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Cuspidal Local systems and Graded Hecke Algebras, II

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ABSTRACT. In this paper, we classify the simple modules of a graded Hecke algebra (an infinitesimal version of an affine Hecke algebra) attached in [L3] to a cuspidal local system on a nilpotent class of a simple Lie algebra. The main techniques used are equivariant homology and perverse sheaves.

This paper is a sequel to [L3].

Let G be a reductive connected algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. To G one can associate an associative algebra $\mathbf{H}$ over $\mathbf{C}[r]$ (r is an indeterminate) as follows: as a $\mathbf{C}[r]$-module, we have $\mathbf{H} = \mathbf{C}[r] \otimes \mathbf{S} \otimes \mathbf{C}[W]$ where $\mathbf{S}$ is the symmetric algebra of the dual $\mathfrak{t}^*$ of a fixed Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and W is the corresponding Weyl group. We assume that we have fixed a system of simple roots $(\alpha_i)_{i \in I}$ in $\mathfrak{t}^*$ with corresponding simple reflections $(s_i)_{i\in I}$ in W. The multiplication of $\mathbf{H}$ is defined by the following rules:

(a) $\mathbf{S} \to \mathbf{H}, \xi \mapsto 1 \otimes \xi \otimes 1$, is an algebra homomorphism;
(b) $\mathbf{C}[W] \to \mathbf{H}, w \mapsto 1 \otimes 1 \otimes w$, is an algebra homomorphism;
(c) $(1 \otimes \xi \otimes 1)(1 \otimes 1 \otimes w) = 1 \otimes \xi \otimes w$ for $\xi \in \mathbf{S}, w \in W$;
(d) $(1 \otimes 1 \otimes s_i)(1 \otimes \xi \otimes 1) - (1 \otimes {}^{s_i}\xi \otimes 1)(1 \otimes 1 \otimes s_i) = 2r \otimes \frac{\xi - {}^{s_i}\xi}{\alpha_i} \otimes 1$

for $\xi \in \mathbf{S}$ and any $i \in I$;
here ${}^{s_i}\xi$ is given by the standard W-action on $\mathbf{S}$. This definition appeared in [L3]; it contains as a special case the algebra considered in [D], which corresponds to $G = GL_n$ and $r = 1$.

The algebra $\mathbf{H}$ is called a *graded Hecke algebra*. As shown in [L4], $\mathbf{H}$ can be regarded as a graded algebra associated to a filtration of the affine Hecke algebra of G; moreover, it is possible to essentially recover the representation theory of one algebra from that of the other algebra.

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Now the simple modules of the affine Hecke algebra (with parameter assumed to be of infinite order and with G assumed to have simply connected derived group) have been studied in [KL], [G] and their classification in terms of the geometry of G (Deligne-Langlands conjecture) was obtained in [KL].

From this one can deduce (using the results of [L4] mentioned above) the classification of the simple modules of $\mathbf{H}$ (on which r acts as multiplication by the scalar $r_0 \neq 0$) in terms of the geometry of $\mathfrak{g}$. Namely, the indexing set turns out to be the set of all triples $(\sigma, \mathcal{O}, \mathcal{E})$ where σ is a semisimple element of $\mathfrak{g}$, (up to G-conjugacy), $\mathcal{O}$ is an orbit of the centralizer $Z_G(\sigma)$ in $\{y \in \mathfrak{g} | [\sigma, y] = 2r_0 y\}$ and $\mathcal{E}$ is an irreducible $Z_G(\sigma)$-equivariant local system (up to isomorphism) on $\mathcal{O}$, which satisfies certain specific restrictions.

Note that the approach of [KL] was in the framework of equivariant K-homology, while that in [G] was partly in equivariant K-homology and partly in intersection cohomology.

One of the aims of this paper is to give a new proof of the classification of the simple modules of $\mathbf{H}$ (with parameter $r_0 \neq 0$), in the framework of equivariant homology and intersection cohomology (no K-theory is used). One advantage of this approach is that equivariant homology mixes well with intersection cohomology, while equivariant K-homology does not.

However, there is a more serious reason for preferring a proof without K-theory. Namely, in [L3], $\mathbf{H}$ was only one member (the simplest one) of a a finite collection $\{\mathbf{H}_1, \mathbf{H}_2, \dots, \mathbf{H}_p\}$ of graded Hecke algebras attached to cuspidal local systems on nilpotent classes for various Levi subalgebras of parabolic subalgebras. (The algebra $\mathbf{H} = \mathbf{H}_1$ considered earlier corresponds to the case where the Levi subalgebra is $\mathfrak{t}$ and the nilpotent class is $\{0\}$.) These more general algebras are defined as in (a)-(d), except that now $\mathfrak{t}$ is the centre of the Levi subalgebra, W is the normalizer in G of $\mathfrak{t}$ modulo the centralizer in G of $\mathfrak{t}$ and in (d), $2r$ is replaced by $c_i r$ where c_i are certain integers depending on i. Our method applies equally well to the algebras $\mathbf{H}_k$; but in this case, it is the only known method (the K-theoretic approach seems to run into serious difficulties in the general case).

Let $Irr\mathbf{H}_k(r_0)$ be the set of isomorphism classes of simple $\mathbf{H}_k$-modules on which r acts as multiplication by the scalar $r_0 \neq 0$. In this paper we prove that $\sqcup_k Irr\mathbf{H}_k(r_0)$ is naturally in bijection with the set of all triples $(\sigma, \mathcal{O}, \mathcal{E})$ as above in which, this time, the irreducible G-equivariant local system $\mathcal{E}$ is completely unrestricted.

(This result is in some sense a q-analogue of the generalized Springer correspondence [L1], in which the set of pairs $(\mathcal{O}', \mathcal{E}')$ (where $\mathcal{O}'$ a nilpotent class of $\mathfrak{g}$ and $\mathcal{E}'$ is an irreducible G-equivariant local system on $\mathcal{O}'$) is put in 1-1 correspondence with the disjoint union of the sets of the irreducible representations of the various W considered above. This contains as a special case the usual Springer correspondence [Spr] in which only the full Weyl group enters and $\mathcal{E}'$ is restricted in a suitable sense.)

Our proof can be divided into three, quite different parts.

The first part establishes a connection between certain triples $(\sigma, \mathcal{O}, \mathcal{E})$ arising from geometry as above and the simple modules of the algebras of higher endomorphisms of certain objects (perverse sheaves) in equivariant derived categories. This part of the proof is close to the intersection cohomology part of Ginzburg's approach; for example, the use of the convolution in §2 and the statements of 5.4, 10.3 are inspired by results in [G],[CG]. The main difference is that in our case the geometry is more complicated, the varieties involved being less smooth; another difference is the systematic use of the equivariant derived categories. (The apparition of equivariant derived categories in the present context answers a question that Soergel asked me.)

The second part of the proof establishes an isomorphism between the higher endomorphism algebras mentioned above and the graded Hecke algebras. This part of the proof relies on results of [L3].

The third part of the proof shows that all triples $(\sigma, \mathcal{O}, \mathcal{E})$ as above are obtained. This relies on results in [L6].

Contents

1. Equivariant derived category

1.1. In this paper, all algebraic varieties (or, simply, varieties) are assumed to be over $\mathbf{C}$ and all algebraic groups are assumed to be affine. The Zariski closure of a subset X of an algebraic variety will be denote by $cl(X)$.

For any algebraic group $\mathfrak{G}$, we denote by $\mathfrak{G}^0$ the identity component of $\mathfrak{G}$, by $Z_{\mathfrak{G}}$ the centre of $\mathfrak{G}$ and by $U_{\mathfrak{G}}$ the unipotent radical of $\mathfrak{G}^0$; we set $\bar{\mathfrak{G}} = \mathfrak{G}/U_{\mathfrak{G}}$. For a subset $S \subset \mathfrak{G}$ we set

$$Z_{\mathfrak{G}}(S) = \{g \in \mathfrak{G} | gsg^{-1} = s \quad \forall s \in S\}, \quad N_{\mathfrak{G}}(S) = \{g \in \mathfrak{G} | gSg^{-1} = S\}.$$

The Lie algebra of $\mathfrak{G}$ is denoted by Lie $\mathfrak{G}$. We write $\pi_{\mathfrak{G}}$: Lie $\mathfrak{G} \to$ Lie $\bar{\mathfrak{G}}$ for the canonical map. We set $\mathfrak{z}_{\mathfrak{G}} =$ Lie $Z_{\mathfrak{G}}$. The adjoint action of $\mathfrak{G}$ on Lie $\mathfrak{G}$ is denoted by Ad. For any $x \in$ Lie $\mathfrak{G}$ we set $Z_{\mathfrak{G}}(x) = \{g \in \mathfrak{G} | Ad(g)x = x\}$. For $x, x' \in$ Lie $\mathfrak{G}$, we set

$$Z_{\mathfrak{G}}(x,x') = \{\dot{g} \in \mathfrak{G} | Ad(g)x = x, Ad(\tilde{g})x' = x'\}.$$

For an algebraic variety X we consider the bounded derived category of the abelian category of $\mathbf{C}$-sheaves over X and we denote by $\mathcal{D}X$ its full subcategory whose objects are the complexes with constructible cohomology sheaves. The j-th cohomology sheaf of $A \in \mathcal{D}X$ is denoted by $\mathcal{H}^j A$; its stalk at $x \in X$ is denoted by $\mathcal{H}^j_x A$. The shift functor $[n] : \mathcal{D}X \to \mathcal{D}X$ (for $n \in \mathbf{Z}$) satisfies $\mathcal{H}^j(A[n]) = \mathcal{H}^{j+n}A$.

If $\mathcal{L}$ is a constructible sheaf on X (for example, a local system) we shall regard $\mathcal{L}$ as an object of $\mathcal{D}X$ concentrated in degree 0. This applies in particular to the constant sheaf $\mathbf{C}$ on X (which will be sometimes denoted by $\mathbf{C}_X$).

The dual of a vector space or of a local system will be denoted by an upper index * (exception: $\mathbf{C}^*$ denotes $\mathbf{C} - \{0\}$).

1.2. If $A, B \in \mathcal{D}X$ and $j \in \mathbf{Z}$, we denote by $\mathrm{Hom}^j_{\mathcal{D}X}(A,B)$ the vector space $\dots = \mathrm{Hom}_{\mathcal{D}X}(A, B[j]) = \mathrm{Hom}_{\mathcal{D}X}(A[1], B[j+1]) = \mathrm{Hom}_{\mathcal{D}X}(A[2], B[j+2]) = \dots$

1.3. If $f : X \to Y$ is a morphism of varieties, there are standard functors (commuting with $[n]$):

$f^* : \mathcal{D}Y \to \mathcal{D}X$ (inverse image), $f^! : \mathcal{D}Y \to \mathcal{D}X$, $f_* : \mathcal{D}X \to \mathcal{D}Y$ (direct image), $f_! : \mathcal{D}X \to \mathcal{D}Y$ (direct image with compact support).

We have canonical adjunction isomorphisms

(a) $\mathrm{Hom}_{\mathcal{D}Y}(f_!A, B) = \mathrm{Hom}_{\mathcal{D}X}(A, f^!B)$, $\mathrm{Hom}_{\mathcal{D}X}(f^*B, A) = \mathrm{Hom}_{\mathcal{D}Y}(B, f_*A)$

for $A \in \mathcal{D}X, B \in \mathcal{D}Y$. Under these isomorphisms, the identity maps $f^*B \to f^*B$, $f^!B \to f^!B$, $f_*A \to f_*A$, $f_!A \to f_!A$ correspond to canonical maps

(b) $f_!f^!B \to B \to f_*f^*B$, $\quad f^*f_*A \to A \to f^!f_!A$.

Note that

(c) $f_! = f_*$ if f is proper,

(d) $f^! = f^*[2d]$ if f is smooth with fibres of pure dimension d.

(We will ignore Tate twists.)

1.4. For $A \in \mathcal{D}X$, we set

(a) $$H^j(X,A) = \mathcal{H}^j(p_*A), \quad H^j_c(X,A) = \mathcal{H}^j(p_!A)$$

where $p : X \to pt$ is the unique morphism from X to the point pt and we identify a sheaf over pt with its stalk (a $\mathbf{C}$-vector space).

1.5. Let $D : \mathcal{D}X \to \mathcal{D}X^{opp}$ be Verdier duality. Then

(a) $$DD = 1;$$

(b) $$Df_! = f_*D, Df^! = f^*D.$$

(f as in 1.3.) It follows that

(c) $$H^j(X, DA) = \mathrm{Hom}(H^{-j}_c(X,A), \mathbf{C})$$

for $A \in \mathcal{D}X$ and $j \in \mathbf{Z}$. For any $n \in \mathbf{Z}$, we have canonically

(d) $$D(A[n]) = (DA)[-n].$$

1.6. Recall that, if X, Y are varieties, we have the external tensor product

$$\boxtimes : \mathcal{D}X \times \mathcal{D}Y \to \mathcal{D}(X \times Y), \quad A, B \mapsto A \boxtimes B.$$

For $A \in \mathcal{D}X, B \in \mathcal{D}Y$ and $n \in \mathbf{Z}$, we have

(a) $$A[n] \boxtimes B = A \boxtimes B[n] = (A \boxtimes B)[n].$$

Let $\otimes : \mathcal{D}X \times \mathcal{D}X \to \mathcal{D}X$ be the functor given by $A \otimes B = \delta^*(A \boxtimes B)$ where $\delta : X \to X \times X$ is the diagonal. For $B \in \mathcal{D}X$, there are canonical isomorphisms

(b) $$B \otimes \mathbf{C}_X = B = \mathbf{C}_X \otimes B$$

and canonical morphisms

(c) $$DB \otimes B \to D\mathbf{C}_X,$$

(c') $$\mathbf{C}_X \to B \otimes DB,$$

which correspond to each other under D. Let $A, B \in \mathcal{D}X$. A morphism $\alpha : A \to B$ induces a morphism $D(DB \otimes B) \to D(DB \otimes A)$ (and a morphism $D(DA \otimes A) \to D(DB \otimes A)$), hence a linear map $H^0(X, D(DB \otimes B)) \to H^0(X, D(DB \otimes A))$ (resp. $H^0(X, D(DA \otimes A)) \to H^0(X, D(DB \otimes A))$). Composing with the linear map $H^0(X, \mathbf{C}_X) \to H^0(X, D(DB \otimes B))$ (resp. $H^0(X, \mathbf{C}_X) \to H^0(X, D(DA \otimes A))$) induced by (c'), we obtain a linear map $H^0(X, \mathbf{C}_X) \to H^0(X, D(DB \otimes A))$ (the same one in both variants). If $p : X \to pt$ is the map to a point, we have a canonical map $\mathbf{C} \to p_*p^*\mathbf{C} = p_*\mathbf{C}_X$; this induces a linear map $\mathbf{C} \to H^0(pt, p_*\mathbf{C}_X) = H^0(X, \mathbf{C}_X)$; composing this with the previous linear map, we obtain a linear map $\mathbf{C} \to H^0(X, D(DB \otimes A))$. The image of 1 under this map is an element $\rho(\alpha) \in H^0(X, D(DB \otimes A))$. It is known that $\alpha \mapsto \rho(\alpha)$ is an isomorphism

(d) $$\mathrm{Hom}_{\mathcal{D}X}(A,B) \xrightarrow{\sim} H^0(X, D(DB \otimes A)).$$

Replacing here B by $B[j]$ we see that we have canonically

(e) $$\mathrm{Hom}^j_{\mathcal{D}X}(A,B) \xrightarrow{\sim} H^j(X, D(DB \otimes A))$$

for any $j \in \mathbf{Z}$. If $f : X \to Y$ is a morphism of varieties we have canonically

(f) $$f^*(A \otimes B) = f^*A \otimes f^*B \text{ for } A, B \in \mathcal{D}Y,$$

(g) $$f_!(A \otimes f^*B) = (f_!A) \otimes B \text{ for } A \in \mathcal{D}X, B \in \mathcal{D}Y.$$

Let $A, B, C \in \mathcal{D}X$ and let $\phi \in \mathrm{Hom}_{\mathcal{D}X}(C, A)$. From the definitions it follows that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}X}(A,B) & \longrightarrow & H^0(X, D(DB \otimes A)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}X}(C,B) & \longrightarrow & H^0(X, D(DB \otimes C)) \end{array}$$

where the horizontal maps are isomorphisms as in (d), the left vertical map sends $g \in \mathrm{Hom}_{\mathcal{D}X}(A,B)$ to the composition $g\phi$ and the right vertical map is induced by $1 \otimes \phi : DB \otimes C \to DB \otimes A$.

Similarly, if we are given $A, B, C \in \mathcal{D}X$ and $\psi \in \mathrm{Hom}_{\mathcal{D}X}(B, C)$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}X}(A,B) & \longrightarrow & H^0(X, D(DB \otimes A)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}X}(A,C) & \longrightarrow & H^0(X, D(DC \otimes A)) \end{array}$$

where the horizontal maps are isomorphisms as in (d), the left vertical map sends $g \in \mathrm{Hom}_{\mathcal{D}X}(A,B)$ to the composition ψg and the right vertical map is induced by $D\psi \otimes 1 : DC \otimes A \to DB \otimes A$.

1.7. For $A, B \in \mathcal{D}X$ and $k \in \mathbf{Z}$, we have a natural bilinear form (cup product)

$$\oplus_{j,j';j+j'=k} H^j(X, A) \times H^{j'}(X, B) \to H^k(X, A \otimes B).$$

(By the Künneth formula, the left hand side may be interpreted as $H^k(X \times X, A \boxtimes B)$ and then our map is induced by the diagonal inclusion $X \to X \times X$.)

1.8. Vector bundles. Let $p : E \to X$ be an n-dimensional vector bundle with zero section $i : X \to E$. For any $A \in \mathcal{D}X$ we define an element $\kappa_E \in \mathrm{Hom}^{2n}_{\mathcal{D}X}(A, A)$ as follows. Note that $p_!\mathbf{C}_E = \mathbf{C}_X[-2n]$. Hence, if $A' = p^*A \in \mathcal{D}E$, we have

$$p_!A' = p_!(\mathbf{C}_E \otimes p^*A) = (p_!\mathbf{C}_E) \otimes A = \mathbf{C}_X[-2n] \otimes A = A[-2n].$$

Applying $p_!$ to the canonical morphism $A' \to i_*i^*A' = i_!i^*A' = i_!A$, we obtain a morphism $p_!A' \to p_!i_!A = A$. Composing this with the isomorphism $p_!A' = A[-2n]$ defined above, we obtain a morphism $A[-2n] \to A$. This is the same as an element $\kappa_E \in \mathrm{Hom}^{2n}_{\mathcal{D}X}(A, A)$.

1.9. Now let $\mathfrak{G}$ be an algebraic group. By definition, a $\mathfrak{G}$-*variety* is an algebraic variety with a given (algebraic) action of $\mathfrak{G}$. A $\mathfrak{G}$-*morphism* between two $\mathfrak{G}$-varieties is, by definition, a morphism which commutes with the $\mathfrak{G}$-actions. Note that the product of two $\mathfrak{G}$-varieties is a $\mathfrak{G}$-variety with the diagonal action of $\mathfrak{G}$. A $\mathfrak{G}$-variety X is said to be *free* if there exists a principal $\mathfrak{G}$-fibration $X \to \mathfrak{G}\backslash X$. Let $Sf(\mathfrak{G})$ be the category whose objects are smooth, free $\mathfrak{G}$-varieties of pure dimension and whose morphisms are the smooth $\mathfrak{G}$-morphisms with fibres of constant pure dimension.

If X is a $\mathfrak{G}$-variety and $\Gamma \in Sf(\mathfrak{G})$, then $\Gamma \times X$ is a free $\mathfrak{G}$-variety and we set ${}_\Gamma X = \mathfrak{G}\backslash(\Gamma \times X)$. Clearly, any morphism $\phi : \Gamma \to \Gamma'$ in $Sf(\mathfrak{G})$ induces a morphism of varieties $\bar\phi : {}_\Gamma X \to {}_{\Gamma'}X$.

1.10. Let X be a $\mathfrak{G}$-variety. There are several definitions of the equivariant derived category $\mathcal{D}_\mathfrak{G}X$. One of them, which I learned from Deligne more than 10 years ago, is based on the simplicial description of a classifying space of $\mathfrak{G}$. Recently, another definition was given by Bernstein and Lunts [**BL**], based on the possibility of approximating the classifying space of $\mathfrak{G}$ by *smooth* varieties. The same idea was used in [**L3**] to define equivariant homology. For this reason, it will be convenient to follow (at least in part) the approach of [**BL**]. An object $A \in \mathcal{D}_\mathfrak{G}X$ is a collection of objects

$${}_\Gamma A \in \mathcal{D}({}_\Gamma X),$$

one for each $\Gamma \in Sf(\mathfrak{G})$, together with isomorphisms

$$\tau^A_\phi : \bar\phi^*{}_{\Gamma'}A \xrightarrow{\sim} {}_\Gamma A \text{ (in } \mathcal{D}({}_\Gamma X))$$

for any morphism $\phi : \Gamma \to \Gamma'$ in $Sf(\mathfrak{G})$, such that, given the morphisms $\Gamma \xrightarrow{\phi} \Gamma' \xrightarrow{\phi'} \Gamma''$ in $Sf(\mathfrak{G})$, we have $\tau^A_\phi \circ \bar\phi^*\tau^A_{\phi'} = \tau^A_{\phi'\phi}$. A morphism $\sigma : A \to B$ in $\mathcal{D}_\mathfrak{G}X$ is a collection of morphisms ${}_\Gamma\sigma : {}_\Gamma A \to {}_\Gamma B$ for each $\Gamma \in Sf(\mathfrak{G})$ such that

$${}_\Gamma\sigma \circ \tau^A_\phi = \tau^B_\phi \circ \bar\phi^*({}_{\Gamma'}\sigma)$$

for any $\phi : \Gamma \to \Gamma'$ in $Sf(\mathfrak{G})$. Composition of morphisms is defined in the obvious way.

We will sometimes consider an object of $\mathcal{D}_\mathfrak{G}X$ without specifying the isomorphisms τ^A_ϕ (when these isomorphisms are the obvious ones).

It is clear that, when $\mathfrak{G} = \{1\}$ we may identify $\mathcal{D}_\mathfrak{G}X = \mathcal{D}X$.

Returning to a general $\mathfrak{G}$, we define for any $n \in \mathbf{Z}$ a functor $[n] : \mathcal{D}_\mathfrak{G}X \to \mathcal{D}_\mathfrak{G}X$ as follows. For $A \in \mathcal{D}_\mathfrak{G}X$, we define $A[n] = ({}_\Gamma(A[n])), \tau^{A[n]}_\phi) \in \mathcal{D}_\mathfrak{G}X$ by ${}_\Gamma(A[n]) = ({}_\Gamma A)[n]$ and $\tau^{A[n]}_\phi$ is obtained from τ^A_ϕ by shift. We have thus defined $[n]$ on objects; the definition on morphisms is the obvious one.

If $A, B \in \mathcal{D}_\mathfrak{G}X$ and $j \in \mathbf{Z}$, we have canonically

(a) $\cdots = \mathrm{Hom}_{\mathcal{D}_\mathfrak{G}X}(A, B[j]) = \mathrm{Hom}_{\mathcal{D}_\mathfrak{G}X}(A[1], B[j+1])$
$= \mathrm{Hom}_{\mathcal{D}_\mathfrak{G}X}(A[2], B[j+2]) = \dots$

We denote the vector space (a) by $\mathrm{Hom}^j_{\mathcal{D}_\mathfrak{G}X}(A, B)$. We set

$$\mathrm{Hom}^\bullet_{\mathcal{D}_\mathfrak{G}X}(A, B) = \oplus_{j\in\mathbf{Z}}\mathrm{Hom}^j_{\mathcal{D}_\mathfrak{G}X}(A, B).$$

If $A, B, C \in \mathcal{D}_\mathfrak{G}X$, we have a natural pairing

$$\mathrm{Hom}^\bullet_{\mathcal{D}_\mathfrak{G}X}(B, C) \times \mathrm{Hom}^\bullet_{\mathcal{D}_\mathfrak{G}X}(A, B) \to \mathrm{Hom}^\bullet_{\mathcal{D}_\mathfrak{G}X}(A, C)$$

defined as follows. Let $h \in \mathrm{Hom}^j_{\mathcal{D}_\mathfrak{G}X}(B, C)$ and $h' \in \mathrm{Hom}^{j'}_{\mathcal{D}_\mathfrak{G}X}(A, B)$; we regard h and h' as morphisms $B[j'] \to C[j+j']$ and $A \to B[j']$ respectively. Then the pairing takes h, h' to the element of $\mathrm{Hom}^{j+j'}_{\mathcal{D}_\mathfrak{G}X}(A, C)$ corresponding to the composition $h \circ h' : A \to C[j+j']$.

In particular, $\mathrm{Hom}^\bullet_{\mathcal{D}_\mathfrak{G}X}(B, B)$ is an associative $\mathbf{C}$-algebra (naturally $\mathbf{Z}$-graded and with unit element) and $\mathrm{Hom}^\bullet_{\mathcal{D}_\mathfrak{G}X}(A, B)$ is naturally a left ($\mathbf{Z}$-graded) $\mathrm{Hom}^\bullet_{\mathcal{D}_\mathfrak{G}X}(B, B)$-module.

If X is a $\mathfrak{G}$-variety and $h : \mathfrak{G}' \to \mathfrak{G}$ is a homomorphism of algebraic groups, then X can be also regarded as a $\mathfrak{G}'$-variety and we have a functor $\mathcal{D}_\mathfrak{G}X \to \mathcal{D}_{\mathfrak{G}'}X$. To an object $({}_\Gamma A, \dots)$ of $\mathcal{D}_\mathfrak{G}X$ it attaches an object $({}_{\Gamma'}A', \dots)$ of $\mathcal{D}_{\mathfrak{G}'}X$ where, for $\Gamma' \in Sf(\mathfrak{G}')$ we define ${}_{\Gamma'}A' = {}_\Gamma A$ with $\Gamma = \mathfrak{G} \times_{\mathfrak{G}'} \Gamma' \in Sf(\mathfrak{G})$. (Then ${}_{\Gamma'}X = {}_\Gamma X$.)

In particular (taking $h : \{1\} \to \mathfrak{G}$ to be the obvious homomorphism), we obtain a functor $\mathcal{D}_\mathfrak{G}X \to \mathcal{D}X$.

On the other hand, if X is a $\mathfrak{G}$-variety and $\mathfrak{G} \to \mathfrak{G}'$ is an imbedding of algebraic groups, we have a canonical identification [**BL, 2.6.3**]

(b) $\mathcal{D}_{\mathfrak{G}'}(\mathfrak{G}' \times_\mathfrak{G} X) = \mathcal{D}_\mathfrak{G}X.$

1.11. Let X, Y be $\mathfrak{G}$-varieties and let $f : X \to Y$ be a $\mathfrak{G}$-morphism. For any $\Gamma \in Sf(\mathfrak{G})$, f induces a morphism ${}_\Gamma f : {}_\Gamma X \to {}_\Gamma Y$ by $(\gamma, x) \mapsto (\gamma, f(x))$. We define functors

$$f^*, f^! : \mathcal{D}_\mathfrak{G}Y \to \mathcal{D}_\mathfrak{G}X, \quad f_*, f_! : \mathcal{D}_\mathfrak{G}X \to \mathcal{D}_\mathfrak{G}Y$$

(commuting with $[n]$) as follows. For $A \in \mathcal{D}_\mathfrak{G}Y$, we define $f^*A = ({}_\Gamma(f^*A), \tau^{f^*A}_\phi)$ in $\mathcal{D}_\mathfrak{G}X$ by ${}_\Gamma(f^*A) = ({}_\Gamma f)^*({}_\Gamma A)$ and $\tau^{f^*A}_\phi$ by the composition

$$\bar\phi^*({}_{\Gamma'}f)^*({}_{\Gamma'}A) = ({}_\Gamma f)^*\bar\phi^*({}_\Gamma A) \to ({}_\Gamma f)^*({}_\Gamma A)$$

(the last map is $({}_\Gamma f)^*\tau^A_\phi$). Here the fact that ϕ is smooth is not used.

Now $f^!A$ is defined in exactly the same way, replacing $f^*, ({}_\Gamma f)^*, ({}_{\Gamma'}f)^*$ by $f^!, ({}_\Gamma f)^!, ({}_{\Gamma'}f)^!$ throughout. (The first equality in the definition of $\tau^{f^!A}_\phi$ holds since ϕ is smooth.)

For $B \in \mathcal{D}_{\mathfrak{G}}X$ we define $f_!B = ({}_{\Gamma}(f_!B), \tau_\phi^{f_!B}) \in \mathcal{D}_{\mathfrak{G}}Y$ by ${}_{\Gamma}(f_!B) = ({}_{\Gamma}f)_!({}_{\Gamma}B)$ and $\tau_\phi^{f_!B}$ by the composition

$$\bar{\phi}^*({}_{\Gamma'}f)_!({}_{\Gamma'}B) = ({}_{\Gamma}f)_!\bar{\phi}^*({}_{\Gamma'}B) \to ({}_{\Gamma}f)_!({}_{\Gamma}B)$$

(the last map is $({}_{\Gamma}f)_!\tau_\phi^B$). Here the fact that ϕ is smooth is not used. Now f_*A is defined in exactly the same way, replacing $f_!, ({}_{\Gamma}f)_!, ({}_{\Gamma'}f)_!$ by $f_*, ({}_{\Gamma}f)_*, ({}_{\Gamma'}f)_*$ throughout. (The first equality in the definition of $\tau_\phi^{f_*B}$ holds since ϕ is smooth.)

We have thus defined $f^*, f^!, f_*, f_!$ on objects; the definitions on morphisms are the obvious ones. The analogues of 1.3(a),(b),(c),(d) continue to hold in the $\mathfrak{G}$-equivariant context.

1.12. Let X, Y be two $\mathfrak{G}$-varieties. We define a functor $\boxtimes : \mathcal{D}_{\mathfrak{G}}X \times \mathcal{D}_{\mathfrak{G}}Y \to \mathcal{D}_{\mathfrak{G}}(X \times Y)$.

For $A \in \mathcal{D}_{\mathfrak{G}}X, B \in \mathcal{D}_{\mathfrak{G}}Y$, we define $A \boxtimes B = ({}_{\Gamma}(A\boxtimes B), \dots) \in \mathcal{D}_{\mathfrak{G}}(X \times Y)$ where ${}_{\Gamma}(A\boxtimes B)$ is the inverse image of ${}_{\Gamma}A \boxtimes {}_{\Gamma}B$ under the canonical map ${}_{\Gamma}(X \times Y) \to {}_{\Gamma} \times {}_{\Gamma}Y$. We have thus defined $\boxtimes$ on objects; the definition on morphisms is the obvious one.

We define a functor $\otimes : \mathcal{D}_{\mathfrak{G}}X \times \mathcal{D}_{\mathfrak{G}}X \to \mathcal{D}_{\mathfrak{G}}X$ by $A \otimes B = \delta^*(A\boxtimes B)$ where $\delta : X \to X \times X$ is the diagonal.

1.13. We define a functor $D : \mathcal{D}_{\mathfrak{G}}X \to \mathcal{D}_{\mathfrak{G}}X^{opp}$ (Verdier duality) as follows. For $A \in \mathcal{D}_{\mathfrak{G}}X$, we define $DA = ({}_{\Gamma}DA, \tau_\phi^{DA}) \in \mathcal{D}_{\mathfrak{G}}X$ by ${}_{\Gamma}DA = D({}_{\Gamma}A)[-2\dim \mathfrak{G}\backslash\Gamma]$ and τ_ϕ^{DA} by the composition

$$\bar{\phi}^*D({}_{\Gamma'}A)[-2\dim \mathfrak{G}\backslash\Gamma'] = D(\bar{\phi}^*{}_{\Gamma'}A[2\dim\mathfrak{G}\backslash\Gamma - 2\dim\mathfrak{G}\backslash\Gamma'])[-2\dim\mathfrak{G}\backslash\Gamma']$$
$$= D(\bar{\phi}^*{}_{\Gamma'}A)[-2\dim\mathfrak{G}\backslash\Gamma] \to D({}_{\Gamma}A)[-2\dim\mathfrak{G}\backslash\Gamma];$$

the last map is $D((\tau_\phi^A)^{-1})[-2\dim\mathfrak{G}\backslash\Gamma]$. (The first equality in the definition of τ_ϕ^{DA} holds since ϕ is smooth with fibres of pure dimension $\dim\mathfrak{G}\backslash\Gamma - \dim\mathfrak{G}\backslash\Gamma'$.)

We have thus defined D on objects; the definition on morphisms is the obvious one. The analogues of 1.5(a),(b),(d) continue to hold in the $\mathfrak{G}$-equivariant context.

The object $({}_{\Gamma}A, \dots)$ of $\mathcal{D}_{\mathfrak{G}}X$ where ${}_{\Gamma}A = \mathbf{C}$ for any $\Gamma \in Sf(\mathfrak{G})$ will be denoted by $\mathbf{C}$ or $\mathbf{C}_X$. For any $B \in \mathcal{D}_{\mathfrak{G}}X$, we have canonically $B \otimes \mathbf{C}_X = B$. (This is given by a collection of isomorphisms like 1.6(b) for each $\Gamma \in Sf(\mathfrak{G})$.) Moreover, we have a canonical morphism

(a) $\qquad DB \otimes B \to D\mathbf{C}_X;$

this is given by a collection of isomorphisms like 1.6(c), with the appropriate shift, for each $\Gamma \in Sf(\mathfrak{G})$.

1.14. For any $j \in \mathbf{Z}$, we define a functor $H^j_{\mathfrak{G}}(X,)$ from $\mathcal{D}_{\mathfrak{G}}X$ to the category of finite dimensional $\mathbf{C}$-vector spaces.

Let $A \in \mathcal{D}_{\mathfrak{G}}X$. The vector space $H^j_{\mathfrak{G}}(X, A)$ is defined as follows. An element of $H^j_{\mathfrak{G}}(X, A)$ is, by definition, a collection of elements $({}_{\Gamma}\rho \in H^j({}_{\Gamma}X, {}_{\Gamma}A))_{\Gamma \in Sf(\mathfrak{G})}$ such that for any morphism $\phi : \Gamma \to \Gamma'$ in $Sf(\mathfrak{G})$, the composition

$$H^j({}_{\Gamma'}X, {}_{\Gamma'}A) \xrightarrow{a} H^j({}_{\Gamma'}X, \bar{\phi}_*\bar{\phi}^*{}_{\Gamma'}A) = H^j({}_{\Gamma}X, \bar{\phi}^*{}_{\Gamma'}A) \xrightarrow{b} H^j({}_{\Gamma}X, {}_{\Gamma}A)$$

takes ${}_{\Gamma'}\rho$ to ${}_{\Gamma}\rho$. Here a is induced by the adjunction morphism ${}_{\Gamma'}A \to \bar{\phi}_*\bar{\phi}^*{}_{\Gamma'}A$ and b is induced by τ_ϕ^A. This defines the functor $H^j_{\mathfrak{G}}(X,)$ on objects. The definition on mophisms is the obvious one. We set $H^{\cdot}_{\mathfrak{G}}(X, A) = \oplus_j H^j_{\mathfrak{G}}(X, A)$.

