism of W_J (it takes s_j to $s_{\tau_{J\cup k}(j)}$ for any $j \in J$), and hence it maps w_0^J to itself. Thus, we have $w_0^{J\cup k} w_0^J = w_0^J w_0^{J\cup k}$, and we denote this element of W by σ_k. The previous equality shows that $\sigma_k^2 = 1$.

(a) For any $j \in J$ we have $\sigma_k(\alpha_j) = \alpha_{\tau_{J\cup k}\tau_J(j)}$ and $\tau_{J\cup k}\tau_J(j) \in J$.

This shows that

(b) σ_k maps the subspace V_J into itself.

Let W^* be the subgroup of W generated by $\{\sigma_k \mid k \in K\}$. By (b), any $w \in W^*$ maps V_J into itself and hence induces an automorphism of V/V_J. Thus, we obtain a group homomorphism $W^* \to GL(V/V_J)$.

Lemma 2.5. This homomorphism is injective. □

Proof. We first prove the following statement.

(a) *Let $w \in W$ be such that $wx = x \bmod V_J$ for any $x \in V$. Then $w \in W_J$.*

We argue by induction on the length n of w. For $n = 0$, the result is clear. Now assume that $n \geq 1$. Let $V_{\geq 0} = \sum_{i \in I} \mathbf{N}\alpha_i$. We can write $w = w's_i$ where $i \in I$ and $w' \in W$ has length $n - 1$. Then $-w(\alpha_i) \in V_{\geq 0}$ and $\alpha_i \in V_{\geq 0}$. By assumption, $-w(\alpha_i) + \alpha_i \in V_J$. Clearly, if the sum of two vectors in $V_{\geq 0}$ is in V_J, then both those vectors are in V_J. Thus, $\alpha_i \in V_J$. It follows that $i \in J$. Using this and our assumption on w, we see that for any $x \in V$, we have

$$w'x - x = ws_i(x) - x = w(x - \langle h_i, x\rangle\alpha_i) - x \equiv x - \langle h_i, x\rangle\alpha_i - x = -\langle h_i, x\rangle\alpha_i \equiv 0$$

where $\equiv$ means equality mod V_J. Thus, the induction hypothesis is applicable to w'. We see that $w' \in W_J$, and hence $w = w's_i \in W_J$. This proves (a). To complete the proof of the lemma, it is now enough to show that $W^* \cap W_J = \{1\}$. Let $w \in W^* \cap W_J$. Since $w \in W^*$, we see from 2.4 (a) that there exists a permutation $j \to \bar{j}$ of J such that

(b) $w(\alpha_j) = \alpha_{\bar{j}}$ for all $j \in J$.

Clearly, an element $w \in W_J$ which satisfies (b) must be 1. The lemma is proved. ■

2.6

Let V'^K be the linear span of P (that is, the linear subspace of V' spanned by $\{\varpi_k \mid k \in K\}$). Note that $\langle,\rangle$ induces a perfect bilinear pairing

(a) $\langle,\rangle : V'^K \times (V/V_J) \to \mathbf{R}$.

It follows that, if $w \in W^*$, then $w : V' \to V'$ maps V'^K into itself. (The restriction of w to V'^K is the contragredient of $w : V/V_J \to V/V_J$ with respect to (a).) Thus, restriction from V' to V'^K defines a homomorphism $W^* \to GL(V'^K)$. Using 2.5, we see that this homomorphism is injective.

Lemma 2.7. For any $k \in K$, $\sigma_k : V'^K \to V'^K$ maps P into itself, and its restriction to P is a reflection in the hyperplane (in $\mathfrak{H}(P)$) given by $\{x' \in P \mid \langle x', \alpha_k\rangle = 0\}$. □

Proof. We first show that

(a) $\sigma_k(\varpi_k/n_k) \in P$.

Since $\sigma_k(H) = H$, it suffices to show that, for $j \in J$, the expression $\langle \sigma_k \varpi_k, \alpha_j \rangle$ is zero. This expression equals

$$\langle \varpi_k, \sigma_k \alpha_j \rangle = \langle \varpi_k, \alpha_{\tau_{J \cup k} \tau_J(j)} \rangle = \delta_{k, \tau_{J \cup k} \tau_J(j)},$$

and this is zero since $\tau_{J \cup k} \tau_J(j) \in J$ and $k \notin J$. Next we show that

(b) $\sigma_k(\varpi_i / n_i) = \varpi_i / n_i$ for $i \in K$, $i \neq k$.

Indeed, for $l \in I$ we have $\sigma_k \alpha_l = \alpha_l \bmod \sum_{j \in J \cup k} \mathbf{Z} \alpha_j$, and hence $\langle \varpi_i, \sigma_k \alpha_l \rangle = \delta_{li}$; thus, $\langle \sigma_k \varpi_i - \varpi_i, \alpha_l \rangle = \langle \varpi_i, \sigma_k \alpha_l - \alpha_l \rangle = \delta_{li} - \delta_{li} = 0$. It follows that $\sigma_k \varpi_i = \varpi_i$ and (b) holds.

Since P is spanned as an affine space by the vectors ϖ_i / n_i for $i \in K$, we see from (a) and (b) that $\sigma_k(P) = P$. Since $\{x' \in P \mid \langle x', \alpha_k \rangle = 0\}$ is spanned as an affine space by the vectors ϖ_i / n_i with $i \in K, i \neq k$, we see from (b) that σ_k is the identity on this affine space.

Next we show that

(c) $\sigma_k(\alpha_k) |_P = -\alpha_k |_P$.

(Here we regard elements of V as functions on V′ via $\langle \, , \rangle$.) For this, we first observe that $w_0^{J \cup k}(\alpha_k) = -\alpha_k$. Indeed, there exists an involution $\tau' : J \cup k \to J \cup k$ such that $w_0^{J \cup k}(\alpha_i) = -\alpha_{\tau'(i)}$ for all $i \in J \cup k$; this involution maps J onto itself since J is excellent. Hence it necessarily maps k to k. On the other hand, $w_0^J(\alpha_k) = \alpha_k \bmod \sum_{j \in J} \mathbf{Z} \alpha_j$, and hence $w_0^J(\alpha_k) |_P = \alpha_k |_P$ and (c) follows.

It remains to show that $\sigma_k : P \to P$ is not the identity map. Let $x' \in P$ be such that $\sigma_k(x') = x'$. Then, by (c),

$$\langle x', \alpha_k \rangle = -\langle x', \sigma_k \alpha_k \rangle = -\langle \sigma_k x', \alpha_k \rangle = -\langle x', \alpha_k \rangle$$

so that $\langle x', \alpha_k \rangle = 0$. This shows that $\sigma_k : P \to P$ is not the identity map. The lemma is proved. ∎

2.8

We regard the affine space H as an euclidean one, using the form (,) on the space of translations H^0 of H. Then P, being an affine subspace of H, has an induced euclidean structure. The action of W on H preserves the euclidean structure, and hence any $w \in W$ such that $w(P) = P$, preserves the euclidean structure of P. In particular, for $k \in K$, $\sigma_k : P \to P$ preserves the euclidean structure of P, and hence, by 2.7, it is an orthogonal reflection in P.

By 2.6, W^* may be identified with the subgroup of the group of automorphisms of the euclidean affine space P generated by the $\sigma_k : P \to P$ with $k \in K$. For any $\mathbf{h} \in \mathfrak{H}(P)$, we denote by $S_{\mathbf{h}}$ the orthogonal reflection in P with fixed point set $\mathbf{h}$.

Since $\sigma_k : P \to P$ permutes among themselves the subspaces in $\mathfrak{H}(P)$, it must also permute among themselves the P-chambers. It follows that W^* acts naturally on the set $\mathfrak{H}(P)$ and on the set of P-chambers. Clearly, if $\mathbf{h} \in \mathfrak{H}(P)$ and $w \in W^*$, then

(a) $wS_{\mathbf{h}}w^{-1} = S_{w\mathbf{h}}$.

Lemma 2.9. W^* acts transitively on the set of P-chambers. □

Proof. Let C' be a P-chamber and let $C = C_K$. It suffices to show that

(a) $C' = w(C)$ for some $w \in W^*$.

To prove this, we may assume that $C' \neq C$. Let $C = C_0, C_1, \dots, C_s = C'$, $D_0, D_1, \dots, D_{s-1}$ be as in 2.3 (b) with $s \geq 1$ as small as possible. We argue by induction on s. There exists $w' \in W^*$ such that $C_{s-1} = w'(C)$; this follows from the induction hypothesis if $s \geq 2$ and is trivial if $s = 1$ (take $w' = 1$). Let $\mathbf{h} \in \mathfrak{H}(P)$ be such that $D_{s-1} \subset \mathbf{h}$. Then $S_{\mathbf{h}}(C_{s-1}) = C_s$. Since $w'^{-1}(C_{s-1}) = C$ and D_{s-1} is contained in the closure of C_{s-1}, it follows that $w'^{-1}D_{s-1}$ is contained in the closure of C. Hence $w'^{-1}\mathbf{h}$ is a wall of C. Hence there exists $k \in K$ such that $w'^{-1}\mathbf{h} = \{x' \in P \mid \langle x', \alpha_k\rangle = 0\}$, and using 2.8 (a), we deduce $w'^{-1}S_{\mathbf{h}}w' = \sigma_k$. Thus, $S_{\mathbf{h}} = w'\sigma_k w'^{-1} \in W^*$. We have $C' = S_{\mathbf{h}}(C_{s-1}) = S_{\mathbf{h}}w'(C)$ and $S_{\mathbf{h}}w' \in W^*$. This proves the induction step of (a) as well as the initial step $s = 1$. The lemma is proved. ■

Lemma 2.10. (a) If $\mathcal{H} \in \mathfrak{H}(P)$, then $S_{\mathbf{h}} \in W^*$.

(b) If $\mathbf{h}, \mathbf{h}'$ belong to $\mathfrak{H}(P)$, then so does $S_{\mathbf{h}}(\mathbf{h}')$. □

Proof. We can find a P-chamber C' such that $\mathbf{h}$ is a wall of C'. By 2.9, we have $C' = w(C_K)$ for some $w \in W^*$. Then $w^{-1}\mathbf{h}$ is a wall of C_K, and hence there exists $k \in K$ such that $w^{-1}\mathbf{h} = \{x' \in P \mid \langle x', \alpha_k\rangle = 0\}$, and using 2.8 (a), we deduce $w^{-1}S_{\mathbf{h}}w = \sigma_k$. It follows that $S_{\mathbf{h}} \in W^*$ and (a) is proved. Now (b) follows from (a) since W^* permutes the set $\mathfrak{H}(P)$. ■

2.11

Since C_K is an open simplex in P and $\cup_{w\in W^*}w(C_K)$ is dense in P (by 2.9), it follows that W^* is infinite. Note that the reflection group W^* acts properly on P (since this action is the restriction of the proper action of W on H). Using this and 2.10 (b), we see that we may apply [NB, Ch. V, §3, 2, Th. 1], and we deduce the following:

(a) W^* *is an irreducible Coxeter group on the generators* $\{\sigma_k \mid k \in K\}$.

(The irreducibility follows from [NB, Ch. V, Ex. 1, p. 128], since in our case the chamber C_K is an open simplex.) Recall that $P^0 = \{x' \in H^0 \mid x' + P = P\}$ (a linear subspace of H^0). Let

$$\mathcal{L}' = \{x' \in P^0 \mid \text{there exists } w \in W^* \text{ with } x' + z = w(z) \text{ for all } z \in P\}.$$

Then $\mathcal{L}'$ is a lattice in P^0 [NB, Ch. VI, p. 180], and we may apply [NB, Ch. VI, §2, 5, Prop. 8] to deduce the following.

(b) *There exists a basis* $\tilde{\alpha}_k \in V/V_J (k \in K)$ *and vectors* $\tilde{h}_k \in V'^K (k \in K)$ *such that*

$\langle \tilde{h}_k, \tilde{\alpha}_{k'} \rangle = \tilde{a}_{kk'}$ *for* $k, k' \in K$,

where $(\tilde{a}_{kk'})$ *is an untwisted affine Cartan matrix indexed by* $K \times K$, $\sigma_k(x') = x' - \langle x', \tilde{\alpha}_k \rangle \tilde{h}_k$ *for all* $k \in K$, *and* $P = \{x' \in V'^K \mid \langle x', \tilde{D} \rangle = 1\}$. *(We define* $\tilde{n}_k, \tilde{r}_k (k \in K)$ *in terms of* $(\tilde{a}_{kk'})$ *in the same way as* n_i, r_i *were defined in terms of* (a_{ij}), *in 2.1, and we set* $\tilde{D} = \sum_{k \in K} \tilde{n}_k \tilde{\alpha}_k \in V/V_J$.*)*

Let $\tilde{\varpi}_k \in V'^K$ be defined by $\langle \tilde{\varpi}_k, \tilde{\alpha}_{k'} \rangle = \delta_{kk'}$ for $k, k' \in K$. We show that $(\tilde{a}_{kk'})$, $\tilde{\varpi}_k$, $\tilde{\alpha}_k$, $\tilde{h}_k$ are uniquely defined by the requirements above. Since the Coxeter group corresponding to $(\tilde{a}_{kk'})$ is W^*, the Coxeter graph of $(\tilde{a}_{kk'})$ is uniquely determined by the Coxeter group W. The corresponding Dynkin graph is then uniquely determined by the requirement that $(\tilde{a}_{kk'})$ is untwisted. Hence $(\tilde{a}_{kk'})$ is itself uniquely determined. It follows that the integers $\tilde{n}_k$ are uniquely determined. Now the vectors $\tilde{\varpi}_k/\tilde{n}_k$ and ϖ_k/n_k are both characterized by the property that they belong to P and are fixed by $\sigma_{k'}$ for all $k' \neq k$. Hence $\tilde{\varpi}_k/\tilde{n}_k = \varpi_k/n_k$ so that $\tilde{\varpi}_k$ is uniquely determined: we have

$$\tilde{\varpi}_k = (1/z_k)\varpi_k \quad \text{where } z_k = n_k/\tilde{n}_k.$$

Now $\tilde{\alpha}_k \in V'^K$ is determined uniquely from the equations $\langle \tilde{\varpi}_{k'}, \tilde{\alpha}_k \rangle = \delta_{kk'}$: we see that

$$\tilde{\alpha}_k = z_k \alpha_k \mod V_J.$$

Finally, $\tilde{h}_k \in V'^K$ is given by

$$\tilde{h}_k = \sum_{k' \in K} \tilde{a}_{kk'} \tilde{\varpi}_{k'},$$

and hence it is uniquely determined.

The procedure above allows us to compute explicitly, in any given case, the matrix $(\tilde{a}_{kk'})$ and the vectors $\tilde{\alpha}_k, \tilde{h}_k$. For this we only need to know explicitly the structure of the Coxeter group W^*; this is given by the following formula for the order $m_{kk'}$ for the order of $\sigma_k \sigma_{k'} \in W^*$ $(k \neq k'$ in $K)$:

$$\text{(c)} \quad m_{kk'} = \frac{2(l(w_0^{J \cup k \cup k'}) - l(w_0^J))}{l(w_0^{J \cup k}) + l(w_0^{J \cup k'}) - 2l(w_0^J)};$$

l is as in 2.4. (When $K = \{k, k'\}$, then $w_0^{J \cup k \cup k'}$ is not defined; in this case we use the convention $l(w_0^{J \cup k \cup k'}) = \infty$, so that $m_{kk'} = \infty$.) The proof of (c) is the same as that of [L1, 5.9.1]; it will be omitted. A formula of the same type as (c) appeared, in connection with relative root systems, in the paper of Tits [T, 2.5.3].

Lemma 2.12. For any $k \in K$, the ratio $z_k = n_k/\tilde{n}_k$ is an integer. □

Proof. We use the following well-known property of affine Weyl groups in two different situations.

(a) Given $i \in I$ and $a \in \mathbf{R}$, the set $\{x' \in H \mid \langle x', \alpha_i \rangle = a\}$ belongs to $\mathfrak{H}(H)$ if and only if $a \in \mathbf{Z}$.

(b) Given $k \in K$ and $a \in \mathbf{R}$, the set $\{x' \in P \mid \langle x', \tilde{\alpha}_k \rangle = a\}$ belongs to $\mathfrak{H}(P)$ if and only if $a \in \mathbf{Z}$.

Now let $k \in K$. By (a), the set $\{x' \in H \mid \langle x', \alpha_k \rangle = 1\}$ belongs to $\mathfrak{H}(H)$. Intersecting this set with P gives us a member of $\mathfrak{H}(P)$. Thus, $\{x' \in P \mid \langle x', \alpha_k \rangle = 1\}$ belongs to $\mathfrak{H}(P)$. Equivalently, $\{x' \in P \mid \langle x', (\tilde{n}_k/n_k)\tilde{\alpha}_k \rangle = 1\}$ belongs to $\mathfrak{H}(P)$. Using (b), it follows that $n_k/\tilde{n}_k$ is an integer. ■

2.13

We say that $k \in K$ is *extraordinary* if the Coxeter graph of W^* is of type $\tilde{C}_n$ (notation of [NB, p. 199]; convention: $\tilde{C}_1 = \tilde{A}_1$) and k corresponds to one of the two extremal points of the Coxeter graph. Otherwise, $k \in K$ is said to be *ordinary*.

Let p_k be the generator ≥ 1 of the subgroup $\sum_{k' \in K} \tilde{a}_{k'k}\mathbf{Z}$ of $\mathbf{Z}$. One checks easily that

(a) $p_k = 2$ if k is extraordinary and $p_k = 1$ if k is ordinary.

We partition K into two subsets $K^\sharp, K^\flat$ by the following requirements.

(i) If $k \in K$ is ordinary, then $k \in K^\sharp$.

(ii) If $k \neq k'$ in K are extraordinary and $z_k, z_{k'}$ are odd, then $k \in K^\sharp, k' \in K^\sharp$.

(iii) If $k \neq k'$ in K are extraordinary and either z_k or $z_{k'}$ is even, then $k \in K^\flat$, $k' \in K^\flat$.

2.14

For $k \in K$, we define $\bar{z}_k \in \mathbf{R}_{>0}$ as follows. If $k \in K^\sharp$, then $\bar{z}_k = z_k$. If $k \in K^\flat$, then $\bar{z}_k = z_k/2$. We have

(a) $\langle \mathcal{L}', \bar{z}_k \alpha_k \rangle \subset \mathbf{Z}$.

To prove this, we will use the following fact:

(b) *$\mathcal{L}'$ is precisely the subgroup of V' generated by* $\{\tilde{h}_k \mid k \in K\}$.

This follows from the definitions, using known properties of affine Weyl groups.

From (b) we see that, to prove (a), it suffices to check that $\bar{z}_k(\tilde{n}_k/n_k)\tilde{a}_{k'k} \in \mathbf{Z}$ for all $k' \in K$. Using now the definition of z_k (2.11) and p_k (see 2.13), we see that it suffices to check that $\bar{z}_k p_k / z_k \in \mathbf{Z}$ or equivalently that $p_k \in \mathbf{Z}$ for $K^\sharp$ and $p_k/2 \in \mathbf{Z}$ for $K^\flat$. This is obvious.

We will now modify the affine Cartan matrix $(\tilde{a}_{kk'})$ and the vectors $\tilde{\alpha}_k, \tilde{h}_k$ as follows. For $k \in K$, we set

$$\hat{\alpha}_k = \bar{z}_k \alpha_k = (\bar{z}_k/z_k)\tilde{\alpha}_k \in V/V_J, \qquad \hat{h}_k = (z_k/\bar{z}_k)\tilde{h}_k \in V'^K.$$

For $k, k' \in K$, we set

$$\hat{a}_{kk'} = \langle \hat{h}_k, \hat{\alpha}_{k'} \rangle = \frac{z_k \bar{z}_{k'}}{z_{k'} \bar{z}_k} \tilde{a}_{kk'}.$$

In other words, we have

(c) $\hat{\alpha}_k = \tilde{\alpha}_k$, and $\hat{h}_k = \tilde{h}_k$ if $k \in K^\sharp$; $\hat{\alpha}_k = (1/2)\tilde{\alpha}_k$, and $\hat{h}_k = 2\tilde{h}_k$ if $k \in K^\flat$;

(d) $\hat{a}_{kk'} = \tilde{a}_{kk'}$ if $k, k' \in K^\sharp$ or if $k, k' \in K^\flat$; $\hat{a}_{kk'} = 2\tilde{a}_{kk'}$ if $k \in K^\flat, k' \in K^\sharp$; $\hat{a}_{kk'} = (1/2)\tilde{a}_{kk'}$ if $k \in K^\sharp, k' \in K^\flat$.

Lemma 2.15. $(\hat{a}_{kk'})_{k,k' \in K}$ is a (not necessarily untwisted) affine Cartan matrix. □

Proof. From the formulas just written, we see that if $(\tilde{a}_{kk'})$ is of type $\neq \tilde{C}_n$, then $(\hat{a}_{kk'}) = (\tilde{a}_{kk'})$; if $(\tilde{a}_{kk'})$ is of type $\tilde{C}_n$, then $(\hat{a}_{kk'})$ is the transpose of $(\tilde{a}_{kk'})$. The lemma follows. ■

2.16

Besides the affine Cartan matrices $(\tilde{a}_{kk'}), (\hat{a}_{kk'})$, we define an affine Cartan matrix $({}^*a_{kk'})$ as follows. We define vectors ${}^*h_k (k \in K)$ in V'^K and ${}^*\alpha_k (k \in K)$ in V/V_J by ${}^*h_k = z_k \tilde{h}_k, {}^*\alpha_k = \alpha_k \bmod V_J$; for $k, k' \in K$, we set

$${}^*a_{kk'} = \langle {}^*h_k, {}^*\alpha_{k'} \rangle = \frac{z_k}{z_{k'}} \tilde{a}_{kk'}.$$

Lemma 2.17. $({}^*a_{kk'})$ is a (not necessarily untwisted) affine Cartan matrix. □

Proof. We first prove that, if $k, k' \in K$, then

(a) ${}^*a_{kk'} \in \mathbf{Z}$.

By definition, we have $\sigma_k(\tilde{\alpha}_{k'}) = \tilde{\alpha}_{k'} - \tilde{a}_{kk'}\tilde{\alpha}_k$, and hence $z_{k'}\sigma_k(\alpha_{k'}) = z_{k'}\alpha_{k'} - z_k \tilde{a}_{kk'}\alpha_k$ and

(b) $\sigma_k(\alpha_{k'}) = \alpha_{k'} - (z_k \tilde{a}_{kk'}/z_{k'})\alpha_k$

(equalities in V/V_J). Since $\sigma_k \in W$, we have $\sigma_k(\alpha_{k'}) \in \sum_{i \in I} \mathbf{Z}\alpha_i$ (equality in V). Comparing with (b), we deduce that $z_k \tilde{a}_{kk'}/z_{k'} \in \mathbf{Z}$. This proves (a).

The other required properties of $({}^*a_{kk'})$ follow immediately from the corresponding properties of $(\tilde{a}_{kk'})$. ■

Lemma 2.18. Given $k \in K$, we have $\hat{h}_k \in \mathcal{L}' - 2\mathcal{L}'$ if $k \in K^\sharp$, and $\hat{h}_k \in 2\mathcal{L}'$ if $k \in K^\flat$. □

Proof. Indeed, by 2.14 (c), it suffices to show that, for $k \in K$, we have $\tilde{h}_k \notin 2\mathcal{L}'$. Assume that this is not so. Then, by 2.14 (b), we have $\tilde{h}_k = 2\sum_{k' \in K} f_{k'}\tilde{h}_{k'}$ where $f_{k'}$ are integers. This, together with the equation $\sum_{k'} \tilde{r}_{k'}\tilde{h}_{k'} = 0$, implies that $\tilde{r}_{k'} = 2$ (see 2.11 (b)) for $k' \neq k$. Since $(\tilde{a}_{kk'})$ is untwisted, it follows that $\tilde{n}_{k'}$ is even for all $k' \neq k$. It is well known that this is impossible for an untwisted affine Cartan matrix. ■

Lemma 2.19. Let $k, k' \in K$ be such that $\tilde{a}_{kk'} = \tilde{a}_{k'k} = -1$. Then $z_k = z_{k'}$. Moreover, $\bar{z}_k = \bar{z}_{k'}$. □

Proof. We have ${}^*a_{kk'}{}^*a_{k'k} = \tilde{a}_{kk'}\tilde{a}_{k'k} = 1$. Since $({}^*a_{kl})$ is an affine Cartan matrix, it follows that ${}^*a_{kk'} = {}^*a_{k'k} = -1$, so that $z_k/z_{k'} = 1$. Under our assumption, k and k' are ordinary (see 2.13) and hence $\bar{z}_k = z_k$ and similarly $\bar{z}_{k'} = z_{k'}$. The lemma is proved. ∎

2.20

Let R be the set of all vectors in V of the form $w(\alpha_i)$ for some $w \in W$ and some $i \in I$. Let ${}^*R$ be the set of all vectors in V/V_J of the form $w({}^*\alpha_k)$ for some $w \in W^*$ and some $k \in K$. Let $\tilde{R}$ be the set of all vectors in V/V_J of the form $w(\tilde{\alpha}_k)$ for some $w \in W^*$ and some $k \in K$. Let $\hat{R}$ be the set of all vectors in V/V_J of the form $w(\hat{\alpha}_k)$ for some $w \in W^*$ and some $k \in K$.

