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On Some Degenerate Principal Series Representations of $U(n, n)$

Soo Teck Lee

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511, Republic of Singapore

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We study the infinitesimal action of $\mathfrak{u}(n, n)$ on the degenerate principal series representations of $U(n, n)$ associated with a parabolic subgroup which stabilizes a maximal isotropic subspace of $\mathbf{C}^{2n}$ relative to the indefinite Hermitian form with signature (n, n). This allows us in particular to determine the reducibility, socle series, and unitarity of these representations. © 1994 Academic Press, Inc.

1. Introduction

In his study of the representations of $SL(2, \mathbf{R})$ [2], Bargmann decomposed the representation space into a sum of eigenspaces for the compact subgroup $SO(2, \mathbf{R})$ of $SL(2, \mathbf{R})$ and computed explicitly how the Lie algebra of $SL(2, \mathbf{R})$ transformed these eigenspaces. Recently, Howe and Tan [12] extended this idea to some degenerate principal series representations of the more complicated groups $O(p, q)$, $U(p, q)$, and $Sp(p, q)$. Their work essentially involves two steps:

(1) Replace $SO(2, \mathbf{R})$ by a maximal compact subgroup K of the group G and decompose a representation of G into a sum of K-types.

(2) Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Determine the action of $\mathfrak{p}$ on the K-types.

The object of this paper is to apply the above procedure to the study of a family of degenerate principal series representations of the group $U(n, n)$ which we shall now describe. Let $\langle \cdot, \cdot \rangle$ be the indefinite Hermitian form with signature (n, n) on $\mathbf{C}^{2n}$ given by

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i \bar{v}_i - \sum_{j=n+1}^{2n} u_j \bar{v}_j, \qquad u = (u_1, ..., u_{2n}),\ v = (v_1, ..., v_{2n}) \in \mathbf{C}^{2n}.$$

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As usual, $\bar{z}$ denotes the complex conjugate of $z \in \mathbf{C}$. Then $U(n, n)$ is the isometry group of $\langle \cdot, \cdot \rangle$. Let $\{\varepsilon_1, ..., \varepsilon_{2n}\}$ be the standard basis of $\mathbf{C}^{2n}$, then $\{\varepsilon_1 + \varepsilon_{n+1}, ..., \varepsilon_n + \varepsilon_{2n}\}$ spans a subspace V in $\mathbf{C}^{2n}$ and V is maximal totally isotropic with respect to the form $\langle \cdot, \cdot \rangle$. Let P be the stabilizer of the flag $\{0, V, \mathbf{C}^{2n}\}$ in $U(n, n)$. Then P is a maximal parabolic subgroup of $U(n, n)$. Let $\mathscr{B}$ be the basis $\{\varepsilon_1 + \varepsilon_{n+1}, ..., \varepsilon_n + \varepsilon_{2n}, \varepsilon_1 - \varepsilon_{n+1}, ..., \varepsilon_n - \varepsilon_{2n}\}$ of $\mathbf{C}^{2n}$. Let $SH(n)$ be the additive group of $n \times n$ skew Hermitian matrices. For $a \in GL(n, \mathbf{C})$ and $b \in SH(n)$, we shall denote the elements in $U(n, n)$ with matrix representations $\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ relative to the basis $\mathscr{B}$ by m_a and n_b, respectively. Let

$$M = \{m_a : a \in GL(n, \mathbf{C})\}, \qquad N = \{n_b : b \in SH(n)\}.$$

Then $M \cong GL(n, \mathbf{C})$, $N \cong SH(n)$ are subgroups of $U(n, n)$ and $P = MN$. For $\alpha, \beta \in \mathbf{C}$ with $\alpha - \beta \in \mathbf{Z}$, we let $\chi_{\alpha,\beta} : P \to \mathbf{C}^\times$ be the character given by

$$\chi_{\alpha,\beta}(p) = \chi_{\alpha,\beta}(m_a n_b) = (\det a)^\alpha \, (\overline{\det a})^\beta, \qquad p = m_a n_b \in P.$$

We shall study the family $\{\mathrm{Ind}_P^{U(n,n)} \chi_{\alpha,\beta}\}_{\alpha - \beta \in \mathbf{Z}}$ of (smoothly) induced representations. The representation space for $\mathrm{Ind}_P^{U(n,n)} \chi_{\alpha,\beta}$ is

$$\{f \in C^\infty(U(n, n)) : f(gp) = \chi_{\alpha,\beta}(p)^{-1} f(g),\ \forall g \in U(n, n),\ p \in P\}$$

and on it $U(n, n)$ acts by left translation:

$$(g \cdot f)(h) = f(g^{-1}h), \qquad g, h \in U(n, n).$$

The family $\{\mathrm{Ind}_P^{U(n,n)} \chi_{\alpha,\beta}\}_{\alpha - \beta \in \mathbf{Z}}$ is known as *the degenerate principal series representations of* $U(n, n)$ *associated with* P. These representations can be realised as spaces of smooth functions in which the highest weight vectors for the maximal compact subgroup $K = U(n) \times U(n)$ of $U(n, n)$ can be explicitly described. In carrying out Step 2 above, we shall compute explicitly the action of $\mathfrak{p}$ on these highest weight vectors. We then use this information to determine the reducibility, socle series, and unitarity of these representations.

The degenerate series representations for the universal covering group of $SU(2, 2)$ have been studied by Speh in [20]. Johnson has also studied the degenerate series of a class of groups which includes $SU(n, n)$ in [14]. Later Johnson [15] and Sahi [18, 19] studied independently these representations in the more general setting of Hermitian symmetric spaces of tube type. Our method is different from and potentially more general than all of the above and we hope that our results present a more explicit picture on the module structure of these representations.

The outline of this paper is as follows. In Section 2, we identify each of these representations with a function space $\mathscr{S}^{\alpha,\beta}(X^{oo})$. We then decompose $\mathscr{S}^{\alpha,\beta}(X^{oo})$ into a sum of K-types and give an explicit description of a highest weight vector in each K-type. In Section 3, we obtain explicit formulas for the highest weight vectors in the tensor products $\mathbf{C}^n \otimes V_\lambda$ and $\mathbf{C}^{n*} \otimes V_\lambda$ of $\mathfrak{gl}_n(\mathbf{C})$ modules. Here V_λ is an irreducible $\mathfrak{gl}_n(\mathbf{C})$ module with highest weight λ. In Section 4, we study how a certain element F_{mj} in the universal enveloping algebra of $\mathfrak{gl}_n(\mathbf{C})$ which occurs in the formulas in Section 3 transforms the $\mathfrak{gl}_n(\mathbf{C})$ highest weight vectors in the polynomial algebra $P(M_n(\mathbf{C}))$. We then deduce the corresponding results for the highest weight vectors of the K-types of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. This allows us in Section 5 to determine the action of $\mathfrak{p}$ on these highest weight vectors. We then deduced the reducibility and socle series of these representations in Section 6. In Section 7, we give a diagrammatic representation of the module structure based on the conventions given in [1]. In Section 8, we draw diagrams to show the "barrier lines" in the K-type plane for the case $n = 2$. This provides an alternative graphical representation for the module structure of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. Finally in Section 9, we determine which of the subquotients of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ can define unitary representations.

2. The $U(n, n)$ Modules $\mathscr{S}^{\alpha,\beta}(X^{oo})$ and Their K-Types

In this section, we shall identify the representation $\mathrm{Ind}_P^{U(n,n)} \chi_{-\beta,-\alpha}$ with a function space $\mathscr{S}^{\alpha,\beta}(X^{oo})$ on which computations on the module structure can be carried out. We then decompose $\mathscr{S}^{\alpha,\beta}(X^{oo})$ into a sum of K-types and give an explicit description of the highest weight vector in each K-type.

Let $M_{2n,n}(\mathbf{C})$ be the space of all $2n \times n$ complex matrices. Let $U(n, n)$ act on it by

$$(g, x) \to (g^{-1})^t x, \qquad g \in U(n, n),\ x \in M_{2n,n}(\mathbf{C}). \tag{2.1}$$

The product $(g^{-1})^t x$ is matrix multiplication. Let $x_o = \begin{pmatrix} I_n \\ -I_n \end{pmatrix}$ and X^{oo} be the $U(n, n)$ orbit of x_o. Here I_n denotes the $n \times n$ identity matrix. The stabilizer of x_o in $U(n, n)$ is easily seen to be N. Consequently,

$$X^{oo} \cong U(n, n)/N$$

as a homogeneous space for $U(n, n)$.

For $\alpha, \beta \in \mathbf{C}$ with $\alpha - \beta \in \mathbf{Z}$, we let

$$\mathscr{S}^{\alpha,\beta}(X^{oo}) = \{f \in C^\infty(X^{oo}) : f(xa) = (\det a)^\alpha \, (\overline{\det a})^\beta f(x),$$
$$\forall x \in X^{oo},\ a \in GL(n, \mathbf{C})\}.$$

$U(n,n)$ acts on $\mathscr{S}^{\alpha,\beta}(X^{oo})$ by

$$(g\cdot f)(x)=f(g^t x), \qquad g\in U(n,n),\ x\in X^{oo}. \tag{2.2}$$

The products xa and $g^t x$ are matrix multiplications.

We now see that $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is isomorphic to $\mathrm{Ind}_P^{U(n,n)}\chi_{-\beta,-\alpha}$ as a $U(n,n)$ module. For $f\in\mathscr{S}^{\alpha,\beta}(X^{oo})$, let $\tilde{f}\colon U(n,n)\to\mathbf{C}$ be given by

$$\tilde{f}(g)=f[(g^{-1})^t x_o], \qquad g\in U(n,n).$$

Then since for each $a\in GL(n,\mathbf{C})$, $m_a x_o=x_o(a^*)^{-1}$ as a map from $\mathbf{C}^n$ to $\mathbf{C}^{2n}$, one can check that $\tilde{f}\in\mathrm{Ind}_P^{U(n,n)}\chi_{-\beta,-\alpha}$. Hence $f\to\tilde{f}\colon\mathscr{S}^{\alpha,\beta}(X^{oo})\to\mathrm{Ind}_P^{U(n,n)}\chi_{-\beta,-\alpha}$ is an isomorphism.

Let $K_1=\{(\begin{smallmatrix}u&0\\0&I\end{smallmatrix}): u\in U(n)\}$, $K_2=\{(\begin{smallmatrix}I&0\\0&u\end{smallmatrix}): u\in U(n)\}$, and $K=\{(\begin{smallmatrix}u_1&0\\0&u_2\end{smallmatrix}): u_1,u_2\in U(n)\}$. Then K_1, K_2, and K are subgroups of $U(n,n)$ and $K_1\cong K_2\cong U(n)$ and $K=K_1\times K_2$, hence is isomorphic to $U(n)\times U(n)$. Recall that the Cartan involution (see [16]) $\theta\colon U(n,n)\to U(n,n)$ is given by $\theta(g)=(\overline{g^{-1}})^t$, $g\in U(n,n)$. Since $K=\{g\in U(n,n)\colon\theta(g)=g\}$, K is a maximal compact subgroup of $U(n,n)$ (again see [16]).

As usual, $\mathfrak{u}(n)$ shall denote the Lie algebra of all $n\times n$ skew Hermitian complex matrices. Let $\mathfrak{k}_1,\mathfrak{k}_2$, and $\mathfrak{k}$ be the Lie algebras of K_1, K_2, and K, respectively. Then $\mathfrak{k}_1\cong\mathfrak{k}_2\cong\mathfrak{u}(n)$ and $\mathfrak{k}=\mathfrak{k}_1\oplus\mathfrak{k}_2\cong\mathfrak{u}(n)\oplus\mathfrak{u}(n)$. For a real Lie algebra $\mathfrak{g}$, we shall denote its complexification by $\mathfrak{g}_{\mathbf{C}}$ and the enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$ by $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$. Then $\mathfrak{k}_{\mathbf{C}}=\mathfrak{k}_{1\mathbf{C}}\oplus\mathfrak{k}_{2\mathbf{C}}\cong\mathfrak{gl}_n(\mathbf{C})\oplus\mathfrak{gl}_n(\mathbf{C})$. For $1\leqslant k, l\leqslant 2n$, let $e_{k,l}\in\mathfrak{gl}_{2n}(\mathbf{C})$ be the element with 1 at its (k,l)-th position and 0 elsewhere. For $1\leqslant i,j\leqslant n$, let $e^{(1)}_{i,j}=e_{i,j}$ and $e^{(2)}_{i,j}=e_{n+i,n+j}$. Then

$$e^{(1)}_{i,j}\to e_{i,j}, \qquad e^{(2)}_{i,j}\to e_{i,j} \tag{2.3}$$

clearly define isomorphisms for $\mathfrak{k}_{1\mathbf{C}}\cong\mathfrak{gl}_n(\mathbf{C})$ and $\mathfrak{k}_{2\mathbf{C}}\cong\mathfrak{gl}_n(\mathbf{C})$ respectively. These isomorphisms extend in the usual way to isomorphisms for their respective enveloping algebras. For $F\in\mathscr{U}(\mathfrak{gl}_n(\mathbf{C}))$, we shall denote its images in $\mathscr{U}(\mathfrak{k}_{1\mathbf{C}})$ and $\mathscr{U}(\mathfrak{k}_{2\mathbf{C}})$ under the above isomorphisms by $F^{(1)}$ and $F^{(2)}$ respectively.

A straightforward calculation shows that

$$P\cap K=\left\{\begin{pmatrix}u&0\\0&u\end{pmatrix}: u\in U(n)\right\}.$$

Hence $P\cap K\cong U(n)$. Let $\nu\colon P\cap K\to\mathbf{C}^\times$ be defined by

$$\nu\begin{pmatrix}u&0\\0&u\end{pmatrix}=\det u, \qquad u\in U(n).$$

We note that $\chi_{-\beta,-\alpha}|_{P\cap K}=\nu^{\alpha-\beta}$. Thus as a representation for K,

$$\mathscr{S}^{\alpha,\beta}(X^{oo})|_K\cong\mathrm{Ind}_P^{U(n,n)}\chi_{-\beta,-\alpha}|_K\cong\mathrm{Ind}_{P\cap K}^K\nu^{\alpha-\beta}. \tag{2.4}$$

Let S^1 denote the group of complex numbers of modulus 1. Let

$$T=\{\mathrm{diag}(t_1,t_2,\ldots,t_n)\colon t_j\in S^1,\ \forall 1\leqslant j\leqslant n\}\subseteq U(n).$$

Then T is a maximal torus of $U(n)$. Every character φ of T has the form

$$\varphi[\mathrm{diag}(t_1,t_2,\ldots,t_n)]=t_1^{\lambda_1}\cdots t_n^{\lambda_n},$$

where $\lambda_1,\ldots,\lambda_n$ are integers (see [3]). Hence we can identify φ with the integral points $\lambda=(\lambda_1,\ldots,\lambda_n)$. We say that λ is a *dominant weight* if $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n$. Let Λ^+ denote the set of all dominant weights. Then, by the Cartan–Weyl highest weight theory [21], the (equivalence classes of) irreducible representations of $U(n)$ are parametrized by the set Λ^+. Thus if $\lambda\in\Lambda^+$, the corresponding representation $\rho^\lambda\colon U(n)\to GL(U^\lambda)$ is the unique (up to equivalence) irreducible representation of $U(n)$ with highest weight λ. Now the Lie algebra of $U(n)$ is $\mathfrak{u}(n)$ and $\mathfrak{u}(n)_{\mathbf{C}}\cong\mathfrak{gl}_n(\mathbf{C})$. Thus by passing from the representation ρ^λ to its differential at the identity of $U(n)$ and complexifying, we obtain a representation of $\mathfrak{gl}_n(\mathbf{C})$ on U^λ. We shall also denote this representation by ρ^λ.

For convenience, we introduce the following notations:

$$\lambda^*=(-\lambda_n,-\lambda_{n-1},\ldots,-\lambda_1), \qquad \forall\lambda=(\lambda_1,\ldots,\lambda_n)\in\Lambda^+,$$

$$e_j=(\overbrace{0,\ldots,0,1}^{j},0,\ldots,0), \qquad \forall 1\leqslant j\leqslant n,$$

$$\mathbf{1}=(1,1,\ldots,1),$$

$$k\mathbf{1}=(k,k,\ldots,k), \qquad \forall k\in\mathbf{Z}.$$

We note that ρ^{λ^*} is isomorphic to $(\rho^\lambda)^*$, the contragredient representation of ρ^λ (see Exercise 15.50 of [6]).

Next we let T_1 and T_2 be the images of T in K_1 and K_2, respectively. If ψ is a character of $T_1\times T_2$, then $\psi(a,b)=\varphi_1(a)\,\varphi_2(b)$ where φ_1 and φ_2 are the restrictions of ψ to T_1 and T_2, respectively. If φ_1 and φ_2 correspond to the integral points λ and μ, then we represent ψ by the ordered pair (λ,μ).

Since $K=K_1\times K_2$ and each $K_i\cong U(n)$, all irreducible representations of K are of the form $\rho^\lambda\otimes\rho^\mu$ for some λ and $\mu\in\Lambda^+$ (see [3]). $\rho^\lambda\otimes\rho^\mu$ has K-highest weight (λ,μ). As in the case of $U(n)$, $\rho^\lambda\otimes\rho^\mu$ becomes a module for $\mathfrak{gl}_n(\mathbf{C})\oplus\mathfrak{gl}_n(\mathbf{C})=(\mathfrak{u}(n)\oplus\mathfrak{u}(n))_{\mathbf{C}}$ in the usual way.

Let $\mathscr{S}^{\alpha,\beta}(X^{oo})_K$ be the space of K-finite vectors in $\mathscr{S}^{\alpha,\beta}(X^{oo})$.

LEMMA 2.5. *We have*

$$\mathcal{S}^{\alpha,\beta}(X^{oo})_K = \sum_{\lambda \in \Lambda^+} V_\lambda,$$

where V_λ is an irreducible K-submodule isomorphic to $\rho^\lambda \otimes \rho^{\lambda^* + (\alpha-\beta)\mathbf{1}}$.

Proof. The proof follows by a straightforward application of Frobenius reciprocity to (2.4) (see [3]).

We shall give an explicit description of the K-highest weight vector in each V_λ. For $1 \leqslant k \leqslant 2n$, $1 \leqslant l \leqslant n$, let E_{kl} be the standard matrix element of $M_{2n,n}(\mathbf{C})$ with 1 at its (k,l)-th position and 0 elsewhere. For each pair $1 \leqslant i, j \leqslant n$, let $z_{i,j}, w_{i,j}\colon M_{2n,n}(\mathbf{C}) \to \mathbf{C}$ be the linear functionals defined as follows:

$$z_{ij}(E_{kl}) = \begin{cases} 1 & \text{if} \quad i=k,\ j=l \\ 0 & \text{otherwise} \end{cases}$$

$$w_{ij}(E_{kl}) = \begin{cases} 1 & \text{if} \quad i+n=k,\ j=l \\ 0 & \text{otherwise.} \end{cases}$$

In other words, z_{ij}, w_{ij} are coordinates for $M_{2n,n}(\mathbf{C})$. By restricting the scalars to $\mathbf{R}$, $M_{2n,n}(\mathbf{C})$ forms a real vector space which we shall denote by $M_{2n,n}(\mathbf{C})_\mathbf{R}$. Let $P_\mathbf{R}(M_{2n,n}(\mathbf{C}))$ be the polynomial algebra on the complexification of $M_{2n,n}(\mathbf{C})_\mathbf{R}$. Then these z_{ij}, w_{ij} together with their complex conjugates $\bar{z}_{ij}$, $\bar{w}_{ij}$ generate $P_\mathbf{R}(M_{2n,n}(\mathbf{C}))$.

The action (2.1) induces an action of $U(n,n)$ on $P_\mathbf{R}(M_{2n,n}(\mathbf{C}))$ given by

$$(g \cdot p)(x) = p(g^t x), \qquad p \in P_\mathbf{R}(M_{2n,n}(\mathbf{C})),\ g \in U(n,n),\ x \in M_{2n,n}(\mathbf{C}).$$

By differentiating this action along one-parameter subgroups of $U(n,n)$, we obtain an action of $\mathfrak{u}(n,n)$ on $P_\mathbf{R}(M_{2n,n}(\mathbf{C}))$. This induces an action of the complexified Lie algebra $\mathfrak{u}(n,n)_\mathbf{C} \cong \mathfrak{gl}_{2n}(\mathbf{C})$ on $P_\mathbf{R}(M_{2n,n}(\mathbf{C}))$. Explicitly, the action by a basis of $\mathfrak{gl}_{2n}(\mathbf{C})$ is given as follows:

$$\begin{aligned} e_{kl}^{(1)} &= \sum_{j=1}^n \left(z_{kj} \frac{\partial}{\partial z_{lj}} - \bar{z}_{lj} \frac{\partial}{\partial \bar{z}_{kj}} \right), \\ e_{kl}^{(2)} &= \sum_{j=1}^n \left(w_{kj} \frac{\partial}{\partial w_{lj}} - \bar{w}_{lj} \frac{\partial}{\partial \bar{w}_{kj}} \right), \\ p_{kl}^+ = e_{k,n+l} &= \sum_{j=1}^n \left(z_{kj} \frac{\partial}{\partial w_{lj}} + \bar{w}_{lj} \frac{\partial}{\partial \bar{z}_{kj}} \right), \\ p_{kl}^- = e_{n+l,k} &= \sum_{j=1}^n \left(w_{lj} \frac{\partial}{\partial z_{kj}} + \bar{z}_{kj} \frac{\partial}{\partial \bar{w}_{lj}} \right). \end{aligned} \tag{2.6}$$

Now for each $1 \leqslant j \leqslant n$, we let

$$\gamma_j = \begin{vmatrix} z_{11} & \cdots & z_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ z_{j1} & \cdots & z_{jn} \\ w_{11} & \cdots & w_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ w_{n-j,1} & \cdots & w_{n-j,n} \end{vmatrix}. \tag{2.7}$$

If we write $z_i = (z_{i1}, z_{i2}, ..., z_{in})$, $w_i = (w_{i1}, w_{i2}, ..., w_{in})$, then

$$\gamma_j = \det(z_1, ..., z_j, w_1, ..., w_{n-j}).$$

We also let $\det w = \det(w_1, ..., w_n)$ and $\det \bar{w} = \det(\bar{w}_1, ..., \bar{w}_n)$. Note that these functions belong to $P_\mathbf{R}(M_{2n,n}(\mathbf{C}))$ and we know explicitly how the complexified Lie algebras $\mathfrak{k}_\mathbf{C} \cong (\mathfrak{u}_n \oplus \mathfrak{u}_n)_\mathbf{C} \cong \mathfrak{gl}_n(\mathbf{C}) \oplus \mathfrak{gl}_n(\mathbf{C})$ and $\mathfrak{u}(n,n)_\mathbf{C} \cong \mathfrak{gl}_{2n}(\mathbf{C})$ act on functions on $M_{2n,n}(\mathbf{C})$. The restriction of these functions to X^{oo} are certainly smooth so that we can compute the action of $\mathfrak{k}_\mathbf{C}$ and $\mathfrak{gl}_{2n}(\mathbf{C})$ on these restrictions as if they are polynomials on $M_{2n,n}(\mathbf{C})$. For this reason, we shall also denote by γ_j the restriction of γ_j to X^{oo}.

We shall now describe the action of $\mathfrak{gl}_{2n}(\mathbf{C})$ on the function γ_j by using (2.6). $e_{kl}^{(1)}$ will replace the row z_l by z_k and $\bar{z}_k$ by $-\bar{z}_l$. Similarly, $e_{kl}^{(2)}$ will replace w_l by w_k and $\bar{w}_k$ by $-\bar{w}_l$. Hence for $1 \leqslant t \leqslant j-1$, $e_{t,t+1}^{(1)} \gamma_j = 0$ because there are two identical rows in the determinant defining $e_{t,t+1}^{(1)} \cdot \gamma_j$. Similarly, $e_{t,t+1}^{(2)} \gamma_j = 0$, $1 \leqslant t \leqslant n-j-1$. Since γ_j does not contain $z_{j+1}, ..., z_n, w_{n-j+1}, ..., w_n$, we also have $e_{t,t+1}^{(1)} \gamma_j = 0$ and $e_{i,i+1}^{(2)} \gamma_j = 0$ for $j \leqslant t \leqslant n-1$ and $n-j \leqslant i \leqslant n-1$. Moreover, γ_j is a weight vector for K of weight

$$((\overbrace{1, ..., 1}^{j}, 0, ..., 0), (\overbrace{1, ..., 1}^{n-j}, 0, ..., 0)).$$

Therefore γ_j is a K-highest weight vector. Also, $\det z$, $\det w$, $\det \bar{z}$, and $\det \bar{w}$ and their reciprocals (which are well defined on X^{oo}) are K-highest weight vectors.