Let m be an integer ≥ 1. We say that a variety Y is *m-acyclic* if it is irreducible and $H^j(Y, \mathbf{C}) = 0$ for $j = 1, 2, \dots, m$. Let $Sf_m(\mathfrak{G})$ be the full subcategory of $Sf(\mathfrak{G})$ whose objects are the $\Gamma \in Sf(\mathfrak{G})$ that are m-acyclic. Note that for any $m \geq 1$ we can find $\Gamma \in Sf_m(\mathfrak{G})$ (see for example **[L3, 1.1]**). We show that

(a) *If $j \in \mathbf{Z}$, $A \in \mathcal{D}_{\mathfrak{G}}X$, and m is sufficiently large, then for any $\Gamma_1 \in Sf_m(\mathfrak{G})$, the map $H^j_{\mathfrak{G}}(X, A) \to H^j({}_{\Gamma_1}X, {}_{\Gamma_1}A)$ given by $({}_{\Gamma}\rho \in H^j({}_{\Gamma}X, {}_{\Gamma}A))_{\Gamma \in Sf(\mathfrak{G})} \to {}_{\Gamma_1}\rho$, is an isomorphism.*

Let $\rho \in H^j({}_{\Gamma_1}X, {}_{\Gamma_1}A)$. Let $\Gamma \in Sf(\mathfrak{G})$. Then the two projections $p : \Gamma_1 \times \Gamma \to \Gamma_1$ and $p' : \Gamma_1 \times \Gamma \to \Gamma$ are morphisms in $Sf(\mathfrak{G})$. They induce natural maps

$$H^j({}_{\Gamma_1}X, {}_{\Gamma_1}A) \to H^j({}_{\Gamma_1\times\Gamma}X, {}_{\Gamma_1\times\Gamma}A), \quad H^j({}_{\Gamma}X, {}_{\Gamma}A) \to H^j({}_{\Gamma_1\times\Gamma}X, {}_{\Gamma_1\times\Gamma}A),$$

(as in the definition of $H^j_{\mathfrak{G}}(X,)$). Let ρ' be the image of ρ under the first of these maps. Now the second map is an isomorphism since the fibres of ${}_{\Gamma_1\times\Gamma}X \to {}_{\Gamma}X$ are isomorphic to Γ_1 hence are m-acyclic and m is large (depending on j, A); hence ρ' is the image under this map of a unique element ${}_{\Gamma}\rho$. It is easy to check that the collection $({}_{\Gamma}\rho \in H^j({}_{\Gamma}X, {}_{\Gamma}A))_{\Gamma \in Sf(\mathfrak{G})}$ belongs to $H^j_{\mathfrak{G}}(X, A)$ and that the resulting map $H^j({}_{\Gamma_1}X, {}_{\Gamma_1}A) \to H^j_{\mathfrak{G}}(X, A)$ $(\rho \mapsto {}_{\Gamma}\rho)$ is the inverse of the one defined in (a).

From the definitions we see that, if $f : X \to Y$ is a $\mathfrak{G}$-equivariant morphism, then

(b) $\qquad H^j_{\mathfrak{G}}(X, A) = H^j_{\mathfrak{G}}(Y, f_*A).$

1.15. Let $A, B \in \mathcal{D}_{\mathfrak{G}}X$. By definition, to give an element of $\mathrm{Hom}_{\mathcal{D}_{\mathfrak{G}}X}(A, B)$ is the same as to give a collection of elements ${}_{\Gamma}\sigma \in \mathrm{Hom}_{\mathcal{D}_{\Gamma}X}({}_{\Gamma}A, {}_{\Gamma}B)$ (one for each $\Gamma \in Sf(\mathfrak{G})$) which are compatible with the isomorphisms τ_ϕ^A, τ_ϕ^B. Using 1.6(d), we see that this is the same as to give a collection of elements in

$$H^0({}_{\Gamma}X, D(D({}_{\Gamma}B) \otimes {}_{\Gamma}A)) = H^0({}_{\Gamma}X, {}_{\Gamma}(D(DB \otimes A)))$$

(one for each $\Gamma \in Sf(\mathfrak{G})$) satisfying certain compatibility conditions. Using the commutative diagrams in 1.6, we see that the compatibility conditions express precisely the fact that our collection is an element of $H^0_{\mathfrak{G}}(X, D(DB \otimes A))$. Thus we have canonically

(a) $\qquad \mathrm{Hom}_{\mathcal{D}_{\mathfrak{G}}X}(A, B) = H^0_{\mathfrak{G}}(X, D(DB \otimes A)).$

Let $\Gamma \in Sf_m(\mathfrak{G})$ where m is large (depending on A, B). We have

(b) $H^0_{\mathfrak{G}}(X, D(DB \otimes A)) = H^0({}_{\Gamma}X, {}_{\Gamma}(D(DB \otimes A))) = H^0({}_{\Gamma}X, D(D({}_{\Gamma}B) \otimes {}_{\Gamma}A)) = \mathrm{Hom}(H^0_c({}_{\Gamma}X, D({}_{\Gamma}B) \otimes {}_{\Gamma}A), \mathbf{C})$.

1.16. In this subsection we assume that $\mathfrak{G}$ is connected. Let $\mathcal{M}X[j]$ be the full subcategory of $\mathcal{D}X$ whose objects are of the form $A'[j]$ where A' is a perverse sheaf on X. If $A \in \mathcal{M}X[j]$ is $\mathfrak{G}$-equivariant (that is, $A[-j]$ is a perverse sheaf which is $\mathfrak{G}$-equivariant in the sense of **[L2]**), we regard A as an object $({}_{\Gamma}A, \dots)$ of $\mathcal{D}_{\mathfrak{G}}X$, where ${}_{\Gamma}A$ is the object of $\mathcal{M}({}_{\Gamma}X)[j - \dim\mathfrak{G}\backslash\Gamma]$ whose inverse image under the canonical map $\Gamma \times X \to {}_\Gamma X$ is equal to the inverse image of A under the canonical map $\Gamma \times X \to X$.

If $A, B \in \mathcal{M}X[j]$ are $\mathfrak{G}$-equivariant, then

(a) $\quad \mathrm{Hom}_{\mathcal{D}_{\mathfrak{G}}X}(A,B) = \mathrm{Hom}_{\mathcal{D}X}(A,B).$

Indeed, in the setup of 1.15(a),(b), we have

$$\mathrm{Hom}_{\mathcal{D}_{\mathfrak{G}}X}(A,B) = H^0_{\mathfrak{G}}(X, D(DB\otimes A)) = H^0({}_\Gamma X, D(D({}_\Gamma A)\otimes {}_\Gamma A)))$$
$$= \mathrm{Hom}_{\mathcal{D}({}_\Gamma X)}({}_\Gamma A, {}_\Gamma B) = \mathrm{Hom}_{\mathcal{D}X}(A,B)$$

(the last equality holds by [BBD, 4.2.5]; here the connectedness of $\mathfrak{G}$ is used). On the other hand, we have

(b) $\quad \mathrm{Hom}^n_{\mathcal{D}_{\mathfrak{G}}X}(A,B) = 0$ if $n < 0$.

Indeed, again in the setup of 1.15(a),(b) we have

$$\mathrm{Hom}^n_{\mathcal{D}_{\mathfrak{G}}X}(A,B) = \mathrm{Hom}_{\mathcal{D}_{\mathfrak{G}}X}(A, B[n]) = H^0_{\mathfrak{G}}(X, D(D(B[n]\otimes A)))$$
$$= H^0({}_\Gamma X, D(D({}_\Gamma B[n])\otimes {}_\Gamma A))) = \mathrm{Hom}_{\mathcal{D}({}_\Gamma X)}({}_\Gamma A, {}_\Gamma B[n])$$
$$= \mathrm{Hom}_{\mathcal{D}({}_\Gamma X)}({}_\Gamma A[-j+\dim G\backslash\Gamma], {}_\Gamma B[-j+\dim G\backslash\Gamma][n]) = 0$$

since ${}_\Gamma A[-j+\dim \mathfrak{G}\backslash\Gamma], {}_\Gamma B[-j+\dim\mathfrak{G}\backslash\Gamma]$ are perverse sheaves on ${}_\Gamma X$ and the perverse sheaves on ${}_\Gamma X$ form an abelian subcategory of $\mathcal{D}({}_\Gamma X)$ (see [BBD]).

1.17. Let $\mathcal{L}$ be a $\mathfrak{G}$-equivariant local system on X. We regard $\mathcal{L}$ as an object $({}_\Gamma\mathcal{L},\dots)$ of $\mathcal{D}_{\mathfrak{G}}X$ where ${}_\Gamma\mathcal{L}$ is the local system on ${}_\Gamma X$ defined as in [L3, 1.1]. (Here, ${}_\Gamma\mathcal{L}$ is considered as an object of $\mathcal{D}({}_\Gamma X)$ as in 1.1.) Hence $H^j_{\mathfrak{G}}(X,\mathcal{L})$ is well defined for any $j \in \mathbf{Z}$ (*equivariant cohomology*). We set

$$H^{\mathfrak{G}}_j(X,\mathcal{L}) = H^{j-2\dim X}_{\mathfrak{G}}(X, D(\mathcal{L}^*))$$

(*equivariant homology*). (If X is empty, then $\dim X$ is not defined and we interpret the right hand side of the previous equality as zero.) We write

$$H^{\mathfrak{G}}_\cdot(X,\mathcal{L}) = \oplus_{j\in\mathbf{Z}} H^{\mathfrak{G}}_j(X,\mathcal{L}).$$

If $\Gamma \in Sf_m(\mathfrak{G})$ and m is large enough (depending on $\mathcal{L}, j$), we have

(a) $\quad H^j_{\mathfrak{G}}(X,\mathcal{L}) = H^j({}_\Gamma X, {}_\Gamma\mathcal{L}),$

(b) $\quad H^{\mathfrak{G}}_j(X,\mathcal{L}) = \mathrm{Hom}(H_c^{2\dim {}_\Gamma X - j}({}_\Gamma X, {}_\Gamma\mathcal{L}^*), \mathbf{C});$

(a) is obtained directly from 1.14(a); (b) is obtained using the following string of equalities

$$H^{\mathfrak{G}}_j(X,\mathcal{L}) = H^{j-2\dim X}_{\mathfrak{G}}(X, D(\mathcal{L}^*)) = H^{j-2\dim X}({}_\Gamma X, {}_\Gamma(D(\mathcal{L}^*)))$$
$$= H^{j-2\dim X - 2\dim\mathfrak{G}\backslash\Gamma}({}_\Gamma X, D({}_\Gamma\mathcal{L}^*))$$
$$= \mathrm{Hom}(H_c^{-j+2\dim X+2\dim\mathfrak{G}\backslash\Gamma}({}_\Gamma X, {}_\Gamma\mathcal{L}^*), \mathbf{C}).$$

(a) shows that the definition of equivariant cohomology given above coincides with the one of Borel [B]; (b) shows that the definition of equivariant homology given above coincides with the one in [L3, 1.1].

1.18. Let X, Y be $\mathfrak{G}$-varieties and let $A \in \mathcal{D}_{\mathfrak{G}}X, B \in \mathcal{D}_{\mathfrak{G}}Y$. We define a bilinear form

(a) $\quad \oplus_{j+j'=k} H^j_{\mathfrak{G}}(X,A)\times H^{j'}_{\mathfrak{G}}(Y,B) \to H^k_{\mathfrak{G}}(X\times Y, A\boxtimes B).$

It is obtained by composing the Künneth isomorphism

$$\oplus_{j+j'=k} H^j({}_\Gamma X, {}_\Gamma A)\otimes H^{j'}({}_\Gamma X, {}_\Gamma B) \xrightarrow{\sim} H^k({}_\Gamma X\times {}_\Gamma Y, {}_\Gamma A\boxtimes {}_\Gamma B)$$

with the map

$$H^k({}_\Gamma X\times {}_\Gamma Y, {}_\Gamma A\boxtimes {}_\Gamma B) \to H^k({}_\Gamma(X\times Y), {}_\Gamma(A\boxtimes B))$$

(inverse image under the obvious map ${}_\Gamma(X\times Y)\to {}_\Gamma X\times {}_\Gamma Y$), for $\Gamma\in Sf_m(\mathfrak{G})$ with large enough m. Similarly, for $A, B\in\mathcal{D}_{\mathfrak{G}}X$, we define

(b) $\quad \oplus_{j+j'=k} H^j_{\mathfrak{G}}(X,A)\times H^{j'}_{\mathfrak{G}}(X,B)\to H^k_{\mathfrak{G}}(X, A\otimes B).$

It is given by the cup product $H^j({}_\Gamma X, {}_\Gamma A)\times H^{j'}({}_\Gamma X, {}_\Gamma B)\to H^{j+j'}({}_\Gamma X, {}_\Gamma A\otimes {}_\Gamma B)$ (see 1.7) for $\Gamma\in Sf_m(\mathfrak{G})$ with large enough m. For $A = \mathbf{C}$, (b) becomes

(c) $\quad H^j_{\mathfrak{G}}(X,\mathbf{C})\otimes H^{j'}_{\mathfrak{G}}(X,B)\to H^{j+j'}_{\mathfrak{G}}(X,B).$

Applying this in the setup of 1.17 with $B=\mathcal{L}$ or $B = D(\mathcal{L}^*)$, we obtain natural linear maps

(d) $\quad H^j_{\mathfrak{G}}(X,\mathbf{C})\otimes H^{j'}_{\mathfrak{G}}(X,\mathcal{L})\to H^{j+j'}_{\mathfrak{G}}(X,\mathcal{L}),$

(e) $\quad H^j_{\mathfrak{G}}(X,\mathbf{C})\otimes H^{\mathfrak{G}}_{j'}(X,\mathcal{L})\to H^{\mathfrak{G}}_{j+j'}(X,\mathcal{L}),$

which coincide with the maps in [L3, 1.3(a),(b)]. In particular, (d) makes $H^\cdot_{\mathfrak{G}}(X,\mathbf{C})$ into a graded $\mathbf{C}$-algebra and $H^\cdot_{\mathfrak{G}}(X,\mathcal{L}), H^{\mathfrak{G}}_\cdot(X,\mathcal{L})$ into graded $H^\cdot_{\mathfrak{G}}(X,\mathbf{C})$-modules.

For $A, B\in\mathcal{D}_{\mathfrak{G}}X$, we have

(f) $\quad \mathrm{Hom}^\cdot_{\mathcal{D}_{\mathfrak{G}}X}(A,B) = H^\cdot_{\mathfrak{G}}(X, D(DB\otimes A))$

(this is deduced from 1.15(a) by shift). Taking here $A = B = \mathbf{C}_X$, we deduce

(g) $\quad \mathrm{Hom}^\cdot_{\mathcal{D}_{\mathfrak{G}}X}(\mathbf{C}_X,\mathbf{C}_X) = H^\cdot_{\mathfrak{G}}(X,\mathbf{C}),$

which respects the algebra structures. For general A, B, we note that $H^\cdot_{\mathfrak{G}}(X, D(DB\otimes A))$ is a $H^\cdot_{\mathfrak{G}}(X,\mathbf{C})$-module (by (c) above) while $\mathrm{Hom}^\cdot_{\mathcal{D}_{\mathfrak{G}}X}(A,B)$ is naturally a $\mathrm{Hom}^\cdot_{\mathcal{D}_{\mathfrak{G}}X}(\mathbf{C}_X,\mathbf{C}_X)$-module. (The product of $\phi:\mathbf{C}_X\to\mathbf{C}_X[n]$ with $\psi: A\to B[m]$ is $\phi\otimes\psi: A\to B[m+n]$ where we use $A = \mathbf{C}_X\otimes A, B = \mathbf{C}_X\otimes B$.) These two module structures on the two sides of (f) are compatible with each other under (f) and (g).

1.19. $\mathfrak{G}$-equivariant vector bundles. Let $p: E\to X$ be a $\mathfrak{G}$-equivariant n-dimensional vector bundle with zero section $i: X\to E$. Let $A\in\mathcal{D}_{\mathfrak{G}}X$. For any $\Gamma\in Sf(\mathfrak{G})$, p induces an n-dimensional vector bundle ${}_\Gamma E\to {}_\Gamma X$, to which we associate an element $\kappa_{{}_\Gamma E}\in\mathrm{Hom}^{2n}_{\mathcal{D}({}_\Gamma X)}({}_\Gamma A, {}_\Gamma A)$ as in 1.8. The collection of these elements (for various $\Gamma\in Sf(\mathfrak{G})$) may be regarded as an element

$$\kappa_E\in\mathrm{Hom}^{2n}_{\mathcal{D}_{\mathfrak{G}}X}(A,A),$$

or equivalently as a morphism $A[-2n]\to A$ in $\mathcal{D}_{\mathfrak{G}}X$. This induces for any j, a linear map $H^{j-2n}_{\mathfrak{G}}(X,A)\to H^j_{\mathfrak{G}}(X,A)$. From the definitions we see that this map is multiplication by a well-defined element $c(E)\in H^{2n}_{\mathfrak{G}}(X,\mathbf{C})$, which is independent of A.

1.20. Functoriality. Let $f: X\to Y$ be a $\mathfrak{G}$-morphism and let $B\in\mathcal{D}_{\mathfrak{G}}Y$. We have canonical maps (in (b)-(d), we asume that $B = \mathcal{L}$ is a local system):

(a) $f^*: H^j_{\mathfrak{G}}(Y,B)\to H^j_{\mathfrak{G}}(X, f^*B)$;

(b) $f_!: H^{\mathfrak{G}}_{j+2\dim X-2\dim Y}(X, f^*\mathcal{L})\to H^{\mathfrak{G}}_j(Y,\mathcal{L})$, if f is proper;

(c) $f^*: H_j^{\mathfrak{G}}(Y,\mathcal{L}) \to H_j^{\mathfrak{G}}(X, f^*\mathcal{L})$, if f is a locally trivial fibration with smooth fibres of pure dimension;

(d) $f^*: H_j^{\mathfrak{G}}(Y,\mathcal{L}) \to H^{\mathfrak{G}}_{j+2\dim X-\dim Y}(X, f^*\mathcal{L})$, if f is an open imbedding.

In each of the four cases above, the map is induced by the adjunction morphism $1 \to f_*f^*$ as follows:

(a) $$H^j_{\mathfrak{G}}(Y,B) \to H^j_{\mathfrak{G}}(Y, f_*f^*B) = H^j_{\mathfrak{G}}(X, f^*B);$$

(b) $$\begin{aligned} &H_{\mathfrak{G}}^{j-2\dim Y}(X, Df^*\mathcal{L}^*) = H_{\mathfrak{G}}^{j-2\dim Y}(Y, f_*Df^*\mathcal{L}^*) \\ &= H_{\mathfrak{G}}^{j-2\dim Y}(Y, Df_*f^*\mathcal{L}^*) \to H_{\mathfrak{G}}^{j-2\dim Y}(Y, D\mathcal{L}^*); \end{aligned}$$

(c) $$\begin{aligned} &H_{\mathfrak{G}}^{j-2\dim Y}(Y, D\mathcal{L}^*) \to H_{\mathfrak{G}}^{j-2\dim Y}(Y, f_*f^*D\mathcal{L}^*) \\ &= H_{\mathfrak{G}}^{j-2\dim X}(Y, f_*f^!D\mathcal{L}^*) \\ &= H_{\mathfrak{G}}^{j-2\dim X}(Y, f_*Df^*\mathcal{L}^*) = H_{\mathfrak{G}}^{j-2\dim X}(X, Df^*\mathcal{L}^*); \end{aligned}$$

(d) $$\begin{aligned} &H_{\mathfrak{G}}^{j-2\dim Y}(Y, D\mathcal{L}^*) \to H_{\mathfrak{G}}^{j-2\dim Y}(Y, f_*f^*D\mathcal{L}^*) \\ &= H_{\mathfrak{G}}^{j-2\dim Y}(Y, f_*f^!D\mathcal{L}^*) \\ &= H_{\mathfrak{G}}^{j-2\dim Y}(Y, f_*Df^*\mathcal{L}^*) = H_{\mathfrak{G}}^{j-2\dim Y}(X, Df^*\mathcal{L}^*). \end{aligned}$$

(The maps (a),(b),(c),(d) are the same as those in [**L3, 1.4**], except that the map (c) is defined in [**L3, 1.4**] without assuming that the fibres of f are smooth.)

We shall write $H^j_{\mathfrak{G}}, H^{\cdot}_{\mathfrak{G}}$ instead of $H^j_{\mathfrak{G}}(pt,\mathbf{C}), H^{\cdot}_{\mathfrak{G}}(pt,\mathbf{C})$, where the point is regarded as a $\mathfrak{G}$-variety in the obvious way. As we have seen above, $H^{\cdot}_{\mathfrak{G}}$ is a graded algebra. The obvious map $p: X \to pt$ induces an algebra homomorphism $p^*: H^{\cdot}_{\mathfrak{G}} \to H^j_{\mathfrak{G}}(X,\mathbf{C})$. Hence, in the setup of 1.18, the graded $H^{\cdot}_{\mathfrak{G}}(X,\mathbf{C})$-modules $H^{\cdot}_{\mathfrak{G}}(X,\mathcal{L}), H^{\mathfrak{G}}_{\cdot}(X,\mathcal{L})$ may be regarded as graded $H^{\cdot}_{\mathfrak{G}}$-modules.

In the same way, for $A, B \in \mathcal{D}_{\mathfrak{G}}X$, the $H^{\cdot}_{\mathfrak{G}}(X,\mathbf{C})$-module $H^{\cdot}_{\mathfrak{G}}(X, D(DB\otimes A))$ may be regarded as a $H^{\cdot}_{\mathfrak{G}}$-module. Using the identification 1.18(f) and

(e) $$\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}pt}(\mathbf{C},\mathbf{C}) = H^{\cdot}_{\mathfrak{G}}$$

(the special case of 1.18(g) for $X = pt$), we can interpret this as a $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}pt}(\mathbf{C},\mathbf{C})$-module structure on $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(A,B)$. This can be also obtained from the $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(\mathbf{C}_X,\mathbf{C}_X)$-module structure on $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(A,B)$, described at the end of 1.18, via the algebra homomorphism $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}pt}(\mathbf{C},\mathbf{C}) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(\mathbf{C}_X,\mathbf{C}_X)$ induced by the inverse image map under $X \to pt$.

1.21. Trivial action of $\mathfrak{G}$. Let $\mathcal{L}$ be a $\mathfrak{G}$-equivariant local system on X. Assume that $\mathfrak{G}$ is connected and that $\mathfrak{G}$ acts trivially on X. We have a canonical isomorphism (as $H^{\cdot}_{\mathfrak{G}}$-modules):

(a) $$H^{\mathfrak{G}}_{\cdot}(X,\mathcal{L}) \xrightarrow{\sim} H^{\cdot}_{\mathfrak{G}} \otimes_{\mathbf{C}} H^{\{1\}}_{\cdot}(X,\mathcal{L}).$$

Indeed, let $j \in \mathbf{Z}$ and let $\Gamma \in Sf_m(\mathfrak{G})$ where m is large. We have ${}_{\Gamma}X = \mathfrak{G}\backslash(\Gamma \times X) = (\mathfrak{G}\backslash\Gamma) \times X$; here we use our assumption that $\mathfrak{G}$ acts trivially on X. Since $\mathfrak{G}$ is connected, the local system ${}_{\Gamma}\mathcal{L}^*$ on ${}_{\Gamma}X$ is necessarily the inverse image of $\mathcal{L}^*$ under the second projection $(\mathfrak{G}\backslash\Gamma) \times X \to X$. Using this and the Künneth formula, we see that

$$\begin{aligned} &H_c^{2\dim {}_{\Gamma}X-j}({}_{\Gamma}X, {}_{\Gamma}\mathcal{L}^*) = H_c^{2\dim {}_{\Gamma}X-j}((\mathfrak{G}\backslash\Gamma)\times X, {}_{\Gamma}\mathcal{L}^*) \\ &= \oplus_{j'+j''=j} H_c^{2\dim {}_{\Gamma}pt-j'}((\mathfrak{G}\backslash\Gamma),\mathbf{C}) \otimes_{\mathbf{C}} H_c^{2\dim X-j''}(X,\mathcal{L}^*). \end{aligned}$$

Taking dual spaces in both sides we obtain

$$H_j^{\mathfrak{G}}(X,\mathcal{L}) = \oplus_{j'+j''=j} H_{j'}^{\mathfrak{G}}(pt,\mathbf{C}) \otimes H_{j''}^{\{1\}}(X,\mathcal{L}) = \oplus_{j'+j''=j} H^{j'}_{\mathfrak{G}} \otimes H_{j''}^{\{1\}}(X,\mathcal{L}),$$

where we have used the canonical isomorphism $H^{j'}_{\mathfrak{G}} \xrightarrow{\sim} H^{\mathfrak{G}}_{j'}(pt,\mathbf{C})$, see [**L3, 1.8(c)**]. Our claim follows.

1.22. A number of results in 1.1-1.8 have obvious analogues in the $\mathfrak{G}$-equivariant context. We will sometimes refer to these results in the equivariant case without further comment.

2. Convolution

2.0. Classically, convolution was a way to organize functions on a certain space as an associative algebra, using an integral. Gradually, functions were replaced by more geometric objects: constructible sheaves (several authors in the early 1980's), coherent sheaves, or classes in K-theory (Kashiwara-Tanisaki), cohomology classes (Ginzburg); in these cases, the integral was replaced by a direct image. The results in this section are extensions of results stated in [G],[CG]: we have weaker smoothness assumptions and we work in an equivariant context.

2.1. Let X be a $\mathfrak{G}$-variety and let $B \in \mathcal{D}_{\mathfrak{G}}X$. We have a cartesian diagram

$$\begin{array}{ccc} X^2 & \xleftarrow{i_0} & X \\ {\scriptstyle i_0^2}\downarrow & & \downarrow{\scriptstyle i_1} \\ X^4 & \xleftarrow{b_0} & X^3 \end{array}$$

where $i_0: X \to X^2 = X \times X, i_1: X \to X^3 = X \times X \times X$ are the diagonal imbeddings and $i_0^2(x,y) = (x,x,y,y)$, $b_0(x,y,z) = (x,y,y,z)$. We define a map

$$H^{\cdot}_{\mathfrak{G}}(X, D(DB\otimes B)) \times H^{\cdot}_{\mathfrak{G}}(X, D(DB\otimes B)) \to H^{\cdot}_{\mathfrak{G}}(X, D(DB\otimes B))$$

as a composition

$$\begin{aligned} &H^{\cdot}_{\mathfrak{G}}(X, D(DB\otimes B)) \times H^{\cdot}_{\mathfrak{G}}(X, D(DB\otimes B)) \\ &\to H^{\cdot}_{\mathfrak{G}}(X^2, D(DB\otimes B)\boxtimes D(DB\otimes B)) \\ &= H^{\cdot}_{\mathfrak{G}}(X^2, i_0^{2!}(B\boxtimes DB\boxtimes B\boxtimes DB)) \to H^{\cdot}_{\mathfrak{G}}(X^2, i_0^{2!}b_{0*}b_0^*(B\boxtimes DB\boxtimes B\boxtimes DB)) \\ &= H^{\cdot}_{\mathfrak{G}}(X^2, i_{0*}i_1^!(B\boxtimes(DB\otimes B)\boxtimes DB)) \to H^{\cdot}_{\mathfrak{G}}(X^2, i_{0*}i_1^!(B\boxtimes DC\boxtimes DB)) \\ &= H^{\cdot}_{\mathfrak{G}}(X^2, i_{0*}i_0^!(B\boxtimes DB)) = H^{\cdot}_{\mathfrak{G}}(X^2, i_{0*}D(DB\otimes B)) = H^{\cdot}_{\mathfrak{G}}(X, D(DB\otimes B)), \end{aligned}$$

where the first arrow is as in 1.18(a), the second arrow is induced by the morphism $1 \to b_{0*}b_0^*$ (see 1.3(b)) and the third arrow is induced by the canonical map $DB\otimes B \to DC$ (see 1.13(a)).

LEMMA 2.2. *Consider the isomorphism* $Hom^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B) \xrightarrow{\sim} H^{\cdot}_{\mathfrak{G}}(X, D(DB \otimes B))$ *(a special case of 1.18(f)). Under this isomorphism, the natural algebra structure on* $Hom^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B)$ *(see 1.10) corresponds to the algebra structure on* $H^{\cdot}_{\mathfrak{G}}(X, D(DB \otimes B))$ *defined by the pairing in 2.1.*

The proof is left to the reader.

2.3. Let $\dot{X} \xrightarrow{f} X$ be a proper morphism of $\mathfrak{G}$-varieties. We have a commutative diagram

(a)
$$\begin{array}{ccccc} \ddot{X}^2 & \xleftarrow{a} & \dddot{X} & \xrightarrow{s} & \ddot{X} \\ {\scriptstyle i^2}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle i}\downarrow \\ \dot{X}^4 & \xleftarrow{b} & \dot{X}^3 & \xrightarrow{c} & \dot{X}^2 \end{array}$$

where
$\ddot{X} = \dot{X} \times_X \dot{X} = \{(x,y) \in \dot{X}^2 | f(x) = f(y)\}$,
$\dddot{X} = \dot{X} \times_X \dot{X} \times_X \dot{X} = \{(x,y,z) \in \dot{X}^3 | f(x) = f(y) = f(z)\}$,
i, r are the obvious imbeddings, $i^2(x,y) = (x,x,y,y)$ and
$a(x,y,z) = ((x,y),(y,z)), b(x,y,z) = (x,y,y,z)$,
$s(x,y,z) = (x,z), c(x,y,z) = (x,z)$.
The left square in (a) is cartesian. Let $A \in \mathcal{D}_{\mathfrak{G}}\dot{X}$. Then $DA \in \mathcal{D}_{\mathfrak{G}}\dot{X}$ is well defined and we may form
$A \boxtimes DA \in \mathcal{D}_{\mathfrak{G}}\dot{X}^2$, $A \boxtimes (DA \otimes A) \boxtimes DA \in \mathcal{D}_{\mathfrak{G}}\dot{X}^3$, $A \boxtimes DC \boxtimes DA \in \mathcal{D}_{\mathfrak{G}}\dot{X}^3$,
$A \boxtimes DA \boxtimes A \boxtimes DA \in \mathcal{D}_{\mathfrak{G}}\dot{X}^4$.
We define a map
(b) $\quad H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA)) \times H^{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA)) \to H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA))$
as the composition

$$H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA)) \times H^{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA)) \xrightarrow{p_1} H^{\cdot}_{\mathfrak{G}}(\dddot{X}, r^!(A \boxtimes (DA \otimes A) \boxtimes DA))$$
$$\xrightarrow{p_2} H^{\cdot}_{\mathfrak{G}}(\dddot{X}, r^!(A \boxtimes DC \boxtimes DA)) \xrightarrow{p_3} H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA))$$

where p_1 is the composition

$$H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA)) \times H^{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA)) \to H^{\cdot}_{\mathfrak{G}}(\ddot{X}^2, i^{2!}(A \boxtimes DA \boxtimes A \boxtimes DA))$$
$$\to H^{\cdot}_{\mathfrak{G}}(\ddot{X}^2, i^{2!} b_* b^* (A \boxtimes DA \boxtimes A \boxtimes DA)) = H^{\cdot}_{\mathfrak{G}}(\ddot{X}^2, a_* r^!(A \boxtimes (DA \otimes A) \boxtimes DA))$$
$$= H^{\cdot}_{\mathfrak{G}}(\dddot{X}, r^!(A \boxtimes (DA \otimes A) \boxtimes DA))$$

(the morphism in the first line is as in 1.18(a); the morphism in the second line is induced by $1 \to b_* b^*$ as in 1.3(b)); p_2 is induced by $DA \otimes A \to DC$ as in 1.13(a); p_3 is the composition

$$H^{\cdot}_{\mathfrak{G}}(\dddot{X}, r^!(A \boxtimes DC \boxtimes DA)) = H^{\cdot}_{\mathfrak{G}}(\dddot{X}, s^! i^!(A \boxtimes DA)) = H^{\cdot}_{\mathfrak{G}}(\ddot{X}, s_! s^! i^!(A \boxtimes DA))$$
$$\to H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA));$$

(the last morphism is induced by $s_! s^! \to 1$ as in 1.3(b); we have used the equality $s_* = s_!$).

2.4. Let $B \in \mathcal{D}_{\mathfrak{G}}X$ be defined by $B = f_! A$. Since f is proper, we have $DB = f_! DA$. We have a cartesian diagram

$$\begin{array}{ccc} \ddot{X} & \xrightarrow{q} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle i_0}\downarrow \\ \dot{X}^2 & \xrightarrow{f \times f} & X^2 \end{array}$$

where $q(x,y) = f(x) = f(y)$. We define an isomorphism
(a) $\quad H^{\cdot}_{\mathfrak{G}}(X, D(DB \otimes B)) = H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA))$
as the composition

$$H^{\cdot}_{\mathfrak{G}}(X, D(DB \otimes B)) = H^{\cdot}_{\mathfrak{G}}(X, D(f_! DA \otimes f_! A))$$
$$= H^{\cdot}_{\mathfrak{G}}(X, i_0^!(f \times f)_*(A \boxtimes DA)) = H^{\cdot}_{\mathfrak{G}}(X, q_* i^!(A \boxtimes DA)) = H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA)).$$

LEMMA 2.5. *The algebra structure on* $H^{\cdot}_{\mathfrak{G}}(X, D(DB \otimes B))$ *defined in 2.1 corresponds, under the isomorphism 2.4(a), to the algebra structure on*

$$H^{\cdot}_{\mathfrak{G}}(\ddot{X}, i^!(A \boxtimes DA))$$

defined in 2.3(b).