Lemma 2.21. (a) Let $w \in W^*$ and $k, k' \in K$ be such that $w(\tilde{\alpha}_k) = \tilde{\alpha}_{k'}$ in V/V_J. Then $z_k = z_{k'}, \bar{z}_k = \bar{z}_{k'}$.

(b) There are well-defined functions $\alpha \mapsto z_\alpha, \alpha \mapsto \bar{z}_\alpha$ from $\tilde{R}$ to $\mathbf{R}_{>0}$ which are constant on the orbits of W^* on $\tilde{R}$ and satisfy $z_{\tilde{\alpha}_k} = z_k$, $\bar{z}_{\tilde{\alpha}_k} = \bar{z}_k$ for all $k \in K$.

(c) $\alpha \mapsto (1/z_\alpha)\alpha$ is a well-defined bijection $\tilde{R} \xrightarrow{\sim} {}^*R$.

(d) $\alpha \mapsto (\bar{z}_\alpha/z_\alpha)\alpha$ is a well-defined bijection $\tilde{R} \xrightarrow{\sim} \hat{R}$. □

Proof. The assumption of (a) implies that $w\sigma_k = \sigma_{k'}w$ (in W^*). Let $l^* : W^* \to \mathbf{N}$ be the length function of the Coxeter group W^*. If $l^*(w\sigma_k) = l^*(w) + 1$, then the equality $w\sigma_k = \sigma_{k'}w$ holds in the braid monoid attached to W^*. If $l^*(w\sigma_k) = l^*(w) - 1$, then, setting $w' = w\sigma_k$, we have the equality $w'\sigma_k = \sigma_{k'}w'$ in the same braid monoid. From 2.19, we see that there is a well-defined homomorphism from the braid monoid of W^* into $\mathbf{N}$ (regarded as a monoid for addition) which takes the value z_k at σ_k for any $k \in K$. Applying this homomorphism to the equality $w\sigma_k = \sigma_{k'}w$ or $w'\sigma_k = \sigma_{k'}w'$, we obtain $z_k = z_{k'}$. An entirely similar proof shows that $\bar{z}_k = \bar{z}_{k'}$. This proves (a). Now (b) follows immediately from (a); (c) and (d) follow immediately from the definitions. ∎

2.22

Let

$$\rho : V/V_J \to V/(V_J + \mathbf{R}D) = (V/V_J)/\mathbf{R}\tilde{D}$$

be the canonical map. ($\tilde{D} = \sum_{k \in K} \tilde{n}_k \tilde{\alpha}_k \in V/V_J$ as in 2.11.) Note that $\langle , \rangle$ induces a perfect pairing $P^0 \times V/(V_J + \mathbf{R}D) \to \mathbf{R}$.

Let $\tilde{\mathcal{R}} = \rho(\tilde{R}), \mathcal{R} = \rho(\hat{R}), {}^*\mathcal{R} = \rho({}^*R)$. Let $\tilde{\mathcal{R}}'$ be the set of all vectors in P^0 of the form $w(\tilde{h}_k)$ for some $w \in W^*$ and some $k \in K$. Let $\mathcal{R}'$ be the set of all vectors in P^0 of the form $w(\hat{h}_k)$ for some $w \in W^*$ and some $k \in K$. Let ${}^*\mathcal{R}'$ be the set of all vectors in P^0 of the form $w({}^*h_k)$ for some $w \in W^*$ and some $k \in K$.

From general properties of affine root systems, we see that

(a) $(\tilde{\mathcal{R}} \subset V/(V_J + \mathbf{R}D), \tilde{\mathcal{R}}' \subset P^0)$,

(b) $(\mathcal{R} \subset V/(V_J + \mathbf{R}D), \mathcal{R}' \subset P^0)$,

(c) $({}^*\mathcal{R} \subset V/(V_J + \mathbf{R}D), {}^*\mathcal{R}' \subset P^0)$

are ordinary root systems. (The bijection $\tilde{\mathcal{R}} \to \tilde{\mathcal{R}}'$ is characterized by the property that it is W^*-equivariant and takes $\rho(\tilde{\alpha}_k)$ to $\tilde{h}_k$; the bijection $\mathcal{R} \to \mathcal{R}'$ is characterized by the property that it is W^*-equivariant and takes $\rho(\hat{\alpha}_k)$ to $\hat{h}_k$; the bijection ${}^*\mathcal{R} \to {}^*\mathcal{R}'$ is characterized by the property that it is W^*-equivariant and takes $\rho({}^*\alpha_k)$ to ${}^*h_k$. Moreover, the root system (a) is reduced, since $(\tilde{a}_{kk'})$ is untwisted. In addition,

(d) *the root system (b) is reduced*,

since either the corresponding affine Cartan matrix, or its transpose, is untwisted (see the proof of 2.15).

However, the root system (c) is not necessarily reduced.

Lemma 2.23. (a) For any $\beta \in \mathcal{R}$, there is a unique number $f(\beta) \in \mathbf{R}_{>0}$ such that $f(\beta)\beta \in \tilde{\mathcal{R}}$.

(b) The map $\beta \to f(\beta)\beta$ from $\mathcal{R}$ to $\tilde{\mathcal{R}}$ is surjective; moreover, $\beta, \beta' \in \mathcal{R}$ are mapped to the same element if and only if $\beta' \in \mathbf{R}_{>0}\beta$.

(c) Let $\beta \in \mathcal{R}$. Choose $\alpha \in \tilde{R}$ such that $\rho((\tilde{z}_\alpha/z_\alpha)\alpha) = \beta$ (see 2.21 (d)). Then $f(\beta) = z_\alpha/\tilde{z}_\alpha$. Hence $f(\beta) \in \{1, 2\}$. □

Proof. We prove (a). Let $\beta \in \mathcal{R}$. Choose $\alpha' \in \hat{R}$ such that $\rho(\alpha') = \beta$. Let $\alpha \in \tilde{R}$ be such that $\alpha' = (\tilde{z}_\alpha/z_\alpha)\alpha$ (see 2.21 (d)). Then $\rho(\alpha) = (z_\alpha/\tilde{z}_\alpha)\beta \in \tilde{\mathcal{R}}$. Hence there exists $\beta' \in \tilde{\mathcal{R}} \cap \mathbf{R}_{>0}\beta$ (necessarily unique since $\tilde{\mathcal{R}}$ is reduced); (a) follows.

We prove (b). Let $\beta' \in \tilde{\mathcal{R}}$. Choose $\alpha \in \tilde{R}$ such that $\rho(\alpha) = \beta'$. Then $(\tilde{z}_\alpha/z_\alpha)\alpha \in \hat{R}$ (see 2.21 (d)), and hence $\rho((\tilde{z}_\alpha/z_\alpha)\alpha) = (\tilde{z}_\alpha/z_\alpha)\beta' \in \mathcal{R}$. Hence there exists $\beta \in \mathcal{R} \cap \mathbf{R}_{>0}\beta'$. This proves surjectivity of the map in (b). The assertion about the fibres is immediate.

We prove (c). In the setup of (c), we have $\rho(\alpha) \in \mathbf{R}_{>0}f(\beta)\beta$. Since $\rho(\alpha)$ and $f(\beta)\beta$ are in $\tilde{\mathcal{R}}$ (which is reduced), it follows that $\rho(\alpha) = f(\beta)\beta$; (c) follows. ■

2.24

Let $\mathcal{R}^\flat$ be the set of all $\beta \in \mathcal{R}$ such that $f(\beta) = 2$. Let $\mathcal{R}^\sharp$ be the set of all $\beta \in \mathcal{R}$ such that $f(\beta) = 1$. From 2.23 we see that we have a partition $\mathcal{R} = \mathcal{R}^\flat \sqcup \mathcal{R}^\sharp$.

Lemma 2.25. Let $\beta \in \mathcal{R}$ and let h_β be the corresponding element of $\mathcal{R}'$.

(a) We have $\beta \in \mathcal{R}^\flat$ if and only if $h_\beta \in 2\mathcal{L}'$.

(b) We have $\beta \in \mathcal{R}^\sharp$ if and only if $h_\beta \in \mathcal{L}' - 2\mathcal{L}'$. □

Proof. Replacing β by $w\beta$ for some $w \in W^*$, we may assume that there exists a $k \in K$ such that $\beta = \rho(\hat{\alpha}_k), h_\beta = \hat{h}_k$. In this case, the result follows from 2.18 (a) and 2.23. ■

Proposition 2.26. The following conditions for $w \in W$ are equivalent:

(a) $w \in W^* \cdot W_J$.

(b) $w \in N(W_J)$ (the normalizer of W_J in W).

(c) $w(P) \subset P$. □

Proof. Since J is excellent, the generators σ_k normalize W_J. It follows that $W^* \subset N(W_J)$ so that (a) $\Longrightarrow$ (b). Clearly, if $y \in W_J$ acts on V as a reflection, then the (-1)-eigenspace of $y : W \to W$ is contained in V_J. (Use the fact that y induces the identity map $V/V_J \to V/V_J$.) Now, if $w \in N(W_J)$ and $j \in J$, then $ws_jw^{-1} \in W_J$ acts on V as a reflection, and hence $w(\alpha_j)$ (which is contained in the (-1)-eigenspace of ws_jw^{-1}) must be contained in V_J. It follows that, if $w \in N(W_J)$, then $w(V_J) \subset V_J$. Since V'^K is the annihilator of V_J in V', it follows that $w : V' \to V'$ maps V'^K into itself. We certainly have $w(H) = H$. Since $P = V'^K \cap H$, it follows that $w(P) \subset P$. Thus, (b)$\Longrightarrow$(c).

Now let $w \in W$ be such that $w(P) \subset P$. Then $w : P \to P$ permutes among themselves the subspaces in $\mathfrak{H}(P)$, and hence it permutes among themselves the P-chambers. Thus, $w(C_K)$ is a P-chamber. Since W^* acts transitively on the set of P-chambers, there exists a $w' \in W^*$ such that $w(C_K) = w'(C_K)$. Let $w'' = w'^{-1}w$. Then $w'' \in W$ satisfies $w''(C_K) = C_K$. It follows that w'' permutes among themselves the extremal points of C_K. These extremal points are zero-dimensional facets in H of mutually distinct types, and w'' must map a facet of H into a facet of the same type. It follows that w'' maps each extremal point of C_K to itself. Since these extremal points form a basis of V'^K, it follows that $w''x' = x'$ for all $x' \in V'^K$. Taking contragredients, we deduce that $w''x = x \bmod V_J$ for all $x \in V$. Using now 2.5 (a), we deduce that $w'' \in W_J$. Thus (c)$\Longrightarrow$(a). The proposition is proved. ■

Corollary 2.27. Let H^0_Z be the set of all $x' \in H^0$ such that there exists a $w \in W$ with $x' + z = w(z)$ for all $z \in H$. We have $H^0_Z \cap P^0 \subset \mathcal{L}'$. □

Proof. Let $x' \in H^0_Z \cap P^0$. We can find $w \in W$ such that $x' + z = w(z)$ for all $z \in H$. Since $x' \in P^0$, we have $x' + P \subset P$, and hence $w(P) \subset P$. By 2.26, we then have $w = w'w''$ for some $w' \in W^*, w'' \in W_J$. Clearly, W_J acts trivially on P. Hence for $z' \in P$ we have $x' + z' = w'w''(z') = w'(z')$. By the definition of $\mathcal{L}'$, this means that $x' \in \mathcal{L}'$. The corollary is proved. ■

2.28

We shall need a variant of the results in 2.11. (The proofs are along the same lines as those in 2.11 and will be omitted.) Namely, in the setup of 2.1, we now assume that we are given a permutation $u : I \xrightarrow{\sim} I$ such that $a_{u(i),u(j)} = a_{ij}$ for all $i, j \in I$. Then $n_{u(i)} = n_i, d_{u(i)} = d_i$ for $i \in I$.

Let $u : W \to W$ be the isomorphism given by $u(s_i) = s_{u(i)}$ for all $i \in I$.

In the setup of 2.2, we define a linear map $u : V' \to V'$ by $u(\varpi_i) = \varpi_{u(i)}$ for all $i \in I$.

A subset J of I is said to be u-excellent if $J \neq I, u(J) = J$ and if, for any $J' \subset I$ such that $J' \neq I, u(J') = J'$, we have $\tau_{J'}(J) = J$, with the notation of 2.4. (In the case where u is the identity map, this reduces to the definition of an excellent subset; see 2.4.) Let us fix a u-excellent subset J of I. Let $K = I - J$, and let $\underline{K}$ be the set of u-orbits on K. We assume that $\underline{K}$ contains at least two elements.

For any $\underline{k} \in \underline{K}$ we have $w_0^{J\cup\underline{k}} w_0^J = w_0^J w_0^{J\cup\underline{k}}$, and we denote this element of W by $\sigma_{\underline{k}}$. We have $\sigma_{\underline{k}} \in W^u$ and $\sigma_{\underline{k}}^2 = 1$.

Now $\sigma_{\underline{k}} : V' \to V'$ maps P^u into itself, and the restriction of $\sigma_{\underline{k}}$ to P^u is the orthogonal reflection in P^u (with the metric induced by P) whose fixed-point set is the affine hyperplane $\{x' \in P^u \mid \langle x', \alpha_k \rangle = 0\}$ where k is any element of $\underline{k}$.

Hence, if W^* is the subgroup of W generated by $\{\sigma_{\underline{k}} \mid \underline{k} \in \underline{K}\}$, then W^* acts naturally by affine isometries on P^u. The action of W^* on P^u is faithful, and it realizes W^* as an affine Weyl group with Coxeter generators $\{\sigma_{\underline{k}} \mid \underline{k} \in \underline{K}\}$. The subset $C_K \cap P^u$ is a fundamental domain for the W^*-action on P^u.

If $\underline{k}, \underline{k}'$ are distinct u-orbits on K, then the order $m_{\underline{k}\underline{k}'}$ of $\sigma_{\underline{k}}\sigma_{\underline{k}'} \in W^*$ is given by

$$\text{(a)}\quad m_{\underline{k}\underline{k}'} = \frac{2(l(w_0^{J\cup\underline{k}\cup\underline{k}'}) - l(w_0^J))}{l(w_0^{J\cup\underline{k}}) + l(w_0^{J\cup\underline{k}'}) - 2l(w_0^J)}.$$

(When $K = \underline{k} \cup \underline{k}'$, then $w_0^{J\cup\underline{k}\cup\underline{k}'}$ is not defined; in this case we use the convention $l(w_0^{J\cup\underline{k}\cup\underline{k}'}) = \infty$, so that $m_{\underline{k}\underline{k}'} = \infty$.)

Lemma 2.29. Let NW_J be the normalizer of W_J in W and let $NW_J^u = NW_J \cap W^u$.

(a) W^* is precisely the set of all $w \in NW_J^u$ such that w has minimal length in its W_J-coset.

(b) For any $w \in W^*$ there exists a sequence $\underline{k}_1, \underline{k}_2, \dots, \underline{k}_p$ in $\underline{K}$ such that $w = \sigma_{\underline{k}_1}\sigma_{\underline{k}_2}\cdots\sigma_{\underline{k}_p}$ and $l(w) = l(\sigma_{\underline{k}_1}) + l(\sigma_{\underline{k}_2}) + \cdots + l(\sigma_{\underline{k}_p})$. (Recall that l is the length function of W.) Moreover, $w = \sigma_{\underline{k}_1}\sigma_{\underline{k}_2}\cdots\sigma_{\underline{k}_p}$ is necessarily a reduced expression in the Coxeter group W^*. □

The proof is along the same lines as that of [L1, 5.10 (ii)].

2.30

Assume now that J is a u-stable subset of I such that $I \neq J$ and such that u has exactly one orbit on $I - J$. Then NW_J, NW_J^u are defined as in 2.29. The following version of Lemma 2.29 continues to hold:

(a) $NW_J^u = W_J \cap W^u$.

3 The root system $\mathcal{R}_\tau$

3.1

In this section we preserve the assumptions of 2.4. Recall from 2.22 (d) that the root system $(\mathcal{R} \subset V/(V_J + \mathbf{R}D), \mathcal{R}' \subset P^0)$ is *reduced.*

Let $\mathcal{T}_c = P^0/\mathcal{L}'$ (a compact torus). Given $\tau \in \mathcal{T}_c$, we define $\mathcal{R}_\tau$ to be the set of all $\beta \in \mathcal{R}$ such that

(a) $\langle \dot\tau, \beta\rangle \in (1/f(\beta))\mathbf{Z}$.

Here $\langle , \rangle : P^0 \times V/(V_J + \mathbf{R}D) \to \mathbf{R}$ is the natural pairing, $f(\beta) \in \{1, 2\}$ is as in 2.23, and $\dot\tau$ is a representative of τ in P^0. (The condition (a) does not depend on the choice of representative, since $f(\beta)\beta \in \tilde{\mathcal{R}}$ and $\langle \mathcal{L}', \tilde{\mathcal{R}}\rangle \subset \mathbf{Z}$.) Let $\mathcal{R}'_\tau$ be the subset of $\mathcal{R}'$ corresponding to $\mathcal{R}_\tau$ under the canonical bijection $\mathcal{R} \xrightarrow{\sim} \mathcal{R}'$.

Lemma 3.2. $(\mathcal{R}_\tau \subset V/(V_J + \mathbf{R}D), \mathcal{R}'_\tau \subset P^0)$ is a (reduced) root system. □

Proof. Taking in account 2.22 (d), we see that we must only verify the following statement.

Let $\beta, \beta' \in \mathcal{R}_\tau$ *and let* s_β *be the reflection in* P^0 *or in* $V/(V_J + \mathbf{R}D)$ *determined by* β *and by* h_β *(the element of* $\mathcal{R}'_\tau$ *corresponding to* β*). Then* $s_\beta(\beta') \in \mathcal{R}_\tau$.

First note that $f(s_\beta\beta') = f(\beta')$ (since s_β comes from an element of W^* and the function f is clearly constant on orbits of W^*). We have

$$\begin{aligned}\langle \dot\tau, s_\beta(\beta')\rangle &= \langle s_\beta\dot\tau, \beta'\rangle = \langle \dot\tau - \langle\dot\tau, \beta\rangle h_\beta, \beta'\rangle = \langle\dot\tau, \beta'\rangle - \langle\dot\tau, \beta\rangle\langle h_\beta, \beta'\rangle \\ &\in \frac{1}{f(\beta')}\mathbf{Z} - \frac{\langle h_\beta, \beta'\rangle}{f(\beta)}\mathbf{Z} = \frac{1}{f(s_\beta(\beta'))}\mathbf{Z} - \frac{\langle h_\beta, \beta'\rangle}{f(\beta)}\mathbf{Z}.\end{aligned}$$

Now $\langle (1/f(\beta))h_\beta, \beta'\rangle \in \mathbf{Z}$, since $h_\beta \in f(\beta)\mathcal{L}'$ (see 2.25) and $\langle \mathcal{L}', \mathcal{R}\rangle \subset \mathbf{Z}$, by the definition of $\hat{R}, \mathcal{R}$. Hence

$$\langle \dot\tau, s_\beta(\beta')\rangle \in \frac{1}{f(s_\beta(\beta'))}\mathbf{Z} - \mathbf{Z} \subset \frac{1}{f(s_\beta(\beta'))}\mathbf{Z}.$$

The lemma is proved. ■

3.3

In the remainder of this section we fix an element k_J *of* $\{k \in K \mid \tilde n_k = 1\}$. (Such k_J certainly exists.)

Consider the (free) action of $\mathcal{L}'$ on P given by $x' : z \to x' + z$. We can form the quotient $\mathcal{L}'\backslash P$ (a compact manifold). Since the action of $\mathcal{L}'$ on P described above is a restriction of the W^*-action on P, we see that there is an induced W^* action on $\mathcal{L}'\backslash P$.

We define a map $p^J : P \to \mathcal{T}_c$ by $x' \mapsto x' - \tilde{\omega}_{k_j} \bmod \mathcal{L}'$. This is well defined since $x' - \tilde{\omega}_{k_j} \in P^0$. We have an isomorphism of manifolds

(a) $\mathcal{L}' \backslash P \xrightarrow{\sim} \mathcal{T}_c$

induced by p^J.

Lemma 3.4. p^J is compatible with the natural actions of W^* on P and on $\mathcal{T}_c$. □

Proof. It suffices to show that, for any $w \in W^*$, we have $w(\tilde{\omega}_{k_j}) - \tilde{\omega}_{k_j} \in \mathcal{L}'$. We may assume that $w = \sigma_k$ for some $k \in K$; in this case,

$$w(\tilde{\omega}_{k_j}) - \tilde{\omega}_{k_j} = -\langle \tilde{\omega}_{k_j}, \tilde{\alpha}_k \rangle \tilde{h}_k = \delta_{k_j,k} \tilde{h}_k \in \mathcal{L}'.$$ ■

3.5

We consider the collection of subsets of $\mathcal{T}_c$ which are images (necessarily one-to-one) of facets of H, under p^J. These subsets form a partition of $\mathcal{T}_c$. Any subset in this partition has a well-defined type (a nonempty subset $S \subset K$), namely, the unique S such that the subset is the image under p^J of a facet of type S in H; we then say that the subset is a facet of type S of $\mathcal{T}_c$. In particular, $p(C_S)$ is a facet of type S in $\mathcal{T}_c$.

Lemma 3.6. Let $\bar{C}_K$ be the closure of C_K in P. For any W^*-orbit in $\mathcal{T}_c$, there exists a unique vector $x' \in \bar{C}_K$ such that $p^J(x')$ is in that orbit. □

Proof. This follows from 3.3 (a), from 3.4, and from the following well-known property of the affine reflection group W^* (in P): Any W^*-orbit on P meets $\bar{C}_K$ in exactly one point. ■

3.7

In the remainder of this section we fix $\tau \in p^J(\bar{C}_K)$.

We denote by x' the unique element of $\bar{C}_K$ such that $\tau = p^J(x')$. We have $x' \in C_S$ for a well-defined nonempty subset S of K. Let W^*_{K-S} be the subgroup of W^* generated by $\{\sigma_k \mid k \in K - S\}$, and let

$${}^*R_{K-S} = {}^*R \cap \sum_{k \in K-S} \mathbf{Z}({}^*\alpha_k);$$

$$\tilde{R}_{K-S} = \tilde{R} \cap \sum_{k \in K-S} \mathbf{Z}(\tilde{\alpha}_k);$$

$$\hat{R}_{K-S} = \hat{R} \cap \sum_{k \in K-S} \mathbf{Z}(\hat{\alpha}_k).$$

Lemma 3.8. (a) The bijection $\tilde{R} \xrightarrow{\sim} {}^*R$ $(\tilde{\alpha} \mapsto (1/z_{\tilde{\alpha}})\tilde{\alpha})$ restricts to a bijection $\tilde{R}_{K-S} \xrightarrow{\sim} {}^*R_{K-S}$.

(b) The bijection $\tilde{R} \xrightarrow{\sim} \hat{R}$ $(\tilde{\alpha} \mapsto (\bar{z}_{\tilde{\alpha}}/z_{\tilde{\alpha}})\tilde{\alpha})$ restricts to a bijection $\tilde{R}_{K-S} \xrightarrow{\sim} \hat{R}_{K-S}$. □

Proof. We prove (a). Let $\tilde{\alpha} \in \tilde{R}$, $\alpha \in {}^*R$ be related by $\alpha = (1/z_{\tilde{\alpha}})\tilde{\alpha}$. We must show that the conditions (c) and (d) below are equivalent:

(c) $\tilde{\alpha} = \sum_{k \in K-S} u_k \tilde{\alpha}_k$ for some integers u_k;

(d) $\alpha = \sum_{k \in K-S} u'_k {}^*\alpha_k$ for some integers u'_k.