LEMMA 2.8. *For each $\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda^+$, the function on X^{oo}*

$$\xi_\lambda = \gamma_1^{l_1} \gamma_2^{l_2} \cdots \gamma_n^{l_n} (\det w)^{\alpha - \lambda_1} (\det \bar{w})^\beta$$

where $l_1 = \lambda_1 - \lambda_2$, $l_2 = \lambda_2 - \lambda_3$, ..., $l_{n-1} = \lambda_{n-1} - \lambda_n$, $l_n = \lambda_n$, is a K-highest weight vector in V_λ.

Proof. First we note that the restriction of each γ_j to X^{oo} is not identically zero and that $\gamma_j\binom{zg}{wg} = (\det g)\,\gamma_j\binom{z}{w}$ for all $\binom{z}{w} \in X^{oo}$ and $g \in GL(n, \mathbf{C})$. Since K is acting by algebra automorphisms on $C^\infty(X^{oo})$, any product of highest weight vectors is again a highest weight vector. Likewise, a product of $GL(n, \mathbf{C})$-eigenvectors is another $GL(n, \mathbf{C})$-eigenvector. We leave it to the reader to check the weight and homogeneity are correct. ∎

3. Highest Weight Vectors in Tensor Products

Recall that for a dominant weight $\mu \in \Lambda^+$ of $U(n)$, $\rho^\mu: U(n) \to GL(U^\mu)$ is the irreducible representation of $U(n)$ with highest weight μ. As we have remarked before, U^μ becomes a module for $\mathfrak{gl}_n(\mathbf{C})$ in a natural way. We also denote the corresponding action of $\mathfrak{gl}_n(\mathbf{C})$ on U^μ by ρ^μ.

We also recall that $\mathfrak{gl}_n(\mathbf{C})$ acts on $\mathbf{C}^n$ by multiplication from the left. Let $\mathbf{C}^{n*}$ denote the dual of $\mathbf{C}^n$. Then as $\mathfrak{gl}_n(\mathbf{C})$-modules we have $\mathbf{C}^n \cong U^{(1,0,\dots,0)}$ and $\mathbf{C}^{n*} \cong U^{(0,\dots,0,-1)}$. We consider the tensor products $\mathbf{C}^n \otimes U^\mu$ and $\mathbf{C}^{n*} \otimes U^\mu$. The weights occurring in $\mathbf{C}^n$ are e_j, $1 \leqslant j \leqslant n$, and each is of multiplicity one. Hence (see [24, page 231]) the possible highest weights occuring in $\mathbf{C}^n \otimes U^\mu$ are $\mu + e_j$ and each occurs with multiplicity at most one. Similarly, all the possible highest weights occuring in $\mathbf{C}^{n*} \otimes U^\mu$ are $\mu - e_j$ for $1 \leqslant j \leqslant n$ and each is of multiplicity at most one. In this section, we shall derive two explicit expressions for these highest weight vectors.

We first consider $\mathbf{C}^n \otimes U^\lambda$. Let $A = \{a_{ij}: 1 \leqslant i, j \leqslant k\}$ be a $k \times k$ matrix with its entries a_{ij} taken from an associative algebra $\mathscr{A}$. Then the *determinant* of A is defined as

$$|A| = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(k)k}.$$

Here S_k denotes the symmetric group on k letters and for $\sigma \in S_k$, $\operatorname{sgn}(\sigma)$ denotes the sign of σ. Recall that $e_{i,j}$ denotes the element of $\mathfrak{gl}_n(\mathbf{C})$ with 1 at its (i, j) position and 0 elsewhere. For $1 \leqslant m < j \leqslant n$, let F_{mj} be the element in $\mathscr{U}(\mathfrak{gl}_n(\mathbf{C}))$ given by

$$F_{mj} = \begin{vmatrix} e_{j,j-1} & h_{j-1,j}+1 & & & 0 \\ e_{j,j-2} & e_{j-1,j-2} & h_{j-2,j}+2 & & \\ \vdots & \vdots & \vdots & & \\ e_{j,m+1} & e_{j-1,m+1} & e_{j-2,m+1} & \cdots & h_{m+1,j}+j-m-1 \\ e_{j,m} & e_{j-1,m} & e_{j-2,m} & \cdots & e_{m+1,m} \end{vmatrix}$$

$$= \sum_I (-1)^{j-m-l+1} e_{j,i_l} e_{i_l,i_{l-1}} \cdots e_{i_1,m} \prod_{\substack{a \notin I \\ m+1 \leqslant a \leqslant j-1}} (h_{aj} + j - a), \tag{3.1}$$

where $h_{kj} = e_{k,k} - e_{j,j}$ and $I = \{i_1 < \cdots < i_l\}$ in the last sum runs over all subsets of $\{m+1, \dots, j-1\}$. We note that under the adjoint action, each $e_{j,i_l} e_{i_l,i_{l-1}} \cdots e_{i_1,m}$ is a weight vector in $\mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))$ with weight

$$(e_j - e_{i_l}) + (e_{i_l} - e_{i_{l-1}}) + \cdots + (e_{i_1} - e_m) = e_j - e_m.$$

Hence F_{mj} is also a weight vector of weight $e_j - e_m$. We also set

$$F_{mm} = 1,$$

the identity element in $\mathscr{U}(\mathfrak{gl}_n(\mathbf{C}))$. For convenience, we let, for $1 \leqslant p < q \leqslant n$,

$$H(j; p, q) = \prod_{a=p}^{q} (h_{aj} + j - a), \qquad H^-(j; p, q) = \prod_{a=p}^{q} (h_{aj} + j - a - 1). \tag{3.2}$$

Lemma 3.3. *For $1 \leqslant m < j \leqslant n$,*

$$F_{mj} = \sum_{t=m+1}^{j} (-1)^{m+t+1} F_{tj} e_{t,m} H(j; m+1, t-1).$$

Proof. This is obtained by expanding F_{mj} along its last row. ∎

For $X \in \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))$, we denote by $\mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))X$ the left ideal of $\mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))$ generated by X. Then for $Y, Z \in \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))$, we set

$$Y \equiv Z \mod \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))X \Leftrightarrow Y - Z \in \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))X.$$

In this case, we say that *Y is congruent to Z modulo $\mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))X$.*

Lemma 3.4. *Let $1 \leqslant k \leqslant n-1$ and $1 \leqslant m < j \leqslant n$.*

(a) *If $1 \leqslant k < m$ or $j < k \leqslant n-1$, then $[e_{k,k+1}, F_{mj}] = 0$.*

(b) *If $k = m$, then*

$$e_{m,m+1} F_{mj} = F_{m+1,j}(h_{m,j} + j - m - 1) + \Big\{ F_{m+1,j} e_{m+1,m} + \sum_{t=m+2}^{j} (-1)^{m+t+1} \times F_{tj} e_{t,m} (h_{m+1,j} + j - m)\, H(j; m+2, t-1) \Big\} e_{m,m+1}.$$

In particular, $e_{m,m+1} F_{mj} \equiv F_{m+1,j}(h_{m,j} + j - m - 1) \bmod \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))\, e_{m,m+1}$.

(c) *If $m < k \leqslant j$, then $e_{k,k+1} F_{mj} \equiv 0 \bmod \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))\, e_{k,k+1}$.*

Proof. If $1 \leqslant k < m$ or $j < k \leqslant n-1$, then $e_{k,k+1}$ commutes with every entry of F_{mj}. Hence (a) follows.

For (b), by Lemma 3.3, we have

$$e_{m,m+1} F_{mj} = \sum_{t=m+1}^{j} (-1)^{m+t+1} e_{m,m+1} F_{tj} e_{t,m} H(j; m+1, t-1).$$

By (a), $e_{m,m+1} F_{tj} = F_{tj} e_{m,m+1}$ for $m+1 \leqslant t \leqslant j$. Hence we obtain

$$e_{m,m+1} F_{mj} = F_{m+1,j}(h_{m,m+1} + e_{m+1,m} e_{m,m+1}) + \sum_{t=m+2}^{j} (-1)^{m+t+1} \times F_{tj}(-e_{t,m+1} + e_{t,m} e_{m,m+1})\, H(j; m+1, t-1).$$

Now $e_{m,m+1} H(j; m+1, t-1) = (h_{m+1,j} + j - m)\, H(j; m+2, t-1)\, e_{m,m+1}$. Thus

$$\begin{aligned} e_{m,m+1} F_{mj} = {} & F_{m+1,j}(h_{m,m+1} + e_{m+1,m} e_{m,m+1}) \\ & + \left\{ \sum_{t=(m+1)+1}^{j} (-1)^{(m+1)+t+1} F_{tj} e_{t,m+1} H(j; m+2, t-1) \right\} \\ & \times (h_{m+1,j} + j - m - 1) + \sum_{t=m+2}^{j} (-1)^{t+m+1} \\ & \times F_{tj} e_{t,m} (h_{m+1,j} + j - m)\, H(j; m+2, t-1)\, e_{m,m+1}. \end{aligned}$$

By Lemma 3.3,

$$\sum_{t=(m+1)+1}^{j} (-1)^{(m+1)+t+1} F_{tj} e_{t,m+1} H(j; m+2, t-1) = F_{m+1,j}.$$

Hence we have

$$\begin{aligned} e_{m,m+1} F_{mj} = {} & F_{m+1,j} h_{m,m+1} + F_{m+1,j}(h_{m+1,j} + j - m - 1) \\ & + \left\{ F_{m+1,j} e_{m+1,m} + \sum_{t=m+2}^{j} (-1)^{t+m+1} \right. \\ & \left. \times F_{tj} e_{t,m} (h_{m+1,j} + j - m)\, H(j; m+2, t-1) \right\} e_{m,m+1} \\ = {} & F_{m+1,j}(h_{m,j} + j - m - 1) \\ & + \left\{ F_{m+1,j} e_{m+1,m} + \sum_{t=m+2}^{j} (-1)^{t+m+1} \right. \\ & \left. \times F_{tj} e_{t,m} (h_{m+1,j} + j - m)\, H(j; m+2, t-1) \right\} e_{m,m+1}. \end{aligned}$$

This proves (b).

Finally we prove (c) by an induction on m in the order $m = j-1, \ldots, 1$. First we note that since $e_{j,j+1} e_{j,j-1} = e_{j,j-1} e_{j,j+1}$, (c) holds for $m = j-1$.

We now assume $1 \leqslant m \leqslant j-2$ and $m < k \leqslant j$. By Lemma 3.3, (a) and (b), we have

$$\begin{aligned} e_{k,k+1} F_{mj} = {} & \sum_{t=m+1}^{k-1} (-1)^{m+t+1} e_{k,k+1} F_{tj} e_{t,m} H(j; m+1, t-1) \\ & + (-1)^{m+k+1} \left\{ F_{k+1,j}(h_{k,j} + j - k - 1) + \left[F_{k+1,j} e_{k+1,k} \right. \right. \\ & + \sum_{t=k+2}^{j} (-1)^{m+t+1} F_{tj} e_{t,k} (h_{k+1,j} + j - k) \\ & \left. \left. \times H(j; k+2, t-1) \right] e_{k,k+1} \right\} e_{k,m} H(j; m+1, k-1) \\ & + (-1)^{m+k} F_{k+1,j} e_{k,k+1} e_{k+1,m} H(j; m+1, k) \\ & + \sum_{s=k+2}^{j} (-1)^{m+s+1} F_{s,j} e_{k,k+1} e_{s,m} H(j; m+1, s-1). \end{aligned}$$

For $m+1 \leqslant t \leqslant k$, we clearly have

$$e_{k,k+1} e_{t,m} H(j; m+1, t-1) = e_{t,m} H(j; m+1, t-1)\, e_{k,k+1}. \tag{3.5}$$

By induction hypothesis, for $m+1 \leqslant t \leqslant k-1$, we have $e_{k,k+1} F_{tj} \equiv 0 \mod \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))\, e_{k,k+1}$. Thus there exists $X_t \in \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))$ such that $e_{k,k+1} F_{tj} = X_t e_{k,k+1}$. Hence for $m+1 \leqslant t \leqslant k-1$,

$$\begin{aligned} e_{k,k+1} F_{tj} e_{t,m} H(j; m+1, t-1) & = X_t e_{k,k+1} e_{t,m} H(j; m+1, t-1) \\ & \equiv 0 \qquad \mod \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))\, e_{k,k+1} \end{aligned}$$

by (3.5), so that

$$\begin{aligned} & \sum_{t=m+1}^{k-1} (-1)^{m+t+1} e_{k,k+1} F_{tj} e_{t,m} H(j; m+1, t-1) \\ & \quad \equiv 0 \qquad \mod \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))\, e_{k,k+1}. \end{aligned}$$

Equation (3.5) also implies

$$\Big\{F_{k+1,j}e_{k+1,k}+\sum_{t=k+2}^{j}(-1)^{m+t+1}F_{tj}e_{t,k}(h_{k+1,j}+j-k)$$
$$\times H(j;k+2,t-1)\Big\}e_{k,k+1}e_{k,m}H(j;m+1,k-1)$$
$$\equiv 0 \qquad \operatorname{mod} \mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))\,e_{k,k+1}.$$

Next we note that

$$e_{k,k+1}e_{k+1,m}H(j;m+1,k)$$
$$=e_{k,m}H(j;m+1,k)+e_{k+1,m}H(j;m+1,k-1)\,(h_{k,j}+j-k-1)\,e_{k,k+1}.$$

Thus

$$(-1)^{m+k}F_{k+1,j}e_{k,k+1}e_{k+1,m}H(j;m+1,k)$$
$$\equiv(-1)^{m+k}F_{k+1,j}e_{k,m}H(j;m+1,k) \qquad \operatorname{mod} \mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))\,e_{k,k+1}.$$

Similarly we see that for $k+2\leqslant s\leqslant j$,

$$e_{k,k+1}e_{s,m}H(j;m+1,s-1)\equiv 0 \qquad \operatorname{mod} \mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))\,e_{k,k+1},$$

so that

$$\sum_{s=k+2}^{j}(-1)^{m+s+1}F_{s,j}e_{k,k+1}e_{s,m}H(j;m+1,s-1)$$
$$\equiv 0 \qquad \operatorname{mod} \mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))\,e_{k,k+1}.$$

Hence

$$e_{k,k+1}F_{mj}\equiv\{(-1)^{m+k+1}F_{k+1,j}(h_{k,j}+j-k-1)\,e_{k,m}H(j;m+1,k-1)$$
$$+(-1)^{m+k}F_{k+1,j}e_{k,m}H(j;m+1,k)\}$$
$$\operatorname{mod} \mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))\,e_{k,k+1}$$
$$\equiv 0 \qquad \operatorname{mod} \mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))\,e_{k,k+1}. \quad \blacksquare$$

PROPOSITION 3.6. *Let $\mu=(\mu_1,...,\mu_n)\in\Lambda^+$ be such that $\mu_{j-1}>\mu_j$. Let*

$$\zeta_j=\sum_{m=1}^{j}(-1)^{m+1}F_{mj}\cdot[\varepsilon_m\otimes H(j;1,m-1)\cdot u]$$

and

$$\tilde{\zeta}_j=\sum_{m=1}^{j}(-1)^{m+1}\varepsilon_m\otimes[F_{mj}H^-(j;1,m-1)\cdot u],$$

where u is a highest weight vector in U^μ. Then both ζ_j and $\tilde{\zeta}_j$ are highest weight vectors in $\mathbf{C}^n\otimes U^\mu$ of weight $\mu+e_j$.

Proof. Let $\zeta_j=\sum_{m=1}^{j}b_{mj}F_{mj}\cdot(\varepsilon_m\otimes u)$ where b_{mj} for $1\leqslant m\leqslant j$ are constants to be determined.

Since F_{mj} has weight e_j-e_m in $\mathscr{U}(\mathrm{gl}_n(\mathbf{C}))$, it is easy to see that ζ_j is a weight vector of weight $\mu+e_j$. Thus ζ_j is a highest weight vector if and only if $e_{k,k+1}\cdot\zeta_j=0$ for $1\leqslant k\leqslant n-1$. We shall derive the conditions on the constants $b_{m,j}$ for which this happens.

Let $j\leqslant k\leqslant n-1$. Then by Lemma 3.4 (a) and (c), there exists $X_m\in\mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))$, $1\leqslant m\leqslant j-1$, for which $e_{k,k+1}F_{mj}=X_m e_{k,+1}$. It follows that

$$e_{k,k+1}\cdot\zeta_j=\sum_{m=1}^{j-1}b_{mj}X_m e_{k,k+1}\cdot(\varepsilon_m\otimes u)=0.$$

Next we let $1\leqslant k\leqslant j-1$. By Lemma 3.4(a), for $k+2\leqslant m\leqslant j$,

$$e_{k,k+1}F_{mj}\cdot(\varepsilon_m\otimes u)=F_{mj}e_{k,k+1}\cdot(\varepsilon_m\otimes u)=0.$$

By Lemma 3.4 (b) and since $e_{k,k+1}(\varepsilon_k\otimes u)=0$, we have

$$e_{k,k+1}F_{kj}\cdot(\varepsilon_k\otimes u)=F_{k+1,j}(h_{k,j}+j-k-1)\cdot(\varepsilon_k\otimes u).$$

By Lemma 3.4 (a), we have

$$e_{k,k+1}F_{k+1,j}\cdot(\varepsilon_{k+1}\otimes u)=F_{k+1,j}e_{k,k+1}\cdot(\varepsilon_{k+1}\otimes u)=F_{k+1,j}(\varepsilon_k\otimes u).$$

By Lemma 3.4 (c), for $1\leqslant m\leqslant k-1$, $e_{k,k+1}F_{mj}\equiv 0 \operatorname{mod} \mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))\,e_{k,k+1}$. Thus for each $1\leqslant m\leqslant k-1$, there exists $X_m\in\mathscr{U}(\mathrm{gl}_{2n}(\mathbf{C}))$ such that $e_{k,k+1}F_{mj}=X_m e_{k,k+1}$. Consequently,

$$e_{k,k+1}F_{mj}\cdot(\varepsilon_m\otimes u)=X_m e_{k,k+1}\cdot(\varepsilon_m\otimes u)=0,$$

and we obtain

$$e_{k,k+1}\cdot\zeta_j=b_{kj}F_{k+1,j}(h_{k,j}+j-k-1)\cdot(\varepsilon_k\otimes u)+b_{k+1,j}F_{k+1,j}\cdot(\varepsilon_k\otimes u)$$
$$=\{b_{kj}(\mu_k-\mu_j+j-k)+b_{k+1,j}\}\,F_{k+1,j}\cdot(\varepsilon_k\otimes u).$$

Hence $e_{k,k+1}\cdot\zeta_j=0$ if $b_{k+1,j}=-(\mu_k-\mu_j+j-k)\,b_{k,j}$. Thus if we put $b_{1j}=1$, then we obtain

$$b_{mj}=(-1)^{m+1}\prod_{a=1}^{m-1}(\mu_a-\mu_j+j-a)\qquad(2\leqslant m\leqslant j).$$

We also observe that $b_{mj}u=(-1)^{m+1}H(j;1,m-1)\cdot u$ for $1\leqslant m\leqslant j$. This shows that $\zeta_j=\sum_{m=1}^{j}(-1)^{m+1}F_{mj}\cdot(\varepsilon_m\otimes H(j;1,m-1)u)$ is a highest weight vector.

The proof for ζ_j being a highest weight vector is similar. ∎

Remarks. (1) Under the condition $\mu_{j-1}>\mu_j$, the highest weight $\mu+e_j$ occurs with multiplicity exactly one. Consequently $\tilde{\zeta}_j$ must be a multiple of ζ_j. In fact, by expanding the terms in ζ_j, one can show that $\zeta_j=\tilde{\zeta}_j$. However, we do not really need this fact and the proof for it is somewhat lengthy. We have therefore made a weaker assertion in the proposition.

(2) We also observe that the expressions for ζ_j and $\tilde{\zeta}_j$ closely resemble the expansion of F_{mj} along its last row in Lemma 3.3.

Next we shall derive the analogous formulas for the module $\mathbf{C}^{n*}\otimes U^\mu$. Let

$$w=\begin{pmatrix}0 & & 1\\ & 1 & \\ 1 & \cdots & 0\end{pmatrix}\in GL(n,\mathbf{C}).$$

Note that $w^2=I_n$. Hence the map $\tau:\mathrm{gl}_n(\mathbf{C})\to\mathrm{gl}_n(\mathbf{C})$ given by

$$\tau(X)=-wX^tw,\qquad\forall X\in\mathrm{gl}_n(\mathbf{C})$$

is a Lie algebra automorphism. τ extends to an automorphism of $\mathscr{U}(\mathrm{gl}_n(\mathbf{C}))$ in the usual way. With this automorphism, we can define another action $\tilde{\rho}^\mu$ of $\mathrm{gl}_n(\mathbf{C})$ on U^μ as follows:

$$\tilde{\rho}^\mu(X)\cdot v=\rho^\mu[\tau(X)]\cdot v,\qquad\forall X\in\mathrm{gl}_n(\mathbf{C}),\ v\in U^\mu.$$

Then $\tilde{\rho}^\mu$ is irreducible since ρ^μ is. Let $u\in U^\mu$ be the highest weight vector for ρ^μ. Since $\tau(\mathfrak{n}^+)\subseteq\mathfrak{n}^+$, $\tau(\mathfrak{n}^+)\cdot u=\mathfrak{n}^+\cdot u=0$. That is, u is also the highest weight vector for $\tilde{\rho}^\mu$. Moreover, if $h=\mathrm{diag}(h_1,\ldots,h_n)$, then

$$\tilde{\rho}^\mu(h)\cdot u=\rho^\mu(\tau(h))\cdot u=(-\mu_1h_n-\cdots-\mu_nh_1)\cdot u.$$

This shows that u is of weight $\mu^*=(-\mu_n,\ldots,-\mu_1)$ with respect to $\tilde{\rho}^\mu$. Hence $(U^\mu,\tilde{\rho}^\mu)\cong U^{\mu^*}$. Thus there is a $\mathrm{gl}_n(\mathbf{C})$ isomorphism $\phi_\mu:(U^\mu,\tilde{\rho}^\mu)\to U^{\mu^*}$.

Recall that $\{\varepsilon_1,\ldots,\varepsilon_n\}$ denotes the standard basis of $\mathbf{C}^n$. Let $\{\varepsilon_1^*,\ldots,\varepsilon_n^*\}$ be the basis of $\mathbf{C}^{n*}$ dual to $\{\varepsilon_1,\ldots,\varepsilon_n\}$. We have $\mathbf{C}^n\cong U^{(1,0,\ldots,0)}$. The map $\phi=\phi_{(1,0,\ldots,0)}:\mathbf{C}^n\to\mathbf{C}^{n*}$ can be chosen to be

$$\phi(\varepsilon_r)=\varepsilon_{n-r+1}^*,\qquad 1\leqslant r\leqslant n.$$

Let $\rho=\rho^{(1,0,\ldots,0)}$ and $\rho^*=\rho^{(0,\ldots,0,-1)}$ be the standard actions of $\mathrm{gl}_n(\mathbf{C})$ on $\mathbf{C}^n$ and $\mathbf{C}^{n*}$ respectively. Let $(\rho\otimes\rho^\mu)^\sim=(\rho\otimes\rho^\mu)\circ\tau$ and $\lambda=\mu^*$. Then the map $\varphi:\mathbf{C}^n\otimes U^\mu\to\mathbf{C}^{n*}\otimes U^\lambda$ given by

$$\varphi(\varepsilon_r\otimes v)=\varepsilon_{n-r+1}^*\otimes\phi_\mu(v)\tag{3.7}$$

is a $\mathrm{gl}_n(\mathbf{C})$ isomorphism for $(\mathbf{C}^n\otimes U^\mu,(\rho\otimes\rho^\mu)^\sim)\cong\mathbf{C}^{n*}\otimes U^\lambda$. Thus for $X\in\mathscr{U}(\mathrm{gl}_n(\mathbf{C}))$,

$$\varphi[(\rho\otimes\rho^\mu)(X)\cdot(\varepsilon_r\otimes v)]=[(\rho^*\otimes\rho^\lambda)(\tau X)]\cdot(\varepsilon_{n-r+1}^*\otimes\phi_\mu(v)).$$

The highest weight vectors in $\mathbf{C}^{n*}\otimes U^\lambda$ are the images of the highest weight vectors in $\mathbf{C}^n\otimes U^\lambda$ under φ. Thus we can obtain analogous formulas for them using the above relation.