The proof is based on the cartesian diagram

$$\begin{array}{ccc} \dot{X} \times \ddot{X} \times \dot{X} & \longrightarrow & \dot{X}^4 \\ \downarrow & & \downarrow \\ X^3 & \xrightarrow{b_0} & X^4 \end{array}$$

where the upper horizontal map is $(x,(y,z),u) \mapsto (x,y,z,u)$, the left vertical map is $(x,(y,z),u) \mapsto (f(x), q(y,z), f(u))$, and the right vertical map is
$$(x,y,z,u) \mapsto (f(x), f(y), f(z), f(u)).$$
We omit further details.

3. Study of a subvariety

3.1. In this section we assume that $\mathfrak{G}$ is a connected algebraic group and that $\mathfrak{H}$ is a closed connected subgroup of $\mathfrak{G}$.

If V is an $\mathfrak{G}$-variety and $K \in \mathcal{D}_{\mathfrak{G}}V$, we can consider the shift $K[n] \in \mathcal{D}_{\mathfrak{G}}V$ not only for an integer n, but more generally, for a function $n : V \to Z$ that is constant on the connected components of V; namely, we apply on each component the usual shift by the integer equal to the value of n on that component. (The category $\mathcal{D}_{\mathfrak{G}}V$ decomposes naturally into direct sum of analogous categories for the various connected components of V.)

3.2. Let Z be a smooth $\mathfrak{H}$-variety and let $\bar{Z}$ be a smooth, $\mathfrak{H}$-stable, closed subvariety of Z. Let $j' : \bar{Z} \to Z$ be the inclusion. Let $\mathcal{L}$ be a $\mathfrak{H}$-equivariant local system on Z and let $\bar{\mathcal{L}} = j'^* \mathcal{L}$. Let $d : Z \to \mathbf{Z}$ (resp. $\bar{d} : \bar{Z} \to \mathbf{Z}$) be the function which attaches to a point of Z (resp. $\bar{Z}$) the dimension of the connected component of Z (resp. $\bar{Z}$) containing that point. The restriction of d to $\bar{Z}$ is denoted again by d. Let $\delta = d - \bar{d} : \bar{Z} \to \mathbf{Z}$.

We denote by E the normal bundle of $\bar{Z}$ in Z. This is a $\mathfrak{H}$-equivariant vector bundle on $\bar{Z}$ whose fibre dimension is given by δ. Applying the construction in 1.19 separately on each connected component of $\bar{Z}$, we obtain a morphism $\kappa_E : \bar{\mathcal{L}}^*[-2\delta] \to \bar{\mathcal{L}}^*$ in $\mathcal{D}_{\mathfrak{H}}\bar{Z}$. Note that $j'^!\mathcal{L} = \bar{\mathcal{L}}[-2\delta], j'^!\mathcal{L}^* = \bar{\mathcal{L}}^*[-2\delta]$. We define a diagram in $\mathcal{D}_{\mathfrak{H}}Z$:

(a)
$$\begin{tikzcd} j'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta]) \arrow[r] \arrow[d] & \mathcal{L} \otimes \mathcal{L}^* \arrow[d] \\ j'_!\mathbf{C}[-2\delta] \arrow[r] & \mathbf{C} \end{tikzcd}$$

as follows. The upper horizontal arrow is
$$j'_!(\bar{\mathcal{L}}[-2\delta]\otimes\bar{\mathcal{L}}^*[-2\delta]) = j'_!\bar{\mathcal{L}}[-2\delta]\otimes j'_!\bar{\mathcal{L}}^*[-2\delta] = j'_!j'^!\mathcal{L}\otimes j'_!j'^!\mathcal{L}^* \to \mathcal{L}\otimes\mathcal{L}^*,$$
(the last map is obtained by using twice $j'_!j'^! \to 1$); the lower horizontal arrow is induced by $j'_!j'^! \to 1$, the left vertical arrow is obtained by applying $j'_!$ to the composition
$$\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta] \xrightarrow{1\otimes\kappa_E} \bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^* \to \mathbf{C}[-2\delta]$$
(the last map being the natural pairing) and the right vertical arrow is again the natural pairing.

LEMMA 3.3. *The diagram 3.2(a) is commutative.*

We define a diagram

(a)
$$\begin{tikzcd} j'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta]) \arrow[r, "a"] \arrow[d] & j'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*) \arrow[r] \arrow[d, "b"] & j'_!\mathbf{C}[-2\delta] \arrow[d] \\ \mathcal{L} \otimes \mathcal{L}^* \arrow[r, "1"] & \mathcal{L} \otimes \mathcal{L}^* \arrow[r] & \mathbf{C} \end{tikzcd}$$

All maps other than a, b are as above. By definition, $a = j'_!(1 \otimes \kappa_E)$ and b is induced by $j'_!j'^! \to 1$ (as in 1.3(b)). (Note that $\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^* = j'^!(\mathcal{L} \otimes \mathcal{L}^*)$.)

It is enough to check that the diagram (a) is commutative. The right square of (a) is clearly commutative. It remains to check that the left square of (a) is commutative. By a standard argument we may assume that j' is the zero section of a vector bundle $p : Z \to \bar{Z}$. In addition, we may assume that $\bar{Z}$ is connected. We may identify $\mathcal{L}^* = p^*\bar{\mathcal{L}}^*$. Let $T = \{(z, z') \in Z \times Z | p(z) = p(z')\}$. We may regard T as a vector bundle over $\bar{Z}$ under $p' : (z, z') \mapsto p(z) = p(z')$. Let $\mathcal{F} = p'^*(\bar{\mathcal{L}} \otimes \bar{\mathcal{L}}^*)$, a local system on T. We have a cartesian diagram

(b)
$$\begin{tikzcd} \bar{Z} \arrow[r, "j'"] \arrow[d, "j'"'] & Z \arrow[d, "g"] \\ Z \arrow[r, "h"] & T \end{tikzcd}$$

where $g(z) = (z, j'p(z))$, $h(z) = (j'p(z), z)$. Let $g' = g \circ j' : \bar{Z} \to T$. The map $g'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta]) \to \mathcal{F}$ induced by $g'_!g'^! \to 1$ is equal to the composition

$$g'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta]) = g_!j'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta]) \to g_!(\mathcal{L} \otimes \mathcal{L}^*[-2\delta]) \to \mathcal{F}$$
where the first arrow is induced by $j'_!j'^! \to 1$ and the second arrow is induced by $g_!g^! \to 1$. Applying h^*, we see that the map $h^*g'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta]) \to h^*\mathcal{F}$ is equal to the composition
$$h^*g'_!(\bar{\mathcal{L}}[-2\delta]\otimes\bar{\mathcal{L}}^*[-2\delta]) = h^*g_!j'_!(\bar{\mathcal{L}}[-2\delta]\otimes\bar{\mathcal{L}}^*[-2\delta]) \to h^*g_!(\mathcal{L}\otimes\mathcal{L}^*[-2\delta]) \to h^*\mathcal{F}.$$

Using the cartesian diagram (b), we have
$$h^*g'_!(\bar{\mathcal{L}}[-2\delta] \otimes \bar{\mathcal{L}}^*[-2\delta]) = j'_!\bar{\mathcal{L}} \otimes j'_!\bar{\mathcal{L}}^*, \quad h^*\mathcal{F} = \mathcal{L} \otimes \mathcal{L}^*,$$
$$h^*g_!(\mathcal{L} \otimes \mathcal{L}^*) = j'_!j'^*(\mathcal{L} \otimes \mathcal{L}^*) = j'_!(\bar{\mathcal{L}} \otimes \bar{\mathcal{L}}^*).$$
The commutativity of the left square in the diagram (a) follows. The lemma is proved.

3.4. We now assume that we are given a commutative diagram of $\mathfrak{H}$-varieties
$$\begin{tikzcd} \dot{\bar{X}} \arrow[r, "\dot{j}"] \arrow[d, "\bar{f}"'] & \dot{X} \arrow[d, "f"] \\ \bar{X} \arrow[r, "j"] & X \end{tikzcd}$$
where

(a) $f, \bar{f}$ are proper and $j, \dot{j}$ are closed imbeddings.

Moreover, we assume that we are given an open dense smooth $\mathfrak{H}$-subvariety Z of $\dot{X}$ such that

(b) $\bar{Z} := \dot{\bar{X}} \cap Z$ is an open dense smooth subvariety of $\dot{\bar{X}}$ and a submanifold of Z;

(c) $\dot{X}, Z$ are connected of dimension d;

(d) the connected components of $\dot{\bar{X}}$ are the same as the irreducible components of $\dot{\bar{X}}$ (hence they are exactly the closures of the connected components of $\bar{Z}$).

Let $j' : \bar{Z} \to Z$ be the restriction of $\dot{j}$, and let $u : Z \to \dot{X}, \bar{u} : \bar{Z} \to \dot{\bar{X}}$ be the inclusions. We assume, furthermore, that we are given an $\mathfrak{H}$-equivariant local system $\mathcal{L}$ on Z such that, if $\bar{\mathcal{L}} = j'^*\mathcal{L}$ (a $\mathfrak{H}$-equivariant local system on $\bar{Z}$), we have

(e) $u_!\mathcal{L}^* = u_*\mathcal{L}^*$;

(f) $\bar{u}_!\bar{\mathcal{L}}^* = \bar{u}_*\bar{\mathcal{L}}^*$.

Let $\bar{d}$ be the function which attaches to each point of $\dot{\bar{X}}$ the dimension of the connected component of $\dot{\bar{X}}$ containing that point and let $\delta = d - \bar{d} : \dot{\bar{X}} \to \mathbf{Z}$. We set
$$A = u_!\mathcal{L}^* = u_*\mathcal{L}^* \in \mathcal{D}_{\mathfrak{H}}\dot{X}, \quad \bar{A} = \bar{u}_!\bar{\mathcal{L}}^* = \bar{u}_*\bar{\mathcal{L}}^* \in \mathcal{D}_{\mathfrak{H}}\dot{\bar{X}}, \quad \bar{A}^\dagger = \bar{A}[-2\delta].$$
Using the definitions and the fact that $Z, \bar{Z}$ are smooth, we see that
$$DA = A[2d], D\bar{A} = \bar{A}[2\bar{d}].$$
It follows that
$$\bar{A} = \dot{j}^*A, D(\bar{A}^\dagger) = \dot{j}^*(DA), D\bar{A} = \dot{j}^!DA, \bar{A}^\dagger = \dot{j}^!A.$$
As in 2.3, to $f : \dot{X} \to X$ we associate the commutative diagram 2.3(a). Replacing

$f : \dot{X} \hookrightarrow X$ by $\bar{f} : \bar{\dot{X}} \to \dot{X}$ we obtain from 2.3(a) a commutative diagram

$$\begin{tikzcd}
\bar{\dot{X}}^2 \arrow[d, "\bar{r}^2"'] & \ddot{\bar{X}} \arrow[l, "\bar{a}"'] \arrow[r, "\bar{s}"] \arrow[d, "\bar{r}"] & \bar{\dot{X}} \arrow[d, "\bar{i}"] \\
\bar{\dot{X}}^4 & \bar{\dot{X}}^3 \arrow[l, "\bar{b}"'] \arrow[r, "\bar{c}"] & \bar{\dot{X}}^2
\end{tikzcd}$$

We define a linear map

$$\Psi : H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \to H^{\cdot}_{\mathfrak{H}}(\dot{X}, i^!(A \boxtimes DA))$$

as the composition

$$H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) = H^{\cdot}_{\mathfrak{H}}(\dot{X}, \ddot{j}_* \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) = H^{\cdot}_{\mathfrak{H}}(\dot{X}, i^!(\dot{j} \times \dot{j})_*(\tilde{A}^\dagger \boxtimes D\tilde{A}))$$
$$= H^{\cdot}_{\mathfrak{H}}(\dot{X}, i^!(\dot{j} \times \dot{j})_!(\dot{j} \times \dot{j})^!(A \boxtimes DA)) \to H^{\cdot}_{\mathfrak{H}}(\dot{X}, i^!(A \boxtimes DA)).$$

(The last morphism is induced by $(\dot{j} \times \dot{j})_!(\dot{j} \times \dot{j})^! \to 1$, see 1.3(b); $\dot{j} \times \dot{j} : \bar{\dot{X}}^2 \to \dot{X}^2$ is induced by $\dot{j}$ and $\ddot{j} : \bar{\dot{X}} \to \dot{X}$ is the restriction of $\dot{j} \times \dot{j}$.)

As in 3.2, we denote by E the normal bundle of $\tilde{Z}$ in Z. Recall that we have a morphism $\kappa_E : \tilde{\mathcal{L}}^*[-2\delta] \to \tilde{\mathcal{L}}^*$ in $\mathcal{D}_{\mathfrak{H}}\tilde{Z}$. (We identify $\delta : \dot{X} \to \mathbf{Z}$ with its restriction to $\tilde{Z}$.) Applying $\tilde{u}_!$, we obtain a morphism $\tilde{u}_! \kappa_E : \tilde{A}^\dagger \to \tilde{A}$ in $\mathcal{D}_{\mathfrak{H}}\bar{\dot{X}}$. Tensoring with the identity map of $D\tilde{A}$, we obtain a morphism $\tilde{u}_!\kappa_E \boxtimes 1 : \tilde{A}^\dagger \boxtimes D\tilde{A} \to \tilde{A} \boxtimes D\tilde{A}$ in $\mathcal{D}_{\mathfrak{H}}\bar{\dot{X}}^2$. This induces a linear map

$$\Phi : H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \to H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A} \boxtimes D\tilde{A})).$$

PROPOSITION 3.5. *(a) The following diagram is commutative:*

$$\begin{tikzcd}
H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \times H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A} \boxtimes D\tilde{A})) \arrow[r, "\bar{\alpha}"] & H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \arrow[dd, "\Psi"] \\
H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \times H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \arrow[u, "1\times\Phi"] \arrow[d, "\Psi\times\Psi"] & \\
H^{\cdot}_{\mathfrak{H}}(\dot{X}, i^!(A \boxtimes DA)) \times H^{\cdot}_{\mathfrak{H}}(\dot{X}, i^!(A \boxtimes DA)) \arrow[r, "\alpha"] & H^{\cdot}_{\mathfrak{H}}(\dot{X}, i^!(A \boxtimes DA))
\end{tikzcd}$$

where α is as in 2.3(b), and $\bar{\alpha}$ is defined similarly, in terms of $\bar{f} : \bar{\dot{X}} \to \dot{X}$ and $\tilde{A}$ instead of $f : \dot{X} \to X$ and A.

(b) The following diagram is commutative:

$$\begin{tikzcd}
H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \times H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A} \boxtimes D\tilde{A})) \arrow[r, "\bar{\alpha}"] \arrow[d, "\Phi\times 1"] & H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \arrow[d, "\Phi"] \\
H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A} \boxtimes D\tilde{A})) \times H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A} \boxtimes D\tilde{A})) \arrow[r, "\bar{\alpha}"] & H^{\cdot}_{\mathfrak{H}}(\bar{\dot{X}}, \bar{i}^!(\tilde{A} \boxtimes D\tilde{A}))
\end{tikzcd}$$

To simplify notation, throughout the proof we write H instead of $H^{\cdot}_{\mathfrak{H}}$. The proof of (b) is routine. We prove (a). We define

$$\Psi' : H(\ddot{\bar{X}}, \bar{r}^!(\tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}^\dagger) \boxtimes D\tilde{A})) \to H(\ddot{X}, r^!(A \boxtimes (DA \otimes A) \boxtimes DA))$$

as the composition

$$H(\ddot{\bar{X}}, \bar{r}^!(\tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}^\dagger) \boxtimes D\tilde{A})) = H(\ddot{X}, \dddot{j}_* \bar{r}^!(\tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}^\dagger) \boxtimes D\tilde{A}))$$
$$= H(\ddot{X}, r^!(\dot{j} \times \dot{j} \times \dot{j})_!(\tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}^\dagger) \boxtimes D\tilde{A}))$$
$$= H(\ddot{X}, r^!(\dot{j}_!\tilde{A}^\dagger \boxtimes (\dot{j}_! D\tilde{A} \otimes \dot{j}_!\tilde{A}^\dagger) \boxtimes \dot{j}_! D\tilde{A}))$$
$$= H(\ddot{X}, r^!(\dot{j}_!\dot{j}^! A \boxtimes (\dot{j}_!\dot{j}^! DA \otimes \dot{j}_!\dot{j}^! A) \boxtimes \dot{j}_!\dot{j}^! DA))$$
$$\to H(\ddot{X}, r^!(A \boxtimes (DA \otimes A) \boxtimes DA))$$

and

$$\Psi'' : H(\ddot{\bar{X}}, \bar{r}^!(\tilde{A}^\dagger \boxtimes DC \boxtimes D\tilde{A})) \to H(\ddot{X}, r^!(A \boxtimes DC \boxtimes DA))$$

as the composition

$$H(\ddot{\bar{X}}, \bar{r}^!(\tilde{A}^\dagger \boxtimes DC \boxtimes D\tilde{A})) = H(\ddot{X}, \dddot{j}_* \bar{r}^!(\tilde{A}^\dagger \boxtimes DC \boxtimes D\tilde{A}))$$
$$= H(\ddot{X}, r^!(\dot{j} \times \dot{j} \times \dot{j})_!(\tilde{A}^\dagger \boxtimes DC \boxtimes D\tilde{A})) = H(\ddot{X}, r^!(\dot{j}_!\tilde{A}^\dagger \boxtimes \dot{j}_! DC \boxtimes \dot{j}_! D\tilde{A}))$$
$$= H(\ddot{X}, r^!(\dot{j}_!\dot{j}^! A \boxtimes \dot{j}_!\dot{j}^! DC \boxtimes \dot{j}_!\dot{j}^! DA)) \to H(\ddot{X}, r^!(A \boxtimes DC \boxtimes DA))$$

(here we use the morphism $\dot{j}_!\dot{j}^! \to 1$, see 1.3(b); $\dot{j} \times \dot{j} \times \dot{j} : \bar{\dot{X}}^3 \to \dot{X}^3$ is induced by $\dot{j}$ and $\dddot{j} : \ddot{\bar{X}} \to \ddot{X}$ is the restriction of $\dot{j} \times \dot{j} \times \dot{j}$; furthermore, we use $\dot{j}^! DC = DC$). Let

$$\Phi' : H(\ddot{\bar{X}}, \bar{r}^!(\tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}^\dagger) \boxtimes D\tilde{A})) \to H(\ddot{\bar{X}}, \bar{r}^!(\tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}) \boxtimes D\tilde{A}))$$

be the linear map induced by

$1 \boxtimes (1 \otimes \tilde{u}_!\kappa_E) \boxtimes 1 : \tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}^\dagger) \boxtimes D\tilde{A} \to \tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A} \boxtimes D\tilde{A}$

(see 3.4). If p_1, p_2, p_3 are as in 2.3 and $\bar{p}_1, \bar{p}_2, \bar{p}_3$ are the analogous maps defined in terms of $\bar{f} : \bar{\dot{X}} \to X$ and $\tilde{A}$ instead of $f : \dot{X} \to X$ and A, we have $\bar{p}_1 \cdot (1 \times \Phi) = \Phi' \cdot \bar{p}_1$ as maps

$H(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \times H(\bar{\dot{X}}, \bar{i}^!(\tilde{A}^\dagger \boxtimes D\tilde{A})) \to H(\ddot{\bar{X}}, \bar{r}^!(\tilde{A}^\dagger \boxtimes (D\tilde{A} \otimes \tilde{A}) \boxtimes D\tilde{A}))$

and we see that to prove the commutativity of the diagram in (a), it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
H(\tilde{\ddot{X}},\tilde{\imath}^!(\tilde{A}^\dagger\boxtimes D\tilde{A}))\times H(\tilde{\ddot{X}},\tilde{\imath}^!(\tilde{A}^\dagger\boxtimes D\tilde{A})) \arrow[r, "\Psi\times\Psi"] \arrow[d, "\tilde{p}_1"] & H(\ddot{X},i^!(A\boxtimes DA))\times H(\ddot{X},i^!(A\boxtimes DA)) \arrow[d, "p_1"] \\
H(\tilde{\ddot{X}},\tilde{r}^!(\tilde{A}^\dagger\boxtimes(D\tilde{A}\otimes\tilde{A}^\dagger)\boxtimes D\tilde{A})) \arrow[r, "\Psi'"] \arrow[d, "\tilde{p}_2\cdot\Psi'"] & H(\ddot{X},r^!(A\boxtimes(DA\otimes A)\boxtimes DA)) \arrow[d, "p_2"] \\
H(\tilde{\ddot{X}},\tilde{r}^!(\tilde{A}^\dagger\boxtimes DC\boxtimes D\tilde{A})) \arrow[r, "\Psi''"] \arrow[d, "\tilde{p}_3"] & H(\ddot{X},r^!(A\boxtimes DC\boxtimes DA)) \arrow[d, "p_3"] \\
H(\tilde{\ddot{X}},\tilde{\imath}^!(\tilde{A}^\dagger\boxtimes D\tilde{A})) \arrow[r, "\Psi"] & H(\ddot{X},i^!(A\boxtimes DA))
\end{tikzcd}$$

Now the commutativity of the upper square and that of the lower square are checked in a routine manner; we are thus reduced to checking the commutativity of the middle square. Since

$$H(\tilde{\ddot{X}},\tilde{r}^!(\tilde{A}^\dagger\boxtimes(D\tilde{A}\otimes\tilde{A}^\dagger)\boxtimes D\tilde{A})) = H(\ddot{X},\ddot{j}_*\tilde{r}^!(\tilde{A}^\dagger\boxtimes(D\tilde{A}\otimes\tilde{A}^\dagger)\boxtimes D\tilde{A}))$$
$$= H(\ddot{X},r^!(\dot{j}\times\dot{j}\times\dot{j})_!(\tilde{A}^\dagger\boxtimes(D\tilde{A}\otimes\tilde{A}^\dagger)\boxtimes D\tilde{A})),$$

$$H(\tilde{\ddot{X}},\tilde{r}^!(\tilde{A}^\dagger\boxtimes DC\boxtimes D\tilde{A})) = H(\ddot{X},\ddot{j}_*\tilde{r}^!(\tilde{A}^\dagger\boxtimes DC\boxtimes D\tilde{A}))$$
$$= H(\ddot{X},r^!(\dot{j}\times\dot{j}\times\dot{j})_!(\tilde{A}^\dagger\boxtimes DC\boxtimes D\tilde{A})),$$

we see that we are reduced to verifying the commutativity of the following diagram which induces the middle square above

$$\begin{tikzcd}
(\dot{j}\times\dot{j}\times\dot{j})_!(\tilde{A}^\dagger\boxtimes(D\tilde{A}\otimes\tilde{A}^\dagger)\boxtimes D\tilde{A}) \arrow[r] \arrow[d] & A\boxtimes(DA\otimes A)\boxtimes DA \arrow[d] \\
(\dot{j}\times\dot{j}\times\dot{j})_!(\tilde{A}^\dagger\boxtimes DC\boxtimes D\tilde{A}) \arrow[r] & A\boxtimes DC\boxtimes DA
\end{tikzcd}$$

This diagram is obtained by taking external tensor products of the following three diagrams:

(c)
$$\begin{tikzcd}
\dot{j}_!\tilde{A}^\dagger \arrow[r] \arrow[d, "1"] & A \arrow[d, "1"] \\
\dot{j}_!\tilde{A}^\dagger \arrow[r] & A
\end{tikzcd}$$

(d)
$$\begin{tikzcd}
\dot{j}_!(D\tilde{A}\otimes\tilde{A}^\dagger) \arrow[r] \arrow[d] & DA\otimes A \arrow[d] \\
\dot{j}_!DC \arrow[r] & DC
\end{tikzcd}$$

(e)
$$\begin{tikzcd}
\dot{j}_!D\tilde{A} \arrow[r] \arrow[d, "1"] & DA \arrow[d, "1"] \\
\dot{j}_!D\tilde{A} \arrow[r] & DA
\end{tikzcd}$$

The diagrams (c),(e) have the horizontal arrows induced by $\dot{j}_!\dot{j}^! \to 1$ and are obviously commutative. Diagram (d) can be decomposed into two squares

$$\begin{tikzcd}
u_!\dot{j}'_!(\tilde{\mathcal{L}}[2\tilde{d}]\otimes\tilde{\mathcal{L}}^*[-2\delta]) \arrow[r] \arrow[d] & u_!(\mathcal{L}[2d]\otimes\mathcal{L}^*) \arrow[d] \\
u_!\dot{j}'_!DC \arrow[r] \arrow[d] & u_!DC \arrow[d] \\
\dot{j}_!DC \arrow[r] & u_!DC
\end{tikzcd}$$

where the lower square has horizontal arrows induced by $\dot{j}'_!\dot{j}'^! \to 1$ and $\dot{j}_!\dot{j}^! \to 1$ and vertical arrows induced by $\tilde{u}_!\tilde{u}^! \to 1$ and $u_!u^! \to 1$ (hence is commutative) and the upper square is obtained by applying $u_![2d]$ to the diagram 3.2(a), hence is again commutative, by Lemma 3.3. Thus, diagram (d) is commutative. The proposition is proved.

4. Localization

4.0. In this section we prove a localization theorem in equivariant homology and we give some applications. (The prototype of a localization theorem is the one of G. Segal [Se] in K-theory.)

4.1. Free $\mathfrak{G}$-actions. Assume that $\mathfrak{G}$ acts freely on a variety X. Let $\mathcal{L}$ be a $\mathfrak{G}$-equivariant local system on X. Then $\mathcal{L}$ is the inverse image of a well defined local system $\bar{\mathcal{L}}$ on $\mathfrak{G}\backslash X$ under the orbit map $X \to \mathfrak{G}\backslash X$ (as in **[L3, 1.1]**). We have a canonical isomorphism (as $H^{\cdot}_{\mathfrak{G}}$-modules):

(a) $$H^{\{1\}}_{\cdot}(\mathfrak{G}\backslash X,\bar{\mathcal{L}}) \xrightarrow{\sim} H^{\mathfrak{G}}_{\cdot}(X,\mathcal{L}).$$

Indeed, let $j \in \mathbf{Z}$ and let $\Gamma \in Sf_m(\mathfrak{G})$ where m is large. We have a fibration
$$_\Gamma X = \mathfrak{G}\backslash(\Gamma\times X) \to \mathfrak{G}\backslash X$$
with fibres isomorphic to Γ. As in **[L3, 1.2]**, this fibration gives rise to an isomorphism
$$H_c^{2\dim {_\Gamma X}-j}({_\Gamma X},{_\Gamma\mathcal{L}^*}) \xrightarrow{\sim} H_c^{2\dim \mathfrak{G}\backslash X-j}(\mathfrak{G}\backslash X,\bar{\mathcal{L}}^*).$$
Taking dual spaces we obtain (a).

4.2. Note that the adjoint action of $\mathfrak{G}$ on Lie $\mathfrak{G}$ induces an action of $\mathfrak{G}$ on Lie $\bar{\mathfrak{G}}$ (see 1.1), hence an action of $\mathfrak{G}$ on the algebra $S^{\cdot}(\text{Lie }\bar{\mathfrak{G}}^*)$ of polynomial functions Lie $\bar{\mathfrak{G}} \to \mathbf{C}$. The algebra of $\mathfrak{G}$-invariant elements of $S^{\cdot}(\text{Lie }\bar{\mathfrak{G}}^*)$ is canonically isomorphic to the algebra $H^{\cdot}_{\mathfrak{G}}$ (see **[L3, 1.11]**). For any semisimple element $\sigma \in$ Lie $\mathfrak{G}$, we denote by $\chi_\sigma : S^{\cdot}(\text{Lie }\bar{\mathfrak{G}}^*) \to \mathbf{C}$ the algebra homomorphism given by evaluating a polynomial function on Lie $\bar{\mathfrak{G}}^*$ at the image of σ

under the canonical map Lie $\mathfrak{G} \to$ Lie $\bar{\mathfrak{G}}^*$. We may regard χ_σ as a $\mathbf{C}$-algebra homomorphism $H^{\cdot}_{\mathfrak{G}} \to \mathbf{C}$ via the isomorphism mentioned above; let $\mathcal{I}^{\mathfrak{G}}_\sigma$ be its kernel.

Note that $\sigma \mapsto \chi_\sigma$ gives a 1-1 correspondence between the set of semisimple elements in Lie $\mathfrak{G}$ (up to the adjoint action of $\mathfrak{G}$) and the set of $\mathbf{C}$-algebra homomorphisms $H^{\cdot}_{\mathfrak{G}} \to \mathbf{C}$.

For σ as above, we denote by $H^{\cdot}_{\mathfrak{G},\sigma}$ (resp. $\hat{H}^{\cdot}_{\mathfrak{G},\sigma}$) the localization (resp. completion) of $H^{\cdot}_{\mathfrak{G}}$ with respect to the maximal ideal $I^{\mathfrak{G}}_\sigma$.

4.3. *In the remainder of this section, we assume that a semisimple element $\sigma \in$ Lie $\mathfrak{G}$ has been fixed and that $\mathfrak{G}$ is connected.*

Let $\mathfrak{H} = Z_{\mathfrak{G}}(\sigma)$. This is a closed, connected subgroup of $\mathfrak{G}$. Let $\mathcal{T} = \mathcal{T}_\sigma$ be the smallest torus in $\mathfrak{G}$ whose Lie algebra contains σ. We denote by $X^{\mathcal{T}}$ the fixed point set of $\mathcal{T}$ on X. Then $X^{\mathcal{T}}$ is stable under the action of $\mathfrak{H}$ on X. Note that

(a) *the ring homomorphism $\hat{H}^{\cdot}_{\mathfrak{G},\sigma} \to \hat{H}^{\cdot}_{\mathfrak{H},\sigma}$ induced by the homomorphism $H^{\cdot}_{\mathfrak{G}} \to H^{\cdot}_{\mathfrak{H}}$ (as in* [**L3, 1.4(g)**]*) is an isomorphism.*

To prove this, we may assume by [**L3, 1.4(h)**] that $\mathfrak{G}$ is reductive; then $\mathfrak{H}$ is automatically reductive. Let $\mathfrak{h}$ be a Cartan subalgebra of Lie $\mathfrak{G}$ containing σ. Then (a) is a consequence of the following known fact: the natural map from $\mathfrak{h}$ modulo the Weyl group of $\mathfrak{G}$ to $\mathfrak{h}$ modulo the Weyl group of $\mathfrak{H}$ is étale from a neighbourhood of σ in the first space to a neighbourhood of σ in the second space.

We have a homomorphism of $H^{\cdot}_{\mathfrak{H}}$-modules $H^{\mathfrak{H}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L}) \xrightarrow{j_!} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L})$ induced by the inclusion $j : X^{\mathcal{T}} \to X$. Applying $H^{\cdot}_{\mathfrak{H},\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}}$ to $j_!$, we obtain a homomorphism denoted again by $j_!$:

(b) $$H^{\cdot}_{\mathfrak{H},\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L}) \xrightarrow{j_!} H^{\cdot}_{\mathfrak{H},\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L}).$$

PROPOSITION 4.4. *(a) If $\mathfrak{G}$ is a torus, then the homomorphism 4.3(b) is an isomorphism. If, moreover, $H^{odd}_c(X, \mathcal{L}^*) = 0$, then $H^{odd}_c(X^{\mathfrak{G}}, \mathcal{L}^*) = 0$.*

(b) If $\mathfrak{H} = \mathfrak{G}$ and $H^{odd}_c(X, \mathcal{L}^*) = 0$, then the homomorphism 4.3(b) is an isomorphism.

(c) If $H^{odd}_c(X, \mathcal{L}^*) = 0$, we have a canonical isomorphism
$$\hat{H}^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(X, \mathcal{L}) \xrightarrow{\sim} \hat{H}^{\cdot}_{\mathfrak{H},\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L}).$$

We prove the first statement of (a). In this case we have $\mathfrak{G} = \mathfrak{H}$. The long exact sequence connecting $H^{\mathfrak{G}}_{\cdot}(X, \mathcal{L}), H^{\mathfrak{G}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L}), H^{\mathfrak{G}}_{\cdot}(X - X^{\mathcal{T}}, \mathcal{L})$ (see [**L3, 1.5**]), together with the exactness of localization, shows that it suffices to show that
$$H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(X - X^{\mathcal{T}}, \mathcal{L}) = 0.$$
Thus, we may assume that $X^{\mathcal{T}} = \emptyset$ and we must prove that

(d) $$H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(X, \mathcal{L}) = 0.$$

We can find a finite chain of closed $\mathfrak{G}$-stable subvarieties $\emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_t = X$ such that, for $m = 1, 2, \ldots, t$, the isotropy group in $\mathfrak{G}$ of a point $p \in X_m - X_{m-1}$ is a subgroup $\mathfrak{G}_m$ of $\mathfrak{G}$ depending only on m, not on p, and

such that the action of $\mathfrak{G}/\mathfrak{G}_m$ on $X_m - X_{m-1}$ is free. We show by induction on m that $H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(X_m, \mathcal{L}) = 0$.

This is clear when $m = 0$. Assume now that $m \geq 1$. We may assume as an induction hypothesis that $H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(X_{m-1}, \mathcal{L}) = 0$. If we knew that
$$H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(X_m - X_{m-1}, \mathcal{L}) = 0,$$
then, from the long exact sequence [**L3, 1.5**] connecting
$$H^{\mathfrak{G}}_{\cdot}(X_m, \mathcal{L}), H^{\mathfrak{G}}_{\cdot}(X_{m-1}, \mathcal{L}), H^{\mathfrak{G}}_{\cdot}(X_m - X_{m-1}, \mathcal{L})$$
together with the exactness of localization, it would follow that $H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(X_m, \mathcal{L}) = 0$, and the induction step would be established. Thus, we are reduced to proving (d) under the assumption that the isotropy group of any point in X is a fixed subgroup $\mathfrak{G}'$ of $\mathfrak{G}$, that $\mathfrak{G}/\mathfrak{G}'$ acts freely on X and that $\sigma \notin$ Lie $\mathfrak{G}'$.

Let $\Gamma \in Sf_m(\mathfrak{G})$ where m is large. We have an obvious fibration $\mathfrak{G}\backslash(\Gamma \times X) \to \mathfrak{G}\backslash X$ with fibres isomorphic to $\mathfrak{G}'\backslash\Gamma$. We consider the Leray spectral sequence of this fibration in cohomology with compact support with coefficients in ${}_\Gamma\mathcal{L}^*$. It suffices to show that the E_∞ term of this spectral sequence is zero (in a certain range) after we apply to it $H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}}$. This follows from the statement that the E_2 term of this spectral sequence is zero (in the same range) after applying to it $H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}}$. This in turn follows from a vanishing statement concerning the fibres of our fibration. Thus we are reduced to the case where $\mathfrak{G}$ acts transitively on X with isotropy group $\mathfrak{G}'$ as above. In this case,
$$H^{\mathfrak{G}}_{\cdot}(X, \mathcal{L}) = H^{\mathfrak{G}'}_{\cdot}(pt, \mathcal{L})$$
(see [**L3, 1.6**]) and it is clear that $H^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}'}_{\cdot}(pt, \mathcal{L}) = 0$ if $\sigma \notin$ Lie $\mathfrak{G}'$. This completes the proof of the first statement of (a). The second statement of (a) is proved by a method analogous to that in [**KL, end of 4.6**], using $\mathbf{C}^*$-equivariant homology instead of $\mathbf{C}^*$-equivariant K-homology.

