

DEGENERATE AFFINE HECKE-CLIFFORD ALGEBRAS AND TYPE Q LIE SUPERALGEBRAS

DAVID HILL, JONATHAN R. KUJAWA, AND JOSHUA SUSSAN

ABSTRACT. We construct the finite dimensional simple integral modules for the (degenerate) affine Hecke-Clifford algebra (AHCA), $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. Our construction includes an analogue of Zelevinsky's segment representations, a complete combinatorial description of the simple calibrated $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -modules, and a classification of the simple integral $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -modules. Our main tool is an analogue of the Arakawa-Suzuki functor for the Lie superalgebra $\mathfrak{q}(n)$.

1. INTRODUCTION

1.1. Throughout this paper, we will work over the ground field $\mathbb{C}$. As is well known, the symmetric group, S_d , has a non-trivial *central extension*:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widehat{S}_d \longrightarrow S_d \longrightarrow 1.$$

The double cover $\widehat{S}_d$ is generated by elements $\zeta, \hat{s}_1, \dots, \hat{s}_{d-1}$, where ζ is central, $\zeta^2 = 1$, and the $\hat{s}_i$ satisfy the relations $\hat{s}_i \hat{s}_{i+1} \hat{s}_i = \hat{s}_{i+1} \hat{s}_i \hat{s}_{i+1}$ and $\hat{s}_j \hat{s}_i = \zeta \hat{s}_i \hat{s}_j$ for admissible i and j satisfying $|i - j| > 1$. The *projective* or *spin* representations of S_d are the linear representations of $\widehat{S}_d$ which factor through $\mathbb{C}\widehat{S}_d/(\zeta + 1)$. This paper is a study of some structures arising from the projective representation theory of symmetric groups.

The double cover $\widehat{S}_d$ suffers a defect: it is difficult to define parabolic induction, see [43, Section 4]. Since the inductive approach to the study of linear representations of the symmetric group is so effective, it is preferable to study the *Sergeev algebra* $\mathcal{S}(d)$ introduced in [41, 29], which provides a natural fix to this problem. As a vector space, $\mathcal{S}(d) = \mathcal{C}\ell(d) \otimes \mathbb{C}S_d$, where $\mathcal{C}\ell(d)$ is the 2^d -dimensional Clifford algebra with generators $c_1, \dots, c_d$ subject to the relations $c_i^2 = -1$ and $c_i c_j = -c_j c_i$ for $i \neq j$, and $\mathbb{C}S_d$ is the group algebra of S_d . Let $s_i = (i, i + 1) \in S_d$ be the i th basic transposition, and identify $\mathcal{C}\ell(d)$ and $\mathbb{C}S_d$ with the subspaces $\mathcal{C}\ell(d) \otimes 1$ and $1 \otimes \mathbb{C}S_d$ respectively. Multiplication is defined so that $\mathcal{C}\ell(d)$ and $\mathbb{C}S_d$ are subalgebras, and $wc_i = c_{w(i)}w$ for all $1 \leq i \leq d$ and $w \in S_d$. The Sergeev algebra admits a natural definition of parabolic induction and the projective representation theory of the symmetric group can be recovered from that of $\mathcal{S}(d)$, [5, Theorem 3.4].

Additionally, the Sergeev algebra is a *superalgebra*, and plays the role of the symmetric group for a super version of Schur-Weyl duality known as Sergeev duality in honor of A. N. Sergeev who extended the classical theorem of Schur and Weyl [41]. If $V = \mathbb{C}^{n|n}$ is the standard representation of the Lie superalgebra $\mathfrak{q}(n)$, then both $\mathcal{S}(d)$ and $\mathfrak{q}(n)$ act on the tensor product $V^{\otimes d}$ and each algebra

Date: May 5, 2009.

2000 *Mathematics Subject Classification.* Primary 20C08, 20C25; Secondary 17B60, 17B20, 17B37.

Research of the second author was partially supported by NSF grant DMS-0734226. Research of the first and third author was partially supported by NSF EMSW21-RTG grant DMS-0354321.

is the commutant algebra of the other. In particular, there exists an isomorphism of superalgebras

$$\mathcal{S}(d) \rightarrow \text{End}_{\mathfrak{q}(n)}(V^{\otimes d}).$$

The algebra $\mathcal{S}(d)$ admits an affinization, $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$, called the (degenerate) affine Hecke-Clifford algebra (AHCA). The affine Hecke-Clifford algebra was introduced by Nazarov in [29] and studied in [29, 6, 24, 49]. As a vector space, $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d) = \mathcal{P}_d[x] \otimes \mathcal{S}(d)$, where $\mathcal{P}_d[x] = \mathbb{C}[x_1, \dots, x_d]$. We identify $\mathcal{P}_d[x]$ and $\mathcal{S}(d)$ with the subspaces $\mathcal{P}_d[x] \otimes 1$ and $1 \otimes \mathcal{S}(d)$. Multiplication is defined so that these are subalgebras, $c_i x_j = x_j c_i$ if $j \neq i$, $c_i x_i = -x_i c_i$, $s_i x_j = x_j s_i$ if $j \neq i, i+1$, and

$$s_i x_i = x_{i+1} s_i - 1 + c_i c_{i+1}.$$

In addition to $\mathcal{S}(d)$ being a subalgebra of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$, there also exists a natural surjection $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d) \twoheadrightarrow \mathcal{S}(d)$ obtained by mapping $x_1 \mapsto 0$, $c_i \mapsto c_i$ and $s_i \mapsto s_i$. Therefore, the representation theory of the AHCA contains that of the Sergeev algebra.

Surprisingly little is explicitly known about the representation theory of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$, in contrast with its linear counterpart, the (degenerate) affine Hecke algebra $\mathcal{H}^{\text{aff}}(d)$. The most significant contribution to the projective theory is from [6, 24], which describe the Grothendieck group of the full subcategory of *integral* $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -modules in terms of the crystal graph associated to a maximal nilpotent subalgebra of $\mathfrak{b}_{\infty}$ (or, more generally, $A_{2\ell}^{(2)}$ if working over a field of odd prime characteristic $2\ell - 1$). We will return to this important topic later on.

The algebra $\mathcal{H}^{\text{aff}}(d)$ has been studied for many years. Of particular interest are those modules for $\mathcal{H}^{\text{aff}}(d)$ which admit a generalized weight space decomposition with respect to the polynomial generators. It is known that among these modules it is enough to consider those for which the generalized eigenvalues of the polynomial generators are integers, cf. [24, §7.1]. These are known as *integral modules*. As discovered in [29], the appropriate analogue of integral modules for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ are those which admit a generalized weight space decomposition with respect to the x_i^2 , and the generalized eigenvalues of the x_i^2 are of the form $q(a) := a(a+1)$, $a \in \mathbb{Z}$.

The finite dimensional, irreducible, integral modules for $\mathcal{H}^{\text{aff}}(d)$ were classified by Zelevinsky in [51] via combinatorial objects known as multisegments. A segment is an interval $[a, b] \in \mathbb{Z}$. To each segment $[a, b]$ with $d = b - a + 1$, Zelevinsky associates a 1-dimensional $\mathcal{H}^{\text{aff}}(d)$ -module $\mathbb{C}_{[a, b]}$ defined from the trivial representation of $\mathbb{C}S_d$ by letting x_1 act by the scalar a . A multisegment may be regarded as a pair of compositions $(\beta, \alpha) = ((b_1, \dots, b_n), (a_1, \dots, a_n)) \in \mathbb{Z}^n \times \mathbb{Z}^n$, with $d_i = b_i - a_i \geq 0$. If $d = d_1 + \dots + d_n$, Zelevinsky associates to the multisegment (β, α) a *standard cyclic* $\mathcal{H}^{\text{aff}}(d)$ -module

$$\mathcal{M}(\beta, \alpha) = \text{Ind}_{\mathcal{H}^{\text{aff}}(d_1) \otimes \dots \otimes \mathcal{H}^{\text{aff}}(d_n)}^{\mathcal{H}^{\text{aff}}(d)} \mathbb{C}_{[a_1, b_1-1]} \boxtimes \dots \boxtimes \mathbb{C}_{[a_n, b_n-1]}.$$

To explain the classification, let $P = \mathbb{Z}^n$ be the weight lattice associated to $\mathfrak{gl}_n(\mathbb{C})$, P^+ the dominant weights, and $\rho = (n-1, \dots, 1, 0)$. Additionally, define the weights

$$P_{\geq 0}(d) = \{\mu \in \mathbb{Z}_{\geq 0}^n \mid \mu_1 + \dots + \mu_n = d\} \quad \text{and} \quad P^+[\lambda] = \{\mu \in P \mid \mu_i \geq \mu_j \text{ whenever } \lambda_i = \lambda_j\}.$$

Given $\lambda \in P^+$, let

$$\mathcal{B}_d[\lambda] = \{\mu \in P^+[\lambda] \mid \lambda - \mu \in P_{\geq 0}(d)\}, \tag{1.1.1}$$

and

$$\mathcal{A}_d = \{(\lambda, \mu) \mid \lambda \in P^+, \text{ and } \mu \in \mathcal{B}_d[\lambda + \rho]\}.$$

Then, the set $\{\mathcal{L}(\beta, \alpha) \mid (\beta, \alpha) \in \mathcal{A}_d\}$ is a complete list of irreducible integral $\mathcal{H}^{\text{aff}}(d)$ -modules.

In the case of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$, the situation is more subtle. To describe this, fix a segment $[a, b]$. The obvious analogue of the trivial representation of $\mathbb{C}S_d$ is the 2^d -dimensional basic spin representation $\mathcal{C}\ell_d = \mathcal{C}\ell(d).1$ of $\mathcal{S}(d)$. If $a = 0$, the action of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ factors through $\mathcal{S}(d)$ and it can be checked that $\mathcal{C}\ell_d$ is the desired segment representation. If $a \neq 0$, it is not immediately obvious how to proceed. Inspiration comes from a *rank 1* application of the functor described below. We define a module structure on the *double* of $\mathcal{C}\ell_d$: $\hat{\Phi}_{[a,b]} = \Phi_a \otimes \mathcal{C}\ell_d$, where Φ_a is a 2-dimensional Clifford algebra. The module $\hat{\Phi}_{[a,b]}$ is not irreducible, but decomposes as a direct sum of irreducibles $\Phi_{[a,b]}^+ \oplus \Phi_{[a,b]}^-$, where $\Phi_{[a,b]}^+$ and $\Phi_{[a,b]}^-$ are isomorphic via an *odd* isomorphism. Let $\Phi_{[a,b]}$ denote one of these simple summands. Now, given a multisegment (λ, μ) , with $\lambda_i - \mu_i = d_i$ and $d = d_1 + \cdots + d_n$, we define the standard cyclic module

$$\mathcal{M}(\lambda, \mu) = \text{Ind}_{\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d_1) \otimes \cdots \otimes \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d_n)}^{\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)} \Phi_{[\mu_1, \lambda_1 - 1]} \circledast \cdots \circledast \Phi_{[\mu_n, \lambda_n - 1]},$$

where $\circledast$ is an analogue of the outer tensor product adapted for superalgebras, see section 2 below.

A weight $\lambda \in P$ is called typical if $\lambda_i + \lambda_j \neq 0$ for all $i \neq j$. Let

$$P^{++} = \{\lambda \in P^+ \mid \lambda_1 \geq \cdots \geq \lambda_n, \text{ and } \lambda_i + \lambda_j \neq 0 \text{ for all } i \neq j\}$$

be the set of dominant typical weights. We prove

Theorem. *Assume that $\lambda \in P^{++}$ and $\mu \in \mathcal{B}_d[\lambda]$. Then, $\mathcal{M}(\lambda, \mu)$ has a unique simple quotient, denoted $\mathcal{L}(\lambda, \mu)$.*

In the special case where the multisegment (λ, μ) corresponds to skew shapes (i.e. $\lambda, \mu \in P^+$), the associated $\mathcal{H}^{\text{aff}}(d)$ -modules are called calibrated. The calibrated representations may also be characterized as those modules on which the polynomial generators act semisimply, and were originally classified by Cherednik in [9]. In [35], Ram gives a complete combinatorial description of the calibrated representations of $\mathcal{H}^{\text{aff}}(d)$ in terms of skew shape tableaux and provides a complete classification (see also [25] for another combinatorial model).

The projective analogue of the skew shapes are the shifted skew shapes which have appeared already in [42, 43] and correspond to when λ and μ are *strict* partitions. As in the linear case, these are the modules for which the x_i act semisimply. In the spirit of [35], we prove that

Theorem. *For each shifted skew shape λ/μ , where λ and μ are strict partitions such that λ contains μ , there is an irreducible $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module $H^{\lambda/\mu}$. Every irreducible, calibrated $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module is isomorphic to exactly one such $H^{\lambda/\mu}$.*

The $H^{\lambda/\mu}$ are constructed directly using the combinatorics of shifted skew shapes. Furthermore, we show that $H^{\lambda/\mu} \cong \mathcal{L}(\lambda, \mu)$. We would also like to point out that Wan, [48], has recently obtained a classification of the calibrated representations for $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ over any arbitrary algebraically closed field of characteristic not equal to 2.

The appearance of the weight lattice for $\mathfrak{gl}_n(\mathbb{C})$ in the representation theory of $\mathcal{H}^{\text{aff}}(d)$ is explained by a work of Arakawa and Suzuki who introduced in [1] a functor from the BGG category $\mathcal{O}(\mathfrak{gl}_n)$ to

the category of finite dimensional representations of $\mathcal{H}^{\text{aff}}(d)$. The authors proved that the functor maps Verma modules to the standard modules or zero. Using the Kazhdan-Lusztig conjecture together with the results of [14], they proved that simple objects in $\mathcal{O}(\mathfrak{gl}_n)$ are mapped by the functor to simple modules or zero. In [44], Suzuki avoided the Kazhdan-Lusztig conjecture, and proved that the functor maps simples to simples using Zelevinsky's classification together with the existence of a nonzero $\mathcal{H}^{\text{aff}}(d)$ -contravariant form on certain standard modules, see [36]. In [45], Suzuki was able to avoid the results of Zelevinsky and independently reproduce the classification via a careful analysis of the standard modules. For a complete explanation of the functor in type A , we refer the reader to [31].

The functor and related constructions have had numerous applications in various areas of representation theory. This includes the study of affine Braid groups and Hecke algebras [31], Yangians [23], the centers of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_n$ [4], finite W-algebras [8], and the proof of Broué's abelian defect conjecture for symmetric groups by Chuang and Rouquier via $\mathfrak{sl}_2$ categorification [11].

We define an analogous functor from the category $\mathcal{O}(\mathfrak{q}(n))$ to the category of finite dimensional modules for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. The construction of this functor relies on the following key result:

Theorem. *Let M be a $\mathfrak{q}(n)$ -supermodule. Then, there exists a homomorphism*

$$\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d) \rightarrow \text{End}_{\mathfrak{q}(n)}(M \otimes V^{\otimes d}).$$

To define the functor, let $\mathfrak{q}(n) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the triangular decomposition of $\mathfrak{q}(n)$. For each $\lambda \in P$, the functor

$$F_\lambda : \mathcal{O}(\mathfrak{q}(n)) \rightarrow \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)\text{-mod}$$

is defined by

$$F_\lambda M = \{ m \in M \mid \mathfrak{n}^+.m = 0 \text{ and } hv = \lambda(h)v \text{ for all } h \in \mathfrak{h} \}.$$

The functor F_λ is exact when $\lambda \in P^{++}$.

The dimension of the highest weight space of a Verma module in $\mathcal{O}(\mathfrak{q}(n))$ is generally greater than one. A consequence of this is that the functor maps a Verma module to a direct sum of the same standard module. A simple object in $\mathcal{O}(\mathfrak{q}(n))$ is mapped to a direct sum of the same simple module or else zero. Determining when a simple object is mapped to something non-zero is a more difficult question than in the non-super case and we have only partial results in this direction. The main difficulty is a lack of information about the category $\mathcal{O}(\mathfrak{q}(n))$. The category of finite dimensional representations of $\mathfrak{q}(n)$ has been studied by Penkov and Serganova [32, 33, 34]; they give a character formula for all finite dimension simple $\mathfrak{q}(n)$ -modules. Using other methods, Brundan [3] has also studied this category, and has even obtained some (conjectural) information about the whole category $\mathcal{O}(\mathfrak{q}(n))$ via the theory of crystals. The most useful information, however, comes from Gorelik [15], who defines the Shapovalov form for Verma modules and calculates the linear factorization of its determinant.

In various works by Ariki, Grojnowski, Vazirani, and Kleshchev [2, 17, 47, 24] it was shown that there is an action of $U(\mathfrak{gl}_\infty)$ on the direct sum of Grothendieck groups of the categories of integral $\mathcal{H}^{\text{aff}}(d)$ -modules, for all d . This gives another type of classification of the simple integral modules as nodes on the crystal graph associated to a maximal nilpotent subalgebra of $\mathfrak{gl}_\infty$. In

[5], Brundan and Kleshchev show there is a classification of the simple integral modules for $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ parameterized by the nodes of the crystal graph associated to a maximal nilpotent subalgebra of $\mathfrak{b}_\infty$, see also [24].

In [27], Leclerc studied dual canonical bases of the quantum group $\mathcal{U}_q(\mathfrak{g})$ for various finite dimensional simple Lie algebras $\mathfrak{g}$ via embeddings of the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{n})$ of a maximal nilpotent subalgebra $\mathfrak{n} \subseteq \mathfrak{g}$ in the *quantum shuffle algebra*. To describe the quantum shuffle algebra associated to $\mathfrak{g}$ of rank r , let $\mathcal{F}$ be the free associative algebra on the letters $[0], \dots, [r-1]$, and let $[i_1, i_2, \dots, i_k] := [i_1] \cdot [i_2] \cdots [i_k]$. Then, the quantum shuffle algebra is the algebra $(\mathcal{F}, *)$, where

$$[i_1, \dots, i_k] * [i_{k+1}, \dots, i_{k+\ell}] = \sum_{\sigma} q^{-e(\sigma)} [i_{\sigma(1)}, \dots, i_{\sigma(k+\ell)}],$$

where the sum is over all minimal length coset representatives in $S_{k+\ell}/(S_k \times S_\ell)$, and $e(\sigma)$ is some explicit function of σ . There exists an *injective* homomorphism $\Psi : \mathcal{U}_q(\mathfrak{n}) \hookrightarrow \mathcal{F}$ satisfying $\Psi(xy) = \Psi(x) * \Psi(y)$ for all $x, y \in \mathcal{U}_q(\mathfrak{n})$. Let $\mathcal{W} = \Psi(\mathcal{U}_q(\mathfrak{n}))$.

The ordering $[0] < [1] < \dots < [r-1]$ yields two total ordering on words in $\mathcal{F}$: One the standard lexicographic ordering reading from *left to right*, and the other the *costandard* lexicographic ordering reading from *right to left*. These orderings give rise to special words in $\mathcal{F}$ called Lyndon words, and every word has a canonical factorization as a non-increasing product of Lyndon words. In [27], Leclerc uses the standard ordering, while we use the costandard ordering. It is easy to translate between results using one ordering as opposed to the other. However, in our situation, choosing the costandard ordering leads to some significant differences in the *shape* of Lyndon words. We will explain this shortly.