Assume first that (d) holds. We have $\tilde{\alpha} = \sum_{k \in K} u_k \tilde{\alpha}_k$ for some uniquely defined integers u_k. (This uses the definition of $\tilde{R}$ and the fact that $\tilde{a}_{kl}$ are integers.) Hence $z_{\tilde{\alpha}}\alpha = \sum_{k \in K} u_k z_k {}^*\alpha_k$. Comparing with (d), we see that $u_k = 0$ if $k \in S$ so that (c) holds. Conversely, assume that (c) holds. We have $\alpha = \sum_{k \in K} u'_k {}^*\alpha_k$ for some uniquely defined integers u'_k. (This uses the definition of ${}^*R$ and the fact that ${}^*a_{kk'}$ are integers.) Hence $\tilde{\alpha} = \sum_{k \in K} u'_k z_{\tilde{\alpha}} z_k^{-1} \tilde{\alpha}_k$. Comparing with (c), we see that $u'_k = 0$ if $k \in S$ so that (d) holds. Thus, the equivalence of (c) and (d) is established and (a) is proved. The proof of (b) is entirely similar. ■

Lemma 3.9. Let $\tilde{\mathcal{R}}_\tau$ be the set of all $\tilde{\beta} \in \tilde{\mathcal{R}}$ such that $\langle \dot{\tau}, \tilde{\beta} \rangle \in \mathbf{Z}$ ($\dot{\tau}$ as in 3.1). The restriction of the map $\rho : V/V_J \to V/(V_J + \mathbf{R}D)$ defines a bijection $\tilde{R}_{K-S} \xrightarrow{\sim} \tilde{\mathcal{R}}_\tau$. □

Proof. Let $\tilde{\beta} \in \tilde{\mathcal{R}}$. Choose $\tilde{\alpha} \in \tilde{R}$ such that $\rho(\tilde{\alpha}) = \tilde{\beta}$. Note that $\tilde{\alpha}$ is uniquely determined up to addition of an integer multiple of $\tilde{D} = \sum_{k \in K} \tilde{n}_k \tilde{\alpha}_k$. The condition that $\tilde{\beta} \in \tilde{\mathcal{R}}_\tau$ is $\langle \dot{\tau}, \tilde{\alpha} \rangle \in \mathbf{Z}$; replacing $\dot{\tau}$ by $x' - \tilde{\omega}_{k_j}$, this may be also written $\langle x' - \tilde{\omega}_{k_j}, \tilde{\alpha} \rangle \in \mathbf{Z}$, and this is equivalent to the condition

(a) $\langle x', \tilde{\alpha} \rangle \in \mathbf{Z}$

(since $\langle \tilde{\omega}_{k_j}, \tilde{\alpha} \rangle$ is automatically an integer). Since $\tilde{\alpha} \in \tilde{R}$, we have

$$\tilde{\alpha} = \sum_{k \in K} f_k \tilde{\alpha}_k$$

where f_k are integers. Condition (a) is the same as the condition that x' belongs to $\mathbf{h}' = \{y \in P \mid \langle y, \tilde{\alpha} \rangle = n\}$ (a member of $\mathfrak{H}(P)$) for some $n \in \mathbf{Z}$. Since $x' \in C_S$, this is equivalent to the condition that $C_S \subset \mathbf{h}'$ and also to the condition that the vertices $\{\tilde{\omega}_s/\tilde{n}_s \mid s \in S\}$ of the simplex C_S are all contained in $\mathbf{h}'$. This condition can be expressed as $\langle \tilde{\omega}_s/\tilde{n}_s, \tilde{\alpha} \rangle = n$ for all $s \in S$, or as $f_s = \tilde{n}_s n$ for all $s \in S$. We see that condition (a) is equivalent to

(b) $\tilde{\alpha} \in n\tilde{D} + \sum_{k \in K-S} \mathbf{Z}\tilde{\alpha}_k$

where n is some integer. Replacing $\tilde{\alpha}$ by $\tilde{\alpha} - n\tilde{D}$ (which is again an element of $\tilde{R}$ that projects to $f(\beta)\beta$), we see that we may achieve that

$$\tilde{\alpha} = \sum_{k \in K-S} u_k \tilde{\alpha}_k \quad \text{for some integers } u_k,$$

that is, $\tilde{\alpha} \in \tilde{R}_{K-S}$. (Now the ambiguity in the choice of $\tilde{\alpha}$ is removed.) The lemma follows. ■

Lemma 3.10. The restriction of the map $\rho : V/V_J \to V/(V_J + \mathbf{R}D)$ defines a bijection $\hat{R}_{K-S} \xrightarrow{\sim} \mathcal{R}_\tau$. □

Proof. The map $\beta \to f(\beta)\beta$ from $\mathcal{R}$ to $\tilde{\mathcal{R}}$ is a bijection, since $\mathcal{R}$ is reduced (see 2.22 (d) and 2.23 (b)). This map clearly restricts to a bijection $\mathcal{R}_\tau \xrightarrow{\sim} \tilde{\mathcal{R}}_\tau$. We have a diagram

$$\begin{array}{ccc} \hat{R}_{K-S} & \longrightarrow & \tilde{R}_{K-S} \\ \downarrow & & \downarrow \\ \mathcal{R}_\tau & \longrightarrow & \tilde{\mathcal{R}}_\tau \end{array}$$

where the lower horizontal map is the bijection considered above, the upper horizontal map is the inverse of the bijection in 3.8 (b), the right vertical map is induced by ρ (a bijection, by 3.9), and the left vertical map is the unique bijection which makes the diagram commutative. From 2.23 (c) it follows that the left vertical map is the restriction of ρ. The lemma follows. ■

Lemma 3.11. The vectors $\{\rho(\hat{\alpha}_k) \mid k \in K - S\}$ form a system of simple roots for $\mathcal{R}_\tau$. □

This follows immediately from 3.10.

3.12

Assume that $k \in K^\flat \cap (K - S)$. We set $\epsilon(k) = (-1)^m$, where $m \in \mathbf{Z}$ is given by $\langle \dot{\tau}, \rho(\hat{\alpha}_k)\rangle = m/2$. (The parity of m is independent of the choice of $\dot{\tau}$.)

Lemma 3.13. (a) If $k = k_J$, then $\epsilon(k) = -1$.
(b) If $k \neq k_J$, then $\epsilon(k) = 1$. □

Proof. Let $k \in K^\flat \cap (K - S)$. Assume that
(c) either $k = k_J$ and $\epsilon(k_J) = 1$, or $k \neq k_J$ and $\epsilon(k_J) = -1$.
We will show that this leads to a contradiction. We have

$$\langle \tilde{\varpi}_{k_J}, \hat{\alpha}_k\rangle = \delta_{kk_J}\bar{z}_k/n_{k_J} = \delta_{kk_J}\bar{z}_k/z_{k_J},$$

which is zero if $k \neq k_J$ and is $\bar{z}_{k_J}/z_{k_J} = 1/2$ if $k = k_J$. This, together with the assumption (c), shows that

$$-\langle \tilde{\varpi}_{k_J}, \hat{\alpha}_k\rangle = \langle x' - \tilde{\varpi}_{k_J}, \hat{\alpha}_{k_J}\rangle - \frac{1}{2} \bmod \mathbf{Z},$$

which implies $\langle x', \hat{\alpha}_k\rangle \in (1/2)\mathbf{Z} - \mathbf{Z}$. Since $\hat{\alpha}_k = (1/2)\tilde{\alpha}_k$, we deduce $\langle x', \tilde{\alpha}_k\rangle \in \mathbf{Z} - 2\mathbf{Z}$. This implies, as in the proof of 3.9, that
(d) $\tilde{\alpha}_k \in n\tilde{D} + \sum_{k' \in K-S} \mathbf{Z}\tilde{\alpha}_{k'}$

where n is defined by $\langle x', \tilde{\alpha}_k \rangle = n$. (Hence n is an odd integer.) On the other hand, since $k \in K - S$, from (d) we deduce that $n = 0$. (Otherwise, $\tilde{D}$ would be a linear combination of the vectors $\{\tilde{\alpha}_{k'} \mid k' \in K - S\}$, which is impossible, since $K - S \neq K$.) Since $n = 0$ is not odd, we have a contradiction. The lemma is proved. ■

Lemma 3.14. The sets ${}^*R_{K-S}$, $\{\alpha \in V/V_J \mid \alpha = w({}^*\alpha_k) \text{ for some } w \in W^*_{K-S}, k \in K - S\}$ coincide. □

Proof. Clearly, the second set in the lemma is contained in ${}^*R_{K-S}$. Conversely, let $\alpha \in {}^*R_{K-S}$. Then $\{y \in P \mid \langle y, \alpha \rangle = 0\}$ belongs to $\mathfrak{H}(P)$, and hence there is a unique reflection $\sigma_\alpha : P \to P$ (in W^*) whose fixed-point set is this hyperplane. We have $\alpha \in V_{I-S}/V_J \subset V/V_J$, and hence $\langle y, \alpha \rangle = 0$ for any $y \in C_S$. Thus, σ_α is the identity map on C_S. Using known properties of affine reflection groups (applied to W^* acting on P), we deduce that $\sigma_\alpha \in W^*_{K-S}$ and that σ_α is conjugate in W^*_{K-S} to a standard generator of W^*_{K-S}. Thus, we can find $w \in W^*_{K-S}$ and $k \in K - S$ such that $\sigma_\alpha = w\sigma_k w^{-1}$. This implies $\alpha = w(\alpha_k)$. The lemma is proved. ■

Lemma 3.15. (a) The restriction of ρ (see 2.22) defines an injective map ${}^*R_{K-S} \to {}^*\mathcal{R}$. Let ${}^*\mathcal{R}_{K-S}$ be the image of this map, and let ${}^*\mathcal{R}'_{K-S}$ be the subset of ${}^*\mathcal{R}'$ corresponding to ${}^*\mathcal{R}_{K-S}$ under the natural bijection ${}^*\mathcal{R} \to {}^*\mathcal{R}'$ (see 2.22).

(b) $({}^*R_{K-S} \subset V/V_J, {}^*\mathcal{R}'_{K-S} \subset V'^K)$ is an ordinary (reduced) root system, for which $\{{}^*\alpha_k \mid k \in K - S\}$ is a system of simple roots and W^*_{K-S} is the Weyl group.

(c) $({}^*\mathcal{R}_{K-S} \subset V/(V_J + \mathbf{R}D), {}^*\mathcal{R}'_{K-S} \subset P^0)$ is an ordinary (reduced) root system, for which $\{\rho({}^*\alpha_k) \mid k \in K - S\}$ is a system of simple roots and W^*_{K-S} is the Weyl group. □

Proof. We prove (a). Let $\alpha, \alpha' \in {}^*R_{K-S}$ be such that $\rho(\alpha) = \rho(\alpha')$. Then $\alpha - \alpha' = zD$ for some $z \in \mathbf{R}$ (equality in V/V_J). Expanding both sides of the last equality with respect to the basis $\{{}^*\alpha_k \mid k \in K\}$ of V/V_J and looking at the coefficient of ${}^*\alpha_s$ for $s \in S$, we see that $z = 0$. This proves (a).

To prove (b), it suffices to verify the following statement.

Let $\alpha, \alpha' \in {}^*R_{K-S}$, and let s_α be the reflection in V/V_J or in V'^K determined by α and by h_α (the element of ${}^*\mathcal{R}'_{K-S}$ corresponding to $\rho(\alpha)$). Then $s_\alpha(\alpha') \in {}^*R_{K-S}$.

By 3.14, we have $\alpha = w({}^*\alpha_k)$ for some $w \in W^*_{K-S}$ and some $k \in K - S$. Since ${}^*R_{K-S}$ is W^*_{K-S}-stable, we see that there is no loss of generality if we assume $\alpha = \alpha_k$ with $k \in K - S$, so that $s_\alpha = \sigma_k$. Since $\sigma_k(\alpha') \in {}^*R$, we see that it suffices to show that σ_k maps $\sum_{k' \in K-S} \mathbf{Z}({}^*\alpha_{k'})$ into itself. Hence it suffices to check that $\sigma_k(\alpha'_k) \in \sum_{k' \in K-S} \mathbf{Z}({}^*\alpha_{k'})$ if $k, k' \in K - S$. But $\sigma_k(\alpha'_k)$ is an integral linear combination of $\alpha_k, \alpha_{k'}$. Thus our statement is verified and (b) is proved.

Now (c) follows immediately from (a) and (b). ■

4 Recollections on affine Hecke algebras

4.1

Let $E, \check{E}$ be two finite-dimensional real vector spaces in duality $\langle,\rangle : \check{E} \times E \to \mathbf{R}$, and let $\underline{R}, \underline{\check{R}}$ be subsets of $E, \check{E}$ (resp.) with a given bijection $\underline{R} \xrightarrow{\sim} \underline{\check{R}}$ such that

(a) $(\underline{R} \subset E, \underline{\check{R}} \subset \check{E})$

is an ordinary (reduced) root system. Let $\mathcal{W}$ be the (finite) Weyl group of this root system (a subgroup of $GL(E)$ and of $GL(\check{E})$). Assume that we are also given

(b) a lattice $X \subset E$ and a lattice $\check{X} \subset \check{E}$ such that $\underline{R} \subset X, \underline{\check{R}} \subset \check{X}$ and such that $\langle,\rangle$ defines a perfect pairing $\check{X} \times X \to \mathbf{Z}$;

(c) a function $\lambda : \underline{R} \to \mathbf{N}$, which is constant on the orbits of $\mathcal{W}$ on $\underline{R}$;

(d) a function $\lambda^* : \underline{R}^\flat \to \mathbf{N}$, which is constant on the orbits of $\mathcal{W}$ on $\underline{R}^\flat$.

Here, $\underline{R}^\flat$ is the set of all $\alpha \in \underline{R}$ such that the corresponding element of $\underline{\check{R}}$ belongs to $2\check{X}$.

We recall (cf. [L11]) the definition of the *affine Hecke algebra* $\mathbf{H} = \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X}$ attached to the root system (a), the lattices (b), and the functions (c) and (d).

We choose a system of simple roots $(\alpha_u)_{u\in U}$ in $\underline{R}$; let $(h_u)_{u\in U}$ be the corresponding elements of $\underline{\check{R}}$. For each $u \in U$, let $s_u : X \to X$ (in $\mathcal{W}$) be the reflection defined by h_u, α_u. These reflections are Coxeter generators of $\mathcal{W}$; let $l : \mathcal{W} \to \mathbf{N}$ be the corresponding length function. Let v be an indeterminate. By definition, $\mathbf{H}$ is the associative $\mathbf{C}[v, v^{-1}]$-algebra with 1 defined by the generators $T(w), w \in \mathcal{W}$ and $\theta(x), x \in X$ and by relations

$$T(w)T(w') = T(ww')$$

for all $w, w' \in \mathcal{W}$ such that $l(ww') = l(w) + l(w')$; and

$$(T(s_u) + 1)(T(s_u) - v^{2\lambda(\alpha_u)}) = 0$$

for all $u \in U$;

$$\theta(x)\theta(x') = \theta(x + x')$$

for all $x, x' \in X$;

$$\theta(x)T(s_u) - T(s_u)\theta(s_u(x)) = (v^{2\lambda(\alpha_u)} - 1)\frac{\theta(x) - \theta(s_u(x))}{1 - \theta(-\alpha_u)}$$

for $x \in X$ and $u \in U$ such that $h_u \in \check{X} - 2\check{X}$; and

$$\theta(x)T(s_u) - T(s_u)\theta(s_u(x))$$
$$= ((v^{2\lambda(\alpha_u)} - 1) + \theta(-\alpha_u)(v^{\lambda(\alpha_u)+\lambda^*(\alpha_u)} - v^{\lambda(\alpha_u)-\lambda^*(\alpha_u)}))\frac{\theta(x) - \theta(s_u(x))}{1 - \theta(-2\alpha_u)}$$

for $x \in X$ and $u \in U$ such that $h_u \in 2\check{X}$.

Note that $\theta(0) = 1$. The definition of $\mathbf{H}$ given above depends apparently on the choice of a system of simple roots. Let us choose another system of simple roots; it is of the form $\{\alpha'_u = w(\alpha_u) \mid u \in \mathcal{U}\}$ for a unique $w \in \mathcal{W}$. Let $\mathbf{H}'$ be the algebra defined like $\mathbf{H}$, but in terms of the new system of simple roots. There is a unique algebra isomorphism $\mathbf{H} \xrightarrow{\sim} \mathbf{H}'$ such that

$$T(s_u) \mapsto T(ws_uw^{-1}) \text{ for all } u \in \mathcal{U}$$

and

$$\theta(x) \mapsto \theta(w(x)) \text{ for all } x \in X.$$

Thus, $\mathbf{H}$ and $\mathbf{H}'$ may be canonically identified, and we see that the algebra $\mathbf{H}$ is canonically defined, independently of the choice of a system of simple roots.

4.2

For any ring A we denote by $\operatorname{Irr} A$ the set of isomorphism classes of simple A-modules. Let $\mathbf{T} = \mathbf{C}^* \otimes \check{X}$ be an algebraic torus with a natural action of $\mathcal{W}$. As in [L11, 10.2 (a)], we have a canonical partition

(a) $\operatorname{Irr} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X} = \sqcup_{(\mathcal{W}t,v_0)\in(\mathcal{W}\backslash\mathbf{T})\times\mathbf{C}^*} \operatorname{Irr}_{\mathcal{W}t,v_0} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X}$

obtained by specifying the action of the centre of $\mathbf{H}$ on a simple $\mathbf{H}$-module; for example, the parameter v_0 is given by the scalar by which v acts. For $v_0 \in \mathbf{C}^*$, we set

$$\operatorname{Irr}_{v_0} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X} = \sqcup_{\mathcal{W}t\in\mathcal{W}\backslash\mathbf{T}} \operatorname{Irr}_{\mathcal{W}t,v_0} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X}.$$

Then

(b) $\operatorname{Irr} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X} = \sqcup_{v_0\in\mathbf{C}^*} \operatorname{Irr}_{v_0} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X}$.

4.3

Let us fix $v_0 \in \mathbf{R}_{>0}$. Let $v_0^{\mathbf{Z}}$ be the subgroup of $\mathbf{C}^*$ generated by v_0.

Note that any $\alpha \in \underline{R}$ defines a homomorphism $\mathbf{T} \to \mathbf{C}^*, z \otimes x \mapsto z^{\langle x,\alpha\rangle}$, which we denote again by α.

For a fixed $t \in \mathbf{T}$, let $\underline{R}^t$ be the set of all $\alpha \in \underline{R}$ such that

$$\alpha(t) \in v_0^{\mathbf{Z}} \quad \text{if } \alpha \in \underline{R} - \underline{R}^\flat \qquad \text{and} \qquad \alpha(t) \in \pm v_0^{\mathbf{Z}} \quad \text{if } \alpha \in \underline{R}^\flat.$$

Then $\underline{R}^t$, together with the corresponding elements in $\check{\underline{R}}$, forms a root system in $(E, \check{E})$. Its Weyl group $\mathcal{W}^t$ is the subgroup of $\mathcal{W}$ generated by the reflections with respect to roots in $\underline{R}^t$. Let $\mathcal{W}'^t$ be the set consisting of those $w \in \mathcal{W}$ such that $w(t) = w_1(t)$ for some $w_1 \in \mathcal{W}^t$. Equivalently, this is the subgroup of $\mathcal{W}$ generated by $\mathcal{W}^t$ and $\mathcal{U}^t = \{w \in \mathcal{W} \mid wt = t\}$ (a normal subgroup of $\mathcal{W}$).

4.4

Assumption. In the remainder of this section, the following assumption will be in force.

The abelian group $\check{X}$ *is generated by* $\underline{\check{R}}$*, together with* $\{y \in \check{X} \mid 2y \in \underline{\check{R}}\}$.

Lemma 4.5. We have $\mathcal{U}^t \subset \mathcal{W}^t$. Hence $\mathcal{W}'^t = \mathcal{W}^t$. □

Proof. By [L11, 1.7], it is enough to consider the following two special cases:

(a) $\underline{R}^\flat = \emptyset$,

(b) $\underline{R}^\flat \neq \emptyset$ and the root system $\underline{R} \subset E$ is irreducible.

In the case (a), Assumption 4.4 shows that the root system $\underline{R} \subset E$ is of simply connected type; in that case, by a result of Steinberg, the group $\mathcal{U}^t$ is generated by reflections with respect to roots $\alpha \in \underline{R}$ such that $\chi_\alpha(t) = 1$; these roots are contained in $\underline{R}^t$, and the result follows. In the case (b), the root system is that of SO_{2n+1}, and the result follows by a direct (easy) computation. ■

4.6

Using the equality $\mathcal{W}'^t = \mathcal{W}^t$ and [L11, 8.6], we obtain a bijection

$$\mathrm{Irr}_{\mathcal{W}^t, v_0} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X} \xrightarrow{\sim} \mathrm{Irr}_{\mathcal{W}^t, v_0} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R}^t,X}.$$

(The restrictions $\lambda|_{\underline{R}^t}, \lambda^*|_{\underline{R}^t \cap \underline{R}^\flat}$ are denoted again by λ, λ^*.)

4.7

As a Lie group, we have $\mathbf{T} = \mathbf{T}_c \times \mathbf{T}_h$, where

$$\mathbf{T}_c = \{z \in \mathbf{C}^*; |z| = 1\} \otimes \check{X} \text{ and } \mathbf{T}_h = \mathbf{R}_{>0} \otimes \check{X}.$$

We can therefore write uniquely $t = t_c t_h$ where $t_c \in \mathbf{T}_c, t_h \in \mathbf{T}_h$.

Let $\dot{\underline{R}}$ be the set of those $\alpha \in \underline{R}$ such that $\alpha(t_c) = 1$ if $\alpha \in \underline{R} - \underline{R}^\flat$ and $\alpha(t_c) = \pm 1$ if $\alpha \in \underline{R}^\flat$. Let $\check{\dot{\underline{R}}}$ be the subset of $\underline{\check{R}}$ corresponding to $\dot{\underline{R}}$ under the natural bijection $\underline{R} \leftrightarrow \underline{\check{R}}$. Then

(a) $(\dot{\underline{R}} \subset E, \check{\dot{\underline{R}}} \subset \check{E})$

is a root system. Replacing $\underline{R}$ by $\dot{\underline{R}}$ in the definition of $\mathcal{W}, \underline{R}^t, \mathcal{W}^t, \mathcal{W}'^t, \mathcal{U}^t$ we obtain $\dot{\mathcal{W}}$, $\dot{\underline{R}}^t$, $\dot{\mathcal{W}}^t$, $\dot{\mathcal{W}}'^t$, $\dot{\mathcal{U}}^t$. Note that

$$\dot{\underline{R}}^t = \underline{R}^t, \dot{\mathcal{W}}^t = \mathcal{W}^t.$$

Indeed, if $\alpha(t) \in v_0^{\mathbf{Z}}$, then automatically, $\alpha(t_c) = 1$ (since $v_0 \in \mathbf{R}_{>0}$); similarly, if $\alpha(t) \in \pm v_0^{\mathbf{Z}}$, then automatically, $\alpha(t_c) = \pm 1$.

We show that we again have $\dot{\mathcal{W}}'^{t} = \dot{\mathcal{W}}^{t}$ (even though Lemma 4.5 may not be applicable). Indeed, we again have $\dot{\mathcal{W}}'^{t} = \dot{\mathcal{W}}^{t}\dot{\mathcal{U}}^{t}$ and it is obvious that $\dot{\mathcal{U}}^{t} \subset \mathcal{U}^{t}$. By Lemma 4.5 we have $\mathcal{U}^{t} \subset \mathcal{W}^{t}$; hence $\dot{\mathcal{U}}^{t} \subset \dot{\mathcal{W}}^{t}$ and $\dot{\mathcal{W}}'^{t} = \dot{\mathcal{W}}^{t}$.