We prove (b). Let T be a maximal torus of $\mathfrak{H}$ such that $\sigma \in$ Lie T. Then $\mathcal{T} \subset T$. Our assumption on the vanishing of $H^{odd}_c(X, \mathcal{L}^*)$ implies the vanishing of $H^{odd}_c(X^{\mathcal{T}}, \mathcal{L}^*)$ (see (a)). Hence we may apply [**L3, 7.5**] and deduce
$$H^{\cdot}_T \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L}) = H^T_{\cdot}(X, \mathcal{L}), \quad H^{\cdot}_T \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L}) = H^T_{\cdot}(X^{\mathcal{T}}, \mathcal{L}).$$
Applying $H^{\cdot}_{T,\sigma} \otimes_{H^{\cdot}_T}$ to these equalities and using (a) for T, we deduce that
$$H^{\cdot}_{T,\sigma} \otimes_{H^{\cdot}_T} (H^{\cdot}_T \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L})) = H^{\cdot}_{T,\sigma} \otimes_{H^{\cdot}_T} (H^{\cdot}_T \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L})),$$
or equivalently
$$H^{\cdot}_{T,\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L}) = H^{\cdot}_{T,\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L}).$$
Applying $\mathbf{C} \otimes_{H^{\cdot}_{T,\sigma}}$ (where we use the unique algebra homomorphism $H^{\cdot}_{T,\sigma} \to \mathbf{C}$), we obtain
$$\mathbf{C} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L}) = \mathbf{C} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L}).$$
By [**L3, 7.2**], $H^{\cdot}_{\mathfrak{H},\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X^{\mathcal{T}}, \mathcal{L}) \xrightarrow{j_!} H^{\cdot}_{\mathfrak{H},\sigma} \otimes_{H^{\cdot}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L})$ is a homomorphism between finitely generated projective modules over the local algebra $H^{\cdot}_{\mathfrak{H},\sigma}$. Hence to prove that it is an isomorphism, it is enough to show that it is so after applying to it $\mathbf{C} \otimes_{H^{\cdot}_{\mathfrak{H}}}$. But this has just been checked. (b) is proved.

We prove (c). The homomorphism $H^{\mathfrak{G}}_{\cdot}(X, \mathcal{L}) \to H^{\mathfrak{H}}_{\cdot}(X, \mathcal{L})$ (as in [**L3, 1.4(f)**]) and the algebra homomorphism $H^{\cdot}_{\mathfrak{G}} \to H^{\cdot}_{\mathfrak{H}}$ (as in [**L3, 1.4(g)**]) give rise to a homomorphism of $H^{\bullet}_{\mathfrak{H}}$-modules $H^{\bullet}_{\mathfrak{H}} \otimes_{H^{\bullet}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\bullet}(X, \mathcal{L}) \to H^{\mathfrak{H}}_{\bullet}(X, \mathcal{L})$, which is an isomorphism, by [**L3, 7.5**]. Applying $\hat{H}^{\bullet}_{\mathfrak{H},\sigma} \otimes_{H^{\bullet}_{\mathfrak{H}}}$ we obtain an isomorphism

$$\hat{H}^{\bullet}_{\mathfrak{H},\sigma} \otimes_{H^{\bullet}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\bullet}(X, \mathcal{L}) \xrightarrow{\sim} \hat{H}^{\bullet}_{\mathfrak{H},\sigma} \otimes_{H^{\bullet}_{\mathfrak{H}}} H^{\mathfrak{H}}_{\bullet}(X, \mathcal{L}).$$

Using now $\hat{H}^{\bullet}_{\mathfrak{G},\sigma} \xrightarrow{\sim} \hat{H}^{\bullet}_{\mathfrak{H},\sigma}$ (see 4.3(a)), we obtain the isomorphism in (c). The proposition is proved.

COROLLARY 4.5. *Assume that $\mathfrak{G} = \mathfrak{H}$. Let $p : E \to Y$ be a $\mathfrak{H}$-equivariant vector bundle of dimension n over the connected $\mathfrak{H}$-variety Y. Let $\mathcal{L}'$ be a $\mathfrak{G}$-equivariant local system on Y such that $H^{odd}_c(Y, \mathcal{L}'^*) = 0$. Assume that the action of T on E leaves stable each fibre of E and its fixed point set on any fibre of E consists of 0 alone. Then the $H^{\bullet}_{\mathfrak{H}}$-linear map*

$$H^{\mathfrak{H}}_{\bullet}(Y, \mathcal{L}'[-2n]) \to H^{\mathfrak{H}}_{\bullet}(Y, \mathcal{L}')$$

induced by $\kappa_E : \mathcal{L}'[-2n] \to \mathcal{L}'$, becomes an isomorphism after applying $H^{\bullet}_{\mathfrak{H},\sigma} \otimes_{H^{\bullet}_{\mathfrak{H}}}$.

Our map is the composition

$$H^{\mathfrak{H}}_{\bullet}(Y, \mathcal{L}'[-2n]) \xrightarrow{v_!} H^{\mathfrak{H}}_{\bullet}(E, p^*\mathcal{L}') \xrightarrow{v^*} H^{\mathfrak{H}}_{\bullet}(Y, \mathcal{L}')$$

where $v : Y \to E$ is the inclusion of the zero section. The second map v^* is an isomorphism (see [**L3, 1.4(e)**]); the first map $v_!$ becomes an isomorphism after applying $H^{\bullet}_{\mathfrak{H},\sigma} \otimes_{H^{\bullet}_{\mathfrak{H}}}$. (This follows from 4.4(b) applied to $X = E$ and to $p^*\mathcal{L}'$; note that our assumptions imply that $H^{odd}_c(E, p^*\mathcal{L}^*) = 0$.) The corollary follows.

4.6. In the remainder of this section, the following hypotheses are made. We are given a proper morphism of $\mathfrak{G}$-varieties

$$f : \dot{X} \to X,$$

an open dense smooth $\mathfrak{G}$-subvariety Z of $\dot{X}$ and an $\mathfrak{G}$-equivariant local system $\mathcal{L}$ on Z, with finite monodromy (this finiteness assumption will not play a role in this section, but it will be needed in the next section). Let $\bar{X}$ be the fixed point set of the torus T in X. (Recall that T is the smallest torus in $\mathfrak{G}$ whose Lie algebra contains σ. Note that T is contained in the centre of $\mathfrak{H} = Z_{\mathfrak{G}}(\sigma)$.) Let $\bar{Z}$ be the fixed point set of T in Z. Let $\bar{\dot{X}}$ be the closure of $\bar{Z}$ in $\dot{X}$. Let $\bar{f} : \bar{\dot{X}} \to \bar{X}$ be the restriction of f. Note that $\bar{X}, \bar{\dot{X}}$ are $\mathfrak{H}$-stable and that $\bar{f}$ is a proper $\mathfrak{H}$-morphism. Let

$$j : \bar{X} \to X, \dot{j} : \bar{\dot{X}} \to \dot{X}, j' : \bar{Z} \to Z, u : Z \to \dot{X}, \bar{u} : \bar{Z} \to \bar{\dot{X}}$$

be the inclusions. Let $\bar{\mathcal{L}} = j'^*\mathcal{L}$ (a $\mathfrak{H}$-equivariant local system on $\bar{Z}$, with finite monodromy). We assume that

(a) $\bar{Z}$ is an open dense subvariety of $\bar{\dot{X}}$;
(b) $\dot{X}, Z$ are connected of dimension d;
(c) the connected components of $\bar{\dot{X}}$ are the same as the irreducible components of $\bar{\dot{X}}$ (hence they are exactly the closures of the connected components of $\bar{Z}$);
(d) $u_!\mathcal{L}^* = u_*\mathcal{L}^*$;
(e) $\bar{u}_!\bar{\mathcal{L}}^* = \bar{u}_*\bar{\mathcal{L}}^*$.

Thus, conditions 3.4(a)-(f) are satisfied. Hence the definitions of 3.4 are applicable in the present context. In particular, $\ddot{X}, \bar{\ddot{X}}, A, \bar{A}, \delta, \bar{d}, \bar{A}^{\dagger}, \Phi, \Psi$ are defined as in 3.4.

Let $Z' = (Z \times Z) \cap \ddot{X}$. Let $\mathcal{E}$ be the local system on Z' given by the restriction of $\mathcal{L} \boxtimes \mathcal{L}^*$. Let $\bar{Z}' = (\bar{Z} \times \bar{Z}) \cap \bar{\ddot{X}}$; this is the fixed point set of the natural T-action on Z'. Let $\bar{\mathcal{E}}$ be the local system on $\bar{Z}'$ given by the restriction of $\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}^*$. We assume that

(f) $H^{odd}_c(Z', \mathcal{E}^*) = 0$.

This implies that

(g) $H^{odd}_c(\bar{Z}', \bar{\mathcal{E}}^*) = 0$.

(See 4.4(a). We have used the fact that $\bar{Z}'$ is the fixed point set of the natural T-action on Z'.)

Let E be the normal bundle of $\bar{Z}$ in Z (naturally, a $\mathfrak{H}$-equivariant vector bundle). The action of T on E is necessarily fibre preserving, and the fixed point set of T on any fibre of E consists of the zero vector only. Hence, if E' is the inverse image of E under the first projection $\bar{Z}' \to \bar{Z}$, then E' is naturally a $\mathfrak{H}$-equivariant vector bundle, the action of T on E' is necessarily fibre preserving, and the fixed point set of T on any fibre of E' consists of the zero vector only.

We have the following result.

LEMMA 4.7. *The $H^{\bullet}_{\mathfrak{H}}$-linear map*

$$\Phi : H^{\bullet}_{\mathfrak{H}}(\bar{\ddot{X}}, \bar{i}^!(\bar{A}^{\dagger} \boxtimes D\bar{A})) \to H^{\bullet}_{\mathfrak{H}}(\bar{\ddot{X}}, \bar{i}^!(\bar{A} \boxtimes D\bar{A}))$$

becomes an isomorphism after tensoring with $H^{\bullet}_{\mathfrak{H},\sigma}$ over $H^{\bullet}_{\mathfrak{H}}$.

Let $i' : \bar{Z}' \to \bar{Z} \times \bar{Z}$ be the inclusion. We have a commutative diagram

$$\begin{array}{ccc}
H^{\bullet}_{\mathfrak{H}}(\bar{Z}', \bar{i}'^!(\bar{\mathcal{L}}^*[-2\delta] \boxtimes \bar{\mathcal{L}}[2\bar{d}])) & \xrightarrow{\Phi_1} & H^{\bullet}_{\mathfrak{H}}(\bar{Z}', \bar{i}'^!(\bar{\mathcal{L}}^* \boxtimes \bar{\mathcal{L}}[2\bar{d}])) \\
\downarrow & & \downarrow \\
H^{\bullet}_{\mathfrak{H}}(\bar{\ddot{X}}, \bar{i}^!(\bar{A}[-2\delta] \boxtimes D\bar{A})) & \xrightarrow{\Phi} & H^{\bullet}_{\mathfrak{H}}(\bar{\ddot{X}}, \bar{i}^!(\bar{A} \boxtimes D\bar{A}))
\end{array}$$

where Φ_1 is induced by

$$\kappa_{E'} : \bar{i}'^!(\bar{\mathcal{L}}^*[-2\delta] \boxtimes \bar{\mathcal{L}}[2\bar{d}]) \to \bar{i}'^!(\bar{\mathcal{L}}^* \boxtimes \bar{\mathcal{L}}[2\bar{d}])$$

and the vertical maps are isomorphisms induced by the inclusion $\bar{Z}' \to \bar{\ddot{X}}$. (Recall that $\bar{A} = \bar{u}_!\bar{\mathcal{L}}^*, D\bar{A} = \bar{u}_!\bar{\mathcal{L}}[2\bar{d}]$.) It remains to show that Φ_1 is an isomorphism after tensoring with $H^{\bullet}_{\mathfrak{H},\sigma}$ over $H^{\bullet}_{\mathfrak{H}}$. But this follows by applying Corollary 4.5 to each connected component of $\bar{Z}'$. (4.5 is applicable, by 4.6(g).) The lemma is proved.

4.8. We consider three algebras over $H^{\bullet}_{\mathfrak{H}}$. The map

(a) $$H^{\bullet}_{\mathfrak{H}}(\ddot{X}, i^!(A \boxtimes DA)) \times H^{\mathfrak{H}}_{\bullet}(\ddot{X}, i^!(A \boxtimes DA)) \to H^{\bullet}_{\mathfrak{H}}(\ddot{X}, i^!(A \boxtimes DA))$$

(a special case of 2.3(b)) defines on $H^{\bullet}_{\mathfrak{H}}(\ddot{X}, i^!(A \boxtimes DA))$ a structure of algebra over $H^{\bullet}_{\mathfrak{H}}$. The analogous map

(b) $$H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A})) \times H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A})) \to H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A}))$$
(a special case of 2.3(b)) defines on $H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A}))$ a structure of algebra over $H^{\cdot}_{\mathfrak{h}}$. Finally, the composition
$$H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}^{\dagger}\boxtimes D\bar{A})) \times H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}^{\dagger}\boxtimes D\bar{A})) \xrightarrow{1\times\Phi}$$
(c) $$H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}^{\dagger}\boxtimes D\bar{A})) \times H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A})) \to H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}^{\dagger}\boxtimes D\bar{A}))$$

(the last map as in (b)) defines on $H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}^{\dagger}\boxtimes D\bar{A}))$ a structure of algebra over $H^{\cdot}_{\mathfrak{h}}$. From Proposition 3.5, we see that the maps
$$H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A})) \xleftarrow{\Phi} H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}^{\dagger}\boxtimes D\bar{A})) \xrightarrow{\Psi} H^{\cdot}_{\mathfrak{h}}(\ddot{X}, i^{!}(A\boxtimes DA))$$
are algebra homomorphisms. From 4.4(b) and 4.7, we see that these become in fact algebra isomorphisms after tensoring with $H^{\cdot}_{\mathfrak{h},\sigma}$ over $H^{\cdot}_{\mathfrak{h}}$. (In order to apply 4.4(b), we note that the map Ψ may be identified with the map $H^{\mathfrak{h}}_{\cdot}(\bar{Z}', \bar{\mathcal{E}}) \to H^{\mathfrak{h}}_{\cdot}(Z', \mathcal{E})$ induced by the inclusion $\bar{Z}' \to Z'$; we also use 4.6(f).) Hence we have the following result.

PROPOSITION 4.9. *The map $\Psi\Phi^{-1}$ defines an isomorphism*
$$H^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A})) \xrightarrow{\sim} H^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} H^{\cdot}_{\mathfrak{h}}(\ddot{X}, i^{!}(A\boxtimes DA))$$
of algebras over $H^{\cdot}_{\mathfrak{h},\sigma}$.

4.10. Let $B = f_! A$. We identify
$$H^{\cdot}_{\mathfrak{h}}(\ddot{X}, i^{!}(A\boxtimes DA)) = H^{\cdot}_{\mathfrak{h}}(X, D(DB\otimes B)) = \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B)$$
as algebras, using 2.4(a), 1.18(f), 2.5, 2.2. Similarly, replacing $\ddot{X}, A$ by $\ddot{\bar{X}}, \bar{A}$, we identify
$$H^{\cdot}_{\mathfrak{h}}(\ddot{\bar{X}}, \bar{i}^{!}(\bar{A}\boxtimes D\bar{A})) = H^{\cdot}_{\mathfrak{h}}(\bar{X}, D(D\bar{B}\otimes\bar{B})) = \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B})$$
as algebras. (Here, $\bar{B} = \bar{f}_!\bar{A}$.) Hence the isomorphism in 4.9 gives rise to an algebra isomorphism

(a) $$H^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B}) \xrightarrow{\sim} H^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B).$$

(We regard $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B)$ and $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B})$ as modules over $H^{\cdot}_{\mathfrak{h}} = \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}pt}(\mathbf{C},\mathbf{C})$, as in the end of 1.20.)

4.11. Note that B may be regarded either as an object of $\mathcal{D}_{\mathfrak{G}}X$ or as an object of $\mathcal{D}_{\mathfrak{h}}X$, and we have an obvious algebra homomorphism $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B)$. This induces an algebra homomorphism
$$H^{\cdot}_{\mathfrak{h}} \otimes_{H^{\cdot}_{\mathfrak{G}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B).$$
This is an isomorphism since (using twice 1.18(f)) it may be identified with
$$H^{\cdot}_{\mathfrak{h}} \otimes_{H^{\cdot}_{\mathfrak{G}}} H^{\mathfrak{G}}_{\cdot}(Z',\mathcal{E}) \to H^{\mathfrak{h}}_{\cdot}(Z',\mathcal{E})$$
which is known to be an isomorphism (see [**L3**, **7.5**] and 4.6(f)). Applying $\hat{H}^{\cdot}_{\mathfrak{h},\sigma}\otimes_{H^{\cdot}_{\mathfrak{h}}}$ we obtain an isomorphism
$$\hat{H}^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B) \to \hat{H}^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B).$$
Using now $\hat{H}^{\cdot}_{\mathfrak{G},\sigma} \xrightarrow{\sim} \hat{H}^{\cdot}_{\mathfrak{h},\sigma}$ (see 4.3(a)), we obtain an isomorphism
$$\hat{H}^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B) \to \hat{H}^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B).$$
We combine this with the isomorphism

$$\hat{H}^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B}) \xrightarrow{\sim} \hat{H}^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}X}(B,B)$$
obtained from 4.10(a), by applying $\hat{H}^{\cdot}_{\mathfrak{h},\sigma}\otimes_{H^{\cdot}_{\mathfrak{h},\sigma}}$ and we deduce the following result.

PROPOSITION 4.12. *There is a natural isomorphism of algebras*
$$\hat{H}^{\cdot}_{\mathfrak{G},\sigma} \otimes_{H^{\cdot}_{\mathfrak{G}}} \mathit{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B) \xrightarrow{\sim} \hat{H}^{\cdot}_{\mathfrak{h},\sigma} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathit{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B}).$$

4.13. Let $\mathbf{mod}_\sigma \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B)$, (resp. $\mathbf{mod}_\sigma \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B})$) be the category of $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B)$-modules (resp. $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B})$-modules) of finite dimension over $\mathbf{C}$, in which some power of the maximal ideal $I^{\mathfrak{G}}_{\sigma}$ (resp. $I^{\mathfrak{h}}_{\sigma}$) of $H^{\cdot}_{\mathfrak{G}}$ (resp. of $H^{\cdot}_{\mathfrak{h}}$) acts as zero.

PROPOSITION 4.14. *The categories*
$$\mathbf{mod}_\sigma \mathit{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B) \text{ and } \mathbf{mod}_\sigma \mathit{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B})$$
are naturally equivalent.

The categories in question may be naturally identified with the categories of modules over the algebras $\hat{H}_{\mathfrak{G},\sigma}\otimes_{H^{\cdot}_{\mathfrak{G}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B,B), \hat{H}_{\mathfrak{h},\sigma}\otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B})$ (respectively), of finite dimension over $\mathbf{C}$ which are continuous in the appropriate adic topology; since these algebras are naturally isomorphic (see 4.12), the proposition follows.

5. A study of simple modules

5.1. In this section we preserve the assumptions of 4.3 as well as the those of 4.6.

The group homomorphism $\mathcal{T} \to \{1\}$ gives rise to a functor $\mathcal{D}\bar{X} \to \mathcal{D}_{\mathcal{T}}\bar{X}$ (see 1.10), since $\mathcal{T}$ acts trivially on $\bar{X}$. This gives rise to a $\mathbf{C}$-algebra homomorphism
$$\mathrm{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B},\bar{B}) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathcal{T}}\bar{X}}(\bar{B},\bar{B})$$
which extends uniquely to a $H^{\cdot}_{\mathcal{T}}$-algebra homomorphism

(a) $$H^{\cdot}_{\mathcal{T}} \otimes_{\mathbf{C}} \mathrm{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B},\bar{B}) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathcal{T}}\bar{X}}(\bar{B},\bar{B}).$$

This is in fact an isomorphism. Indeed, using 1.18(f), we can interpret it as the inverse of the isomorphism $H^{\mathcal{T}}_{\cdot}(\bar{Z}',\bar{\mathcal{E}}) \xrightarrow{\sim} H^{\cdot}_{\mathcal{T}} \otimes_{\mathbf{C}} H^{\{1\}}_{\cdot}(\bar{Z}',\bar{\mathcal{E}})$ (see 1.21(a)). On the other hand, the inclusion $\mathcal{T} \to \mathfrak{h}$ gives us a functor $\mathcal{D}_{\mathfrak{h}}\bar{X} \to \mathcal{D}_{\mathcal{T}}\bar{X}$ (see 1.10). This defines an algebra homomorphism
$$\mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B}) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathcal{T}}\bar{X}}(\bar{B},\bar{B}),$$
which extends uniquely to a homomorphism of $H^{\cdot}_{\mathcal{T}}$-algebras

(b) $$H^{\cdot}_{\mathcal{T}} \otimes_{H^{\cdot}_{\mathfrak{h}}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{h}}\bar{X}}(\bar{B},\bar{B}) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathcal{T}}\bar{X}}(\bar{B},\bar{B}).$$

This is in fact an isomorphism. Indeed, using 1.18(f), we can interpreted it as the isomorphism $H_T^{\cdot} \otimes_{H_{\mathfrak{H}}^{\cdot}} H^{\mathfrak{H}}_{\cdot}(\bar{Z}', \bar{\mathcal{E}}) \xrightarrow{\sim} H^T_{\cdot}(\bar{Z}', \bar{\mathcal{E}})$ (see [**L3, 7.5**] and 4.6(f)). Combining the isomorphisms (a),(b), we obtain an isomorphism of $H_T^{\cdot}$-algebras

(c) $$H_T^{\cdot} \otimes_{H_{\mathfrak{H}}^{\cdot}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{H}}\bar{X}}(\bar{B}, \bar{B}) \xrightarrow{\sim} H_T^{\cdot} \otimes_{\mathbf{C}} \mathrm{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B}, \bar{B}).$$

We shall regard $\mathbf{C}$ as a $H_{\mathfrak{H}}^{\cdot}$-algebra via the algebra homomorphism $\chi_\sigma : H_{\mathfrak{H}}^{\cdot} \to \mathbf{C}$ (as in 4.2), as a $H_T^{\cdot}$-algebra via the composition $H_T^{\cdot} \to H_{\mathfrak{H}}^{\cdot} \to \mathbf{C}$, and as a $\hat{H}_{\mathfrak{H},\sigma}$-algebra via the algebra homomorphism $\hat{H}_{\mathfrak{H},\sigma} \to \mathbf{C}$ defined by χ_σ. Applying $\mathbf{C} \otimes_{H_T^{\cdot}}$ to (c), we obtain an isomorphism of $\mathbf{C}$-algebras

(d) $$\mathbf{C} \otimes_{H_{\mathfrak{H}}^{\cdot}} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{H}}\bar{X}}(\bar{B}, \bar{B}) \xrightarrow{\sim} \mathrm{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B}, \bar{B}).$$

Let $\mathbf{mod}\mathrm{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B}, \bar{B})$ be the category of $\mathrm{Hom}^{\cdot}_{\mathcal{D}X}(\bar{B}, \bar{B})$-modules of finite dimension over $\mathbf{C}$.

PROPOSITION 5.2. *The set of isomorphism classes of simple objects of* $\mathbf{mod}\mathit{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B}, \bar{B})$ *is naturally in bijection with the set of isomorphism classes of simple objects of* $\mathbf{mod}_\sigma \mathit{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{H}}\bar{X}}(\bar{B}, \bar{B})$) *and hence (by 4.14) with the set of isomorphism classes of simple objects of* $\mathbf{mod}_\sigma \mathit{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{G}}X}(B, B)$.

This follows from 5.1(d), since the ideal $I_\sigma^{\mathfrak{H}}$ must act as zero on a simple object of $\mathbf{mod}_\sigma \mathrm{Hom}^{\cdot}_{\mathcal{D}_{\mathfrak{H}}\bar{X}}(\bar{B}, \bar{B})$),

5.3. Applying the decomposition theorem [**BBD**] to the proper map $\bar{f} : \bar{\bar{X}} \to \bar{X}$ and to the complex $\bar{A} = \tilde{u}_! \bar{\mathcal{L}}^{\bullet}$ on $\bar{Z}$ (a direct sum of shifts of perverse sheaves of geometric origin, by 4.6(e) and by the finiteness of monodromy of $\bar{\mathcal{L}}$) we see that $\bar{B} = \bar{f}_! \bar{A} \in \mathcal{D}\bar{X}$ is a semisimple complex or, in other words, there exists an isomorphism

(a) $$\bar{B} \xrightarrow{\sim} \oplus_{n \in \mathbf{Z}} \bar{B}^n[-n]$$

where $\bar{B}^n$ is the n-th perverse cohomology sheaf of $\bar{B}$, and each $\bar{B}^n$ is a semisimple perverse sheaf on $\bar{X}$. We choose a specific isomorphism as in (a).

LEMMA 5.4. *(Compare [**CG, 8.6.9**].) There is a natural 1-1 correspondence between the set of isomorphism classes of simple modules of the finite dimensional* $\mathbf{C}$*-algebra* $\mathcal{A} := \mathit{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B}, \bar{B})$ *and the set* $\mathfrak{J}$ *consisting of the isomorphism classes of simple perverse on* $\bar{X}$*, for which some shift is a direct summand of* $\bar{B}$.

We have $\mathcal{A} = \oplus_{n,n',k} \mathrm{Hom}_{\mathcal{D}\bar{X}}(\bar{B}^n[-n], \bar{B}^{n'}[-n'+k])$, so that $\mathcal{A} = \oplus_{s \in \mathbf{Z}} \mathcal{A}_s$, where

$$\mathcal{A}_s = \oplus_{n,n',k; n-n'+k=s} \mathrm{Hom}_{\mathcal{D}\bar{X}}(\bar{B}^n[-n], \bar{B}^{n'}[-n'+k]).$$

It is clear that $\mathcal{A}_s \cdot \mathcal{A}_{s'} \subset \mathcal{A}_{s+s'}$ for all s, s', under the algebra structure of $\mathcal{A}$. Note that $\mathcal{A}_s = 0$ for $s < 0$, since $\mathrm{Hom}^0_{\mathcal{D}\bar{X}}(\bar{B}^n, \bar{B}^{n'}[s]) = 0$ for $s < 0$ (a property of perverse sheaves, see 1.16). Hence projection onto the $s = 0$ summand is an algebra homomorphism

$$\mathcal{A} \to \mathcal{A}_0 = \oplus_{n,n'} \mathrm{Hom}^0_{\mathcal{D}\bar{X}}(\bar{B}^n[-n], \bar{B}^{n'}[-n]) = \oplus_{n,n'} \mathrm{Hom}^0_{\mathcal{D}\bar{X}}(\bar{B}^n, \bar{B}^{n'})$$
$$= \mathrm{Hom}^0_{\mathcal{D}\bar{X}}(\oplus_n \bar{B}^n, \oplus_{n'} \bar{B}^{n'}) = \mathrm{Hom}^0_{\mathcal{D}\bar{X}}(\bar{B}', \bar{B}'),$$

where $\bar{B}' := \oplus_{n \in \mathbf{Z}} \bar{B}^n$. Now $\bar{B}'$ is a semisimple perverse sheaf on $\bar{X}$ (since each $\bar{B}^n$ is so); it follows that $\mathcal{A}_0$ is a semisimple algebra. Since the kernel of the homomorphism $\mathcal{A} \to \mathcal{A}_0$ is the two-sided ideal $\oplus_{s>0} \mathcal{A}_s$ which consists of nilpotent elements, it follows that this two-sided ideal is precisely the radical of the finite dimensional algebra $\mathcal{A}$. Let $(P_j)_{j \in \mathfrak{J}}$ be a (finite) collection of simple perverse sheaves on $\bar{X}$ such that any simple constituent of the perverse sheaf $\bar{B}'$ is isomorphic to exactly one P_j. Then $\mathrm{Hom}^0_{\mathcal{D}\bar{X}}(P_j, \bar{B}')$ is naturally a left $\mathrm{Hom}^0_{\mathcal{D}\bar{X}}(\bar{B}', \bar{B}')$-module (via composition of morphisms) or, in other words, an $\mathcal{A}_0$-module. It is clear that

$$\mathbf{P}_j := \mathrm{Hom}^0_{\mathcal{D}\bar{X}}(P_j, \bar{B}') \text{ (for } j \in \mathfrak{J})$$

is a complete collection of mutually non-isomorphic simple $\mathcal{A}_0$-modules. These modules can be regarded as $\mathcal{A}$-modules, via the algebra homomorphism $\mathcal{A} \to \mathcal{A}_0$ considered above, and they clearly provide a complete collection of mutually non-isomorphic simple $\mathcal{A}$-modules. The lemma is proved.

6. Stratification of $\mathfrak{g}$

6.0. In the remainder of this paper, G will denote a fixed reductive, connected algebraic group. Let $\mathfrak{g} = \mathrm{Lie}\, G$. Let $\mathfrak{g}_N$ be the set of nilpotent elements in $\mathfrak{g}$.

$G \times \mathbf{C}^*$ acts on $\mathfrak{g}$ and on $\mathfrak{g}_N$ by $(g, t) : y \mapsto t^{-2} Ad(g) y$. For $y \in \mathfrak{g}_N$ we denote by $\bar{Z}_G(y)$ the stabilizer of y in $G \times \mathbf{C}^*$. Thus,

$$\bar{Z}_G(y) = \{(g, t) \in G \times \mathbf{C}^* | Ad(g) y = t^2 y\}.$$

6.1. We will define a partition of $\mathfrak{g}$ into finitely many constructible subsets stable under the adjoint action of G. Let $\mathbf{A}$ be the set of all pairs (L, C) where L is a Levi subgroup of a parabolic subgroup of G and C is a nilpotent L-orbit in Lie L. There is a natural G-action on $\mathbf{A}$ (conjugation on both factors); the set $G \backslash \mathbf{A}$ of G-orbits is finite. For any L as above we set $W_L = N_G(L)/L$ (a finite group). Let $(L, C) \in \mathbf{A}$. Recall that $\mathfrak{z}_L = \mathrm{Lie}\, Z_L$. For any $x \in \mathfrak{z}_L$, we have $L \subset Z_G(x)$. We set

$$\mathfrak{z}_{L,*} = \{x \in \mathfrak{z}_L | Z_G(x) = L\} \text{ (an open dense subset of } \mathfrak{z}_L),$$
$$Y_{L,C} = \cup_{g \in G} Ad(g)(\mathfrak{z}_{L,*} + C),$$
$$\tilde{Y}_{L,C} = \{(x, gL) \in \mathfrak{g} \times G/L | Ad(g^{-1}) x \in \mathfrak{z}_{L,*} + C\}.$$

We have a morphism of algebraic varieties $\tilde{Y}_{L,C} \to \mathfrak{g}$ (first projection) whose image is precisely $Y_{L,C}$. Moreover the fibres of the restriction map $\tilde{Y}_{L,C} \to Y_{L,C}$ are exactly the orbits of the free action of the finite group $\{hL \in N_G(L)/L | Ad(h)(C) = C\}$ on $\tilde{Y}_{L,C}$ (translation on the second component). Since $\tilde{Y}_{L,C}$ is an irreducible variety of dimension

$$\dim(G/L) + \dim \mathfrak{z}_L + \dim C,$$

it follows that $Y_{L,C}$ is an irreducible constructible subset of $\mathfrak{g}$ of dimension

$\dim(G/L)+\dim\mathfrak{z}_L+\dim C$. Clearly, $Y_{L,C}$ depends only on the G-orbit of (L,C) in A, and we have a partition into finitely many pieces

$$\mathfrak{g}=\sqcup_{(L,C)\in G\backslash \mathrm{A}} Y_{L,C}.$$

LEMMA 6.2. *Let $(L,C)\in \mathrm{A}$. Let $cl(C)$ be the closure of C in Lie L. Choose a parabolic subgroup P of G such that L is a Levi subgroup of P. Let $\mathfrak{n}=$ Lie U_P. Let $Y'_{L,C}=\cup_{g\in G}Ad(g)(\mathfrak{z}_L+cl(C)+\mathfrak{n})$. Then the closure of $Y_{L,C}$ in $\mathfrak{g}$ is equal to $Y'_{L,C}$.*

The morphism of algebraic varieties

(a) $$\{(x,gP)\in\mathfrak{g}\times G/P|Ad(g^{-1})x\in\mathfrak{z}_L+cl(C)+\mathfrak{n}\}\to\mathfrak{g}$$

given by the first projection, is proper, since G/P is complete and $\mathfrak{z}_L+cl(C)+\mathfrak{n}$ is a closed P-invariant subset of Lie P. It follows that its image, $Y'_{L,C}$ is a closed subset of $\mathfrak{g}$. The first variety in (a) is irreducible of dimension

$$\dim(G/P)+\dim\mathfrak{z}_L+\dim C+\dim\mathfrak{n}=\dim G/L+\dim\mathfrak{z}_L+\dim C$$

(use the second projection). It follows that $Y'_{L,C}$ is an irreducible variety of dimension $\le\dim G/L+\dim\mathfrak{z}_L+\dim C$. Since $Y_{L,C}$ is a (constructible) subset of $Y'_{L,C}$ which is irreducible of dimension equal to $\dim G/L+\dim\mathfrak{z}_L+\dim C$, it follows that $Y_{L,C}$ is dense in $Y'_{L,C}$. This completes the proof.

6.3. Remark. Let $Y'_{L,C,N}$ be the set of nilpotent elements in $Y'_{L,C}$. As in the previous lemma, $Y'_{L,C,N}$ is the image of the map

(a) $$\{(x,gP)\in\mathfrak{g}\times G/P|Ad(g^{-1})x\in cl(C)+\mathfrak{n}\}\to\mathfrak{g}$$

given by the first projection. The first variety in (a) is irreducible (use the second projection). It follows that $Y'_{L,C,N}$ is an irreducible variety.