Bases for $\mathcal{W}$ are parameterized by certain words called *good words*. A *good word* is a nonincreasing product of *good Lyndon word* which have been studied in [26, 37, 38, 39]. The good Lyndon words are in 1-1 correspondence with the positive roots, Δ^+ , of $\mathfrak{g}$, and the (standard or costandard) lexicographic ordering on good Lyndon words gives rise to a convex ordering on Δ^+ . The convex ordering on Δ^+ gives rise to a PBW basis for $\mathcal{U}_q(\mathfrak{n})$, which in turn gives a multiplicative basis $\{E_g^* = (E_{l_k}^*) * \dots * (E_{l_1}^*)\}$ for $\mathcal{W}$ labeled by good words $g = l_1 \cdots l_k$, where $l_1 \geq \dots \geq l_k$ are good Lyndon words. Additionally, the bar involution on $\mathcal{U}_q(\mathfrak{n})$ gives rise to a bar involution on $\mathcal{W}$, and hence, a *dual canonical basis* $\{b_g^*\}$ labeled by good words. The transition matrix between the basis $\{E_g^*\}$ and $\{b_g^*\}$ is triangular and, in particular, $b_l^* = E_l^*$ for each good Lyndon word l . In what follows, let $\underline{w}$ denote the specialization at $q = 1$ of an element $w \in \mathcal{W}$.

For $\mathfrak{g}$ of type $A_\infty = \varinjlim A_r$, good Lyndon words are labelled by segments $[a, b]$, and there is no difference between the standard and costandard ordering. In this case, for a good Lyndon word l , $\underline{E}_l^* = l$. The Mackey theorem for $\mathcal{H}^{\text{aff}}(d)$ (see section 3.4) implies that the formal character of a standard module $\mathcal{M}(\beta, \alpha)$ is given by $\underline{E}_g^*$, where g is the good word $[\alpha_1, \dots, \beta_1 - 1, \dots, \alpha_n, \dots, \beta_n - 1]$. A much deeper fact, proved by Ariki in [2], is that the character of the simple module $\mathcal{L}(\beta, \alpha)$ is given by the dual canonical basis element $\underline{b}_g^*$.

Leclerc also studied the Lie algebra $\mathfrak{b}_r$ of type B_r , and hence that of type $B_\infty = \varinjlim B_r$. The good Lyndon words for $\mathfrak{b}_r$ with respect to the standard ordering are segments $[i, \dots, j]$, $0 \leq i \leq j < r$, and *double segments* $[0, \dots, j, 0, \dots, k]$, $0 \leq j < k < r$ (cf. [27, §8.2]). In this case, when $l = [i, \dots, j]$ is

a segment, $\underline{b}_l^* = [i, \dots, j] = \text{ch } \Phi_{[i,j]}$. However, when $l = [0, \dots, j, 0, \dots, k]$ is a double segment

$$\underline{b}_l^* = 2[0] \cdot ([0, \dots, j] * [1, \dots, k]). \quad (1.1.2)$$

When we adopt the costandard ordering, the picture becomes much more familiar. Indeed, the good Lyndon words are of the form $[i, \dots, j]$ $0 \leq i < j < r$ and $[j, \dots, 0, 0, \dots, k]$, $0 \leq j < k < r$! In particular, they correspond to weights of the segment representations $\Phi_{[i,j]}$ and $\Phi_{[-j-1,k]}$ respectively. Moreover, for $l = [j, \dots, 0, 0, \dots, k]$

$$\underline{b}_l^* = 2[j, \dots, 0, 0, \dots, k] = \text{ch } \Phi_{[-j-1,k]}.$$

Leclerc conjectures [27, Conjecture 52] that for each good word g of *principal degree* d , there exists a simple $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -module with character given by b_g^* . We are not yet able to confirm the conjecture for general good words. However, the combinatorial construction of $H^{\lambda/\mu}$ immediately implies Leclerc's conjecture for calibrated representations (cf. [27, Proposition 51] and Corollary 5.1.5). Additionally, for each good Lyndon word l (with respect to the costandard ordering), there is a simple module with character b_l^* .

Also, an application of the functor F_λ gives a representation theoretic interpretation of (1.1.2) above. Indeed, let $\lambda = (k+1, j+1)$ and $\alpha = (1, -1)$. Then,

$$\text{ch } \mathcal{L}(\lambda, -\alpha) = 2[0] \cdot ([0, \dots, j] * [1, \dots, k]).$$

Finally, the analysis of good Lyndon words leads to a classification of simple integral modules for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. Indeed, recall the set (1.1.1), and let

$$\mathcal{B}_d = \{(\lambda, \mu) \mid \lambda \in P^{++}, \text{ and } \mu \in \mathcal{B}_d(\lambda)\}.$$

Then,

Theorem. *The following is a complete list of pairwise non-isomorphic simple modules for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$:*

$$\{ \mathcal{L}(\lambda, \mu) \mid (\lambda, \mu) \in \mathcal{B}_d \}.$$

We believe this paper may serve as a starting point for future investigations into categorification theories associated to non-simply laced Dynkin diagrams. In particular, we hope that the functor introduced here will play a role in showing that the 2-category for $\mathfrak{b}_\infty$, introduced by Khovanov-Lauda and independently by Rouquier, acts on $\mathcal{O}(\mathfrak{q}(n))$, see [20, 21, 22, 40]. Additionally, in [50], Wang and Zhao initiated a study of super analogues of W -algebras. This functor should be useful for studying these W -superalgebras along the lines of [7, 8].

In [3], Brundan studied the category of finite dimensional modules for $\mathfrak{q}(n)$ via Kazhdan-Lusztig theory. Among the finite dimensional $\mathfrak{q}(n)$ -modules are the polynomial representations, which correspond under our functor to calibrated representations. Other modules in this category are those associated to *rational* weights, i.e. strict partitions with negative parts allowed. The functor should map these modules to interesting $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -modules. These should be investigated. It would also be interesting to compare the Kazhdan-Lusztig polynomials in [3] to those appearing in [27].

We now briefly outline the paper. In section 2, we review some basic notion of super representation theory. In section 3 we define the degenerate AHCA and review some of its properties which may also be found in [24]. The standard modules and their irreducible quotients are introduced in

section 4. The classification of the calibrated representations are given in section 5. In section 6 we review some basic notions about category $\mathcal{O}(\mathfrak{q}(n))$ which may be found in [3, 15]. Next, in section 7 the functor is developed along with its properties. Finally, in section 8 a classification of simple modules is obtained.

1.2. Acknowledgments. The work presented in this paper was begun while the second author visited the Mathematical Sciences Research Institute in Berkeley, CA. He would like to thank the administration and staff of MSRI for their hospitality and especially the organizers of the “Combinatorial Representation Theory” and “Representation Theory of Finite Groups and Related Topics” programs for providing an exceptionally stimulating semester.

We would like to thank Mikhail Khovanov for suggesting we consider a super analogue of the Arakawa-Suzuki functor. We would also like to thank Bernard Leclerc for pointing out [27], as well as Monica Vazirani and Weiqiang Wang for some useful comments.

2. (ASSOCIATIVE) SUPERALGEBRAS AND THEIR MODULES

We now review some basics of the theory of superalgebras, following [5, 6, 24]. The objects in this theory are $\mathbb{Z}_2$ -graded. Throughout the exposition, we will make definitions for homogeneous elements in this grading. These definitions should always be extended by linearity. Also, we often choose to not write the prefix *super*. As the paper progresses this term may be dropped; however, we will always point out when we are explicitly ignoring the $\mathbb{Z}_2$ -grading.

A vector superspace is a $\mathbb{Z}_2$ -graded $\mathbb{C}$ -vector space $V = V_0 \oplus V_1$. Given a nonzero homogeneous vector $v \in V_i$, let $p(v) = \bar{i} \in \mathbb{Z}_2$ be its *parity*. Given a superspace V , let ΠV be the superspace obtained by reversing the parity. That is, $\Pi V_i = V_{i+1}$. A supersubspace of V is a *graded* subspace $U \subseteq V$. That is, $U = (U \cap V_0) \oplus (U \cap V_1)$. Observe that U is a supersubspace if, and only if, U is stable under the map $v \mapsto (-1)^{p(v)}v$ for homogeneous vectors $v \in V$.

Given two superspaces V, W , the direct sum $V \oplus W$ and tensor product $V \otimes W$ satisfy $(V \oplus W)_{\bar{i}} = V_{\bar{i}} \oplus W_{\bar{i}}$ and

$$(V \otimes W)_{\bar{i}} = \bigoplus_{\bar{j} + \bar{k} = \bar{i}} V_{\bar{j}} \otimes W_{\bar{k}}.$$

We may regard $\text{Hom}_{\mathbb{C}}(V, W)$ as a superspace by setting $\text{Hom}_{\mathbb{C}}(V, W)_{\bar{i}}$ to be the set of all homogeneous linear maps of degree $\bar{i}$. That is, linear maps $\varphi : V \rightarrow W$ such that $\varphi(V_{\bar{j}}) \subseteq W_{\bar{j} + \bar{i}}$. Finally, $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a superspace, where $\mathbb{C} = \mathbb{C}_{\bar{0}}$.

Now, a superalgebra is a vector superspace A that has the structure of an associative, unital algebra such that $A_{\bar{i}}A_{\bar{j}} \subseteq A_{\bar{i} + \bar{j}}$. A superideal of A is a two sided ideal of A that is also a supersubspace of A . A superalgebra homomorphism $\varphi : A \rightarrow B$ is an even (i.e. grading preserving) linear map which is also an algebra homomorphism. Observe that since φ is even, its kernel, $\ker \varphi$, is a superideal of A . Finally, given superalgebras A and B , their tensor product $A \otimes B$ is a superalgebra with product given by

$$(a \otimes b)(a' \otimes b') = (-1)^{p(a')p(b)}(aa' \otimes bb'). \quad (2.0.1)$$

We now turn our attention to supermodules. Given a superalgebra A , let $A\text{-smod}$ denote the category of all finite dimensional A -supermodules, and $A\text{-mod}$ be the category of A -modules in the usual ungraded sense. An object in $A\text{-smod}$ is a $\mathbb{Z}_2$ -graded left A -module $M = M_0 \oplus M_1$ such that

$A_{\bar{i}}M_{\bar{j}} \subseteq M_{\bar{i}+\bar{j}}$. A homomorphism of A -supermodules M and N is a map of vector superspaces $f : M \rightarrow N$ satisfying $f(am) = (-1)^{p(a)p(f)}af(m)$ when f is homogeneous. A submodule of an A -supermodule M will always be a supersubspace of M . An A -supermodule M is called irreducible if it contains no proper nontrivial subsupermodules.

The supermodule M may or may not remain irreducible when regarded as an object in $A\text{-mod}$. If M remains irreducible as an A -module, it is called *absolutely irreducible*, and if it decomposes, it is called *self associate*. Alternatively, absolutely irreducible supermodules are said to be irreducible of type $\mathbb{M}$, while self associate supermodules are irreducible of type $\mathbb{Q}$. When $M \in A\text{-smod}$ is self associate, there exists an odd A -smod homomorphism θ_M which interchanges the two irreducible components of M as an object in $A\text{-mod}$.

Now, let A and B be superalgebras, $M \in A\text{-smod}$ and $N \in B\text{-smod}$. The vector superspace $M \otimes N$ has the structure of an $A \otimes B$ -supermodule via the action is given by

$$(a \otimes b)(m \otimes n) = (-1)^{p(b)p(m)}(am \otimes bn) \quad (2.0.2)$$

for homogeneous $b \in B$ and $m \in M$. This is called the outer tensor product of M and N and is denoted $M \boxtimes N$.

Unlike the classical situation, it may happen that the outer tensor product of irreducible supermodules is no longer irreducible. This only happens when both modules are self associate. To see this, let $M \in A\text{-smod}$ and $N \in B\text{-smod}$ be self associate, and recall the odd homomorphisms θ_M and θ_N . Then, $\theta_M \otimes \theta_N : M \boxtimes N \rightarrow M \boxtimes N$, is an even automorphism of $M \boxtimes N$ that squares to -1 . Hence $M \boxtimes N$ decomposes as direct sum of two $A \otimes B$ -supermodules, namely the $(\pm\sqrt{-1})$ -eigenspaces. These two summands are absolutely irreducible and isomorphic under the odd isomorphism $\Theta_{M,N} := \theta_M \otimes \text{id}_N$, see [5, Lemma 2.9] and [6, Section 2-b]. When M and N are irreducible, define the (irreducible) $A \otimes B$ -module $M \circledast N$ by the formula

$$M \boxtimes N = \begin{cases} M \circledast N, & \text{if either } M \text{ or } N \text{ is of type } \mathbb{M}; \\ (M \circledast N) \oplus \Theta_{M,N}(M \circledast N), & \text{if both } M \text{ and } N \text{ are of type } \mathbb{Q}. \end{cases} \quad (2.0.3)$$

When $M = M' \oplus M''$, define $M \circledast N = (M' \circledast N) \oplus (M'' \circledast N)$.

Finally, let $A\text{-smod}_{\text{ev}}$ be the abelian subcategory of $A\text{-smod}$ with the same objects, but only *even* morphisms. Then, the Grothendieck group $K(A\text{-smod})$ is the quotient of the Grothendieck group $K(A\text{-smod}_{\text{ev}})$ modulo the relation $M - \Pi M$ for every A -supermodule M . We would like to emphasize again that we allow odd morphisms and, therefore, $M \cong \Pi M$ in the original category.

3. THE DEGENERATE AFFINE HECKE-CLIFFORD ALGEBRA

In this section we define the algebra which is the principle object of study in this paper and summarize the results we will require in what follows. Many of the results may be found in [24], however, we include them here in an effort to make this paper self contained and readable to a wider audience.

3.1. The Algebra. Let $\mathcal{C}\ell(d)$ denote the Clifford algebra over $\mathbb{C}$ with generators $c_1, \dots, c_d$, and relations

$$c_i^2 = -1, \quad c_i c_j = -c_j c_i \quad 1 \leq i \neq j \leq d. \quad (3.1.1)$$

Then $\mathcal{C}\ell(d)$ is a superalgebra by declaring the generators $c_1, \dots, c_d$ to all be of degree $\bar{1}$.

Let S_d be the symmetric group on d letters with Coxeter generators $s_1, \dots, s_{d-1}$ and relations

$$s_i^2 = 1 \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad s_i s_j = s_j s_i \quad (3.1.2)$$

for all admissible i and j such that $|i - j| > 1$. The group algebra of the symmetric group, $\mathbb{C}S_d$, is a superalgebra by viewing it as concentrated in degree $\bar{0}$; that is, $(\mathbb{C}S_d)_{\bar{0}} = \mathbb{C}S_d$.

The *Sergeev algebra* is given by setting

$$\mathcal{S}(d) = \mathcal{C}\ell(d) \otimes \mathbb{C}S_d$$

as a vector superspace and declaring $\mathcal{C}\ell(d) \cong \mathcal{C}\ell(d) \otimes 1$ and $\mathbb{C}S_d \cong 1 \otimes \mathbb{C}S_d$ to be subsuperalgebras. The Clifford generators $c_1, \dots, c_d$ and Coxeter generators $s_1, \dots, s_{d-1}$ are subject to the mixed relation

$$s_i c_i = c_{i+1} s_i, \quad s_i c_{i+1} = c_i s_i, \quad s_i c_j = c_j s_i, \quad (3.1.3)$$

for all admissible i and j such that $j \neq i, i + 1$.

The algebra of primary interest in this paper is the (*degenerate*) *affine Hecke-Clifford algebra*, AHCA. It is given as

$$\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d) = \mathcal{P}_d[x] \otimes \mathcal{S}(d)$$

as a vector superspace, where $\mathcal{P}_d[x] := \mathbb{C}[x_1, \dots, x_d]$ is the polynomial ring in d variables and is viewed as a superalgebra concentrated in degree $\bar{0}$. Multiplication is defined so that $\mathcal{S}(d) \cong 1 \otimes \mathcal{S}(d)$ and $\mathcal{P}_d[x] \cong \mathcal{P}_d[x] \otimes 1$ are subsuperalgebras. The generators of these two subalgebras are subject to the mixed relations

$$c_i x_i = -x_i c_i, \quad c_j x_i = x_i c_j, \quad 1 \leq i \neq j \leq d, \quad (3.1.4)$$

and

$$s_i x_i = x_{i+1} s_i - 1 + c_i c_{i+1}, \quad s_i x_j = x_j s_i \quad (3.1.5)$$

for $1 \leq i \leq d - 1$, $1 \leq j \leq d$, $j \neq i, i + 1$.

Note that relation (3.1.5) differs from the corresponding relation in [6, 24]. This is because in (3.1.1) we choose $c_i^2 = -1$, following [30, 41, 42], whereas in *loc. cit.* the authors take $c_i^2 = 1$. The resulting algebras are isomorphic and the only effect of this convention is that this change of sign has to be taken into account when comparing formulae.

It will be useful to consider another decomposition

$$\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d) \cong \mathcal{A}(d) \otimes \mathbb{C}S_d, \quad (3.1.6)$$

where $\mathcal{A}(d)$ is the subalgebra generated by $\mathcal{C}\ell(d)$ and $\mathcal{P}_d[x]$. As a superspace

$$\mathcal{A}(d) \cong \mathcal{P}_d[x] \otimes \mathcal{C}\ell(d). \quad (3.1.7)$$

We have the following PBW-type theorem for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. Given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{Z}_2^d$, set $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $c^\varepsilon = c_1^{\varepsilon_1} \cdots c_d^{\varepsilon_d}$. Then,

Theorem 3.1.1. [24, Theorem 14.2.2] *The set $\{x^\alpha c^\varepsilon w \mid \alpha \in \mathbb{Z}_{\geq 0}^d, \varepsilon \in \mathbb{Z}_2^d, w \in S_d\}$ forms a basis for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$.*

3.2. Some (Anti)Automorphisms. The superalgebra $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ admits an automorphism $\sigma : \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d) \rightarrow \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ given by

$$\sigma(s_i) = -s_{d-i}, \quad \sigma(c_i) = c_{d+1-i}, \quad \sigma(x_i) = x_{n+1-i}. \quad (3.2.1)$$

It also admits an antiautomorphism $\tau : \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d) \rightarrow \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ given by

$$\tau(s_i) = s_i, \quad \tau(c_i) = -c_i, \quad \tau(x_i) = x_i.$$

Note that, for superalgebras, antiautomorphism means that, for any homogeneous $x, y \in \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$,

$$\tau(xy) = (-1)^{p(x)p(y)} \tau(y) \tau(x). \quad (3.2.2)$$

3.3. Weights and Integral Modules. We now introduce the class of integral $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -modules. It is these modules which are the main focus of the paper. To this end, for each $a \in \mathbb{C}$, define

$$q(a) = a(a+1). \quad (3.3.1)$$

By [24, Theorem 14.3.1], the center of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ consists of symmetric polynomials in $x_1^2, \dots, x_d^2$. Let $\mathcal{P}_d[x^2] = \mathbb{C}[x_1^2, \dots, x_d^2] \subset \mathcal{P}_d[x]$. A *weight* is an algebra homomorphism

$$\zeta : \mathcal{P}_d[x^2] \rightarrow \mathbb{C}.$$

It is often convenient to identify a weight ζ with the d -tuple of complex numbers $\zeta = (\zeta(x_1^2), \dots, \zeta(x_d^2)) \in \mathbb{C}^d$.