Using the last equality and [L11, 8.6], for $\dot{\underline{R}}$ instead of $\underline{R}$, we obtain a natural bijection

$$\mathrm{Irr}_{\dot{\mathcal{W}}t,v_0}\mathbf{H}^{\lambda,\lambda^*}_{\dot{\underline{R}},X} \xrightarrow{\sim} \mathrm{Irr}_{\dot{\mathcal{W}}'^{t},v_0}\mathbf{H}^{\lambda,\lambda^*}_{\dot{\underline{R}}^{t},X} = \mathrm{Irr}_{\dot{\mathcal{W}}^{t}t,v_0}\mathbf{H}^{\lambda,\lambda^*}_{\dot{\underline{R}}^{t},X}.$$

Combining this with the bijection in 4.6 gives a natural bijection

(b) $\mathrm{Irr}_{\mathcal{W}t,v_0}\mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X} \xrightarrow{\sim} \mathrm{Irr}_{\dot{\mathcal{W}}t,v_0}\mathbf{H}^{\lambda,\lambda^*}_{\dot{\underline{R}},X}$.

(The restriction of λ, λ^* to $\dot{\underline{R}}$ is denoted again by λ, λ^*.)

4.8

We recall the definition of the *graded Hecke algebra* $\bar{\mathbf{H}} = \bar{\mathbf{H}}^{\mu}_{\underline{R},E}$ associated to the root system 4.1 (a) and to a function $\mu : \underline{R} \to \mathbf{Z}$ which is constant on the orbits of $\mathcal{W}$ on $\underline{R}$. (See [L11, 4.4].)

Let r be an indeterminate. We write $E_{\mathbf{C}}, \check{E}_{\mathbf{C}}$ instead of $\mathbf{C} \otimes_{\mathbf{R}} E, \mathbf{C} \otimes_{\mathbf{R}} \check{E}$. Choose a system of simple roots α_u $(u \in U)$ in $\underline{R}$; let $h_u \in {}'\check{R}$, $s_u \in \mathcal{W}$ be as in 4.1. By definition, $\bar{\mathbf{H}}$ is the associative $\mathbf{C}[r]$-algebra with 1 defined by the generators $t_w (w \in \mathcal{W})$ and $f (f \in E_{\mathbf{C}})$ and the relations

(a) $t_w t_{w'} = t_{ww'}$ for $w, w' \in \mathcal{W}$; $t_1 = 1$;

(b) $ff' = f'f$ for $f, f' \in E_{\mathbf{C}}$;

(c) $f t_{s_u} - t_{s_u}(s_u(f)) = \mu(\alpha_u)\langle h_u, f\rangle r$

for $f \in E_{\mathbf{C}}$ and $u \in U$.

Again, the definition of $\bar{\mathbf{H}}$ given above depends apparently on the choice of a system of simple roots. Let us choose another system of simple roots; it is of the form $\{\alpha'_u = w(\alpha_u) \mid u \in U\}$ for a unique element w of $\mathcal{W}$. Let $\bar{\mathbf{H}}'$ be the algebra defined like $\bar{\mathbf{H}}$, but in terms of the new system of simple roots. There is a unique algebra isomorphism $\bar{\mathbf{H}} \xrightarrow{\sim} \bar{\mathbf{H}}'$ such that

$$t(w') \mapsto t(ww'w^{-1}) \quad \text{for all } w' \in \mathcal{W}$$

and

$$f \mapsto w(f) \quad \text{for all } f \in E_{\mathbf{C}}.$$

Thus, $\bar{\mathbf{H}}$ and $\bar{\mathbf{H}}'$ may be canonically identified, and we see that the algebra $\bar{\mathbf{H}}$ is canonically defined, independently of the choice of a system of simple roots.

Now the algebra homomorphisms from the centre of $\bar{\mathbf{H}}$ to $\mathbf{C}$ are naturally indexed by $(\mathcal{W}\backslash \check{E}_{\mathbf{C}}) \times \mathbf{C}$; see [L11, 4.5]. Hence we have a natural partition

(d) $\operatorname{Irr} \bar{\mathbf{H}}^{\mu}_{\underline{R},E} = \sqcup_{(\mathcal{W}x, r_0') \in (\mathcal{W}\backslash \check{E}_{\mathbf{C}}) \times \mathbf{C}} \operatorname{Irr}_{\mathcal{W}x, r_0'} \bar{\mathbf{H}}^{\mu}_{\underline{R},E}$

analogous to the partition 4.2 (a); it is obtained by specifying the action of the centre of $\bar{\mathbf{H}}^{\mu}_{\underline{R},E}$ on a simple $\bar{\mathbf{H}}^{\mu}_{\underline{R},E}$-module. For example, the parameter r_0' is given by the scalar by which r acts.

To simplify notation, we shall also write $\operatorname{Irr}_{x, r_0'} \bar{\mathbf{H}}^{\mu}_{\underline{R},E}$ instead of $\operatorname{Irr}_{\mathcal{W}x, r_0'} \bar{\mathbf{H}}^{\mu}_{\underline{R},E}$ (where $x \in \check{E}_{\mathbf{C}}$). For any $r_0 \in \mathbf{C}$, we set

(e) $\operatorname{Irr}_{r_0} \bar{\mathbf{H}}^{\mu}_{\underline{R},E} = \sqcup_{\mathcal{W}x \in \mathcal{W}\backslash \check{E}_{\mathbf{C}}} \operatorname{Irr}_{\mathcal{W}x, r_0} \bar{\mathbf{H}}^{\mu}_{\underline{R},E}$.

4.9

Let us return to our fixed $v_0 \in \mathbf{R}_{>0}$ and $t = t_c t_h \in \mathbf{T}$ (see 4.3 and 4.7). We write $v_0 = e^{r_0}$ where $r_0 \in \mathbf{R}$, and we define

$$\log : \mathbf{T}_h = \mathbf{R}_{>0} \otimes \check{X} \to \mathbf{R} \otimes \check{X}$$

as the isomorphism of Lie groups induced by the isomorphism $\mathbf{R}_{>0} \to \mathbf{R}$, inverse to the exponential.

We consider the graded Hecke algebra $\bar{\mathbf{H}}^{\mu}_{\dot{\underline{R}},E}$ attached to the root system 4.7 (a) and to the function $\mu : \dot{\underline{R}} \to \mathbf{Z}$ given by

(a) $\mu(\alpha) = 2\lambda(\alpha)$ if $\alpha \in \dot{\underline{R}} - \underline{R}^{\flat}$,

(b) $\mu(\alpha) = \lambda(\alpha) + \alpha(t_c)\lambda^*(\alpha)$ if $\alpha \in \dot{\underline{R}} \cap \underline{R}^{\flat}$.

From the definition of $\dot{\underline{R}}$ and from [L11, 3.15] we see that t_c is a $\dot{\mathcal{W}}$-invariant element of $\mathbf{T}$. Hence μ is constant on the orbits of $\dot{\mathcal{W}}$ on $\dot{\underline{R}}$. Thus, the $\mathbf{C}[r]$-algebra $\bar{\mathbf{H}}^{\mu}_{\dot{\underline{R}},E}$ is well defined. We have a canonical bijection

$$\operatorname{Irr}_{\dot{\mathcal{W}}t, v_0} \mathbf{H}^{\lambda,\lambda^*}_{\dot{\underline{R}},X} \xrightarrow{\sim} \operatorname{Irr}_{\dot{\mathcal{W}} \log t_h, r_0} \bar{\mathbf{H}}^{\mu}_{\dot{\underline{R}},E}.$$

This is induced by an isomorphism of the completions of the algebras $\mathbf{H}^{\lambda,\lambda^*}_{\dot{\underline{R}},X}, \bar{\mathbf{H}}^{\mu}_{\dot{\underline{R}},E}$ with respect to the maximal ideals of the centres defined by $(\dot{\mathcal{W}}t, v_0)$, $(\dot{\mathcal{W}} \log t_h, r_0)$ respectively; this isomorphism is constructed by repeating word-for-word the proof of [L11, 9.3]. Composing with the bijection 4.7 (b), we obtain a natural bijection

(c) $\operatorname{Irr}_{\mathcal{W}t, v_0} \mathbf{H}^{\lambda,\lambda^*}_{\underline{R},X} \xrightarrow{\sim} \operatorname{Irr}_{\dot{\mathcal{W}} \log t_h, r_0} \bar{\mathbf{H}}^{\mu}_{\dot{\underline{R}},E}$.

4.10 Errata to [L11]

In the statement of Lemma 1.7, omit "uniquely". In 8.1, the definition of $\tilde{\mathcal{W}}_0^t$, replace $w(t) = t$ by $w(t) = t \bmod \mathcal{T}\langle v_0 \rangle$. In 8.8 (a), replace $1_{s_\alpha}(c)$ by $1_{s_\alpha(c)}$. In 8.15 (a) and (b), replace ${}_c\hat{H}_{c'}$ by ${}_c\tilde{H}_{c'}$.

5 Affine Hecke algebras and algebraic groups over C

In §1 we saw how affine Hecke algebras (with not necessarily equal parameters) arise in the representation theory of a simple p-adic group. In this section we shall see that such affine Hecke algebras also appear in a geometrical context, related to algebraic groups over $\mathbf{C}$ and cuspidal character sheaves.

5.1

We preserve the notation of §2. We now fix an index $0 \in I$ such that $n_0 = 1$.

Recall that $H^0_{\mathbf{Z}}$ is the set of all $x' \in H^0$ such that there exists a $w \in W$ with $x' + z = w(z)$ for all $z \in H$. (Note that w is uniquely determined by x' and $x' \mapsto w$ is an injective group homomorphism $H^0_{\mathbf{Z}} \to W$. The image of this homomorphism is a normal abelian subgroup of W, and the quotient $\bar{W} = W/H^0_{\mathbf{Z}}$ is a finite group.) From the theory of affine Weyl groups, it is well known that $H^0_{\mathbf{Z}}$ is the lattice in H^0 generated by $\{h_i \mid i \in I\}$. Consider the dual lattice $H^{0*}_{\mathbf{Z}} = \{x \in V/\mathbf{R}D \mid \langle H^0_{\mathbf{Z}}, x\rangle \in \mathbf{Z}\}$ in $V/\mathbf{R}D$. Let $\bar{R} \subset H^{0*}_{\mathbf{Z}}$ be the image of R under the canonical map $V \to V/\mathbf{R}D$. Let $\bar{\alpha}_i \in \bar{R}$ be the image of $\alpha_i \in R$.

Let $H^0_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} H^0$. Let $T = H^0_{\mathbf{C}}/H^0_{\mathbf{Z}}$ (a complex algebraic torus). We identify $H^0_{\mathbf{Z}} = \mathrm{Hom}(\mathbf{C}^*, T)$ (as abelian groups) by associating to $x' \in H^0_{\mathbf{Z}}$ the homomorphism $\mathbf{C}^* \to T$ given by $e^{2\pi\sqrt{-1}z} \mapsto zx' \bmod H^0_{\mathbf{Z}}$ (with $z \in \mathbf{C}$). We identify $H^{0*}_{\mathbf{Z}} = \mathrm{Hom}(T, \mathbf{C}^*)$ (as abelian groups) by associating to $x \in H^{0*}_{\mathbf{Z}}$ the homomorphism $T \to \mathbf{C}^*$ induced by $x' \mapsto e^{2\pi\sqrt{-1}\langle x', x\rangle}$ (from $H^0_{\mathbf{C}} \to \mathbf{C}^*$). Then the pairing $\langle\, ,\rangle : H^0_{\mathbf{Z}} \times H^{0*}_{\mathbf{Z}} \to \mathbf{Z}$ becomes the standard pairing $\mathrm{Hom}(\mathbf{C}^*, T) \times \mathrm{Hom}(T, \mathbf{C}^*) \to \mathbf{Z}$.

5.2

We can imbed T as a maximal torus of a simply connected almost simple algebraic group G over $\mathbf{C}$, in such a way that $h_i \in \mathrm{Hom}(\mathbf{C}^*, T)$ and $\bar{\alpha}_i \in \mathrm{Hom}(T, \mathbf{C}^*)$ (for $i \in I - \{0\}$) are a system of simple coroots and simple roots of G with respect to T; then $\bar{R}$ becomes the set of roots of G with respect to T. Moreover, G and the imbedding are determined uniquely up to a unique isomorphism.

The natural W-action on H^0 (and $H^0_{\mathbf{C}}$) leaves the subgroup $H^0_{\mathbf{Z}}$ stable, and hence it induces a W-action on T. The subgroup $H^0_{\mathbf{Z}}$ of W acts trivially on H^0 and hence the W-action on T factors through an action of the finite quotient $\bar{W} = W/H^0_{\mathbf{Z}}$. This coincides with the Weyl group of G with respect to T with its natural action on T.

We have a direct product decomposition (as a Lie group) $T = T_c \times T_h$, given by multiplication, where $T_c = H^0/H^0_{\mathbf{Z}}$ is the maximal compact torus of T and $T_h = \mathbf{R}_{>0} \otimes H^0_{\mathbf{Z}}$. Note that T_c, T_h are stable under the action of W (or $\bar{W}$).

An element of G is said to be *compact* (resp. *hyperbolic*) if it is in the G-conjugacy class of an element of T_c (resp. T_h). It is well known that any semisimple element s of G can be written uniquely in the form $s = s_c s_h$ where $s_c \in G$ is compact, $s_h \in G$ is hyperbolic and $s_c s_h = s_h s_c$.

Let $\mathfrak{g}$ be the Lie algebra of G. An element of $\mathfrak{g}$ is said to be *compact* (resp. *hyperbolic*) if it is in the Ad(G)-orbit of an element of Lie T_c (resp. Lie T_h). It is well known that any semisimple element ξ of $\mathfrak{g}$ can be written uniquely in the form $\xi = \xi_c + \xi_h$ where $\xi_c \in \mathfrak{g}$ is compact, $\xi_h \in \mathfrak{g}$ is hyperbolic, and $[\xi_c, \xi_h] = 0$.

The exponential map $\exp : \mathfrak{g} \to G$ restricts to a bijection from the set of hyperbolic elements in $\mathfrak{g}$ to the set of hyperbolic elements in G; the inverse of this bijection is denoted by log.

Consider the (free) action of $H^0_{\mathbf{Z}}$ on H given by $x' : z \to x' + z$. We can form the quotient $H^0_{\mathbf{Z}} \backslash H$ (a compact manifold). Since the action of $H^0_{\mathbf{Z}}$ on H described above is a restriction of the W-action on H, we see that there is an induced W- (or $\bar{W}$-) action on $H^0_{\mathbf{Z}} \backslash H$. Let $p : H \to T_c$ be the map given the composition $H \to H^0 \to T_c$. (The first map is $x' \mapsto x' - \varpi_0$, and the second map is the canonical homomorphism.) We have an isomorphism of manifolds

(a) $H^0_{\mathbf{Z}} \backslash H \xrightarrow{\sim} T_c$

induced by p. Now p is compatible with the W-actions on H and on T_c since $w(\varpi_0) - \varpi_0 \in H^0_{\mathbf{Z}}$ for any $w \in W$. Hence the isomorphism (a) is compatible with the W- (or $\bar{W}$-) actions.

5.3

For any $t \in T$ we denote by $\bar{R}_t$ the set of all $\alpha \in \bar{R}$ such that $\alpha(t) = 1$. (We identify as above α with a homomorphism $T \to \mathbf{C}^*$.)

For a subset $J' \subset I$ with $J' \neq I$, we set $\bar{R}_{J'} = \bar{R} \cap \sum_{j \in J'} \mathbf{Z}\bar{\alpha}_j$.

Lemma 5.4. Let K' be a nonempty subset of I, let $J' = I - K'$, and let $t \in p(C_{K'})$. Then $\bar{R}_t = \bar{R}_{J'}$. □

Proof. The proof is very similar to that of 3.9. Let $\tilde{t} \in C_{K'}$ be such that $p(\tilde{t}) = t$. Let $\alpha \in \bar{R}$. Then α is the image of an element $\tilde{\alpha} \in R$ under $V \to V/\mathbf{R}D$. We have $\tilde{\alpha} = \sum_{i \in I} f_i \alpha_i$ where $f_i \in \mathbf{Z}$. The condition that $\alpha(t) = 1$ is equivalent to the condition that $\langle \tilde{t}, \tilde{\alpha} \rangle \in \mathbf{Z}$, and hence to the condition that for some $n \in \mathbf{Z}$, $\tilde{t}$ is contained in $\mathbf{h} = \{x' \in H \mid \langle x', \tilde{\alpha} \rangle = n\}$ (which belongs to $\mathfrak{H}(H)$). Since $\tilde{t} \in C_{K'}$, this is equivalent to the condition that $C_{K'} \subset \mathbf{h}$ and also to the condition that the vertices $\{\varpi_k/n_k \mid k \in K'\}$ of the simplex $C_{K'}$ are all contained in $\mathbf{h}$. This condition can be expressed as $\langle \varpi_k/n_k, \tilde{\alpha} \rangle = n$ for all $k \in K'$, or as $f_k = n_k n$ for all $k \in K'$. We see that the condition that $\alpha(t) = 1$ is equivalent to the condition that $\tilde{\alpha} - nD \in \sum_{j \in J'} \mathbf{Z}\alpha_j$ for some $n \in \mathbf{Z}$. Replacing $\tilde{\alpha}$ by $\tilde{\alpha} - nD$, we see that $\tilde{\alpha}$ can be chosen so that $\tilde{\alpha} \in \sum_{j \in J'} \mathbf{Z}\alpha_j$ and then there is no ambiguity in the choice. This establishes a bijection between $\bar{R}_t$ and the set $R \cap \sum_{j \in J'} \mathbf{Z}\alpha_j$. The last set is naturally in bijection with $\bar{R}_{J'}$ via the restriction of $V \to V/\mathbf{R}D$. The lemma follows. ∎

5.5

The centralizer $Z_G(t)$ of $t \in T$ in G is the connected reductive subgroup of G generated by T and by the root subgroups of G corresponding to roots $\alpha \in \bar{R}^t$. For a subset $J' \subset I$ with $J' \neq I$, we denote by $G_{J'}$ the connected reductive subgroup of G generated by T and by the root subgroups of G corresponding to roots in $\bar{R}_{J'}$. Using 5.4, we see that, if J' is a subset of I with $J' \neq I$ and $t \in p(C_{K'})$ where $K' = I - J'$, then $Z_G(t) = G_{J'}$. In particular, $Z_G(t)$ depends only on J' and not on t.

The relationship between centralizers of semisimple elements of G and affine Weyl groups goes back to Borel and de Siebenthal [BS].

5.6

We denote the Lie algebra of G by $\mathfrak{g}$. For a subset $J' \subset I$ with $J' \neq I$, we denote the Lie algebra of $G_{J'}$ by $\mathfrak{g}_{J'}$.

In the remainder of this section, we fix a subset J *of* I, *with* $J \neq I$, *a nilpotent* G_J*-orbit* $\mathcal{C}$ *on the Lie algebra* $\mathfrak{g}_J$, *and an irreducible, cupidal* G_J*-equivariant local system* $\mathcal{F}$ *on* $\mathcal{C}$.

The possible J are listed explicitly in each case in the tables in §7 (they correspond to the boxed elements of I in those tables); this list is easily obtained from the classification of cuspidal local systems [L10]. From this we see that

the subset J *of* I *is necessarily excellent.*

Let $y_0 \in \mathcal{C}$. We can find a semisimple element d in the derived subalgebra of $\mathfrak{g}_J$ such that $[d, y_0] = 2y_0$. Let $\theta : \mathbf{C}^* \to \mathrm{Der}\, G_J$ (derived group of G_J) be the homomorphism of algebraic groups given by $\theta(b) = \exp(bd)$ for all $b \in \mathbf{C}^*$. Then $\mathrm{Ad}(\theta(b))y_0 = b^2 y_0$ for all $b \in \mathbf{C}^*$. From [L5, 2.8] it follows easily that the pair (y_0, θ) is uniquely determined up to the action G_J.

5.7 The function c_k

Let $K = I - J$.

Until the end of 5.16, we further assume that K *has at least two elements.*

Then the discussion in §2 is applicable in the present case. The discussion in §3 is also applicable in the present case, since the assumption in 3.1 is now automatically satisfied, as we see by examining the tables in §7.

For $k \in K$, we have $G_J \subset G_{J \cup k}$; in fact, G_J is a Levi subgroup of a maximal parabolic subgroup of $G_{J \cup k}$. Let $\mathfrak{n}$ be the nil-radical of a parabolic subalgebra of $\mathfrak{g}_{J \cup k}$ with Levi subalgebra $\mathfrak{g}_J$.

If $y \in \mathcal{C}$, then $\mathrm{ad}(y) : \mathfrak{g} \to \mathfrak{g}$ maps $\mathfrak{n}$ into itself; let c_k be the unique integer ≥ 2 such that $\mathrm{ad}(y)^{c(k)-1} : \mathfrak{n} \to \mathfrak{n}$ is zero and $\mathrm{ad}(y)^{c(k)-2} : \mathfrak{n} \to \mathfrak{n}$ is nonzero. This is clearly independent of the choice of $\mathfrak{n}$ and y.

5.8

Consider the group homomorphism $\mathcal{L}' \to W^*$ (notation of 2.4 and 2.11) given by $x' \mapsto w'$ where $w' \in W^*$ is such that $x' + z' = w'(z')$ for all $z' \in P$. Note that w' is uniquely determined by x' (see 2.5), and that our homomorphism is injective. The image of this homomorphism is a normal abelian subgroup of W^* and the quotient $\mathcal{W} = W^*/\mathcal{L}'$ is a finite group. This is, in fact, the Weyl group of the root system $(\mathcal{R} \subset V/(V_J + \mathbf{R}D), \mathcal{R}' \subset P^0)$ (see 2.22 (b)).

Lemma 5.9. (a) If $\mathcal{R}^\flat \neq \emptyset$, then $\mathcal{R}^\flat$ is a single $\mathcal{W}$-orbit and $K^\flat$ consists of exactly two elements k, k'.

(b) There is a unique $\mathcal{W}$-invariant function $\lambda : \mathcal{R} \to \mathbf{N}$ such that $\lambda(\rho(\hat{\alpha}_k)) = \bar{z}_k c_k/2$ for all $k \in K^\sharp$ and whose restriction to $\mathcal{R}^\flat$ (if nonempty) is constant with value $(\bar{z}_k c_k + \bar{z}_{k'} c_{k'})/2$, where k, k' are as in (a).

(c) If $\mathcal{R}^\flat \neq \emptyset$, there is a unique $\mathcal{W}$-invariant function $\lambda^* : \mathcal{R}^\flat \to \mathbf{N}$, which is constant with value $(\bar{z}_k c_k - \bar{z}_{k'} c_{k'})/2$, where k, k' (as in (a)) are ordered in such a way that $\bar{z}_k c_k \geq \bar{z}_{k'} c_{k'}$.

□

Then (a) follows from the tables in §7; (b) follows from the following two statements:

If $k, k' \in K$ *are such that* $\sigma_k \sigma_{k'}$ *has order* 3 *in* W^*, *then* $c_k = c_{k'}$.

If $k \neq k'$ *in* K *are extraordinary and* $\mathcal{R}^\flat = \emptyset$, *then* $c_k = c_{k'}$.

(Both these statements follow from the tables in §7.) Item (c) is obvious. The fact that the values of λ, λ^* are integers rather than rational numbers is verified case by case from the tables in §7. ■

5.10

Recall that in 3.3 we have chosen an element k_J of $\{k \in K \mid \tilde{n}_k = 1\}$. We want to make this choice a little more precise, in the case where $K^\flat \neq \emptyset$. In this case, $\{k \in K \mid \tilde{n}_k = 1\} = K^\flat$

has exactly two elements, and we choose k_J so that

$$\bar{z}_{k_J} c_{k_J} \leq \bar{z}_k c_k$$

where k is the unique element of $K^\flat$ distinct from k_J.

5.11

Consider the root system $(\mathcal{R} \subset V/(V_J + \mathbf{R}D), \mathcal{R}' \subset P^0)$, the lattice $\mathcal{L}' \subset P^0$ (see 2.11), the dual lattice

$$\mathcal{L} = \{x \in V/(V_J + \mathbf{R}D) \mid \langle \mathcal{L}', x \rangle \in \mathbf{Z}\},$$

and the functions $\lambda : \mathcal{R} \to \mathbf{N}, \lambda^* : \mathcal{R}^\flat \to \mathbf{N}$ as in 5.9. Note that $\mathcal{R}' \subset \mathcal{L}'$ and $\mathcal{R} \subset \mathcal{L}$ (using the definitions). Hence, to the data above, one can associate an affine Hecke algebra $\mathbf{H}_{\mathcal{R},\mathcal{L}}^{\lambda,\lambda^*}$ as in 4.1. Assumption 4.4 is satisfied in the present case: the abelian group $\mathcal{L}'$ is generated by $\{\hat{h}_k \mid k \in K^\sharp\} \cup \{\hat{h}_k/2 \mid k \in K^\flat\}$. An equivalent statement is that $\mathcal{L}'$ is generated by $\{\tilde{h}_k \mid k \in K\}$ (see 2.14 (b)). Hence the results in §4 are applicable to this affine Hecke algebra.