For $1\leqslant p<q\leqslant n$, let S_{pq} be the element in $\mathscr{U}(\mathrm{gl}_n(\mathbf{C}))$ given by

$$S_{pq}=\begin{vmatrix}e_{p+1,p} & -(h_{p,p+1}+1) & & & 0\\ e_{p+2,p} & e_{p+2,p+1} & -(h_{p,p+2}+2) & & \\ \vdots & \vdots & \vdots & & \\ e_{q-1,p} & e_{q-1,p+1} & e_{q-1,p+2} & \cdots & -(h_{p,q-1}+q-1-p)\\ e_{q,p} & e_{q,p+1} & e_{q,p+2} & \cdots & e_{q,q-1}\end{vmatrix}$$

$$=\sum_{\substack{I=\{i_1<\cdots<i_l\}\\ \subseteq\{p+1,\ldots,q-1\}}}e_{i_1,p}e_{i_2,i_1}\cdots e_{i_l,i_{l-1}}e_{q,i_l}\prod_{\substack{a\notin I\\ p+1\leqslant a\leqslant q-1}}(h_{p,a}+a-p).\tag{3.8}$$

We also set $S_{pp}=1$, the identity element in $\mathscr{U}(\mathrm{gl}_n(\mathbf{C}))$.

Remark. The element S_{pq} has been studied by Carter and Lusztig in [5] and by Carter in [4]. They call it a "*lowering operator.*" In fact, Carter studied the commutator relations between S_{pq} and what he called "raising operators" in [4]. This is more general than our Lemma 3.4.

A straightforward calculation using the fact that $\tau(e_{p,q})=-e_{n-q+1,n-p+1}$, $\forall 1\leqslant p,q\leqslant n$, gives

$$\tau(F_{mj})=(-1)^{j-m}S_{n-j+1,n-m+1},\qquad 1\leqslant m<j\leqslant n.\tag{3.9}$$

PROPOSITION 3.10. *Let $\lambda \in \Lambda^+$ and let v be a highest weight vector in U^λ. If $\lambda_j > \lambda_{j+1}$, then*

$$v_j = \sum_{m=j}^{n} S_{jm} \cdot \left[\varepsilon_m^* \otimes \prod_{a=m+1}^{n} (h_{j,a} + a - j) \cdot v \right],$$

$$\tilde{v}_j = \sum_{m=j}^{n} \varepsilon_m^* \otimes \left[S_{jm} \prod_{a=m+1}^{n} (h_{ja} + a - j - 1) \right] \cdot v$$

are highest weight vectors in $\mathbf{C}^{n*} \otimes U^\lambda$ of weight $\lambda - e_j$.

Proof. Recall the map φ given in (3.7) and that, by Proposition 3.6, ζ_k and $\tilde{\zeta}_k$ are highest weight vectors in $\mathbf{C}^n \otimes U^\mu$ of weight $\mu + e_k$. Thus $\varphi(\zeta_k)$ and $\varphi(\tilde{\zeta}_k)$ are highest weight vectors in $\mathbf{C}^{n*} \otimes U^\lambda$ of weight $\lambda - e_{n-k+1}$. Using (3.9), we obtain

$$\varphi(\zeta_k) = \sum_{r=1}^{k} (-1)^{r+1} (-1)^{k-r} S_{n-k+1, n-r+1}$$
$$\times \left[\varepsilon_{n-r+1}^* \otimes \prod_{a=1}^{r-1} (-h_{n-a+1, n-k+1} + k - a) \cdot v \right],$$

where $v = \phi_\mu(u)$ is a highest weight vector in U^λ.

Now set $j = n - k + 1$, $m = n - r + 1$, and $b = n - a + 1$. Then

$$\varphi(\zeta_k) = (-1)^{k+1} \sum_{m=j}^{n} S_{jm} \cdot \left[\varepsilon_m^* \otimes \prod_{b=m+1}^{n} (h_{j,b} + b - j) \cdot v \right].$$

Since $v_j = (-1)^{k+1} \varphi(\zeta_k)$, v_j is a highest weight vector in $\mathbf{C}^{n*} \otimes U^\lambda$ of weight $\lambda - e_j$. The proof for $\tilde{v}_j$ being a highest weight vector is similar. ∎

4. THE EFFECT OF F_{mj} ON HIGHEST WEIGHT VECTORS

In this section, we shall study how F_{mj} transforms the $GL(n, \mathbf{C})$ highest weight vectors in the polynomial algebra $P(M_n(\mathbf{C}))$. We then show that this result can be carried over to the highest weight vectors of the K-types in $\mathcal{S}^{\alpha,\beta}(X^{oo})$. This will play a very crucial rule in our main computation in the next section.

Let L and R be the actions of $GL(n, \mathbf{C})$ on $M_n(\mathbf{C})$ given by

$$L(g)T = (g^{-1})^t T, \qquad (g \in GL(n, \mathbf{C}),\ T \in M_n(\mathbf{C}))$$
$$R(g)T = Tg^{-1}.$$

These actions induce actions of $GL(n, \mathbf{C})$ on the polynomial algebra $P(M_n(\mathbf{C}))$ by algebra automorphisms in a standard way. We shall also denote these induced actions by L and R, respectively. For $1 \leqslant k, l \leqslant n$, we let E_{kl} be the element in $M_n(\mathbf{C})$ with 1 at its (k, l)-th entry and 0 elsewhere. Then $\{E_{kl}: 1 \leqslant k, l \leqslant n\}$ is a basis of $M_n(\mathbf{C})$. We let $\{t_{k,l}: 1 \leqslant k, l \leqslant n\}$ be the corresponding dual basis. Then the actions by the basis $\{e_{ij}\}$ of the Lie algebra $\mathrm{gl}_n(\mathbf{C})$ of $GL(n, \mathbf{C})$ are given by the operators

$$L(e_{ij}) = \sum_{k=1}^{n} t_{ik} \frac{\partial}{\partial t_{jk}},$$

$$R(e_{ij}) = \sum_{k=1}^{n} t_{ki} \frac{\partial}{\partial t_{kj}}.$$

Since the actions L and R commute with each other, they give rise to a joint action $L \times R$ of $GL(n, \mathbf{C}) \times GL(n, \mathbf{C})$ on $P(M_n(\mathbf{C}))$. Under the action $L \times R$, we have a decomposition (see [11])

$$P(M_n(\mathbf{C})) \cong \sum_{\lambda \in \Lambda_0^+} \rho^\lambda \otimes \rho^\lambda,$$

where $\Lambda_0^+ = \{\lambda \in \Lambda^+ : \lambda_n \geqslant 0\}$. For $1 \leqslant j \leqslant n$, we set

$$\tilde{\gamma}_j = \begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1j} \\ t_{21} & t_{22} & \cdots & t_{2j} \\ \vdots & \vdots & \cdots & \vdots \\ t_{j1} & t_{j2} & \cdots & t_{jj} \end{vmatrix}.$$

For $\lambda \in \Lambda_0^+$, let $\boldsymbol{l} = \boldsymbol{l}(\lambda) = (l_1, ..., l_n) \in \mathbf{Z}^n$ be given by

$$l_j = \lambda_j - \lambda_{j+1},\ 1 \leqslant j \leqslant n-1, \qquad l_n = \lambda_n. \tag{4.1}$$

Then

$$\tilde{\gamma}^{\boldsymbol{l}} = \tilde{\gamma}_1^{l_1} \tilde{\gamma}_2^{l_2} \cdots \tilde{\gamma}_n^{l_n} \tag{4.2}$$

is a joint $GL(n, \mathbf{C}) \times GL(n, \mathbf{C})$ highest weight vector in the $\rho^\lambda \otimes \rho^\lambda$ isotypic component of $P(M_n(\mathbf{C}))$.

Let U be the standard unipotent subgroup of $GL(n, \mathbf{C})$, i.e., it consists of all elements with 1 at all its diagonal entries and 0 at all entries below the diagonal. Let

$$\mathcal{A} = P(M_n(\mathbf{C}))^{I \times U} = \{ f \in P(M_n(\mathbf{C})) : R(u) f = f,\ \forall u \in U \}.$$

Then every element of $\mathscr{A}$ is a highest weight vector for $(GL(n, \mathbf{C}), R)$. Moreover, $\mathscr{A}$ is a module for $(GL(n, \mathbf{C}), L)$ which admits a decomposition

$$\mathscr{A} = \sum_{\lambda \in \Lambda_0^+} \mathscr{A}_\lambda \cong \sum_{\lambda \in \Lambda_0^+} \rho^\lambda \otimes (\rho^\lambda)^U \cong \sum_{\lambda \in \Lambda_0^+} \rho^\lambda \tag{4.3}$$

into irreducible submodules. The polynomial $\tilde{\gamma}^{1(\lambda)}$ given in (4.2) is a highest weight vector in $\mathscr{A}_\lambda$. We also note that $\mathscr{A}$ is a graded algebra graded by Λ_0^+ and $\mathscr{A}_\lambda$ is the space of elements homogeneous of degree λ. In fact, if $f \in \mathscr{A}_\lambda$ and $g \in \mathscr{A}_\mu$, then since $(GL(n, \mathbf{C}), R)$ acts by algebra automorphisms on $P(M_n(\mathbf{C}))$, fg is a highest weight vector for $(GL(n, \mathbf{C}), R)$ of weight $\lambda + \mu$. Consequently $fg \in \mathscr{A}_{\lambda+\mu}$.

Our object is to compute $F_{mj}(\tilde{\gamma}^l)$ in $\mathscr{A}$. Before we proceed, we want to express F_{mj} in a form more convenient for our computations.

LEMMA 4.4. *For* $1 \leqslant m < j \leqslant n$,

$$F_{mj} = \sum_{t=m+1}^{j} (-1)^{m+t+1} e_{t,m} F_{tj} H^-(j; m+1, t-1).$$

Proof. We shall prove the lemma by an induction on m in the order $m = j-1, j-2, ..., 1$.

For $m = j-1$, $F_{j-1,j} = e_{j,j-1}$. Thus the lemma holds for this case.

Next we assume $1 \leqslant m < j-1$. By Lemma 3.3,

$$\begin{aligned}
F_{mj} &= \sum_{t=m+1}^{j} (-1)^{m+t+1} F_{tj} e_{t,m} H(j; m+1, t-1) \\
&= (-1)^{m+j+1} e_{j,m} H(j; m+1, j-1) + \sum_{t=m+1}^{j-1} (-1)^{m+t+1} \\
&\quad \times \left\{ \sum_{l=t+1}^{j} (-1)^{t+l+1} e_{l,t} F_{lj} H^-(j; t+1, l-1) \right\} e_{t,m} H(j; m+1, t-1) \\
&\qquad \text{(by induction hypothesis)} \\
&= (-1)^{m+j+1} e_{j,m} H(j; m+1, j-1) + \sum_{t=m+1}^{j-1} (-1)^{m+t+1} \\
&\quad \times \sum_{l=t+1}^{j} (-1)^{t+l+1} (e_{t,m} e_{l,t} + e_{l,m}) \\
&\quad \times F_{l,j} H^-(j; t+1, l-1) H(j; m+1, t-1)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{m+j+1} e_{j,m} H(j; m+1, j-1) \\
&\quad + \sum_{t=m+1}^{j-1} (-1)^{m+t+1} e_{t,m} F_{tj} H(j; m+1, t-1) \\
&\qquad \text{(by induction hypothesis)} \\
&\quad + \sum_{t=m+1}^{j-1} \sum_{l=t+1}^{j} (-1)^{m+l} e_{l,m} F_{lj} H^-(j; t+1, l-1) H(j; m+1, t-1) \\
&= e_{m+1,m} F_{m+1,j} + \sum_{t=m+2}^{j} (-1)^{m+t+1} e_{t,m} F_{tj} \\
&\quad \times H(j; m+1, t-1) - \sum_{l=m+2}^{j} (-1)^{m+l+1} e_{l,m} F_{l,j} \\
&\quad \times \sum_{t=m+1}^{l-1} H(j; m+1, t-1) H^-(j; t+1, l-1) \\
&= e_{m+1,m} F_{m+1,j} + \sum_{t=m+2}^{j} (-1)^{m+t+1} e_{t,m} F_{tj} \\
&\quad \times \left\{ H(j; m+1, t-1) - \sum_{s=m+1}^{t-1} H(j; m+1, s-1) H^-(j; s+1, t-1) \right\}.
\end{aligned}$$

Now recall the identity

$$\prod_{i=1}^{N} (x_i - 1) = \prod_{i=1}^{N} x_i - \sum_{s=0}^{N-1} \prod_{a=1}^{s} x_a \prod_{b=s+2}^{N} (x_b - 1)$$

for a sequence $\{x_i\}_{i=1}^N$ in an associative algebra. Hence

$$\begin{aligned}
&\left\{ H(j; m+1, t-1) - \sum_{s=m+1}^{t-1} H(j; m+1, s-1) H^-(j; s+1, t-1) \right\} \\
&\qquad = H^-(j; m+1, t-1).
\end{aligned}$$

It follows that

$$\begin{aligned}
F_{mj} &= e_{m+1,m} F_{m+1,j} + \sum_{t=m+2}^{j} (-1)^{m+t+1} e_{t,m} F_{tj} H^-(j; m+1, t-1) \\
&= \sum_{t=m+1}^{j} (-1)^{m+t+1} e_{t,m} F_{tj} H^-(j; m+1, t-1). \qquad \blacksquare
\end{aligned}$$

LEMMA 4.5. *For* $1 \leqslant m \leqslant k \leqslant j-2$,

$$\tilde{\gamma}_{j-1}(e_{j,m}\cdot\tilde{\gamma}_k) - \sum_{t=k+1}^{j-1} (e_{j,t}\cdot\tilde{\gamma}_{j-1})(e_{t,m}\cdot\tilde{\gamma}_k) = (e_{j,m}\cdot\tilde{\gamma}_{j-1})\tilde{\gamma}_k.$$

Proof. Let $z=(z_{ab})\in M_n(\mathbf{C})$ be such that $e_{j,m}\cdot\tilde{\gamma}_{j-1}(z)\neq 0$ and consider the following system of linear equations in the unknowns $x_1, ..., x_{m-1}, x_{m+1}, ..., x_j$:

$$x_1\tilde{z}_1 + \cdots + x_{m-1}\tilde{z}_{m-1} + x_j\tilde{z}_j + x_{m+1}\tilde{z}_{m+1} + \cdots + x_{j-1}\tilde{z}_{j-1} = \tilde{z}_m,$$

where

$$\tilde{z}_l = \begin{pmatrix} z_{l1} \\ \vdots \\ z_{l_{j-1}} \end{pmatrix}, \qquad 1 \leqslant l \leqslant j.$$

By Cramer's rule, for $t=1, 2, ..., m-1, m+1, ..., j-1$, we have

$$x_t = \frac{\det(\tilde{z}_1, ..., \tilde{z}_{t-1}, \tilde{z}_m, \tilde{z}_{t+1}, ..., \tilde{z}_{m-1}, \tilde{z}_j, \tilde{z}_{m+1}, ..., \tilde{z}_j)}{\det(\tilde{z}_1, ..., \tilde{z}_{m-1}, \tilde{z}_j, \tilde{z}_{m+1}, ..., \tilde{z}_{j-1})}$$
$$= -\frac{e_{j,t}\cdot\tilde{\gamma}_{j-1}}{e_{j,m}\cdot\tilde{\gamma}_{j-1}}$$

and

$$x_j = \frac{\tilde{\gamma}_{j-1}}{e_{j,m}\cdot\tilde{\gamma}_{j-1}}.$$

Thus if we write

$$\hat{z}_l = \begin{pmatrix} z_{l1} \\ \vdots \\ z_{l_k} \end{pmatrix},$$

then noting that $k \leqslant j-2$,

$$\tilde{\gamma}_k = \det\left(\hat{z}_1, ..., \hat{z}_{m-1}, \sum_{\substack{1\leqslant t\leqslant j \\ t\neq m}} x_t\hat{z}_t, \hat{z}_{m+1}, ..., \hat{z}_k\right)$$
$$= \sum_{t=k+1}^{j-1} -\frac{(e_{j,t}\cdot\tilde{\gamma}_{j-1})(e_{t,m}\cdot\tilde{\gamma}_k)}{e_{j,m}\cdot\tilde{\gamma}_{j-1}} + \frac{\tilde{\gamma}_{j-1}(e_{j,m}\cdot\tilde{\gamma}_k)}{e_{j,m}\cdot\tilde{\gamma}_{j-1}}.$$

Since the set of $z\in M_n(\mathbf{C})$ with $e_{j,m}\cdot\tilde{\gamma}_{j-1}(z)\neq 0$ is Zariski dense in $M_n(\mathbf{C})$, the lemma follows. ∎

We shall need the following notation in our next proposition. For $1\leqslant j\leqslant n$, $1\leqslant p<q\leqslant n$, and $\lambda\in\Lambda^+$, we set

$$\lambda(j; p, q) = \prod_{a=p}^{q} (\lambda_a - \lambda_j + j - a - 1). \tag{4.6}$$

Recall from (3.2) that $H^-(j; p, q) = \prod_{a=p}^{q} (h_{a,j} + j - a - 1)$. Then $H^-(j; p, q)\cdot\tilde{\gamma}^{1(\lambda)} = \lambda(j; p, q)\,\tilde{\gamma}^{1(\lambda)}$ and

$$\lambda(j; p, q)\,\lambda(j; q+1, r) = \lambda(j; p, r). \tag{4.7}$$

We also let $\{\varepsilon_1, ..., \varepsilon_n\}$ be the standard basis of $\mathbf{Z}^n$ so that $\mathbf{l}\pm\varepsilon_j$ shall denote the element of $\mathbf{Z}^n$ with its jth coordinate $l_j\pm 1$ and other coordinates the same as those of $\mathbf{l}$.

PROPOSITION 4.8. *For* $1\leqslant m\leqslant j-1$,

$$F_{mj}\cdot\tilde{\gamma}^{\mathbf{l}(\lambda)} = (-1)^{j+m+1}\lambda(j; m, j-1)(e_{jm}\cdot\tilde{\gamma}_{j-1})\cdot\tilde{\gamma}^{\mathbf{l}-\varepsilon_{j-1}}.$$

In particular, if $l_{j-1}=0$, *then* $F_{mj}(\tilde{\gamma}^{\mathbf{l}})=0$.

Proof. We shall prove the proposition by induction on m in the order $j-1, j-2, ..., 1$. The case $m=j-1$ can be verified easily. We let $1\leqslant m\leqslant j-2$. By Lemma 4.4, we have

$$\begin{aligned}
F_{mj}(\tilde{\gamma}^{\mathbf{l}}) &= (-1)^{m+j+1}\lambda(j; m+1, j-1)\,e_{j,m}\cdot\tilde{\gamma}^{\mathbf{l}} \\
&\quad + \sum_{t=m+1}^{j-1} (-1)^{m+t+1}\lambda(j; m+1, t-1)\,e_{t,m}F_{tj}\cdot\tilde{\gamma}^{\mathbf{l}} \\
&= (-1)^{m+j+1}\lambda(j; m+1, j-1)\sum_{k=m}^{j-1} l_k(e_{j,m}\cdot\gamma_k)\,\tilde{\gamma}^{\mathbf{l}-\varepsilon_k} \\
&\quad + \sum_{t=m+1}^{j-1} (-1)^{m+t+1}\lambda(j; m+1, t-1)\,e_{t,m} \\
&\quad \times\{(-1)^{t+j+1}\lambda(j; t, j-1)(e_{j,t}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{\mathbf{l}-\varepsilon_{j-1}}\} \\
&\qquad \text{(by induction hypothesis)} \\
&= (-1)^{m+j+1}\lambda(j; m+1, j-1)\sum_{k=m}^{j-1} l_k(e_{j,m}\cdot\gamma_k)\,\tilde{\gamma}^{\mathbf{l}-\varepsilon_k} \\
&\quad + (-1)^{m+j}\lambda(j; m+1, j-1)\sum_{t=m+1}^{j-1} e_{t,m} \\
&\quad \cdot\{(e_{j,t}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{\mathbf{l}-\varepsilon_{j-1}}\} \qquad \text{(by 4.7)}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{m+j+1}\lambda(j; m+1, j-1) \sum_{k=m}^{j-1} l_k(e_{j,m}\cdot\tilde{\gamma}_k)\,\tilde{\gamma}^{l-\varepsilon_k} \\
&\quad + (-1)^{m+j}\lambda(j; m+1, j-1)\left\{\sum_{t=m+1}^{j-1}(-e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}}\right. \\
&\quad \left. + \sum_{t=m+1}^{j-1}\sum_{k=m}^{t-1} l_k(e_{j,t}\cdot\tilde{\gamma}_{j-1})(e_{t,m}\cdot\tilde{\gamma}_k)\,\tilde{\gamma}^{l-\varepsilon_{j-1}-\varepsilon_k}\right\} \\
&= (-1)^{m+j+1}\lambda(j; m+1, j-1)(j-m-1)(e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}} \\
&\quad + (-1)^{m+j+1}\lambda(j; m+1, j-1)\left\{\sum_{k=m}^{j-1} l_k(e_{j,m}\cdot\tilde{\gamma}_k)\,\tilde{\gamma}^{l-\varepsilon_k}\right. \\
&\quad \left. - \sum_{k=m}^{j-2} l_k \sum_{t=k+1}^{j-1}(e_{j,t}\cdot\tilde{\gamma}_{j-1})(e_{t,m}\cdot\tilde{\gamma}_k)\,\tilde{\gamma}^{l-\varepsilon_{j-1}-\varepsilon_k}\right\}.
\end{aligned}
$$

By Lemma 4.5, the content in the parentheses $\{\ \}$ reads

$$
\begin{aligned}
&\sum_{k=m}^{j-2} l_k\left\{\tilde{\gamma}_{j-1}(e_{j,m}\cdot\tilde{\gamma}_k) - \sum_{t=k+1}^{j-1}(e_{j,t}\cdot\tilde{\gamma}_{j-1})(e_{t,m}\cdot\tilde{\gamma}_k)\right\} \\
&\qquad \times \tilde{\gamma}^{l-\varepsilon_{j-1}-\varepsilon_k} + l_{j-1}(e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}} \\
&\quad = \sum_{k=m}^{j-2} l_k(e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}} + l_{j-1}(e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}} \\
&\quad = (\lambda_m - \lambda_j)(e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
F_{mj}\cdot(\tilde{\gamma}^l) &= (-1)^{m+j+1}\lambda(j; m+1, j-1)(\lambda_m - \lambda_j + j - m - 1) \\
&\quad \times (e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}} \\
&= (-1)^{m+j+1}\lambda(j; m, j-1)(e_{j,m}\cdot\tilde{\gamma}_{j-1})\,\tilde{\gamma}^{l-\varepsilon_{j-1}}. \quad \blacksquare
\end{aligned}
$$

We shall now deduce the effect of F_{mj} on the highest weight vectors of the K-types in $\mathscr{S}^{\alpha,\beta}(X^{oo})$. Recall that for $1 \leqslant j \leqslant n$, the function γ_j given in (2.7) is a joint highest weight vector for $U(n)\times U(n)$ (hence for $GL(n,\mathbf{C})\times GL(n,\mathbf{C})$) in $C^\infty(X^{oo})$. For clarity we shall denote the action by the first and the second copy of $GL(n,\mathbf{C})$ by σ_1 and σ_2 on $C^\infty(X^{oo})$, respectively. For $\lambda \in \Lambda_0^+$, the function

$$\gamma^{l(\lambda)} = \gamma_1^{l_1}\gamma_2^{l_2}\cdots\gamma_n^{l_n}$$

is a joint $GL(n,\mathbf{C})\times GL(n,\mathbf{C})$ highest weight vector with weight $(\lambda, \lambda_1\mathbf{1}+\lambda^*)$. Let $\mathfrak{A}_\lambda$ be the irreducible $(GL(n,\mathbf{C}),\sigma_1)$ submodule of $C^\infty(X^{oo})$ generated by γ^l. Let $\mathfrak{A} = \bigoplus_{\lambda\in\Lambda_0^+}\mathfrak{A}_\lambda$. Then every element of $\mathfrak{A}$ is a highest weight vector for $(GL(n,\mathbf{C}),\sigma_2)$. By a similar reasoning as in the case of $\mathscr{A}$, $\mathfrak{A}$ is graded algebra by Λ_0^+ and $(GL(n,\mathbf{C}),\sigma_1)$ acts by algebra automorphisms of $\mathfrak{A}$.

LEMMA 4.9. *Let $T: \mathscr{A} \to \mathfrak{A}$ be the $GL(n,\mathbf{C})$ module map specified by*

$$T(\tilde{\gamma}^{l(\lambda)}) = \gamma^{l(\lambda)}, \qquad \forall \lambda \in \Lambda_0^+.$$

Then T is an algebra isomorphism.