6.4. Let $(L,C),(L',C')$ be two pairs in A. We say that $(L',C')\preceq(L,C)$ if (after replacing (L',C') by a G-conjugate) the statements (a),(b) below hold:

(a) $L\subset L'$;

(b) C' is contained in the closure of the nilpotent L'-orbit in Lie L' induced (cf. [**LS**]) by C.

By the transitivity of induction of nilpotent orbits [**LS**], $\preceq$ is a preorder on A; it induces a partial order on $G\backslash\mathrm{A}$, denoted again by $\preceq$.

PROPOSITION 6.5. *The closure of $Y_{L,C}$ in $\mathfrak{g}$ is equal to*

$$\sqcup_{(L',C')\in G\backslash \mathrm{A};(L',C')\preceq(L,C)}Y_{L',C'}.$$

We use the notation of Lemma 6.2. According to that lemma, the closure of $Y_{L,C}$ is $Y'_{L,C}$. Let $x\in Y'_{L,C}$ and let $(L',C')\in\mathrm{A}$ be such that $x\in Y_{L',C'}$. We want to show that

(a) $Y_{L',C'}\subset Y'_{L,C}$;

(b) $(L',C')\preceq(L,C)$.

Since $Y'_{L,C},Y_{L',C'}$ are stable under the adjoint action, we may assume, by replacing x by a G-conjugate, that $x\in\mathfrak{z}_L+cl(C)+\mathfrak{n}$. Replacing also (L',C') by a G-conjugate, we may assume in addition that $x\in\mathfrak{z}_{L',*}+C'$. Let $x=x_s+x_n$ be the Jordan decomposition of x (x_s is semisimple, x_n is unipotent). From our assumption we have $x_s\in\mathfrak{z}_{L',*},x_n\in C'$. Moreover, x_s is a semisimple element of Lie P, hence it is contained in the Lie algebra of some Levi subgroup of P. Replacing (L,C) by a suitable U_P-conjugate, we may therefore assume in addition that $x_s\in\mathfrak{z}_L$ (then, necessarily, $x_n\in cl(C)+\mathfrak{n}$). Hence $L\subset Z_G(x_s)$. But $Z_G(x_s)=L'$, hence $L\subset L'$ and $\mathfrak{z}_{L'}\subset\mathfrak{z}_L$. Let $z\in\mathfrak{z}_{L'},\gamma\in C'$. Since $x_n\in C'$, we have $\gamma=Ad(g)x_n$ for some $g\in L'$. Note that $Ad(g)z=z$. Hence

$$\begin{aligned}z+\gamma&=z+Ad(g)x_n\\&=Ad(g)(z+x_n)\in Ad(g)(\mathfrak{z}_{L'}+x_n)\subset Ad(g)(\mathfrak{z}_L+cl(C)+\mathfrak{n})\subset Y'_{L,C}.\end{aligned}$$

It follows that $\mathfrak{z}_{L'}+C'\subset Y'_{L,C}$. Hence $Y_{L',C'}\subset Y'_{L,C}$. Note also that $[x_s,x_n]=0$ and $Z_G(x_s)=L'$ imply that $x_n\in$ Lie L'. Since $x_n\in cl(C)+\mathfrak{n}$, we have $x_n\in cl(C)+(\mathfrak{n}\cap\text{Lie }L')$. This shows that x_n is contained in the closure of the nilpotent L'-orbit in Lie L' induced by C (note that $\mathfrak{n}\cap$Lie L' is the nil-radical of the parabolic subalgebra Lie $P\cap$Lie L' of Lie L' with Levi subalgebra Lie L). Thus, (a) and (b) are verified.

Conversely, assume that we are given $(L',C')\in\mathrm{A}$ such that $(L',C')\preceq(L,C)$. We show that $Y_{L',C'}\subset Y'_{L,C}$. We may assume that conditions (a),(b) of 6.4 are satisfied. Then $L\subset L'$ and $Q=P\cap L'$ is a parabolic subgroup of L' with Levi subgroup L. Moreover, C' is contained in $\cup_{g\in L'}Ad(g)(cl(C)+\text{Lie }U_Q)$. We have

$$\begin{aligned}\mathfrak{z}_{L'}+C'&\subset\cup_{g\in L'}Ad(g)(\mathfrak{z}_{L'}+cl(C)+\text{Lie }U_Q)\\&\subset\cup_{g\in L'}Ad(g)(\mathfrak{z}_L+cl(C)+\mathfrak{n})\subset Y'_{L,C},\end{aligned}$$

so that $Y_{L',C'}\subset Y'_{L,C}$. The proposition is proved.

PROPOSITION 6.6. *(a) $Y_{L,C}$ is a locally closed (irreducible) subvariety of $\mathfrak{g}$.*

(b) $Y_{L,C}$ is a smooth variety.

(a) follows from 6.5, using the following general fact, whose proof is immediate. Assume that we are given a partition of an algebraic variety V into finitely many constructible subsets V_j where the index j runs through a finite set J with a given partial order $\preceq$. Assume further that, for any $j\in J$, the closure of V_j is the union $\cup_{j';j'\preceq j}V_{j'}$. Then each V_j is locally closed in V.

We now prove (b). We consider the morphism $Y_{L,C}\to G/N_G(L)$ given by $x\mapsto gN_G(L)$ where $Ad(g^{-1})x\in\mathfrak{z}_{L,*}+C$. This is a G-equivariant fibration over the homogeneous space $G/N_G(L)$ whose fibre at $N_G(L)$ is $\mathfrak{z}_{L,*}+C$, which is non-singular. (b) follows.

6.7. Let A_0 be the subset of A consisting of those $(L,C)\in\mathrm{A}$ such that there exists an irreducible L-equivariant local system $\mathcal{E}$ on C which is *cuspidal* in the sense of [**L3, 2.2**].

LEMMA 6.8. *Let $(L,C),(\tilde{L},\tilde{C})$ be two pairs in A_0.*

(a) Assume that there exists an isomorphism of algebraic groups $f:L\xrightarrow{\sim}\tilde{L}$ whose differential carries C to $\tilde{C}$. Then there exists $g\in G$ such that $\tilde{L}=gLg^{-1}$ and $\tilde{C}=Ad(g)C$.

(b) For any $g\in N_G(L)$, we have $Ad(g)C=C$.

(c) If P_1, P_2 are two parabolic subgroups of G which have L as Levi subgroup, then $P_2 = nP_1n^{-1}$ for some $n \in N_G(L)$.

This follows from the classification of cuspidal local systems [**L3, 2.8**].

6.9. Let $(L, C) \in \mathbf{A}_0$. A pair $(L', C') \in \mathbf{A}$ is said to be (L, C)-regular if $L \subset L'$ and C' is the nilpotent L'-orbit in Lie L' induced by C. A G-orbit in $\mathbf{A}$ is said to be (L, C)-regular if it contains some (L, C)-regular pair.

LEMMA 6.10. *If $(L_1, C_1) \in \mathbf{A}$ belongs to a (L, C)-regular G-orbit and if $L \subset L_1$, then (L_1, C_1) is (L, C)-regular.*

Let (L', C') be an (L, C)-regular pair in the G-orbit of (L_1, C_1). Let $g \in G$ be such that $(L_1, C_1) = (gL'g^{-1}, Ad(g)C')$. Since (L', C') is (L, C)-regular, it follows that (L_1, C_1) is $(gLg^{-1}, Ad(g)C)$-regular. Now $(gLg^{-1}, Ad(g)C)$ and (L, C) can be regarded as elements of the set $\mathbf{A}_0$ (defined in terms of L_1 rather than G) and we may apply to them Lemma 6.8(a) for L_1 instead of G (we take f in that Lemma given by conjugation by g, which does not necessarily preserve L_1). We deduce that there exists $g' \in L_1$ such that $g'gLg^{-1}g'^{-1} = L$ and $Ad(g')Ad(g)C = C$. Since (L_1, C_1) is $(gLg^{-1}, Ad(g)C)$-regular, we see by conjugating by g', that (L_1, C_1) is a (L, C)-regular pair. The lemma is proved.

6.11. Let $(L, C) \in \mathbf{A}_0$. We set $T_{L,C} = \cup_{(L',C')} Y_{L',C'}$, union over all (L, C)-regular G-orbits in $\mathbf{A}$.

PROPOSITION 6.12. *$T_{L,C}$ is an open dense subset of the closure of $Y_{L,C}$ in $\mathfrak{g}$.*

In view of Lemma 6.5, it suffices to prove the following statement.

Assume that $(L', C') \in \mathbf{A}$ is (L, C)-regular and that $(L'', C'') \in \mathbf{A}$ satisfies $(L', C') \preceq (L'', C'') \preceq (L, C)$. Then the G-orbit of (L'', C'') is (L, C)-regular.

We can find $(L_1, C_1), (L_2, C_2)$ in $\mathbf{A}$ such that (L_1, C_1) is in the G-orbit of (L', C'), (L_2, C_2) is in the G-orbit of (L'', C''), $L \subset L_2 \subset L_1$, C_2 is contained in the closure of the nilpotent L_2-orbit C_2' on Lie L_2 induced by C and C_1 is contained in the closure of the nilpotent L_1-orbit C_1' on Lie L_1 induced by C_2. It follows that

(a) $\operatorname{codim}_{\text{Lie } L_2} C_2 \geq \operatorname{codim}_{\text{Lie } L_2} C_2' = \operatorname{codim}_{\text{Lie } L} C$
with equality only if $C_2 = C_2'$, and

(b) $\operatorname{codim}_{\text{Lie } L_1} C_1 \geq \operatorname{codim}_{\text{Lie } L_1} C_1' = \operatorname{codim}_{\text{Lie } L_2} C_2$
with equality only if $C_1 = C_1'$.

From Lemma 6.10 it follows that (L_1, C_1) is (L, C)-regular. Hence C_1 is the nilpotent L_1-orbit on Lie L_1 induced by C, so that

(c) $\operatorname{codim}_{\text{Lie } L_1} C_1 = \operatorname{codim}_{\text{Lie } L} C$.

Combining this with (a),(b), we see that the inequalities in (a) and (b) must be equalities, so that $C_2 = C_2'$ and $C_1 = C_1'$. The equality $C_2 = C_2'$ means that the pair (L_2, C_2) is (L, C)-regular. Thus, the G-orbit of (L'', C'') is (L, C)-regular. The proposition is proved.

6.13. Let $(L, C) \in \mathbf{A}_0$. Let $P, \mathfrak{n}, Y'_{L,C}$ be as in 6.2. Let $x \in Y'_{L,C}$. Choose $g \in G$ such that

(a) $Ad(g^{-1})x \in z + cl(C) + \mathfrak{n}$ where $z \in \mathfrak{z}_L$.

LEMMA 6.14. *(a) The W_L-orbit of z depends only on x, not on the choice of g. Hence $x \mapsto$ (W_L-orbit of z) is a well defined map $d' : Y'_{L,C} \to W_L \backslash \mathfrak{z}_L$.*

(b) If $z, z' \in \mathfrak{z}_L$ are such that $z' = Ad(h')z$ for some $h' \in G$, then z, z' are in the same W_L-orbit.

To prove (a), it is enough to verify the following statement.

(c) *Assume that $y, y' \in$ Lie P and $z, z' \in \mathfrak{z}_L$ are such that $y - z, y' - z'$ are nilpotent elements and $y' = Ad(h)(y)$ for some $h \in G$. Then z, z' are in the same W_L-orbit.*

Let $y = y_s + y_n, y' = y_s' + y_n'$ be the Jordan decompositions of y, y'. Since y_s, y_s' are semisimple elements of Lie P, there exist Levi subgroups L_1, L_1' of P such that $y_s \in$ Lie $L_1, y_s' \in$ Lie L_1'. We have $L_1 = u_1 L u_1^{-1}, L_1' = u_1' L u_1'^{-1}$ where $u_1, u_1' \in U_P$. Then $y - Ad(u_1)z$ is a nilpotent element in Lie P. Let $\rho :$ Lie $P \to$ Lie L_1 be the canonical projection. Then $\rho(y) - Ad(u_1)z$ is a nilpotent element of Lie L_1. (Note that $Ad(u_1)z$ belongs to the centre of Lie L_1.) Hence the semisimple part of $\rho(y) - Ad(u_1)z$ (which is $\rho(y_s) - Ad(u_1)z$) is zero. Since $y_s \in$ Lie L_1, we have $\rho(y_s) = y_s$. Thus, $y_s = Ad(u_1)z$. Similarly, $y_s' = Ad(u_1')z'$. From $y' = Ad(h)y$ it follows that $y_s' = Ad(h)y_s$. Hence $Ad(u_1')z' = Ad(h)Ad(u_1)z$. Thus, $z' = Ad(h')z$ for some $h' \in G$. Thus, the proof of (a) is reduced to that of (b).

We prove (b). Let $M = Z_G(z')$ and let $\tilde{L} = h'Lh'^{-1}, \tilde{C} = Ad(h')C$. Then $(\tilde{L}, \tilde{C})$ and (L, C) can be regarded as elements of $\mathbf{A}_0$ (defined in terms of M rather than G) and we may apply to them Lemma 6.8 for M instead of G (we take f in that lemma given by conjugation by h', which does not necessarily preserve M). We deduce that there exists $h'' \in M$ such that $h''Lh''^{-1} = \tilde{L}, Ad(h'')C = \tilde{C}$. Then $n =: h''^{-1}h' \in N_G(L)$ and $Ad(n)z = Ad(h''^{-1})Ad(h')z = Ad(h''^{-1})z' = z'$. Thus, z, z' are in the same W_L-orbit. The lemma is proved.

6.15. Remark. In the case where $(L', C') \in \mathbf{A}$ is (L, C)-regular, the restriction of the map $d' : Y'_{L,C} \to W_L \backslash \mathfrak{z}_L$ to $Y_{L',C'}$ can be described as follows. Let $x \in Y_{L',C'}$. Choose $g_1 \in G$ such that $Ad(g_1^{-1})x \in z_1 + C'$ where $z_1 \in \mathfrak{z}_{L',\bullet}$. Then $d'(x)$ is the W_L-orbit of z_1. To verify this, it suffices to check that, if $g \in G$ and $z \in \mathfrak{z}_L$ satisfy $Ad(g^{-1})x \in z + cl(C) + \mathfrak{n}$, then z, z_1 are in the same W_L-orbit. (Note that $\mathfrak{z}_{L'} \subset \mathfrak{z}_L$ since $L \subset L'$, hence $z_1 \in \mathfrak{z}_L$.) But this follows from 6.14(c) applied to $y = Ad(g^{-1})x$ and $y' = Ad(g_1^{-1})x$.

6.16. We preserve the setup of 6.11. Let $P, \mathfrak{n}$ be as in 6.2. We set

$$\tilde{T} = \{(x, gP) \in T_{L,C} \times G/P | Ad(g^{-1})x \in \mathfrak{z}_L + C + \mathfrak{n}\}.$$

We consider the commutative diagram

$$\begin{tikzcd} \tilde{T} \arrow[r, "a"] \arrow[d] & \mathfrak{z}_L \arrow[d] \\ T_{L,C} \arrow[r, "d"] & W_L\backslash\mathfrak{z}_L \end{tikzcd}$$

where the vertical maps are the obvious ones, d is the restriction of $d' : Y'_{L,C} \to W_L\backslash\mathfrak{z}_L$ in 6.14(a), and $a(x, gP) \in \mathfrak{z}_L$ is given by $Ad(g^{-1})x \in a(x, gP) + C + \mathfrak{n}$.

PROPOSITION 6.17. *This commutative diagram is cartesian.*

We shall verify this only at the level of sets. It suffices to verify the following statement.

Let $(L', C') \in \mathbf{A}$ be (L, C)-regular. Let $x \in Y_{L',C'}, z \in \mathfrak{z}_L, z' \in \mathfrak{z}_{L',*}, g \in G$ be such that $Ad(g^{-1})x \in z' + C'$ and z, z' are in the same W_L-orbit. Then there is a unique coset $g'P \in G/P$ such that $Ad(g'^{-1})x \in z + C + \mathfrak{n}$.

Let $n \in N_G(L)$ be such that $Ad(n^{-1})z' = z$. By Lemma 6.10, $(n^{-1}L'n, Ad(n^{-1})C')$ is (L, C)-regular. Hence by replacing g, L', C' by gn, $n^{-1}L'n, Ad(n^{-1})C'$, we are reduced to verifying the following statement.

Let $(L', C') \in \mathbf{A}$ be (L, C)-regular. Let $x \in \mathfrak{g}, z \in \mathfrak{z}_{L',*}, g \in G$ be such that $Ad(g^{-1})x \in z + C'$. Then there is a unique coset $g'P \in G/P$ such that $Ad(g'^{-1})x \in z + C + \mathfrak{n}$.

Let $Q = P \cap L'$. Note that Q is a parabolic subgroup of L' with Levi subgroup L. Let C'_0 be the unique open Q-orbit in $C + \text{Lie } U_Q$. Then $C'_0 \subset C'$. Replacing g by gg' for some $g' \in L'$, we may assume that $Ad(g^{-1})x \in z + C'_0$. Replacing x by $Ad(g^{-1})x$ we may assume in addition that $g = 1$. We are therefore reduced to verifying the following statement.

Let $x \in \mathfrak{g}, z \in \mathfrak{z}_{L',*}$ be such that

(a) $x \in z + C'_0$.

Then $g' \in G$ satisfies

(b) $Ad(g'^{-1})x \in z + C + \mathfrak{n}$,

if and only if $g' \in P$.

The fact that (b) holds for $g' \in P$ is clear, since $C'_0 \subset C + \mathfrak{n}$. We prove the converse. We consider $g' \in G$ such that (b) holds; we must prove that $g' \in P$. Let $P' = g'Pg'^{-1}$. We have $x \in \text{Lie } P'$. Let $x = x_s + x_n$ be the Jordan decomposition of x. Since x_s is a semisimple element of Lie P', there exists a Levi subgroup M of P' such that $x_s \in \text{Lie } M$. From (b) we see that the image of x in Lie P'/Lie $U_{P'}$ is a central element plus a nilpotent element. It follows that x_s is a central element of Lie M. From (a) we see that $x_s = z$. Thus, z is a central element of Lie M. Since $L' = Z_G(z)$, it follows that $M \subset L'$. From this we deduce that $Q' = P' \cap L'$ is a parabolic subgroup of L' with Levi subgroup M. Next we note that $g'Lg'^{-1}$ and M are two Levi subgroups of P' hence $M = ug'Lg'^{-1}u^{-1}$ for some $u \in U_{P'}$. Let $C_M = ug'Cg'^{-1}u^{-1}$. Applying Lemma 6.8 to L' instead of G, and to $(L, C), (M, C_M)$, we see that $M = rLr^{-1}$ for some $r \in L'$. Since Q', Q are parabolic subgroups of L' with Levi subgroups M, L, we see from Lemma 6.8(c) that r can be assumed to satisfy in addition $Q' = rQr^{-1}$. Applying $Ad(g')$ to (b) we see that $x \in \mathfrak{z}_M + C_M + \text{Lie } U_{P'}$. From (a), we have $x \in \text{Lie } L'$. It follows that

$$x \in (\mathfrak{z}_M + C_M + \text{Lie } U_{P'}) \cap \text{Lie } L' = \mathfrak{z}_M + C_M + \text{Lie } U_{Q'}.$$

By (a), the semisimple part of x is z, in the centre of Lie L'; it follows that $x \in z + C_M + \text{Lie } U_{Q'}$. Applying $Ad(r^{-1})$, we deduce $Ad(r^{-1})x \in z + C + \text{Lie } U_Q$. The last equality and (a) can be rewritten as follows:

$$x - z \in C'_0, \quad Ad(r^{-1})(x - z) \in C + \text{Lie } U_Q.$$

(Recall that $r \in L'$). This implies, by [**L3, 2.14**], that $r \in Q$. We then have $Q' = Q$, or equivalently, $P' \cap L' = P \cap L'$. In particular, P' has L as a Levi subgroup. Using Lemma 6.8(c), it follows that $P' = nPn^{-1}$ for some $n \in N_G(L)$. Since $P' = g'Pg'^{-1}$, we have $g' = np$ for some $p \in P$. Replacing g' by $g'p^{-1}$, we see that we may assume that $g' = n$. From (b) we can deduce $x \in Ad(n^{-1})z + C + \text{Lie } U_{P'}$. As before, from this we deduce $x \in Ad(n^{-1})z + C + \text{Lie } U_{Q'}$ and, using the fact that the semisimple part of x is z (in the centre of Lie L'), we see that $Ad(n^{-1})z = z$. Thus $n \in Z_G(z) = L'$. From $n \in L', P' = nPn^{-1}, P' \cap L' = P \cap L'$, we deduce $n(P \cap L')n^{-1} = P \cap L'$. Since the parabolic subgroup $P \cap L'$ of L' is self-normalizing in L', it follows that $n \in P \cap L'$. In particular, $g' = n \in P$. The proposition is proved.

7. W-action on certain perverse sheaves

7.1. In the rest of this paper, we fix a G-orbit $\mathbf{o}$ in $\mathbf{A}_0$ (see 6.7). Let $\mathbf{o}'$ be the set of all triples (P, L, C) where $(L, C) \in \mathbf{o}$ and P is a parabolic subgroup of G with Levi subgroup L. Let $\mathfrak{P}$ be the set of all P such that $(P, L, C) \in \mathbf{o}'$ for some $(L, C) \in \mathbf{o}$. For $P \in \mathfrak{P}$, we define $C_P \subset \text{Lie } \bar{P}$ as the image of C under the canonical map $\pi_P : \text{Lie } P \to \text{Lie } \bar{P}$ where $(L, C) \in \mathbf{o}$ is such that $(P, L, C) \in \mathbf{o}'$.

From Lemma 6.8(c) it follows that the natural action of G on $\mathbf{o}'$ by conjugation on all factors is transitive. It follows that $\mathfrak{P}$ is a single G-conjugacy class of parabolic subgroups of G and, for $P \in \mathfrak{P}$, the nilpotent $\bar{P}$-orbit C_P in Lie $\bar{P}$ depends only on P, and not on the choice of (L, C) above. Let

$$\mathcal{V} = \cup_{P \in \mathfrak{P}} \pi_P^{-1}(\mathfrak{z}_{\bar{P}} + cl(C_P))$$

($cl(C_P)$ is the closure of C_P in Lie $\bar{P}$);

$$\mathcal{V}_{op} = \cup_{(L,C) \in \mathbf{o}}(\mathfrak{z}_{L,*} + C);$$

$$\mathcal{V}_R = \cup_{(L',C')}(\mathfrak{z}_{L',*} + C')$$

(union over all $(L', C') \in \mathbf{A}$ that are (L, C)-regular for some $(L, C) \in \mathbf{o}$);

$$\mathcal{V}_N = \mathcal{V} \cap \mathfrak{g}_N = \cup_{P \in \mathfrak{P}} \pi_P^{-1}(cl(C_P));$$

$$\mathcal{V}_{RN} = \mathcal{V}_R \cap \mathfrak{g}_N.$$

Clearly,

$$\mathcal{V} = Y'_{L,C}, \mathcal{V}_{op} = Y_{L,C}, \mathcal{V}_R = T_{L,C}, \mathcal{V}_N = Y'_{L,C,N}$$

(as in 6.2, 6.1, 6.11, 6.3) for $(L, C) \in \mathbf{o}$, in a more invariant description. We have a diagram with a cartesian square

$$\begin{array}{ccccc} & & \mathcal{V}_{RN} & \longrightarrow & \mathcal{V}_N \\ & & \downarrow & & \downarrow \\ \mathcal{V}_{op} & \longrightarrow & \mathcal{V}_R & \longrightarrow & \mathcal{V} \end{array}$$

in which the maps are obvious inclusions. Note the following facts (where $(L, C) \in \mathbf{o}$):

(a) $\mathcal{V}$ *is an irreducible closed subvariety of* $\mathfrak{g}$, *of dimension* $\dim(G/L) + \dim \mathfrak{z}_L + \dim C$;

(b) $\mathcal{V}_R$ *is an open dense subvariety of* $\mathcal{V}$;

(c) $\mathcal{V}_{op}$ *is an open dense smooth subvariety of* $\mathcal{V}_R$;

(d) $\mathcal{V}_N$ *is an irreducible closed subvariety of* $\mathcal{V}$ *of dimension* $\dim(G/L)+\dim C$;

(e) $\mathcal{V}_{RN}$ *is equal to the nilpotent G-orbit in* $\mathfrak{g}$ *induced from L by C; it is an open dense subvariety of* $\mathcal{V}_N$.

(a),(b),(c) follow from 6.2, 6.1, 6.12, 6.6. The first assertion of (e) follows from definitions. From properties of induced nilpotent orbits it follows that the codimension of $\mathcal{V}_{RN}$ in $\mathfrak{g}$ equals the codimension of C in L, hence $\dim \mathcal{V}_{RN} = \dim(G/L) + \dim C$. The fact that $\mathcal{V}_{RN}$ is open in $\mathcal{V}_N$ follows from the fact that $\mathcal{V}_R$ is open in $\mathcal{V}$. Since $\mathcal{V}_N$ is irreducible (see 6.3), $\mathcal{V}_{RN}$ is dense in $\mathcal{V}_N$ and $\dim \mathcal{V}_N = \dim \mathcal{V}_{RN}$. Thus (d) and (e) hold.

7.2. Let

$$\dot{\mathfrak{g}} = \{(x, P) \in \mathfrak{g} \times \mathfrak{P} | x \in \pi_P^{-1}(\mathfrak{z}_{\bar{P}} + C_P)\}$$

and let $\dot{\mathcal{V}}, \dot{\mathcal{V}}_{op}, \dot{\mathcal{V}}_R, \dot{\mathcal{V}}_N, \dot{\mathcal{V}}_{RN}$ be the inverse images of $\mathcal{V}, \mathcal{V}_{op}, \mathcal{V}_R, \mathcal{V}_N, \mathcal{V}_{RN}$ (respectively) under the first projection $pr_1 : \dot{\mathfrak{g}} \to \mathfrak{g}$. Clearly, $\dot{\mathfrak{g}} = \dot{\mathcal{V}}$ and

$$\dot{\mathcal{V}}_N = \{(x, P) \in \mathfrak{g}_N \times \mathfrak{P} | x \in \pi_P^{-1}(C_P)\}.$$

From the diagram in 7.1, we obtain, by taking inverse images under $pr_1 : \dot{\mathfrak{g}} \to \mathfrak{g}$, a diagram with a cartesian square

$$\begin{array}{ccccc} & & \dot{\mathcal{V}}_{RN} & \longrightarrow & \dot{\mathcal{V}}_N \\ & & \downarrow & & \downarrow \\ \dot{\mathcal{V}}_{op} & \longrightarrow & \dot{\mathcal{V}}_R & \longrightarrow & \dot{\mathcal{V}} \end{array}$$

in which the maps are obvious inclusions.

7.3. The group W. Let $\underline{\mathfrak{P}}$ be the set of all pairs (P, L) where $P \in \mathfrak{P}$ and L is a Levi subgroup of P. We consider the vector bundle over $\underline{\mathfrak{P}}$ whose fibre at (P, L) is the vector space $\mathfrak{z}_L$. This is a G-equivariant vector bundle in a natural way; let $\mathfrak{z}$ be the space of sections of this vector bundle which are G-invariant. For each $(P, L) \in \underline{\mathfrak{P}}$ we have an obvious linear map $\mathfrak{z} \to \mathfrak{z}_L$ (evaluation at (P, L)). One checks easily that this is an isomorphism. We denote its inverse by $a_{P,L} : \mathfrak{z}_L \xrightarrow{\sim} \mathfrak{z}$.

The diagonal action of G on $\mathfrak{P} \times \mathfrak{P}$ has only finitely many orbits; an orbit is said to be *good* if it consists of pairs (P, P') such that P, P' have a common Levi subgroup. Let W be the set of good G-orbits on $\mathfrak{P} \times \mathfrak{P}$. If $\mathcal{O}, \mathcal{O}'$ are two good G-orbits on $\mathfrak{P} \times \mathfrak{P}$, we define $\mathcal{O}''$ to be the set of all $(P_1, P_3) \in \mathfrak{P} \times \mathfrak{P}$ with the following property: there exists $P_2 \in \mathfrak{P}$ such that $(P_1, P_2) \in \mathcal{O}, (P_2, P_3) \in \mathcal{O}'$ and P_1, P_2, P_3 have a common Levi subgroup. Then $\mathcal{O}''$ is a good G-orbit on $\mathfrak{P} \times \mathfrak{P}$ and $\mathcal{O}, \mathcal{O}' \mapsto \mathcal{O}''$ defines a group structure on W.

For any $(P, L) \in \underline{\mathfrak{P}}$, we define $b_{P,L} : W_L \to W$ by $n \mapsto G$-orbit of (P, nPn^{-1}). This is an isomorphism of groups. There is a natural action of the finite group W on the vector space $\mathfrak{z}$, which under $a_{P,L}, b_{P,L}$ corresponds to the *Ad*-action of W_L on $\mathfrak{z}_L$ (for any (P, L)).

7.4. The set I. We consider the set of all parabolic subgroups of G which contain strictly some parabolic subgroup in $\mathfrak{P}$ and are minimal with this property. G acts on this set by conjugation. Our set decomposes into a disjoint union of G-orbits: $\sqcup_{i \in I} \mathfrak{P}_i$; here I is a finite set that indexes the set of orbits. Given $i \in I$, we consider the set of all pairs $(P_1, P_2) \in \mathfrak{P} \times \mathfrak{P}$ such that P_1, P_2 contain a common Levi subgroup and the parabolic subgroup generated by P_1, P_2 belongs to $\mathfrak{P}_i$. This is a good G-orbit in $\mathfrak{P} \times \mathfrak{P}$ hence it is an element of W, denoted by s_i.

The elements s_i for $i \in I$ generate W and W together with the generators s_i is a finite Coxeter group (see [**L1, 9.2**]).

7.5. Let $\mathfrak{z}_*$ be the open subset of $\mathfrak{z}$ corresponding to $\mathfrak{z}_{L,*} \subset \mathfrak{z}_L$ under $a_{P,L} : \mathfrak{z}_L \xrightarrow{\sim} \mathfrak{z}$. Note that W acts freely on $\mathfrak{z}_*$.

To $(x, P) \in \dot{\mathcal{V}}$ we associate $z \in \mathfrak{z}$ as follows. We choose L such that $(P, L) \in \underline{\mathfrak{P}}$ and define z by the conditions

$$z' \in \mathfrak{z}_L, \pi_P(x - z') \in C_P, a_{P,L}(z') = z.$$

Note that z is independent of the choice of L. We thus have a morphism

(a) $$a' : \dot{\mathcal{V}} \to \mathfrak{z}, \quad a'(x, P) = z.$$

We denote by $\bar{a} : \dot{\mathcal{V}}_R \to \mathfrak{z}$ the restriction of a' to $\dot{\mathcal{V}}_R$. Then $\bar{a}$ is part of a commutative diagram

(b) $$\begin{array}{ccc} \dot{\mathcal{V}}_R & \xrightarrow{\bar{a}} & \mathfrak{z} \\ \downarrow & & \downarrow \\ \mathcal{V}_R & \xrightarrow{\bar{d}} & W\backslash\mathfrak{z} \end{array}$$

where the vertical maps are the obvious ones and $\bar{d}$ is uniquely determined. As a consequence of 6.17, this diagram is in fact cartesian. By a routine argument we see that

(c) $$\dot{\mathcal{V}}_{op} = a'^{-1}(\mathfrak{z}_*) = \bar{a}^{-1}(\mathfrak{z}_*) \text{ and } \mathcal{V}_{op} = \bar{d}^{-1}(W\backslash\mathfrak{z}_*).$$

Hence we have a cartesian diagram

(d) $$\begin{array}{ccc} \dot{\mathcal{V}}_{op} & \longrightarrow & \mathfrak{z}_* \\ \downarrow & & \downarrow \\ \mathcal{V}_{op} & \longrightarrow & W\backslash\mathfrak{z}_* \end{array}$$

where the vertical maps are the obvious ones and the horizontal maps are the restrictions of $\tilde a, \tilde d$ in (b). From the definitions we see that

(e) $\dot{\mathcal{V}}_{RN} = \tilde a^{-1}(0)$ and $\mathcal{V}_{RN} = \tilde d^{-1}(0)$.

It follows that

(f) *the canonical map $\dot{\mathcal{V}}_{RN} \to \mathcal{V}_{RN}$ is an isomorphism.*

LEMMA 7.6. *(a) $\dot{\mathcal{V}}$ is a smooth irreducible algebraic variety.*
(b) $\dot{\mathcal{V}}_R$ is an open dense subvariety of $\dot{\mathcal{V}}$.
(c) $\dot{\mathcal{V}}_{op}$ is an open dense subvariety of $\dot{\mathcal{V}}_R$.

Using the second projection $\dot{\mathcal{V}} \to \mathfrak{P}$ and the fact that for $P \in \mathfrak{P}$, the variety $\mathfrak{z}_{\bar P} + C_P$ is smooth, irreducible, we see that (a) holds. From the cartesian diagram 7.5(b), we see that $\dot{\mathcal{V}}_R$ is non-empty; $\dot{\mathcal{V}}_R$ is open in $\dot{\mathcal{V}}$ since $\dot{\mathcal{V}}_R$ is the inverse image of $\mathcal{V}_R$ under $\dot{\mathcal{V}} \to \mathcal{V}$ and $\mathcal{V}_R$ is open in $\mathcal{V}$ (see 7.1(b)). This proves (b). The proof of (c) is like that of (b).

7.7. On the fibre product $\mathcal{V}_R \times_{(W\backslash\mathfrak{z})} \mathfrak{z}$ we have a natural action of W. This is the restriction of the W-action on $\mathcal{V}_R \times \mathfrak{z}$ which is the identity on the first factor and the obvious action (see 7.3) on the second factor. We can transport this W-action to $\dot{\mathcal{V}}_R$ under the isomorphism $\dot{\mathcal{V}}_R \xrightarrow{\sim} \mathcal{V}_R \times_{(W\backslash\mathfrak{z})} \mathfrak{z}$ given by the cartesian diagram 7.5(b). We obtain a W-action on $\dot{\mathcal{V}}_R$. Since this W-action is obtained from the one on $\mathfrak{z}$ by change of base, we see that

(a) *the canonical map $W\backslash\dot{\mathcal{V}}_R \to \mathcal{V}_R$ induced by $\dot{\mathcal{V}}_R \to \mathcal{V}_R$, is an isomorphism of algebraic varieties.*

We denote the W-action on $\dot{\mathcal{V}}_R$ by $w : q \mapsto \rho_w(q)$. From 7.5(e) we see that

(b) $\rho_w(q) = q$ for all $w \in W$ and $q \in \dot{\mathcal{V}}_{RN}$.