5.12

Let $\underline{W}$ be a Coxeter group with simple reflections $\{s_m \mid m \in M\}$, with a given function $L : M \to \mathbf{N}$, which takes equal values on any two simple reflections that are conjugate, and let Γ be a group with a given action on $\underline{W}$ preserving the group structure, the generating set, and the function L.

Following Iwahori and Matsumoto, to these data one can associate an algebra over $\mathbf{C}[v, v^{-1}]$ with generators $\{T_{s_m} \mid m \in M\} \cup \{T_\gamma \mid \gamma \in \Gamma\}$ and the following relations:

(a) $(T_{s_m} + 1)(T_{s_m} - v^{2L(m)}) = 0$ for $m \in M$;

(b) $T_{s_m} T_{s_{m'}} T_{s_m} = \cdots = T_{s_{m'}} T_{s_m} T_{s_{m'}} \cdots$ where both products have a number of factors equal to the order of $s_m s_{m'}$ in W (if that order is finite and $m \neq m'$);

(c) $T_\gamma T_{\gamma'} = T_{\gamma\gamma'}$ if $\gamma, \gamma' \in \Gamma$; $T_1 = 1$;

(d) $T_\gamma T_{s_m} = T_{\gamma(s_m)} T_\gamma$ if $m \in M, \gamma \in \Gamma$.

We say that this algebra is given by an *IM-presentation*.

Now the affine Hecke algebra $\mathbf{H}_{\mathcal{R},\mathcal{L}}^{\lambda,\lambda^*}$ admits an IM-presentation that we now describe. First we define an untwisted affine Cartan matrix $(a^\dagger_{kk'})_{k,k' \in K}$ as follows.

From the definitions, we see that $\{\hat{h}_k \mid k \in K - \{k_J\}\}$ is a system of simple coroots for $\mathcal{R}'$. The negative of the highest coroot in $\mathcal{R}'$ with respect to this system of simple coroots is denoted by $\alpha^\dagger_{k_J}$.

Let $h^\dagger_{k_J}$ be the element of $\mathcal{R}$ corresponding to $\alpha^\dagger_{k_J}$ under the canonical bijection $\mathcal{R}' \xrightarrow{\sim} \mathcal{R}$. For $k \in K - \{k_J\}$, we set $\alpha^\dagger_k = \hat{h}_k$ and $h^\dagger_k = \hat{\alpha}_k$. For $k, l \in K$, we set $a^\dagger_{kl} = \langle \alpha^\dagger_l, h^\dagger_k \rangle$. Clearly,

$(a^\dagger_{kl})_{k,l\in K}$ *is an untwisted affine Cartan matrix.*

The affine Weyl group corresponding to this affine Cartan matrix is denoted by $\underline{W}$. This is a Coxeter group on generators $\{\underline{\sigma}_k \mid k \in K\}$ of order 2 such that, for $k \neq l$ in K, the order of $\underline{\sigma}_k\underline{\sigma}_l$ is equal to 2, if $a^\dagger_{kl} = a^\dagger_{lk} = 0$; to 3, if $a^\dagger_{kl}a^\dagger_{lk} = 1$; to 4, if $a^\dagger_{kl}a^\dagger_{lk} = 2$; to 6, if $a^\dagger_{kl}a^\dagger_{lk} = 3$; and to ∞, if $a^\dagger_{kl}a^\dagger_{lk} = 4$.

Let $\mathcal{L}_0$ be the subgroup of $\mathcal{L}$ generated by $\mathcal{R}$. The quotient $\mathcal{L}/\mathcal{L}_0$ (a finite abelian group) acts in a standard way as a group of automorphisms of the Coxeter group $\underline{W}$.

There is a unique function $L : K \to \mathbf{N}$ such that the following hold:

(i) $L(k) = \lambda(\rho(\hat{\alpha}_k))$ if $k \in K - \{k_J\}$;

(ii) $L(k_J) = \lambda(\rho(\hat{\alpha}_{k'}))$ if $k_J \in K^\sharp$ and $k' \in K - \{k_J\}$ is such that $\alpha^\dagger_{k'}$ is in the same W^*-orbit as $\alpha^\dagger_{k_J}$;

(iii) $L(k_J) = \lambda^*(\rho(\hat{\alpha}_{k'}))$ if $k_J \in K^\flat$ and $k' \in K^\flat$ is such that $k' \neq k_J$.

The algebra given by the IM-presentation attached to the Coxeter group $\underline{W}$, to the function L, and to $\Gamma = \mathcal{L}/\mathcal{L}_0$ is canonically isomorphic to $\mathbf{H}^{\lambda,\lambda^*}_{\mathcal{R},\mathcal{L}}$. (This is shown in [L11, §3] by a construction generalizing an earlier one of Bernstein and Zelevinsky.)

5.13

Let $\nu_0 \in \mathbf{R}_{>0}$. We set $P^0_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} P^0$. We consider the algebraic torus

$$\mathcal{T} = P^0_{\mathbf{C}}/\mathcal{L}' = \mathbf{C}^* \otimes \mathcal{L}'.$$

The last equality is given by $P^0_{\mathbf{C}}/\mathcal{L}' = (\mathbf{C} \otimes \mathcal{L}')/(\mathbf{Z} \otimes \mathcal{L}') = (\mathbf{C}/\mathbf{Z}) \otimes \mathcal{L}' = \mathbf{C}^* \otimes \mathcal{L}'$, where we use the isomorphism $\mathbf{C}/\mathbf{Z} \xrightarrow{\sim} \mathbf{C}^*, u \mapsto e^{2\pi\sqrt{-1}u}$.

We have a direct product decomposition (as a Lie group) $\mathcal{T} = \mathcal{T}_c \times \mathcal{T}_h$, given by multiplication, where $\mathcal{T}_c = P^0/\mathcal{L}'$ (see 3.1) is the maximal compact torus of $\mathcal{T}$ and $\mathcal{T}_h = \mathbf{R}_{>0} \otimes \mathcal{L}'$.

We define a map $t \mapsto t^\psi$ from $p^J(\bar{C}_K) \subset \mathcal{T}_c$ to T_c by $t^\psi = p(x')$ where $x' \in \bar{C}_K, p^J(x') = t$. (Recall that $p^J : P \to \mathcal{T}_c$ is defined in terms of our chosen k_J.) For example, we have $1^\psi = \varpi_{k_J} - \varpi_0 \bmod H^0_{\mathbf{Z}}$.

A complete set of representatives for the $\bar{W}$-orbits on T is provided by

(a) $\sqcup_{S\subset I, S\neq\emptyset} p(C_S) \cdot \exp_{T_h}(B_S)$

where $B_S = \{x \in H^0 \mid \langle x, \alpha_i \rangle \geq 0\ \forall i \in I - S\}$ and $\exp_{T_h} : H^0 = \mathbf{R} \otimes H^0_{\mathbf{Z}} \to T_h = \mathbf{R}_{>0} \otimes H^0_{\mathbf{Z}}$ is induced by $\exp : \mathbf{R} \to \mathbf{R}_{>0}$.

Similarly, a complete set of representatives for the $\mathcal{W}$-orbits on $\mathcal{T}$ is provided by

(b) $\sqcup_{S\subset K, S\neq\emptyset} p^J(C_S)\cdot \exp_{\mathcal{T}_h}(B_S^J)$

where

$$B_S^J = \{x \in P^0 \mid \langle x,\alpha_i\rangle \geq 0\ \forall i\in K-S\}$$
$$= \{x \in H^0 \mid \langle x,\alpha_i\rangle \geq 0\ \forall i\in K-S, \langle x,\alpha_i\rangle = 0\ \forall i \in J\} \subset B_S$$

and $\exp_{\mathcal{T}_h} : P^0 = \mathbf{R}\otimes\mathcal{L}' \to \mathcal{T}_h = \mathbf{R}_{>0}\otimes\mathcal{L}'$ is induced by $\exp : \mathbf{R}\to\mathbf{R}_{>0}$.

5.14

We fix an orbit for the natural action of $\mathcal{W}$ (or W^*) on $\mathcal{T}$. This orbit contains a unique element t in the set 5.13 (b): we have $t = t_c t_h$ where $t_c \in p^J(C_S)$ and $t_h \in \exp_{\mathcal{T}_h}(B_S^J)$ for a well-defined $S\subset K, S\neq\emptyset$. Note that $\mathcal{W}_{t_c}$ coincides with the Weyl group of the root system $(\mathcal{R}_{t_c} \subset V/(V_J + \mathbf{R}D), \mathcal{R}'_{t_c} \subset P^0)$ defined as in 3.1 and 3.2, for $\tau = t_c$. (This is seen by an argument like that in the proof of 4.5.) As in 4.9, we define $\mu : \mathcal{R}_{t_c} \to \mathbf{Z}$ by

$$\mu(\alpha) = 2\lambda(\alpha) \text{ if } \alpha\in\mathcal{R}_{t_c} - \mathcal{R}^\flat$$

$$\mu(\alpha) = \lambda(\alpha) + \alpha(t_c)\lambda^*(\alpha) \text{ if } \alpha\in\mathcal{R}^\flat.$$

Then the graded Hecke algebra $\bar{\mathbf{H}}^{\mu}_{\mathcal{R}_{t_c}, V/(V_J+\mathbf{R}D)}$ is well defined (see 4.8), and we have a canonical bijection (see 4.9 (c))

(a) $\mathrm{Irr}_{\mathcal{W}t,v_0}\,\mathbf{H}^{\lambda,\lambda^*}_{\mathcal{R},\mathcal{L}} \xrightarrow{\sim} \mathrm{Irr}_{\mathcal{W}_{t_c}\log t_h, r_0}\,\bar{\mathbf{H}}^{\mu}_{\mathcal{R}_{t_c}, V/(V_J+\mathbf{R}D)}$.

Here $r_0\in\mathbf{R}$ is defined by $v_0 = e^{r_0}$, and $\log t_h \in P^0$ is defined by $\exp_{\mathcal{T}_h}(\log t_h) = t_h$.

As in 3.7 (for $\tau = t_c$), we denote by x' the unique element of C_S such that $p^J(x') = t_c$.

Lemma 5.15. $\mu : \mathcal{R}_{t_c} \to \mathbf{Z}$ (in 5.13) is the unique function that is constant on the orbits of $\mathcal{W}_{t_c}$ and satisfies

(a) $\mu(\rho(\hat{\alpha}_k)) = \bar{z}_k c_k$ for all $k\in K-S$. □

Proof. Recall that the vectors $\{\rho(\hat{\alpha}_k) \mid k\in K-S\}$ form a system of simple roots for $\mathcal{R}_{t_c}$ (see 3.11); hence we only have to show that μ satisfies (a). Assume first that $k\in K-S, k\notin K^\flat$. Then, from the definitions,

$$\mu(\rho(\hat{\alpha}_k)) = 2\lambda(\rho(\hat{\alpha}_k)) = \bar{z}_k c_k,$$

as desired. Assume next that $k\in K-S, k\in K^\flat, k = k_J$, and let $k_1\in K^\flat$ be the unique element such that $k_1\neq k_J$. From the definitions,

$$\mu(\rho(\alpha_k)) = \lambda(\rho(\alpha_k)) + \alpha_k(t_c)\lambda^*(\rho(\alpha_k))$$
$$= \frac{1}{2}(\bar{z}_{k_1}c_{k_1} + \bar{z}_{k_J}c_{k_J}) - \frac{1}{2}(\bar{z}_{k_1}c_{k_1} - \bar{z}_{k_J}c_{k_J}) = \bar{z}_{k_J}c_{k_J} = \bar{z}_k c_k.$$

(We have used the equality $\alpha_k(t_c) = -1$; see 3.13 (a).) Finally, assume that $k \in K - S, k \in K^\flat, k \neq k_J$. From the definitions,

$$\begin{aligned}\mu(\rho(\alpha_k)) &= \lambda(\rho(\alpha_k)) + \alpha_k(t_c)\lambda^*(\rho(\alpha_k)) \\ &= \frac{1}{2}(\bar z_k c_k + \bar z_{k_J} c_{k_J}) + \frac{1}{2}(\bar z_k c_k - \bar z_{k_J} c_{k_J}) = \bar z_k c_k.\end{aligned}$$

(We have used the equality $\alpha_k(t_c) = 1$; see 3.13 (b).) The lemma is proved. ■

5.16

The root systems

$$({}^*\mathcal{R}_{K-S} \subset V/(V_J + \mathbf{R}D), {}^*\mathcal{R}'_{K-S} \subset P^0), \quad (\mathcal{R}_{t_c} \subset V/(V_J + \mathbf{R}D), \mathcal{R}'_{t_c} \subset P^0)$$

have sets of simple roots $\{\rho({}^*\alpha_k) \mid k \in K - S\}, \{\rho(\hat\alpha_k) \mid k \in K - S\}$ respectively (see 3.11 and 3.15), and these are related by $\rho(\hat\alpha_k) = \bar z_k \rho({}^*\alpha_k)$ for all k (see 2.14). The corresponding simple coroots are again proportional. It follows that the Weyl groups of these two root systems have the same simple reflections and hence they coincide. Thus, we have

(a) $\mathcal{W}^*_{K-S} = \mathcal{W}_{t_c}$.

It also follows that there is a unique function $\nu : {}^*\mathcal{R}_{K-S} \to \mathbf{N}$ (necessarily constant on the orbits of $\mathcal{W}^*_{K-S}$) such that $\mathcal{R}_{t_c} = \{\nu(\alpha)\alpha \mid \alpha \in {}^*\mathcal{R}_{K-S}\}$; moreover, we have $\nu({}^*\alpha_k) = \bar z_k$ for all $k \in K - S$. Let $\mu' : {}^*\mathcal{R}_{K-S} \to \mathbf{N}$ be the function defined by $\mu'(\alpha) = \mu(\nu(\alpha)\alpha)/\nu(\alpha)$. This function is again constant on the orbits of $\mathcal{W}^*_{K-S}$ and is characterized by the equality

$$\mu'({}^*\alpha_k) = c_k \quad \text{for all } k \in K - S.$$

(See 5.15.) Let $\bar{\mathbf{H}}$ be the graded Hecke algebras attached to the root system $(\mathcal{R}_{t_c} \subset V/(V_J + \mathbf{R}D), \mathcal{R}'_{t_c} \subset P^0)$ and the function $\mu : \mathcal{R}_{t_c} \to \mathbf{N}$. Let $\bar{\mathbf{H}}'$ be the graded Hecke algebras attached to the root system $({}^*\mathcal{R}_{K-S} \subset V/(V_J + \mathbf{R}D), {}^*\mathcal{R}'_{K-S} \subset P^0)$ and the function $\mu' : {}^*\mathcal{R}_{K-S} \to \mathbf{N}$. We have an algebra isomorphism $\bar{\mathbf{H}} \xrightarrow{\sim} \bar{\mathbf{H}}'$, which is the identity on the generators (see 4.8); here we use the fact that the two root systems above have the same Weyl group. We observe that the relation 4.8 (c) is the same for both algebras, in view of the identity

$$\mu(\rho(\hat\alpha_k))\langle \hat h_k, f\rangle = \mu'(\rho({}^*\alpha_k))\langle {}^*h_k, f\rangle,$$

which states that $\bar z_k c_k \langle \hat h_k, f\rangle = c_k \langle \bar z_k \hat h_k, f\rangle$. The isomorphism $\bar{\mathbf{H}} \xrightarrow{\sim} \bar{\mathbf{H}}'$ induces a bijection

(b) $\mathrm{Irr}_{\mathcal{W}_{t_c} \log t_h, r_0} \bar{\mathbf{H}}^{\mu}_{\mathcal{R}_{t_c}, \mathcal{V}/(V_J+\mathbf{R}D)} \xrightarrow{\sim} \mathrm{Irr}_{\mathcal{W}^*_{K-S} \log t_h, r_0} \bar{\mathbf{H}}^{\mu'}_{{}^*\mathcal{R}_{K-S}, \mathcal{V}/(V_J+\mathbf{R}D)}$.

5.17

We identify in the obvious way $H^0_{\mathbb{C}}$ with the Lie algebra $\mathfrak{t}$ of T. Let $\mathfrak{z}_J$ be the Lie algebra of the centre of G_J. This is a subspace of $\mathfrak{t}$. Then the subspace $P^0_{\mathbb{C}}$ of $H^0_{\mathbb{C}}$ is identified with the subspace $\mathfrak{z}_J$ of $\mathfrak{t}$, and the dual space $\mathbf{C}\otimes_{\mathbf{R}} \mathcal{V}/(V_J+\mathbf{R}D)$ of $P_{\mathbf{C}}^0$ is identified with the dual $\mathfrak{z}_J^*$ of $\mathfrak{z}_J$. From 3.14 we see that ${}^*R_{K-S}$ is the set of indivisible roots of the (relative) root system (in $\mathfrak{z}_J^*$) attached in [L10, 2.4 and 2.5] to G_{I-S} and to its subgroup G_J (a Levi subgroup of a parabolic subgroup of G_{I-S}.)

Note also that $G_{I-S} = Z_G(t_c^\psi)$ where t_c^ψ is as in 5.13. (This follows from 5.5 since $t_c^\psi \in C_S$.)

Let $\bar{\mathbf{H}}(Z_G(t_c^\psi), G_J, \mathcal{C}, \mathcal{F})$ be the graded Hecke algebra attached by the topological method of [L14] to the reductive group $G_{I-S} = Z_G(t_c^\psi)$, to its subgroup G_J (a Levi subgroup of a parabolic subgroup), and to the irreducible cuspidal local system $\mathcal{F}$ on $\mathcal{C} \subset \mathfrak{g}_J$ (see 5.6).

Using the definitions, we see that we may identify canonically

$$\bar{\mathbf{H}}(Z_G(t_c^\psi), G_J, \mathcal{C}, \mathcal{F}) = \bar{\mathbf{H}}^{\mu'}_{{}^*\mathcal{R}_{K-S}, V/(V_J+\mathbf{R}D)}$$

as graded algebras. We shall write $\mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F})$ instead of $\mathbf{H}^{\lambda,\lambda^*}_{\mathcal{R},\mathcal{L}}$.

We now take the compositions of the bijections 5.14 (a) and 5.16 (b) and use the identifications above. We obtain a natural bijection

(a) $\mathrm{Irr}_{\mathcal{W}t, v_0} \mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Irr}_{\log t_h, r_0} \bar{\mathbf{H}}(Z_G(t_c^\psi), G_J, \mathcal{C}, \mathcal{F})$.

5.18

We now drop the assumption that K has at least two elements (5.7). In the case where K has exactly one element, the algebra $\mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F})$ is defined to be $\mathbf{C}[v, v^{-1}]$ and the algebra $\bar{\mathbf{H}}(Z_G(t_c^\psi), G_J, \mathcal{C}, \mathcal{F})$ is $\mathbf{C}[r]$. We have $\mathcal{T} = \{1\}$, and we interpret $\mathcal{W}t$ in 5.17 (a) as $\{1\}$ and $W^*_{K-S} \log t_h$ as $\{0\}$. The bijection 5.17 (a) continues to hold; it is a bijection between two sets with one element.

5.19

We put together the bijections 5.17 (a) for various $(J, \mathcal{C}, \mathcal{F})$ as in 5.6 (including the case where $S - J$ has exactly one element; see 5.18) and for various t in the set 5.13 (b). We obtain canonical identifications

$$\begin{aligned}
&\sqcup_{J,\mathcal{C},\mathcal{F}} \mathrm{Irr}_{v_0} \mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F})\\
&= \sqcup_{J,\mathcal{C},\mathcal{F}} \sqcup_{S\subset I-J, S\neq\emptyset} \sqcup_{t_c \in p^J(C_S)} \sqcup_{t_h \in B_S^J} \mathrm{Irr}_{\mathcal{W}(t_c t_h), v_0} \mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F})\\
&= \sqcup_{J,\mathcal{C},\mathcal{F}} \sqcup_{S\subset I-J, S\neq\emptyset} \sqcup_{t_c \in p^J(C_S)} \sqcup_{t_h \in B_S^J} \mathrm{Irr}_{\log t_h, r_0} \bar{\mathbf{H}}(Z_G(t_c^\psi), G_J, \mathcal{C}, \mathcal{F})\\
&= \sqcup_{J,\mathcal{C},\mathcal{F}} \sqcup_{S\subset I-J, S\neq\emptyset} \sqcup_{t_c^\psi \in p(C_S)} \sqcup_{t_h \in B_S^J} \mathrm{Irr}_{\log t_h, r_0} \bar{\mathbf{H}}(Z_G(t_c^\psi), G_J, \mathcal{C}, \mathcal{F})\\
&= \sqcup_{S\subset I, S\neq\emptyset} \sqcup_{t' \in p(C_S)} \sqcup_{J\subset I-S,\mathcal{C},\mathcal{F}} \sqcup_{t_h \in B_S^J} \mathrm{Irr}_{\log t_h, r_0} \bar{\mathbf{H}}(Z_G(t'), G_J, \mathcal{C}, \mathcal{F})
\end{aligned}$$

(We have made the change of variable $t_c \mapsto t' = t_c^{\psi}$; note that this is a bijection $p^J(C_S) \mapsto p(C_S)$, since $p^J : C_S \to p^J(C_S)$ and $p : C_S \to p(C_S)$ are bijections.)

5.20

Applying the results in [L14, §8] to the group $Z_G(t') = G_{I-S}$ for a fixed $t' \in p(C_S)$, we obtain a canonical bijection between the set

(a) $\sqcup_{J \subset I-S, \mathcal{C}, \mathcal{F}} \mathrm{Irr}_{r_0} \bar{\mathbf{H}}(Z_G(t'), G_J, \mathcal{C}, \mathcal{F}) = \sqcup_{J \subset I-S, \mathcal{C}, \mathcal{F}} \sqcup_x \mathrm{Irr}_{x, r_0} \bar{\mathbf{H}}(Z_G(t'), G_J, \mathcal{C}, \mathcal{F})$

(where x runs over the set of orbits on $\mathfrak{z}_J$ of the normalizer of G_J in G_{I-S}) and

(b) the set of triples $(\xi, y, \mathcal{V})$ (modulo the natural action of $Z_G(t') = G_{I-S}$) where
ξ is a semisimple element of Lie $Z_G(t')$,
y is a nilpotent element of Lie $Z_G(t')$ such that $[\xi, y] = 2r_0 y$, and
$\mathcal{V}$ is an irreducible representation (up to isomorphism) of the group of components of the centralizer $Z_{Z_G(t')}(\xi, y)$.

Under this correspondence, an element in $\mathrm{Irr}_{x, r_0} \bar{\mathbf{H}}(Z_G(t'), G_J, \mathcal{C}, \mathcal{F})$ (with $x \in \mathfrak{z}_J$) corresponds to the class of a triple $(\xi, y, \mathcal{V})$ with $\xi = x + d$ where $d \in \mathfrak{g}_J$ is attached to $(J, \mathcal{C})$ as in 5.6.

Since d is a hyperbolic element commuting with x, we see that x is hyperbolic if and only if ξ is hyperbolic (in Lie G_{I-S}). Hence the bijection above restricts to a bijection between the subset of (a) defined by the condition that x is hyperbolic and the subset of (b) defined by the condition that ξ is hyperbolic. Thus, we obtain a bijection between the set

$$\sqcup_{J \subset I-S, \mathcal{C}, \mathcal{F}} \sqcup_{x \in B_S^J} \mathrm{Irr}_{x, r_0} \bar{\mathbf{H}}(Z_G(t'), G_J, \mathcal{C}, \mathcal{F})$$

and the set of classes of triples $(\xi, y, \mathcal{V})$ as in (b), with ξ hyperbolic.

Combining this with the identification in 5.19, we obtain a bijection between the set $\sqcup_{(J, \mathcal{C}, \mathcal{F}) \in \bar{\mathfrak{S}}} \mathrm{Irr}\, \mathbf{H}_{v_0}(G, G_J, \mathcal{C}, \mathcal{F})$ and the set of quadruples $(t', \xi, y, \mathcal{V})$ where $t' \in p(\bar{C}_I)$ and $(\xi, y, \mathcal{V})$ is as in (b) (with ξ hyperbolic).