Proof. For $\lambda, \mu \in \Lambda_0^+$, the maps

$$
\begin{aligned}
\mathscr{A}_\lambda \otimes \mathscr{A}_\mu &\to \mathscr{A}_{\lambda+\mu} \to \mathfrak{A}_{\lambda+\mu} \\
f\otimes h &\to fh \to T(fh)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{A}_\lambda \otimes \mathscr{A}_\mu &\to \mathfrak{A}_\lambda \otimes \mathfrak{A}_\mu \to \mathfrak{A}_{\lambda+\mu} \\
f\otimes h &\to Tf \otimes Tg \to (Tf)(Tg)
\end{aligned}
$$

both are $GL(n,\mathbf{C})$ module maps sending $\tilde{\gamma}^{l(\lambda)}\otimes\tilde{\gamma}^{l(\mu)}$ to $\gamma^{l(\lambda+\mu)}$. Since such a map is unique, we must have $T(fh) = T(f)\,T(g)$. $\blacksquare$

We now recall the $\mathfrak{gl}_n(\mathbf{C})\oplus\mathfrak{gl}_n(\mathbf{C})$ highest weight vector ξ_λ in the K-type V_λ of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ as given in Lemma 2.8. Note that $\xi_\lambda = \gamma^{l(\lambda)}(\det w)^{\alpha-\lambda_1}(\det\bar{w})^\beta$.

COROLLARY 4.10. *For $1 \leqslant m \leqslant j-1$,*

$$F_{mj}\cdot\xi_\lambda = (-1)^{j+m+1}\lambda(j; m, j-1)(e_{j,m}\cdot\gamma_{j-1})\,\gamma^{l(\lambda)-\varepsilon_{j-1}}(\det w)^{\alpha-\lambda_1}(\det\bar{w})^\beta.$$

Proof. This is a straightforward computation based on Proposition 4.8 and Lemma 4.9. $\blacksquare$

Remark. In the author's dissertation [17], only the above corollary is proved. The more general results in this section were suggested by Roger Howe.

5. TRANSITION OF K-TYPES

In this section, we shall use the results obtained in Sections 3 and 4 to study how the enveloping algebra $\mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))$ transforms the highest weight vectors of the K-types in $\mathscr{S}^{\alpha,\beta}(X^{oo})$.

Let $\mathfrak{p} = \{ \left(\begin{smallmatrix} 0 & a \\ (\bar{a})^t & 0 \end{smallmatrix}\right) : a \in \mathfrak{gl}_n(\mathbf{C}) \}$. Then $\mathfrak{u}(n, n) = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{u}(n, n)$. Since $\mathfrak{p}$ is invariant under the adjoint action of K, its complexification

$$\mathfrak{p}_{\mathbf{C}} = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} : a, b \in \mathfrak{gl}_n(\mathbf{C}) \right\}$$

is a K module. For a given $\lambda \in \Lambda^+$, we consider the K-module map $m_\lambda : \mathfrak{p}_{\mathbf{C}} \otimes V_\lambda \to \mathscr{S}^{\alpha, \beta}(X^{oo})_K$ given by

$$m_\lambda(p \otimes u) = p \cdot u, \qquad \forall p \in \mathfrak{p}_{\mathbf{C}}, \qquad u \in V_\lambda.$$

If ψ is a highest weight vector in $\mathfrak{p}_{\mathbf{C}} \otimes V_\lambda$ such that its weight occurs as the highest weight of some K-type V_η in $\mathscr{S}^{\alpha, \beta}(X^{oo})$ and if $m_\lambda(\psi) \neq 0$, then since every K-type in $\mathscr{S}^{\alpha, \beta}(X^{oo})$ occurs with multiplicity exactly one, $m_\lambda(\psi)$ is a highest weight vector in V_η. In this case we see that vectors in V_λ can be transformed to vectors in V_η by the action of the universal enveloping algebra $\mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))$. Thus knowing $m_\lambda(\psi)$ allows us to deduce the module structure of $\mathscr{S}^{\alpha, \beta}(X^{oo})$. In this section, we shall compute $m_\lambda(\psi)$ for all such highest weight vectors ψ in $\mathfrak{p}_{\mathbf{C}} \otimes V_\lambda$.

We recall that $K = K_1 \times K_2 \cong U(n) \times U(n)$. Let $\mathfrak{p}^+ = \{ \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) : a \in \mathfrak{gl}_n(\mathbf{C}) \}$, $\mathfrak{p}^- = \{ \left(\begin{smallmatrix} 0 & 0 \\ a & 0 \end{smallmatrix}\right) : a \in \mathfrak{gl}_n(\mathbf{C}) \} \subseteq \mathfrak{gl}_{2n}(\mathbf{C})$. Then $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are K-submodules of $\mathfrak{p}_{\mathbf{C}}$. Let $P^+ : \mathfrak{p}^+ \to \mathbf{C}^n \otimes \mathbf{C}^{n*}$ and $P^- : \mathfrak{p}^- \to \mathbf{C}^{n*} \otimes \mathbf{C}^n$ be the linear maps given by

$$\begin{aligned} P^+(p_{kl}^+) &= P^+(e_{k, n+l}) = \varepsilon_k \otimes \varepsilon_l^*, \\ P^-(p_{kl}^-) &= P^-(e_{n+l, k}) = \varepsilon_k^* \otimes \varepsilon_l. \end{aligned} \tag{5.1}$$

Then P^+ and P^- are K-module isomorphisms.

We now fix $\alpha, \beta \in \mathbf{C}$ with $\alpha - \beta \in \mathbf{Z}$ and $\lambda \in \Lambda^+$. We recall from Lemma 2.8 the function ξ_λ and that it is a highest weight vector in the K-type V_λ in $\mathscr{S}^{\alpha, \beta}(X^{oo})$. We also recall as a K-module, $V_\lambda \cong \rho^\lambda \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}$. Let u be a K_1 highest weight vector of ρ^λ and u' be a K_2-highest weight vector in $\rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}$. Then $u \otimes u'$ is a K-highest weight vector in $\rho^\lambda \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}$. It follows that there is a K-isomorphism

$$L : V_\lambda \to \rho^\lambda \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}} \tag{5.2}$$

such that $L(\xi_\lambda) = u \otimes u'$. Let

$$\begin{aligned} Q^+ &: \mathfrak{p}^+ \otimes V_\lambda \to (\mathbf{C}^n \otimes \rho^\lambda) \otimes (\mathbf{C}^{n*} \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}) \\ Q^- &: \mathfrak{p}^- \otimes V_\lambda \to (\mathbf{C}^{n*} \otimes \rho^\lambda) \otimes (\mathbf{C}^n \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}), \end{aligned} \tag{5.3}$$

respectively, be the compositions of the isomorphisms

$$\begin{aligned} \mathfrak{p}^+ \otimes V_\lambda &\overset{p^+ \otimes L}{\cong} (\mathbf{C}^n \otimes \mathbf{C}^{n*}) \otimes (\rho^\lambda \otimes \rho^{\lambda + (\alpha - \beta)\mathbf{1}}) \\ &\cong (\mathbf{C}^n \otimes \rho^\lambda) \otimes (\mathbf{C}^{n*} \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}) \\ \mathfrak{p}^- \otimes V_\lambda &\overset{P^- \otimes L}{\cong} (\mathbf{C}^{n*} \otimes \mathbf{C}^n) \otimes (\rho^\lambda \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}) \\ &\cong (\mathbf{C}^{n*} \otimes \rho^\lambda) \otimes (\mathbf{C}^n \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}), \end{aligned} \tag{5.4}$$

where the second isomorphism in each case above "switches the second component with the third" in the tensor products. Here $\mathbf{C}^n \otimes \rho^\lambda$ and $\mathbf{C}^{n*} \otimes \rho^\lambda$ are tensor products of K_1-modules whereas $\mathbf{C}^{n*} \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}$ and $\mathbf{C}^{n*} \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}$ are tensor products of K_2-modules.

Since $\mathfrak{p}_{\mathbf{C}} \otimes V_\lambda = (\mathfrak{p}^+ \otimes V_\lambda) \otimes (\mathfrak{p}^- \otimes V_\lambda)$, by (5.3), all the possible K-highest weights in $\mathfrak{p} \otimes V_\lambda$ are $(\lambda \pm e_j, \lambda^* + (\alpha - \beta)\mathbf{1} \mp e_k)$, $1 \leqslant j, k \leqslant n$. However these highest weights occur in $\mathscr{S}^{\alpha, \beta}(X^{oo})$ if and only if $k = n - j + 1$. Consequently we only need to determine the images under m_λ of those highest weight vectors in $\mathfrak{p}_{\mathbf{C}} \otimes V_\lambda$ of weight $(\lambda \pm e_j, \lambda^* + (\alpha - \beta)\mathbf{1} \mp e_{q - j + 1})$, $1 \leqslant j \leqslant n$.

We now fix $1 \leqslant j \leqslant n$ and shall first study the transition from V_λ to $V_{\lambda + e_j}$. Recall the notations introduced in (2.3). By Proposition 3.6 and 3.10,

$$\begin{aligned} \phi &= \sum_{m=1}^{j} (-1)^{m+1} \varepsilon_m \otimes \left[F_{mj}^{(1)} \prod_{a=1}^{m-1} (h_{a,j}^{(1)} + j - a - 1) \right] \cdot u \\ \phi' &= \sum_{m=n-j+1}^{n} S_{n-j+1, m}^{(2)} \cdot \left[\varepsilon_m^* \otimes \prod_{a=m+1}^{n} (h_{n-j+1, a}^{(2)} + a - n + j - 1) \cdot u' \right] \end{aligned} \tag{5.5}$$

are highest weight vectors in $\mathbf{C}^n \otimes \rho^\lambda$ and $\mathbf{C}^{n*} \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}}$ of weight $\lambda + e_j$ and $\lambda^* + (\alpha - \beta)\mathbf{1} - e_{n-j+1}$, respectively. Thus $\phi \otimes \phi'$ is a K-highest weight vector in $(\mathbf{C}^n \otimes \rho^\lambda) \otimes (\mathbf{C}^{n*} \otimes \rho^{\lambda^* + (\alpha - \beta)\mathbf{1}})$ of weight $(\lambda + e_j, \lambda^* + (\alpha - \beta)\mathbf{1} - e_{n-j+1})$. Let

$$\begin{aligned} \psi_{\lambda + e_j} &= (Q^+)^{-1} (\phi \otimes \phi') \\ \eta_{\lambda + e_j} &= (Q^+)^{-1} (\phi \otimes (\varepsilon_{\eta - j + 1}^* \otimes u')). \end{aligned} \tag{5.6}$$

Since Q^+ is a K-isomorphism, $\psi_{\lambda + e_j}$ is a highest weight vector in $\mathfrak{p}^+ \otimes V_\lambda$ of weight $(\lambda + e_j, \lambda^* + (\alpha - \beta)\mathbf{1} - e_{n-j+1})$. Observe that $\eta_{\lambda + e_j}$ is a highest weight vector for K_1 and has the same K-weights as $\psi_{\lambda + e_j}$.

LEMMA 5.7.

$$m_\lambda(\psi_{\lambda+e_j}) = \prod_{a=1}^{j-1} (\lambda_a - \lambda_j + j - a)\, m_\lambda(\eta_{\lambda+e_j}).$$

Proof. The weight of u' is

$$\lambda^* + (\alpha-\beta)\,\mathbf{1} = (-\lambda_n + \alpha - \beta, -\lambda_{n-1} + \alpha - \beta, ..., -\lambda_1 + \alpha - \beta)$$

so that

$$h^{(2)}_{n-j+1,a} \cdot u' = [-\lambda_j + \alpha - \beta - (-\lambda_{n-a+1} + \alpha - \beta)]\, u' = (\lambda_{n-a+1} - \lambda_j)\, u'.$$

We also note that in $\mathbf{C}^{n*} \otimes \rho^{\lambda^* + (\alpha-\beta)\mathbf{1}}$,

$$e^{(2)}_{n-j+1,m} \cdot (\varepsilon^*_{n-j+1} \otimes u') = -\varepsilon^*_m \otimes u'$$

for all $n-j+2 \leqslant m \leqslant n$. Thus

$$\begin{aligned}\phi' = &- \sum_{m=n-j+2}^{n} \prod_{a=m+1}^{n} (\lambda_{n-a+1} - \lambda_j + a - n + j - 1)\, S^{(2)}_{n-j+1,m} \\ &\times e^{(2)}_{n-j+1,m} \cdot (\varepsilon^*_{n-j+1} \otimes u') \\ &+ \prod_{a=n-j+2}^{n} (\lambda_{n-a+1} - \lambda_j + a - n + j - 1)(\varepsilon^*_{n-j+1} \otimes u'),\end{aligned}$$

so that

$$\begin{aligned}\psi_{\lambda+e_j} = &- \sum_{m=n-j+2}^{n} \prod_{a=m+1}^{n} (\lambda_{n-a+1} - \lambda_j + a - n + j - 1)\, S^{(2)}_{n-j+1,m} \\ &\times e^{(2)}_{n-j+1,m} \cdot \eta_{\lambda+e_j} \\ &+ \prod_{a=n-j+2}^{n} (\lambda_{n-a+1} - \lambda_j + a - n + j - 1) \cdot \eta_{\lambda+e_j}.\end{aligned}$$

Since m_λ is a K-module map,

$$\begin{aligned}m_\lambda(\psi_{\lambda+e_j}) = &- \sum_{m=n-j+2}^{n} \prod_{a=m+1}^{n} (\lambda_{n-a+1} - \lambda_j + a - n + j - 1)\, S^{(2)}_{n-j+1,m} \\ &\times e^{(2)}_{n-j+1,m} \cdot m_\lambda(\eta_{\lambda+e_j}) \\ &+ \prod_{a=n-j+2}^{n} (\lambda_{n-a+1} - \lambda_j + a - n + j - 1) \cdot m_\lambda(\eta_{\lambda+e_j}).\end{aligned}$$

Now $\eta_{\lambda+e_j}$ is a K_1-highest weight vector of weight $\lambda+e_j$. This highest weight for K_1 only occurs in $V_{\lambda+e_j}$. Thus $m_\lambda(\eta_{\lambda+e_j}) \in V_{\lambda+e_j}$. Moreover, $\eta_{\lambda+e_j}$ has the same K-weight as the highest weight vector $\xi_{\lambda+e_j}$ in $V_{\lambda+e_j}$. Hence either $m_\lambda(\eta_{\lambda+e_j}) = 0$ or $m_\lambda(\eta_{\lambda+e_j})$ must be a nonzero multiple of $\xi_{\lambda+e_j}$. In either case, we have for $n-j+2 \leqslant m \leqslant n$,

$$S^{(2)}_{n-j+1,m} e^{(2)}_{n-j+1,m} \cdot m_\lambda(\eta_{\lambda+e_j}) = 0.$$

Hence the lemma follows. ∎

By this lemma, it is sufficient to compute $m_\lambda(\eta_{\lambda+e_j})$. A straightforward computation using (5.5) shows that

$$m_\lambda(\eta_{\lambda+e_j}) = \sum_{m=1}^{j} (-1)^{m+1}\, p^+_{m,n-j+1} F^{(1)}_{mj} \prod_{a=1}^{m-1} (h^{(1)}_{a,j} + j - a - 1) \cdot \xi_\lambda. \qquad (5.8)$$

From now on until we study the transition from V_λ to $V_{\lambda-e_j}$, we shall be only dealing with operators from $\mathfrak{p}^+$ and $\mathscr{U}(\mathfrak{k}_{1\mathbf{C}})$. Hence until then we shall omit the superscript (1) and write e_{kl}, F_{mj}, and $h_{a,j}$ for $e^{(1)}_{kl}$, $F^{(1)}_{mj}$, and $h^{(1)}_{a,j}$, respectively.

LEMMA 5.9. *For* $1 \leqslant k \leqslant j-2$,

$$\gamma_{j-1}(p^+_{j,n-j}\gamma_k) - \sum_{m=k+1}^{j-1} (e_{j,m} \cdot \gamma_{j-1})(p^+_{m,n-j+1}\gamma_k) = (-1)^{n-j} \gamma_j \gamma_k.$$

Proof. The proof is similar to that of Lemma 4.5. For $\binom{z}{w} \in M_{2n,n}(\mathbf{C})$ with $\gamma_j\binom{z}{w} \neq 0$, we apply Cramer's rule to the system of equations with unknowns $x_1, ..., x_j, y_1, ..., y_{n-j}$:

$$x_1 z_1 + \cdots + x_j z_j + y_1 w_1 + \cdots + y_{n-j} w_{n-j} = w_{n-j+1}. \quad ∎$$

For convenience, we modify our notations that were introduced in (4.1) and (4.2) slightly. For $\lambda \in \Lambda^+$, $\boldsymbol{l} = \boldsymbol{l}(\lambda)$ shall denote a $n+1$ tuple (instead of n tuple) of numbers given by

$$l_0 = \alpha - \lambda_1, \qquad l_j = \lambda_j - \lambda_{j+1}, \qquad 1 \leqslant j \leqslant n-1, \qquad l_n = \lambda_n,$$

and we set $\gamma_0 = \det w$. Thus

$$\gamma^{\boldsymbol{l}} = \gamma_0^{l_0} \gamma_1^{l_1} \gamma^{l_2} \cdots \gamma_n^{l_n}$$

and $\xi_\lambda = \gamma^{\boldsymbol{l}} (\det \bar{w})^\beta$.

PROPOSITION 5.10.

$$m_\lambda(\eta_{\lambda+e_j}) = (-1)^{n+1}(\alpha+\lambda_j+j-1)\prod_{a=1}^{j-1}(\lambda_a-\lambda_j+j-a-1)\,\xi_{\lambda+e_j}.$$

Proof. By (5.8) and Corollary 4.10,

$$\begin{aligned}
m_\lambda(\eta_{\lambda+e_j}) &= \sum_{m=1}^{j}(-1)^{m+1}p^+_{m,n-j+1}F_{mj}H^-(j;1,m-1)\cdot\xi_\lambda\\
&= \sum_{m=1}^{j-1}(-1)^{m+1}\lambda(j;1,m-1)\,p^+_{m,n-j+1}F_{mj}\cdot\xi_\lambda\\
&\quad+(-1)^{j+1}\lambda(j;1,j-1)\,p^+_{j,n-j+1}\cdot\xi_\lambda\\
&= \sum_{m=1}^{j-1}(-1)^{m+1}\lambda(j;1,m-1)\,p^+_{m,n-j+1}\\
&\quad\times\{(-1)^{m+j+1}\lambda(j;m,j-1)(e_{j,m}\cdot\gamma_{j-1})\,\gamma^{l-\varepsilon_{j-1}}(\det\bar w)^\beta\}\\
&\quad+(-1)^{j+1}\lambda(j;1,j-1)\sum_{k=0}^{j-1}l_k(p^+_{j,n-j+1}\cdot\gamma_k)\,\gamma^{l-\varepsilon_k}(\det\bar w)^\beta\\
&= (-1)^j\lambda(j;1,j-1)\sum_{m=1}^{j-1}[(p^+_{m,n-j+1}e_{j,m}\cdot\gamma_{j-1})\,\gamma^{l-\varepsilon_{j-1}}(\det\bar w)^\beta\\
&\quad+\sum_{k=0}^{m-1}l_k(e_{j,m}\cdot\gamma_{j-1})(p^+_{m,n-j+1}\cdot\gamma_k)\,\gamma^{l-\varepsilon_{j-1}-\varepsilon_k}](\det\bar w)^\beta\\
&\quad+(-1)^{j+1}\lambda(j;1,j-1)\left[\sum_{k=0}^{j-2}l_k(p^+_{j,n-j+1}\gamma_k)\,\gamma^{l-\varepsilon_k}(\det\bar w)^\beta\right.\\
&\quad\left.+\,l_{j-1}(p^+_{j,n-j+1}\cdot\gamma_{j-1})\,\gamma^{l-\varepsilon_{j-1}}(\det\bar w)^\beta.\right]
\end{aligned}$$

Observe that $p^+_{m,n-j+1}e_{j,m}\cdot\gamma_{j-1}=(-1)^{n-j+1}\gamma_j$ and $p^+_{j,n-j+1}\cdot\gamma_{j-1}=(-1)^{n-j}\gamma_j$. Hence

$$\begin{aligned}
m_\lambda(\eta_{\lambda+e_j}) &= (-1)^{n+1}\lambda(j;1,j-1)(j-1)\,\xi_{\lambda+e_j}+(-1)^j\lambda(j;1,j-1)\\
&\quad\times\sum_{k=0}^{j-2}l_k\sum_{m=k+1}^{j-1}(e_{j,m}\cdot\gamma_{j-1})(p^+_{m,n-j+1}\cdot\gamma_k)\,\gamma^{l-\varepsilon_{j-1}-\varepsilon_k}(\det\bar w)^\beta\\
&\quad+(-1)^{j+1}\lambda(j;1,j-1)\sum_{k=0}^{j-2}l_k(p^+_{j,n-j+1}\cdot\gamma_k)\,\gamma^{l-\varepsilon_k}(\det\bar w)^\beta\\
&\quad+(-1)^{n+1}\lambda(j;1,j-1)\,l_{j-1}\,\xi_{\lambda+e_j}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{n+1}\lambda(j;1,j-1)(\lambda_{j-1}+j-1)\,\xi_{\lambda+e_j}\\
&\quad+(-1)^{j+1}\lambda(j;1,j-1)\sum_{k=0}^{j-2}l_k[\gamma_{j-1}(p^+_{j,n-j+1}\cdot\gamma_k)\\
&\quad-\sum_{m=k+1}^{j-1}(e_{j,m}\cdot\gamma_{j-1})(p^+_{m,n-j+1}\cdot\gamma_k)]\,\gamma^{l-\varepsilon_{j-1}-\varepsilon_k}(\det\bar w)^\beta\\
&= (-1)^{n+1}\lambda(j;1,j-1)(l_{j-1}+j-1)\,\xi_{\lambda+e_j}\\
&\quad+(-1)^{j+1}\lambda(j;1,j-1)(-1)^{n-j}\left(\sum_{k=0}^{j-2}l_k\right)\xi_{\lambda+e_j}\\
&\qquad\text{(by Lemma 5.9)}\\
&= (-1)^{n+1}(\alpha-\lambda_j+j-1)\,\lambda(j;1,j-1)\,\xi_{\lambda+e_j}. \quad ∎
\end{aligned}$$

COROLLARY 5.11.

$$\begin{aligned}
m_\lambda(\psi_{\lambda+e_j}) &= (-1)^{n+1}(\alpha-\lambda_j+j-1)\prod_{a=1}^{j-1}(\lambda_a-\lambda_j+j-a)\\
&\quad\times(\lambda_a-\lambda_j+j-a-1)\,\xi_{\lambda+e_j}.
\end{aligned}$$

Proof. This follows from Proposition 5.10 and Lemma 5.7. ∎

Next we shall study the transition from V_λ to $V_{\lambda-e_j}$. The K-highest weight of $V_{\lambda-e_j}$ is $(\lambda-e_j,\lambda^*+(\alpha-\beta)\mathbf{1}+e_{n-j+1})$. If we set $\mu=\lambda^*+(\alpha-\beta)\mathbf{1}$ and $r=n-j+1$, then

$$(\lambda-e_j,\lambda^*+(\alpha-\beta)\mathbf{1}+e_{n-j+1})=(\mu^*+(\alpha-\beta)\mathbf{1}-e_{n-r+1},\mu+e_r).$$

This suggests that we may obtain transition formula from V_λ to $V_{\lambda-e_j}$ by "interchanging the roles of K_1 and K_2."

We recall that

$$K=\left\{\begin{pmatrix}k_1&0\\0&k_2\end{pmatrix}: k_1,k_2\in U(n)\right\}.$$

Let $r: K\to K$ be the automorphism given by

$$\tau\left[\begin{pmatrix}k_1&0\\0&k_2\end{pmatrix}\right]=\begin{pmatrix}k_2&0\\0&k_1\end{pmatrix}.$$

Then by composing actions with τ, we obtain new actions of K on $\mathscr{S}^{\alpha,\beta}(X^{oo})$, $\mathfrak{p}^+$, $\mathfrak{p}^-$, $\mathfrak{p}^+\oplus V_\lambda$, and $\mathfrak{p}^-\otimes V_\lambda$.

Let σ be the restriction to K of the action (2.2) of $U(n, n)$ on $\mathscr{S}^{\alpha,\beta}(X^{oo})$. As remarked above, $\sigma \circ \tau$ defines a representation of K on $\mathscr{S}^{\alpha,\beta}(X^{oo})$. For $f \in \mathscr{S}^{\alpha,\beta}(X^{oo})$, let $\tilde{f}\binom{z}{w} = f\binom{w}{z}$, $\forall \binom{z}{w} \in X^{oo}$. Then clearly $\tilde{f} \in \mathscr{S}^{\alpha,\beta}(X^{oo})$ and the map $T: f \to \tilde{f}$ defines a K-isomorphism for $(\mathscr{S}^{\alpha,\beta}(X^{oo}), \sigma) \cong (\mathscr{S}^{\alpha,\beta}(X^{oo}), \sigma \circ \tau)$. Recall that for $\lambda \in \Lambda^+$, we have $\xi_\lambda = \gamma_1^{\lambda_1 - \lambda_2} \cdots \gamma_{n-1}^{\lambda_{n-1} - \lambda_n} \gamma_n^{\lambda_n} (\det w)^{\alpha - \lambda_1} (\det \bar{w})^{\beta}$. By using the identity $\det z \det \bar{z} = \det w \det \bar{w}$ on X^{oo}, we obtain

$$\tilde{\xi}_\lambda = c\xi_{\lambda^* + (\alpha-\beta)\mathbf{1}}, \tag{5.12}$$

where $c = c(\lambda)$ is some power of (-1). It follows that if $f \in V_\lambda$, then $\tilde{f} \in V_{\lambda^* + (\alpha-\beta)\mathbf{1}}$.