Given an $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -supermodule M and a weight ζ , define the ζ *weight space*,

$$M_\zeta = \{m \in \mathcal{M} \mid x_i^2 m = q(\zeta(x_i^2)) m \text{ for all } i = 1, \dots, d\},$$

and the *generalized ζ weight space*,

$$M_\zeta^{\text{gen}} = \left\{ m \in \mathcal{M} \mid (x_i^2 - q(\zeta(x_i^2)))^k m = 0 \text{ for } k \gg 0 \text{ and all } i = 1, \dots, d \right\}.$$

Observe that if $M_\zeta^{\text{gen}} \neq 0$, then $M_\zeta \neq 0$.

Following [6], say that an $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module M is *integral* if

$$M = \bigoplus_{\zeta} M_\zeta^{\text{gen}}$$

and $M_\zeta^{\text{gen}} \neq 0$ implies $\zeta(x_i^2) \in \mathbb{Z}$ for $i = 1, \dots, d$.

Let $\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ denote the full subcategory of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -smod of finite dimensional *integral* modules for the degenerate AHCA. Unless stated otherwise, all $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -modules will be integral by assumption.

3.4. The Mackey Theorem. In this section we review the Mackey Theorem for integral $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}$ -modules. Refer to [24] for details.

Let $\mu = (\mu_1, \dots, \mu_k)$ be a composition of d . Define the parabolic subgroup $S_\mu = S_{\mu_1} \times \dots \times S_{\mu_k} \subseteq S_d$, and parabolic subalgebra $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu) := \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu_1) \otimes \dots \otimes \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu_k) \subseteq \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$. Define the functor

$$\text{Ind}_\mu^d : \text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu) \rightarrow \text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d), \quad \text{Ind}_\mu^d M = \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d) \otimes_{\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu)} M.$$

This functor is left adjoint to $\text{Res}_\mu^d : \text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d) \rightarrow \text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu)$. Also, given a composition $\nu = (\nu_1, \dots, \nu_\ell)$ of d , which is a refinement of μ (i.e. there exist $0 = i_1 \leq \dots \leq i_{k+1} = \ell$ such that $\nu_{i_j} + \dots + \nu_{i_{j+1}-1} = \mu_j$), define Ind_ν^μ and Res_ν^μ in the obvious way.

Now, let μ and ν be compositions of d , and let $D_{\mu,\nu}$ denote the set of minimal length $S_\mu \backslash S_d / S_\nu$ -double coset representatives and $D_\nu = D_{(1^d),\nu}$. Let $w \in D_{\mu,\nu}$. The following lemma is standard.

Lemma 3.4.1. *Let $\nu = (\nu_1, \dots, \nu_n)$ be a composition of d , and set $a_i = \nu_1 + \dots + \nu_{i-1} + 1$ and $b_i = \nu_1 + \dots + \nu_i$. If $w \in D_\nu$ and $a_i \leq k < k' \leq b_i$ for some i , then $w(k) < w(k')$.*

It is known that $S_\mu \cap wS_\nu w^{-1}$ and $w^{-1}S_\mu w \cap S_\nu$ are parabolic subgroups of S_d . Hence we may define compositions $\mu \cap w\nu$ and $w^{-1}\mu \cap \nu$ by the formulae

$$S_\mu \cap w^{-1}S_\nu w = S_{\mu \cap w\nu} \quad \text{and} \quad w^{-1}S_\mu w \cap S_\nu = S_{w^{-1}\mu \cap \nu}.$$

Moreover, the map $\sigma \mapsto w\sigma w^{-1}$ induces a length preserving isomorphism $S_{\mu \cap w\nu} \rightarrow S_{w^{-1}\mu \cap \nu}$.

Using this last fact, it can be proved that for each $w \in D_{\mu,\nu}$ there exists an algebra isomorphism

$$\varphi_{w^{-1}} : \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu \cap w\nu) \rightarrow \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(w^{-1}\mu \cap \nu)$$

given by $\varphi_{w^{-1}}(\sigma) = w^{-1}\sigma w$, $\varphi_{w^{-1}}(c_i) = c_{w^{-1}(i)}$ and $\varphi_{w^{-1}}(x_i) = x_{w^{-1}(i)}$ for $1 \leq i \leq d$ and $\sigma \in S_{\mu \cap w\nu}$. If M is a left $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\mu \cap w\nu)$ -supermodule, let ${}^w M$ denote the $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(w^{-1}\mu \cap \nu)$ -supermodule obtained by twisting the action with the isomorphism $\varphi_{w^{-1}}$. We have the following ‘‘Mackey Theorem’’:

Theorem 3.4.2. [24, Theorem 14.2.5] *Let M be an $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(\nu)$ -supermodule. Then $\text{Res}_\mu^d \text{Ind}_\nu^d M$ admits a filtration with subquotients isomorphic to*

$$\text{Ind}_{\mu \cap w\nu}^\mu {}^w (\text{Res}_{w^{-1}\mu \cap \nu}^\nu M),$$

one for each $w \in D_{\mu,\nu}$. Moreover the subquotients can be taken in any order refining the Bruhat order on $D_{\mu,\nu}$. In particular, $\text{Ind}_{\mu \cap \nu}^\mu \text{Res}_{\mu \cap \nu}^\nu M$ appears as a subsupermodule.

3.5. Characters. Following [24, Chapter 16], we now describe the notion of characters for integral $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -supermodules.

Recall the subsuperalgebra $\mathcal{A}(d) \subseteq \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ defined in (3.1.7). When $d = 1$ and $a \in \mathbb{Z}$ there exists a 2-dimensional simple $\mathcal{A}(1)$ -module

$$\mathcal{L}(a) = \mathcal{C}_\ell(1)1_a = \mathbb{C}1_a \oplus \mathbb{C}c_1.1_a,$$

which is free as a $\mathcal{C}_\ell(1)$ -module satisfying

$$x_1.1_a = \sqrt{q(a)}1_a.$$

The $\mathbb{Z}_2$ -grading on $\mathcal{L}(a)$ is given by setting $p(1_a) = \bar{0}$.

Observe that $\mathcal{L}(a) \cong \mathcal{L}(-a-1)$ and that by replacing $\sqrt{q(a)}$ with $-\sqrt{q(a)}$ in the action of x_1 yields an isomorphic supermodule under the odd isomorphism $1_a \mapsto c_1.1_a$. A direct calculation verifies that this module is of type M if $a \neq 0$ and of type Q if $a = 0$.

Now, $\mathcal{A}(d) \cong \mathcal{A}(1) \otimes \dots \otimes \mathcal{A}(1)$. Hence, applying (2.0.3) we obtain a simple $\mathcal{A}(d)$ -module $\mathcal{L}(a_1) \otimes \dots \otimes \mathcal{L}(a_d)$. Given $(a_1, \dots, a_d) \in \mathbb{Z}_{\geq 0}^d$, let

$$\gamma_0(a_1, \dots, a_d) = |\{i \mid a_i = 0\}|. \quad (3.5.1)$$

We have

Lemma 3.5.1. [24, Lemma 16.1.1] *The set*

$$\{\mathcal{L}(a_1) \otimes \cdots \otimes \mathcal{L}(a_d) \mid (a_1, \dots, a_d) \in \mathbb{Z}_{\geq 0}^d\}$$

is a complete set of pairwise non-isomorphic irreducible integral $\mathcal{A}(d)$ -modules.

The module $\mathcal{L}(a_1) \otimes \cdots \otimes \mathcal{L}(a_d)$ is of type M if γ_0 is even and of type Q if γ_0 is odd. Moreover,

$$\dim \mathcal{L}(a_1) \otimes \cdots \otimes \mathcal{L}(a_d) = 2^{n - \lfloor \gamma_0/2 \rfloor}$$

where $\gamma_0 = \gamma_0(a_1, \dots, a_d)$ as above.

Restriction to the subalgebra $A(d) = \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}((1^d)) \subseteq \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ defines a functor from $\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ to $\mathcal{A}(d)\text{-mod}$. The map obtained by applying this functor and passing to the Grothendieck group of the category $\mathcal{A}(d)\text{-mod}$ yields a map

$$\text{ch} : \text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d) \rightarrow K(\mathcal{A}(d)\text{-mod})$$

defined by

$$\text{ch } M = [\text{Res}_{1^d}^d M]$$

where $[X]$ is the image of an $\mathcal{A}(d)$ -module, X , in $K(\mathcal{A}(d)\text{-mod})$. The image $\text{ch } M$ is called the *formal character* of the $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module M .

The following fundamental result is given in [24, Theorem 17.3.1].

Lemma 3.5.2. *The induced map on Grothendieck rings*

$$\text{ch} : K(\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)) \rightarrow K(\mathcal{A}(d)\text{-mod})$$

is injective.

For convenience of notation, set

$$[a_1, \dots, a_d] = [\mathcal{L}(a_1) \otimes \cdots \otimes \mathcal{L}(a_d)].$$

The following lemma describes how to calculate the character of $M \otimes N$ in terms of the characters of M and N , and is a special case of the Mackey Theorem:

Lemma 3.5.3. [24, Shuffle Lemma] *Let $K \in \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(k)$ and $M \in \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(m)$ be simple, and assume that*

$$\text{ch } K = \sum_{\underline{i} \in \mathbb{Z}_{\geq 0}^k} r_{\underline{i}} [i_1, \dots, i_k] \quad \text{and} \quad \text{ch } M = \sum_{\underline{j} \in \mathbb{Z}_{\geq 0}^m} s_{\underline{j}} [j_1, \dots, j_m].$$

Then,

$$\text{ch } \text{Ind}_{m,k}^{m+k} K \otimes M = \sum_{\underline{i}, \underline{j}} r_{\underline{i}} s_{\underline{j}} [i_1, \dots, i_k] * [j_1, \dots, j_m]$$

where

$$[i_1, \dots, i_k] * [i_{k+1}, \dots, i_{k+m}] = \sum_{w \in D_{(m,k)}} [w(i_1), \dots, w(i_{k+m})].$$

3.6. Duality. Now, given an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -module M , we obtain a new module M^σ by twisting the action of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ by σ . That is, define a new action, $*$, on M by $x * m = \sigma(x).m$ for all $x \in \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. We have

Lemma 3.6.1. [24, Lemma 14.6.1] *If M is an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(k)$ -module and N is an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(\ell)$ -module, then*

$$(\text{Ind}_{k,\ell}^{k+\ell} M \otimes N)^\sigma \cong \text{Ind}_{k,\ell}^{k+\ell} M^\sigma \otimes N^\sigma.$$

If M is an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -module, with character

$$\text{ch } M = \sum_{\underline{i} \in \mathbb{Z}_{\geq 0}^d} r_{\underline{i}}[i_1, \dots, i_d],$$

then Lemma 3.6.1 implies that

$$\text{ch } M^\sigma = \sum_{\underline{i} \in \mathbb{Z}_{\geq 0}^d} r_{\underline{i}}[i_d, \dots, i_1].$$

3.7. Contravariant Forms. Let M be in $\text{Rep } \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. A bilinear form $(\cdot, \cdot) : M \otimes M \rightarrow \mathbb{C}$ is called a contravariant form if

$$(x.v, v') = (v, \tau(x).v')$$

for all $x \in \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ and $v, v' \in M$.

Lemma 3.7.1. *Let M be in $\text{Rep } \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ equipped with a contravariant form $(\cdot, \cdot)$. Then*

$$M_\eta \perp M_\zeta^{\text{gen}} \quad \text{unless } \eta = \zeta.$$

Proof. Assume $\eta \neq \zeta$, and let $v \in M_\eta$ and $v' \in M_\zeta^{\text{gen}}$. Choose i such that $q(\eta(x_i^2)) \neq q(\zeta(x_i^2))$, and $N \gg 0$ such that

$$(x_i^2 - q(\zeta(x_i^2)))^N . v' = 0.$$

Then

$$\begin{aligned} (q(\eta(x_i^2)) - q(\zeta(x_i^2)))^N (v, v') &= ((x_i^2 - q(\zeta(x_i^2)))^N . v, v') \\ &= (v, \tau((x_i^2 - q(\zeta(x_i^2)))^N) . v') \\ &= (v, (x_i^2 - q(\zeta(x_i^2)))^N . v') = 0 \end{aligned}$$

showing that $(v, v') = 0$. □

3.8. Intertwiners. Define the intertwiner

$$\phi_i = s_i(x_i^2 - x_{i+1}^2) + (x_i + x_{i+1}) - c_i c_{i+1}(x_i - x_{i+1}). \quad (3.8.1)$$

Given an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -supermodule M , we understand that $\phi_i \mathcal{M}_\zeta^{\text{gen}} \subseteq \mathcal{M}_{s_i(\zeta)}^{\text{gen}}$. Moreover, a straightforward calculation gives

$$\phi_i^2 = 2x_i^2 + 2x_{i+1}^2 - (x_i^2 - x_{i+1}^2)^2. \quad (3.8.2)$$

The following lemma now directly follows (see also [24]).

Lemma 3.8.1. *Assume that Y is in $\text{Rep } \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$, and $v \in Y$ satisfies $x_i.v = \sqrt{q(a)}v$ and $x_{i+1}.v = \sqrt{q(b)}v$ for some $a, b \in \mathbb{Z}$. Then, $\phi_i^2.v \neq 0$ unless $q(a) = q(b+1)$ or $q(a) = q(b-1)$.*

4. STANDARD MODULES

We construct a family of standard modules which are an analogue of Zelevinsky's construction for the degenerate affine Hecke algebra. The key ingredient is to define certain irreducible supermodules for a parabolic subalgebra of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$; the so-called segment representations. The standard modules are then obtained by inducing from the outer tensor product of these modules.

4.1. Segment Representations. We begin by constructing a family of irreducible $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -supermodules that are analogues of Zelevinsky's segment representations for the degenerate affine Hecke algebra. To begin, define the 2^d -dimensional $\mathcal{S}(d)$ -supermodule

$$\mathcal{C}\ell_d = \text{Ind}_{S_d}^{\mathcal{S}(d)} \mathbb{C}\mathbf{1}, \quad (4.1.1)$$

where $\mathbb{C}\mathbf{1}$ is the trivial representation of S_d . That is, $\mathcal{C}\ell_d = \mathcal{C}\ell(d).\mathbf{1}$, where the cyclic vector $\mathbf{1}$ satisfies

$$w.\mathbf{1} = \mathbf{1}, \quad w \in S_d.$$

This is often referred to as the *basic spin representation* of $\mathcal{S}(d)$.

Introduce algebra involutions $\epsilon_i : \mathcal{C}\ell(d) \rightarrow \mathcal{C}\ell(d)$ by $\epsilon_i(c_j) = (-1)^{\delta_{ij}} c_j$ for $1 \leq i, j \leq d$. The elements ϵ_i act on $\mathcal{C}\ell_d$ by $\epsilon_i.\mathbf{1} = \mathbf{1}$ for $1 \leq i \leq d$ and, more generally, $\epsilon_i.s\mathbf{1} = \epsilon_i(s)\mathbf{1}$ for $1 \leq i \leq d$. Also, note that the operators ϵ_i commute with each other.

For each $a \in \mathbb{Z}$, define the Clifford algebra

$$\Phi_a = \begin{cases} \mathbb{C}\langle \varphi \rangle / (\varphi^2 - a), & \text{if } a \neq 0; \\ \mathbb{C}\langle \varphi \rangle / (\varphi), & \text{if } a = 0. \end{cases} \quad (4.1.2)$$

The $\mathbb{Z}_2$ -grading on Φ_a is given by declaring $p(\varphi) = \bar{1}$.

Given a pair of integers $a \leq b$ define the *segment*

$$[a, b] = \{a, a+1, \dots, b\}.$$

Given a segment $[a, b]$ with $b - a + 1 = d \in \mathbb{Z}_{\geq 0}$, define the $\Phi_a \otimes \mathcal{S}(d)$ -module

$$\hat{\Phi}_{[a,b]} = \Phi_a \boxtimes \mathcal{C}\ell_d. \quad (4.1.3)$$

Of course, when $d = 0$ the segment $[a, a-1] = \emptyset$, and $\hat{\Phi}_{\emptyset} = \Phi_a \otimes \mathbb{C}$.

For $i = 1, \dots, d$ let s_{ij} denote the transposition (ij) , and

$$\mathcal{L}_i = \sum_{j < i} (1 - c_j c_i) s_{ij} \quad (4.1.4)$$

be the *i*th Jucys-Murphy element (cf. [24, (13.22)]).

Proposition 4.1.1. *Let $[a, b]$ be a segment with $b - a + 1 = d$. Then,*

- (i) *The vector space $\hat{\Phi}_{[a,b]}$ is an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -module with $s_i.v = (1 \otimes s_i).v$, $c_i.v = (1 \otimes c_i).v$ and*

$$\begin{aligned} x_i.v &= (a \otimes \epsilon_i + 1 \otimes \mathcal{L}_i - \varphi \otimes c_i).v \\ &= \left(a \otimes \epsilon_i + \sum_{k < i} 1 \otimes (1 - c_k c_i) s_{ki} - \varphi \otimes c_i \right).v, \end{aligned}$$

for all $v \in \hat{\Phi}_{[a,b]}$.

(ii) The action of $\mathcal{P}_d[x^2]$ on $\hat{\Phi}_{[a,b]}$ is determined by

$$x_i^2 \cdot (\varphi^\delta \otimes \mathbf{1}) = q(a+i-1)\varphi^\delta \otimes \mathbf{1}, \quad \delta \in \{0, 1\}, \quad i = 1, \dots, d.$$

Proof. (i) The fact that this is an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -module is an easy check which we leave to the reader.

(ii) To check the action of x_i^2 , observe that

$$x_i \cdot \mathbf{1} \otimes \mathbf{1} = \left(a + i - 1 - \sum_{j < i} c_j c_i \right) \cdot \mathbf{1} \otimes \mathbf{1} + c_i \cdot \varphi \otimes \mathbf{1}$$

and

$$x_i \cdot \varphi \otimes \mathbf{1} = \left(a + i - 1 - \sum_{j < i} c_j c_i \right) \cdot \varphi \otimes \mathbf{1} + a c_i \cdot \mathbf{1} \otimes \mathbf{1}.$$

Now, the result follows using the commutation relations for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. $\square$

Remark 4.1.2. In fact, we need not consider all $a, b \in \mathbb{Z}$. Given any segment $[a, b]$, consider the module $\hat{\Phi}_{[a,b]}^\sigma$ obtained by twisting the action of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ by the automorphism σ as described in Section 3.6. Note that when $b \neq -1$,

$$\hat{\Phi}_{[a,b]}^\sigma \cong \hat{\Phi}_{[-b-1, -a-1]}.$$

When $b = -1$, $\hat{\Phi}_{[a,-1]}^\sigma \cong \hat{\Phi}_{[0,-a-1]}^{\oplus 2}$. In particular, for $b \neq 0$, $\hat{\Phi}_{[-(b+1), b-1]}^\sigma \cong \hat{\Phi}_{[-b,b]}$, and $\hat{\Phi}_{[-1,-1]}^\sigma \cong \hat{\Phi}_{[0,0]}^{\oplus 2}$. Therefore, it is enough to describe the modules

- (1) $\hat{\Phi}_{[a,b]}$, $0 \leq a \leq b$, and
- (2) $\hat{\Phi}_{[-a,b]}$, $0 < a \leq b$.