Here, $\mathbf{H}_{v_0}(G, G_J, \mathcal{C}, \mathcal{F})$ denotes the quotient algebra of $\mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F})$ by the two-sided ideal generated by $(v - v_0)$, so that $\mathrm{Irr}_{v_0} \mathbf{H}(G, G_J, \mathcal{C}, \mathcal{F}) = \mathrm{Irr}\, \mathbf{H}_{v_0}(G, G_J, \mathcal{C}, \mathcal{F})$. Moreover, $\bar{\mathfrak{S}}$ denotes the set of all triples $(J, \mathcal{C}, \mathcal{F})$ where J is a proper subset of I and $\mathcal{F}$ is an irreducible cuspidal G_J-equivariant local system on a nilpotent G_J-orbit $\mathcal{C}$ in $\mathfrak{g}_J$ (up to isomorphism).

Setting $s = t' \exp(\xi)$ (so that $t' = s_c, \exp(\xi) = s_h$), we obtain a natural bijection between the set $\sqcup_{(J, \mathcal{C}, \mathcal{F}) \in \bar{\mathfrak{S}}} \mathrm{Irr}\, \mathbf{H}_{v_0}(G, G_J, \mathcal{C}, \mathcal{F})$ and the set of triples $(s, y, \mathcal{V})$ where
s is a semisimple element of G such that $s_c \in p(\bar{C}_I)$,
y is a nilpotent element of $\mathfrak{g}$ such that $\mathrm{Ad}(s) y = v_0^2 y$,
$\mathcal{V}$ is an irreducible representation (up to isomorphism) of the group of components of the centralizer group $Z_G(s, y)$;

here, two triples $(s, y, \mathcal{V}), (s', y', \mathcal{V}')$ are regarded as equal if $s_c = s'_c$ and $(s_h, y, \mathcal{V}), (s'_h, y', \mathcal{V}')$ are in the same $Z_G(s_c)$-orbit.

Note that for any semisimple element s of G, the compact part s_c is conjugate to a unique element of $p(\bar{C}_I)$ (see 5.13). Hence the bijection above can be reformulated as follows.

Theorem 5.21. Let $v_0 \in \mathbf{R}_{>0}$, and let $\mathfrak{G}(v_0)$ be the set consisting of all triples $(s, y, \mathcal{V})$ (modulo the natural action of G) where

s is a semisimple element of G,

y is a nilpotent element of $\mathfrak{g}$ such that $\mathrm{Ad}(s)y = v_0^2 y$,

$\mathcal{V}$ is an irreducible representation (up to isomorphism) of the group of components of the centralizer group $Z_G(s, y)$.

There is a natural bijection

(a) $\mathfrak{G}(v_0) \leftrightarrow \sqcup_{(J,\mathcal{C},\mathcal{F})\in\bar{\mathfrak{G}}} \mathrm{Irr}\,\mathbf{H}_{v_0}(G, G_J, \mathcal{C}, \mathcal{F})$. □

See the introduction for historical remarks.

5.22

Theorem 5.21 holds in particular for $v_0 = 1$. In that case, the set in the right-hand side of 5.21 (a) consists of the irreducible representations of a finite collection of affine Weyl groups (possibly of extended type), while the set $\mathfrak{G}(1)$ may be identified (via $(s, y, \mathcal{V}) \mapsto (s\exp(y), \mathcal{V})$) with the set of all pairs $(g, \mathcal{V})$ (modulo the natural action of G) where g is an element of G and $\mathcal{V}$ is an irreducible representation (up to isomorphism) of the group of components of the centralizer group $Z_G(g)$.

5.23

We have natural partitions

$$\mathfrak{G}(v_0) = \sqcup_\chi \mathfrak{G}(v_0)_\chi, \qquad \bar{\mathfrak{G}} = \sqcup_\chi \bar{\mathfrak{G}}_\chi$$

where χ runs through $S'(I) = \mathrm{Hom}(Z_G, \mathbf{C}^*)$. (The centre of G is denoted by Z_G.) Namely, $\bar{\mathfrak{G}}_\chi$ denotes the subset of $\bar{\mathfrak{G}}$ consisting of those $(J, \mathcal{C}, \mathcal{F})$ such that Z_G acts on each stalk of $\mathcal{F}$ via the character χ (using the G_J-equivariance of $\mathcal{F}$ and the inclusion $Z_G \subset G_J$); $\mathfrak{G}(v_0)_\chi$ denotes the subset of $\mathfrak{G}(v_0)$ consisting of those $(s, y, \mathcal{V})$ such that Z_G acts on $\mathcal{V}$ via the character χ (using the obvious homomorphism $Z_G \to Z_G(s, y)$).

The triples $(J, \mathcal{C}, \mathcal{F})$ in $\bar{\mathfrak{G}}_\chi$ are called *geometric diagrams of type* χ. From the definitions it is clear that the bijection in 5.21 restricts, for any $\chi \in S'(I)$, to a bijection

(a) $\mathfrak{G}(v_0)_\chi \leftrightarrow \sqcup_{(J,\mathcal{C},\mathcal{F})\in\bar{\mathfrak{G}}_\chi} \mathrm{Irr}\,\mathbf{H}_{v_0}(G, G_J, \mathcal{C}, \mathcal{F})$.

6 The arithmetic/geometric correspondence

6.1

We now consider simultaneously the simply connected, almost simple algebraic group G over $\mathbf{C}$ (as in 5.2) and a connected, adjoint, simple algebraic group $\mathbf{G}$, defined and split over $\tilde{\mathbf{K}}$ (see 1.1), which is identified with the group of its $\tilde{\mathbf{K}}$-rational points; we assume that $\mathbf{G}$ is of type dual to that of G (in the sense of Langlands).

To G corresponds an untwisted affine Cartan matrix with set of indices I (as in 5.1). To $\mathbf{G}$ corresponds a Coxeter group (an affine Weyl group) $\mathbf{W}'$ (see 1.10), or equivalently, an untwisted affine Cartan matrix with set of indices $\mathbf{I}$. It is known that we may identify

(a) $S'(I) = S(\mathbf{I})$

as (finite) abelian groups ($S(\mathbf{I})$ as in 1.11 and $S'(I)$ as in 5.23).

6.2

For any $u \in \tilde{S}(\mathbf{I})$ (see 1.11), we can find (and will select) a $\mathbf{K}$-rational structure on $\mathbf{G}$ with Frobenius map F_u (see 1.1) such that the permutation of $\mathbf{I}$ induced by F_u (as in 1.12) is equal to u. (This follows from the results of Bruhat and Tits [BT].) Let q be as in 1.1.

Theorem 6.3. Let $u \in S(\mathbf{I})$ and let $\chi \in S'(I)$ be the corresponding element (see 6.1 (a)). There exists a one-to-one correspondence

(a) $\bar{\mathfrak{G}}_\chi \leftrightarrow \bar{\mathfrak{A}}_u$

(where $\bar{\mathfrak{G}}_\chi$ is as in 5.23 and $\bar{\mathfrak{A}}_u$ is as in 1.22), or equivalently, between the set of geometric diagrams of type χ and the set of arithmetic diagrams of type u, such that, if the geometric diagram $(J, \mathcal{C}, \mathcal{F}) \in \bar{\mathfrak{G}}_\chi$ corresponds to the arithmetic diagram $(\mathbf{J}, \mathbf{E}, \psi) \in \bar{\mathfrak{A}}_u$, then the affine Hecke algebras $\mathbf{H}_{\sqrt{q}}(G, G_J, \mathcal{C}, \mathcal{F})$ and $\mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E})$ have the same IM-presentation (hence are isomorphic). $\square$

This is proved by examining the tables in §7, where the list of arithmetic diagrams of type u and that of geometric diagrams of type χ is given and the two types of diagrams are matched.

6.4

Let u, χ be as in 6.3. We consider the bijections

$$\mathcal{U}(\mathbf{G}^{F_u}) \leftrightarrow \sqcup_{(\mathbf{J},\mathbf{E},\psi)\in\bar{\mathfrak{A}}_u} \operatorname{Irr} \mathcal{H}'(\mathbf{I}, \mathbf{J}, u, \mathbf{E}) \leftrightarrow \sqcup_{(J,\mathcal{C},\mathcal{F})\in\bar{\mathfrak{G}}_\chi} \operatorname{Irr} \mathbf{H}_{\sqrt{q}}(G, G_J, \mathcal{C}, \mathcal{F}) \leftrightarrow \mathfrak{G}(\sqrt{q})_\chi$$

where the first bijection is as in 1.22 (a), the second bijection is given by 6.3, and the third bijection is as in 5.23 (a).

Corollary 6.5. The composition of the bijections in 6.4 is a bijection

(a) $\mathfrak{U}(\mathbf{G}^{F_u}) \leftrightarrow \mathfrak{G}(\sqrt{q})_\chi$.

Taking a disjoint union over $u \in S(\mathbf{I})$, we obtain a bijection

(b) $\sqcup_{u \in S(\mathbf{I})} \mathfrak{U}(\mathbf{G}^{F_u}) \leftrightarrow \mathfrak{G}(\sqrt{q})$. □

The left-hand side of (b) is a disjoint union of the sets of unipotent representations of the various inner forms of a split **K**-form of **G**; recall that the right-hand side of (b) is the set consisting of all triples $(s, y, \mathcal{V})$ (modulo the natural action of G) where

s is a semisimple element of G,

y is a nilpotent element of $\mathfrak{g}$ such that $\mathrm{Ad}(s)y = qy$,

$\mathcal{V}$ is an irreducible representation (up to isomorphism) of the group of components of the centralizer group $Z_G(s, y)$.

The corollary above is a further confirmation of the Langlands philosophy, which predicts that irreducible representations of the various inner forms of a split **K**-form of **G** may be parametrized in terms of G. For further historical remarks, see the introduction.

6.6

The bijection 6.3 (a) is closely related to the bijection of [L8, IV, 17.8.3], which associates to any character sheaf of G a semisimple class in the dual group of G and some additional data. We will explain this in a special case. Assume that $(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{G}_\chi$ is such that $I - J$ has exactly one element, and let $(\mathbf{J}, \mathbf{E}, \psi) \in \mathfrak{A}_u$ be the corresponding triple under 6.3 (a).

Let $t \in p(C_J)$ (as in 5.5). We identify $\mathcal{C}$ with $t\exp(\mathcal{C})$ using $g \mapsto t\exp(g)$; we may therefore regard $\mathcal{F}$ as a local system on $t\exp(\mathcal{C})$. This extends uniquely to a G-equivariant local system on the G-conjugacy class in G containing $t\exp(\mathcal{C})$. Extending this by zero on the boundary of that conjugacy class, we obtain a cuspidal character sheaf $\tilde{\mathcal{F}}$ on G. Under [L8, IV, 17.8.3], to this cuspidal character sheaf corresponds a certain local system $\mathcal{L}$, which may be interpreted as a semisimple conjugacy class in the complex dual group, and the type of the centralizer of an element of this semisimple class may be interpreted as a type of a parahoric subgroup of **G**, or as a subset **J** of **I**. This is the same as the **J** considered above. The additional data attached in [L8, IV, 17.8.3] to $\tilde{\mathcal{F}}$ may be interpreted as a unipotent representation of the $\mathbf{F}_q$-points of the "reductive quotient" of our parahoric subgroup. This is the same as **E** above.

A similar relationship holds in the general case.

6.7

Note that the triples $(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{G}$ are closely related to the data $(L, \Sigma, \mathcal{E})$ in [L8, II, 8.1.1], which are used to construct the "admissible complexes" (that is, the character sheaves) on G. Namely, if $(L, \Sigma, \mathcal{E})$ is given, then we pick $x \in \Sigma$ and we write $x = x'x''$ with $x' \in L$ semisimple and $x'' \in L$ unipotent. We may assume that the centralizer of x' in G is G_J. Let $\mathcal{C}$ be the conjugacy class of x'' in G_J. Then $\mathcal{E}$ defines a local system $\mathcal{F}$ on $\mathcal{C}$, and we have $(J, \mathcal{C}, \mathcal{F}) \in \mathfrak{G}$.

6.8

In the rest of this section, we explain the tables of §7. Those tables contain a complete list of the arithmetic diagrams corresponding to various $u \in S(\mathbf{I})$, a complete list of the various geometric diagrams corresponding to various $\chi \in S'(I)$, and the matching between them.

Let $\tilde{S}(I)$ be the group of permutations of I defined like $\tilde{S}(\mathbf{I})$ in 1.11. In §7, there is one table for any $u \in S(\mathbf{I})$ (or equivalently, any $\chi \in S'(I)$) and any $\tilde{S}(I)$-orbit on the set of proper subsets J of I such that J can be completed to a geometric diagram $(J, \mathcal{C}, \mathcal{F}) \in \bar{\mathfrak{G}}_\chi$.

In each table we indicate one particular representative J of that orbit by a name (like $\tilde{E}_8/E_7A_1$, where $\tilde{E}_8$ refers to the affine Dynkin diagram with vertices I, and E_7A_1 refers to the subset J) and by a picture of the pair $J \subset I$ (the vertices in J are the vertices of I inside boxes); in the classical types, the picture attached to a name can be found in 6.10. This is accompanied by the words

geometric, $a \times b$

where "geometric" means that this is part of a geometric diagram (that is, attached to G), a is the number of subsets of I in $\tilde{S}(I)$-orbit of J, and b is the number of pairs $(\mathcal{C}, \mathcal{F})$ such that $(J, \mathcal{C}, \mathcal{F}) \in \bar{\mathfrak{G}}_\chi$. Thus, the table accounts for ab geometric diagrams.

Next, the table specifies ab arithmetic diagrams which are matched with the ab geometric diagrams. (We do not show how, but we use as guiding principle the method of 6.6.) Thus, we give some information on $u \in S(\mathbf{I})$ (ord(u) denotes the order of u). If our extended Dynkin diagram with vertices **I** is of type $\tilde{D}_n$ with $n \geq 4$, we say that u is of type 1 if the permutation u is a product of two 2-cycles; we say that u is of type 2 if $n > 4$, u has a single fixed point, and ord(u) = 4 (for n odd) and ord(u) = 2 (for n even). We indicate a representative **J** for a $\Omega^u = \Omega$-orbit on the set of proper u-stable subsets of **I** such that **J** can be completed to an arithmetic diagram $(\mathbf{J}, \mathbf{E}, \psi)$ of type u.

J is indicated by a name (like $\tilde{E}_8/E_7$, where $\tilde{E}_8$ refers to the affine Dynkin diagram with vertices **I**, and E_7 refers to the subset **J**) and by a picture of the pair $\mathbf{J} \subset \mathbf{I}$ (the vertices in **J** are the vertices of **I** inside boxes); in the classical types the picture attached to a name can be found in 6.10. This is accompanied by the words

arithmetic, $a' \times b'$

where "arithmetic" means that this is part of an arithmetic diagram (that is, attached to $\mathbf{G}$), a' is the number of elements of $\bar{\Omega}_1^u$ (see 1.15 and 1.20), and b' is the cardinal of a subset of $\mathcal{U}^0(\mathbf{J}, u)$ (see 1.21) that is relevant for the table in question.

Note that the same subset $\mathbf{J}$ may appear in more than one table, and the sum of the b' corresponding to each of these tables (with fixed u) is equal to the cardinal of $\mathcal{U}^0(\mathbf{J}, u)$. In the classical types, we have always $b' = 1$; in the exceptional types when $\mathcal{U}^0(\mathbf{J}, u)$ may have many elements, we indicate which subset of $\mathcal{U}^0(\mathbf{J}, u)$ is relevant for a given table by specifying (at the end of the table) the names (from [L6]) of the representations $\mathbf{E}$ in the table, whenever such names are available. Each table accounts for $a'b'$ arithmetic diagrams. We have $ab = a'b'$.

Next, each table contains a $\flat - \sharp$-*diagram*. This is the affine Dynkin diagram corresponding to the (not necessarily untwisted) affine Cartan matrix $(\hat{a}_{kk'})$ associated as in 1.14 to the subset J of I. Its vertices are indexed by $K = I - J$ (if K has at least two elements); the diagram is empty if K has exactly one element. Each vertex is marked with $\flat$ or $\sharp$ according to whether it belongs to $K^\flat$ or $K^\sharp$. Moreover, for each such vertex k we specify a datum of form $A \times B$ where $A = c_k$ (see 5.7) and $B = \bar{z}_k$ (see 2.14).

Next, each table contains a diagram (called H.A., for Hecke algebra) describing the IM-presentation of an affine Hecke algebra. This is obtained from the $\flat - \sharp$-diagram (hence ultimately from the geometric diagram) by the procedure in 5.9, 5.11, and 5.12, or from the arithmetic diagram by the procedure in 1.18 and 1.19. (The two procedures give the same result.) (See 6.11 for a further discussion of affine Hecke algebras.)

6.9 Conventions

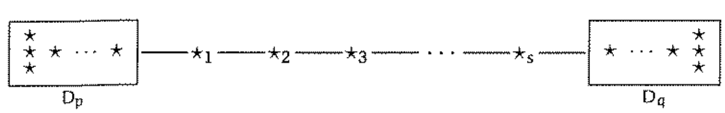
For $a = 0$ or 1, we interpret A_{a-1} as the empty graph. For $a = 0$ or $a = 1$, we interpret D_a as the empty graph. A graph of the form

$$\star \Rightarrow \star\!-\!\!-\!\!- \cdots -\!\!-\!\!-\star-\!\!-\!\!- \Leftarrow \star \text{ or } \star \Leftarrow \star-\!\!-\!\!- \cdots -\!\!-\!\!-\star-\!\!-\!\!- \Rightarrow \star$$

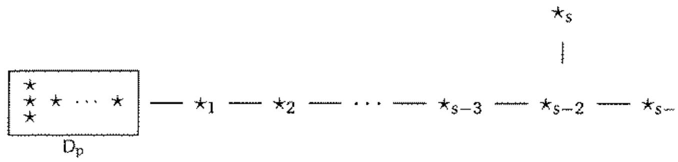
with $s \geq 1$ vertices is considered to be $\star \overset{\infty}{-\!\!-} \star$ when $s = 2$ and $\emptyset$ when $s = 1$.

6.10 Some diagrams

$\tilde{D}_n/(D_p \times D_q)$

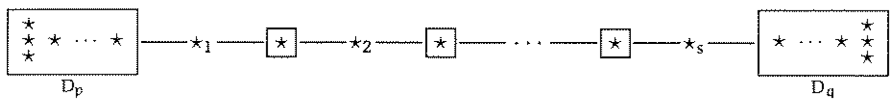


$\tilde{D}_n/D_p$

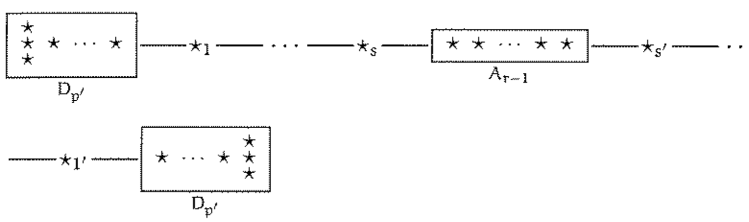


$\tilde{D}_n/\emptyset$

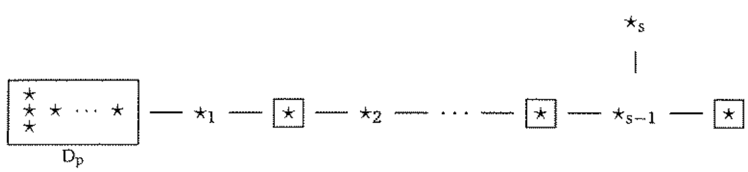
$\tilde{D}_n/(D_p \times D_q \times A_1^{s-1}), p + q + 2(s-1) = n$



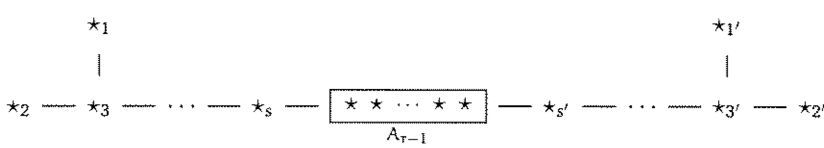
$\tilde{D}_n/(D_{p'} \times D_{p'} \times A_{r-1})$



$\tilde{D}_n/(D_p \times A_1^{s-1}), p + 2(s-1) = n$

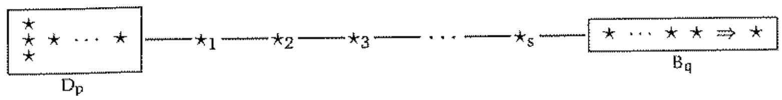


$\tilde{D}_n/A_{r-1}$

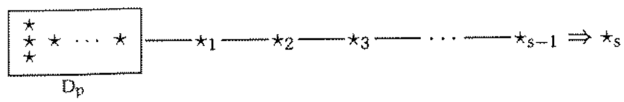


$\tilde{D}_n/A_1^s, 2(s-1) = n$

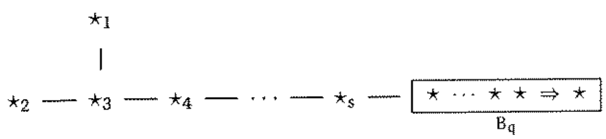
$\tilde{B}_n/(D_p \times B_q)$



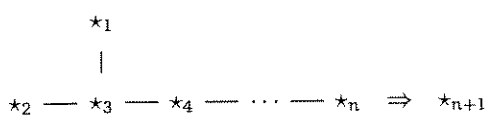
$\tilde{B}_n/D_p$



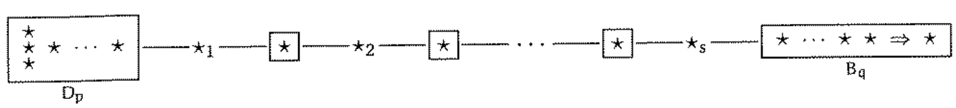
$\tilde{B}_n/B_q$



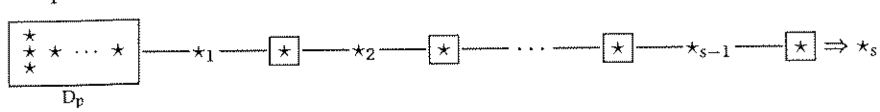
$\tilde{B}_n/\emptyset$



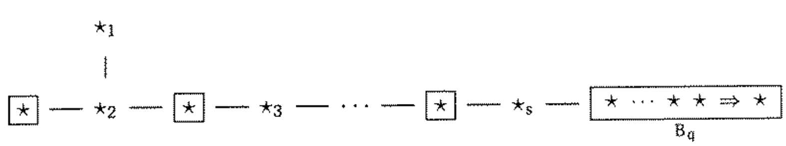
$\tilde{B}_n/(D_p \times B_q \times A_1^{s-1}), p + q + 2(s-1) = n$



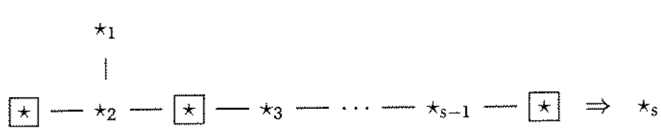
$\tilde{B}_n/(D_p \times A_1^{s-1}), p + 2(s-1) = n$



$\tilde{B}_n/(B_p \times A_1^{s-1}), p + 2(s-1) = n$



$\tilde{B}_n/A_1^{s-1}, 2(s-1) = n$



$\tilde{C}_n/(C_{p'} \times C_{q'})$

$$\underset{C_{p'}}{\boxed{\star \Rightarrow \star \;\star \cdots \star}} — \star_1 — \star_2 — \star_3 — \cdots — \star_s — \underset{C_{q'}}{\boxed{\star \cdots \star \;\star \Leftarrow \star}}$$

$\tilde{C}_n/(C_{p'} \times C_{p'} \times A_{r-1})$

$$\underset{C_{p'}}{\boxed{\star \Rightarrow \star \;\star \cdots \star}} — \star_1 — \cdots — \star_s — \underset{A_{r-1}}{\boxed{\star \;\star \cdots \star \;\star}} — \star_{s'} — \cdots$$

$$— \star_{1'} — \underset{C_{p'}}{\boxed{\star \cdots \star \;\star \Leftarrow \star}}$$

6.11 Notation for affine Hecke algebras

The affine Hecke algebras (in the IM-presentation; see 5.12) that are needed in the tables in §7 will be described by specifying the corresponding extended Dynkin diagram and the appropriate parameters at each vertex.