Next recall that we have K-isomorphisms (5.1) $P^+: \mathfrak{p}^+ \cong \mathbf{C}^n \otimes \mathbf{C}^{n*}$ and $P^-: \mathfrak{p}^- \cong \mathbf{C}^{n*} \otimes \mathbf{C}^n$, and $P^+(p_{kl}^+) = \varepsilon_k \otimes \varepsilon_l^*$ and $P^-(p_{kl}^-) = \varepsilon_k^* \otimes \varepsilon_l$. We denote the adjoint action of K on $\mathfrak{p}^+$ and $\mathfrak{p}^-$ by Ad. Thus $\mathrm{Ad} \circ \tau$ defines another action of K on $\mathfrak{p}^+$. In fact, if we set $(p_{kl}^-)^\sim = p_{kl}^+$ and extend it to a linear isomorphism of $\mathfrak{p}^-$ onto $\mathfrak{p}^+$, then the map $p \to \tilde{p}$ defines a K-isomorphism for $(\mathfrak{p}^-, \mathrm{Ad}) \cong (\mathfrak{p}^+, \mathrm{Ad} \circ \tau)$. Now let $\lambda \in \Lambda^+$ and set $\mu = \lambda^* + (\alpha - \beta)\mathbf{1}$. We define a map $d: \mathfrak{p}^- \otimes V_\lambda \to \mathfrak{p}^+ \otimes V_\mu$ by

$$d(p \otimes f) = \tilde{p} \otimes \tilde{f}, \qquad \forall p \in \mathfrak{p}^-, \qquad f \in V_\lambda.$$

Then it is easy to see that d is an K-isomorphism for

$$(\mathfrak{p}^- \otimes V_\lambda, \mathrm{Ad} \otimes \sigma) \cong (\mathfrak{p}^+ \otimes V_\mu, ((\mathrm{Ad} \otimes \sigma) \circ \tau)).$$

LEMMA 5.13. *The following diagram is commutative*:

$$\begin{array}{ccc} \mathfrak{p}^- \otimes V_\lambda & \xrightarrow{m_\lambda} & \mathscr{S}^{\alpha,\beta}(X^{oo}) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle T} \\ \mathfrak{p}^+ \otimes V_\mu & \xrightarrow[m_\mu]{} & \mathscr{S}^{\alpha,\beta}(X^{oo}). \end{array}$$

Proof. Let $p \in \mathfrak{p}^-$ and $f \in V_\lambda$. Then

$$Tm_\lambda(p \otimes f) = (p, f)^\sim$$

and

$$m_\mu d(p \otimes f) = m_\mu(\tilde{p} \otimes \tilde{f}) = \tilde{p} \cdot \tilde{f}.$$

Thus we need to show $(p \cdot f)^\sim = \tilde{p} \cdot \tilde{f}$. Using (2.6), one can easily check that this holds for the basis $\{p_{kl}^- : 1 \leqslant k, l \leqslant n\}$ of $\mathfrak{p}^-$. ∎

By (5.2), (5.3), and Proposition 3.6 and 3.10,

$$\begin{aligned} \psi_{\lambda - e_j} &= Q^- \left\{ \sum_{t=j}^{n} \left[\varepsilon_t^* \otimes S_{jt}^{(1)} \prod_{a=t+1}^{n} (h_{j,a}^{(1)} + a - j - 1) \cdot u \right] \right. \\ &\quad \left. \otimes \sum_{m=1}^{r} \left[(-1)^{m+1} \varepsilon_m \otimes F_{mr}^{(2)} \prod_{b=1}^{m-1} (h_{b,r}^{(2)} + r - b - 1) \cdot u' \right] \right\} \qquad (5.14) \\ &= \sum_{t=j}^{n} \sum_{m=1}^{r} (-1)^{m+1} p_{tm}^- \otimes S_{jt}^{(1)} \prod_{a=t+1}^{n} (h_{j,a}^{(1)} + a - j - 1) \\ &\quad \times F_{m,r}^{(2)} \prod_{b=1}^{m-1} (h_{b,r}^{(2)} + r - b - 1) \cdot \xi_\lambda \end{aligned}$$

is a K-highest weight vector of weight $(\lambda - e_j, \mu + e_r)$.

PROPOSITION 5.15.

$$\begin{aligned} m_\lambda(\psi_{\lambda - e_j}) &= (\beta + \lambda_j + n - j) \prod_{a=j+1}^{n} (\lambda_j - \lambda_a + a - j) \\ &\quad \times (\lambda_j - \lambda_a + a - j - 1)\, \xi_{\lambda - e_j}. \end{aligned}$$

Proof. We have

$$\begin{aligned} d(\psi_{\lambda - e_j}) &= \sum_{t=j}^{n} \sum_{m=1}^{r} (-1)^{m+1} p_{mt}^+ \otimes T \left[S_{jt}^{(1)} \prod_{a=t+1}^{n} (h_{j,a}^{(1)} + a - j - 1) \right. \\ &\quad \left. \times F_{mr}^{(2)} \prod_{b=1}^{m-1} (h_{b,r}^{(2)} + r - b - 1) \cdot \xi_\lambda \right] \\ &= \sum_{t=j}^{n} \sum_{m=1}^{r} (-1)^{m+1} p_{mt}^+ \otimes S_{j,t}^{(2)} \prod_{a=t+1}^{n} (h_{j,a}^{(2)} + a - j - 1) \\ &\quad \times F_{mr}^{(1)} \prod_{b=1}^{m-1} (h_{b,r}^{(1)} + r - b - 1) \cdot \tilde{\xi}_\lambda. \end{aligned}$$

Now by (5.12), $\tilde{\xi}_\lambda = c\xi_\mu$ where $c = c(\lambda)$ is some power of (-1). On the other hand,

$$\begin{aligned} \psi_{\mu + e_r} &= \sum_{t=j}^{n} \sum_{m=1}^{r} (-1)^{m+1} p_{mt}^+ \otimes S_{jt}^{(2)} \prod_{a=t+1}^{n} (h_{j,a}^{(2)} + a - j - 1) \\ &\quad \times F_{mr}^{(1)} \prod_{b=1}^{m-1} (h_{b,r}^{(1)} + r - b - 1) \cdot \xi_\mu \end{aligned}$$

is a highest weight vector in $\mathfrak{p}^+ \otimes V_\mu$ of weight $(\mu + e_r, \mu^* + (\alpha - \beta)\mathbf{1} - e_j)$. Consequently, $d(\psi_{\lambda - e_j}) = c\psi_{\mu + e_r}$ so that by Corollary 5.11

$$\begin{aligned} m_\mu d(\psi_{\lambda - e_j}) &= c m_\mu(\psi_{\mu + e_r}) \\ &= c(-1)^{n+1} (\alpha - \mu_r + r - 1) \prod_{a=1}^{r-1} (\mu_a - \mu_r + r - a) \\ &\quad \times (\mu_a - \mu_r + r - a - 1)\, \xi_{\mu + e_r}. \end{aligned}$$

Now $\tilde{\xi}_{\lambda - e_j} = c(-1)^{n+1} \xi_{\mu + e_r}$. Hence by Lemma 5.13 we have

$$\begin{aligned} m_\lambda(\psi_{\lambda - e_j}) &= T^{-1} m_\mu d(\psi_{\lambda - e_j}) \\ &= T^{-1} \Big\{ [\alpha - (-\lambda_j + \alpha - \beta) + n - j + 1 - 1] \\ &\quad \times \prod_{a=1}^{r-1} (\lambda_{n-r+1} - \lambda_{n-a+1} + r - a) \\ &\quad \times (\lambda_{n-r+1} - \lambda_{n-a+1} + r - a - 1)\, \tilde{\xi}_{\lambda - e_j} \Big\} \\ &= (\beta + \lambda_j + n - j) \prod_{b=j+1}^{n} (\lambda_j - \lambda_b + b - j) \\ &\quad \times (\lambda_j - \lambda_b + b - j - 1)\, \xi_{\lambda - e_j}. \quad \blacksquare \end{aligned}$$

6. Reducibility, Scole Series, and Module Diagrams

In this section, we shall use the results obtained in the previous section to study the module structure of $\mathscr{S}^{\alpha, \beta}(X^{oo})$. In particular, we shall determine which $\mathscr{S}^{\alpha, \beta}(X^{oo})$ are irreducible. For each $\mathscr{S}^{\alpha, \beta}(X^{oo})$ which is reducible we shall describe its composition series and scole series.

By the results of Harish-Chandra (see, for instance, Proposition 1.2.5 of [7]), we know that each $U(n, n)$ submodule of $\mathscr{S}^{\alpha, \beta}(X^{oo})$ is given by the closure of the sum of a collection of K-types invariant under $\mathfrak{p}_{\mathbf{C}}$. In particular, this also implies that $\mathscr{S}^{\alpha, \beta}(X^{oo})$ is irreducible as an $U(n, n)$ module if and only if $\mathscr{S}^{\alpha, \beta}(X^{oo})_K$ is irreducible as a module of $\mathfrak{gl}_{2n}(\mathbf{C})$. From now on we shall abuse notation and shall not distinguish a $\mathfrak{gl}_{2n}(\mathbf{C})$ invariant subspace in $\mathscr{S}^{\alpha, \beta}(X^{oo})_K$ from the $U(n, n)$ submodule in $\mathscr{S}^{\alpha, \beta}(X^{oo})$ it defines.

Corollary 6.1. *Let $\lambda \in \Lambda^+$ and $1 \leqslant j \leqslant n$. Then*

(i) *$V_{\lambda + e_j}$ is contained in the $U(n, n)$ submodule of $\mathscr{S}^{\alpha, \beta}(X^{oo})$ generated by V_λ if and only if $\alpha - \lambda_j + j - 1 \neq 0$. If $\alpha - \lambda_j + j - 1 = 0$, then the subspace $\sum_{\mu_j \leqslant \lambda_j} V_\mu$ is a submodule of $\mathscr{S}^{\alpha, \beta}(X^{oo})$.*

(ii) *$V_{\lambda - e_j}$ is contained in the $U(n, n)$ submodule of $\mathscr{S}^{\alpha, \beta}(X^{oo})$ generated by V_λ if and only if $\beta + \lambda_j + n - j \neq 0$. If $\beta + \lambda_j + n - j = 0$, then the subspace $\sum_{\mu_j \geqslant \lambda_j} V_\mu$ is a submodule of $\mathscr{S}^{\alpha, \beta}(X^{oo})$.*

Proof. The if part follows from Corollary 5.11 and Proposition 5.15. We prove the only if part.

Assume that $\alpha - \lambda_j + j - 1 = 0$. For a subspace $W \subseteq \mathscr{S}^{\alpha, \beta}(X^{oo})_K$, we let

$$\mathfrak{p}(W) = \{p_1 w_1 + \cdots + p_r w_r : r \geqslant 1, p_j \in \mathfrak{p}_{\mathbf{C}}, w_j \in W, \forall 1 \leqslant j \leqslant r\}$$

and

$$\mathfrak{p}^{k+1}(W) = \mathfrak{p}(\mathfrak{p}^k(W)), \qquad \forall k \geqslant 1.$$

We consider the case $W = V_\lambda$. Since $[\mathfrak{k}_{\mathbf{C}}, \mathfrak{p}_{\mathbf{C}}] \subseteq \mathfrak{p}_{\mathbf{C}}$, each $\mathfrak{p}^k(V_\lambda)$ is a K-module. In fact, $\mathfrak{p}^k(V_\lambda)$ is a finite sum of K-types. Since $\mathfrak{gl}_{2n}(\mathbf{C}) = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}$, the submodule $\mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))\, V_\lambda$ generated by V_λ is $V_\lambda \oplus \sum_{k \geqslant 1} \mathfrak{p}^k(V_\lambda)$. We claim that for all $k \geqslant 1$, if $V_\mu \subseteq \mathfrak{p}^k(V_\lambda)$, then $\mu_j \leqslant \lambda_j$. We see this by an induction on k. First we note that by Corollary 5.11, $V_{\lambda + e_j} \not\subseteq \mathfrak{p}(V_\lambda)$. The case $k = 1$ follows from this. Assume that the claim holds for $\mathfrak{p}^k(V_\lambda)$. Let $V_\eta \subseteq \mathfrak{p}^k(V_\lambda)$. If $\eta_j < \lambda_j$, then $(\eta + e_j)_j = \eta_j + 1 \leqslant \lambda_j$. If $\eta_j = \lambda_j$, then $\alpha - \eta_j + j - 1 = \alpha - \lambda_j + j - 1 = 0$. Thus by Corollary 5.11, $V_{\eta + e_j} \cap \mathfrak{p}(V_\eta) = 0$. We also note that for $r \neq j$, $(\eta + e_r)_j = \eta_j \leqslant \lambda_j$. Thus for every $V_\mu \subseteq \mathfrak{p}^{k+1}(V_\lambda)$, $\mu_j \leqslant \lambda_j$. This proves the claim.

It follows from the claim that $V_{\lambda + e_j} \cap \mathscr{U}(\mathfrak{gl}_{2n}(\mathbf{C}))\, V_\lambda = 0$. Hence (i) holds. The proof for (ii) is similar. $\blacksquare$

We shall represent the K-type V_λ by the integral point $\lambda = (\lambda_1, ..., \lambda_n)$ in $\mathbf{R}^n$. Let $x_1, ..., x_n$ be the standard coordinates of $\mathbf{R}^n$. By the above corollary, the hyperplanes $l_j^\alpha : x_j = \alpha + j - 1$ and $l_j^\beta : x_j = -(\beta + n - j)$ are "barriers" to the action of $\mathfrak{p}_{\mathbf{C}}$ in the following sense: any K-type V_λ with λ on l_j^α cannot be transformed by $\mathfrak{p}_{\mathbf{C}}$ to $V_{\lambda + e_j}$ and any K type V_μ with μ on l_j^β cannot be transformed by $\mathfrak{p}_{\mathbf{C}}$ to $V_{\mu - e_j}$.

Note that the parameters α and β of the representation space $\mathscr{S}^{\alpha, \beta}(X^{oo})$ satisfy $\alpha - \beta \in \mathbf{Z}$. Hence either we have both $\alpha \in Z$ and $\beta \in \mathbf{Z}$ or both $\alpha \notin \mathbf{Z}$ and $\beta \notin \mathbf{Z}$.

Theorem 6.2. *$\mathscr{S}^{\alpha, \beta}(X^{oo})$ is irreducible if and only if $\alpha \notin \mathbf{Z}$ and $\beta \notin \mathbf{Z}$.*

Proof. Assume that $\alpha, \beta \notin \mathbf{Z}$. Let $\lambda, \mu \in \Lambda^+$. We shall construct a "polygonal line" connecting λ and μ. Since $\alpha, \beta \notin \mathbf{Z}$, the hyperplanes l_j^α and

l_j^β for $1 \leqslant j \leqslant n$ do not contain any K-type. Consequently they will not obstruct the movement of K-types along this polygonal line. Thus starting from the K-type V_λ, successive applications of operators from $\mathfrak{p}$ will transform V_λ to the K-types along this line and eventually to V_μ. It follows from this that $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is irreducible.

We shall define a sequence $(\eta^{(m)})_{m=0}^n$ in $\mathbf{Z}^n$ inductively as follows. Let $1 \leqslant i_1 < i_2 < \cdots < i_a \leqslant n$ and $n \geqslant j_1 > j_2 > \cdots > j_b \geqslant 1$ be such that $\lambda_{i_u} \leqslant \mu_{i_u}$ and $\lambda_{j_v} > \mu_{j_v}$, $\forall 1 \leqslant u \leqslant a$, $1 \leqslant v \leqslant b$, and $\{i_1, ..., i_a\} \cup \{j_1, ..., j_b\} = \{1, 2, ..., n\}$. Set $\eta^{(0)} = \lambda$. For $1 \leqslant m \leqslant a$, we set

$$\eta_k^{(m)} = \begin{cases} \mu_{i_m} & k = i_m \\ \eta_k^{(m-1)} & k \neq i_m \end{cases} \tag{6.3}$$

and for $a+1 \leqslant m \leqslant b$,

$$\eta_k^{(m)} = \begin{cases} \mu_{j_{m-a}} & k = j_{m-a} \\ \eta_k^{(m-1)} & k \neq j_{m-a}. \end{cases} \tag{6.4}$$

That is, we successively replace those coordinates of λ less than those of μ by those of μ from the front and replace others from the rear. In this way we can assure that $\eta^{(m)} \in \Lambda^+$, $\forall 0 \leqslant m \leqslant n$. We note that $\eta^{(n)} = \mu$ and

$$\eta^{(m)} = \begin{cases} \eta^{(m-1)} + (\mu_{i_m} - \lambda_{i_m})\, e_{i_m} & 1 \leqslant m \leqslant a \\ \eta^{(m-1)} - (\lambda_{j_{m-a}} - \mu_{j_{m-a}})\, e_{j_{m-a}} & a+1 \leqslant m \leqslant b. \end{cases}$$

Thus there are nonnegative integers $l_1, ..., l_n$ such that $V_{\eta^{(m)}} \subseteq \mathfrak{p}^{l_m}(V_{\eta^{(m-1)}})$, $\forall 1 \leqslant m \leqslant n$. It follows that $V_\mu \subseteq \mathfrak{p}^r(V_\lambda)$ where $r = \sum_{j=1}^n l_j$. Similarly $V_\lambda \subseteq \mathfrak{p}^d(V_\mu)$ for some $d \geqslant 0$. This proves the if part.

Conversely assume $\alpha, \beta \in \mathbf{Z}$. Take any $\lambda \in \Lambda^+$ with $\lambda_j = \alpha + j - 1$. Then by Corollary 6.1, the submodule $\sum_{\mu_j \leqslant \lambda_j} V_\mu$ does not contain $V_{\lambda + e_j}$, hence is a proper submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. ∎

Next we want to study the module structure of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ when it is reducible. In this case, the barrier hyperplanes l_j^α and l_j^β for $1 \leqslant j \leqslant n$ play an important role in describing the submodules of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. We note that the "gap" between l_j^α and l_j^β is independent of j for all $1 \leqslant j \leqslant n$; i.e.,

$$[\alpha + j - 1] - [-(\beta + n - j)] = \alpha + \beta + n - 1.$$

We shall first give a complete analysis for the case $\alpha + \beta + n - 1 \geqslant 0$, then a parallel analysis for the case $\alpha + \beta + n - 1 < 0$.

Case 1: $\alpha + \beta + n - 1 \geqslant 0$. In this case, the barriers in the λ_j-direction can be visualized as in Fig. 1.

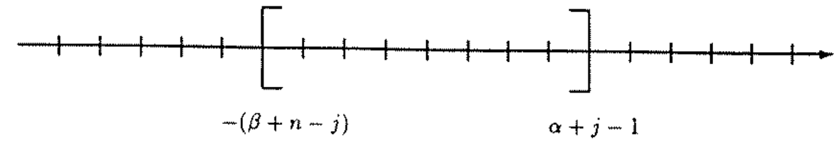

FIGURE 1

Since $\alpha + \beta + n - 1 \geqslant 0$, the λ_j-coordinates of the points on l_j^α are always greater or equal to those of l_j^β. The symbol [in Fig. 1 means that the transition of K-types from "left to right" is permissible but the converse is not. The symbol] means the same for the other direction. We now set

$$\begin{aligned} A_1^j &= \{\lambda \in \Lambda^+ : \lambda_j < -(\beta + n - j)\}, \\ A_2^j &= \{\lambda \in \Lambda^+ : -(\beta + n - j) \leqslant \lambda_j \leqslant \alpha + j - 1\}, \\ A_3^j &= \{\lambda \in \Lambda^+ : \lambda_j > \alpha + j - 1\}. \end{aligned} \tag{6.5}$$

Then $\Lambda^+ = A_1^j \cup A_2^j \cup A_3^j$ and the union is disjoint.

For each sequence $a = (a_1, ..., a_n)$ with $a_j \in \{1, 2, 3\}$, $\forall 1 \leqslant j \leqslant n$, we let

$$R_a = A_{a_1}^1 \cap A_{a_2}^2 \cap \cdots \cap A_{a_n}^n. \tag{6.6}$$

Clearly the set of the nonempty R_a forms a partition for Λ^+. If $R_a \neq \varnothing$, then we call the subspace $\sum_{\lambda \in R_a} V_\lambda$ a *constituent* of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. For convenience, we shall also denote this subspace by R_a. It will be clear from the context whether we are refering to R_a as a subset of Λ^+ or as a subspace of $\mathscr{S}^{\alpha,\beta}(X^{oo})$.

LEMMA 6.7. *If $\lambda, \mu \in R_a$, then V_λ and V_μ generate the same $U(n, n)$-submodule in $\mathscr{S}^{\alpha,\beta}(X^{oo})$.*

Proof. It suffices to observe that for $\lambda, \mu \in R_a$, the sequence $(\eta^{(m)})_{m=0}^n$ in (6.3) and (6.4) is contained in R_a. ∎

By this lemma, any $U(n, n)$ submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is given by a sum of constituents. Thus we should study the K-types in a constituent collectively. Our problem is therefore reduced to determining which R_a's are nonempty and how $\mathfrak{p}_{\mathbf{C}}$ transform them.

Let $D = \{a = (a_1, ..., a_n) : a_j \in \{1, 2, 3\}, \forall j, a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n\}$. It can be checked that the dominant condition (i.e., $\lambda \in \Lambda^+$ satisfies $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$) implies that $A_{a_j}^j \cap A_{a_{j+1}}^{j+1} = \varnothing$ whenever $a_j < a_{j+1}$, $\forall j$. Consequently, if $a \notin D$ then $R_a = \varnothing$. Hence the collection of R_a's can be reduced to $\{R_a : a \in D\}$.

LEMMA 6.8. *For $a \in D$, let $l_2(a)$ denote the number of entries of a which are equal to 2. Let $\alpha + \beta + n - 1 = i$ where $i \in \mathbf{Z}_+ \cup \{0\}$.*

1. *If $i \geqslant n-1$, then $R_a \neq \varnothing$ for all $a \in D$.*
2. *If $0 \leqslant i \leqslant n-2$, then $R_a \neq \varnothing$ if and only if $l_2(a) \leqslant i+1$.*

We shall postpone the proof of this lemma till we have stated our main theorems. One way to visualise Lemma 6.8 is the following. We arrange all the constituents in a number of rows. The first row consists of all R_a with $l_2(a)=0$. The second row consists of all R_a with $l_2(a)=1$, and the same applies for other rows. Then for $\alpha+\beta+n-1 \geqslant n-1$, we have the configuration shown in Fig. 2.

By (2) of Lemma 6.8 the configuration for $\alpha+\beta+n-1=i$ with $0 \leqslant i \leqslant n-2$ can be obtained by removing the lowest $(n-i-1)$ rows from Fig. 2.

By our notations used in Corollary 6.1, $\mathfrak{p}(R_a)$ will denote the subspace of $\mathscr{S}^{\alpha,\beta}(X^{oo})_K$ spanned by $\bigcup_{\lambda \in R_a} \mathfrak{p}(V_\lambda)$.

For $s, t \in \mathbf{Z}_+ \cup \{0\}$, let

$$a(s,t) = (\overbrace{3, \ldots, 3}^{s}, 2, \ldots, 2, \overbrace{1, \ldots, 1}^{t}) \in D.$$

LEMMA 6.9. *Let $\alpha-\beta+n-1=i$, $i \in \mathbf{Z}_+ \cup \{0\}$.*

(a) *If $i \geqslant n-1$, then for $b \neq a(s,t)$,*

$$\mathfrak{p}(R_{a(s,t)}) \cap R_b \neq \varnothing$$
$$\Leftrightarrow \begin{cases} b = a(s-1,t) \quad \text{or} \quad b = a(s,t-1) & \text{if} \quad s \geqslant 1, \quad t \geqslant 1 \\ b = a(s-1,0) & \text{if} \quad s \geqslant 1, \quad t = 0 \\ b = a(0,t-1) & \text{if} \quad s = 0, \quad t \geqslant 1 \end{cases}$$

and $\mathfrak{p}(R_{a(0,0)}) \subseteq R_{a(0,0)}$.