The following result describes $\hat{\Phi}_{[a,b]}$ at the level of characters.

Proposition 4.1.3. Let $[a, b]$ be a segment with $a, b \geq 0$. Then,

- (1) if $0 \leq a \leq b$, then

$$\text{ch } \hat{\Phi}_{[a,b]} = \begin{cases} [a, \dots, b], & \text{if } a = 0; \\ 2[a, \dots, b], & \text{if } a \neq 0; \end{cases}$$

- (2) if $0 < a \leq b$, then

$$\text{ch } \hat{\Phi}_{[-a,b]} = 4[a-1, \dots, 1, 0, 0, 1, \dots, b]$$

Proof. The action of x_i^2 commutes with $\mathcal{C}\ell(d)$ and $\hat{\Phi}_{[a,b]} = \mathcal{C}\ell(d) \cdot (1 \otimes \mathbf{1}) + \mathcal{C}\ell(d) \cdot (\varphi \otimes \mathbf{1})$. Therefore, applying Proposition 4.1.1(2), we deduce in both cases that the x_i^2 act by the prescribed eigenvalues. The result now follows from the dimension formula in Lemma 3.5.1. $\square$

Let $\varphi \hat{\mathbf{1}}_{[a,b]} = \varphi \otimes \mathbf{1}$ and $\hat{\mathbf{1}}_{[a,b]} = 1 \otimes \mathbf{1}$. Also, in what follows, we omit the tensor symbols. For example, we write

$$a\epsilon_i + \mathcal{L}_i - \varphi c_i := a \otimes \epsilon_i + 1 \otimes \mathcal{L}_i - \varphi \otimes c_i.$$

Definition 4.1.4. Let $a \in \mathbb{Z}$ and $\kappa_1, \dots, \kappa_d \in \mathbb{R}$ satisfy $\kappa_i^2 = q(a+i-1)$ where $d = b-a+1$. Given a subset $S \subseteq \{1, \dots, d\}$ define the element $X_S \in \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ by

$$X_S = \prod_{i \notin S} (x_i + \kappa_i).$$

Observe that X_S is only defined up to the choices of sign for $\kappa_1, \dots, \kappa_d$.

Lemma 4.1.5. *Let $[a, b]$ be a segment with $d = b - a + 1$. Assume that either $-a \notin \{1, \dots, d\}$ and S is arbitrary, or assume that $-a \in \{1, \dots, d\}$ and either $-a + 1 \in S$ or $-a \in S$. Then $X_S \cdot \hat{\mathbf{1}}_{[a, b]} \neq 0$.*

Proof. Let $\hat{\mathbf{1}} = \hat{\mathbf{1}}_{[a, b]}$. By Proposition 4.1.1(i),

$$x_k \cdot v = (a\epsilon_k + \mathcal{L}_k - \varphi c_k) \cdot v.$$

Let $\{d_1 > d_2 > \dots > d_\ell\} = \{1, \dots, d\} \setminus S$. Since the x_i mutually commute,

$$\begin{aligned} X_S \cdot \hat{\mathbf{1}} &= (x_{d_1} + \kappa_{d_1}) \cdots (x_{d_\ell} + \kappa_{d_\ell}) \cdot \hat{\mathbf{1}} \\ &= (a\epsilon_{d_1} + \kappa_{d_1} + \mathcal{L}_{d_1} - \varphi c_{d_1}) \cdots (a\epsilon_{d_\ell} + \kappa_{d_\ell} + \mathcal{L}_{d_\ell} - \varphi c_{d_\ell}) \cdot \hat{\mathbf{1}} \\ &= ((a + \kappa_{d_1}) + \mathcal{L}_{d_1} - \varphi c_{d_1}) \cdots ((a + \kappa_{d_\ell}) + \mathcal{L}_{d_\ell} - \varphi c_{d_\ell}) \cdot \hat{\mathbf{1}}. \end{aligned}$$

The last equality follows since $\epsilon_k \mathcal{L}_j = \mathcal{L}_j \epsilon_k$ if $k > j$. Now,

$$\begin{aligned} X_S \cdot \hat{\mathbf{1}} &= \left(\left(a + \kappa_{d_1} + \sum_{j < d_1} s_{jd_1} \right) + \left(\sum_{j < d_1} s_{jd_1} c_j - \varphi \right) c_{d_1} \right) \cdots \\ &\quad \cdots \left(\left(a + \kappa_{d_\ell} + \sum_{j < d_\ell} s_{jd_\ell} \right) + \left(\sum_{j < d_\ell} s_{jd_\ell} c_j - \varphi \right) c_{d_\ell} \right) \cdot \hat{\mathbf{1}} \\ &= \prod_{i \notin S} (a + i - 1 + \kappa_i) \cdot \hat{\mathbf{1}} + (\star) \cdot \hat{\mathbf{1}} \end{aligned} \tag{4.1.5}$$

where $(\star) = p'(c) - \varphi p''(c)$, where $p'(c) \in \mathcal{C}\ell(d)_{\bar{0}}$, $p''(c) \in \mathcal{C}\ell(d)_{\bar{1}}$, and $p'(c)$ has no constant term. Therefore, if either $a \geq 0$, or $-a + 1 \in S$, $X_S \cdot \hat{\mathbf{1}} \neq 0$.

Now, assume $-a + 1 \in \{1, \dots, d\}$, and $-a + 1 \notin S$, but $a \in S$. Observe that $\kappa_{-a+1} = \kappa_{-a} = 0$. Now,

$$x_{-a} \cdot \hat{\mathbf{1}} = \left(-1 - \sum_{j < -a} c_j c_{-a} - \varphi c_{-a} \right) \cdot \hat{\mathbf{1}} = -c_{-a} c_{-a+1} x_{-a+1} \cdot \hat{\mathbf{1}}. \tag{4.1.6}$$

Let $R = S \cup \{-a + 1\}$ and $T = R \setminus \{-a\}$. Then,

$$X_S \cdot \hat{\mathbf{1}} = X_R x_{-a+1} \cdot \hat{\mathbf{1}} = c_{-a} c_{-a+1} X_R x_{-a} \cdot \hat{\mathbf{1}} = c_{-a} c_{-a+1} X_T \cdot \hat{\mathbf{1}} \neq 0.$$

Finally, if $d = -a$, then in (4.1.5), $d_1 = -a$ and it is clear that the coefficient of $c_{-a-1} c_{-a}$ is nonzero. $\square$

Lemma 4.1.6. *If $i \notin S$, then $x_i X_S \cdot \hat{\mathbf{1}} = \kappa_i X_S \cdot \hat{\mathbf{1}}$.*

Proof. Since $x_i^2 \cdot \hat{\mathbf{1}} = q(a - i + 1) \hat{\mathbf{1}} = \kappa_i^2 \hat{\mathbf{1}}$,

$$x_i(x_i + \kappa_i) \cdot \hat{\mathbf{1}} = (x_i^2 + \kappa_i x_i) \cdot \hat{\mathbf{1}} = \kappa_i(\kappa_i + x_i) \cdot \hat{\mathbf{1}},$$

so the result follows because the x_i commute. $\square$

Lemma 4.1.7. *If $i, i + 1 \notin S$ and $i \neq -a$, then*

$$s_i X_S \cdot \hat{\mathbf{1}} = \left(\frac{\kappa_{i+1} + \kappa_i}{2(a + i)} + \frac{\kappa_{i+1} - \kappa_i}{2(a + i)} c_i c_{i+1} \right) X_S \cdot \hat{\mathbf{1}}.$$

Proof. Let $w := X_S \cdot \hat{\mathbf{1}}$, and recall the intertwining element ϕ_i . By character considerations $\phi_i \cdot \hat{\Phi}_{[a,b]} = \{0\}$. In particular,

$$\begin{aligned} 0 &= \phi_i \cdot w \\ &= (s_i(x_i^2 - x_{i+1}^2) + (x_i + x_{i+1}) - c_i c_{i+1}(x_i - x_{i+1})) \cdot w \\ &= -2(a+i)s_i \cdot w + ((\kappa_{i+1} + \kappa_i) + (\kappa_{i+1} - \kappa_i)c_i c_{i+1}) \cdot w. \end{aligned}$$

Hence, the result. $\square$

We can now describe the irreducible segment representations of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$.

Theorem 4.1.8. *The following holds:*

- (i) *The module $\hat{\Phi}_{[0,d-1]}$ is an irreducible $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -module of type Q.*
- (ii) *Assume $0 < a \leq b$. The module $\hat{\Phi}_{[a,b]}$ has a submodule $\hat{\Phi}_{[a,b]}^+ = \mathcal{C}\ell(d) \cdot w$, where $w = X_\emptyset \cdot \hat{\mathbf{1}}$. Moreover, if $w' = (x_1 - \kappa_1)X_{\{1\}} \cdot \hat{\mathbf{1}}$, and $\hat{\Phi}_{[a,b]}^- = \mathcal{C}\ell(d) \cdot w'$, then*

$$\hat{\Phi}_{[a,b]} = \hat{\Phi}_{[a,b]}^+ \oplus \hat{\Phi}_{[a,b]}^-.$$

The submodules $\hat{\Phi}_{[a,b]}^\pm$ are simple modules of type M.

- (iii) *If $0 < a \leq b$, the $\hat{\Phi}_{[-a,b]}$ has a submodule $\hat{\Phi}_{[-a,b]}^+ = \mathcal{C}\ell(d)w \oplus \mathcal{C}\ell(d)\bar{w}$, where*

$$w = -(1 + \sqrt{-1}c_a c_{a+1})X_{\{a+1\}} \cdot \hat{\mathbf{1}} \quad \text{and} \quad \bar{w} = s_a w.$$

Moreover, if

$$w' = -(1 - \sqrt{-1}c_a c_{a+1})X_{\{a+1\}} \cdot \hat{\mathbf{1}}, \quad \bar{w}' = s_a w',$$

and $\hat{\Phi}_{[-a,b]}^- = \mathcal{C}\ell(d)w' \oplus \mathcal{C}\ell(d)\bar{w}'$, then

$$\hat{\Phi}_{[-a,b]} = \hat{\Phi}_{[-a,b]}^+ \oplus \hat{\Phi}_{[-a,b]}^-.$$

The submodules $\hat{\Phi}_{[-a,b]}^\pm$ are simple of type M.

Proof. (i) First, we deduce that $\hat{\Phi}_{[0,d-1]}$ is irreducible by character considerations. It has two *non-homogeneous* submodules:

$$\mathcal{C}\ell(d)(\sqrt{-d} + (c_1 + \cdots + c_d)) \cdot \hat{\mathbf{1}}_{[0,d-1]} \quad \text{and} \quad \mathcal{C}\ell(d)(\sqrt{-d} - (c_1 + \cdots + c_d)) \cdot \hat{\mathbf{1}}_{[0,d-1]}.$$

These vector spaces are clearly stable under the action of $\mathcal{S}(d)$. Since x_1 acts by zero on these vector spaces, the action of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ factors through $\mathcal{S}(d)$ and thus these vector spaces are $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -submodules. Therefore $\hat{\Phi}_{[0,d-1]}$ is of type Q (cf. Section 2).

(ii) Let $\hat{\mathbf{1}} = \hat{\mathbf{1}}_{[a,b]}$, $w = X_\emptyset \cdot \hat{\mathbf{1}}$ and $\hat{\Phi}_{[a,b]}^+ = \mathcal{C}\ell(d) \cdot w$. By Lemma 4.1.5, $w \neq 0$. Now, Lemmas 4.1.6 and 4.1.7 together imply that $\hat{\Phi}_{[a,b]}^+$ is a submodule.

It now remains to show that $\hat{\Phi}_{[a,b]} = \hat{\Phi}_{[a,b]}^+ \oplus \hat{\Phi}_{[a,b]}^-$, where $\hat{\Phi}_{[a,b]}^-$ is as in the statement of the proposition. To this end, assume that $w' \in \hat{\Phi}_{[a,b]}^+$. That is, there exists $p(c) \in \mathcal{C}\ell(d)$ such that $p(c) \cdot w = w'$. Write

$$p(c) = \sum_{\varepsilon} a_{\varepsilon} c^{\varepsilon},$$

where the sum is over $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{Z}_2^d$. Then, for $1 \leq i \leq d$,

$$(-1)^{\delta_{1i}} w' = \frac{1}{\kappa_i} x_i \cdot w' = \frac{1}{\kappa_i} x_i \left(\sum_{\varepsilon} a_{\varepsilon} c^{\varepsilon} \right) \cdot w = \left(\sum_{\varepsilon} (-1)^{\varepsilon_i} a_{\varepsilon} c^{\varepsilon} \right) \cdot w,$$

where (of course) the δ on the left of the equal sign is the Kronecker delta. This forces $p(c) = rc_1 + s$ for complex numbers r and s . Since w' is even, $r = 0$ implying that $w' = sw$ which is impossible.

(iii) We deal with $\hat{\Phi}_{[-a,b]}^+$, the proposed submodule $\Phi_{[-a,b]}^-$ being similar. Let $w = -(1 + \sqrt{-1}c_ac_{a+1})X_{\{a+1\}}.\hat{\mathbf{1}}$, $\bar{w} = s_a.w$, and $\hat{\Phi}_{[a,b]}^+ = \mathcal{C}\ell(d).w + \mathcal{C}\ell(d).\bar{w}$. The proof of Lemma 4.1.5 shows that

$$X_{\{a+1\}}.\hat{\mathbf{1}} = \prod_{\substack{1 \leq i \leq d \\ i \neq a+1}} (a+i-1+\kappa_i).\hat{\mathbf{1}} + (\star).\hat{\mathbf{1}}$$

where $(\star) = p'(c) - \varphi p''(c)$ where $p'(c) \in \mathcal{C}\ell(d)_{\bar{0}}$, $p''(c) \in \mathcal{C}\ell(d)_{\bar{1}}$, and $p'(c)$ has no constant term. It is also easy to see that $p'(c)$ and $p''(c)$ have coefficients in $\mathbb{R}$. We conclude from this that $w \neq 0$. Note that by definition, $c_ac_{a+1}.w = -\sqrt{-1}w$.

Lemma 4.1.6 shows that for $i \neq a, a+1$, $x_i.w = \kappa_i w$. Moreover,

$$x_a.w = -(1 - \sqrt{-1}c_ac_{a+1})x_a X_{\{a+1\}}.\hat{\mathbf{1}} = 0.$$

Also, $x_a.\hat{\mathbf{1}} = -c_ac_{a+1}x_{a+1}.\hat{\mathbf{1}}$ (see the computation (4.1.6) for details). Thus,

$$w = -\sqrt{-1}(1 + \sqrt{-1}c_ac_{a+1})X_{\{a\}}.\hat{\mathbf{1}} \quad (4.1.7)$$

so $x_{a+1}.w = 0$. As for $\bar{w} = s_a.w$, $x_i.\bar{w} = \kappa_i \bar{w}$ for $i \neq a, a+1$. Using commutation relations, we compute

$$x_a.\bar{w} = x_a s_a.w = (s_a x_{a+1} - 1 - c_a c_{a+1}).w = -(1 + \sqrt{-1})w. \quad (4.1.8)$$

Similarly,

$$x_{a+1}.\bar{w} = (1 + \sqrt{-1})w. \quad (4.1.9)$$

We now turn to the action of the symmetric group. First, for $i \neq a-1, a+1$, Lemma 4.1.7 shows that $s_i.w \in \hat{\Phi}_{[a,b]}^+$. Also by Lemma 4.1.7,

$$s_{a-1}X_{\{a+1\}}.\hat{\mathbf{1}} = \frac{\kappa_{a-1}}{2}(c_{a-1}c_a - 1)X_{\{a+1\}}.\hat{\mathbf{1}}.$$

Thus,

$$\begin{aligned} s_{a-1}.w &= -\frac{\kappa_{a-1}}{2}(1 + \sqrt{-1}c_{a-1}c_{a+1})(c_{a-1}c_a - 1)X_{\{a+1\}}.\hat{\mathbf{1}} \\ &= -\frac{\kappa_{a-1}}{2}(1 + c_{a-1}c_a + \sqrt{-1}c_{a-1}c_{a+1} - \sqrt{-1}c_ac_{a+1})X_{\{a+1\}}.\hat{\mathbf{1}} \\ &= \frac{\kappa_{a-1}}{2}(c_{a-1}c_a - 1).w. \end{aligned}$$

Similarly, by (4.1.7) and Lemma 4.1.7,

$$s_{a+1}.w = \frac{\kappa_{a+2}}{2}(1 + c_{a+1}c_{a+2}).w.$$

Now, for $i \neq a-1, a+1$, $s_i s_a = s_a s_i$. Hence, by Lemma 4.1.7

$$s_i.\bar{w} = \left(\frac{\kappa_{i+1} + \kappa_i}{2(a+i)} + \frac{\kappa_{i+1} - \kappa_i}{2(a+i)}c_i c_{i+1} \right).\bar{w}. \quad (4.1.10)$$

To deduce the action of s_{a-1} and s_a on $\bar{w}$, we proceed as in the proof of Lemma 4.1.7. Recall again the intertwining elements ϕ_{a-1} and ϕ_{a+1} . By character considerations, we deduce that

$\phi_{a-1}.\bar{w} = 0 = \phi_{a+1}.\bar{w}$. Unlike in lemma 4.1.6, in this case the action of x_a (resp. x_{a+1}) is given by (4.1.8) (resp. (4.1.9)). Thus,

$$s_{a-1}.\bar{w} = \frac{(1 + \sqrt{-1})}{2}(1 + c_{a-1}c_a).w - \frac{\kappa_{a-1}}{2}(1 - c_{a-1}c_a).\bar{w} \quad (4.1.11)$$

and

$$s_{a+1}.\bar{w} = \frac{(1 - \sqrt{-1})}{2}(1 - c_{a+1}c_{a+2}).w + \frac{\kappa_{a+2}}{2}(1 + c_{a+1}c_{a+2}).\bar{w}. \quad (4.1.12)$$

It is easy to see that $\hat{\Phi}_{[-a,b]} = \hat{\Phi}_{[-a,b]}^+ + \hat{\Phi}_{[-a,b]}^-$ since $\frac{1}{2}(w + w') = X_{\{a\}}.\hat{\mathbf{1}}$ is a cyclic vector for $\hat{\Phi}_{[-a,b]}$. As in part (ii), it is easy to see that if $w' = p(c)w + r(c)s_aw$ where $p(c)$ and $r(c)$ are polynomials in the Clifford generators, that $p(c) = \lambda_1 + \lambda_2c_ac_{a+1}$ and $r(c) = \lambda_3 + \lambda_4c_ac_{a+1}$ for some complex numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Noting that $c_ac_{a+1}w = -\sqrt{-1}w$ gives that all the coefficients are zero.