It is always understood that the group Γ in the IM-presentation (see 5.12) is the group of all special permutations of the set of vertices of our extended Dynkin diagram, except for

(a) $\tilde{C}_n^{sc}[{}_a b_c]$ (see below), and

(b) $\tilde{C}_1 = \tilde{A}_1$ with the L-function having distinct values at the two vertices for which $\Gamma = \{1\}$. In classical types we introduce some notation for the various affine Hecke algebras.

$\tilde{A}_n[a], n \geq 1$:

$$\cdots — a — a — a — \cdots,$$

a cyclic graphs with $n+1$ vertices.

$\tilde{D}_n[a], n \geq 4$:

$$\begin{array}{ccccccc} & a & & & a & & \\ & | & & & | & & \\ a — & a & — & \cdots — & a & — & a, \end{array}$$

with $n+1$ vertices.

$\tilde{B}_n[a_b]$:

$$\begin{array}{ccccccc} & a & & & & & \\ & | & & & & & \\ a — & a & — a — & \cdots — & a & \Rightarrow & b, \end{array}$$

with $n+1$ vertices, if $n \geq 3$;

$$a \Rightarrow b \Leftarrow a$$

if $n = 2$;

$$a \overset{\infty}{\text{——}} a$$

if $n = 1$.

$\tilde{C}_n[{}_a b_a]$, $n \geq 1$:

$$a \Rightarrow b \text{——} b \text{——} b \text{——} \cdots \text{——} b \Leftarrow a,$$

with $n+1$ vertices, if $n \geq 2$;

$$a \overset{\infty}{\text{——}} a,$$

if $n = 1$.

$\tilde{C}_n^{sc}[{}_a b_c]$, $n \geq 1$:

$$a \Rightarrow b \text{——} b \text{——} b \text{——} \cdots \text{——} b \Leftarrow c,$$

with $n+1$ vertices, if $n \geq 2$;

$$a \overset{\infty}{\text{——}} c,$$

if $n = 1$.

We shall identify the algebras

$$\tilde{C}_n^{sc}[{}_0 b_c] = \tilde{B}_n[b_c]$$

in the standard way, for $n \geq 1$.

7 Tables

7.1

$\tilde{E}_8/E_8$ (geometric, 1×1)

∅

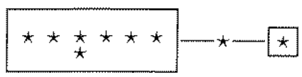
$\tilde{E}_8/E_8, u = 1$ (arithmetic, 1×1)

H.A. : ∅

E : $E_8^{II}[1]$

7.2

$\tilde{E}_8/E_7A_1$ (geometric, 1×1)



∅

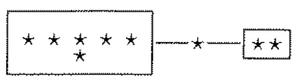
$\tilde{E}_8/E_8, u = 1$ (arithmetic, 1×1)

H.A. : ∅

E : $E_8[-1]$

7.3

$\tilde{E}_8/E_6A_2$ (geometric, 1×2)



∅

$\tilde{E}_8/E_8, u = 1$ (arithmetic, 1×2)

H.A. : ∅

E : $E_8[\theta], E_8[\theta^2]$

7.4

$\tilde{E}_8/D_8$ (geometric, 1×1)

∅

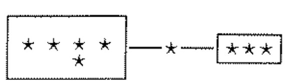
$\tilde{E}_8/E_8, u = 1$ (arithmetic, 1×1)

H.A. : ∅

E : $E_8^{I}[1]$

7.5

$\tilde{E}_8/D_5A_3$ (geometric, 1×2)



∅

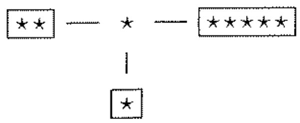
$\tilde{E}_8/E_8, u = 1$ (arithmetic, 1×2)

H.A. : ∅

E : $E_8[i], E_8[-i]$

7.6

$\tilde{E}_8/A_5A_2A_1$ (geometric, 1×2)



$\emptyset$

$\tilde{E}_8/E_8, u = 1$ (arithmetic, 1×2)

H.A. : $\emptyset$

E : $E_8[-\theta], E_8[-\theta^2]$

7.7

$\tilde{E}_8/A_4A_4$ (geometric, 1×4)

$\emptyset$

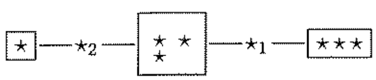
$\tilde{E}_8/E_8, u = 1$ (arithmetic, 1×4)

H.A. : $\emptyset$

E : $E_8[\zeta], E_8[\zeta^2], E_8[\zeta^3], E_8[\zeta^4]$

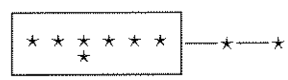
7.8

$\tilde{E}_8/A_3A_3A_1$ (geometric, 1×2)



$\flat_2^{7\times 2} \overset{\infty}{—} \flat_1^{8\times 2}$

$\tilde{E}_8/E_7, u = 1$ (arithmetic, 1×2)

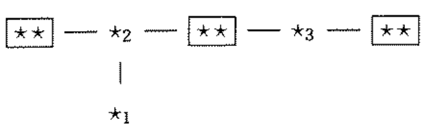


H.A. : $\mathbf{1}_0 \overset{\infty}{—} \mathbf{15}_1$

E : $E_7[\xi], E_7[-\xi]$

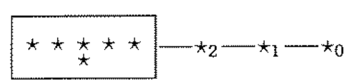
7.9

$\tilde{E}_8/A_2A_2A_2$ (geometric, 1×2)



$\sharp_1^{2\times 1} \Lleftarrow \sharp_2^{6\times 3} — \sharp_3^{6\times 3}$

$\tilde{E}_8/E_6, u = 1$ (arithmetic, 1×2)

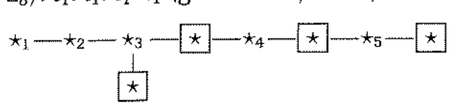


H.A. : $\mathbf{1}_0 — \mathbf{1}_1 \Rrightarrow \mathbf{9}_2$

E : $E_6[\theta], E_6[\theta^2]$

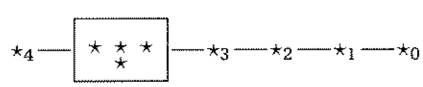
7.10

$\tilde{E}_8/A_1A_1A_1A_1$ (geometric, 1×1)



$\sharp_1^{2\times 1} — \sharp_2^{2\times 1} \Leftarrow \sharp_3^{4\times 2} — \sharp_4^{4\times 2} — \sharp_5^{4\times 2}$

$\tilde{E}_8/D_4, u = 1$ (arithmetic, 1×1)



H.A. : $\mathbf{1}_0 — \mathbf{1}_1 — \mathbf{1}_2 \Rightarrow \mathbf{4}_3 — \mathbf{4}_4$

7.11

$\tilde{E}_8/\emptyset$ (geometric, 1×1)

$$\begin{array}{cccccccccccccccc} \star_1 & - & \star_3 & - & \star_4 & - & \star_5 & - & \star_6 & - & \star_7 & - & \star_8 & - & \star_0 \\ & & & & | & & & & & & & & & & \\ & & & & \star_2 & & & & & & & & & & \end{array}$$

$$\begin{array}{cccccccccccccccc} \sharp_1^{2\times 1} & - & \sharp_3^{2\times 1} & - & \sharp_4^{2\times 1} & - & \sharp_5^{2\times 1} & - & \sharp_6^{2\times 1} & - & \sharp_7^{2\times 1} & - & \sharp_8^{2\times 1} & - & \sharp_0^{2\times 1} \\ & & & & | & & & & & & & & & & \\ & & & & \sharp_2^{2\times 1} & & & & & & & & & & \end{array}$$

$\tilde{E}_8/\emptyset, u = 1$ (arithmetic, 1×1)

$$\begin{array}{cccccccccccccccc} \star_1 & - & \star_3 & - & \star_4 & - & \star_5 & - & \star_6 & - & \star_7 & - & \star_8 & - & \star_0 \\ & & & & | & & & & & & & & & & \\ & & & & \star_2 & & & & & & & & & & \end{array}$$

H.A. : $\mathbf{1}_1 - \mathbf{1}_3 - \mathbf{1}_4 - \mathbf{1}_5 - \mathbf{1}_6 -$
$$\begin{array}{ccccc} & & & & \mathbf{1}_4 \\ & & & & | \\ & & & & \mathbf{1}_2 \\ \mathbf{1}_7 & - & \mathbf{1}_8 & - & \mathbf{1}_0 \end{array}$$

7.12

$\tilde{E}_7/E_7$ (geometric, 2×1)

$$\star - \boxed{\begin{array}{cccccc} \star & \star & \star & \star & \star & \star \\ & & \star & & & \end{array}}$$

$\emptyset$

$\tilde{E}_7/E_6, \mathrm{ord}(u) = 2$ (arithmetic, 2×1)

$$\star - \boxed{\begin{array}{ccccc} \star & \star & \star & \star & \star \\ & & \star & & \end{array}} - \star$$

H.A. : $\emptyset$

7.13

$\tilde{E}_7/A_5A_2$ (geometric, 2×2)

$$\boxed{\begin{array}{cccc} \star & \star & \star & \star \\ & & & \star \end{array}} - \star - \boxed{\star\star}$$

$\emptyset$

$\tilde{E}_7/E_6, u \neq 1$ (arithmetic, 2×2)

$$\star - \boxed{\begin{array}{ccccc} \star & \star & \star & \star & \star \\ & & \star & & \end{array}} - \star$$

H.A. : $\emptyset$

7.14

$\tilde{E}_7/A_3A_3A_1$ (geometric, 1×2)

$$\begin{array}{ccccc} \boxed{\star\star\star} & - & \star & - & \boxed{\star\star\star} \\ & & | & & \\ & & \boxed{\star} & & \end{array}$$

$\emptyset$

$\tilde{E}_7/E_7, u = 1$ (arithmetic, 1×2)

$$\star - \boxed{\begin{array}{cccccc} \star & \star & \star & \star & \star & \star \\ & & \star & & & \end{array}}$$

H.A. : $\emptyset$

7.15

$\tilde{E}_7/A_3A_3$ (geometric, 1×2)

$$\begin{array}{ccccc} \boxed{\star\star\star} & - & \star_1 & - & \boxed{\star\star\star} \\ & & | & & \\ & & \star_2 & & \end{array}$$

$\flat_1^{8\times 2} \overset{\infty}{-} \flat_2^{2\times 1}$

$\tilde{E}_7/A_5, \mathrm{ord}(u) = 2$ (arithmetic, 2×1)

$$\begin{array}{ccccc} \star_0 & - & \boxed{\star\star\star\star\star} & - & \star_0 \\ & & | & & \\ & & \star_1 & & \end{array}$$

H.A. : $\mathbf{9}_1 \overset{\infty}{-} \mathbf{7}_0$

7.16

$\tilde{E}_7/A_2A_2A_2$ (geometric, 1×2)

$$\boxed{\star\star} — \star_1 — \boxed{\begin{smallmatrix}\star\\ \star\end{smallmatrix}} — \star_2 — \boxed{\star\star}$$

$$\natural_1^{6\times 3} \overset{\infty}{—} \natural_2^{6\times 3}$$

$\tilde{E}_7/E_6, u = 1$ (arithmetic, 1×2)

$$\star — \boxed{\star\ \star\ \underset{\star}{\star}\ \star\ \star} — \star$$

H.A. : $\mathbf{9} \overset{\infty}{—} \mathbf{9}$

7.17

$\tilde{E}_7/A_1A_1A_1A_1$ (geometric, 1×1)

$$\begin{array}{ccccccccccccc} \boxed{\star} & — & \star_1 & — & \boxed{\star} & — & \star_2 & — & \boxed{\star} & — & \star_4 & — & \boxed{\star} \\ &&&&&& | &&&&&& \\ &&&&&& \star_3 &&&&&& \end{array}$$

$$\begin{array}{ccccc} & & \natural_4^{4\times 2} & & \\ & & | & & \\ \natural_1^{4\times 2} & — & \natural_2^{4\times 2} & \Rightarrow & \natural_3^{2\times 1} \end{array}$$

$\tilde{E}_7/D_4, u = 1$ (arithmetic, 1×1)

$$\star_0 — \star_1 — \boxed{\star\ \underset{\star}{\star}\ \star} — \star_2 — \star_3$$

H.A. : $\mathbf{1}_0 \Rightarrow \mathbf{4}_1 — \mathbf{4}_2 \Leftarrow \mathbf{1}_3$

7.18

$\tilde{E}_7/A_1A_1A_1$ (geometric, 2×1)

$$\begin{array}{ccccccccccccc} \star_5 & — & \star_4 & — & \star_3 & — & \star_2 & — & \boxed{\star} & — & \star_1 & — & \boxed{\star} \\ &&&&&& | &&&&&& \\ &&&&&& \boxed{\star} &&&&&& \end{array}$$

$$\natural_5^{2\times 1} — \natural_4^{2\times 1} — \natural_3^{2\times 1} \Rightarrow \natural_2^{4\times 1} — \natural_1^{4\times 1}$$

$\tilde{E}_7/\emptyset, \mathrm{ord}(u) = 2$ (arithmetic, 2×1)

$$\begin{array}{ccccccccccccc} \star_0 & — & \star_1 & — & \star_2 & — & \star_3 & — & \star_{2'} & — & \star_{1'} & — & \star_{0'} \\ &&&&&& | &&&&&& \\ &&&&&& \star_4 &&&&&& \end{array}$$

H.A. : $\mathbf{1}_4 — \mathbf{1}_3 \Leftarrow \mathbf{2}_2 — \mathbf{2}_1 — \mathbf{2}_0$

7.19

$\tilde{E}_7/\emptyset$ (geometric, 1×1)

$$\begin{array}{ccccccccccccc} \star_0 & — & \star_1 & — & \star_3 & — & \star_4 & — & \star_5 & — & \star_6 & — & \star_7 \\ &&&&&& | &&&&&& \\ &&&&&& \star_2 &&&&&& \end{array}$$

$$\begin{array}{ccccccccccccc} \natural_0^{2\times 1} & — & \natural_1^{2\times 1} & — & \natural_3^{2\times 1} & — & \natural_4^{2\times 1} & — & \natural_5^{2\times 1} & — & \natural_6^{2\times 1} & — & \natural_7^{2\times 1} \\ &&&&&& | &&&&&& \\ &&&&&& \natural_2^{2\times 1} &&&&&& \end{array}$$

$\tilde{E}_7/\emptyset, u = 1$ (arithmetic, 1×1)

$$\begin{array}{ccccccccccccc} \star_0 & — & \star_1 & — & \star_3 & — & \star_4 & — & \star_5 & — & \star_6 & — & \star_7 \\ &&&&&& | &&&&&& \\ &&&&&& \star_2 &&&&&& \end{array}$$

H.A. : $$\begin{array}{ccccccccccccc} \mathbf{1}_0 & — & \mathbf{1}_1 & — & \mathbf{1}_3 & — & \mathbf{1}_4 & — & \mathbf{1}_5 & — & \mathbf{1}_6 & — & \mathbf{1}_7 \\ &&&&&& | &&&&&& \\ &&&&&& \mathbf{1}_2 &&&&&& \end{array}$$

7.20

$\tilde{E}_6/E_6$ (geometric, 3×1)

$$\begin{array}{c} \boxed{\star\ \star\ \underset{\star}{\star}\ \star\ \star} \\ | \\ \star \end{array}$$

$\emptyset$

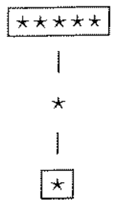
$\tilde{E}_6/D_4, \mathrm{ord}(u) = 3$ (arithmetic, 3×1)

$$\begin{array}{ccccc} \star & — & \boxed{\star\ \underset{\star}{\star}\ \star} & — & \star \\ & & | & & \\ & & \star & & \end{array}$$

H.A. : $\emptyset$

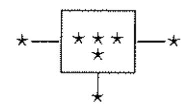
7.21

$\tilde{E}_6/A_5A_1$ (geometric, 3×1)



$\emptyset$

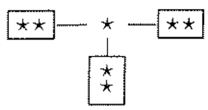
$\tilde{E}_6/D_4$, ord(u) $= 3$ (arithmetic, 3×1)



H.A. : $\emptyset$

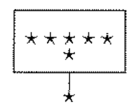
7.22

$\tilde{E}_6/A_2A_2A_2$ (geometric, 1×2)



$\emptyset$

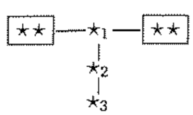
$\tilde{E}_6/E_6, u = 1,$ (arithmetic, 1×2)



H.A. : $\emptyset$

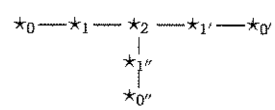
7.23

$\tilde{E}_6/A_2A_2$ (geometric, 3×1)



$\sharp_1^{6\times 1} \;<\equiv\; \sharp_2^{2\times 1} - \sharp_3^{2\times 1}$

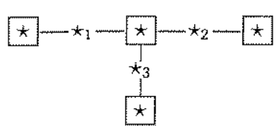
$\tilde{E}_6/\emptyset$, ord(u) $= 3$ (arithmetic, 3×1)



H.A. : $\mathbf{3}_0 - \mathbf{3}_1 \;\equiv>\; \mathbf{1}_2$

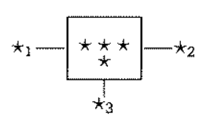
7.24

$\tilde{E}_6/A_1A_1A_1A_1$ (geometric, 1×1)



$\sharp_1^{4\times 2}$ — $\sharp_2^{4\times 2}$
\ $\sharp_3^{4\times 2}$ /

$\tilde{E}_6/D_4, u = 1$ (arithmetic, 1×1)



H.A. : $\mathbf{4}_1$ — $\mathbf{4}_2$
\ $\mathbf{4}_3$ /

7.25

$\tilde{E}_6/\emptyset$ (geometric, 1×1)

$$\begin{array}{ccccccccc} \star_1 & - & \star_3 & - & \star_4 & - & \star_5 & - & \star_6 \\ & & & & | & & & & \\ & & & & \star_2 & & & & \\ & & & & | & & & & \\ & & & & \star_0 & & & & \end{array}$$

$$\begin{array}{ccccccccc} \sharp_1^{2\times 1} & - & \sharp_3^{2\times 1} & - & \sharp_4^{2\times 1} & - & \sharp_5^{2\times 1} & - & \sharp_6^{2\times 1} \\ & & & & | & & & & \\ & & & & \sharp_2^{2\times 1} & & & & \\ & & & & | & & & & \\ & & & & \sharp_0^{2\times 1} & & & & \end{array}$$

$\tilde{E}_6/\emptyset, u = 1$ (arithmetic, 1×1)

$$\begin{array}{ccccccccc} \star_1 & - & \star_3 & - & \star_4 & - & \star_5 & - & \star_6 \\ & & & & | & & & & \\ & & & & \star_2 & & & & \\ & & & & | & & & & \\ & & & & \star_0 & & & & \end{array}$$

$$\text{H.A.} : \begin{array}{ccccccccc} 1_1 & - & 1_3 & - & 1_4 & - & 1_5 & - & 1_6 \\ & & & & | & & & & \\ & & & & 1_2 & & & & \\ & & & & | & & & & \\ & & & & 1_0 & & & & \end{array}$$

7.26

$\tilde{F}_4/F_4$ (geometric, 1×1)

$\star - \boxed{\star\star \Rightarrow \star\star}$

$\emptyset$

$\tilde{F}_4/F_4, u = 1$ (arithmetic, 1×1)

$\star - \boxed{\star\star \Rightarrow \star\star}$

H.A. : $\emptyset$

E : $F_4^{II}[1]$

7.27

$\tilde{F}_4/C_3A_1$ (geometric, 1×1)

$\boxed{\star} - \star - \boxed{\star \Rightarrow \star\star}$

$\emptyset$

$\tilde{F}_4/F_4, u = 1$ (arithmetic, 1×1)

$\star - \boxed{\star\star \Rightarrow \star\star}$

H.A. : $\emptyset$

E : $F_4[-1]$

7.28

$\tilde{F}_4/A_2A_2$ (geometric, 1×2)

$\boxed{\star\star} - \star \Rightarrow \boxed{\star\star}$

$\emptyset$

$\tilde{F}_4/F_4, u = 1$ (arithmetic, 1×2)

$\star - \boxed{\star\star \Rightarrow \star\star}$

H.A. : $\emptyset$

E : $F_4[\theta], F_4[\theta^2]$

7.29

$\tilde{F}_4/A_3A_1$ (geometric, 1×2)

$\boxed{\star\star\star} \Rightarrow \star - \boxed{\star}$

$\emptyset$

$\tilde{F}_4/F_4, u = 1$ (arithmetic, 1×2)

$\star - \boxed{\star\star \Rightarrow \star\star}$

H.A. : $\emptyset$

E : $F_4[i], F_4[-i]$

7.30

$\tilde{F}_4/B_4$ (geometric, 1×1)

$\boxed{\star\star\star \Rightarrow \star}$—$\star$

$\emptyset$

$\tilde{F}_4/F_4, u = 1$ (arithmetic, 1×1)

$\star$—$\boxed{\star\star \Rightarrow \star\star}$

H.A. : $\emptyset$

$\mathbf{E}$: $F_4^I[1]$

7.31

$\tilde{F}_4/A_1A_1$ (geometric, 1×1)

$\boxed{\star}$—$\star_1$—$\boxed{\star} \Rightarrow \star_2$—$\star_3$

$\flat_1^{4\times1} \Leftarrow \sharp_2^{3\times2} \Rightarrow \flat_3^{2\times1}$

$\tilde{F}_4/B_2, u = 1$ (arithmetic, 1×1)

$\star$—$\star$—$\boxed{\star \Rightarrow \star}$—$\star$

H.A. : $\mathbf{3}_1 \Rightarrow \mathbf{3}_2 \Leftarrow \mathbf{1}_0$

7.32

$\tilde{F}_4/\emptyset$ (geometric, 1×1)

$\star_5$—$\star_4$—$\star_3 \Rightarrow \star_2$—$\star_1$

$\sharp_5^{2\times1}$—$\sharp_4^{2\times1}$—$\sharp_3^{2\times1} \Rightarrow \sharp_2^{2\times1}$—$\sharp_1^{2\times1}$

$\tilde{F}_4/\emptyset, u = 1$ (arithmetic, 1×1)

$\star_4$—$\star_3 \Leftarrow \star_2$—$\star_1$—$\star_0$

H.A. : $\mathbf{1}_4$—$\mathbf{1}_3 \Leftarrow \mathbf{1}_2$—$\mathbf{1}_1$—$\mathbf{1}_0$

7.33

$\tilde{G}_2/G_2$ (geometric, 1×1)

$\boxed{\star <\!\!\equiv \star}$—$\star$

$\emptyset$

$\tilde{G}_2/G_2, u = 1$ (arithmetic, 1×1)

$\boxed{\star <\!\!\equiv \star}$—$\star$

H.A. : $\emptyset$

7.34

$\tilde{G}_2/A_2$ (geometric, 1×2)

$\star <\!\!\equiv \boxed{\star\star}$

$\emptyset$

$\tilde{G}_2/G_2, u = 1$ (arithmetic, 1×2)

$\boxed{\star <\!\!\equiv \star}$—$\star$

H.A. : $\emptyset$

7.35

$\tilde{G}_2/A_1A_1$ (geometric, 1×1)

$\boxed{\star} <\!\!\equiv \star$—$\boxed{\star}$

$\emptyset$

$\tilde{G}_2/G_2, u = 1$ (arithmetic, 1×1)

$\boxed{\star <\!\!\equiv \star}$—$\star$

H.A. : $\emptyset$

7.36

$\tilde{G}_2/\emptyset$ (geometric, 1×1)

$$\star_1 <\!\equiv \star_2 — \star_3$$

$$\natural_1^{2\times 1} <\!\equiv \natural_2^{2\times 1} — \natural_3^{2\times 1}$$

$\tilde{G}_2/\emptyset, u = 1$ (arithmetic, 1×1)

$$\star_0 — \star_1 <\!\equiv \star_2$$

H.A. : $1_0 — 1_1 <\!\equiv 1_2$

7.37

$\tilde{A}_n/A^b_{a-1}$ (geometric, $a \times 1$)

Here $n \geq 1$, a is an integer greater than or equal to 1 dividing $n+1$, and $b = (n+1)/a$;

$$\cdots — \underset{A_{a-1}}{\boxed{\star\ \star\ \cdots\ \star\ \star}} — \star_1 — \underset{A_{a-1}}{\boxed{\star\ \star\ \cdots\ \star\ \star}} — \star_2 — \underset{A_{a-1}}{\boxed{\star\ \star\ \cdots\ \star\ \star}} — \star_3 — \cdots$$

(a cyclic graph with $n+1$ vertices);

$$\cdots — \natural_1^{2a\times 1} — \natural_2^{2a\times 1} — \natural_3^{2a\times 1} — \cdots$$

(a cyclic graph with b vertices) if $b \geq 2$ and $\emptyset$ if $b = 1$;

$\tilde{A}_n/\emptyset, \operatorname{ord}(u) = a$ (arithmetic, $a \times 1$)

$$\cdots — \star — \star — \star — \cdots$$

(a cyclic graph with $n+1$ vertices).