(b) *If $0 \leqslant i \leqslant n-2$, then for $n-s-t < i+1$ and $b \neq a(s,t)$,*

$$\mathfrak{p}(R_{a(s,t)}) \cap R_b \neq \varnothing$$
$$\Leftrightarrow \begin{cases} b = a(s-1,t) \quad \text{or} \quad b = s(s,t-1) & \text{if} \quad s \geqslant 1, \quad t \geqslant 1 \\ b = a(s-1,0) & \text{if} \quad s \geqslant 1, \quad t = 0 \\ b = a(0,t-1) & \text{if} \quad s = 0, \quad t \geqslant 1 \end{cases}$$

and for $n-s-t=i+1$, $\mathfrak{p}(R_{a(s,t)}) \subseteq R_{a(s,t)}$.

The proof of this lemma will be given later in this section. Lemma 6.9 essentially tells us that if we arrange the constituents as in Fig. 2, then $\mathfrak{p}_{\mathbf{C}}$ will "transform" the constituents from one level to the nearest constituent(s) in the next lower level. Thus repeated application of operators from $\mathfrak{p}_{\mathbf{C}}$ can only transform the constituents "downward" but any "upward transition" is not possible.

$$\begin{array}{ccccccccc}
R_{(3,\ldots,3,3)} & & R_{(3,\ldots,3,1)} & & R_{(3,\ldots,3,1,1)} & & \cdots & & R_{(1,\ldots,1)} \\
& R_{(3,\ldots,3,2)} & & R_{(3,\ldots,3,2,1)} & & \cdots & & R_{(2,1,\ldots,1)} & \\
& & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
& & & \cdot & \cdot & \cdot & & & \\
& & & R_{(3,2,\ldots,2)} & & R_{(2,\ldots,2,1)} & & & \\
& & & & R_{(2,\ldots,2)} & & & &
\end{array}$$

FIGURE 2

We now recall the notion of *socle series* [8]. The *socle* of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is the sum of all irreducible submodules of $\mathscr{S}^{\alpha,\beta}(X^{oo})$, and is written $\operatorname{Soc}(\mathscr{S}^{\alpha,\beta}(X^{oo}))$. The *socle series* of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is the ascending chain

$$\operatorname{Soc}^0(\mathscr{S}^{\alpha,\beta}(X^{oo})) \subseteq \operatorname{Soc}^1(\mathscr{S}^{\alpha,\beta}(X^{oo})) \subseteq \operatorname{Soc}^2(\mathscr{S}^{\alpha,\beta}(X^{oo})) \subseteq \cdots$$

of submodules of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ defined inductively by setting $\operatorname{Soc}^0(\mathscr{S}^{\alpha,\beta}(X^{oo}))=0$ and

$$\operatorname{Soc}^{r+1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))/\operatorname{Soc}^r(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \operatorname{Soc}(\mathscr{S}^{\alpha,\beta}(X^{oo}))/\operatorname{Soc}^r(\mathscr{S}^{\alpha,\beta}(X^{oo})))$$

for all nonnegative integers r. We say that $\mathscr{S}^{\alpha,\beta}(X^{oo})$ has *socle length* m if $\operatorname{Soc}^m(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \mathscr{S}^{\alpha,\beta}(X^{oo})$ but $\operatorname{Soc}^{m-1}(\mathscr{S}^{\alpha,\beta}(X^{oo})) \neq \mathscr{S}^{\alpha,\beta}(X^{oo})$.

THEOREM 6.10. *Let $\alpha+\beta+n-1=i$ where $i \in \mathbf{Z}_+ \cup \{0\}$.*

(1) *If $i \geqslant n-1$, then the socle series for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is given by*

$$\operatorname{Soc}^j(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \begin{cases} \sum_{l_2(a) \geqslant n-j+1} R_a & 1 \leqslant j \leqslant n \\ \mathscr{S}^{\alpha,\beta}(X^{oo}) & j \geqslant n+1. \end{cases}$$

In particular, we have the following

(a) *$\mathscr{S}^{\alpha,\beta}(X^{oo})$ has scole length $n+1$.*

(b) *$\operatorname{Soc}^1(\mathscr{S}^{\alpha,\beta}(X^{oo})) = R_{(2,\ldots,2)}$ is a finite-dimensional irreducible submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. When $i=n-1$, $R_{(2,\ldots,2)}$ only consists of the K-type $V_{\alpha 1}$ which is one dimensional.*

(c) *For $0 \leqslant q \leqslant p \leqslant n$, we set*

$$M_{p,q} = \operatorname{Soc}^p(\mathcal{S}^{\alpha,\beta}(X^{oo})) \oplus \sum_{t=0}^{q} R_{a(p-t,t)}.$$

Then

$$\begin{aligned} 0 &\subseteq M_{0,0} \\ &\subseteq M_{1,0} \subseteq M_{1,1} \\ &\subseteq M_{2,0} \subseteq M_{2,1} \subseteq M_{2,2} \\ &\cdots \\ &\cdots \\ &\cdots \\ &\subseteq M_{n,0} \subseteq M_{n,1} \subseteq M_{n,2} \subseteq \cdots \subseteq M_{n,n} = \mathcal{S}^{\alpha,\beta}(X^{oo}) \end{aligned}$$

is a composition series of $\mathcal{S}^{\alpha,\beta}(X^{oo})$ with $\frac{1}{2}(n+1)(n+2)$ composition factors.

(2) *If $0 \leqslant i \leqslant n-2$, then the socle series of $\mathcal{S}^{\alpha,\beta}(X^{oo})$ is given by*

$$\operatorname{Soc}^j(\mathcal{S}^{\alpha,\beta}(X^{oo})) = \begin{cases} \sum_{l_2(a) \geqslant i-j+2} R_a & 1 \leqslant j \leqslant i+1 \\ \mathcal{S}^{\alpha,\beta}(X^{oo}) & j \geqslant i+2. \end{cases}$$

In particular, we have the following.

(a) *$\mathcal{S}^{\alpha,\beta}(X^{oo})$ has socle length $i+2$.*

(b) *For each $a \in D$ with $l_2(a) = i+1$, R_a is an irreducible $U(n,n)$ submodule of $\mathcal{S}^{\alpha,\beta}(X^{oo})$. Moreover,*

$$\operatorname{Soc}^1(\mathcal{S}^{\alpha,\beta}(X^{oo})) = \sum_{l_2(a) = i+1} R_a.$$

(c) *For $n-i-1 \leqslant p \leqslant n$, $0 \leqslant q \leqslant p$, we set*

$$M_{p,q} = \operatorname{Soc}^{p-(n-i-1)}(\mathcal{S}^{\alpha,\beta}(X^{oo})) \oplus \sum_{t=0}^{q} R_{a(p-t,t)}.$$

Then

$$\begin{aligned} 0 &\subseteq M_{n-i-1,0} \subseteq M_{n-i-1,1} \subseteq \cdots \subseteq M_{n-i-1,n-i-1} \\ &\subseteq M_{n-i,0} \subseteq M_{n-i,1} \subseteq \cdots \subseteq M_{n-i,n-i} \\ &\cdots \\ &\cdots \\ &\cdots \\ &\subseteq M_{n,0} \subseteq M_{n,1} \subseteq \cdots \subseteq M_{n,n} = \mathcal{S}^{\alpha,\beta}(X^{oo}) \end{aligned}$$

is a composition series for $\mathcal{S}^{\alpha,\beta}(X^{oo})$ with $\frac{1}{2}(i+2)(2n-i+1)$ composition factors.

Case 2: $\alpha+\beta+n-1<0$. In this case the λ_j-coordinates of the points on l_j^α are always less than those of l_j^β. We can visualise the barrier along the λ_j-axis as shown in Fig. 3.

We let

$$B_1^j = \{\lambda \in \Lambda^+ : \lambda_j \leqslant \alpha+j-1\}$$
$$B_2^j = \{\lambda \in \Lambda^+ : \alpha+j-1 < \lambda_j < -(\beta+n-j)\}$$
$$B_3^j = \{\lambda \in \Lambda^+ : \lambda_j \geqslant -(\beta+n-j)\}.$$

As in the previous case, for a sequence $(a_1, ..., a_n)$ with $a_j \in \{1, 2, 3\}$, we set $R_a = B_{a_1}^1 \cap B_{a_2}^2 \cap \cdots \cap B_{a_n}^n$. Recall that $D = \{(a_1, ..., a_n) : a_j \in \{1, 2, 3\},\ \forall j,\ a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n\}$. Then for $a \notin D$, $R_a = \varnothing$. Thus we only need to consider the collection $\{R_a : a \in D\}$. If $R_a \neq \varnothing$, then we shall also denote the subspace $\sum_{\lambda \in R_a} V_\lambda$ by R_a and call it *constituent* of $\mathcal{S}^{\alpha,\beta}(X^{oo})$.

PROPOSITION 6.11. *If $\alpha+\beta+n-1=-1$, then $\mathcal{S}^{\alpha,\beta}(X^{oo})$ is a direct sum of $n+1$ irreducible $U(n,n)$ submodules.*

Proof. In this case, for $1 \leqslant j \leqslant n$, the hyperplanes l_j^α and l_j^β are defined respectively by $\lambda_j = \alpha+j-1$ and $\lambda_j = \alpha+j$. Thus l_j^α and l_j^β are 1 unit of distance apart, as shown in Fig. 4.

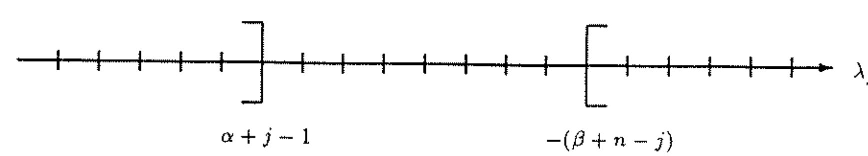

FIGURE 3

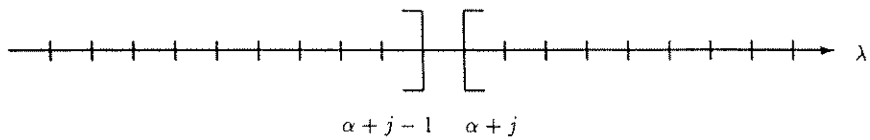

FIGURE 4

In particular, we note that $B_2^j=\varnothing$, $\forall 1\leqslant j\leqslant n$. Consequently $\mathscr{S}^{\alpha,\beta}(X^{oo})$ has exactly $n+1$ constituents, namely,

$$R_{(3,\ldots,3)},\ R_{(3,\ldots,3,1)},\ \ldots,\ R_{(3,1,\ldots,1)},\ R_{(1,\ldots,1)}.$$

Moreover, it is clear from Fig. 4 that the K-types in a constituent can never be transformed by $\mathfrak{p}_{\mathbf{C}}$ to any K-type in another constituent. Therefore each of its constituents forms an irreducible submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. ∎

With reasoning completely analogous to Lemma 6.8, we have the following.

LEMMA 6.12. *Let $\alpha+\beta+n-1=-i$ where $i\in\mathbf{Z}_+$ and $i\geqslant 2$.*

(1) *If $i\geqslant n+1$, then $R_a\neq\varnothing$, $\forall a\in D$.*

(2) *If $2\leqslant i\leqslant n$, then for $a\in D$, $R_a\neq\varnothing$ if and only if $l_2(a)\leqslant i-1$.*

As we have done in Lemma 6.8, we can visualise Lemma 6.12 as follows. For $i\geqslant n+1$, the set of constituents of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is given by $\{R_a: a\in D\}$. We shall arrange them as in Fig. 5.

$R_{(2,\ldots,2)}$

$R_{(3,2,\ldots,2)}$ $R_{(2,\ldots,2,1)}$

. . .

. . . . .

$R_{(3,\ldots,3,2)}$ $R_{(3,\ldots,3,2,1)}$. $R_{(2,1,\ldots,1)}$

$R_{(3,\ldots,3,3)}$ $R_{(3,\ldots,3,1)}$ $R_{(3,\ldots,3,1,1)}$. $R_{(1,\ldots,1)}$

FIGURE 5

The configuration for the case $1\leqslant i\leqslant n$ is obtained by deleting the top $n-i+1$ rows of Fig. 5.

LEMMA 6.13. *Let $\alpha+\beta+n-1=-i$ where $i\in\mathbf{Z}_+$ and $i\geqslant 2$.*

(a) *If $i\geqslant n+1$, then for $b\neq a(s,t)$,*

$$\mathfrak{p}(R_{a(s,t)})\cap R_b\neq\varnothing \Leftrightarrow b=a(s+1,t) \quad \text{or} \quad b=a(s,t+1)$$

and $0\leqslant s+t\leqslant n-1$ and $\mathfrak{p}(R_{a(s,t)})\subseteq R_{a(s,t)}$ if $s+t=n$.

(b) *If $1\leqslant i\leqslant n$, then for $b\neq a(s,t)$,*

$$\mathfrak{p}(R_{a(s,t)})\cap R_b\neq\varnothing \Leftrightarrow b=a(s+1,t) \quad \text{or} \quad b=a(s,t+1)$$

and $n-i+1\leqslant s+t\leqslant n-1$ and $\mathfrak{p}(R_{a(s,t)})\subseteq R_{a(s,t)}$ if $s+t=n$.

Proof. The proof is similar to that of Lemma 6.9.

If we display the constituents as in Fig. 5 then Lemma 6.13 says that $\mathfrak{p}_{\mathbf{C}}$ transforms the constituents downward but not upward.

We now deduce the following theorem from Lemma 6.13. We omit its proof as it is similar to that of Theorem 6.10.

THEOREM 6.14. *Let $\alpha+\beta+n-1=-i$ where $i\in\mathbf{Z}_+$ and $i\geqslant 2$.*

(1) *If $2\leqslant i\leqslant n$, then the scole series of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is given by*

$$\operatorname{Soc}^j(\mathscr{S}^{\alpha,\beta}(X^{oo}))=\begin{cases}\sum_{l_2(a)\leqslant j-1} R_a & 1\leqslant j\leqslant i-1\\ \mathscr{S}^{\alpha,\beta}(X^{oo}) & j\geqslant i.\end{cases}$$

In particular, we have the following:

(a) *$\mathscr{S}^{\alpha,\beta}(X^{oo})$ has socle length i.*

(b) *For each $a\in D$ with $l_2(a)=0$, R_a is an irreducible $U(n,n)$ submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. Moreover, $\operatorname{Soc}^1(\mathscr{S}^{\alpha,\beta}(X^{oo}))=\sum_{l_2(a)=0} R_a$ is a direct sum of $n+1$ irreducible submodules.*

(c) *For* $0 \leqslant p \leqslant i-1$, $0 \leqslant q \leqslant n-p$, *we let*

$$M_{p,q} = \mathrm{Soc}^p(\mathscr{S}^{\alpha,\beta}(X^{oo})) \oplus \sum_{t=0}^{q} R_{a(n-p-t,t)}.$$

Then

$$\begin{aligned} 0 &\subseteq M_{0,0} \subseteq M_{0,1} \subseteq \cdots \subseteq M_{0,n-1} \subseteq M_{0,n} \\ &\subseteq M_{1,0} \subseteq M_{1,1} \subseteq \cdots \subseteq M_{1,n-1} \\ &\cdots \\ &\cdots \\ &\cdots \\ &\subseteq M_{i-1,0} \subseteq M_{i-1,1} \subseteq \cdots \subseteq M_{i-1,n-i+1} = \mathscr{S}^{\alpha,\beta}(X^{oo}) \end{aligned}$$

is a composition series for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ *with* $\frac{1}{2}i(2n-i+3)$ *composition factors.*

(2) *If* $i \geqslant n+1$, *then the socle series of* $\mathscr{S}^{\alpha,\beta}(X^{oo})$ *is given by*

$$\mathrm{Soc}^j(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \begin{cases} \sum_{l_2(a) \leqslant j-1} R_a & 1 \leqslant j \leqslant n \\ \mathscr{S}^{\alpha,\beta}(X^{oo}) & j \geqslant n+1 \end{cases}$$

In particular, we have the following.

(a) $\mathscr{S}^{\alpha,\beta}(X^{oo})$ *has socle length* $n+1$.

(b) *For each* $a \in D$ *with* $l_2(a) = 0$, R_a *is an irreducible* $U(n,n)$ *submodule of* $\mathscr{S}^{\alpha,\beta}(X^{oo})$. *Moreover,* $\mathrm{Soc}^1(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \sum_{l_2(a)=0} R_a$ *is a direct sum of* $n+1$ *irreducible submodules.*

(c) *For* $0 \leqslant p \leqslant n$, $0 \leqslant q \leqslant n-p$, *we let*

$$M_{p,q} = \mathrm{Soc}^p(\mathscr{S}^{\alpha,\beta}(X^{oo})) \oplus \sum_{t=0}^{q} R_{a(n-p-t,t)}.$$

Then

$$\begin{aligned} 0 &\subseteq M_{0,0} \subseteq M_{0,1} \subseteq \cdots \subseteq M_{0,n-1} \subseteq M_{0,n} \\ &\subseteq M_{1,0} \subseteq M_{1,1} \subseteq \cdots \subseteq M_{1,n-1} \\ &\cdots \\ &\cdots \\ &\cdots \\ &\subseteq M_{n-1,0} \subseteq M_{n-1,1} \\ &\subseteq M_{n,0} = \mathscr{S}^{\alpha,\beta}(X^{oo}) \end{aligned}$$

is a composition series for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ *with* $\frac{1}{2}(n+1)(n+2)$ *composition factors.*

We shall now give the proofs for the results we have stated.

LEMMA 6.15. *Let* $\alpha+\beta+n-1 = i$ *where* i *is a nonnegative integer.*

(a) *If* $0 \leqslant i \leqslant n-1$, *then*

$$A_2^k \cap A_2^{k+1} \cap \cdots \cap A_2^{k+l} \begin{cases} \neq \varnothing & \text{if } 0 \leqslant l \leqslant i,\ 1 \leqslant k \leqslant n-l \\ = \varnothing & \text{if } i+1 \leqslant l \leqslant n-1,\ 1 \leqslant k \leqslant n-l. \end{cases}$$

In particular, $\lambda \in A_2^k \cap A_2^{k+1} \cap \cdots \cap A_2^{k+i}$ *if and only if* $\lambda_j = \alpha+k-1$, $\forall k \leqslant j \leqslant k+i$, $1 \leqslant k \leqslant n-i$.

(b) *If* $i \geqslant n$, *then* $A_2^1 \cap A_2^2 \cap \cdots \cap A_2^n \neq \varnothing$.

Proof. For part (a), it suffices to prove $A_2^k \cap \cdots \cap A_2^{k+i} \neq \varnothing$ and $A_2^k \cap \cdots \cap A_2^{k+i+1} = \varnothing$. Note that if $\lambda \in A_2^k$, then $-(\beta+n-k) \leqslant \lambda_k \leqslant \alpha+k-1$. Since $-(\beta+n-k) = -(\beta+n)+k = \alpha+k-i-1$, we have $\alpha+k-i-1 \leqslant \lambda_k \leqslant \alpha+k-1$. Similarly, if $\lambda \in A_2^{k+i}$, then $\alpha+k-1 \leqslant \lambda_{k+i} \leqslant \alpha+k+i-1$. From this we see that $\lambda = (\lambda_1, ..., \lambda_n)$ with $\lambda_j = \alpha+k-1$, $\forall 1 \leqslant j \leqslant n$, is in $A_2^k \cap \cdots \cap A_2^{k+i}$.

Next assume $\lambda \in A_2^k \cap \cdots \cap A_2^{k+i+1}$, then $\lambda_k \leqslant \alpha+k-1$ and $\lambda_{k+i+1} \geqslant \alpha+k$. Hence $\lambda_{k+i+1} > \lambda_k$ which contradicts $\lambda \in \Lambda^+$.

For part (b), observe that $\lambda = (\alpha, ..., \alpha) \in A_2^1 \cap \cdots \cap A_2^n$ if $i \geqslant n$. ∎

We now see that Lemma 6.8 follows as a corollary of Lemma 6.15. Similarly, Lemma 6.12 follows from the following lemma.

LEMMA 6.16. *Let* $\alpha+\beta+n-1 = -i$ *where* $i \in \mathbf{Z}$ *and* $i \geqslant 2$.

(a) *If* $2 \leqslant i \leqslant n$, *then*

$$B_2^k \cap B_2^{k+1} \cap \cdots \cap B_2^{k+l} \begin{cases} \neq \varnothing & \text{if } 0 \leqslant l \leqslant i-2,\ 1 \leqslant k \leqslant n-l \\ = \varnothing & \text{if } i-1 \leqslant l \leqslant n-1,\ 1 \leqslant k \leqslant n-l. \end{cases}$$

In particular, $\lambda \in B_2^k \cap B_2^{k+1} \cap \cdots \cap B_2^{k+i-2}$ *if and only if* $\lambda_j = \alpha+k+i-2$, $\forall k \leqslant j \leqslant k+i-2$, $1 \leqslant k \leqslant n-i+2$.

(b) *If* $i \geqslant n+1$, *then* $B_2^1 \cap B_2^2 \cap \cdots \cap B_2^n \neq \varnothing$.

Proof of Lemma 6.9. We first prove (a). Assume that $s \geqslant 1$, $t \geqslant 1$. Then by Lemma 6.8, $R_{a(s,t)}$, $R_{a(s-1,t)}$, and $R_{a(s,t-1)}$ are constituents of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. It is clear from the definition (6.5) of A_1^j, A_2^j, and A_3^j that $\mathfrak{p}(R_{a(s,t)}) \cap R_{a(s-1,t)} \neq 0$ and $\mathfrak{p}(R_{a(s,t)}) \cap R_{a(s,t-1)} \neq 0$. On the other hand, suppose that $\mathfrak{p}(R_{a(s,t)}) \cap R_b \neq 0$. Then for some K-type V_λ in $R_{a(s,t)}$, $\mathfrak{p}(V_\lambda)$

contains some K-type V_η in R_b. Now the K-types in $\mathfrak{p}(V_\lambda)$ are of the form $V_{\lambda \pm e_j}$ for some $1 \leqslant j \leqslant n$. Assume that $V_\eta = V_{\lambda + e_j}$. If $a(s, t) = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$, then $a_j = 1$, $b_j = 2$, and $a_k = b_k$, $\forall k \neq j$. Moreover, $j \geqslant n - t$. If $j > n - t + 1$, then

$$b = \underbrace{\overbrace{(3, ..., 3, 2, ..., 2, 1, ..., 1, 2}^{j}}_{n-t+1}, 1, ..., 1) \notin D.$$

Hence $j = n - t + 1$ and $b = a(s, t-1)$. Similarly if $V_\eta = V_{\lambda - e_j}$, then $b = a(s-1, t)$. Since $a(0, 0) = (2, 2, ..., 2)$, all the K-types in R_a are not able to move across any barrier. Consequently, $\mathfrak{p}(R_{a(0,0)})$ will not intersect any other constituents and hence must be contained in $R_{a(0,0)}$.

The proof for (b) is similar. ∎

Proof of Theorem 6.10. We shall prove (1) by an induction on j. Since $\mathfrak{gl}_{2n}(\mathbf{C}) = \mathfrak{p}_{\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$, it is immediate from Lemma 6.9 that $R_{a(0,0)}$ is an irreducible $U(n, n)$ submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. If we arrange the constituents of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ as in Fig. 2, then $\mathfrak{p}_{\mathbf{C}}$ only transforms the constituents downward. Thus it is clear that $\mathscr{S}^{\alpha,\beta}(X^{oo})$ has no other irreducible submodules. Hence $\mathrm{Soc}^1(\mathscr{S}^{\alpha,\beta}(X^{oo})) = R_{a(0,0)}$.

Next we let $1 \leqslant j \leqslant n+1$ and assume that

$$\mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \sum_{l_2(a) \geqslant n-j+2} R_a.$$

Let $a \in D$ be such that $l_2(a) = n - j + 1$. Then $a = a(s, j-s-1)$ for some $s \geqslant 0$. By Lemma 6.9,

$$\mathfrak{p}(R_a) \subseteq R_a \oplus R_{a(s-1, j-s-1)} \oplus R_{a(s, j-s-2)}.$$

Since $R_{a(s-1, j-s-1)} \oplus R_{a(s, j-s-2)} \subseteq \mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))$, $(R_a \oplus \mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo})))/\mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))$ is an irreducible submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})/\mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))$. Moreover,

$$\left[\left(\sum_{l_2(a)=n-j+1} R_a\right) + \mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))\right] \Big/ \mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))$$

$$= \sum_{l_2(a)=n-j+1} [R_a + \mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))]/\mathrm{Soc}^{j-1}(\mathscr{S}^{\alpha,\beta}(X^{oo}))$$

as a direct sum of irreducible $U(n, n)$ modules.

Since $\mathrm{Soc}^{n+1}(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \mathscr{S}^{\alpha,\beta}(X^{oo})$, $\mathrm{Soc}^j(\mathscr{S}^{\alpha,\beta}(X^{oo})) = \mathscr{S}^{\alpha,\beta}(X^{oo})$, $\forall j \geqslant n+2$.