Therefore, we are left to show that $\hat{\Phi}_{[-a,b]}^+$ is simple. Indeed, assume $V \subseteq \hat{\Phi}_{[-a,b]}^+$ is a submodule. Then,

$$\text{ch } V = [a - 1, \dots, 0, 0, \dots, b].$$

Let $v = p_1(c).w + p_2(c).\bar{w} \in V$ be a vector satisfying $x_i.v = \kappa_i v$ for all i , where $p_1(c), p_2(c) \in \mathcal{C}\ell(d)$. For $i = 1, 2$, define $p'_i(c)$ by the formulae $x_ap_i(c) = p'_i(c)x_a$. Then,

$$0 = x_a.v = -(1 + \sqrt{-1})p'_2(c).w$$

showing that $p'_2(c) = 0$ (hence, $p_2(c) = 0$). Now, arguing as above with the vector $s_a.v$ shows that $p_1(c) = 0$. $\square$

We can now define the irreducible segment representations which are the key to defining the standard $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -modules.

Definition 4.1.9. Let $a, b \in \mathbb{Z}_{\geq 0}$.

- (1) Let $\Phi_{[0,d-1]} = \hat{\Phi}_{[0,d-1]}^+, \mathbf{1} := X_{\{1\}}.\hat{\mathbf{1}}$, where $\kappa_i = \sqrt{q(i-1)}$.
- (2) If $0 < a \leq b$, let $\Phi_{[a,b]} = \hat{\Phi}_{[a,b]}^+$ in Proposition 4.1.8(ii), with $\kappa_i = +\sqrt{q(a+i-1)}$ for all i , and let $\mathbf{1} := w$.
- (3) If $0 < a \leq b$, let $\Phi_{[-a,b]} = \hat{\Phi}_{[-a,b]}^+$ with $\kappa_i = +\sqrt{q(-a+i-1)}$, $\mathbf{1} := w$ and $\bar{\mathbf{1}} := \bar{w}$.
- (4) If $0 \leq a$, let $\Phi_{[a,a-1]} = \Phi_{\emptyset} = \mathbb{C}$.

4.2. Some Lie Theoretic Notation. It is convenient in this section to introduce some Lie theoretic notation. This section differs from [24] in that the notation defined here is associated to the Lie superalgebra $\mathfrak{q}(n)$ (as opposed to the Kac-Moody algebra $\mathfrak{b}_{\infty}$).

Define the sets $P = \mathbb{Z}^n$, $P_{\geq 0} = \mathbb{Z}_{\geq 0}^n$, and

$$P^+ = \{ \lambda = (\lambda_1, \dots, \lambda_n) \in P \mid \lambda_i \geq \lambda_{i+1} \text{ for all } 1 \leq i \leq n \} \quad (4.2.1)$$

$$P^{++} = \{ \lambda \in P^+ \mid \lambda_i + \lambda_j \neq 0 \text{ for all } 1 \leq i, j \leq n \} \quad (4.2.2)$$

$$P_{\text{rat}}^+ = \{ \lambda \in P^+ \mid \lambda_i = \lambda_{i+1} \text{ implies } \lambda_i = 0 \} \quad (4.2.3)$$

$$P_{\text{poly}}^+ = \{ \lambda \in P_{\text{rat}}^+ \mid \lambda_n \geq 0 \} \quad (4.2.4)$$

$$P_{\geq 0} = \{ \lambda \in P \mid \lambda_i \geq 0 \text{ for all } i \}, \quad (4.2.5)$$

The weights (4.2.1) are called dominant, and (4.2.2) are called dominant typical. A weight $\lambda \in P$ is simply *typical* if $\lambda_i + \lambda_j \neq 0$ for all i, j . The weights (4.2.3) are called rational, (4.2.4) are polynomial, and the set 4.2.5 are simply compositions. For each of the sets $X = P^+, P^{++}, P_{\text{rat}}^+, P_{\text{poly}}^+, P_{\geq 0}$ above, define

$$X(d) = \{\lambda \in X \mid \lambda_1 + \cdots + \lambda_n = d\}.$$

Let $R \subset P$ be the root system of type A_{n-1} . That is, $R = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}$ where α_{ij} is the n -tuple with 1 in the i th coordinate and -1 in the j th coordinate. The positive roots are $R^+ = \{\alpha_{ij} \in R \mid i < j\}$, the root lattice Q is the $\mathbb{Z}$ -span of R , and Q^+ is the $\mathbb{Z}_{\geq 0}$ -span of R^+ . The symmetric group, S_n , acts on P by place permutation. Define the length function $\ell : S_n \rightarrow \mathbb{Z}_{\geq 0}$ in the usual way:

$$\ell(w) = |\{\alpha \in R^+ \mid w(\alpha) \in -R^+\}|.$$

Equivalently, $\ell(w)$ is the number of simple transpositions occurring in a reduced expression for w . Write $w \rightarrow y$ if $y = s_\alpha w$ for some $\alpha \in R^+$ and $\ell(w) < \ell(y)$. Define the *Bruhat* order on S_n by $w <_b y$ if there exists a sequence $w \rightarrow w_1 \rightarrow \cdots \rightarrow y$. Also, for $\lambda \in P$, define

$$S_n[\lambda] = \{w \in S_n \mid w(\lambda) = \lambda\}, \quad \text{and} \quad R[\lambda] = \{\alpha_{ij} \in R \mid s_{ij}(\lambda) = \lambda\},$$

and define

$$P^+[\lambda] = \{\mu \in P \mid \mu_i \geq \mu_j \text{ if } s_{ij} \in S_n[\lambda]\}, \quad \text{and} \quad P^-[\lambda] = \{\mu \in P \mid \mu_i \leq \mu_j \text{ if } s_{ij} \in S_n[\lambda]\}$$

where $s_{ij} \in S_n$ denotes the transposition (ij) .

4.3. Induced Modules. Using the irreducible segment representations defined above we now define standard representations. Let $\lambda, \mu \in P$ satisfy $\lambda - \mu \in P_{\geq 0}(d)$. Define

$$\widehat{\Phi}(\lambda, \mu) = \widehat{\Phi}_{[\mu_1, \lambda_1 - 1]} \boxtimes \cdots \boxtimes \widehat{\Phi}_{[\mu_n, \lambda_n - 1]}$$

and

$$\Phi(\lambda, \mu) = \Phi_{[\mu_1, \lambda_1 - 1]} \otimes \cdots \otimes \Phi_{[\mu_n, \lambda_n - 1]},$$

and define *standard (cyclic) modules* for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ by

$$\widehat{\mathcal{M}}(\lambda, \mu) = \text{Ind}_{d_1, \dots, d_n}^d \widehat{\Phi}(\lambda, \mu) \tag{4.3.1}$$

and

$$\mathcal{M}(\lambda, \mu) = \text{Ind}_{d_1, \dots, d_n}^d \Phi(\lambda, \mu). \tag{4.3.2}$$

We call the standard modules $\widehat{\mathcal{M}}(\lambda, \mu)$ and $\mathcal{M}(\lambda, \mu)$ *big* and *little*, respectively.

Both the big and little standard modules are cyclic. Let

$$\widehat{\mathbf{1}}_{\lambda, \mu} = 1 \otimes (\widehat{\mathbf{1}} \otimes \cdots \otimes \widehat{\mathbf{1}}) \in \widehat{\mathcal{M}}(\lambda, \mu) \tag{4.3.3}$$

be the distinguished cyclic generator of $\widehat{\mathcal{M}}(\lambda, \mu)$. Fix the following choice of distinguished cyclic generator $\mathbf{1}_{\lambda, \mu} \in \mathcal{M}(\lambda, \mu)$. Let $i_1 < \cdots < i_k$ be such that $\mu_{i_j} = 0$ for all j and $\gamma_0(\mu) = k$. Choose

$$\mathbf{1}_{\lambda, \mu} = \prod_{j=1}^{\lfloor k/2 \rfloor} (1 - \sqrt{-1} c_{i_{2j-1}} c_{i_{2j}}) 1 \otimes (\mathbf{1} \otimes \cdots \otimes \mathbf{1}).$$

Lemma 4.3.1. *Let $\lambda, \mu \in P$ so that $\lambda - \mu \in P_{\geq 0}(d)$. Then,*

$$(i) \dim \widehat{\mathcal{M}}(\lambda, \mu) = \frac{d!}{d_1! \cdots d_n!} 2^{d+n-\gamma_0(\mu)}$$

- (ii) $\dim \mathcal{M}(\lambda, \mu) = \frac{d!}{d_1! \cdots d_n!} 2^{d - \lfloor \frac{\gamma_0(\mu)}{2} \rfloor}$
- (iii) $\widehat{\mathcal{M}}(\lambda, \mu) \cong \mathcal{M}(\lambda, \mu)^{\oplus 2^{n - \lfloor \frac{\gamma_0(\mu)+1}{2} \rfloor}}$.

Proof. (i) The dimension of $\widehat{\mathcal{M}}(\lambda, \mu)$ follows from the definition.

(ii) Use Proposition 4.1.8.

(iii) Since induction commutes with direct sums we have that $\widehat{\mathcal{M}}(\lambda, \mu)$ is a direct sum of copies of $\mathcal{M}(\lambda, \mu)$. A count using (i) and (ii) yields (iii). $\square$

We end this section by recording certain data about the weight spaces and generalized weight spaces of $\mathcal{M}(\lambda, \mu)$ which will be useful later. Define the weight $\zeta_{\lambda, \mu} : \mathcal{P}_d[x^2] \rightarrow \mathbb{C}$ by $f \cdot \mathbf{1}_{\lambda, \mu} = \zeta_{\lambda, \mu}(f) \mathbf{1}_{\lambda, \mu}$ for all $f \in \mathcal{P}_d[x]$. As in §4.2, the symmetric group, S_d , acts on an integral weight $\zeta : \mathcal{P}_d[x^2] \rightarrow \mathbb{C}$ by $w(\zeta)(x_i^2) = \zeta(x_{w(i)}^2)$. Let

$$S_d[\zeta] = \{ w \in S_d \mid w(\zeta) = \zeta \}.$$

Define $\ell(w)$ to be the length of w (i.e. the number of simple transpositions occurring in a reduced expression of w) and recall the definition of the Bruhat order given in section 4.2.

Lemma 4.3.2. *Given $\lambda, \mu \in P$ with $\lambda - \mu \in P_{\geq 0}(d)$,*

- (i) $P(\mathcal{M}(\lambda, \mu)) = \{ w(\zeta_{\lambda, \mu}) \mid w \in D_{\lambda - \mu} \},$
- (ii) *For any $\zeta \in P(\mathcal{M}(\lambda, \mu))$,*

$$\dim \mathcal{M}(\lambda, \mu)_{\zeta}^{\text{gen}} = 2^{d - \lfloor \frac{\gamma_0(\mu)}{2} \rfloor} |\{ w \in D_{\lambda - \mu} \mid w(\zeta) = \zeta \}|.$$

In particular,

$$\dim \mathcal{M}(\lambda, \mu)_{\zeta_{\lambda, \mu}}^{\text{gen}} = 2^{d - \lfloor \frac{\gamma_0(\mu)}{2} \rfloor} |D_{\lambda - \mu} \cap S_d[\zeta_{\lambda, \mu}]|.$$

Proof. (i) This follows directly upon applying the Mackey Theorem to the character map.

(ii) Given $f \in \mathcal{P}_d[x^2]$ and $w \in S_d$, we have the relation

$$fw = w \cdot w^{-1}(f) + \sum_{u <_b w} u C_u f_u$$

where the sum is over $u <_b w$ in the Bruhat order, $C_u \in \mathcal{C}\ell(d)$, $f_u \in \mathcal{P}_d[x]$ and $\deg f_u < \deg f$, see [24, Lemma 14.2.1]. Therefore, if $f \in \mathcal{P}_d[x^2]$, $C \in \mathcal{C}\ell(d)$ and $w \in D_{\lambda - \mu}$,

$$f(wC \cdot \mathbf{1}_{\lambda, \mu}) = w(\zeta_{\lambda, \mu})(f)wC \cdot \mathbf{1}_{\lambda, \mu} + \sum_{u <_b w} u C_u f_u \cdot \mathbf{1}_{\lambda, \mu} \quad (4.3.4)$$

where the sum is over $u \in D_{\lambda - \mu}$. In particular, $wC \cdot \mathbf{1}_{\lambda - \mu} \in \mathcal{M}(\lambda, \mu)_{\zeta_{\lambda, \mu}}^{\text{gen}}$ only if $w \in D_{\lambda - \mu} \cap S_d[\zeta_{\lambda, \mu}]$. Conversely, if $w \in D_{\lambda - \mu} \cap S_d[\zeta_{\lambda, \mu}]$, it is straightforward to see that all u occurring on the right hand side of (4.3.4) also belong to $D_{\lambda - \mu} \cap S_d[\zeta_{\lambda, \mu}]$. This gives the result. $\square$

4.4. Unique Simple Quotients. In general, the standard cyclic module $\mathcal{M}(\lambda, \mu)$ may not have a unique simple head. However, in this subsection, we determine sufficient conditions for this to hold. Throughout this section, keep in mind that $q(a) = q(-a - 1)$ for all $a \in \mathbb{Z}$. We follow closely the strategy in [45]. We begin with some preparatory lemmas.

Lemma 4.4.1. *Let M be an $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module, and ζ a weight of M , then there exists $v \in \mathcal{M}(\lambda, \mu)_\zeta$ such that*

$$x_i.v = \sqrt{q(\zeta(x_i^2))} v$$

for all $i = 1, \dots, d$.

Proof. Choose $0 \neq v_0 \in M_\zeta$. Recall the definition 4.1.4. We adapt this to our current situation by setting $\kappa_i = \sqrt{q(\zeta(x_i^2))}$ and $S = \{i \mid x_i v = -\kappa_i v\}$. Then, $v_1 := X_S.v_0 \in \mathcal{M}_\zeta$ is nonzero and $x_i.v_1 = \pm \kappa_i v_1$ for all i . Now, set

$$v = \left(\prod_{i \in S} c_i \right) v_1.$$

Then, v is nonzero and has the desired properties. $\square$

Therefore, we may define the non-zero subspace

$$M_{\sqrt{\zeta}} = \left\{ m \in M_\zeta \mid x_i.m = \sqrt{q(\zeta(x_i^2))} m \text{ for } i = 1, \dots, d \right\}.$$

We will use the following key lemma repeatedly in this section.

Lemma 4.4.2. *Let Y be in $\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ and $v \in Y_{\sqrt{\zeta}}$ for some weight ζ . Assume that for some $1 \leq i < d-1$, $x_i.v = \sqrt{q(a)}$, $x_{i+1}.v = \sqrt{q(b)}$ where $a, b \in \mathbb{Z}$ and either $q(a) \neq 0$ or $q(b) \neq 0$. Further, if $q(a) = q(b \pm 1)$, assume that*

$$s_{i+1}.v = (\kappa_1 + \kappa_2 c_{i+1} c_{i+2}).v \tag{4.4.1}$$

for some constants $\kappa_1, \kappa_2 \in \mathbb{C}$, not both 0. Then, $v \in \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d). \phi_i.v$.

Proof. First, if $q(a) = q(b) \neq 0$, then using (3.8.1) and Lemma 14.8.1 of [24] we deduce that

$$\phi_i.v = 2q(a)v \neq 0,$$

so the result is trivial. If $q(a) \neq q(b \pm 1)$, then using (3.8.2) we deduce that

$$\phi_i^2.v = (2q(a) - 2q(b) - (q(a) - q(b))^2)v \neq 0$$

and again the result is trivial.

Now, let $\kappa_3 = q(a) - q(b) \neq 0$, $\kappa_4 = \sqrt{q(a)} - \sqrt{q(b)} \neq 0$ and $\kappa_5 = \sqrt{q(a)} + \sqrt{q(b)} > 0$. Then, appealing again to (3.8.1) we have that

$$\phi_i v = (\kappa_3 s_i - \kappa_4 c_i c_{i+1} + \kappa_5)v$$

Let $\mathbf{c}'$ and $\mathbf{c}''$ be two elements of the Clifford algebra. Consider an expression of the form

$$\begin{aligned} (1 + \mathbf{c}' s_{i+1} - \mathbf{c}'' s_i s_{i+1}) \phi_i v = & (\kappa_3 s_i - \kappa_4 c_i c_{i+1} + \kappa_5 + \kappa_3 \mathbf{c}' s_{i+1} s_i \\ & - \kappa_4 \mathbf{c}' c_i c_{i+2} s_{i+1} + \kappa_5 \mathbf{c}' s_{i+1} - \kappa_3 \mathbf{c}'' s_{i+1} s_i s_{i+1} \\ & + \kappa_4 \mathbf{c}'' c_{i+1} c_{i+2} s_i s_{i+1} - \kappa_5 \mathbf{c}'' s_i s_{i+1}) v. \end{aligned}$$

By (4.4.1), this equals

$$\begin{aligned} & (\kappa_3 s_i - \kappa_4 c_i c_{i+1} + \kappa_5 + \kappa_3 \mathbf{c}' s_{i+1} s_i - \kappa_1 \kappa_4 \mathbf{c}' c_i c_{i+2} \\ & - \kappa_2 \kappa_4 \mathbf{c}' c_i c_{i+1} + \kappa_1 \kappa_5 \mathbf{c}' + \kappa_2 \kappa_5 \mathbf{c}' c_{i+1} c_{i+2} - \kappa_1 \kappa_3 \mathbf{c}'' s_{i+1} s_i \\ & - \kappa_2 \kappa_3 \mathbf{c}'' c_i c_{i+1} s_{i+1} s_i + \kappa_1 \kappa_4 \mathbf{c}'' c_{i+1} c_{i+2} s_i - \kappa_2 \kappa_4 \mathbf{c}'' c_i c_{i+1} s_i \\ & - \kappa_1 \kappa_5 \mathbf{c}'' s_i - \kappa_2 \kappa_5 \mathbf{c}'' c_i c_{i+2} s_i) v. \end{aligned}$$

The coefficient of $s_i v$ is

$$\kappa_3 + \kappa_1 \kappa_4 \mathbf{c}'' c_{i+1} c_{i+2} - \kappa_2 \kappa_4 \mathbf{c}'' c_i c_{i+1} - \kappa_1 \kappa_5 \mathbf{c}'' - \kappa_2 \kappa_5 \mathbf{c}'' c_i c_{i+2}.$$

The coefficient of $s_{i+1} s_i v$ is

$$\kappa_3 \mathbf{c}' - \kappa_1 \kappa_3 \mathbf{c}'' - \kappa_2 \kappa_3 \mathbf{c}'' c_i c_{i+1}.$$