7.8. Let $\mathbf{c}$ be the set of all pairs (x, P) where $P \in \mathfrak{P}$ and $x \in C_P$. Clearly, $\mathbf{c}$ is a homogeneous variety of G (for the conjugation action).

ASSUMPTION. *In the remainder of this paper we assume that we are given an irreducible, G-equivariant local system $\mathcal{F}$ on $\mathbf{c}$, such that, for any $P \in \mathfrak{P}$, the inverse image of $\mathcal{F}$ under $C_P \to \mathbf{c}$, $x \mapsto (x, P)$ (an irreducible $\bar P$-equivariant local system on C_P) is cuspidal in the sense of* [**L3, 2.2**].

The irreducibility of the inverse image follows from the fact that the isotropy group of (x, P) in G has the same group of components as the isotropy group of x in $\bar P$; this shows also that $\mathcal{F}$ always exists, under our assumption that $\mathbf{o} \in \mathbf{A}_0$.

Note that $\mathcal{F}^*$ is a local system of the same kind as $\mathcal{F}$.

7.9. Let $\dot{\mathcal{V}} \xrightarrow{s} \mathbf{c}$ be given by $s(x, P) = (x', P)$ where $x' \in C_P, \pi_P(x) - x' \in \mathfrak{z}_{\bar P}$. Then $\mathcal{L} = s^*\mathcal{F}$ is a local system on $\dot{\mathcal{V}}$; we have $\mathcal{L}^* = s^*\mathcal{F}^*$. The restrictions of $\mathcal{L}$ to the open subsets $\dot{\mathcal{V}}_R, \dot{\mathcal{V}}_{op}$ are local systems denoted by $\mathcal{L}_R, \mathcal{L}_{op}$. The restriction of $\mathcal{L}$ to the subvariety $\dot{\mathcal{V}}_{RN}$ is a local system denoted by $\mathcal{L}_{RN}$.

LEMMA 7.10. *(a) The local systems $\mathcal{L}^*, \mathcal{L}_R^*, \mathcal{L}_{op}^*, \mathcal{L}_{RN}^*$ on $\dot{\mathcal{V}}, \dot{\mathcal{V}}_R, \dot{\mathcal{V}}_{op}, \dot{\mathcal{V}}_{RN}$ respectively, are irreducible.*

(b) Let $w \in W$ and let $\rho_w : \dot{\mathcal{V}}_R \to \dot{\mathcal{V}}_R$ be as in 7.7. There is a unique isomorphism $\tau_w : \mathcal{L}_R^* \xrightarrow{\sim} \rho_w^*(\mathcal{L}_R^*)$ of local systems over $\dot{\mathcal{V}}_R$ such that the restriction $\mathcal{L}_{RN}^* \xrightarrow{\sim} \mathcal{L}_{RN}^*$ of τ_w to $\mathcal{L}_{RN}^*$ is the identity. Recall (7.7(b)) that w acts as the identity on $\dot{\mathcal{V}}_{RN}$.

(c) Let $w, w' \in W$. For any $q \in \dot{\mathcal{V}}_R$, the map on stalks $\mathcal{L}_{R,q}^* \xrightarrow{\tau_{ww'}} \mathcal{L}_{R,\rho_{ww'}(q)}^*$ coincides with the composition

$$\mathcal{L}_{R,q}^* \xrightarrow{\tau_{w'}} \mathcal{L}_{R,\rho_{w'}(q)}^* \xrightarrow{\tau_w} \mathcal{L}_{R,\rho_w(\rho_{w'}(q))}^*.$$

We first show that $\mathcal{L}_{op}^*$ is an irreducible local system. It is enough to observe that the fibres of the fibration $\dot{\mathcal{V}}_{op} \to \mathbf{c}$ (restriction of $s : \dot{\mathcal{V}} \to \mathbf{c}$) are connected: these fibres are isomorphic to $\mathfrak{z}_{L,*} \times \mathrm{Lie}\, U_P$, where $(P, L) \in \mathfrak{P}$.

Using 7.6(c),(b) we see that the irreducibility of $\mathcal{L}_{op}^*$ implies the irreducibility of $\mathcal{L}_R^*$ and that of $\mathcal{L}^*$. To prove the irreducibility of $\mathcal{L}_{RN}^*$, it is enough to observe that the fibres of the fibration $\dot{\mathcal{V}}_{RN} \to \mathbf{c}$ (restriction of $s : \dot{\mathcal{V}} \to \mathbf{c}$) are connected: the fibre at (x, P) is isomorphic to an open dense subset of $\mathrm{Lie}\, U_P$, hence is connected. Thus, (a) is proved.

We prove (b). Clearly, (b) is an immediate consequence of (a) and the statement (d) below.

(d) *The local systems $\rho_w^*(\mathcal{L}_R^*), \mathcal{L}_R^*$ over $\dot{\mathcal{V}}_R$ are isomorphic.*

Using the fact that $\dot{\mathcal{V}}_{op}$ is open dense in $\dot{\mathcal{V}}_R$, we see that (d) is a consequence of the following statement.

(e) *The local systems $\rho_w^*(\mathcal{L}_{op}^*), \mathcal{L}_{op}^*$ over $\dot{\mathcal{V}}_{op}$ are isomorphic.*

(We denote the restriction of $\rho_w : \dot{\mathcal{V}}_R \to \dot{\mathcal{V}}_R$ to $\dot{\mathcal{V}}_{op}$ again by ρ_w.)

We prove (e). We choose $(P, L, C) \in \mathbf{o}'$. Let $\mathcal{F}_L^*$ be the inverse image of $\mathcal{F}^*$ under $C \to \mathbf{c}$, $x \mapsto (\pi_P(x), P)$. This is an irreducible L-equivariant cuspidal local system on $\mathbf{c}$. According to [**L1, 9.2(b)**], if $\bar w \in N_G(L)$ represents w, then

(f) $\bar w^*(\mathcal{F}_L^*) \cong \mathcal{F}_L^*$ *as local systems on C.*

($\bar w : C \to C$ denotes conjugation by $\bar w$.) We identify $W = W_L$ and $\tilde Y_{L,C}$ (see 6.1) with $\dot{\mathcal{V}}_{op}$ by $(x, gL) \mapsto (x, gPg^{-1})$. The W-action on $\dot{\mathcal{V}}_{op}$ becomes the W_L-action on $\tilde Y_{L,C}$ given by $n : (x, gL) \mapsto (x, gn^{-1}L)$, where $n \in N_G(L)$. Since $\tilde Y_{L,C}$ is fibered (under the second projection) over the simply connected manifold G/L, with fibre over L equal to $\mathfrak{z}_{L,*} + C$, it is enough to show that the restrictions of $\mathcal{L}_{op}^*, \rho_w^*(\mathcal{L}_{op}^*)$ to this fibre are isomorphic. From the definitions, we see that these restrictions may be identified with $\mathcal{F}_L^*, \bar w^*(\mathcal{F}_L^*)$, pulled back under the canonical projection $\mathfrak{z}_{L,*} + C \to C$; it remains to use (f). Thus, (b) is proved. Now (c) follows immediately from (b). The lemma is proved.

7.11. Let

(a) $K_R = (pr_1)_!(\mathcal{L}_R^*) \in \mathcal{D}\mathcal{V}_R$, where $pr_1 : \dot{\mathcal{V}}_R \to \mathcal{V}_R$ is the first projection;

(b) $K_{op} = (pr_1)_!(\mathcal{L}_{op}^*) \in \mathcal{D}\mathcal{V}_{op}$, where $pr_1 : \dot{\mathcal{V}}_{op} \to \mathcal{V}_{op}$ is the first projection;

(c) $K = (pr_1)_!\mathcal{L}^* \in \mathcal{D}\mathfrak{g}$, where $pr_1 : \dot{\mathfrak{g}} = \dot{\mathcal{V}} \to \mathfrak{g}$ is the first projection.

Note that

(d) the restriction of K to $\mathcal{V}_R$ is canonically K_R; the restriction of K_R to $\mathcal{V}_{op}$ is canonically K_{op}.

PROPOSITION 7.12. *(a) K_R is a constructible sheaf on $\mathcal{V}_R$. The stalk of K_R at $x \in \mathcal{V}_R$ is canonically isomorphic to $\oplus_q \mathcal{F}^*_{s(q)}$ (q runs over the finite set $pr_1^{-1}(x)$ and $s(q) \in \mathbf{c}$ is as in 7.9).*

(b) K_{op} is a local system on $\mathcal{V}_{op}$.

(c) K is the intersection cohomology complex on $\mathfrak{g}$ associated to the local system K_{op} on the open dense smooth subvariety $\mathcal{V}_{op}$ of the closed subvariety $\mathcal{V}$ of $\mathfrak{g}$.

(d) K_R is the intersection cohomology complex on $\mathcal{V}_R$ associated to the local system K_{op} on the open dense smooth subvariety $\mathcal{V}_{op}$ of $\mathcal{V}_R$.

(a) follows from the definitions using the fact that $pr_1 : \dot{\mathcal{V}}_R \to \mathcal{V}_R$ is a finite (ramified) covering (see the cartesian diagram 7.5(b)); (b) follows from the fact that $pr_1 : \dot{\mathcal{V}}_{op} \to \mathcal{V}_{op}$ is a finite unramified covering (see the cartesian diagram 7.5(d)). (c) is proved in **[L3, 3.4]**; the proof is based on the cleanness property **[L2, 23.1]** of cuspidal character sheaves. (d) follows from (c) by restriction to the open subset $\mathcal{V}_R$ of $\mathcal{V}$.

7.13. Let $w \in W$. We define an isomorphism $\tau'_w : K_R \xrightarrow{\sim} K_R$ of constructible sheaves on $\mathcal{V}_R$ as the composition

$$K_R = (pr_1)_!(\mathcal{L}^*_R) \xrightarrow{\sim} (pr_1)_!(\rho_w^*\mathcal{L}^*_R) \xrightarrow{\sim} (pr_1)_!(\mathcal{L}^*_R) = K_R$$

where the first isomorphism is induced by $\tau_w : \mathcal{L}^*_R \xrightarrow{\sim} \rho_w^*\mathcal{L}^*_R$ (see 8.10(b)) and the second isomorphism is obtained from the cartesian diagram

$$\begin{array}{ccc} \dot{\mathcal{V}}_R & \xrightarrow{\rho_w} & \dot{\mathcal{V}}_R \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \mathcal{V}_R & \xrightarrow{1} & \mathcal{V}_R \end{array}$$

From 7.10(c) we see that $\tau'_{ww'} = \tau'_w\tau'_{w'}$ for $w, w' \in W$. Thus, $w \mapsto \tau'_w$ defines an algebra homomorphism $\mathbf{C}[W] \to End_{\mathcal{D}\mathcal{V}_R} K_R$.

PROPOSITION 7.14. *Consider the algebra homomorphisms*

$$\mathbf{C}[W] \to End_{\mathcal{D}\mathcal{V}_R} K_R \to End_{\mathcal{D}\mathcal{V}_{op}} K_{op} \leftarrow End_{\mathcal{D}\mathfrak{g}} K;$$

the first one is defined in 7.13; the second and third one are defined by restriction. All these homomorphisms are isomorphisms.

The second and third homomorphism are isomorphisms by 7.12(c),(d) and the definition of an intersection cohomology complex. It remains to show that the composition $\mathbf{C}[W] \to End_{\mathcal{D}\mathcal{V}_{op}} K_{op}$ is an isomorphism. We first show that this map is injective. It is enough to show that, for any $x \in \mathcal{V}_{op}$, the induced homomorphism from $\mathbf{C}[W]$ to the space of endomorphisms of the stalk of K_{op} at x is injective. By 7.12(a), that stalk is naturally decomposed in a direct sum indexed by a principal homogeneous space of W; from the definitions we see that the W-action on the stalk permutes the summands compatibly with the obvious W-action on the indices. The injectivity of our map follows. To prove surjectivity, it is enough to show that

$$\dim End_{\mathcal{D}\mathcal{V}_{op}} K_{op} \le \sharp W.$$

The passage from the local system $\mathcal{L}^*_{op}$ to the local system K_{op} corresponds to induction from the fundamental group Π' of $\dot{\mathcal{V}}_{op}$ at a point q to the fundamental group Π of $\mathcal{V}_{op}$ at $pr_1(q)$. Note that Π' is a normal subgroup of Π and $\Pi/\Pi' = W$. The endomorphism algebra of the representation of Π induced by an irreducible representation of Π' has dimension $\le \sharp W$, by Mackey's theorem. The proposition follows.

7.15. Note that $G \times \mathbf{C}^*$ acts on $\mathbf{c}$ by $(g,t) : (x,P) \mapsto (t^{-2}Ad(g)x, gPg^{-1})$. (Here we regard $Ad(g)$ as a map $C_P \to C_{gPg^{-1}}$.) The local systems $\mathcal{F}, \mathcal{F}^*$ on $\mathbf{c}$ are automatically $G \times \mathbf{C}^*$-equivariant since they are G-equivariant. (We use the fact that, for $(x,P) \in \mathbf{c}$, the group of components of the stabilizer of (x,P) in G is the same as the group of components of the stabilizer of (x,P) in $G \times \mathbf{C}^*$, since $Z_{\bar{P}}(x), \tilde{Z}_{\bar{P}}(x)$ have the same groups of components.)

Next $G \times \mathbf{C}^*$ acts on $\dot{\mathcal{V}}$ by $(g,t) : (x,P) \mapsto (t^{-2}Ad(g)x, gPg^{-1})$ and the morphism $s : \dot{\mathcal{V}} \to \mathbf{c}$ (see 7.9) commutes with the $G \times \mathbf{C}^*$-actions. It follows that $\mathcal{L} = s^*\mathcal{F}$ and $\mathcal{L}^* = s^*\mathcal{F}^*$ are $G \times \mathbf{C}^*$-equivariant local systems on $\dot{\mathcal{V}}$.

The action of $G \times \mathbf{C}^*$ on $\dot{\mathcal{V}}$ leaves stable the subvarieties $\dot{\mathcal{V}}_N, \dot{\mathcal{V}}_R, \dot{\mathcal{V}}_{RN}, \dot{\mathcal{V}}_{op}$ and is compatible with the action $(g,t) : x \to t^{-2}Ad(g)x$ on $\mathfrak{g}$ under $pr_1 : \dot{\mathfrak{g}} \to \mathfrak{g}$; moreover, this last action on $\mathfrak{g}$ leaves stable the subvarieties $\mathcal{V}_N, \mathcal{V}_R, \mathcal{V}_{RN}, \mathcal{V}_{op}$. In particular, $\mathcal{L}^*_{op}$ (see 7.9) is a $G \times \mathbf{C}^*$-equivariant local system on $\dot{\mathcal{V}}_{op}$ and K_{op} (see 7.11, 7.12(b)) is a $G \times \mathbf{C}^*$-equivariant local system on $\mathcal{V}_{op}$. From 7.12(c) it then follows that $K[\dim \mathcal{V}]$ (see 7.11) is a $G \times \mathbf{C}^*$-equivariant perverse sheaf on $\mathfrak{g}$. Hence, as in 1.16, we may regard K as an object of $\mathcal{D}_{G \times \mathbf{C}^*}\mathfrak{g}$.

We have canonical algebra isomorphisms

(a) $$\mathbf{C}[W] = \mathrm{Hom}_{\mathcal{D}\mathfrak{g}}(K,K) = \mathrm{Hom}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K)$$

(the first equality is given by 7.14, the second equality is given by 1.16(a)).

7.16. We define

$$\ddot{\mathcal{V}} = \dot{\mathcal{V}} \times_{\mathcal{V}} \dot{\mathcal{V}} = \{(x,P,P') \in \mathcal{V} \times \mathfrak{P} \times \mathfrak{P} | (x,P) \in \dot{\mathcal{V}}, (x,P') \in \dot{\mathcal{V}}\}.$$

We define $K^* \in \mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}$ in terms of $\mathcal{L}$ in the same way as K was defined in terms of $\mathcal{L}^*$; then, by a property of intersection cohomology,

$$DK = K^*[2\dim \mathcal{V}].$$

Note that $\mathcal{L}, \mathcal{L}^*$ may be pulled back to local systems on $\ddot{\mathcal{V}}$ using the projections $\ddot{\mathcal{V}} \to \dot{\mathcal{V}}$ given by $(x,P,P') \to (x,P), (x,P,P') \to (x,P')$ respectively. (We will often denote the pull-back of a local system by the same letter as the original local system, to lighten the notation.) Taking tensor product, we obtain a local system $\ddot{\mathcal{L}} := \mathcal{L} \otimes \mathcal{L}^*$ on $\ddot{\mathcal{V}}$.

Note also that the $G \times \mathbf{C}^*$-actions on $\mathcal{V}, \dot{\mathcal{V}}$ induce a $G \times \mathbf{C}^*$-action on $\ddot{\mathcal{V}}$ and that the local system $\ddot{\mathcal{L}}$ on $\ddot{\mathcal{V}}$ is $G \times \mathbf{C}^*$-equivariant.

8. The algebra H

8.0. We introduce some further notation. Let $\mathfrak{s} = \mathrm{Lie}\ SL_2(\mathbf{C})$ with its standard basis

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

such that $[e,f]=h,[h,e]=2e,[h,f]=-2f$. Let J be the set of all Lie algebra homomorphisms $\mathfrak{s}\to\mathfrak{g}$. For $y\in\mathfrak{g}_N$, let J_y be the set of all $\phi\in J$ such that $\phi(e)=y$. By the Morozov-Jacobson-Kostant theorem,

(a) *J_y is non-empty and the unipotent radical of $Z_G^0(y)$ acts simply transitively on J_y.*

For $t\in\mathbf{C}^*$, we set $\mathrm{d}_t=\left(\begin{smallmatrix}t&0\\0&t^{-1}\end{smallmatrix}\right)$. Note that $G\times\mathbf{C}^*$ acts on J by $(g,t):\phi\mapsto Ad(g)\circ\phi\circ Ad(\mathrm{d}_t)^{-1}$. The map $J\to\mathfrak{g}_N$ given by $\phi\to\phi(e)$ commutes with the $G\times\mathbf{C}^*$-actions. For $\phi\in J$, we set

$$Z_G(\phi)=\{g\in G|Ad(g)\circ\phi=\phi\}$$

and we denote by $\tilde{Z}_G(\phi)$ the stabilizer of ϕ in $G\times\mathbf{C}^*$:

$$\tilde{Z}_G(\phi)=\{(g,t)\in G\times\mathbf{C}^*|Ad(g)\circ\phi=\phi\circ Ad(\mathrm{d}_t)\}.$$

Given $\phi\in J$, there is a unique homomorphism of algebraic groups $\hat{\phi}:SL_2(\mathbf{C})\to G$ such that the differential of $\hat{\phi}$ is $\phi:\mathfrak{s}\to\mathfrak{g}$. Clearly,

(b) *the map $(g,t)\to(g\hat{\phi}(\mathrm{d}_t),t)$ defines an isomorphism $Z_G(\phi)\times\mathbf{C}^*\xrightarrow{\sim}\tilde{Z}_G(\phi)$ of algebraic groups.*

A known consequence of (a) is that $Z_G(\phi)$ is a maximal reductive subgroup of $Z_G(\phi(e))$ and that $\tilde{Z}_G(\phi)$ is a maximal reductive subgroup of $\tilde{Z}_G(\phi(e))$.

8.1. We consider the nilpotent G-orbit

$$\mathcal{V}_0=\cup_{(L,C)\in\mathfrak{o}}C$$

in $\mathfrak{g}$. Let $\mathfrak{c}_1$ be the variety consisting of all quadruples (ϕ,P,L,C) where $(P,L,C)\in\mathfrak{o}'$ and $\phi\in J$ satisfies $\phi(\mathfrak{s})\subset\mathrm{Lie}\,L$ and $\phi(e)\in C$. Note that $G\times\mathbf{C}^*$ acts

- on $\mathfrak{c}$ by $(g,t):(x,P)\mapsto(t^{-2}Ad(g)x,gPg^{-1})$,
- on $\mathfrak{c}_1$ by $(g,t):(\phi,P,L,C)\mapsto(Ad(g)\circ\phi\circ Ad(\mathrm{d}_t)^{-1},gPg^{-1},gLg^{-1},gCg^{-1})$.

These actions, and even their restrictions to actions of G (via $g\mapsto(g,1)$) are transitive.

Let $\mathbf{X}$ (resp. $\mathbf{X}_1$) be the set of all isomorphism classes of G-equivariant line bundles over $\mathfrak{c}$ (resp. $\mathfrak{c}_1$). Let $\tilde{\mathbf{X}}$ (resp. $\tilde{\mathbf{X}}_1$) be the set of all isomorphism classes of $G\times\mathbf{C}^*$-equivariant line bundles over $\mathfrak{c}$ (resp. $\mathfrak{c}_1$). Tensor product of line bundles makes $\mathbf{X},\mathbf{X}_1$ and $\tilde{\mathbf{X}},\tilde{\mathbf{X}}_1$ into finitely generated abelian groups. Consider the $G\times\mathbf{C}^*$-equivariant map $\mathfrak{c}_1\to\mathfrak{c}$ given by $(\phi,P,L,C)\mapsto(\pi_P(\phi(e)),P)$. Taking inverse images of G- or $G\times\mathbf{C}^*$-equivariant line bundles under this map, gives us homomorphisms

$$\mathbf{X}_1\leftarrow\mathbf{X},\quad\tilde{\mathbf{X}}_1\leftarrow\tilde{\mathbf{X}}.$$

LEMMA 8.2. *These homomorphisms are isomorphisms.*

Let $(\phi,P,L,C)\in\mathfrak{c}_1$ and let $x=\phi(e)$. Let H,H_1 be the stabilizer in G of $(\pi_P(x),P)\in\mathfrak{c},(\phi,P,L,C)\in\mathfrak{c}_1$ respectively. The inclusion $H_1\to H$ induces a homomorphisms

$$\text{(a)}\qquad \mathrm{Hom}(H_1,\mathbf{C}^*)\leftarrow\mathrm{Hom}(H,\mathbf{C}^*)$$

which may be identified with $\mathbf{X}_1\leftarrow\mathbf{X}$. We show that the homomorphism (a) is an isomorphism. We have $H=Z_L(x)U_P$ and $H_1=Z_L(\phi)$. Since $Z_L(\phi)$ is a maximal reductive subgroup of $Z_L(x)$, we see that

$$\mathrm{Hom}(H,\mathbf{C}^*)=\mathrm{Hom}(Z_L(x),\mathbf{C}^*)=\mathrm{Hom}(H_1,\mathbf{C}^*)$$

so that the homomorphism (a) is an isomorphism.

Let $\tilde{H},\tilde{H}_1$ be the stabilizer in $G\times\mathbf{C}^*$ of $(\pi_P(x),P)\in\mathfrak{c},(\phi,P,L,C)\in\mathfrak{c}_1$ respectively.

We have imbeddings of $\mathbf{C}^*$ into each of the groups $\tilde{H},\tilde{H}_1$ given by $t\mapsto(\hat{\phi}(\mathrm{d}_t),t)$. We identify $\mathbf{C}^*$ with its image under either of these imbeddings. It is clear that H is normal in $\tilde{H}$, that $\tilde{H}=H\mathbf{C}^*=\mathbf{C}^*H$ and that $H\cap\mathbf{C}^*=\{1\}$; moreover, H_1 is normal in $\tilde{H}_1$, $\tilde{H}_1=H_1\mathbf{C}^*=\mathbf{C}^*H_1$ and $H_1\cap\mathbf{C}^*=\{1\}$. Hence our earlier result on H,H_1 implies that the homomorphism

$$\text{(b)}\qquad \mathrm{Hom}(\tilde{H}_1,\mathbf{C}^*)\leftarrow\mathrm{Hom}(\tilde{H}_0,\mathbf{C}^*)$$

induced by the inclusion $H_1\to\tilde{H}_0$, is an isomorphism. Equivalently, the homomorphism $\tilde{\mathbf{X}}_1\leftarrow\tilde{\mathbf{X}}$ is an isomorphism. The lemma is proved.

8.3. We have natural homomorphisms

$$\tilde{\mathbf{X}}\to\mathbf{X},\quad\tilde{\mathbf{X}}_1\to\mathbf{X}_1$$

given by regarding a $G\times\mathbf{C}^*$-equivariant line bundle as a G-equivariant one. (G is imbedded in $G\times\mathbf{C}^*$ by $g\to(g,1)$.) On the other hand, we have homomorphisms

$$\tilde{\mathbf{X}}\to\mathbf{Z},\quad\tilde{\mathbf{X}}_1\to\mathbf{Z}$$

given by the homomorphisms

$$\mathrm{Hom}(H,\mathbf{C}^*)\to\mathrm{Hom}(\mathbf{C}^*,\mathbf{C}^*),\quad \mathrm{Hom}(H_1,\mathbf{C}^*)\to\mathrm{Hom}(\mathbf{C}^*,\mathbf{C}^*)$$

induced by the imbeddings $\mathbf{C}^*\to H,\mathbf{C}^*\to H_1$ in the proof of the previous lemma. (These homomorphisms are independent of the choice of $(\phi,P,L,C)\in\mathfrak{c}_1$ in that proof, since G acts transitively on $\mathfrak{c}_1$.) From the previous proof, we see that these homomorphisms define (canonical) isomorphisms of abelian groups

$$\tilde{\mathbf{X}}\xrightarrow{\sim}\mathbf{X}\oplus\mathbf{Z},\quad\tilde{\mathbf{X}}_1\xrightarrow{\sim}\mathbf{X}_1\oplus\mathbf{Z}.$$

LEMMA 8.4. *Let $(\phi,P,L,C)\in\mathfrak{c}_1$. The homomorphism of abelian groups*

$$\mathbf{X}_1=\mathrm{Hom}(H_1,\mathbf{C}^*)=\mathrm{Hom}(Z_L(\phi),\mathbf{C}^*)\to\mathrm{Hom}(Z_L(\phi)^0,\mathbf{C}^*)$$

(restriction) has finite kernel and finite cokernel.

$Z_L(\phi)^0$ is equal to Z_L^0 (see [**L3, 2.3**]) hence $H_1^0=Z_L(\phi)^0$ is a central torus in $H_1=Z_L(\phi)$. In particular, the commutator subgroup H_1' of H_1 is finite. Hence the identity component of the diagonalizable algebraic group H_1/H_1' is finitely covered by H_1^0 so that in the diagram

$$\mathrm{Hom}(H_1,\mathbf{C}^*)=\mathrm{Hom}(H_1/H_1',\mathbf{C}^*)\xrightarrow{a}\mathrm{Hom}((H_1/H_1')^0,\mathbf{C}^*)\xrightarrow{b}\mathrm{Hom}(H_1^0,\mathbf{C}^*)$$

a is a surjection whose kernel is the (finite) torsion group of $\mathrm{Hom}(H_1/H_1',\mathbf{C}^*)$ and b is an injection with finite cokernel. The lemma follows.

8.5. For any $(\phi,P,L,C)\in\mathfrak{c}_1$, we consider the composition of $\mathbf{C}$-linear isomorphisms

$$\mathbf{C}\otimes\mathbf{X}\xrightarrow{\sim}\mathbf{C}\otimes\mathbf{X}_1\xrightarrow{\sim}\mathbf{C}\otimes\mathrm{Hom}(Z_L(\phi)^0,\mathbf{C}^*)\xrightarrow{\sim}\mathfrak{z}_L^*=\mathfrak{z}^*;$$

the first one is induced by $\mathbf{X}\xrightarrow{\sim}\mathbf{X}_1$ in 8.2; the second one is induced by the map in 8.4; the third one is induced by passage to Lie algebras (here $\mathfrak{z}^*$ is the vector space dual to $\mathfrak{z}$); the fourth one is induced by $a_{P,L}$ (see 7.3). Since G acts transitively on $\mathbf{c}_1$, the composition

(a) $$\mathbf{C} \otimes \mathbf{X} \xrightarrow{\sim} \mathfrak{z}^*$$

is independent of the choice of $(\phi, P, L, C) \in \mathbf{c}_1$. Recall that we have canonically $\tilde{\mathbf{X}} = \mathbf{X} \oplus \mathbf{Z}$ (see 8.3). Tensoring with $\mathbf{C}$ we obtain

(b) $$\mathbf{C} \otimes \tilde{\mathbf{X}} = (\mathbf{C} \otimes \mathbf{X}) \oplus \mathbf{C} \xrightarrow{\sim} \mathfrak{z}^* \oplus \mathbf{C}.$$

This can be also obtained as the composition

$$\mathbf{C} \otimes \tilde{\mathbf{X}} \xrightarrow{\sim} \mathbf{C} \otimes \tilde{\mathbf{X}}_1 \xrightarrow{\sim} \mathbf{C} \otimes \mathrm{Hom}(\tilde{Z}_L(\phi)^0, \mathbf{C}^*) \xrightarrow{\sim} \mathbf{C} \otimes \mathrm{Hom}(Z_L(\phi)^0 \times \mathbf{C}^*, \mathbf{C}^*) \xrightarrow{\sim} \mathfrak{z}_L^* \oplus \mathbf{C} = \mathfrak{z}^* \oplus \mathbf{C},$$

where the first map is induced by $\tilde{\mathbf{X}} \xrightarrow{\sim} \tilde{\mathbf{X}}_1$ in 8.2; the second one is induced by

$$\tilde{\mathbf{X}}_1 = \mathrm{Hom}(\bar{H}_1, \mathbf{C}^*) = \mathrm{Hom}(\tilde{Z}_L(\phi), \mathbf{C}^*) \to \mathrm{Hom}(\tilde{Z}_L(\phi)^0, \mathbf{C}^*)$$

(analogous to 8.4), the third map is induced by $Z_L(\phi) \times \mathbf{C}^* \xrightarrow{\sim} \tilde{Z}_L(\phi)$ (see 8.0(b)) and the fourth map is induced by passage to Lie algebras.

We will denote by r the element $(0,1) \in \mathfrak{z}^* \oplus \mathbf{C}$.

8.6. The root system R. Let $(P, L) \in \underline{\mathfrak{P}}$. For any linear form $\alpha : \mathfrak{z}_L \to \mathbf{C}$ we set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [y, x] = \alpha(y)x \quad \forall y \in \mathfrak{z}_L\}.$$

Then $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{z}_L^*} \mathfrak{g}_\alpha$. Let

$$R = \{\alpha \in \mathfrak{z}_L^* | \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}.$$

It is easy to see that Lie U_P is a sum of $\mathfrak{g}_\alpha$'s ($\alpha \in R$). Let $R^+ = \{\alpha \in R | \mathfrak{g}_\alpha \in \text{Lie } U_P\}$. Then R is a (not necessarily reduced) root system in $\mathfrak{z}_L^*$ and W_L is its Weyl group; R^+ is a system of positive roots for R. (See **[L3, 2.5]**.) Let Π be the corresponding set of simple roots in R. We identify $\mathfrak{z}_L^* = \mathfrak{z}^*$ via $a_{P,L}$ (see 7.3). Hence we may regard R as a (not necessarily reduced) root system in $\mathfrak{z}^*$ (independent of the choice of $(P, L) \in \underline{\mathfrak{P}}$) with Weyl group W, with positive roots R^+ and with simple roots Π.

Given $i \in I$, there is a unique simple root $\alpha_i \in \Pi$ and a unique vector $\check{\alpha}_i \in \mathfrak{z}$ such that $s_i(\xi) = \xi - \xi(\check{\alpha}_i)\alpha_i$ for all $\xi \in \mathfrak{z}^*$ (s_i is as in 7.4). We have $\Pi = \{\alpha_i | i \in I\}$.

8.7. Given $i \in I$, we define an integer $c_i \geq 2$ by the conditions

$$ad(\phi(e))^{c_i - 2} : \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i} \to \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i} \text{ is } \neq 0,$$
$$ad(\phi(e))^{c_i - 1} : \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i} \to \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i} \text{ is } 0,$$

where $\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{2\alpha_i}$ and $\phi(e)$ are defined in terms of a fixed $(\phi, P, L, C) \in \mathbf{c}_1$. (Note that $\mathfrak{g}_{\alpha_i}$ and $\mathfrak{g}_{2\alpha_i}$ are stable under $ad(y)$ for any $y \in \text{Lie } L$.) Using **[L3, 2.8]** we see that, if

$$c_i' = \dim \ker (ad(\phi(e)) : \mathfrak{g}_{\alpha_i} \to \mathfrak{g}_{\alpha_i}), \quad c_i'' = \dim \mathfrak{g}_{2\alpha_i},$$

then

(a) $$c_i'' \in \{0, 1\}, \quad c_i = 2c_i' + c_i''.$$

8.8. We define a map

(a) $$\tilde{\mathbf{X}} \to \mathrm{Hom}^2_{\mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}}(K, K)$$

($K = (pr_1)_! \mathcal{L}^*$ as in 7.11, 7.15) as follows. Consider a $G \times \mathbf{C}^*$-equivariant line bundle E on $\mathbf{c}$ and let $\tilde{E}$ be the inverse image of E under the $G \times \mathbf{C}^*$-morphism $s : \dot{\mathcal{V}} = \dot{\mathfrak{g}} \to \mathbf{c}$ (see 7.9). By the construction in 1.19, $\tilde{E}$ gives rise to a morphism $\kappa_{\tilde{E}} : \mathcal{L}^*[-2] \to \mathcal{L}^*$ in $\mathcal{D}_{G \times \mathbf{C}^*} \dot{\mathfrak{g}}$. Applying the functor $(pr_1)_! : \mathcal{D}_{G \times \mathbf{C}^*} \dot{\mathfrak{g}} \to \mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}$, we obtain a morphism

$$(pr_1)_! \kappa_{\tilde{E}} : K[-2] \to K$$

in $\mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}$ (or, equivalently, an element of $\mathrm{Hom}^2_{\mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}}(K, K)$). By definition, the map (a) is given by $E \mapsto (pr_1)_! \kappa_{\tilde{E}}$.

8.9. We now consider the free associative $\mathbf{C}$-algebra $'\mathbf{H}$ (with unit) on the generators (E) (in 1-1 corespondence with the elements E of $\tilde{\mathbf{X}}$) and s_i (indexed by $i \in I$).