Parts (b), (c), and (d) of (1) are direct consequences of (a).

The proof for (2) is similar. ∎

7. Diagrams for $\mathscr{S}^{\alpha,\beta}(X^{oo})$

In this section, we want to give a diagrammatic representation of the module structure of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. We shall adopt the terminologies and conventions given in Section 2 of [1] which we shall now briefly recall.

Let M be a module for a group or algebra. M is said to be *multiplicity free* if no two of its composition factors are isomorphic. If M is multiplicity free, then a *diagram* for M is a graph $\mathscr{G}$ defined as follows. The vertex set $V(\mathscr{G})$ is the set of all composition factors of M. For S_i and $S_j \in V(\mathscr{G})$, there is a directed edge $e(S_i, S_j)$ from the node S_i to the node S_j if and only if there are submodules U and V of M with $V \subseteq U$ and U/V is a nonsplit extension of S_i by S_j. This means that there is a nonsplit exact sequence

$$0 \to S_j \to U/V \to S_i \to 0.$$

A subset of $V(\mathscr{G})$ is said to be *arrow closed* if whenever a node S_i is in this subset and there is a directed edge $e(S_i, S_j)$, then S_j is also in this subset. For each submodule U of M, let $c(U)$ be the set of those composition factors which occur in U. Then c is an isomorphism of the lattice of submodules of M and a lattice of subsets of $V(\mathscr{G})$. In particular, a subset of $V(\mathscr{G})$ is arrow closed if and only if it is the image of submodule of M under c. Hence one can recover the lattice of submodules from the module diagram for M. We shall use a blackened cricle to represent a node and a line to represent a directed edge. We shall arrange the nodes in such a way that all the edges are directed downward so that there is no need to indicate their directions.

We are now ready to determine the diagram for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ when $\alpha, \beta \in \mathbf{Z}$. First we note that $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is multiplicity free because it is already multiplicity free as a K-module. We now identify each constituent R_a with the composition factor it defines. Then the vertex set of the module diagram can be taken to be the set $\mathfrak{R}$ of all constituents of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. Let U and V be submodules of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ such that $V \subseteq U$. Suppose that for some constituents R_a and R_b, we have a nonsplit exact sequence

$$0 \to R_b \to U/V \to R_a \to 0. \tag{7.1}$$

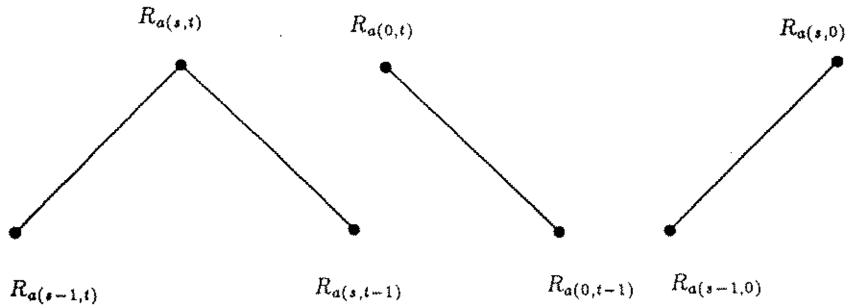

FIGURE 6

By Lemma 6.7, U and V are given as sums of constituents. Since (7.1) is exact, we have $U = V \oplus R_a \oplus R_b$. If $\mathfrak{p}(R_a) \cap R_b = 0$, then $\mathfrak{p}(R_a) \subseteq V \oplus R_a$. Consequently $V \oplus R_a$ is a submodule of U and we have $U/V = (R_a + V)/V \oplus (R_b + V)/V$ and (7.1) splits. On the other hand, if $\mathfrak{p}(R_a) \cap R_b \neq 0$, then the module generated by $(R_a + V)/V$ in U/V is the whole of U/V. Thus (7.1) does not split. Thus there is a directed edge $e(R_a, R_b)$ if and only if $\mathfrak{p}(R_a) \cap R_b \neq 0$ and $a \neq b$. Lemma 6.9 and 6.13 tell us when this occurs.

We shall first examine the case $\alpha + \beta + n - 1 = i$ where i is an integer greater than or equal to $n-1$. Then by Lemma 6.9, there is a directed edge $e(R_{a(s,t)}, R_b)$ if and only if $b = a(s-1, t)$ or $b = (s, t-1)$ if $s \geqslant 1$ and $t \geqslant 1$; $b = a(s-1, 0)$ if $s \geqslant 1$ and $t = 0$; and $b = a(0, t-1)$ if $s = 0$ and $t \geqslant 1$. Graphically we have Fig. 6.

Thus the arrangement of the constituents given in Fig. 2 is a convenient one to represent the diagram for $\mathscr{S}^{\alpha,\beta}(X^{oo})$. It is now clear that the module diagram in this case is an "inverted" Pascal triangle with $n+1$ rows. The diagram from the case $n = 7$ is given in Fig. 7.

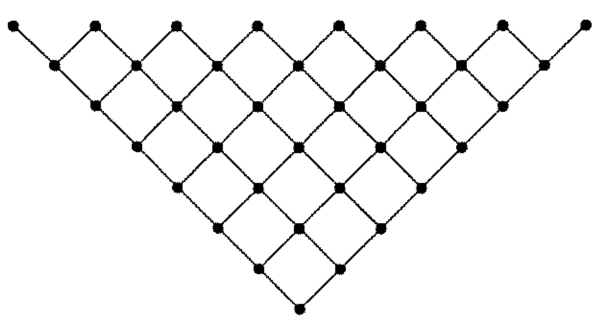

FIGURE 7

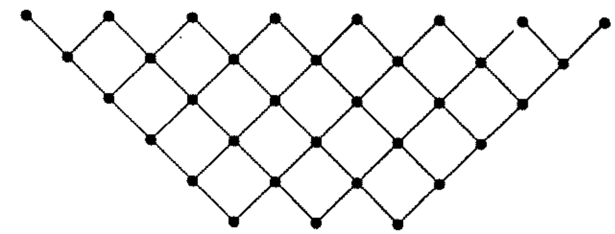

FIGURE 8

FIGURE 9

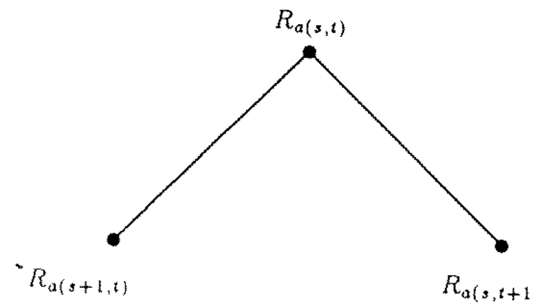

FIGURE 10

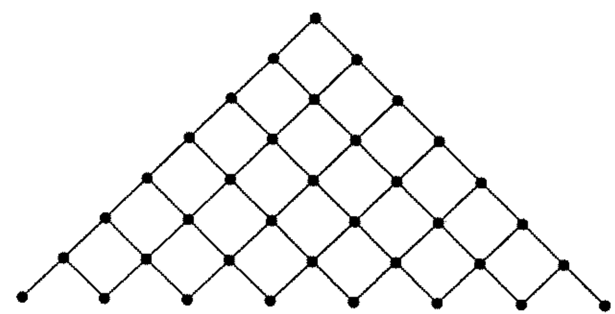

FIGURE 11

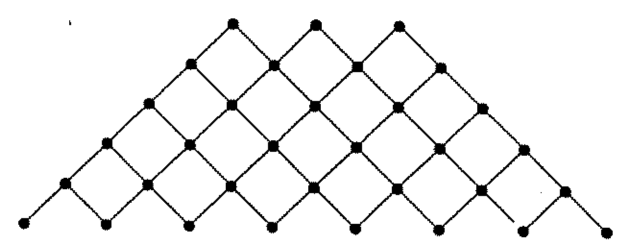

FIGURE 12

The reader can easily modify the dimensions of Fig. 8 to obtain the diagram for other values of n.

If $0 \leqslant i \leqslant n-2$, then the diagram for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ can be obtained from that of the previous case by deleting $(n-i-1)$ rows from the bottom. Thus the diagram for $n=7$ and $i=4$ is given in Fig. 8.

If $\alpha+\beta+n-1=-1$, then we have seen that $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is a direct sum of $n+1$ irreducible submodules. Hence its module diagram is given by $n+1$ nodes with no directed edge. The case $n=7$ is given in Fig. 9.

Next we consider the case $\alpha+\beta+n-1=-i$ where $i \in \mathbf{Z}_+$. By Lemma 6.13, there is a directed edge $e(R_{a(s,t)}, R_b)$ if and only if $s+t \leqslant n-1$ and $b=a(s+1,t)$ or $a(s,t+1)$. Consequently we have the graphical representation in Fig. 10.

Thus in the case $i \geqslant n+1$, the diagram for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is given by an "upright" Pascal triangle with $n+1$ rows, the arrangement of the constituents being given in Fig. 3. The diagram for $n=7$ is given in Fig. 11.

Finally, if $2 \leqslant i \leqslant n$, then the diagram for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is an "upright" Pascal triangle with its top $n-i+1$ rows removed. The diagram for the case $n=7$ and $i=6$ is given in Fig. 12.

8. The Barrier Lines for the Case $n=2$

In the case $n=2$, the K-types of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ can be represented by integral points in $\mathbf{R}^2$ on or below in the line $y=x$. Moreover, the barrier hyperplanes are just lines in $\mathbf{R}^2$. In this section we shall present the graphs of these barrier lines, thus obtaining another graphical representation of the module structure of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. In particular, we shall observe how the gaps between the barrier lines affect the number of composition factors of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. We hope that a comparison of the graphs for the barrier lines and the module diagram in this case will enhance our understanding of the general results in Section 6.

As we shall only consider those $\mathscr{S}^{\alpha,\beta}(X^{oo})$ which are reducible, α and will be assumed to be integers throughout this section. There are four barrier lines (see the explanation after Corollary 6.1), namely

$$l_1^\alpha : x=\alpha, \qquad l_1^\beta : x=-\beta-1, \qquad l_2^\alpha : y=\alpha+1, \qquad l_2^\beta : y=-\beta.$$

Case 1: $\alpha+\beta+1 \geqslant 1$. In this case the points on l_1^α have greater x-coordinates than those on l_1^β. Moreover, the lines l_1^α and l_1^β are at a distance

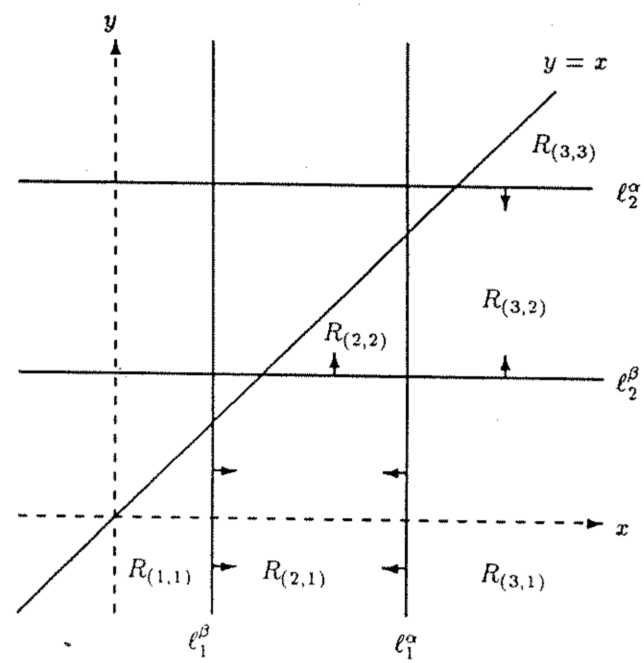

Figure 13

at least 1 unit apart. The same is true for l_2^α and l_2^β. We see from Fig. 13 that the barrier lines partition the set of K-types into six constituents which we call $R_{(1,1)}$, $R_{(2,1)}$, $R_{(3,1)}$, $R_{(2,2)}$, $R_{(3,2)}$, and $R_{(3,3)}$ (see the definition in 6.6). Observe that when the gap between l_1^α and l_1^β is exactly 1 unit, $R_{(2,2)}$ contains exactly one K-type.

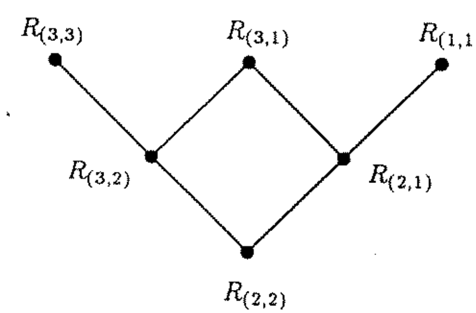

Figure 14

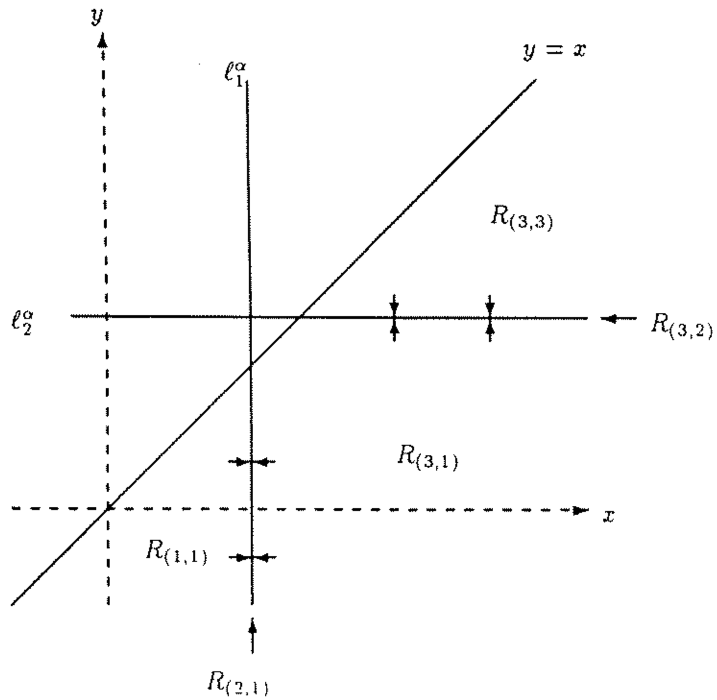

FIGURE 15

The arrows attached to the barrier lines indicate the direction in which the K-type can move. Thus the K-type in $R_{(3,2)}$ and $R_{(2,1)}$ can move into $R_{(2,2)}$ but not the converse. In particular, since the K-types in $R_{(2,2)}$ are trapped by the barrier lines l_1^α and l_2^β, the sum of K-types in $R_{(2,2)}$ forms a submodule of $U(2,2)$. We can now easily deduce the module diagram for $\mathscr{S}^{\alpha,\beta}(X^{oo})$ (Fig. 14) by examining the transition of K-types between other constituents.

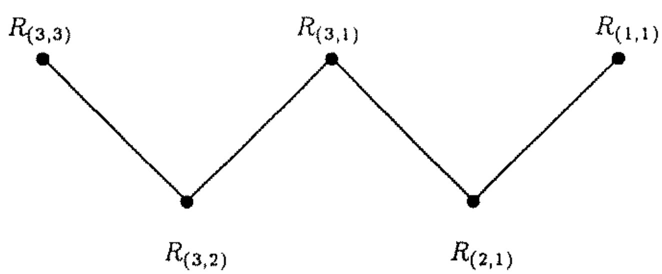

FIGURE 16

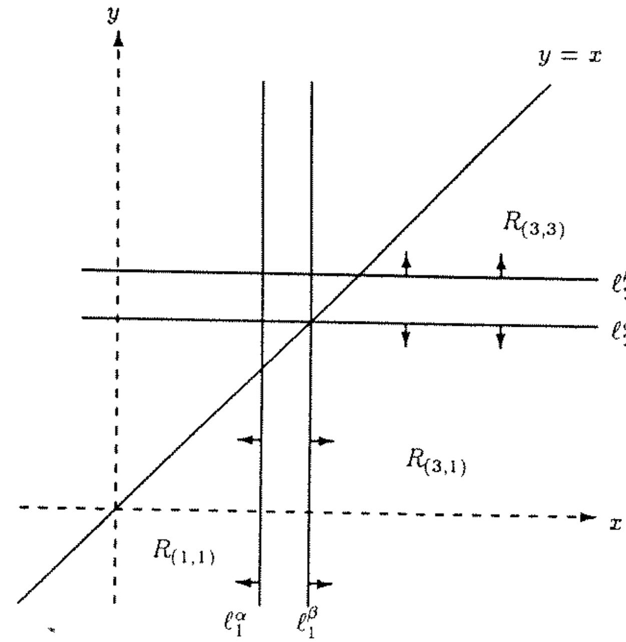

FIGURE 17

Case 2: $\alpha+\beta+1=0$. In this case the lines l_1^α and l_1^β coincide and the lines l_2^α and l_2^β coincide (Fig. 15). Consequently, the constituent $R_{(2,2)}$ collapses. Thus we are left with five constituents (Fig. 16). Note that $R_{(2,1)}$ and $R_{(3,2)}$ are sets of integral points on the lines $x=\alpha$ and $y=\alpha+1$, respectively.

Case 3: $\alpha+\beta+1=-1$. In this case we see that the barrier lines partition the K-types into three constituents and block all attempts to pass them (Figs. 17, 18). Thus each constituent forms a submodule of $\mathscr{S}^{\alpha,\beta}(X^{oo})$. Note that in this case (α,β) is on the unitary axis.

Case 4: $\alpha+\beta+1=-2$. In this case the x-coordinates of the points of l_1^α are 2 units less than those of l_1^β and the number of constituents increases

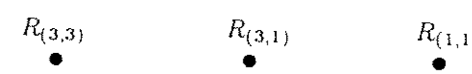

FIGURE 18

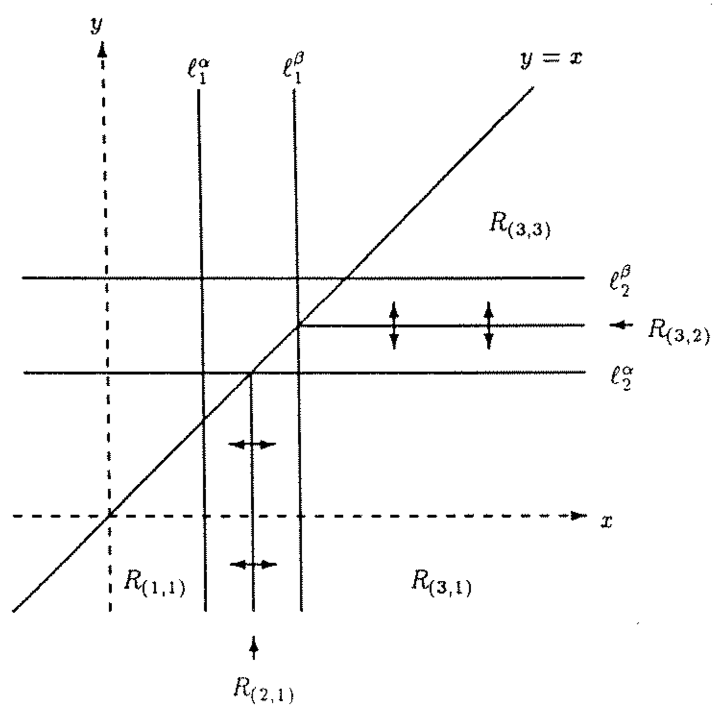

FIGURE 19

to 5 again (Fig. 19). Observer that the module diagram (Fig. 20) is that of Case 2 but with all arrows reversed.

Case 5: $\alpha+\beta+1 \leqslant -3$. In this case the x-coordinates of the points of l_1^β are greater than those of l_1^α and the lines l_1^β and l_1^α are at a distance at least 3 units appart (Fig. 21). We note that the number of constituents increases to 6 and the module diagram is that of case 1 but with all arrows reversed (Fig. 22).

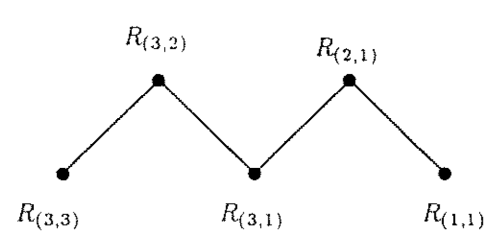

FIGURE 20

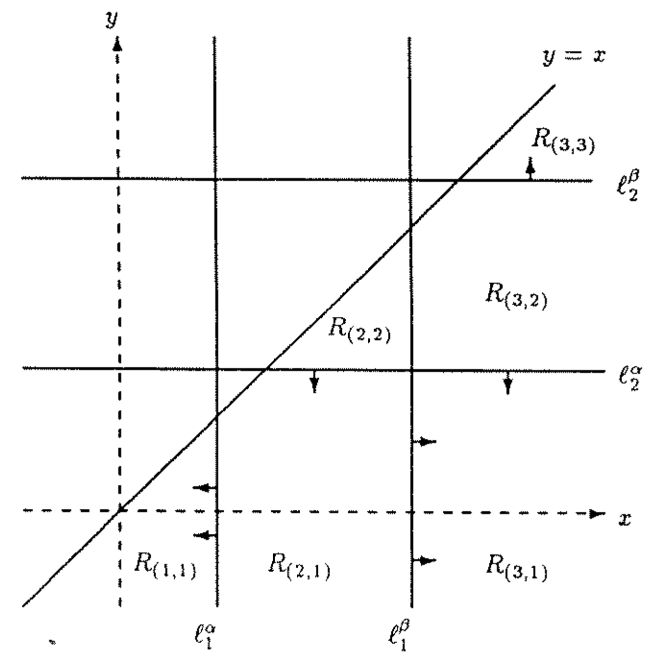

FIGURE 21

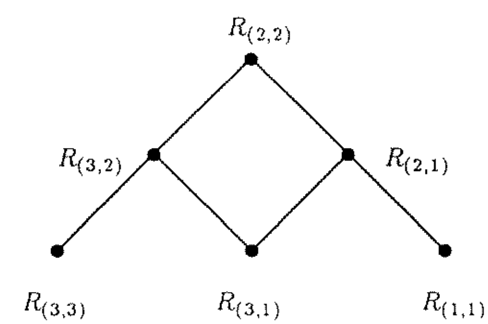

FIGURE 22

9. UNITARITY

In this section, we shall follow the ideas laid down by Howe and Tan in [12] to determine which of the subquotients of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ can define unitary representations.

The modular function P is given by

$$\delta(p)=\delta(m_a n_b)=|\det a|^{2n}, \qquad p=m_a n_b \in P.$$

It follows from the general theory that there is a $U(n,n)$ invariant functional on $S^{-n,-n}(X^{oo}) \cong \operatorname{Ind}_P^{U(n,n)} \delta$ given by

$$l(f)=\int_K f(kx_o)\,dk.$$

For all $f \in \mathscr{S}^{\alpha,\beta}(X^{oo})_K$ and $h \in \mathscr{S}^{-\bar{\beta}-n,-\bar{\alpha}-n}$, $f\bar{h} \in \mathscr{S}^{-n,-n}(X^{oo})$. The map $b(f,h)=l(f\bar{h})$ on $\mathscr{S}^{\alpha,\beta}(X^{oo})_K \times \mathscr{S}^{-\bar{\beta}-n,-\bar{\alpha}-n}(X^{oo})_K$ is sesquilinear and $U(n,n)$ invariant. In particular, if $\alpha+\bar{\beta}=-n$, then b defines a $U(n,n)$ inner product on $\mathscr{S}^{\alpha,\beta}(X^{oo})$. We record this fact in the following proposition.

PROPOSITION 9.1. *If $\alpha+\bar{\beta}=-n$, then the inner product on $\mathscr{S}^{\alpha,\beta}(X^{oo})$ given by*

$$(f_1,f_2)=\int_K f_1(kx_o)\overline{f_2(kx_o)}\,dk$$

is $U(n,n)$ invariant.

Because of Proposition 9.1, we shall call the set of (α,β) satisfying $\alpha+\bar{\beta}=-n$ the *unitary axis*. Hence if $\alpha,\beta \in \mathbf{Z}$ and (α,β) lies on the unitary axis, then $\mathscr{S}^{\alpha,\beta}(X^{oo})$ must be completely reducible. This explains Proposition 6.11.

Remark. For a similar reason, we see that $\mathscr{S}^{-\alpha-n,-\beta-n}(X^{oo})_K$ is isomorphic to the $(\mathfrak{gl}_{2n}(\mathbf{C}),K)$-dual module of $\mathscr{S}^{\alpha,\beta}(X^{oo})_K$. This explains why when $\alpha,\beta \in \mathbf{Z}$, the module diagram (see Section 7) for $\mathscr{S}^{-\alpha-n,-\beta-n}(X^{oo})_K$ can be obtained by reversing the arrows in the diagram of $\mathscr{S}^{\alpha,\beta}(X^{oo})$.