In order to make both of these coefficients zero, set $\mathbf{c}' = \mathbf{c}''(\kappa_1 + \kappa_2 c_i c_{i+1})$ and

$$\mathbf{c}'' = \gamma(\kappa_1 \kappa_5 + \kappa_1 \kappa_4 c_{i+1} c_{i+2} - \kappa_2 \kappa_4 c_i c_{i+1} - \kappa_2 \kappa_5 c_i c_{i+2}),$$

where

$$\gamma = \frac{-\kappa_3}{(\kappa_1^2 + \kappa_2^2)(\kappa_4^2 + \kappa_5^2)}.$$

The coefficient of v is

$$\begin{aligned} & -\kappa_4 c_i c_{i+1} + \kappa_5 - \kappa_1 \kappa_4 \mathbf{c}' c_i c_{i+2} - \kappa_2 \kappa_4 \mathbf{c}' c_i c_{i+1} + \kappa_1 \kappa_5 \mathbf{c}' + \kappa_2 \kappa_5 \mathbf{c}' c_{i+1} c_{i+2} \\ & = -\kappa_4 c_i c_{i+1} + \kappa_5 - \kappa_1 \kappa_4 \mathbf{c}''(\kappa_1 c_i c_{i+2} + \kappa_2 c_{i+1} c_{i+2}) - \kappa_2 \kappa_4 \mathbf{c}''(\kappa_1 c_i c_{i+1} - \kappa_2) \\ & + \kappa_1 \kappa_5 \mathbf{c}''(\kappa_1 + \kappa_2 c_i c_{i+1}) + \kappa_2 \kappa_4 \mathbf{c}''(\kappa_1 c_{i+1} c_{i+2} - \kappa_2 c_i c_{i+2}). \end{aligned}$$

This is equal to

$$\begin{aligned} & \kappa_5 - \kappa_4 c_i c_{i+1} + (-\kappa_1 \kappa_2 \kappa_4 + \kappa_1 \kappa_2 \kappa_5) \mathbf{c}'' c_i c_{i+1} + (-\kappa_1^2 \kappa_4 - \kappa_2^2 \kappa_5) \mathbf{c}'' c_i c_{i+2} \\ & + (-\kappa_1 \kappa_2 \kappa_4 + \kappa_1 \kappa_2 \kappa_5) \mathbf{c}'' c_{i+1} c_{i+2} + (\kappa_2^2 \kappa_4 + \kappa_1^2 \kappa_5) \mathbf{c}'' \\ & = \kappa_5 - \kappa_4 c_i c_{i+1} + (\kappa_1 \kappa_2 \kappa_5 - \kappa_1 \kappa_2 \kappa_4) \gamma (-\kappa_1 \kappa_5 c_i c_{i+2} - \kappa_1 \kappa_4 c_i c_{i+2} - \kappa_2 \kappa_4 - \kappa_2 \kappa_5 c_{i+1} c_{i+2}) \\ & + (-\kappa_1^2 \kappa_4 - \kappa_2^2 \kappa_5) \gamma (-\kappa_1 \kappa_5 c_i c_{i+2} + \kappa_1 \kappa_4 c_i c_{i+1} - \kappa_2 \kappa_5 + \kappa_2 \kappa_4 c_{i+1} c_{i+2}) \\ & + (-\kappa_1 \kappa_2 \kappa_4 + \kappa_1 \kappa_2 \kappa_5) \gamma (-\kappa_1 \kappa_5 c_{i+1} c_{i+2} - \kappa_2 \kappa_4 c_i c_{i+2} + \kappa_1 \kappa_4 + \kappa_2 \kappa_5 c_i c_{i+1}) \\ & + (\kappa_2^2 \kappa_4 + \kappa_1^2 \kappa_5) \gamma (-\kappa_1 \kappa_4 c_{i+1} c_{i+2} + \kappa_2 \kappa_4 c_i c_{i+1} - \kappa_1 \kappa_5 + \kappa_2 \kappa_5 c_i c_{i+2}) \\ & = \kappa_5 + \delta_1 c_i c_{i+1} + \delta_2 c_{i+1} c_{i+2} + \delta_3 c_i c_{i+2} \end{aligned}$$

for some constants $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$.

Thus,

$$\begin{aligned} & (\kappa_5 - \delta_1 c_i c_{i+1} - \delta_2 c_{i+1} c_{i+2} - \delta_3 c_i c_{i+2})(1 + \mathbf{c}' s_{i+1} - \mathbf{c}'' s_i s_{i+1}) \phi_i v \\ & = (\kappa_5 + \delta_1^2 + \delta_2^2 + \delta_3^2) v. \end{aligned}$$

Since $\delta_1^2, \delta_2^2, \delta_3^2 \in \mathbb{R}_{\geq 0}$ and $\kappa_5 > 0$, the result follows. $\square$

Proposition 4.4.3. *Assume that $\lambda \in P^{++}$, $\mu \in P^+[\lambda]$, and $\lambda - \mu \in P_{\geq 0}(d)$. Then,*

$$\mathcal{M}(\lambda, \mu)_{\sqrt{\zeta_{\lambda, \mu}}} = \mathbb{C} \mathbf{1}_{\lambda, \mu}.$$

We begin by proving a special case of the Proposition. Suppose n divides d , and $d/n = b - a$ for some $a, b \in \mathbb{Z}$, $b > 0$. Let $\lambda = (b, \dots, b)$ and $\mu = (a, \dots, a)$ be weights of $\mathfrak{q}(n)$. Set $\mathcal{M}_{a,b,n} = \mathcal{M}(\lambda, \mu)$, and set $\mathbf{1}_{a,b,n} = \mathbf{1}_{\lambda, \mu}$. Let

$$\zeta_{a,b,n} = (a, a+1, \dots, b-1, \dots, a, a+1, \dots, b-1)$$

be a weight for $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ where the sequence $a, a+1, \dots, b-1$ appears n times.

The first goal is to compute the weight space $(\mathcal{M}_{a,b,n})_{\sqrt{\zeta_{a,b,n}}}$.

Set $n = d$ in the definition above so that $b = a + 1$. The resulting module is the Kato module $K(a, \dots, a) = K_a$, where all the x_i^2 act by $q(a)$ on the vector $\mathbf{1}_{a,b,n}$.

The following is [24, Lemma 16.3.2, Theorem 16.3.3].

- Lemma 4.4.4.** (1) *If $a \neq -1$ or 0 , the weight space of $K(a, \dots, a)$ corresponding to $(a, \dots, a)$ with respect to the operators $x_1^2, \dots, x_n^2$ has dimension 2^n . If $a = -1$ or 0 , then the weight space of $K(a, \dots, a)$ corresponding to $(a, \dots, a)$ with respect to the operators $x_1, \dots, x_n$ has dimension $2^{\lfloor \frac{n+1}{2} \rfloor}$.*
- (2) *The module $K(a, \dots, a)$ is equal to its generalized weight space for the weight $(a, \dots, a)$.*
- (3) *The module $K(a, \dots, a)$ is simple of type Q if $a = 0$ and d is odd, and is of type M otherwise.*

Set $m = d/n$. In the set of weights of $\mathcal{M}_{a,b,n}$, there exists a unique anti-dominant weight $\zeta_{a,b,n}^\circ$ that is given by

$$\zeta_{a,b,n}^\circ = (\underbrace{a, \dots, a}_n, \underbrace{a+1, \dots, a+1}_n, \dots, \underbrace{b-1, \dots, b-1}_n).$$

Take an element $\tau \in D_{\lambda-\mu}$ such that $\tau(\zeta_{a,b,n}) = \zeta_{a,b,n}^\circ$. If $a \geq 0$, it is given by $\tau = \omega^1 \cdots \omega^{m-1}$, where $\omega^p = \rho_{n-1}^p \rho_{n-2}^p \cdots \rho_1^p$,

$$\rho_k^p = \xi_{k(p+1)-(k-1)}^p \cdots \xi_{(k(p+1)-1)}^p \xi_{k(p+1)}^p,$$

and, for $1 \leq r \leq d-1$, and $1 \leq p \leq d-r$, $\xi_r^p = s_{r+p-1} \cdots s_{r+1} s_r$.

If $b \leq 0$, then $\tau = \sigma(\omega^1 \cdots \omega^{m-1})$, where σ is the automorphism of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$. Finally, if $a < 0$ and $b > 0$, $\tau = \sigma_{(-a+1)n}(\omega^2 \cdots \omega^{-a}) \omega^{-a+1} \cdots \omega^{m-1}$, where σ_{-a} is the automorphism of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(-a) \subseteq \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ embedded on the left.

Lemma 4.4.5. *The vector $\phi_\tau \mathbf{1}_{a,b,n}$ is a cyclic vector of $\mathcal{M}_{a,b,n}$.*

Proof. This follows from iterated applications of lemma 4.4.2. □

The proof of the following lemma is similar to [45, Lemma A.7], substituting Lemmas 4.4.4 and 4.3.2 appropriately into Suzuki's argument.

Lemma 4.4.6. $(\mathcal{M}_{a,b,n})_{\sqrt{\zeta_{a,b,n}^\circ}} \subseteq \phi_\tau \mathcal{C}\ell(d) \mathbf{1}_{a,b,n}$.

Proof. By an argument similar to the proof of [45, Lemma A.7], we deduce that

$$(\mathcal{M}_{a,b,n})_{\zeta_{a,b,n}^\circ} \cong (K_a)_{a(n)} \otimes (K_{a+1})_{(a+1)(n)} \otimes \cdots \otimes (K_{b-1})_{(b-1)(n)}$$

if $a \geq 0$, and

$$(\mathcal{M}_{a,b,n})_{\zeta_{a,b,n}^\circ} \cong (K_{-a-1})_{(-a-1)(n)} \otimes \cdots \otimes (K_1)_{1(n)} \otimes (K_0)_{0(2n)} \otimes (K_1)_{1(n)} \cdots \otimes (K_{b-1})_{(b-1)(n)}$$

if $a < 0$. Here, $(K_j)_{j(n)}$ is the weight space $K(j, \dots, j)_{(j, \dots, j)}$ of a Kato module. Since

$$(\mathcal{M}_{a,b,n})_{\sqrt{\zeta_{a,b,n}^\circ}} \subseteq (\mathcal{M}_{a,b,n})_{\zeta_{a,b,n}^\circ},$$

we deduce that if $a \geq 0$

$$(\mathcal{M}_{a,b,n})_{\sqrt{\zeta_{a,b,n}^\circ}} = (K_a)_{\sqrt{a(n)}} \otimes (K_{a+1})_{\sqrt{(a+1)(n)}} \otimes \dots \otimes (K_{b-1})_{\sqrt{(b-1)(n)}} \subseteq \mathcal{C}\ell(d) \phi_\tau \mathbf{1}_{a,b,n}.$$

Similarly, if $a < 0$, $(\mathcal{M}_{a,b,n})_{\sqrt{\zeta_{a,b,n}^\circ}} \subseteq \mathcal{C}\ell(d) \phi_\tau \mathbf{1}_{a,b,n}$. $\square$

Proposition 4.4.7. *For the special standard module defined above, $(\mathcal{M}_{a,b,n})_{\sqrt{\zeta_{a,b,n}^\circ}} \subseteq \mathcal{C}\ell(d) \mathbf{1}_{a,b,n}$.*

Proof. For $i = 1, \dots, d$, let $i = jm+r$ where $0 \leq j < n$ and $0 < r < m$. Take any $v \in (\mathcal{M}_{a,b,n})_{\sqrt{\zeta_{a,b,n}^\circ}}$. Lemma 4.4.6 implies that $\phi_\tau v = \phi_\tau z \mathbf{1}$ for some $z \in \mathcal{C}\ell(d)$. Put $v_0 = v - z \mathbf{1}$. Then $\phi_\tau v_0 = 0$. Note that since $r \neq m$, $\phi_i v_0 = 0$ since $s_i(\zeta_{a,b,n})$ is not a weight of $\mathcal{M}_{a,b,n}$.

If $r \neq -a$, we can solve for $s_i v_0$ in the equation $\phi_i v_0 = 0$ to get

$$s_i v_0 = \left(\frac{\kappa_r - \kappa_{r-1}}{-2(a+r)} + \frac{\kappa_r + \kappa_{r-1}}{-2(a+r)} c_i c_{i+1} \right) v_0$$

where $\kappa_r = \sqrt{q(a+r-1)}$.

Similarly, if $r \neq -a$,

$$s_i \mathbf{1}_{a,b,n} = \left(\frac{\kappa_r - \kappa_{r-1}}{-2(a+r)} + \frac{\kappa_r + \kappa_{r-1}}{-2(a+r)} c_i c_{i+1} \right) \mathbf{1}_{a,b,n}.$$

If $r = -a$, then routine calculations from earlier gives that

$$c_i c_{i+1} \mathbf{1}_{a,b,n} = -\sqrt{-1} \mathbf{1}_{a,b,n}.$$

Hence there exists an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -homomorphism $\psi : \mathcal{M}_{a,b,n} \rightarrow \mathcal{M}_{a,b,n}$ such that $\psi(\mathbf{1}_{a,b,n}) = v_0$ if $a \geq 0$ or $b \leq 0$. If $a < 0 < b$, then there is an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -homomorphism $\psi : \mathcal{M}_{a,b,n} \rightarrow \mathcal{M}_{a,b,n}$ such that $\psi(\mathbf{1}_{a,b,n}) = \prod_{0 \leq j < n} (1 + \sqrt{-1} c_{jm-a} c_{jm-a+1}) v_0$

Thus by lemma 4.4.6, the kernel of ψ is equal to $\mathcal{M}_{a,b,n}$. Therefore $v_0 = 0$. Thus $v \in \mathcal{C}\ell(d) \mathbf{1}_{a,b,n}$. $\square$

We now reduce the general case to the special case above. To this end, fix $\lambda \in P^{++}$, $\mu \in P^+[\lambda]$, and $\lambda - \mu \in P_{\geq 0}(d)$. Set $d_i = \lambda_i - \mu_i$, and let $a_i = d_1 + \dots + d_{i-1} + 1$, $b_i = d_1 + \dots + d_i$. Observe that

$$\zeta_{\lambda,\mu}(x_{a_i}^2) = \mu_i \quad \text{and} \quad \zeta_{\lambda,\mu}(x_{b_i}^2) = \lambda_i - 1. \quad (4.4.2)$$

Furthermore, observe that if $a_i \leq c \leq b_i$,

$$\zeta_{\lambda,\mu}(x_c^2) = \zeta_{\lambda,\mu}(x_{b_i}^2) - (b_i - c) \quad \text{and} \quad \zeta_{\lambda,\mu}(x_c^2) = \zeta_{\lambda,\mu}(x_{a_i}^2) + (c - a_i). \quad (4.4.3)$$

Since $\lambda \in P^{++}$ and $\mu \in P^+[\lambda]$, we can find integers $0 = n'_0 < n'_1 < \dots < n'_r = n$, and $0 = n_0 < n_1 < \dots < n_s = n$ such that

$$R[\lambda] = R \cap \sum_{i \neq n'_0, \dots, n'_r} \mathbb{Z} \alpha_i \quad \text{and} \quad R[\lambda] \cap R[\mu] = R \cap \sum_{i \neq n_0, \dots, n_s} \mathbb{Z} \alpha_i.$$

Let

$$I'_p = \{ a_{n'_{p-1}+1}, a_{n'_{p-1}+1} + 1, \dots, b_{n'_p} - 1 \} \quad (p = 1, \dots, r), \quad I' = I'_1 \cup \dots \cup I'_r,$$

and

$$I_p = \{a_{n_{p-1}+1}, a_{n_{p-1}+1} + 1, \dots, b_{n_p} - 1\} \quad (p = 1, \dots, s), \quad I = I_1 \cup \dots \cup I_s.$$

Then, $S_{\lambda-\mu} \subseteq S_I \subseteq S_{I'}$ and

$$S_{I'}/S_{\lambda-\mu} \cong D_{\lambda-\mu} \cap S_{I'} \quad \text{and} \quad S_I/S_{\lambda-\mu} \cong D_{\lambda-\mu} \cap S_I, \quad (\text{cf. §3.4}).$$

Lemma 4.4.8. [45, Lemma A.9] *There is a containment of sets $D_{\lambda-\mu} \cap S_d[\zeta_{\lambda,\mu}] \subset D_{\lambda-\mu} \cap S_I$.*

Let $v \in \mathcal{M}(\lambda, \mu)_{\sqrt{\zeta_{\lambda,\mu}}}$. For each $p \in \{1, \dots, s\}$, we can write $v = \sum_j x_j^{(p)} z_j^{(p)} v_j$ where $v_j \in \Phi(\lambda, \mu)$, $\{x_j^{(p)}\}_j$ are linearly independent elements of $\mathbb{C}[D_{\lambda-\mu} \cap S_{I-I_p}]$ and $z_j^{(p)} \in \mathbb{C}[D_{\lambda-\mu} \cap S_{I_p}]$. Let $\mathcal{P}_d[x^2]_{I_p} = \mathbb{C}[x_i^2 | i \in I_p]$.

Lemma 4.4.9. [45, Lemma A.10] *For $f \in \mathcal{P}_d[x^2]_{I_p}$, $f z_k^{(p)} v_j = \zeta_{\lambda,\mu}(f) z_k^{(p)} v_j$.*

Proof. Observe

$$0 = (f - \zeta_{\lambda,\mu}(f))v = \sum_j x_j^{(p)} (f - \zeta_{\lambda,\mu}(f)) z_j^{(p)} \mathbf{1}_{\lambda,\mu}.$$

Since $S_{I_p} \subset S_d$ is closed with respect to the Bruhat order we have $f z_j^{(p)} \mathbf{1}_{\lambda,\mu} \in \mathbb{C}[D_{\lambda-\mu} \cap S_{I_p}]$. Since $\{x_j^{(p)}\}_j$ are linearly independent, each $(f - \zeta_{\lambda,\mu}(f)) z_j^{(p)} \mathbf{1}_{\lambda,\mu}$ must be 0. $\square$

Proof of Proposition 4.4.3. Let $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(I_p)$ be the subalgebra corresponding to I_p . Note that $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(I_p) \cong \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(|I_p|)$. First note that $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(I_p) v_j \cong \mathcal{M}_{a,b,n_p-n_{p-1}}$ for some a, b . Thus by Proposition 4.4.7, $z_k^{(p)} v_j \in \mathbb{C} \mathbf{1}_{\lambda,\mu}$. Thus, $v \in \mathbb{C}[D_{\lambda-\mu} \cap S_{I-I_p}]$ for any p . It now follows that $v \in \mathbb{C} \mathbf{1}_{\lambda,\mu}$. $\blacksquare$

Theorem 4.4.10. *Assume that $\lambda \in P^{++}$, $\mu \in P^+[\lambda]$, and $\lambda - \mu \in P_{\geq 0}(d)$. Then $\mathcal{M}(\lambda, \mu)$ has a unique simple quotient module, denoted $\mathcal{L}(\lambda, \mu)$.*

Proof. Assume N is a submodule of $\mathcal{M}(\lambda, \mu)$. If $N_{\zeta_{\lambda,\mu}}^{\text{gen}} \neq 0$, then $N_{\sqrt{\zeta_{\lambda,\mu}}} \neq 0$. By the previous lemma, $N \cap \mathcal{C}\ell(d) \mathbf{1}_{\lambda,\mu} \neq \{0\}$, so $\mathbf{1}_{\lambda,\mu} \in N$ because $\mathcal{C}\ell(d) \mathbf{1}_{\lambda,\mu}$ is an irreducible $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(\lambda - \mu)$ -module. Hence, $N = \mathcal{M}(\lambda, \mu)$. It follows that

$$N \subseteq \bigoplus_{\eta \neq \zeta_{\lambda,\mu}} \mathcal{M}(\lambda, \mu)_{\eta}^{\text{gen}}.$$

The sum of all proper submodules satisfies this property. Therefore, $\mathcal{M}(\lambda, \mu)$ has a unique maximal proper submodule and a unique simple quotient. $\square$

Let $\mathcal{R}(\lambda, \mu)$ denote the unique maximal submodule, and define $\mathcal{L}(\lambda, \mu) = \mathcal{M}(\lambda, \mu) / \mathcal{R}(\lambda, \mu)$.