Let $\mathbf{H}$ be the associative $\mathbf{C}$-algebra (with unit) defined by the generators (ξ) (in 1-1 correspondence with the elements ξ of $\mathfrak{z}^* \oplus \mathbf{C}$) and s_i (indexed by $i \in I$) and the following relations:

(a) $(a\xi + a'\xi') = a(\xi) + a'(\xi')$ for any $\xi, \xi' \in \mathfrak{z}^* \oplus \mathbf{C}$ and any $a, a' \in \mathbf{C}$;
(b) $(\xi)(\xi') = (\xi')(\xi)$ for any $\xi, \xi' \in \mathfrak{z}^* \oplus \mathbf{C}$;
(c) $s_i (i \in I)$ satisfy the relations of W;
(d) $s_i(\xi) - ({}^{s_i}\xi)s_i = c_i \xi(\check{\alpha}_i)(r)$ for any $\xi \in \mathfrak{z}^*$ and $i \in I$;
(e) $s_i(r) = (r)s_i$ for any $i \in I$.

(r is as in 8.6. The natural action of W on $\mathfrak{z}^*$ is denoted by $w : \xi \mapsto {}^w\xi$.)

The algebras $'\mathbf{H}, \mathbf{H}$ are graded (by $\mathbf{Z}_{\geq 0}$) by the rule $\deg(\xi) = 2, \deg s_i = 0$.

Using **[L3, 6.3]** it is easy to identify the algebra $\mathbf{H}$ with the algebra denoted $\mathbf{H}$ in **[L3, 6.3]**. There is a unique (surjective) homomorphism of algebras with 1

(f) $$'\mathbf{H} \to \mathbf{H}$$

which carries s_i to s_i and carries (E) to (ξ), where $\xi \in \mathfrak{z}^* \oplus \mathbf{C}$ corresponds to $1 \otimes E \in \mathbf{C} \otimes \tilde{\mathbf{X}}$ under 8.5(b).

We have an algebra isomorphism $\mathbf{H} \xrightarrow{\sim} \mathbf{H}^{opp}$ (opposed algebra) which maps each generator s_i or (ξ) to itself.

8.10. We consider the homomorphism of graded $\mathbf{C}$-algebras with unit

(a) $$'\mathbf{H} \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}}(K, K)$$

which takes (E) to $(pr_1)_! \kappa_{\tilde{E}}$ (see 8.8) and s_i to the image of $s_i \in W$ under $\mathbf{C}[W] = \mathrm{Hom}^0_{\mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}}(K, K)$ (see 7.14).

THEOREM 8.11. *There is a unique isomorphism* $\mathbf{H} \xrightarrow{\sim} \mathrm{Hom}^{\cdot}_{\mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}}(K, K)$ *of graded algebras with unit whose composition with the homomorphism* $'\mathbf{H} \to \mathbf{H}$ *(see 8.9(f)) is equal to the homomorphism 8.10(a).*

The proof is based on results in **[L3]**. In the following computation we fix $j \in \mathbf{Z}$ and we choose $\Gamma \in \mathcal{S}f_m(G \otimes \mathbf{C}^*)$ with large m:

$$\begin{aligned}
\mathrm{Hom}^j_{\mathcal{D}_{G \times \mathbf{C}^*} \mathfrak{g}}(K, K) &= \mathrm{Hom}(H_c^{-j}({}_\Gamma \mathfrak{g}, D({}_\Gamma K) \otimes {}_\Gamma K), \mathbf{C}) \\
&= \mathrm{Hom}(H_c^{-j}({}_\Gamma \mathfrak{g}, {}_\Gamma K^*[2 \dim {}_\Gamma \mathcal{V}] \otimes {}_\Gamma K), \mathbf{C}) \\
&= \mathrm{Hom}(H_c^{-j+2 \dim {}_\Gamma \mathcal{V}}({}_\Gamma \mathfrak{g}, {}_\Gamma K^* \otimes {}_\Gamma K), \mathbf{C}) \\
&= \mathrm{Hom}(H_c^{-j+2 \dim {}_\Gamma \mathcal{V}}({}_\Gamma \ddot{\mathcal{V}}, {}_\Gamma \ddot{\mathcal{L}}^*), \mathbf{C}) = H_j^{G \times \mathbf{C}^*}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}}).
\end{aligned}$$

(Notation of 7.16. We have used the equality $\dim \mathcal{V} = \dim \ddot{\mathcal{V}}$, see [**L3, 3.3**].) Taking now direct sum over j, we obtain a canonical identification

(a) $$\mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K) = H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})$$

as graded vector spaces. We denote by $\mathbf{1}$ the element of $H_0^{G\times\mathbf{C}^*}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})$ corresponding to the identity element in $\mathrm{Hom}^0_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K)$.

Using the definitions, the first commutative diagram in 1.6 (or rather its $G\times\mathbf{C}^*$-equivariant analogue) and the functoriality properties of the operation κ_E in 1.8, we see that for any $E \in \tilde{\mathbf{X}}$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K) \arrow[r, "\sim"] \arrow[d] & H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}}) \arrow[d] \\
\mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K) \arrow[r, "\sim"] & H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})
\end{tikzcd}$$

where the horizontal maps are as in (a), the left vertical map is right multiplication by $(pr_1)_!\kappa_{\tilde{E}} \in \mathrm{Hom}^2_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K)$ and the right vertical map is multiplication by $c(E_1) \in H^2_{G\times\mathbf{C}^*}(\ddot{\mathcal{V}},\mathbf{C})$. Here E_1 is the inverse image of the line bundle $\tilde{E}$ (see 8.8) under the $G\times\mathbf{C}^*$-equivariant map $\ddot{\mathcal{V}} \to \dot{\mathcal{V}}$ $((x,P,P') \mapsto (x,P))$ and $c(E_1)$ is defined as in 1.19.

Similarly, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K) \arrow[r, "\sim"] \arrow[d] & H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}}) \arrow[d] \\
\mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K) \arrow[r, "\sim"] & H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})
\end{tikzcd}$$

where the horizontal maps are as in (a), the left vertical map is right multiplication by s_i and the right vertical map is the map $\Delta(s_i) : H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}}) \to H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})$ defined in [**L3, 4.3**].

By [**L3, §3, §4, 5.1**], $H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})$ is naturally a left $\mathbf{H}$-module; moreover, by [**L3, 4.11**(a), **4.6**], the map

(b) $$\mathbf{H} \to H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})$$

given by $h \mapsto h\mathbf{1}$, is an isomorphism of vector spaces. Using the definitions and the commutative diagrams above we see that the following diagram is commutative:

$$\begin{tikzcd}
{}'\mathbf{H} \arrow[r, "(8.10(a))"] \arrow[d, "(8.9(f))"'] & \mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K) \arrow[d, "(a)"'] \\
\mathbf{H} \arrow[r, "(b)"] & H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}})
\end{tikzcd}$$

Note that the arrows (b) and (a) are isomorphisms of vector spaces. We define an isomorphism of vector spaces $\mathbf{H} \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K)$ as the composition

$$\mathbf{H} \to H^{G\times\mathbf{C}^*}_{\cdot}(\ddot{\mathcal{V}},\ddot{\mathcal{L}}) \to \mathrm{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K,K);$$

(the first arrow is given by (b), the second arrow is the inverse of the one in (a)). This is automatically an algebra isomorphism since the arrows 8.10(a) and

8.9(f) are algebra homomorphisms and the arrow 8.9(f) is surjective. This proves the existence part of the theorem. The uniqueness follows immediately from the surjectivity of the arrow 8.9(f). The theorem is proved.

8.12. Let σ be a semisimple element of $\mathfrak{g}$ and let $r_0 \in \mathbf{C}$. Then (σ, r_0) is a semisimple element of $\mathfrak{g}\oplus\mathbf{C}$ and we set $\mathfrak{H} = Z_{G\times\mathbf{C}^*}(\sigma,r_0) = Z_G(\sigma)\times\mathbf{C}^*$ (a closed, connected subgroup of $G\times\mathbf{C}^*$). Let $\mathcal{T} = \mathcal{T}_{\sigma,r_0}$ be the smallest torus in $G\times\mathbf{C}^*$ whose Lie algebra contains (σ,r_0). We show that the hypotheses of 4.6 are satisfied when we take

$\mathfrak{G} = G\times\mathbf{C}^*$;

$X = \mathfrak{g}$;

$\dot{X} = \{(x,P) \in \mathfrak{g}\times\mathfrak{P} \mid x \in \pi_P^{-1}(\mathfrak{z}_{\bar{P}} + cl(C_P))\}$

($cl(C_P)$ is the closure of C_P in Lie $\bar{P}$);

$f : \dot{X} \to X$, the first projection;

$Z = \dot{\mathcal{V}} = \dot{\mathfrak{g}}$ (an open dense subset of $\dot{X}$);

$\mathcal{L}$ as in 7.9;

$\bar{X}, \bar{\dot{X}}$, the fixed point sets of the torus $\mathcal{T}$ in $X, \dot{X}$

(thus, $\bar{X} = \{x \in \mathfrak{g} \mid [\sigma,x] = 2r_0x\}, \bar{\dot{X}} = \{(x,P) \in \dot{X} \mid [\sigma,x] = 2r_0x, \sigma \in \text{Lie } P\}$);

$\bar{f} : \bar{\dot{X}} \to \bar{X}$, the restriction of f;

$\bar{Z}$, the fixed point set of $\mathcal{T}$ in Z.

Then f is proper since $\mathfrak{P}$ is a projective variety. The fact that $\mathcal{L}$ has finite monodromy is clear from the definition. Property 4.6(b) is obvious. Property 4.6(d) can be easily deduced from [**L3, 2.2(b)**] (see also [**L2, 23.1**]). Property 4.6(f) follows from [**L3, 4.6**]. We prove properties 4.6(a), 4.6(c), 4.6(e) simultaneously. We have

$$\bar{Z} = \{(x,P) \in \mathfrak{g}\times\mathfrak{P} \mid \sigma \in \text{Lie } P, [\sigma,x] = 2r_0x, x \in \pi_P^{-1}(\mathfrak{z}_{\bar{P}} + C_P)\}.$$

For each connected component $\mathcal{O}$ of $\{P \in \mathfrak{P} \mid \sigma \in \text{Lie } P\}$, we denote by $\bar{Z}_{\mathcal{O}}$ the subvariety of $\bar{Z}$ defined by the condition $P \in \mathcal{O}$. Then the varieties $\bar{Z}_{\mathcal{O}}$ are open and closed in $\bar{Z}$, they form a partition of $\bar{Z}$ and their closures in $\dot{X}$ are clearly disjoint. Note also that, for any $\mathcal{O}$ as above, $\mathfrak{H}$ acts naturally on $\mathcal{O}$ and on $\bar{Z}_{\mathcal{O}}$, the first action being transitive. It is therefore enough to verify the following statement.

For fixed $P \in \mathcal{O}$, the (smooth) variety $V := \{x \in \mathfrak{g} \mid [\sigma,x] = 2r_0x, x \in \pi_P^{-1}(\mathfrak{z}_{\bar{P}} + C_P)\}$ is open in its closure $cl(V)$ in $\{x \in \mathfrak{g} \mid [\sigma,x] = 2r_0x\}$; moreover, V is either empty or connected and the local system on V defined by $\mathcal{L}^*$ is such that its direct image under the inclusion $V \to cl(V)$ is zero on the complement of V.

We can choose $(L,C) \in \mathfrak{o}$ such that $(P,L,C) \in \mathfrak{o}'$ and such that $\sigma \in$ Lie L. Let

$$V' = \{x \in \text{Lie } L \mid [\sigma,x] = 2r_0x\} \cap (\mathfrak{z}_L + C),$$
$$V'' = \{x \in \text{Lie } U_P \mid [\sigma,x] = 2r_0x\}.$$

Then $V = V' \times V''$ (using addition). Note that V'' is a vector space. Hence we are reduced to verifying the following statement.

(a) *The variety V' is open in its closure $cl(V')$ in $\{x \in \text{Lie } L \mid [\sigma,x] = 2r_0x\}$;*

moreover, the local system on V' defined by $\mathcal{L}^*$ is such that its direct image under the inclusion $V' \to cl(V')$ is zero on the complement of V'.

We may assume that V' is non-empty.

Assume first that $r_0 = 0$. Then σ commutes with some element of C, hence $\sigma \in \mathfrak{z}_L$ (since any element of C is distinguished). Thus, $V' = \mathfrak{z}_L + C$ and $cl(V') = \mathfrak{z}_L + cl(C)$ so that V' is open in $cl(V')$. We are reduced to showing that C is open in $cl(C)$ (this is well known) and that the local system on C defined by $\mathcal{L}^*$ is such that its direct image under the inclusion $C \to cl(C)$ is zero on the complement of C (see **[L3, 2.2(b)]**).

Next, we assume that $r_0 \neq 0$. Then any vector in $\{x \in \text{Lie } L | [\sigma, x] = 2r_0 x\}$ is automatically nilpotent. Hence $V' = \{x \in \text{Lie } L | [\sigma, x] = 2r_0 x\} \cap C$. Let $\phi \in J$ be such that $\phi(e) \in V'$. Then $[\sigma - r_0\phi(h), \phi(e)] = 0$ hence $\sigma - r_0\phi(h) \in \mathfrak{z}_L$ (using the fact that $\phi(e)$ is distinguished). Hence $V' = \{x \in \text{Lie } L | [\phi(h), x] = 2x\} \cap C$. But this is well known to be an open dense subset of $\{x \in \text{Lie } L | [\phi(h), x] = 2x\}$ and in particular, V' is open in $cl(V')$. In the present case, the vanishing statement in (a) follows from **[L5]**. This completes the verification of properties 4.6(a), 4.6(c), 4.6(e).

8.13. Since $\text{Hom}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K, K)$ is naturally a $H^{\cdot}_{G\times\mathbf{C}^*}$-algebra (see 1.20), we have a natural ring homomorphism $H^{\cdot}_{G\times\mathbf{C}^*} \to \text{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K, K)$ whose image is contained in the centre of $\text{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K, K)$.

(a) *This image is precisely the centre of* $\text{Hom}^{\cdot}_{\mathcal{D}_{G\times\mathbf{C}^*}\mathfrak{g}}(K, K)$.

Indeed, using 8.11 and **[L3, 6.5]** we may identify this centre with the coordinate ring of $(\mathfrak{z}^*/W) \times \mathbf{C}$, or equivalently (using **[L3.1.11]** and the notation of 8.4, 8.0) with the ring $H^{\cdot}_{Z_G(\phi)^0\times\mathbf{C}^*} = H^{\cdot}_{\tilde{Z}_G(\phi)^0}$ (the last equality comes from 8.4(b)) so that our homomorphism from $H^{\cdot}_{G\times\mathbf{C}^*}$ to the centre becomes the ring homomorphism $H^{\cdot}_{G\times\mathbf{C}^*} \to H^{\cdot}_{\tilde{Z}_G(\phi)^0}$ induced by the inclusion $\tilde{Z}_G(\phi)^0 \subset G \times \mathbf{C}^*$. We are then reduced to verifying the following statement.

(b) *The natural map from the variety of semisimple conjugacy classes of $\tilde{Z}_G(\phi)^0$ to the variety of semisimple conjugacy classes of $G \times \mathbf{C}^*$ is injective.* (Compare **[L6, 14.3(a)]**.)

Let **modH** be the category of **H**-modules of finite dimension over **C**. For any algebra homomorphism $\chi : H^{\cdot}_{G\times\mathbf{C}^*} \to \mathbf{C}$, we denote by $\mathbf{mod}_\chi\mathbf{H}$ the full subcategory of **modH** whose objects are the **H**-modules on which some power of the kernel of χ acts as zero. It is clear that

$$\mathbf{modH} = \oplus_\chi \mathbf{mod}_\chi\mathbf{H}.$$

We may index the various χ in the form χ_{σ,r_0} (as in 4.2) where (σ, r_0) is a semisimple element of $\mathfrak{g} \oplus \mathbf{C}$ (modulo the adjoint action of $G \times \mathbf{C}^*$). Then the previous decomposition becomes

(c) $$\mathbf{modH} = \oplus_{\sigma,r_0}\mathbf{mod}_{\sigma,r_0}\mathbf{H}$$

where we write $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ instead of $\mathbf{mod}_{\chi_{\sigma,r_0}}\mathbf{H}$ and (σ, r_0) runs over the semisimple elements of $\mathfrak{g} \oplus \mathbf{C}$ (modulo the adjoint action of $G \times \mathbf{C}^*$).

It follows that the set of irreducible objects of **modH** (up to isomorphism) is

naturally the disjoint union of the sets of irreducible objects of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ (up to isomorphism), where (σ, r_0) is as above. This coincides with the decomposition according to the character by which the centre of **H** acts on a simple module (by (a)).

THEOREM 8.14. *Fix (σ, r_0) as above. We use the notation in 8.12. Let $\bar{A}$ be the direct image of $\mathcal{L}^*|_{\tilde{Z}}$ under the inclusion $\tilde{Z} \to \tilde{X}$ and let $\bar{B} = \bar{f}_!\bar{A} \in \mathcal{D}_{\mathfrak{H}}\bar{X}$. Let $I^{\mathfrak{H}}_{\sigma,r_0}$ be the kernel of the algebra homomorphism $H^{\cdot}_{\mathfrak{H}} \to \mathbf{C}$ (restriction of $\chi_{\sigma,r_0} : H^{\cdot}_{\mathfrak{G}} \to \mathbf{C}$). Let $\mathfrak{J}$ be the set of isomorphism classes of simple perverse on $\{y \in \mathfrak{g} | [\sigma, y] = 2r_0 y\}$, for which some shift is a direct summand of $\bar{B}$.*

(a) The category $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ is naturally equivalent to the category of $Hom^{\cdot}_{\mathcal{D}_{\mathfrak{H}}\bar{X}}(\bar{B}, \bar{B})$-modules of finite dimension over $\mathbf{C}$ in which some power of $I^{\mathfrak{H}}_{\sigma,r_0}$ acts as zero.

(b) The set of isomorphism classes of simple objects of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ is naturally in 1-1 correspondence with the set $\mathfrak{J}$.

By 8.12, the discussion in §4,§5 is applicable. Therefore we may apply 4.14 and (a) follows; we may apply 5.2, 5.4 and (b) follows.

8.15. In the setup of 8.14, we assume that $r_0 \neq 0$. Let $\mathcal{M}$ be the (finite) set consisting of all pairs $(\mathcal{O}, \mathcal{E})$, where $\mathcal{O}$ is a $\mathfrak{H}$-orbit in $\{y \in \mathfrak{g} | [\sigma, y] = 2r_0 y\}$ and $\mathcal{E}$ is an irreducible, $\mathfrak{H}$-equivariant local system on $\mathcal{O}$ (up to isomorphism). The intersection cohomology complex on $\mathfrak{g}_{2r_0}$ associated to $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}$ is denoted by $\mathcal{E}^\sharp$.

Let $\mathcal{M}_{\mathfrak{o},\mathcal{F}}$ be the set of all $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}$ such that $\mathcal{E}$ appears in the local system $\mathcal{H}^n\bar{B}|_{\mathcal{O}}$ for some n. (Recall that $\mathfrak{o}, \mathcal{F}$ are as in 7.1, 7.8.)

Clearly, $\mathcal{M}_{\mathfrak{o},\mathcal{F}}$ may be identified with the set of all $Z_G(\sigma)$-conjugacy classes of pairs (y, ρ) where y is in the $(2r_0)$-eigenspace of $ad(\sigma) : \mathfrak{g} \to \mathfrak{g}$ and ρ is an irreducible representation of the finite group $Z_G(\sigma, y)/Z^0_G(\sigma, y)$ that occurs in the obvious representation of this group on

$$\oplus_n H^n_c(\{P \in \mathfrak{P} | (y, P) \in \dot{\mathfrak{g}}, \sigma \in \text{Lie } P\}, \dot{\mathcal{L}}^*).$$

Let $\mathcal{M}'_{\mathfrak{o},\mathcal{F}}$ be the set of all $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}$ such that some shift of $\mathcal{E}^\sharp$ is a direct summand of $\bar{B} \in \mathcal{D}\mathfrak{g}_{2r_0}$.

PROPOSITION 8.16. *(a) The subsets $\mathcal{M}_{\mathfrak{o},\mathcal{F}}$, for various pairs $\mathfrak{o}, \mathcal{F}$ as in 7.1, 7.8, form a partition of $\mathcal{M}$.*

(b) The subsets $\mathcal{M}'_{\mathfrak{o},\mathcal{F}}$, for various pairs $\mathfrak{o}, \mathcal{F}$ as in 7.1, 7.8, form a partition of $\mathcal{M}$.

The proof, based on results of **[L6]**, will be given in §9.

The partitions of $\mathcal{M}$ defined in (a),(b) coincide, by the following result.

PROPOSITION 8.17. *The following four conditions for $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}$ are equivalent:*

(a) $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}_{\mathfrak{o},\mathcal{F}}$;

(b) $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}'_{\mathfrak{o},\mathcal{F}}$;

(c) *some shift of* $\mathcal{E}^\sharp$ *is a direct summand of* $\tilde{B} \in \mathcal{D}_{\mathfrak{H}}\mathfrak{g}_{2r_0}$;

(d) *Let* $\hat{\mathcal{O}}$ *be the nilpotent* G*-orbit in* $\mathfrak{g}$ *such that* $\mathcal{O} \subset \hat{\mathcal{O}}$. *Then there exists an irreducible* G*-equivariant local system* $\hat{\mathcal{E}}$ *on* $\hat{\mathcal{O}}$ *such that* $\hat{\mathcal{E}}$ *appears in the local system* $\mathcal{H}^n B|_{\hat{\mathcal{O}}}$ *for some* n *and* $\mathcal{E}$ *appears in the local system* $\hat{\mathcal{E}}|_{\mathcal{O}}$.

The proof will be given in §9.

COROLLARY 8.18. *Recall that* $r_0 \neq 0$. *The set of isomorphism classes of simple objects of* $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ *is naturally in 1-1 correspondence with the set* $\mathcal{M}_{\mathfrak{o},\mathcal{F}}$.

This follows immediately from 8.14 and 8.17.

9. Analysis of certain local systems

9.1. We consider a semisimple element $(\sigma, r_0) \in \mathfrak{g} \oplus \mathbf{C}$ such that $r_0 \neq 0$. The vector space $\mathfrak{g}$ has a direct sum decomposition $\mathfrak{g} = \oplus_{a \in \mathbf{C}} \mathfrak{g}_a$ where $\mathfrak{g}_a = \{x \in \mathfrak{g} | [\sigma, x] = ax\}$. Let $\mathfrak{g}^\dagger = \oplus_{n \in \mathbf{Z}} \mathfrak{g}_{2nr_0}$. This is a Lie subalgebra of $\mathfrak{g}$ containing σ. Moreover, $\mathfrak{g}^\dagger$ is the Lie algebra of a connected reductive subgroup $G^\dagger$ of G.

Let J_{σ,r_0} be the set of all $\phi \in J$ such that

$$\phi(e) \in \mathfrak{g}_{2r_0}, \phi(h) \in \mathfrak{g}_0, \phi(f) \in \mathfrak{g}_{-2r_0}.$$

The following is a known graded variant of the Morozov-Jacobson-Kostant theorem 8.0(a).

PROPOSITION 9.2. *For* $y \in \mathfrak{g}_{2r_0}$, *let* $J_{y,\sigma,r_0} = J_y \cap J_{\sigma,r_0}$. *Then*

(a) J_{y,σ,r_0} *is non-empty;*

(b) *if* $\phi, \phi' \in J_{y,\sigma,r_0}$, *then there exists* $g \in Z_G(y, \sigma)$ *such that* $\phi' = Ad(g) \circ \phi$.

Note that J_{y,σ,r_0}, defined in terms of $\mathfrak{g}$, is the same as J_{y,σ,r_0}, defined in terms of $\mathfrak{g}^\dagger$. Therefore, we may assume that $\mathfrak{g} = \mathfrak{g}^\dagger$. In this case, the proof in [**L6, 3.3**] gives the desired result.

9.3. Proof of Proposition 8.16. Let $\mathfrak{g}^\dagger, G^\dagger$ be as in 9.1. We will prove 8.16(b) by reducing to the analogous statement for $G^\dagger$. Note that $\sigma \in \mathfrak{g}^\dagger$, that the $(2r_0)$-eigenspaces of $ad(\sigma)$ on $\mathfrak{g}$ and $\mathfrak{g}^\dagger$ coincide, and that $Z_G(\sigma) = Z_{G^\dagger}(\sigma)$. It follows that the set $\mathcal{M}$ defined in terms of G, σ, r_0 is the same as the set $\mathcal{M}$ defined in terms of $G^\dagger, \sigma, r_0$. Given $\mathfrak{o}, \mathcal{F}$ as in 7.1, 7.8, such that $\tilde{B} \neq 0$, we define $\mathfrak{P}^0$ as the subvariety of $\mathfrak{P}$ consisting of those P such that there exists a Levi subalgebra of Lie P that contains σ and is contained in $\mathfrak{g}^\dagger$.

Note that $\tilde{B}$ is the direct image with compact support of a local system under the first projection

(a) $$\{(y, P) \in \dot{\mathfrak{g}} | y \in \mathfrak{g}_{2r_0}, \sigma \in \text{Lie } P\} \to \mathfrak{g}_{2r_0}.$$

Since we assume that $\tilde{B} \neq 0$, there exists $(y, P) \in \dot{\mathfrak{g}}$ such that $y \in \mathfrak{g}_{2r_0}$ and $\sigma \in \text{Lie } P$. We show that

(b) $$P \in \mathfrak{P}^0.$$

We can choose $(L, C) \in \mathfrak{o}$ such that $(P, L, C) \in \mathfrak{o}'$ and such that $\sigma \in \text{Lie } L$. Let y' be the projection of y in Lie L (with respect to the decomposition Lie $P =$ Lie $L \oplus$ Lie U_P). Since $(y, P) \in \dot{\mathfrak{g}}$, we have $y' \in C$ hence y' is a distinguished

nilpotent element of Lie L. Applying 9.2 to Lie L instead of $\mathfrak{g}$, we can find a Lie algebra homomorphism $\phi : \mathfrak{s} \to \text{Lie } L$ such that $\phi(e) = y$ and $[\sigma, \phi(h)] = 0$. Then $\sigma - r_0\phi(h)$ is a semisimple element of Lie L and $[\sigma - r_0\phi(h), y] = 2r_0 y - 2r_0 y = 0$. Since y is distinguished in Lie L, any semisimple element of Lie L commuting with y must belong to the centre of Lie L. In particular, we have $\sigma - r_0\phi(h) \in \mathfrak{z}_L$. Hence the eigenvalues of $ad(\sigma)$ on Lie L coincide with the eigenvalues of $r_0\phi(h)$ on Lie L and these are known to be integer multiples of $2r_0$. We deduce that Lie $L \subset \mathfrak{g}^\dagger$. This shows that (b) holds.

If $P \in \mathfrak{P}^0$, the intersection $P \cap G^\dagger$ is a parabolic subgroup of $G^\dagger$. We can find $(L, C) \in \mathfrak{o}$ such that $L \subset P \cap G^\dagger$ and $\sigma \subset \text{Lie } L$. Then (L, C) belongs to the analogue of A_0 (for $G^\dagger$ instead of G) and its $G^\dagger$-orbit is independent of the choice of P. Let $\mathfrak{P}^\dagger$ be the conjugacy class of parabolic subgroups of $G^\dagger$ containing $P \cap G^\dagger$; this is again independent of the choice of P (by 6.8(c) for $G^\dagger$). We see that $G^\dagger$ inherits from G a structure $(\mathfrak{o}^\dagger, \mathcal{F}^\dagger)$ like the structure $(\mathfrak{o}, \mathcal{F})$ of G.

Let $\tilde{B}^\dagger \in \mathcal{D}\mathfrak{g}^\dagger_{2r_0} = \mathcal{D}\mathfrak{g}_{2r_0}$ be defined in terms of this structure of $G^\dagger$ and (σ, r_0) in the same same way as $\tilde{B} \in \mathcal{D}\mathfrak{g}_{2r_0}$ was defined in terms of G and (σ, r_0).

From (a), (b), we see that $\tilde{B}$ is the direct image with compact support of a local system $\mathcal{L}_1$ under the first projection $\{(y, P) \in \dot{\mathfrak{g}} | y \in \mathfrak{g}_{2r_0}, P \in \mathfrak{P}^0\} \to \mathfrak{g}_{2r_0}$. This last map can be factored as

$$\{(y, P) \in \dot{\mathfrak{g}} | y \in \mathfrak{g}_{2r_0}, P \in \mathfrak{P}^0\} \xrightarrow{p} \{(y, P') \in \dot{\mathfrak{g}}^\dagger | y \in \mathfrak{g}_{2r_0}, \sigma \in \text{Lie } P'\} \xrightarrow{q} \mathfrak{g}_{2r_0}$$

where $p(y, P) = (y, P \cap G^\dagger), q(y, P') = y$; ($\dot{\mathfrak{g}}^\dagger$ is defined in terms of $G^\dagger, \mathfrak{o}^\dagger, \mathcal{F}^\dagger$ in the same way as $\dot{\mathfrak{g}}$ was defined in terms of $G, \mathfrak{o}, \mathcal{F}$). Thus, $\tilde{B} = q_! p_! \mathcal{L}_1$. By definition, $\tilde{B}^\dagger = q_! \mathcal{L}_2$ where $\mathcal{L}_2$ is a local system such that $p^* \mathcal{L}_2 = \mathcal{L}_1$. Now p is a fibration with fibres isomorphic to a fixed (partial) flag manifold (that is the manifold of all $P \in \mathfrak{P}$ containing a fixed $P' \in \mathfrak{P}'$). Hence $p_! \mathcal{L}_1 = p_! p^* \mathcal{L}_2 = \mathcal{L}_2 \otimes p_! p^* \mathbf{C}$ is a direct sum of shifts of $\mathcal{L}_2$. It follows that

(c) $\tilde{B}$ is isomorphic to a direct sum of shifts of copies of $\tilde{B}^\dagger$.

This shows that the set $\mathcal{M}_{\mathfrak{o},\mathcal{F}}$ (for G) is either empty or is equal to the analogous set $\mathcal{M}_{\mathfrak{o}^\dagger,\mathcal{F}^\dagger}$ (for $G^\dagger$). This argument can be reversed. We see therefore that the proof of 8.16 for G is reduced to the proof of the analogous statement for $G^\dagger$. So we can assume that $G^\dagger = G$ or, equivalently, that the eigenvalues of $ad(\sigma)$ on $\mathfrak{g}$ are integer multiples of $2r_0$. In this case, we can find a homomorphism of algebraic groups $\iota : \mathbf{C}^* \to G$ and an integer $n \neq 0$ such that, for any $k \in \mathbf{Z}$,

$$\mathfrak{g}_{2kr_0} = \mathfrak{g}_{(kn)} := \{x \in \mathfrak{g} | Ad(\iota(t))x = t^{kn}x \quad \forall t \in \mathbf{C}^*\}.$$

Thus, we are in the setup of [**L6**] and the result follows from [**L6, 20.1**]. This proves 8.16(b).

We now prove 8.16(a). It is enough to prove that $\mathcal{M}_{\mathfrak{o},\mathcal{F}} = \mathcal{M}'_{\mathfrak{o},\mathcal{F}}$. As above, it suffices to prove this assuming that $G^\dagger = G$, and that we are in the setup of [**L6**]. The inclusion $\mathcal{M}'_{\mathfrak{o},\mathcal{F}} \subset \mathcal{M}_{\mathfrak{o},\mathcal{F}}$ is obvious. We prove the reverse inclusion. Let $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}_{\mathfrak{o},\mathcal{F}}$. From the definition, we see that there exists $(\mathcal{O}', \mathcal{E}') \in \mathcal{M}'_{\mathfrak{o},\mathcal{F}}$ such that $\mathcal{E}$ appears in $\mathcal{H}^n \mathcal{E}'^\sharp|_{\mathcal{O}'}$ for some n. By 8.16(b), we have $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}'_{\tilde{\mathfrak{o}},\tilde{\mathcal{F}}}$ for some $\tilde{\mathfrak{o}}, \tilde{\mathcal{F}}$ as in 7.1, 7.8. Combining the last two sentences with [**L6, 20.2**], we see that $(\tilde{\mathfrak{o}}, \tilde{\mathcal{F}}) = (\mathfrak{o}, \mathcal{F})$. Hence $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}'_{\mathfrak{o},\mathcal{F}}$. We have proved that

$\mathcal{M}'_{\mathbf{o},\mathcal{F}} = \mathcal{M}_{\mathbf{o},\mathcal{F}}$. Proposition 8.16 is proved.

9.4. Proof of Proposition 8.17. The equivalence of 8.17(a), 8.17(b) has already been proved in 9.3. The equivalence of 8.17(b), 8.17(c) is clear since $\mathcal{E}^\sharp$ is a $\mathfrak{H}$-equivariant perverse sheaf. We show that 8.17(a) and 8.17(d) are equivalent. Let $y \in \mathcal{O}$ and let ρ be the irreducible representation of $Z_G(\sigma,y)/Z_G^0(\sigma,y)$ corresponding to $\mathcal{E}$. Condition 8.17(a) can be reformulated as follows: ρ appears in the natural representation of $Z_G(\sigma,y)/Z_G^0(\sigma,y)$ on

$$\oplus_n H_c^n(\{P \in \mathfrak{P} | (y,P) \in \dot{\mathfrak{g}}, \sigma \in \text{Lie } P\}, \mathcal{L}^*)$$

or, equivalently, in the virtual representation

$$\textstyle\sum_n (-1)^n H_c^n(\{P \in \mathfrak{P} | (y,P) \in \dot{\mathfrak{g}}, \sigma \in \text{Lie } P\}, \mathcal{L}^*)$$

of that finite group. (Note that H_c^{odd} of the variety above is zero, by **[L3, 8.6]** and 4.4(a).)

On the other hand, condition 8.17(d) can be reformulated as follows: ρ appears in the natural representation of $Z_G(\sigma,y)/Z_G^0(\sigma,y)$ on

$$\oplus_n H_c^n(\{P \in \mathfrak{P} | (y,P) \in \dot{\mathfrak{g}}\}, \mathcal{L}^*)$$

or, equivalently, in the virtual representation

$$\textstyle\sum_n (-1)^n H_c^n(\{P \in \mathfrak{P} | (y,P) \in \dot{\mathfrak{g}}, \}, \mathcal{L}^*)$$

of that finite group. (Note that H_c^{odd} of the variety above is zero, by **[L3, 8.6]**.) But the two virtual representations above coincide, by the principle of conservation of Euler characteristic by passage to the fixed point set of a torus action. Thus, 8.17(a) and 8.17(d) are equivalent. Proposition 8.17 is proved.