Next we shall determine the *complementary series*. For this discussion, it is convenient to fix $m \in \mathbf{Z}$ and consider $(\alpha,\beta) \in \mathbf{C}^2$ with $\alpha-\beta=m$. We shall denote the restriction of the inner product on $\mathscr{S}^{\alpha,\beta}(X^{oo})$ given in Proposition 9.1 to each K-type V_λ by $(\cdot,\cdot)_\lambda$; i.e.,

$$(f_1,f_2)_\lambda=\int_K f_1(kx_o)\overline{f_2(kx_o)}\,dk \qquad f_1,f_2 \in V_\lambda. \tag{9.2}$$

We note that $(\cdot,\cdot)_\lambda$ is K-invariant. Since V_λ is an irreducible K-module, any K-invariant inner product on V_λ is some multiple of $(\cdot,\cdot)_\lambda$. Thus if $\langle\cdot,\cdot\rangle$ is a $U(n,n)$ inner product on $\mathscr{S}^{\alpha,\beta}(X^{oo})$, then there exists positive constants $\{c_\lambda\}_{\lambda\in\Lambda^+}$ such that

$$\langle f_1,f_2\rangle=c_\lambda(f_1,f_2)_\lambda, \qquad f_1,f_2 \in V_\lambda. \tag{9.3}$$

Moreover, the K-types are mutually orthogonal with respect to the form $\langle\cdot,\cdot\rangle$. Thus $\langle\cdot,\cdot\rangle$ is completely determined by the constants $\{c_\lambda\}_{\lambda\in\Lambda^+}$.

Now by a theorem of Harish-Chandra [10, Theorem 9], if the Lie algebra $\mathfrak{u}(n,n)$ of $U(n,n)$ acts on $\mathscr{S}^{\alpha,\beta}(X^{oo})_K$ by skew-Hermitain operators with respects to the form $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle X\cdot\varphi,\varphi'\rangle+\langle\varphi,X\cdot\varphi'\rangle=0, \qquad \varphi,\varphi' \in \mathscr{S}^{\alpha,\beta}(X^{oo})_K, \qquad X \in \mathfrak{u}(n,n), \tag{9.4}$$

then $\mathscr{S}^{\alpha,\beta}(X^{oo})_K$ can be completed in exactly one way to a unitary representation of $U(n,n)$. In this case we say the $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is *unitarizable*. We recall that $\mathfrak{u}(n,n)$ has a Cartan decomposition given by $\mathfrak{u}(n,n)=\mathfrak{k}\oplus\mathfrak{p}$. Since $\langle\cdot,\cdot\rangle$ is K-invariant, (9.4) automatically holds for all $X \in \mathfrak{k}$. Thus $\mathscr{S}^{\alpha,\beta}(X^{oo})$ is unitarizable if $\forall p \in \mathfrak{p}$, $\varphi,\varphi' \in \mathscr{S}^{\alpha,\beta}(X^{oo})_K$, the condition

$$\langle p\cdot\varphi,\varphi'\rangle+\langle\varphi,p\cdot\varphi'\rangle=0 \tag{9.5}$$

holds. For each $\lambda \in \Lambda^+$, we let

$$\pi_\lambda : \mathscr{S}^{\alpha,\beta}(X^{oo})_K \to V_\lambda$$

be the canonical projection. Since $\mathscr{S}^{\alpha,\beta}(X^{oo})_K=\sum_{\lambda\in\Lambda^+} V_\lambda$ and the K-types are mutually orthogonal, it is sufficient to take φ,φ' from the K-types. Thus we fix $\lambda \in \Lambda^+$ and take $\varphi \in V_\lambda$. Then for $\eta \in \Lambda^+$, $\pi_\eta(p\cdot\varphi)=0$, $\forall\eta \neq \lambda \pm e_j$, $1 \leqslant j \leqslant n$; that is, $p\cdot\varphi$ have zero projection onto those K-types V_η with $\eta \neq \lambda \pm e_j$. Consequently, (9.5) holds for all $p \in \mathfrak{p}$ and all $\varphi,\varphi' \in \mathscr{S}^{\alpha,\beta}(X^{oo})_K$ if it holds for all $p \in \mathfrak{p}$, $\varphi \in V_\lambda$, $\varphi' \in V_{\lambda\pm e_j}$, $\lambda \in \Lambda^+$, and $1 \leqslant j \leqslant n$. In fact, we can even restrict φ' to just $V_{\lambda+e_j}$. This is because for $\varphi \in V_\lambda$ and $\varphi' \in V_{\lambda-e_j}$, we have

$$\langle p\cdot\varphi,\varphi'\rangle+\langle\varphi,p\cdot\varphi'\rangle=\overline{\langle p\cdot\varphi',\varphi\rangle+\langle\varphi',p\cdot\varphi\rangle}.$$

Now by the orthogonality of the K-types and (9.5),

$$c_{\lambda+e_j}(\pi_{\lambda+e_j}(p\cdot\varphi),\varphi')_{\lambda+e_j}+c_\lambda(\varphi,\pi_\lambda(p,\varphi'))_\lambda=0 \tag{9.6}$$

holds for all $p \in \mathfrak{p}$, $\varphi \in V_\lambda$, $\varphi' \in V_{\lambda+e_j}$, $\lambda \in \Lambda^+$, and $1 \leqslant j \leqslant n$. We shall show that (9.6) gives rise to a recursive relation satisfied by the sequence $\{c_\lambda\}_{\lambda\in\Lambda^+}$.

Let Kx_o be the K-orbit of x_o, i.e.,

$$Kx_o=\left\{\begin{pmatrix}u_1\\u_2\end{pmatrix} \in X^{oo} : u_1,u_2 \in U(n)\right\}.$$

We let K act on $C^\infty(Kx_o)$ by $(k\cdot f)(x)=f(k'x)$ for $x\in Kx_o$ and $k\in K$. Let $r=r_{\alpha,\beta}:\mathscr{S}^{\alpha,\beta}(X^{oo})\to C^\infty(Kx_o)$ be the restriction of functions to Kx_o, i.e., $r(f)=f|_{Kx_o}$. Then r is a K-module map and its image is

$$r(\mathscr{S}^{\alpha,\beta}(X^{oo}))=\{f\in C^\infty(Kx_o): f(xa)=(\det a)^m f(x)\ \forall x\in Kx_o,\ a\in U(n)\}$$

where $m=\alpha-\beta$. For each $\lambda\in\Lambda^+$, we set $U_\lambda=r(V_\lambda)$. U_λ has a K-invariant inner product given by

$$(\phi_1,\phi_2)_\lambda=\int_K \phi_1(kx_o)\,\overline{\phi_2(kx_o)}\,dk,\qquad \phi_1,\phi_2\in U_\lambda. \tag{9.7}$$

On the other hand, we can extend the functions in $r(\mathscr{S}^{\alpha,\beta}(X^{oo}))$ back to X^{oo}. Thus for each $\lambda\in\Lambda^+$, we define a map $E_\lambda=E_{\alpha,\beta,\lambda}:U_\lambda\to V_\lambda\subseteq\mathscr{S}^{\alpha,\beta}(X^{oo})$ by

$$E_\lambda(f)(xa)=(\det a)^\alpha\,(\overline{\det a})^\beta f(x),\qquad x\in Kx_o,\qquad a\in GL(n,\mathbf{C}).$$

Then E_λ is a K-isomorphism so that

$$\mathscr{S}^{\alpha,\beta}(X^{oo})_K=\sum_{\lambda\in\Lambda^+}E_\lambda(U_\lambda)$$

and we have

$$(E_\lambda\phi_1,E_\lambda\phi_2)_\lambda=(\phi_1,\phi_2)_\lambda,\qquad \phi_1,\phi_2\in U_\lambda. \tag{9.8}$$

We now fix $\lambda\in\Lambda^+$. Then the map E_λ induces an obvious isomorphism $\Phi=\Phi_\lambda:\mathfrak{p}_{\mathbf{C}}\otimes V_\lambda\to\mathfrak{p}_{\mathbf{C}}\oplus U_\lambda$ given by

$$\Phi(p\otimes E_\lambda(\phi))=p\otimes\phi,\qquad p\in\mathfrak{p}_{\mathbf{C}},\qquad \phi\in U_\lambda.$$

Recall that as a K-module, $\mathfrak{p}_{\mathbf{C}}\otimes V_\lambda$ breaks up into a sum of irreducible submodules of highest weights $(\lambda\pm e_j,\lambda^*+(\alpha-\beta)\mathbf{1}\mp e_k)$, $1\leqslant j,\ k\leqslant n$. Let $\psi^\pm_{jk}$ be the highest weight vector in $\mathfrak{p}_{\mathbf{C}}\otimes V_\lambda$ of weight $(\lambda\pm e_j,\lambda^*+(\alpha-\beta)\mathbf{1}\mp e_k)$. Then in our previous notations, $\psi_{\lambda\pm e_j}=\psi^\pm_{j,n-j+1}$. For $1\leqslant j\leqslant n$, we let

$$c^+(\lambda,j)=(-1)^{n+1}\prod_{a=1}^{j-1}(\lambda_a-\lambda_j+j-a)(\lambda_a-\lambda_j+j-a-1),$$

$$c^-(\lambda,j)=\prod_{a=j+1}^{n}(\lambda_j-\lambda_a+a-j)(\lambda_j-\lambda_a+a-j-1),$$

and define the K-map $T^\pm_{\lambda,j}:\mathfrak{p}_{\mathbf{C}}\otimes U_\lambda\to U_{\lambda\pm e_j}$ by setting

$$T^\pm_{\lambda,j}\Phi(\psi_{\lambda\pm e_j})=c^\pm(\lambda,j)\,r(\xi_{\lambda\pm e_j}) \tag{9.9}$$

and sending other highest weight vectors in $\mathfrak{p}_{\mathbf{C}}\otimes U_\lambda$ to 0.

LEMMA 9.10. *For $p\in\mathfrak{p}$, $\phi\in U_\lambda$, and $1\leqslant j\leqslant n$, we have*

$$\pi_{\lambda+e_j}(p\cdot E_\lambda(\phi))=(\alpha-\lambda_j+j-1)\,E_{\lambda+e_j}T^+_{\lambda,j}(p\otimes\phi),$$
$$\pi_{\lambda-e_j}(p\cdot E_\lambda(\phi))=(\beta+\lambda_j+n-j)\,E_{\lambda-e_j}T^-_{\lambda,j}(p\otimes\phi).$$

Proof. There are $X^\pm_{kl}\in\mathscr{U}(\mathfrak{k}_{\mathbf{C}})$, $1\leqslant k,\ l\leqslant n$ such that

$$p\otimes\phi=\sum_{k,l}\{X^+_{kl}\Phi(\psi^+_{kl})+X^-_{kl}\Phi(\psi^-_{kl})\}.$$

Then by (9.9),

$$T^\pm_{\lambda,j}(p\otimes\phi)=c^\pm(\lambda,j)\,X^\pm_{j,n-j+1}\,r(\xi_{\lambda\pm e_j}),\qquad \forall 1\leqslant j\leqslant n.$$

It follows that

$$\begin{aligned}
p\cdot E_\lambda(\phi)&=m_\lambda(p\otimes E_\lambda(\phi))\\
&=\sum_{k,l}\{X^+_{kl}m_\lambda(\psi^+_{kl})+X^-_{kl}m_\lambda(\psi^-_{kl})\}\\
&=\sum_j\{(\alpha-\lambda+j-1)\,c^+(\lambda,j)\,X^+_{j,n+1}\xi_{\lambda+e_j}\\
&\quad+(\beta+\lambda_j+n-j)\,c^-(\lambda,j)\,X^-_{j,n-j+1}\xi_{\lambda-e_j}\}\\
&=\sum_j\{(\alpha-\lambda_j+j-1)\,E_{\lambda+e_j}T^+_{\lambda,j}(p\otimes\phi)\\
&\quad+(\beta+\lambda_j+n-j)\,E_{\lambda-e_j}T^-_{\lambda,j}(p\otimes\phi)\}. \qquad\blacksquare
\end{aligned}$$

If the inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{S}^{\alpha,\beta}(X^{oo})$ defined by a sequence of positive constants $\{c_\lambda\}_{\lambda\in\Lambda^+}$ via (9.3) is $U(n,n)$ invariant, then by (9.6), (9.8), and Lemma 9.10, we have for $\phi\in U_\lambda$ and $\phi'\in U_{\lambda+e_j}$,

$$\begin{aligned}
&(\alpha-\lambda_j+j-1)\,c_{\lambda+e_j}(T^+_{\lambda,j}(p\otimes\phi),\phi')_{\lambda+e_j}\\
&\quad+(\bar\beta+\lambda_j+n-j+1)\,c_\lambda(\phi,T^-_{\lambda+e_j,j}(p\otimes\phi'))_\lambda=0.
\end{aligned}\tag{9.11}$$

LEMMA 9.12.

$$(T^+_{\lambda,j}(p\otimes\phi),\phi')_{\lambda+e_j}=(\phi,T^-_{\lambda+e_j,j}(p\otimes\phi'))_\lambda.$$

Proof. We choose (α, β) from the unitary axis with the property $\alpha, \beta \notin \mathbf{Z}$ (and $\alpha - \beta = m$). Then by Proposition 9.1, by putting $c_\lambda = 1$, $\forall \lambda \in \Lambda^+$, we obtain a $U(n, n)$ invariant inner product on $\mathscr{S}^{\alpha, \beta}(X^{oo})$. It follows from (9.11) that

$$(\alpha - \lambda_j + j - 1)(T^+_{\lambda, j}(p \otimes \phi), \phi')_{\lambda + e_j}$$
$$+ (\bar{\beta} + \lambda_j + n - j + 1)(\phi, T^-_{\lambda + e_j}(p \otimes \phi'))_\lambda = 0.$$

Since $\alpha + \bar{\beta} = -n$, $(\bar{\beta} + \lambda_j + n - j + 1) = -(\alpha - \lambda_j + j - 1)$, whence

$$(\alpha + \lambda_j + j - 1)\{(T^+_{\lambda, j}(p \otimes \phi), \phi')_{\lambda + e_j} - (\phi, T^-_{\lambda + e_j, j}(p \otimes \phi'))_\lambda\} = 0.$$

We have chosen $\alpha \notin \mathbf{Z}$ so that $(\alpha + \lambda_j + j - 1) \neq 0$. Hence

$$(T^+_{\lambda, j}(p \otimes \phi), \phi')_{\lambda + e_j} - (\phi, T^-_{\lambda + e_j, j}(p \otimes \phi'))_\lambda = 0. \quad \blacksquare$$

Hence (9.11) is reduced to the folowing equation:

$$(\alpha - \lambda_j + j - 1)\, c_{\lambda + e_j} + (\bar{\beta} + \lambda_j + n - j + 1)\, c_\lambda = 0. \tag{9.13}$$

We now let

$$N_{\lambda, j} = \frac{\alpha - \lambda_j + j - 1}{\bar{\beta} + \lambda_j + n - j + 1} = -\frac{c_\lambda}{c_{\lambda + e_j}}. \tag{9.14}$$

Observation. If $\mathscr{S}^{\alpha, \beta}(X^{oo})$ is irreducible, i.e., when $\alpha, \beta \notin \mathbf{Z}$, then it is unitarizable if and only if the transition coefficients $N_{\lambda, j}$ are all negative. In this case, by setting $c_0 = 1$, one obtains the values of all c_λ by (9.14).

We now let $\tilde{\alpha} = \alpha - (m - n)/2$. Then since $\alpha - \beta = m$, we have

$$N_{\lambda, j} = \frac{\tilde{\alpha} + ((m - n)/2 - \lambda_j + j - 1)}{\bar{\tilde{\alpha}} - ((m - n)/2 - \lambda_j + j - 1)}.$$

Thus a necessary condition for all $N_{\lambda, j}$ to be real is that either $\tilde{\alpha}$ is real or $\tilde{\alpha}$ is purely imaginary. The latter is just the unitary axis. Thus we shall assume that $\tilde{\alpha}$ is real. If $m \equiv n \pmod 2$, then there exists λ and j for which $(m - n)/2 - \lambda_j + j - 1 = 0$. In this case $N_{\lambda, j} = 1 > 0$ so that there is no complementary series. On the other hand, if $m - n \equiv 1 \pmod 2$, then $(m - n)/2 - \lambda_j + j - 1$ is always nonzero and

$$N_{\lambda, j} < 0 \Leftrightarrow \tilde{\alpha}^2 < \left(\frac{m - n}{2} - \lambda_j + j - 1\right)^2. \tag{9.15}$$

Since the minimal value of $|(m - n)/2 - \lambda_j + j - 1|$ is $\frac{1}{2}$, $N_{\lambda, j} < 0$ for all $\lambda \in \Lambda^+$ and $1 \leqslant j \leqslant n$ if and only if $|\tilde{\alpha}| < \frac{1}{2}$. Hence in this case there is a complementary series given by $|\alpha - (m - n)/2| < \frac{1}{2}$.

THEOREM 9.16. *Let $\alpha - \beta = m$. If $m - n \equiv 1 \pmod 2$, then the family $\{\mathscr{S}^{\alpha, \beta}(X^{oo}) : \alpha$ real, $|\alpha - (m - n)/2| < \frac{1}{2}\}$ is unitarizable.*

COROLLARY 9.17. *The spherical case (i.e., $\alpha = \beta$) has a complementary series if and only if n is odd.*

Finally, we want to investigate in the case when $\mathscr{S}^{\alpha, \beta}(X^{oo})$ is reducible which of its constituents possess a $U(n, n)$ invariant inner product. If $\alpha + \beta + n - 1 \geqslant 0$, we note that $N_{\lambda, j} < 0$ if only if $\lambda \in A^j_1 \cup A^j_3$. In the case $\alpha + \beta + n - 1 < 0$, $N_{\lambda, j} < 0$ if and only if $\lambda \in B^j_1 \cup B^j_3$. We can now formulate the following criterion for unitarity.

Criterion for unitarity. Let R_a be a constituent $\mathscr{S}^{\alpha, \beta}(X^{oo})$. Then R_a has a $U(n, n)$ invariant inner product if and only if one of the following holds:

(1) $l_2(a) = 0$.

(2) Suppose that $a = a(s, t)$ with $s + t < n$ and there is a constant $c(a) \in \mathbf{Z}$ such that for all $\lambda \in R_a$, we have $\lambda_k = c(a)$, $\forall s + 1 \leqslant k \leqslant t - 1$.

Condition (1) says that R_a does not intersect any A^j_2 (or B^j_2) so that all $N_{\lambda, j}$ are negative. Condition (2) says that if R_a intersects with some A^j_2 (or B^j_2), then the intersection should be "minimal" or else transition of a K-type in the λ_j-direction will violate unitarity. By Lemma 6.15 and Lemma 6.16 we know when condition (2) above does occur. For $\alpha + \beta + n - 1 = i$, $0 \leqslant i \leqslant n - 1$, the constituents R_a which satisfy condition (2) are those with $l_2(a) = i + 1$. They correspond to the lowest row in the module diagram. For $\alpha + \beta + n - 1 = -i$, $2 \leqslant i \leqslant n$, these consistuents are those with $l_2(a) = i - 1$. They correspond to the first row in the module

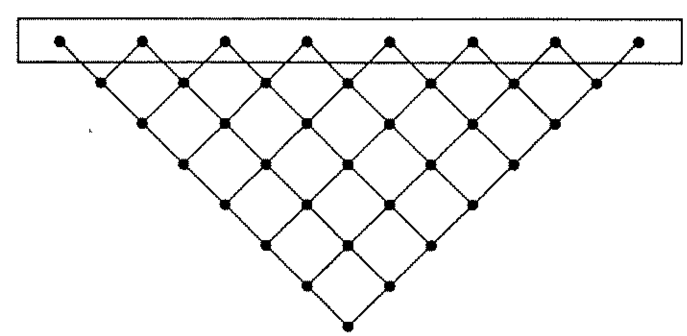

FIGURE 23

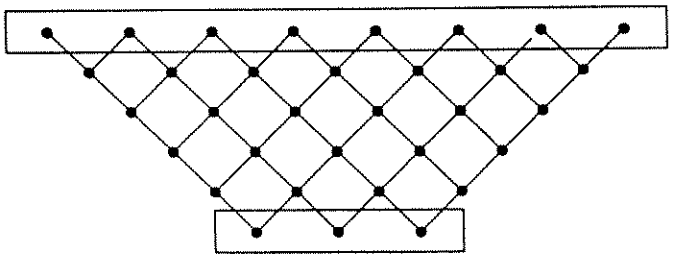

FIGURE 24

diagram. We now summarise these facts in the following theorem. The diagrams given in Figs. 23–26 are for the case $n=7$.

THEOREM 9.18. *Let $\alpha, \beta \in \mathbf{Z}$ and $\alpha+\beta+n-1=i$ where $i \in \mathbf{Z}$.*

(a) *If $i \geqslant n$, then the unitary constituents are those R_a with $l_2(a)=0$* (Fig. 23).

(b) *If $0 \leqslant i \leqslant n-1$, then the unitary constituents are those R_a with $l_2(a)=0$ or $l_2(a)=i+1$* (Fig. 24).

(c) *If $i=-1$, then all of the $n+1$ constituents of $\mathscr{S}^{\alpha,\beta}(X^{oo})$ are unitary. Note that this is part of the unitary axis.*

(d) *If $-(n+1) \leqslant i \leqslant -2$, then the unitary constituents are those R_a with $l_2(a)=0$ or $l_2(a)=-i-1$* (Fig. 25).

(e) *If $i \leqslant -(n+2)$, then the unitary constituents are those R_a with $l_2(a)=0$* (Fig. 26).

The readers may like to identify the unitary constituents for the case $n=2$ in the K-type plane as given in Figs. 13, 15, 17, 19, and 21. For instance, for $\alpha+\beta+1 \geqslant 2$, the three constituents R_{11}, R_{31}, and R_{33} at the "corners" of Fig. 13 are unitary.

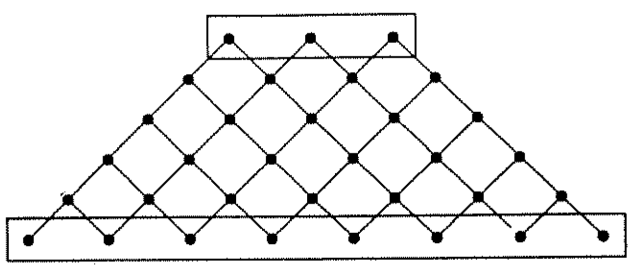

FIGURE 25

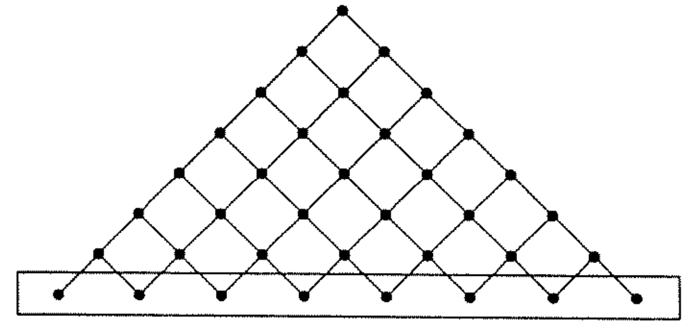

FIGURE 26

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K-Homology, Asymptotic Representations, and Unsuspended E-Theory

Marius Dădărlat

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

AND

Terry A. Loring*

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131

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Connes and Higson defined a bivariant homology theory $E(A, B)$ for separable C^*-algebras. The elements of $E(A, B)$ are taken to be homotopy classes of asymptotic momorphisms from $SA \otimes \mathscr{K}$ to $SB \otimes \mathscr{K}$. In symbols $E(A, B) \cong [[SA, SB \otimes \mathscr{K}]]$. If A is K-nuclear then E-theory agrees with Kasparov's bivariant K-theory. We show that, in many cases, one need not take suspensions to calculate the E-theory group $E(A, B)$. For many A, we show

$$E(A, B) \cong [[A, B \otimes \mathscr{K}]]$$

for all B. Among the A for which this is true are $C_0(X \setminus \{pt\})$ for X with the homotopy type of a finite, connected CW complex. This gives a concrete realization of K-homology, related to the Brown–Douglas–Fillmore description. For example, $K_0(X)$ arises as asymptotic representations,

$$K_0(X) \cong [[C_0(X), \mathscr{K}]].$$

Other A for which our isomorphic holds include the nonunital dimension-drop intervals. In this case, there is no distinction between $*$-homomorphisms and asymptotic morphisms so we have succeeded in classifying all $*$-homomorphisms from a dimension-drop interval to a stable C^*-algebra. This was subsequently used by Elliott (*J. Reine Angew. Math.* **443** (1993), 179–219) in the classification of certain C^*-algebras. The dimension-drop interval may also be used to describe $K_*(X; \mathbf{Z}/n)$ in terms of paths of asymptotic representations of $C_0(X)$. © 1994 Academic Press, Inc.

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