H.A.: $\tilde{A}_{b-1}$ if $b \geq 2$ and $\emptyset$ if $b = 1$.

7.38

$\tilde{D}_n/(D_p \times D_q), 2p = (a+b)^2, 2q = (a-b)^2$ (geometric, 2×1), a even, b even, $a > b > 0$. Here $a^2 + b^2 < n \geq 4$. Let $s = n + 1 - p - q$.

$\tilde{D}_n/(D_{p'} \times D_{q'}), p' = a^2, q' = b^2, u = 1$ (arithmetic, 2×1);

$$\flat_s^{2(a-b)\times 1} \Leftarrow \natural_{s-1}^{2\times 1} — \natural_{s-2}^{2\times 1} — \cdots — \natural_2^{2\times 1} \Rightarrow \flat_1^{2(a+b)\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}^{sc}_{s-1}[2_a 1_{2b}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.39

$\tilde{D}_n/(D_p \times D_q), 2p = (a+b)^2, 2q = (a-b)^2$ (geometric, 2×1), a odd, b odd, $a > b > 0$. Here $a^2 + b^2 < n \geq 4$. Let $s = n + 1 - p - q$.

$\tilde{D}_n/(D_{p'} \times D_{q'}), p' = a^2, q' = b^2$, u of type 1 (arithmetic, 2×1);

$$\flat_s^{2(a-b)\times 1} \Leftarrow \sharp_{s-1}^{2\times 1} —— \sharp_{s-2}^{2\times 1} —— \cdots —— \sharp_2^{2\times 1} \Rightarrow \flat_1^{2(a+b)\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}^{sc}_{s-1}[{}_{2a}1_{2b}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.40

$\tilde{D}_n/(D_p \times D_p), 2p = a^2$ (geometric, 1×1), $a > 0$ even. Let $s = n + 1 - 2p$.

$\tilde{D}_n/D_{p'}, p' = a^2, u = 1$ (arithmetic, 1×1);

$$\flat_s^{2a\times 1} \Leftarrow \sharp_{s-1}^{2\times 1} —— \sharp_{s-2}^{2\times 1} —— \cdots —— \sharp_2^{2\times 1} \Rightarrow \flat_1^{2a\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}^{sc}_{s-1}[{}_0 1_{2a}] = \tilde{B}_{s-1}[1_{2a}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.41

$\tilde{D}_n/D_p, p = 2a^2$ (geometric, 2×1, if $s \geq 2$; 4×1, if $s = 1$) a even, $a > 0$. Here $a^2 < n \geq 4$ and $s = n + 1 - p$.

$\tilde{D}_n/(D_{p'} \times D_{p'}), p' = a^2, u = 1$ (arithmetic, 2×1, if $s \geq 2$; 4×1, if $s = 1$).

$$\begin{tikzcd} \sharp_s^{2\times 1} \arrow[d, dash] & & & \\ \sharp_{s-2}^{2\times 1} \arrow[r, dash] & \cdots \arrow[r, dash] & \sharp_2^{2\times 1} \arrow[r, Rightarrow] & \sharp_1^{4a\times 1} \\ \sharp_{s-1}^{2\times 1} \arrow[u, dash] & & & \end{tikzcd}$$

H.A.: $\tilde{C}_{s-1}[{}_{2a}1_{2a}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.42

$\tilde{D}_n/D_p, p = 2a^2$ (geometric, 2×1, if $s \geq 2$; 4×1, if $s = 1$) a odd, $a > 0$. Here $a^2 < n \geq 4$ and $s = n + 1 - p$.

$\tilde{D}_n/(D_{p'} \times D_{p'}), p' = a^2$, u of type 1 (arithmetic, 2×1, if $s \geq 2$; 4×1, if $s = 1$).

$$\begin{tikzcd} \sharp_s^{2\times 1} \arrow[d, dash] & & & \\ \sharp_{s-2}^{2\times 1} \arrow[r, dash] & \cdots \arrow[r, dash] & \sharp_2^{2\times 1} \arrow[r, Rightarrow] & \sharp_1^{4a\times 1} \\ \sharp_{s-1}^{2\times 1} \arrow[u, dash] & & & \end{tikzcd}$$

H.A.: $\tilde{C}_{s-1}[{}_{2a}1_{2a}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.43

$\tilde{D}_n/\emptyset$ (geometric, 1×1), $n = s$. Here $n \geq 4$.

$\tilde{D}_n/\emptyset, u = 1$ (arithmetic, 1×1).

$$\begin{array}{ccccccccc}
\natural_0^{2\times 1} & & & & & & & & \natural_s^{2\times 1} \\
| & & & & & & & & | \\
\natural_2^{2\times 1} & — & \natural_3^{2\times 1} & — & \cdots & — & \natural_{s-3}^{2\times 1} & — & \natural_{s-2}^{2\times 1} \\
| & & & & & & & & | \\
\natural_1^{2\times 1} & & & & & & & & \natural_{s-1}^{2\times 1}
\end{array}$$

H.A.: $\tilde{D}_n[1]$.

7.44

$\tilde{D}_n/(D_p \times D_q \times A_1^{s-1}), 2p = (1/2)(2a+b)(2a+b+1), 2q = (1/2)(2a-b)(2a-b-1), p+q+(2s-2) = n$ (geometric, 2×2), $a > 0, b \geq 0, s \geq 1, a \equiv n \bmod 2, (1/2)b(b+1) \equiv n \bmod 2, b \neq 2a, b \neq 2a-1$. Our assumptions imply that $p > q > 1$ and $n \geq 5$.

$\tilde{D}_n/(D_{p'} \times D_{p'} \times A_{r-1}), p' = a^2, r = (1/2)(b^2+b)$, u of type 2; (arithmetic, 4×1).

$$\flat_s^{|4a-2b-1|\times 1} \Leftarrow \natural_{s-1}^{4\times 1} — \natural_{s-2}^{4\times 1} — \cdots — \natural_2^{4\times 1} \Rightarrow \flat_1^{(4a+2b+1)\times 1}$$

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{4a}2_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.45

$\tilde{D}_n/(D_p \times A_1^{s-1}), 2p = (1/2)(2a+b)(2a+b+1), p + 2(s-1) = n$ (geometric, 4×1), $a > 0, s \geq 1, a \equiv n \bmod 2, (1/2)b(b+1) \equiv n \bmod 2, b = 2a$ or $b = 2a-1$. Here $n \geq 5$.

$\tilde{D}_n/(D_{p'} \times D_{p'} \times A_{r-1}), p' = a^2, r = (1/2)(b^2+b)$, u of type 2 (arithmetic, 4×1).

$$\flat_s^{2\times \frac{1}{2}} \Leftarrow \natural_{s-1}^{4\times 1} — \natural_{s-2}^{4\times 1} — \cdots — \natural_2^{4\times 1} \Rightarrow \flat_1^{(4a+2b+1)\times 1}$$

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{4a}2_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.46

$\tilde{D}_n/(D_p \times D_p \times A_1^{s-1}), 2p = (1/2)b(b+1), 2p + 2(s-1) = n$ (geometric, 1×2) $b > 0, s \geq 1, 0 \equiv (1/2)b(b+1) \bmod 2$. Here $n \geq 5$.

$\tilde{D}_n/A_{r-1}, r = (1/2)(b^2+b)$ with u of type 2; (arithmetic, 2×1).

$$\flat_s^{(2b+1)\times 1} \Leftarrow \natural_{s-1}^{4\times 1} — \natural_{s-2}^{4\times 1} — \cdots — \natural_2^{4\times 1} \Rightarrow \flat_1^{(2b+1)\times 1}$$

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{0}2_{2b+1}] = \tilde{B}_{s-1}[2_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.47

$\tilde{D}_n/A_1^s$, $2(s-1)=n$ (geometric, 4×1), $s\geq 4$. Here $n\geq 6$.

$\tilde{D}_n/\emptyset$, u of type 2 (arithmetic, 4×1).

$$\sharp_s^{2\times 1} \Rightarrow \sharp_{s-1}^{4\times 1} — \sharp_{s-2}^{4\times 1} — \cdots — \sharp_2^{4\times 1} \Leftarrow \sharp_1^{2\times 1}$$

H.A.: $\tilde{B}_{s-1}[2_1]$.

7.48

$\tilde{B}_n/(D_p\times B_q)$, $2p=(a+b+1)^2$, $2q+1=(a-b)^2$ (geometric, 1×1), $a>b\geq 0$, $a+b+1$ even. Let $s=n+1-p-q$.

$\tilde{C}_n/(C_{p'}\times C_{q'})$, $p'=a(a+1)$, $q'=b(b+1)$, $u=1$ (arithmetic, 1×1).

$$\flat_s^{2(a-b)\times 1} \Leftarrow \sharp_{s-1}^{2\times 1} — \sharp_{s-2}^{2\times 1} — \cdots — \sharp_2^{2\times 1} \Rightarrow \flat_1^{2(a+b+1)\times 1}$$

H.A.: $\tilde{C}_{s-1}^{sc}[2_{a+1}1_{2b+1}]$ if $s\geq 2$ and $\emptyset$ if $s=1$.

7.49

$\tilde{B}_n/(D_p\times B_q)$, $2p=(a-b)^2$, $2q+1=(a+b+1)^2$ (geometric, 1×1), $a>b\geq 0$, $a-b$ even. Let $s=n+1-p-q$.

$\tilde{C}_n/(C_{p'}\times C_{q'})$, $p'=a(a+1)$, $q'=b(b+1)$, $u=1$ (arithmetic, 1×1).

$$\flat_s^{2(a+b+1)\times 1} — \sharp_{s-1}^{2\times 1} — \sharp_{s-2}^{2\times 1} — \cdots — \sharp_2^{2\times 1} — \flat_1^{2(a-b)\times 1}$$

H.A.: $\tilde{C}_{s-1}^{sc}[2_{a+1}1_{2b+1}]$ if $s\geq 2$ and $\emptyset$ if $s=1$.

7.50

$\tilde{B}_n/B_q$, $q=2a^2+2a$ (geometric, 1×1), $a\geq 0$. Let $s=n+1-q$.

$\tilde{C}_n/(C_{p'}\times C_{p'})$, $p'=a(a+1)$, $u=1$ (arithmetic, 1×1).

$$\begin{array}{ccccccccccc} \sharp_s^{2(2a+1)\times 1} & \Leftarrow & \sharp_{s-1}^{2\times 1} & — & \sharp_{s-2}^{2\times 1} & — & \cdots & — & \sharp_3^{2\times 1} & — & \sharp_2^{2\times 1} \\ &&&&&&&& | && \\ &&&&&&&& \sharp_1^{2\times 1} && \end{array}$$

H.A.: $\tilde{C}_{s-1}[2_{a+1}1_{2a+1}]$ if $s\geq 2$ and $\emptyset$ if $s=1$.

7.51

$\tilde{B}_n/(D_p \times B_q \times A_1^{s-1})$, $2p = (1/2)(2a+b+1)(2a+b+2)$, $2q+1 = (1/2)(2a-b)(2a-b+1)$, $p+q+2(s-1) = n$ (geometric, 1×2), $a \geq 0$, $b > 0$, $a+1 \equiv (1/2)b(b-1) \bmod 2$.

$\tilde{C}_n/(C_{p'} \times C_{p'} \times A_{r-1})$, $p' = a^2+a$, $r = (1/2)(b^2+b)$, $\mathrm{ord}(u) = 2$ (arithmetic, 2×1);

$$\flat_s^{|4a-2b+1| \times 1} \Leftarrow \natural_{s-1}^{4\times 1} \text{——} \natural_{s-2}^{4\times 1} \text{——} \cdots \text{——} \natural_2^{4\times 1} \Rightarrow \flat_1^{(4a+2b+3)\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{4a+2}2_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.52

$\tilde{B}_n/(D_p \times B_q \times A_1^{s-1})$, $2p = (1/2)(2a-b)(2a-b+1)$, $2q+1 = (1/2)(2a+b+1)(2a+b+2)$, $p+q+2(s-1) = n$ (geometric, 1×2), $a \geq 0$, $b \geq 0$, $a \equiv (1/2)b(b-1) \bmod 2$, $b \neq 2a$, $b \neq 2a+1$.

$\tilde{C}_n/(C_{p'} \times C_{p'} \times A_{r-1})$, $p' = a^2+a$, $r = (1/2)(b^2+b)$, $\mathrm{ord}(u) = 2$ (arithmetic, 2×1);

$$\flat_s^{(4a+2b+3) \times 1} \Leftarrow \natural_{s-1}^{4\times 1} \text{——} \natural_{s-2}^{4\times 1} \text{——} \cdots \text{——} \natural_2^{4\times 1} \Rightarrow \flat_1^{|4a-2b+1|\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{4a+2}2_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.53

$\tilde{B}_n/(B_q \times A_1^{s-1})$, $2q+1 = (1/2)(2a+b+1)(2a+b+2)$, $q+2(s-1) = n$ (geometric, 2×1), $a \geq 0$, $b = 2a$ or $b = 2a+1$

$\tilde{C}_n/(C_{p'} \times C_{p'} \times A_{r-1})$, $p' = a^2+a$, $r = (1/2)(b^2+b)$, $\mathrm{ord}(u) = 2$ (arithmetic, 2×1).

$$\flat_s^{(4a+2b+3) \times 1} \Leftarrow \natural_{s-1}^{4\times 1} \text{——} \natural_{s-2}^{4\times 1} \text{——} \cdots \text{——} \natural_2^{4\times 1} \Rightarrow \flat_1^{2\times\frac{1}{2}}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{4a+2}2_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.54

$\tilde{C}_n/(C_p \times C_q)$, $p = (1/2)(a+b)(a+b+1)$, $q = (1/2)(a-b-1)(a-b)$ (geometric, 2×1), a even, $a > 0$, $b \geq 0$. Let $s = n+1-p-q$.

$\tilde{B}_n/(D_{p'} \times B_{q'})$, $p' = a^2$, $q' = b^2+b$, $u = 1$ (arithmetic, 2×1).

$$\flat_s^{(2a+2b+1) \times 1} \Leftarrow \natural_{s-1}^{2\times 1} \text{——} \natural_{s-2}^{2\times 1} \text{——} \cdots \text{——} \natural_2^{2\times 1} \Rightarrow \flat_1^{|2a-2b-1|\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{2a}1_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.55

$\tilde{C}_n/(C_p \times C_q), p = (1/2)(a+b)(a+b+1), q = (1/2)(a-b-1)(a-b)$ (geometric, 2×1), a odd; $a > 0, b \geq 0$. Let $s = n+1-p-q$.

$\tilde{B}_n/(D_{p'} \times B_{q'}), p' = a^2, q' = b^2+b, \mathrm{ord}(u) = 2$ (arithmetic, 2×1).

$$\flat_s^{(2a+2b+1)\times 1} \Leftarrow \sharp_{s-1}^{2\times 1} \text{---} \sharp_{s-2}^{2\times 1} \text{---} \cdots \text{---} \sharp_2^{2\times 1} \Rightarrow \flat_1^{|2a-2b-1|\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{C}_{s-1}^{sc}[{}_{2a}1_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

7.56

$\tilde{C}_n/(C_p \times C_p), p = (1/2)b(b+1)$ (geometric 1×1), $b \geq 0$. Let $s = n+1-2p$.

$\tilde{B}_n/B_{q'}, q' = b^2+b, u = 1$ (arithmetic, 1×1).

$$\sharp_s^{(2b+1)\times 1} \Rightarrow \sharp_{s-1}^{2\times 1} \text{---} \sharp_{s-2}^{2\times 1} \text{---} \cdots \text{---} \sharp_2^{2\times 1} \Leftarrow \sharp_1^{(2b+1)\times 1}$$

if $s \geq 2$ and $\emptyset$ if $s = 1$.

H.A.: $\tilde{B}_{s-1}[1_{2b+1}]$ if $s \geq 2$ and $\emptyset$ if $s = 1$.

8 Complements

8.1

Let $\mathbf{G}, \mathbf{I}, G, I$ be as in 6.1. Let $O(G)$ be the (finite) subgroup of the group of all automorphisms of G consisting of the automorphisms which preserve T, the set $\{\check{\alpha}_i \mid i \in I - \{0\}\}$ (see 5.2) and an associated "*épinglage*." We form the semidirect product $\tilde{G} = G \cdot O(G)$ with identity component G. As it is known, the quotient group $\tilde{S}(\mathbf{I})/S(\mathbf{I})$ may be identified with $O(G)$.

In this section, we assume that we are given an element $\vartheta \in O(G) - \{1\}$. Let $\tilde{S}(\mathbf{I})_\vartheta$ be the set of all $u \in \tilde{S}(\mathbf{I})$ such that the corresponding coset in $\tilde{S}(\mathbf{I})/S(\mathbf{I})$ corresponds to $\vartheta \in O(G)$.

One can expect that the following generalization of Corollary 6.5 (supporting the Langlands philosophy) should hold.

There is a natural bijection between $\sqcup_{u \in \tilde{S}(\mathbf{I})_\vartheta} \mathcal{U}(\mathbf{G}^{F_u})$ *and the set of all triples* $(s, y, \mathcal{V})$ *(modulo the natural action of* G*) where* $s \in G\vartheta$ *is semisimple in* $\tilde{G}$, $y \in \mathfrak{g}$ *is nilpotent with* $\mathrm{Ad}(s)y = qy$, *and* $\mathcal{V}$ *is an irreducible representation (up to isomorphism) of the group of components of the centralizer group* $Z_G(s, y)$.

It is likely that the proof of the previous statement will not differ too much from the one in the case $\vartheta = 1$. This is the case at least if we restrict ourselves to the case of irreducible representations with nonzero vectors fixed by an Iwahori subgroup, for a distinguished u in $\tilde{S}(I)_\vartheta$ (see [C], which is modeled on [CG].)

In the remainder of this section, we will sketch the beginning of such a proof, namely, a matching of arithmetic diagrams and geometric diagrams.

8.2

The various arithmetic diagrams of type u (for $u \in \tilde{S}(I)_\vartheta$) and the type and parameters of the corresponding affine Hecke algebras will be listed in 8.3–8.14. In each case, we describe the subset J by specifying the type of the subdiagram corresponding to J. The notation for affine Hecke algebras is as in 6.11.

If the extended Dynkin diagram is of type $\tilde{D}_n$, then we say that u is of type 1 if u is a permutation with exactly one 2-cycle; we say that u is of type 2 if u is not of type 1. (Recall that $u \notin S(I)$.)

8.3 Type $\tilde{A}_n$, $\mathrm{ord}(u) = 2$

J is of type $A_{p'-1} \times A_{q'-1}$ (both components are u-stable) where $p' = a(a+1)/2$, $q' = b(b+1)/2$, and $a + b \equiv (-1)^n \bmod 4$. Note that $p' + q' \equiv n + 1 \bmod 2$.

H.A.: $\tilde{C}^{sc}_{s-1}[{}_{2a+1}2_{2b+1}]$ where $s = ((n+1) - (p' + q' - 2))/2$.

8.4 Type $\tilde{D}_n$, u of type 1

J is of type $D_{p'} \times D_{q'}$, $p' = a^2$, $q' = b^2$, a is odd, b is even, and u acts nontrivially on first factor.

H.A.: $\tilde{C}^{sc}_{s-1}[{}_{2a}1_{2b}]$ where $s = n + 1 - (p' + q')$.

8.5 Type $\tilde{D}_n$, u of type 1

J is of type $D_{p'}$, $p' = a^2$, a is odd, u acts nontrivially.

H.A.: $\tilde{C}^{sc}_{s-1}[{}_0 1_{2b}] = \tilde{B}_{s-1}[1_{2a}]$ where $s = n + 1 - p'$.

8.6 Type $\tilde{D}_n$, u of type 2

J is of type $D_{p'} \times D_{p'} \times A_{r-1}$, $p' = a^2$, $r = (b^2 + b)/2$, $a > 0$, $b \geq 0$, $a \equiv n + 1 \bmod 2$, $(b^2 + b)/2 \equiv n \bmod 2$, and hence $a + 1 \equiv (b^2 + b)/2 \bmod 2$.

H.A.: $\tilde{C}^{sc}_{s-1}[{}_{4a}2_{2b+1}]$ where $s = ((n+1) - 2p' - (r-1))/2$.

8.7 Type $\tilde{D}_n$, u of type 2

J is of type A_{r-1} where $r = (b^2 + b)/2$.

H.A.: $\tilde{B}_{s-1}[2_{2b+1}]$ where $s = ((n+1) - (r-1))/2$.

8.8 Type $\tilde{E}_6$, ord(u) = 2

J is of type E_6, and there are two possible **E** (one the dual of the other).

H.A.: $\emptyset$.

8.9 Type $\tilde{E}_6$, ord(u) = 2

J is of type E_6, **E** is self-dual.

H.A.: $\emptyset$.

8.10 Type $\tilde{E}_6$, ord(u) = 2

J is of type A_5.

H.A.: $\mathbf{1} \overset{\infty}{—} \mathbf{9}$.

8.11 Type $\tilde{E}_6$, ord(u) = 2

$\mathbf{J} = \emptyset$.

H.A.: $\mathbf{2} — \mathbf{2} \Leftarrow \mathbf{1} — \mathbf{1} — \mathbf{1}$.

8.12 Type $\tilde{D}_4$, ord(u) = 3

J is of type D_4.

H.A.: $\emptyset$.

8.13 Type $\tilde{D}_4$, ord(u) = 3

J is of type D_4, **E** is other than that in the previous table.

H.A.: $\emptyset$.

8.14 Type $\tilde{D}_4$, ord(u) = 3

$\mathbf{J} = \emptyset$.

H.A.: $\mathbf{1} — \mathbf{1} \equiv> \mathbf{3}$.

8.15

In our case ($\vartheta \neq 1$), the geometric diagrams are defined as triples $(Z, \mathcal{C}, \mathcal{F})$, where Z is the centralizer in G of a semisimple element in the coset $G\vartheta \subset \tilde{G}$ and $\mathcal{F}$ is an irreducible Z-equivariant cuspidal local system on a nilpotent Z-orbit in the Lie algebra of Z. (A study of the centralizers of semisimple elements in $G\vartheta$ was made in [Ka].)

In each case in 8.3–8.14, one can indicate a geometric diagram which is a candidate to be matched with the arithmetic diagram. We describe the type of Z in each case.

In 8.3 with n odd, Z is of type $D_p \times C_q$ where $2p = (a+b+1)^2/4$, $2q = (a-b-1)(a-b+1)/4$.

In 8.3 with n even, Z is of type $B_p \times C_q$ where $2p+1 = (a+b+1)^2/4$, $2q = (a-b-1)(a-b+1)/4$.

In 8.4, Z is of type $B_p \times B_q$, $2p+1 = (a+b)^2$, $2q+1 = (a-b)^2$.

In 8.5, Z is of type $B_p \times B_p$, $2p+1 = a^2$.

In 8.6, Z is of type $B_p \times B_q \times A_1^{s-1}$, $2p+1 = (1/2)(2a+b)(2a+b+1)$, $2q+1 = (1/2)(2a-b)(2a-b-1)$, $p+q+(2s-2) = n-1$.

In 8.7, Z is of type $B_p \times B_p \times A_1^{s-1}$, $2p+1 = (1/2)b(b+1)$, $2p+(2s-2) = n-1$.

In 8.8, Z is of type $A_2 \times A_2$.

In 8.9, Z is of type F_4.

In 8.10, Z is of type $A_1 \times A_1 \times A_1$.

In 8.11, Z is of type $\emptyset$ (a torus).

In 8.12, Z is of type G_2.

In 8.13, Z is of type $A_1 \times A_1$.

In 8.14, Z is of type $\emptyset$ (a torus).

8.16

We note that the relationship between geometric diagrams and character sheaves of G mentioned in 6.6 and 6.7 generalizes to the present case, provided that we consider character sheaves not on G, but on the coset $G\vartheta$. (Such character sheaves were considered in [L9].)

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