9.5. Recall that in Propositions 8.16, 8.17 we have assumed that $r_0 \neq 0$. However, a version of these propositions holds also in the case where $r_0 = 0$. Assume that (σ, r_0) (as in 8.13) is fixed and $r_0 = 0$. Let $\mathfrak{g}_{0,nil}$ be the variety of nilpotent elements in $\mathfrak{g}_0 = \{y \in \mathfrak{g} | [\sigma, y] = 0\}$, We define $\mathcal{M}$ as in 8.15, except that $\mathcal{O}$ is in addition assumed to be contained in $\mathfrak{g}_{0,nil}$. For $(\mathcal{O},\mathcal{E}) \in \mathcal{M}$, let $\mathcal{E}^\sharp$ be the intersection cohomology complex on $\mathfrak{g}_{0,nil}$ defined by $(\mathcal{O},\mathcal{E})$. We define $\mathcal{M}_{\mathbf{o},\mathcal{F}}$ as in 8.15. Let $\mathcal{M}'_{\mathbf{o},\mathcal{F}}$ be the set of all $(\mathcal{O},\mathcal{E}) \in \mathcal{M}$ such that some shift of $\mathcal{E}^\sharp$ is a direct summand of $\tilde{B}|_{\mathfrak{g}_{0,nil}}$. Then Proposition 8.16, 8.17 and Corollary 8.18 continue to hold for $r_0 = 0$ (in 8.17(d) we must replace $\tilde{B} \in \mathcal{D}_{\mathfrak{H}}\mathfrak{g}_0$ by $\tilde{B}|_{\mathfrak{g}_{0,nil}}$).

The proofs are close to those given for $r_0 \neq 0$. As in 9.3, we reduce ourselves to the case where $\sigma = 0$. In that case, instead of using the results in **[L6]**, we use corresponding results in **[L2]**.

10. Standard modules

10.1. For $C \in \mathcal{D}(pt)$, we write $\mathcal{H}^m C$ instead of $\mathcal{H}^m_{pt} C$. Note that $M(C) := \oplus_m \mathcal{H}^m C$ is naturally a $\text{Hom}^{\cdot}_{\mathcal{D}(pt)}(C,C)$-module. (If $\phi \in \text{Hom}^k_{\mathcal{D}(pt)}(C,C)$ and $\xi \in \mathcal{H}^l C$, then $\phi \cdot \xi \in \mathcal{H}^{l+k} C$ is by definition the image of ξ under the linear map $\mathcal{H}^l C \to \mathcal{H}^{l+k} C$ induced by ϕ.)

10.2. We fix a semisimple element $(\sigma, r_0) \in \mathfrak{g} \oplus \mathbf{C}$ such that $r_0 \neq 0$. We will use the notation of 8.12. Let $\mathcal{A}, \mathcal{A}_s$ be as in §5 (specialized to the context of 8.12). As in §5, we write

$$\bar{B} = \oplus_{n \in \mathbf{Z}} \bar{B}^n[-n], \quad \bar{B}' = \oplus_{n \in \mathbf{Z}} \bar{B}^n, \quad \mathbf{P}_j := \text{Hom}^0_{\mathcal{D}\bar{X}}(P_j, \bar{B}')$$

where $(P_j)_{j \in \mathfrak{J}}$ are the simple constituents of $\bar{B}'$.

Let $y \in \bar{X}$; we denote by $i_y : pt \to \bar{X}$ the map such that $i_y(pt) = y$. The functors $i_y^* : \mathcal{D}\bar{X} \to \mathcal{D}(pt)$ and $i_y^! : \mathcal{D}\bar{X} \to \mathcal{D}(pt)$ define algebra homomorphisms

(a) $\quad i_y^* : \mathcal{A} = \text{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B},\bar{B}) \to \text{Hom}^{\cdot}_{\mathcal{D}(pt)}(i_y^*\bar{B}, i_y^*\bar{B})$,

(b) $\quad i_y^! : \mathcal{A} = \text{Hom}^{\cdot}_{\mathcal{D}\bar{X}}(\bar{B},\bar{B}) \to \text{Hom}^{\cdot}_{\mathcal{D}(pt)}(i_y^!\bar{B}, i_y^!\bar{B})$.

The $\text{Hom}^{\cdot}_{\mathcal{D}(pt)}(i_y^*\bar{B}, i_y^*\bar{B})$-module $M(i_y^*\bar{B})$ (as in 10.1) can be regarded as an $\mathcal{A}$-module, via (a) and the $\text{Hom}^{\cdot}_{\mathcal{D}(pt)}(i_y^!\bar{B}, i_y^!\bar{B})$-module $M(i_y^!\bar{B})$ (as in 10.1) can be regarded as an $\mathcal{A}$-module, via (b). We have

$$M(i_y^*\bar{B}) = \oplus_{n,l} \mathcal{H}^{l-n} i_y^* \bar{B}^n, \quad M(i_y^!\bar{B}) = \oplus_{n,l} \mathcal{H}^{l-n} i_y^! \bar{B}^n,$$

hence

$$M(i_y^*\bar{B}) = \oplus_{s \in \mathbf{Z}} M_s(i_y^*\bar{B}), \quad M(i_y^!\bar{B}) = \oplus_{s \in \mathbf{Z}} M_s(i_y^!\bar{B}),$$

where

$$M_s(i_y^*\bar{B}) = \oplus_n \mathcal{H}^s i^* \bar{B}^n = \mathcal{H}^s i^* \bar{B}', \quad M_s(i_y^!\bar{B}) = \oplus_n \mathcal{H}^s i^! \bar{B}^n = \mathcal{H}^s i^! \bar{B}'.$$

It is clear from the definitions that

$$\mathcal{A}_s M_{s'}(i_y^*\bar{B}) \subset M_{s+s'}(i_y^*\bar{B}), \quad \mathcal{A}_s M_{s'}(i_y^!\bar{B}) \subset M_{s+s'}(i_y^!\bar{B})$$

for all s, s', under the $\mathcal{A}$-module structure of $M_s(i_y^*\bar{B}), M_s(i_y^!\bar{B})$. Since $\mathcal{A}_s = 0$ for $s < 0$, we see that, for any $s \in \mathbf{Z}$,

$$M_{\geq s}(i_y^*\bar{B}) := \oplus_{s' \in \mathbf{Z}; s' \geq s} M_{s'}(i_y^*\bar{B}), \quad M_{\geq s}(i_y^!\bar{B}) := \oplus_{s' \in \mathbf{Z}; s' \geq s} M_{s'}(i_y^!\bar{B})$$

is a $\mathcal{A}$-submodule of $M(i_y^*\bar{B}), M(i_y^!\bar{B})$ (respectively). Hence

$$grM(i_y^*\bar{B}) = \oplus_{s'} M_{\geq s'}(i_y^*\bar{B})/M_{\geq s'+1}(i_y^*\bar{B}),$$
$$grM(i_y^!\bar{B}) = \oplus_{s'} M_{\geq s'}(i_y^!\bar{B})/M_{\geq s'+1}(i_y^!\bar{B})$$

are $\mathcal{A}$-modules. Clearly, on these two $\mathcal{A}$-modules, the radical ideal $\oplus_{s>0}\mathcal{A}_s$ of $\mathcal{A}$ acts as zero; in other words, $\mathcal{A}$ acts through its semisimple quotient $\mathcal{A}_0$.

LEMMA 10.3. *(Compare* **[CG, 8.6.15]**.*) In the Grothendieck group of $\mathcal{A}$-modules of finite dimension over* $\mathbf{C}$, *we have the equalities*

(a) $\quad M(i_y^*\bar{B}) = \oplus_{j \in \mathfrak{J}; l \in \mathbf{Z}} \mathcal{H}^l i_y^* P_j \otimes \mathbf{P}_j$,

(b) $\quad M(i_y^!\bar{B}) = \oplus_{j \in \mathfrak{J}; l \in \mathbf{Z}} \mathcal{H}^l i_y^! P_j \otimes \mathbf{P}_j$,

where $\mathcal{A}$ acts on $\mathcal{H}^l i_y^* P_j \otimes \mathbf{P}_j$ and on $\mathcal{H}^l i_y^! P_j \otimes \mathbf{P}_j$ by $a : \xi \otimes \xi' \mapsto \xi \otimes a\xi'$.

In that Grothendieck group we have $M(i_y^*\bar{B}) = grM(i_y^*\bar{B})$. Since $grM(i_y^*\bar{B})$ and the right hand side of (a) are $\mathcal{A}$-modules which factor through $\mathcal{A}_0$, for the proof of (a) it suffices to show that the $\mathcal{A}_0$-modules $grM(i_y^*\bar{B})$ and $\oplus_{j \in \mathfrak{J}; l \in \mathbf{Z}} \mathcal{H}^l i_y^* P_j \otimes \mathbf{P}_j$ are isomorphic. We have

$$\oplus_{j \in \mathfrak{J}; l \in \mathbf{Z}} \mathcal{H}^l i_y^* P_j \otimes \mathbf{P}_j = \oplus_{j \in \mathfrak{J}; l \in \mathbf{Z}} \mathcal{H}^l i_y^* P_j \otimes \text{Hom}^0_{\mathcal{D}\bar{X}}(P_j, \bar{B}')$$
$$= \oplus_{l \in \mathbf{Z}} \mathcal{H}^l i_y^* (\oplus_{j \in \mathfrak{J}} P_j \otimes \text{Hom}^0_{\mathcal{D}\bar{X}}(P_j, \bar{B}')) = \oplus_{l \in \mathbf{Z}} \mathcal{H}^l i_y^* \bar{B}'$$

(c) $$= \oplus_{l,n \in \mathbf{Z}} \mathcal{H}^l i_y^* \bar{B}^n = grM(i_y^*\bar{B})$$

and using the definitions we see that (c) is an equality of $\mathcal{A}_0$-modules. This proves (a). The proof of (b) is entirely similar.

10.4. Since $\bar{B}^n$ is a $\mathfrak{H}$-equivariant perverse sheaf, the stabilizer $\mathfrak{H}_y$ of y in $\mathfrak{H}$ (that is $\{(g,t) \in G \times \mathbf{C}^* | Ad(g)\sigma = \sigma, Ad(g)y = t^2 y\}$) acts naturally on the stalks $\mathcal{H}^l i_y^* \bar{B}^n, \mathcal{H}^l i_y^! \bar{B}^n$ and these actions factor through the finite group $\mathfrak{H}_y/\mathfrak{H}_y^0$. This induces an action of $\mathfrak{H}_y/\mathfrak{H}_y^0$ on $M(i_y^* \bar{B})$ and on $M(i_y^! \bar{B})$ which commutes with the $\mathcal{A}$-module structure and preserves the filtrations, hence we get actions of $\mathfrak{H}_y/\mathfrak{H}_y^0$ on $grM(i_y^* \bar{B}), grM(i_y^! \bar{B})$ which commute with the $\mathcal{A}_0$-module structure.

Similarly, since P_j is a $\mathfrak{H}$-equivariant perverse sheaf, the group $\mathfrak{H}_y/\mathfrak{H}_y^0$ acts naturally on the stalks $\mathcal{H}^l i_y^* P_j, \mathcal{H}^l i_y^! P_j$. We can then regard $\mathcal{H}^l i_y^*(P_j) \otimes \mathbf{P}_j, \mathcal{H}^l i_y^!(P_j) \otimes \mathbf{P}_j$ as $\mathfrak{H}_y/\mathfrak{H}_y^0$-modules by $z : \xi \otimes \xi' \mapsto (z\xi) \otimes \xi'$. These actions of $\mathfrak{H}_y/\mathfrak{H}_y^0$ clearly commute with the $\mathcal{A}_0$-module structure.

If V, ρ are $(\mathfrak{H}_y/\mathfrak{H}_y^0)$-modules, with ρ irreducible, we set $V^\rho = \mathrm{Hom}_{\mathfrak{H}_y/\mathfrak{H}_y^0}(\rho, V)$.

Note that $M(i_y^* \bar{B})^\rho$ inherits from $M(i_y^* \bar{B})$ an $\mathcal{A}$-module structure and $(grM(i_y^* \bar{B}))^\rho$ inherits from $grM(i_y^* \bar{B})$ an $\mathcal{A}_0$-module structure. From the definitions we see that

$$M(i_y^* \bar{B})^\rho = (grM(i_y^* \bar{B})^\rho$$

in the Grothendieck group of $\mathcal{A}$-modules of finite dimension over $\mathbf{C}$. Moreover, since the equality of $\mathcal{A}_0$-modules in 10.3(c) is compatible with the $\mathfrak{H}_y/\mathfrak{H}_y^0$ actions, we see that

$$(grM(i_y^* \bar{B})^\rho = \oplus_{j \in \mathfrak{J}; l \in \mathbf{Z}} (\mathcal{H}^l i_y^* P_j)^\rho \otimes \mathbf{P}_j$$

as $\mathcal{A}_0$-modules. This proves the first equality in the following proposition (the proof of the second equality is similar).

PROPOSITION 10.5. *Let ρ be as above. In the Grothendieck group of $\mathcal{A}$-modules of finite dimension over $\mathbf{C}$, we have the equalities*

$$M(i_y^* \bar{B})^\rho = \oplus_{j \in \mathfrak{J}, l \in \mathbf{Z}} (\mathcal{H}^l i_y^* P_j)^\rho \otimes \mathbf{P}_j,$$
$$M(i_y^! \bar{B})^\rho = \oplus_{j \in \mathfrak{J}, l \in \mathbf{Z}} (\mathcal{H}^l i_y^! P_j)^\rho \otimes \mathbf{P}_j$$

where $\mathcal{A}$ acts on $(\mathcal{H}^l i_y^* P_j)^\rho \otimes \mathbf{P}_j$ and on $(\mathcal{H}^l i_y^! P_j)^\rho \otimes \mathbf{P}_j$ by $a : \xi \otimes \xi' \mapsto \xi \otimes a\xi'$.

10.6. Let $\mathcal{O}$ be the $\mathfrak{H}$-orbit of y in $\{y \in \mathfrak{g} | [\sigma, y] = 2r_0 y\}$ and let $\mathcal{E}$ be the irreducible $\mathfrak{H}$-equivariant local system on $\mathcal{O}$ determined by y, ρ. Thus, $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}$ (see 8.15). It is clear that the $\mathcal{A}$-modules $M(i_y^* \bar{B})^\rho, M(i_y^! \bar{B})^\rho$ depend only on $(\mathcal{O}, \mathcal{E})$, hence we may denote them by $M_{\mathcal{O},\mathcal{E},*}, M_{\mathcal{O},\mathcal{E},!}$ respectively.

By 8.17, the set $\mathfrak{J}$ may be naturally identified with the subset $\mathcal{M}_{\mathbf{o},\mathcal{F}}$ of $\mathcal{M}$. We shall therefore write $P_{\mathcal{O}',\mathcal{E}'}, \mathbf{P}_{\mathcal{O}',\mathcal{E}'}$ instead of $P_j, \mathbf{P}_j$, where $j \in \mathfrak{J}$ corresponds to $(\mathcal{O}', \mathcal{E}') \in \mathcal{M}_{\mathbf{o},\mathcal{F}}$.

For $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}, (\mathcal{O}', \mathcal{E}') \in \mathcal{M}_{\mathbf{o},\mathcal{F}}$, we denote by $p_{\mathcal{O},\mathcal{E};\mathcal{O}',\mathcal{E}'}$ the multiplicity of $\mathcal{E}$ in the local system $\oplus_{l \in \mathbf{Z}} \mathcal{H}^l P_{\mathcal{O}',\mathcal{E}'}|_{\mathcal{O}}$ and by $p'_{\mathcal{O},\mathcal{E};\mathcal{O}',\mathcal{E}'}$ the multiplicity of $\mathcal{E}^*$ in the local system $\oplus_{l \in \mathbf{Z}} \mathcal{H}^l D P_{\mathcal{O}',\mathcal{E}'}|_{\mathcal{O}}$.

From **[L6, §16, 17.6]**, we see that $p'_{\mathcal{O},\mathcal{E};\mathcal{O}',\mathcal{E}'} = p_{\mathcal{O},\mathcal{E};\mathcal{O}',\mathcal{E}'}$. Hence Proposition 10.5 implies the following result.

COROLLARY 10.7. *For any $\mathcal{O}, \mathcal{E}) \in \mathcal{M}$, we have the following equalities in the Grothendieck group of $\mathcal{A}$-modules of finite dimension over $\mathbf{C}$:*

(a) $$M_{\mathcal{O},\mathcal{E},*} = M_{\mathcal{O},\mathcal{E},!} = \sum p_{\mathcal{O},\mathcal{E};\mathcal{O}',\mathcal{E}'} \mathbf{P}_{\mathcal{O}',\mathcal{E}'}$$

(sum over all $(\mathcal{O}', \mathcal{E}') \in \mathcal{M}_{\mathbf{o},\mathcal{F}}$).

10.8. An $\mathcal{A}$-module of finite dimension over $\mathbf{C}$ can be regarded as an object of $\mathbf{mod}_{\sigma,r_0} \mathrm{Hom}_{\mathcal{D}_{\mathfrak{H}},\tilde{X}}(\bar{B}, \bar{B})$ (by 5.1(d)) and as an object of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ (by 8.14(a)). (Note that σ of §5 is now replaced by (σ, r_0).) In particular, $M_{\mathcal{O},\mathcal{E},*}, M_{\mathcal{O},\mathcal{E},!}$ may be regarded as objects of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ and $\mathbf{P}_{\mathcal{O},\mathcal{E}}$ are simple objects of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$ (for $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}_{\mathbf{o},\mathcal{F}}$). Moreover, 10.7(a) may be regarded as an equality in the Grothendieck group of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$.

COROLLARY 10.9. *(a) $\{\mathbf{P}_{\mathcal{O},\mathcal{E}} | (\mathcal{O}, \mathcal{E}) \in \mathcal{M}_{\mathbf{o},\mathcal{F}}\}$ is a $\mathbf{Z}$-basis of the Grothendieck group of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$.*

(b) $\{M_{\mathcal{O},\mathcal{E},!} | (\mathcal{O}, \mathcal{E}) \in \mathcal{M}_{\mathbf{o},\mathcal{F}}\}$ is a $\mathbf{Z}$-basis of the Grothendieck group of $\mathbf{mod}_{\sigma,r_0}\mathbf{H}$.

(c) If $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}_{\mathbf{o},\mathcal{F}}$ is such that $\mathcal{O}$ is the open $\mathfrak{H}$-orbit in $\{y \in \mathfrak{g} | [\sigma, y] = 2r_0 y\}$, then the $\mathbf{H}$-modules $M_{\mathcal{O},\mathcal{E},!}, \mathbf{P}_{\mathcal{O},\mathcal{E}}$ are isomorphic.

(a) follows from 8.14(b). (b) follows from (a) and from the last sentence in 10.8: indeed, from 10.7 we see that $M_{\mathcal{O},\mathcal{E},!} = 0$ unless $(\mathcal{O}', \mathcal{E}') \in \mathcal{M}_{\mathbf{o},\mathcal{F}}$ and that the matrix expressing the elements in (b) in terms of the basis in (a) is integral, upper triangular, with 1 in diagonal. Under the assumption of (c), the $\mathbf{H}$-modules $M_{\mathcal{O},\mathcal{E},!}, \mathbf{P}_{\mathcal{O},\mathcal{E}}$ define the same element in the Grothendieck group (by 10.7(a)), hence they are isomorphic since $\mathbf{P}_{\mathcal{O},\mathcal{E}}$ is simple.

10.10. The integers $p_{\mathcal{O},\mathcal{E};\mathcal{O}',\mathcal{E}'}$ which enter in 10.7(a) are in principle computable, from the results in **[L6]**. More precisely, those results should be applied not directly to $G, \mathfrak{g}$ but to $(G^\dagger, \mathfrak{g}^\dagger, \iota)$ (as in 9.1, 9.3).

10.11. The $\mathbf{H}$-modules $M_{\mathcal{O},\mathcal{E},!}$ for $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}_{\mathbf{o},\mathcal{F}}$ are called *standard modules.* We want to compare these modules with those defined in **[L3,§8]**. Let $y \in \mathfrak{g}_N$. We set

(a) $$\mathfrak{P}_y = \{P \in \mathfrak{P} | y \in \pi_P^{-1} C_P\}.$$

We may regard this as a subvariety of $\dot{\mathfrak{g}}$ (the inverse image of y under $pr_1 : \dot{\mathfrak{g}} \to \mathfrak{g}$). The subgroup $\bar{Z}_G(y)$ of $G \times \mathbf{C}^*$ (see 6.0) acts on $\mathfrak{P}_y$ (as restriction of the action 7.15 of $G \times \mathbf{C}^*$ on $\dot{\mathfrak{g}}$). Then

(b) $$H_{\cdot}^{\bar{Z}_G^0(y)}(\mathfrak{P}_y, \mathcal{L})$$

may be regarded as a H-module, by **[L3, 8.13]** and as a $\bar{Z}_G(y)/\bar{Z}_G^0(y)$-module, by **[L3, 1.9(a)]**; the two module structures commute, by **[L3, 8.5]**.

Now assume that y satisfies $[\sigma, y] = 2r_0 y$. Then $(\sigma, r_0) \in$ Lie $\bar{Z}_G^0(y)$ defines an algebra homomorphism $\chi_\sigma : H_{\bar{Z}_G^0(y)}^{\cdot} \to \mathbf{C}$ as in 4.2; using this we can regard $\mathbf{C}$ as a $H_{\bar{Z}_G^0(y)}^{\cdot}$-algebra and we form

(c) $$\mathbf{C} \otimes_{H_{\bar{Z}_G^0(y)}^{\cdot}} H_{\cdot}^{\bar{Z}_G^0(y)}(\mathfrak{P}_y, \mathcal{L});$$

this inherits an $\mathbf{H}$-module structure and a $Z_G(\sigma, y)/Z_G^0(\sigma, y)$-module structure from $H_{\cdot}^{\bar{Z}_G^0(y)}(\mathfrak{P}_y, \mathcal{L})$; these two structures commute. If ρ is an irreducible $Z_G(\sigma, y)/Z_G^0(\sigma, y)$-module such that (y, ρ) corresponds to a pair in $\mathcal{M}_{\mathbf{o},\mathcal{F}}$ (see 8.15), we set

(d) $$(\mathbf{C} \otimes_{H_{\bar{Z}_G^0(y)}^{\cdot}} H_{\cdot}^{\bar{Z}_G^0(y)}(\mathfrak{P}_y, \mathcal{L}))^\rho$$

$$= \mathrm{Hom}_{Z_G(\sigma,y)/Z_G^0(\sigma,y)}(\rho, \mathbf{C}\otimes_{H_{\bar{Z}^0_G(y)}} H_\cdot^{\bar{Z}^0_G(y)}(\mathfrak{P}_y, \mathcal{L}));$$

this is then naturally a H-module (a standard module in the sense of [**L3**,§8].

PROPOSITION 10.12. *(a) The* **H**-*module 10.11(d) is isomorphic to the* **H**-*module $M_{\mathcal{O},\mathcal{E}}$ where $\mathcal{O}$ contains y and $\mathcal{E}$ corresponds to ρ.*

(b) The $\mathbf{H}\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$*-module $M(i_y^!\tilde{B})$ is isomorphic to the one in 10.11(c).*

(a) clearly follows from (b). We verify (b). Let $\mathfrak{P}_y^\sigma = \{P\in\mathfrak{P}_y|\sigma\in \text{Lie } P\}$ and let $k:\mathfrak{P}_y^\sigma\to\tilde{Z}$ ($\tilde{Z}$ is as in 8.12) be the inclusion $k(P)=(y,P)$. Let $\pi:\mathfrak{P}_y^\sigma\to\{y\}$ be the obvious map and let $\tilde{f}':\tilde{Z}\to\dot{X}$ be the composition $\tilde{f}\tilde{u}$ (recall that $\tilde{u}:\tilde{Z}\to\dot{X}$ is the inclusion). Note that $k,\pi,i_y,\tilde{f}'$ form a cartesian diagram. Hence we have

$$\begin{aligned} M(i_y^!\tilde{B}) &= H^\cdot(\{y\}, i_y^!\tilde{B}) = H^\cdot(\{y\}, i_y^!\tilde{f}_*\tilde{u}_*\tilde{\mathcal{L}}^*)\\ &= H^\cdot(\{y\}, i_y^!\tilde{f}'_*\tilde{\mathcal{L}}^*) = H^\cdot(\{y\}, \pi_*k^!\tilde{\mathcal{L}}^*)\\ &= H^\cdot(\mathfrak{P}_y^\sigma, k^!\tilde{\mathcal{L}}^*) = H^\cdot(\mathfrak{P}_y^\sigma, Dk^*D\tilde{\mathcal{L}}^*)\\ \text{(c)}\qquad &= H^\cdot(\mathfrak{P}_y^\sigma, Dk^*\tilde{\mathcal{L}}) = H_\cdot^{\{1\}}(\mathfrak{P}_y^\sigma, \tilde{\mathcal{L}}). \end{aligned}$$

(We write $\tilde{\mathcal{L}}$ instead of $k^*\tilde{\mathcal{L}}$.) In the following equalities T is as in 8.12, and we regard $\mathbf{C}$ as an algebra over $H^\cdot_{\bar{Z}^0_G(y)}$, over $H^\cdot_{Z_1}$ (where $Z_1 = Z^0_{\bar{Z}(y)}(\sigma, r_0)$) or over $H^\cdot_T$ (by restriction):

$$\begin{aligned} H_\cdot^{\{1\}}(\mathfrak{P}_y^\sigma,\tilde{\mathcal{L}}) &= \mathbf{C}\otimes_{H_T^\cdot} H_\cdot^T(\mathfrak{P}_y^\sigma,\tilde{\mathcal{L}}) = \mathbf{C}\otimes_{H_{Z_1}^\cdot} H_\cdot^{Z_1}(\mathfrak{P}_y^\sigma,\tilde{\mathcal{L}})\\ \text{(d)}\qquad &= \mathbf{C}\otimes_{H_{Z_1}^\cdot} H_\cdot^{Z_1}(\mathfrak{P}_y,\mathcal{L}) = \mathbf{C}\otimes_{H^\cdot_{\bar{Z}(y)^0}} H_\cdot^{\bar{Z}(y)^0}(\mathfrak{P}_y,\mathcal{L}). \end{aligned}$$

Here, the first equality follows from 1.21(a); the second equality follows from [**L3, 7.5**] using the vanishing of $H_c^{odd}(\mathfrak{P}_y^\sigma,\tilde{\mathcal{L}}^*)$ (which in turn, follows from the vanishing of $H_c^{odd}(\mathfrak{P}_y,\mathcal{L}^*)$ [**L3, 8.6**], together with 4.4(a)); the third equality follows from 4.4(b); the fourth equality follows from [**L3, 7.5**], using the vanishing of $H_c^{odd}(\mathfrak{P}_y,\mathcal{L}^*)$ [**L3, 8.6**]. Combining (c), (d) we see that the two modules in (b) have the same underlying $\mathbf{C}$ vector space. The identification of the module structures can be done along the lines of the proof of 8.11; we omit further details.

10.13. Replacing (σ, r_0) by $(t\sigma, tr_0)$ in the definition of

$$\mathbf{C}\otimes_{H^\cdot_{\bar{Z}(y)^0}} H_\cdot^{\bar{Z}(y)^0}(\mathfrak{P}_y,\mathcal{L})$$

(see 1.11(c)) for any $t\in\mathbf{C}$, we get a family of $\mathbf{H}\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$-modules indexed by $t\in\mathbf{C}$. This can be regarded as the family of fibres of an algebraic vector bundle over $\mathbf{C}$ (here we use [**L3, 7.2**] and [**L3, 8.6**]). We can restrict this to a family of $W\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$-modules indexed by $t\in\mathbf{C}$ (using the obvious imbedding $\mathbf{C}[W]\subset\mathbf{H}$). Since a finite dimensional representation of a finite group cannot be continuously deformed in a non-trivial way, it follows that the $W\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$-module corresponding to $t=1$ is isomorphic to the $W\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$-module corresponding to $t=0$. This last module may be identified with the restriction of the $W\times(Z_G(y)/Z_G^0(y))$-module $H_\cdot^{\{1\}}(\mathfrak{P}_y,\mathcal{L})$ to $W\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$, by the argument giving 10.12(d). Hence the $W\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$-module in 1.11(c) is isomorphic to this restriction. Hence, it is in principle, computable. (The $W\times(Z_G(y)/Z_G^0(y))$-module $H_\cdot^{\{1\}}(\mathfrak{P}_y,\mathcal{L})$ is computable from the theory of generalized Green functions, see [**L2**].)

10.14. A substantial amount of information about the simple **H**-modules can be deduced from information about the standard modules $M_{\mathcal{O},\mathcal{E},!}$, which is often more readily accessible. For example, from 10.12 and 10.13, we know in principle the structure of $M(i_y^!\tilde{B})$ as a $W\times(Z_G(\sigma,y)/Z_G^0(\sigma,y))$-module, hence we know also the structure of $M_{\mathcal{O},\mathcal{E},!}$ as a W-module and from this we then deduce, in principle (using 10.7(a) and 10.10) the structure of $\mathbf{P}_{\mathcal{O},\mathcal{E}}$ as a W-module.

11. Generalized flag manifolds

11.1. The variety $\mathfrak{P}_x = \{P\in\mathfrak{P}|x\in\pi_P^{-1}C_P\}$ (see 10.11(a)) for $x\in\mathcal{V}_0 = \cup_{(L,C)\in\mathbf{o}}C$ is called a *generalized flag manifold.* (In the special case where $\mathbf{o}$ is the set of pairs consisting of a maximal torus and the zero nilpotent orbit, we have $\mathcal{V}_0=\{0\}$ and $\mathfrak{P}_0$ is the usual flag manifold of G.) The following result shows that the generalized flag manifolds share some properties with the usual flag manifolds.

PROPOSITION 11.2. *Let $x\in\mathcal{V}_0$ and let U be the unipotent radical of $Z_G^0(x)$.*

(a) If $(L,C)\in\mathbf{o}$ is such that $x\in C$, then the natural map $Z_L(x)/Z_L^0(x)\to Z_G(x)/Z_G^0(x)$ is a bijection.

(b) The group $Z_G^0(x)$ acts transitively on $\mathfrak{P}_x$ (by conjugation).

(c) If $P\in\mathfrak{P}_x$, then $\beta_P := (P\cap Z_G^0(x))U$ is a Borel subgroup of $Z_G^0(x)$.

(d) The morphism $P\mapsto\beta_P$ from $\mathfrak{P}_x$ to the variety of all Borel subgroups of $Z_G^0(x)$ is a fibration. The fibres are exactly the orbits of the conjugation action of U on $\mathfrak{P}_x$, hence are all affine spaces.

(a) follows from [**L6, 2.7(d)**]. To prove (b), we pick $(P,L,C)\in\mathbf{o}'$ with $x\in C$, and we use the chain of equalities

$$\begin{aligned} \{g\in G|Ad(g^{-1})x\in C+\text{Lie } U_P\} &= \{g\in G|Ad(g^{-1})x\in x+\text{Lie } U_P\}L\\ &= Z_G(x)U_PL = Z_G(x)P = Z_G^0(x)P. \end{aligned}$$

The first and third equality are obvious; the second equality follows from [**Spa, 1.4(i)**]; the fourth equality follows from (a). To prove (c), we may assume by (b), that P is as in the earlier part of the proof. Then the desired result follows from [**L6, 11.7(a)**]. We prove (d). Let $P,P'\in\mathfrak{P}_x$ be such that $\beta_P=\beta_{P'}$. By (b), we have $P'=gPg^{-1}$ for some $g\in Z_G^0(x)$. Hence $\beta_P=\beta_{P'}=g\beta_Pg^{-1}$. Since β_P is self-normalizing in $Z_G^0(x)$, we have $g\in\beta_P$. Thus $g=pu$ where $p\in P\cap Z_G^0(x)$ and $u\in V$. Since U is normal in $Z_G^0(x)$, we have $g=u'p$ where

$u' = pup^{-1} \in U$. We have $P' = u'Pu'^{-1}$ and (d) follows. The proposition is proved.

Errata to [L3]

1.1 (line 2) Replace "action" by "action of G".
1.2 (line 2) Replace -1 by $-i$.
1.6 a) (line 1) Replace last G by G'.
1.13 a) Replace second $\mathcal{L}$ by $\mathcal{L}^*$.
2.1 b) Replace L by G.
2.13 b) (line 2) Replace $2k$ by $2n$.
3.5 h) Replace $\dot{\mathcal{L}} \boxtimes \dot{\mathcal{L}}$ by $\dot{\mathcal{L}} \boxtimes \dot{\mathcal{L}}^*$.
p.170 (line 2,3) Replace $\ddot{\mathfrak{g}}_1, \ddot{\mathfrak{g}}_2$ by $\dot{\mathfrak{g}}_1, \dot{\mathfrak{g}}_2$.
3.9 k) Replace $\dot{\mathfrak{g}}$ by $\ddot{\mathfrak{g}}$.
4.3 e) Replace $\ddot{V}^\Omega$ by $\ddot{V}^P$.
p.176 (line 13) Replace $\nu =$ by $\nu \in$.
4.14 (line 2) should be $(\Delta(s)\Delta(\alpha) + \Delta(\alpha)\Delta(s))(h) = prh$.
4.14(a) (line 1) Replace third $\ddot{g}$ by $\ddot{V}_{RS}$.
4.14(a) (line 2) Replace $\ddot{g}$ by $\ddot{V}_{RS}$.
p.182 (line -11) Replace 4.3 d) by 4.4 a).
p.185 (lines -15, -17) Replace M_0 by M^0.
p.185 (line -17) Replace $\ddot{g}_{NR}$ by $\ddot{V}_{RN}$.
p.186 (line -9) Replace E by E_e.
p.188 (line 8) Replace A, f by A', f'.
p.188 (line 17) Replace a) by f).
p.188 (line 18) Replace ξ by χ.
p.188 (line 20) Replace b) by g).
7.3 c) Replace m by M.
p.191 (line -6) Replace last X by X'.
8.1 c) (line 2) Replace $\mathcal{O}$ by $\tilde{\mathcal{O}}$.
p.193 (line -6) Replace $\dot{\mathcal{L}}^*$ by $\dot{\mathcal{L}}$.
8.1 (last line) Replace $W \times W'$ by $W \times W$.
8.2 (lines 3,4) Replace h'^* by $\dot{h}^*$.
p.201 (line 9) Move "We tensor the previous exact sequence with $\mathbf{C}_{\sigma,r_0}$" two lines up.

8.15 The last part of the proof is incorrect as written (as pointed out by D. Vogan); one can easily correct it by arguing along the lines of the proof in [**KL, 5.13**].

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