5. CLASSIFICATION OF CALIBRATED REPRESENTATIONS

A representation M of the AHCA is called *calibrated* if the polynomial subalgebra $\mathcal{P}_d[x] \subseteq \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ acts semisimply. The main combinatorial object associated to such a representation is the shifted skew shape. Calibrated representations of the affine Hecke algebra were studied and classified in [35]. The main combinatorial object in that case were pairs of skew shapes and content functions. That construction along with [27, Conjecture 52] motivated the construction given here. A proof of a slightly modified version of that conjecture is given here. Leclerc defined a calibrated representation to be one in which $\mathcal{P}_d[x^2]$ acts semisimply. For example, the module $\Phi_{[-1,0]}$ is calibrated in the sense of [27] but x_1, x_2 do not act diagonally in any basis.

5.1. Construction of Calibrated Representations. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_r)$ be two partitions with $\lambda_1 > \dots > \lambda_r > 0$ and $\mu_1 \geq \dots \geq \mu_r$ such that $\mu_i = \mu_{i+1}$ implies $\mu_i = 0$ and $\lambda_i \geq \mu_i$ for all i . To such data, associate a shifted skew shape of boxes where row i has $\lambda_i - \mu_i$ boxes and the leftmost box occurs in position i . Figure 1 illustrates a skew shape for $\lambda = (5, 2, 1)$ and $\mu = (3, 1, 0)$.

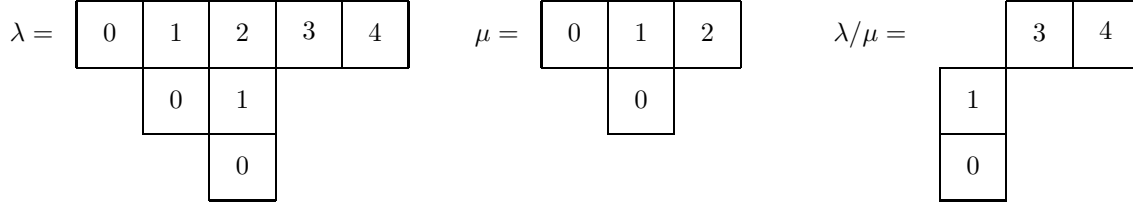


FIGURE 1. Skew Shape filled with contents

A standard filling of a skew shape λ/μ with a total of d boxes is an insertion of the set $\{1, \dots, d\}$ into the boxes of the skew shape such that each box gets exactly one element, each element is used exactly once and the rows are increasing from left to right and the columns are increasing from top to bottom. In a shifted shape, λ , all the boxes will lie above one main diagonal running from northwest to southeast. Each box in this main diagonal will be assigned content 0. The contents of the other boxes will be constant along the diagonals where the contents of the diagonal northeast of its immediate neighbor will be one more than the contents of its immediate neighbor. In a shifted skew shape, λ/μ , the contents are defined as in figure 1.

Given a standard tableaux L for a shifted skew shape λ/μ , let $c(L_i)$ be the contents of the box labeled by i . Thus L gives rise to a d -tuple $c(L) = (c(L_1), \dots, c(L_d))$ called the content reading of λ/μ with respect to L .

Let λ/μ be a shifted skew shape such that λ/μ has d boxes. Set $\kappa_{i,L} = \sqrt{q(c(L_i))}$ and

$$\mathcal{Y}_{i,L} = \sqrt{1 - \frac{1}{(\kappa_{i+1,L} - \kappa_{i,L})^2} - \frac{1}{(\kappa_{i+1,L} + \kappa_{i,L})^2}}.$$

Now to a skew shape λ/μ , associate a vector space $\widehat{H}^{\lambda/\mu} = \oplus_L Cl(d)v_L$ where L ranges over all standard tableaux of shape λ/μ and d is the number of boxes in the shifted skew shape. Define $x_i v_L = \kappa_{i,L} v_L$. Define

$$s_i v_L = \frac{1}{\kappa_{i+1,L} - \kappa_{i,L}} v_L + \frac{1}{\kappa_{i+1,L} + \kappa_{i,L}} c_i c_{i+1} v_L + \mathcal{Y}_{i,L} v_{s_i L}$$

where $v_{s_i L} = 0$ if $s_i L$ is not a standard tableaux.

Proposition 5.1.1. *The action of the x_i and s_i given above endow $\widehat{H}^{\lambda/\mu}$ with the structure of a $\mathcal{H}_{C_\ell}^{\text{aff}}(d)$ -module.*

Proof. We have

$$\begin{aligned}
s_i^2 v_L &= \frac{1}{\kappa_{i+1,L} - \kappa_{i,L}} s_i v_L - \frac{1}{\kappa_{i+1,L} + \kappa_{i,L}} c_i c_{i+1} s_i v_L + \mathcal{Y}_{i,L} s_i v_{s_i L} \\
&= \frac{1}{\kappa_{i+1,L} - \kappa_{i,L}} \left(\frac{1}{\kappa_{i+1,L} - \kappa_{i,L}} v_L + \frac{1}{\kappa_{i+1,L} + \kappa_{i,L}} c_i c_{i+1} v_L + \mathcal{Y}_{i,L} v_{s_i L} \right) \\
&\quad - \frac{c_i c_{i+1}}{\kappa_{i+1,L} + \kappa_{i,L}} \left(\frac{1}{\kappa_{i+1,L} - \kappa_{i,L}} v_L + \frac{1}{\kappa_{i+1,L} + \kappa_{i,L}} c_i c_{i+1} v_L + \mathcal{Y}_{i,L} v_{s_i L} \right) \\
&\quad + \mathcal{Y}_{i,L} \left(\frac{1}{\kappa_{i,L} - \kappa_{i+1,L}} v_{s_i L} + \frac{1}{\kappa_{i+1,L} + \kappa_{i,L}} c_i c_{i+1} v_{s_i L} + \mathcal{Y}_{i,L} v_L \right) \\
&= \left(\frac{1}{(\kappa_{i+1,L} - \kappa_{i,L})^2} + \frac{1}{(\kappa_{i+1,L} + \kappa_{i,L})^2} + \mathcal{Y}_{i,L} \mathcal{Y}_{i,L} \right) v_L = v_L.
\end{aligned}$$

Note that if $v_{s_i L} = 0$, then $\frac{1}{(\kappa_{i+1,L} - \kappa_{i,L})^2} + \frac{1}{(\kappa_{i+1,L} + \kappa_{i,L})^2} = 1$.

Next,

$$s_i x_i v_L = \frac{\kappa_{i,L}}{\kappa_{i+1,L} - \kappa_{i,L}} v_L + \frac{\kappa_{i,L}}{\kappa_{i+1,L} + \kappa_{i,L}} c_i c_{i+1} v_L + \mathcal{Y}_{i,L} v_{s_i L}.$$

On the other hand,

$$x_{i+1} s_i v_L - v_L + c_i c_{i+1} v_L = \frac{\kappa_{i+1,L}}{\kappa_{i+1,L} - \kappa_{i,L}} v_L - \frac{\kappa_{i+1,L}}{\kappa_{i+1,L} + \kappa_{i,L}} c_i c_{i+1} v_L + \mathcal{Y}_{i,L} v_{s_i L} - v_L + c_i c_{i+1} v_L.$$

Thus it is easily seen that

$$s_i x_i v_L = x_{i+1} s_i v_L - v_L + c_i c_{i+1} v_L.$$

We now check the braid relations. To this end, fix $j \in \mathbb{N}$ and set $\kappa_i = \sqrt{j+i}$ for $i \geq 0$.

$$L = \begin{array}{|c|c|c|} \hline i & i+1 & i+2 \\ \hline \end{array}$$

FIGURE 2. Case 1

Case 1: Let L be the standard tableaux given in Figure 2. A calculation gives

$$\begin{aligned}
s_i s_{i+1} s_i v_L &= s_{i+1} s_i s_{i+1} v_L = \left(\frac{1}{(\kappa_3 - \kappa_2)^2 (\kappa_2 - \kappa_1)} - \frac{1}{(\kappa_2 + \kappa_3)^2 (\kappa_1 + \kappa_2)} \right) v_L \\
&\quad + \left(\frac{1}{(\kappa_3^2 - \kappa_2^2) (\kappa_2 + \kappa_1)} + \frac{1}{(\kappa_3^2 - \kappa_2^2)^2 (\kappa_2 - \kappa_1)} \right) c_i c_{i+1} v_L \\
&\quad + \left(\frac{1}{(\kappa_3^2 - \kappa_2^2) (\kappa_2 - \kappa_1)} + \frac{1}{(\kappa_3^2 - \kappa_2^2)^2 (\kappa_2 + \kappa_1)} \right) c_{i+1} c_{i+2} v_L \\
&\quad + \left(\frac{1}{(\kappa_3 - \kappa_2)^2 (\kappa_2 + \kappa_1)} - \frac{1}{(\kappa_2 + \kappa_3)^2 (\kappa_2 - \kappa_1)} \right) c_i c_{i+2} v_L.
\end{aligned}$$

$$L_1 = \begin{array}{|c|c|} \hline i & i+1 \\ \hline i+2 & \\ \hline \end{array}$$

$$L_2 = \begin{array}{|c|c|} \hline i & i+2 \\ \hline i+1 & \\ \hline \end{array}$$

FIGURE 3. Case 2

Case 2: Let L_1 and L_2 be the standard tableaux given in Figure 3. A calculation gives

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_1} &= s_{i+1} s_i s_{i+1} v_{L_1} = \left(\frac{1}{(\kappa_3 - \kappa_2)^2 (\kappa_1 - \kappa_3)} + \frac{1}{(\kappa_2 + \kappa_3)^2 (\kappa_1 + \kappa_3)} \right) v_{L_1} \\
&+ \left(\frac{1}{(\kappa_3^2 - \kappa_2^2) (\kappa_1 - \kappa_3)} - \frac{1}{(\kappa_3^2 - \kappa_2^2)^2 (\kappa_1 + \kappa_3)} \right) c_i c_{i+1} v_{L_1} \\
&+ \left(\frac{1}{(\kappa_2^2 - \kappa_3^2) (\kappa_1 + \kappa_3)} + \frac{1}{(\kappa_3^2 - \kappa_2^2)^2 (\kappa_1 - \kappa_3)} \right) c_{i+1} c_{i+2} v_{L_1} \\
&+ \left(\frac{1}{(\kappa_3 - \kappa_2)^2 (\kappa_1 + \kappa_3)} - \frac{1}{(\kappa_2 + \kappa_3)^2 (\kappa_1 - \kappa_3)} \right) c_i c_{i+2} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_3 - \kappa_2) (\kappa_1 - \kappa_2)} \right) v_{L_2} + \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_3 - \kappa_2) (\kappa_1 + \kappa_2)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_1 - \kappa_2) (\kappa_2 + \kappa_3)} \right) c_{i+1} c_{i+2} v_{L_2} + \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_2 + \kappa_3) (\kappa_1 + \kappa_2)} \right) c_i c_{i+2} v_{L_2}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_2} &= s_{i+1} s_i s_{i+1} v_{L_2} = \left(\frac{1}{(\kappa_1 - \kappa_2)^2 (\kappa_3 - \kappa_1)} + \frac{1}{(\kappa_1 + \kappa_2)^2 (\kappa_1 + \kappa_3)} \right) v_{L_2} \\
&+ \left(\frac{1}{(\kappa_1^2 - \kappa_2^2) (\kappa_3 - \kappa_1)} - \frac{1}{(\kappa_1^2 - \kappa_2^2)^2 (\kappa_1 + \kappa_3)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{-1}{(\kappa_1^2 - \kappa_2^2) (\kappa_1 + \kappa_3)} + \frac{1}{(\kappa_1^2 - \kappa_2^2)^2 (\kappa_3 - \kappa_1)} \right) c_{i+1} c_{i+2} v_{L_2} \\
&+ \left(\frac{1}{(\kappa_1 - \kappa_2)^2 (\kappa_1 + \kappa_3)} + \frac{1}{(\kappa_1 + \kappa_2)^2 (\kappa_3 - \kappa_1)} \right) c_i c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_2}}{(\kappa_3 - \kappa_2) (\kappa_1 - \kappa_2)} \right) v_{L_1} + \left(\frac{\mathcal{Y}_{i+1, L_2}}{(\kappa_1 - \kappa_2) (\kappa_2 + \kappa_3)} \right) c_i c_{i+1} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_2}}{(\kappa_3 - \kappa_2) (\kappa_1 + \kappa_2)} \right) c_{i+1} c_{i+2} v_{L_1} + \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_2 + \kappa_3) (\kappa_1 + \kappa_2)} \right) c_i c_{i+2} v_{L_1}.
\end{aligned}$$



FIGURE 4. Case 3

Case 3: Let L_1 and L_2 be as in figure 4. Then, a calculation analogous to case 2 shows that $s_i s_{i+1} s_i v_{L_1} = s_{i+1} s_i s_{i+2} v_{L_1}$ and $s_i s_{i+1} s_i v_{L_2} = s_{i+1} s_i s_{i+2} v_{L_2}$.

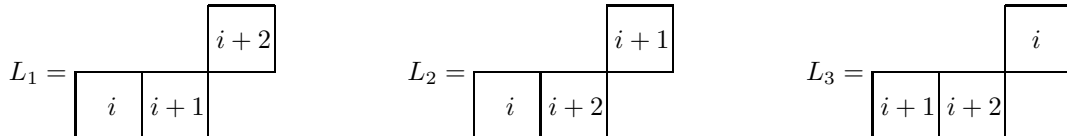


FIGURE 5. Case 4

Case 4: Let L_1, L_2 , and L_3 be the standard tableaux given in Figure 4.

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_1} &= s_{i+1} s_i s_{i+1} v_{L_1} = \left(\frac{1}{(\kappa_1 - \kappa_0)^2 (\kappa_3 - \kappa_1)} + \frac{1}{(\kappa_0 + \kappa_1)^2 (\kappa_1 + \kappa_3)} \right) v_{L_1} \\
&+ \left(\frac{1}{(\kappa_1^2 - \kappa_0^2) (\kappa_3 - \kappa_1)} - \frac{1}{(\kappa_1^2 - \kappa_0^2) (\kappa_1 + \kappa_3)} \right) c_i c_{i+1} v_{L_1} \\
&+ \left(\frac{1}{(\kappa_0^2 - \kappa_1^2) (\kappa_1 + \kappa_3)} - \frac{1}{(\kappa_0^2 - \kappa_1^2) (\kappa_3 - \kappa_1)} \right) c_{i+1} c_{i+2} v_{L_1} \\
&+ \left(\frac{1}{(\kappa_1 - \kappa_0)^2 (\kappa_1 + \kappa_3)} - \frac{1}{(\kappa_0 + \kappa_1)^2 (\kappa_3 - \kappa_1)} \right) c_i c_{i+2} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_1 - \kappa_0) (\kappa_3 - \kappa_0)} \right) v_{L_2} + \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_1 - \kappa_0) (\kappa_0 + \kappa_3)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_0 + \kappa_1) (\kappa_3 - \kappa_0)} \right) c_{i+1} c_{i+2} v_{L_2} + \left(\frac{\mathcal{Y}_{i+1, L_1}}{(\kappa_0 + \kappa_1) (\kappa_0 + \kappa_3)} \right) c_i c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_1} \mathcal{Y}_{i, L_2}}{\kappa_1 - \kappa_0} \right) v_{L_3} + \left(\frac{\mathcal{Y}_{i+1, L_1} \mathcal{Y}_{i, L_2}}{\kappa_0 + \kappa_1} \right) c_{i+1} c_{i+2} v_{L_3}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_2} &= s_{i+1} s_i s_{i+1} v_{L_2} = \left(\frac{1}{(\kappa_3 - \kappa_0)^2 (\kappa_1 - \kappa_3)} + \frac{1}{(\kappa_0 + \kappa_3)^2 (\kappa_1 + \kappa_3)} + \frac{\mathcal{Y}_{i, L_2} \mathcal{Y}_{i, L_3}}{\kappa_1 - \kappa_0} \right) v_{L_2} \\
&+ \left(\frac{1}{(\kappa_3^2 - \kappa_0^2) (\kappa_1 - \kappa_3)} - \frac{1}{(\kappa_3^2 - \kappa_0^2) (\kappa_1 + \kappa_3)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{-1}{(\kappa_3^2 - \kappa_0^2) (\kappa_1 - \kappa_3)} - \frac{1}{(\kappa_3^2 - \kappa_0^2) (\kappa_1 - \kappa_3)} \right) c_{i+1} c_{i+2} v_{L_2} \\
&+ \left(\frac{1}{(\kappa_3 - \kappa_0)^2 (\kappa_1 + \kappa_3)} - \frac{1}{(\kappa_0 + \kappa_3)^2 (\kappa_1 - \kappa_3)} + \frac{\mathcal{Y}_{i, L_2} \mathcal{Y}_{i, L_3}}{\kappa_0 + \kappa_1} \right) c_i c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i, L_2}}{(\kappa_3 - \kappa_0) (\kappa_1 - \kappa_3)} + \frac{\mathcal{Y}_{i, L_2}}{(\kappa_1 - \kappa_0) (\kappa_0 - \kappa_3)} \right) v_{L_3} \\
&+ \left(\frac{-\mathcal{Y}_{i, L_2}}{(\kappa_3 + \kappa_0) (\kappa_1 + \kappa_3)} + \frac{\mathcal{Y}_{i, L_2}}{(\kappa_1 - \kappa_0) (\kappa_0 + \kappa_3)} \right) c_i c_{i+1} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i, L_2}}{(\kappa_3 + \kappa_0) (\kappa_1 - \kappa_3)} - \frac{\mathcal{Y}_{i, L_2}}{(\kappa_1 + \kappa_0) (\kappa_0 + \kappa_3)} \right) c_{i+1} c_{i+2} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i, L_2}}{(\kappa_3 - \kappa_0) (\kappa_1 + \kappa_3)} + \frac{\mathcal{Y}_{i, L_2}}{(\kappa_1 + \kappa_0) (\kappa_0 - \kappa_3)} \right) c_i c_{i+2} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_2}}{(\kappa_1 - \kappa_0) (\kappa_3 - \kappa_0)} \right) v_{L_1} + \left(\frac{\mathcal{Y}_{i+1, L_2}}{(\kappa_1 + \kappa_0) (\kappa_3 - \kappa_0)} \right) c_i c_{i+1} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i+1, L_2}}{(\kappa_3 + \kappa_0) (\kappa_1 - \kappa_0)} \right) c_{i+1} c_{i+2} v_{L_1} + \left(\frac{\mathcal{Y}_{i+1, L_2}}{(\kappa_3 + \kappa_0) (\kappa_1 + \kappa_0)} \right) c_i c_{i+2} v_{L_1}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_3} &= s_{i+1} s_i s_{i+1} v_{L_3} = \left(\frac{1}{(\kappa_3 - \kappa_0)^2 (\kappa_1 - \kappa_0)} + \frac{1}{(\kappa_0 + \kappa_3)^2 (\kappa_0 + \kappa_1)} + \frac{\mathcal{Y}_{i,L_2} \mathcal{Y}_{i,L_3}}{\kappa_1 - \kappa_3} \right) v_{L_3} \\
&+ \left(\frac{1}{(\kappa_0^2 - \kappa_3^2) (\kappa_1 - \kappa_0)} - \frac{1}{(\kappa_0^2 - \kappa_3^2) (\kappa_0 + \kappa_1)} \right) c_i c_{i+1} v_{L_3} \\
&+ \left(\frac{-1}{(\kappa_0^2 - \kappa_3^2) (\kappa_0 + \kappa_1)} + \frac{1}{(\kappa_0^2 - \kappa_3^2) (\kappa_1 - \kappa_0)} \right) c_{i+1} c_{i+2} v_{L_3} \\
&+ \left(\frac{1}{(\kappa_3 - \kappa_0)^2 (\kappa_0 + \kappa_1)} + \frac{1}{(\kappa_0 + \kappa_3)^2 (\kappa_1 - \kappa_0)} + \frac{\mathcal{Y}_{i,L_2} \mathcal{Y}_{i,L_3}}{\kappa_1 + \kappa_3} \right) c_i c_{i+2} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3}}{(\kappa_0 - \kappa_3) (\kappa_1 - \kappa_0)} + \frac{\mathcal{Y}_{i,L_3}}{(\kappa_1 - \kappa_3) (\kappa_3 - \kappa_0)} \right) v_{L_2} \\
&+ \left(\frac{-\mathcal{Y}_{i,L_3}}{(\kappa_3 + \kappa_0) (\kappa_0 + \kappa_1)} + \frac{\mathcal{Y}_{i,L_3}}{(\kappa_1 - \kappa_3) (\kappa_0 + \kappa_3)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3}}{(\kappa_3 + \kappa_0) (\kappa_1 - \kappa_0)} - \frac{\mathcal{Y}_{i,L_3}}{(\kappa_1 + \kappa_3) (\kappa_0 + \kappa_3)} \right) c_{i+1} c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3}}{(\kappa_0 - \kappa_3) (\kappa_0 + \kappa_1)} + \frac{\mathcal{Y}_{i,L_3}}{(\kappa_1 + \kappa_3) (\kappa_3 - \kappa_0)} \right) c_i c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3} \mathcal{Y}_{i+1,L_2}}{\kappa_1 - \kappa_0} \right) v_{L_1} + \left(\frac{\mathcal{Y}_{i,L_3} \mathcal{Y}_{i+1,L_2}}{\kappa_1 + \kappa_0} \right) c_i c_{i+1} v_{L_1}.
\end{aligned}$$

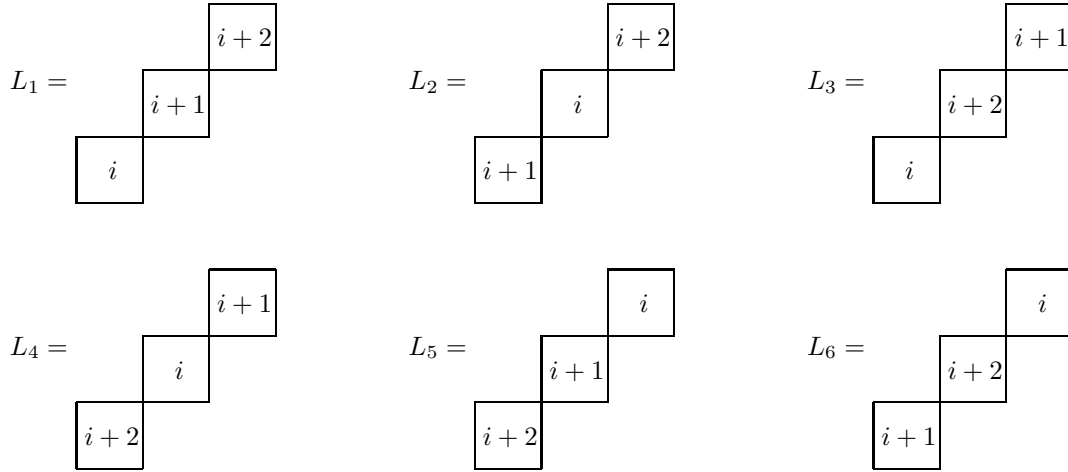


FIGURE 6. Case 5

Case 5: Let L_1, L_2, L_3, L_4, L_5 , and L_6 be given as in Figure 6.

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_1} &= s_{i+1} s_i s_{i+1} v_{L_1} = \left(\frac{1}{(\kappa_2 - \kappa_0)^2 (\kappa_4 - \kappa_2)} + \frac{1}{(\kappa_2 + \kappa_0)^2 (\kappa_4 + \kappa_2)} + \frac{\mathcal{Y}_{i,L_1} \mathcal{Y}_{i,L_2}}{\kappa_4 - \kappa_0} \right) v_{L_1} \\
&+ \left(\frac{1}{(\kappa_2^2 - \kappa_0^2) (\kappa_4 - \kappa_2)} - \frac{1}{(\kappa_2^2 - \kappa_0^2) (\kappa_4 + \kappa_2)} \right) c_i c_{i+1} v_{L_1} \\
&+ \left(\frac{-1}{(\kappa_2^2 - \kappa_0^2) (\kappa_4 + \kappa_2)} + \frac{1}{(\kappa_2^2 - \kappa_0^2) (\kappa_4 - \kappa_2)} \right) c_{i+1} c_{i+2} v_{L_1} \\
&+ \left(\frac{1}{(\kappa_2 - \kappa_0)^2 (\kappa_4 + \kappa_2)} + \frac{1}{(\kappa_2 + \kappa_0)^2 (\kappa_4 - \kappa_2)} + \frac{\mathcal{Y}_{i,L_1} \mathcal{Y}_{i,L_2}}{\kappa_4 + \kappa_0} \right) c_i c_{i+2} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i,L_1}}{(\kappa_2 - \kappa_0) (\kappa_4 - \kappa_2)} + \frac{\mathcal{Y}_{i,L_1}}{(\kappa_4 - \kappa_0) (\kappa_0 - \kappa_2)} \right) v_{L_2} \\
&+ \left(\frac{-\mathcal{Y}_{i,L_1}}{(\kappa_2 + \kappa_0) (\kappa_4 + \kappa_2)} + \frac{\mathcal{Y}_{i,L_1}}{(\kappa_4 - \kappa_0) (\kappa_0 + \kappa_2)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i,L_1}}{(\kappa_2 + \kappa_0) (\kappa_4 - \kappa_2)} - \frac{\mathcal{Y}_{i,L_1}}{(\kappa_4 + \kappa_0) (\kappa_0 + \kappa_2)} \right) c_{i+1} c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i,L_1}}{(\kappa_2 - \kappa_0) (\kappa_4 + \kappa_2)} + \frac{\mathcal{Y}_{i,L_1}}{(\kappa_4 + \kappa_0) (\kappa_0 - \kappa_2)} \right) c_i c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_1}}{(\kappa_2 - \kappa_0) (\kappa_4 - \kappa_0)} \right) v_{L_3} + \left(\frac{\mathcal{Y}_{i+1,L_1}}{(\kappa_2 - \kappa_0) (\kappa_4 + \kappa_0)} \right) c_i c_{i+1} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_1}}{(\kappa_2 + \kappa_0) (\kappa_4 - \kappa_0)} \right) c_{i+1} c_{i+2} v_{L_3} + \left(\frac{\mathcal{Y}_{i+1,L_1}}{(\kappa_2 + \kappa_0) (\kappa_4 + \kappa_0)} \right) c_i c_{i+2} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_1} \mathcal{Y}_{i,L_3}}{\kappa_2 - \kappa_0} \right) v_{L_6} + \left(\frac{\mathcal{Y}_{i+1,L_1} \mathcal{Y}_{i,L_3}}{\kappa_2 + \kappa_0} \right) c_{i+1} c_{i+2} v_{L_6} \\
&+ \left(\frac{(\mathcal{Y}_{i,L_1}) (\mathcal{Y}_{i+1,L_2})}{\kappa_4 - \kappa_2} \right) v_{L_4} + \left(\frac{\mathcal{Y}_{i,L_1} \mathcal{Y}_{i+1,L_2}}{\kappa_4 + \kappa_2} \right) c_i c_{i+1} v_{L_4} + (\mathcal{Y}_{i,L_1} \mathcal{Y}_{i+1,L_2} \mathcal{Y}_{i,L_4}) v_{L_5}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_2} &= s_{i+1} s_i s_{i+1} v_{L_2} = \left(\frac{1}{(\kappa_2 - \kappa_0)^2 (\kappa_4 - \kappa_0)} + \frac{1}{(\kappa_2 + \kappa_0)^2 (\kappa_4 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_1} \mathcal{Y}_{i,L_2}}{\kappa_4 - \kappa_2} \right) v_{L_2} \\
&+ \left(\frac{1}{(\kappa_0^2 - \kappa_2^2) (\kappa_4 - \kappa_0)} - \frac{1}{(\kappa_0^2 - \kappa_2^2) (\kappa_4 + \kappa_0)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{-1}{(\kappa_0^2 - \kappa_2^2) (\kappa_4 + \kappa_0)} + \frac{1}{(\kappa_0^2 - \kappa_2^2) (\kappa_4 - \kappa_0)} \right) c_{i+1} c_{i+2} v_{L_2} \\
&+ \left(\frac{1}{(\kappa_0 - \kappa_2)^2 (\kappa_4 + \kappa_0)} + \frac{1}{(\kappa_2 + \kappa_0)^2 (\kappa_4 - \kappa_0)} + \frac{\mathcal{Y}_{i,L_1} \mathcal{Y}_{i,L_2}}{\kappa_4 + \kappa_2} \right) c_i c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i,L_2}}{(\kappa_0 - \kappa_2) (\kappa_4 - \kappa_0)} + \frac{\mathcal{Y}_{i,L_2}}{(\kappa_4 - \kappa_2) (\kappa_2 - \kappa_0)} \right) v_{L_1} \\
&+ \left(\frac{-\mathcal{Y}_{i,L_2}}{(\kappa_2 + \kappa_0) (\kappa_4 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_2}}{(\kappa_4 - \kappa_2) (\kappa_0 + \kappa_2)} \right) c_i c_{i+1} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i,L_2}}{(\kappa_2 + \kappa_0) (\kappa_4 - \kappa_0)} - \frac{\mathcal{Y}_{i,L_2}}{(\kappa_4 + \kappa_2) (\kappa_0 + \kappa_2)} \right) c_{i+1} c_{i+2} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i,L_2}}{(\kappa_0 - \kappa_2) (\kappa_0 + \kappa_4)} + \frac{\mathcal{Y}_{i,L_2}}{(\kappa_4 + \kappa_2) (\kappa_0 - \kappa_2)} \right) c_i c_{i+2} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_2}}{(\kappa_0 - \kappa_2) (\kappa_4 - \kappa_2)} \right) v_{L_4} + \left(\frac{\mathcal{Y}_{i+1,L_2}}{(\kappa_0 - \kappa_2) (\kappa_4 + \kappa_2)} \right) c_i c_{i+1} v_{L_4} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_2}}{(\kappa_2 + \kappa_0) (\kappa_4 - \kappa_2)} \right) c_{i+1} c_{i+2} v_{L_4} + \left(\frac{\mathcal{Y}_{i+1,L_2}}{(\kappa_2 + \kappa_0) (\kappa_4 + \kappa_2)} \right) c_i c_{i+2} v_{L_4} + \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_1} \mathcal{Y}_{i,L_4}}{\kappa_0 - \kappa_2} \right) v_{L_5} + \left(\frac{\mathcal{Y}_{i+1,L_1} \mathcal{Y}_{i,L_4}}{\kappa_2 + \kappa_0} \right) c_{i+1} c_{i+2} v_{L_5} \\
&+ \left(\frac{\mathcal{Y}_{i,L_2} \mathcal{Y}_{i+1,L_1}}{\kappa_4 - \kappa_0} \right) v_{L_3} + \left(\frac{\mathcal{Y}_{i,L_2} \mathcal{Y}_{i+1,L_1}}{\kappa_4 + \kappa_0} \right) c_i c_{i+1} v_{L_3} + (\mathcal{Y}_{i,L_2} \mathcal{Y}_{i+1,L_1} \mathcal{Y}_{i,L_3}) v_{L_6}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_3} &= s_{i+1} s_i s_{i+1} v_{L_3} = \left(\frac{1}{(\kappa_4 - \kappa_0)^2 (\kappa_2 - \kappa_4)} + \frac{1}{(\kappa_4 + \kappa_0)^2 (\kappa_4 + \kappa_2)} + \frac{\mathcal{Y}_{i,L_3} \mathcal{Y}_{i,L_6}}{\kappa_2 - \kappa_0} \right) v_{L_3} \\
&+ \left(\frac{1}{(\kappa_4^2 - \kappa_0^2)(\kappa_2 - \kappa_4)} - \frac{1}{(\kappa_4^2 - \kappa_0^2)(\kappa_4 + \kappa_2)} \right) c_i c_{i+1} v_{L_3} \\
&+ \left(\frac{-1}{(\kappa_4^2 - \kappa_0^2)(\kappa_4 + \kappa_2)} + \frac{1}{(\kappa_4^2 - \kappa_0^2)(\kappa_2 - \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_3} \\
&+ \left(\frac{1}{(\kappa_4 - \kappa_0)^2 (\kappa_4 + \kappa_2)} + \frac{1}{(\kappa_4 + \kappa_0)^2 (\kappa_2 - \kappa_4)} + \frac{\mathcal{Y}_{i,L_3} \mathcal{Y}_{i,L_6}}{\kappa_0 + \kappa_2} \right) c_i c_{i+2} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3}}{(\kappa_4 - \kappa_0)(\kappa_2 - \kappa_4)} + \frac{\mathcal{Y}_{i,L_3}}{(\kappa_2 - \kappa_0)(\kappa_0 - \kappa_4)} \right) v_{L_6} \\
&+ \left(\frac{-\mathcal{Y}_{i,L_3}}{(\kappa_4 + \kappa_0)(\kappa_4 + \kappa_2)} + \frac{\mathcal{Y}_{i,L_3}}{(\kappa_2 - \kappa_0)(\kappa_0 + \kappa_4)} \right) c_i c_{i+1} v_{L_6} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3}}{(\kappa_4 + \kappa_0)(\kappa_2 - \kappa_4)} - \frac{\mathcal{Y}_{i,L_3}}{(\kappa_0 + \kappa_2)(\kappa_0 + \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_6} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3}}{(\kappa_4 - \kappa_0)(\kappa_2 + \kappa_4)} + \frac{\mathcal{Y}_{i,L_3}}{(\kappa_0 + \kappa_2)(\kappa_0 - \kappa_4)} \right) c_i c_{i+2} v_{L_6} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_3}}{(\kappa_4 - \kappa_0)(\kappa_2 - \kappa_0)} \right) v_{L_1} + \left(\frac{\mathcal{Y}_{i+1,L_3}}{(\kappa_4 - \kappa_0)(\kappa_0 + \kappa_2)} \right) c_i c_{i+1} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_3}}{(\kappa_4 + \kappa_0)(\kappa_2 - \kappa_0)} \right) c_{i+1} c_{i+2} v_{L_1} + \left(\frac{\mathcal{Y}_{i+1,L_3}}{(\kappa_4 + \kappa_0)(\kappa_0 + \kappa_2)} \right) c_i c_{i+2} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_3} \mathcal{Y}_{i,L_1}}{\kappa_4 - \kappa_0} \right) v_{L_2} + \left(\frac{\mathcal{Y}_{i+1,L_3} \mathcal{Y}_{i,L_1}}{\kappa_4 + \kappa_0} \right) c_{i+1} c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i,L_3} \mathcal{Y}_{i+1,L_6}}{\kappa_2 - \kappa_4} \right) v_{L_5} + \left(\frac{\mathcal{Y}_{i,L_3} \mathcal{Y}_{i+1,L_6}}{\kappa_4 + \kappa_2} \right) c_i c_{i+1} v_{L_5} + (\mathcal{Y}_{i,L_3} \mathcal{Y}_{i+1,L_6} \mathcal{Y}_{i,L_3}) v_{L_4}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_4} &= s_{i+1} s_i s_{i+1} v_{L_4} = \left(\frac{1}{(\kappa_4 - \kappa_2)^2 (\kappa_0 - \kappa_4)} + \frac{1}{(\kappa_4 + \kappa_2)^2 (\kappa_4 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_4} \mathcal{Y}_{i,L_5}}{\kappa_0 - \kappa_2} \right) v_{L_4} \\
&+ \left(\frac{1}{(\kappa_4^2 - \kappa_2^2)(\kappa_0 - \kappa_4)} - \frac{1}{(\kappa_4^2 - \kappa_2^2)(\kappa_4 + \kappa_0)} \right) c_i c_{i+1} v_{L_4} \\
&+ \left(\frac{-1}{(\kappa_4^2 - \kappa_2^2)(\kappa_4 + \kappa_0)} + \frac{1}{(\kappa_4^2 - \kappa_2^2)(\kappa_0 - \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_4} \\
&+ \left(\frac{1}{(\kappa_4 - \kappa_2)^2 (\kappa_4 + \kappa_0)} + \frac{1}{(\kappa_4 + \kappa_2)^2 (\kappa_0 - \kappa_4)} + \frac{\mathcal{Y}_{i,L_4} \mathcal{Y}_{i,L_5}}{\kappa_0 + \kappa_2} \right) c_i c_{i+2} v_{L_4} \\
&+ \left(\frac{\mathcal{Y}_{i,L_4}}{(\kappa_4 - \kappa_2)(\kappa_0 - \kappa_4)} + \frac{\mathcal{Y}_{i,L_4}}{(\kappa_0 - \kappa_2)(\kappa_2 - \kappa_4)} \right) v_{L_5} \\
&+ \left(\frac{-\mathcal{Y}_{i,L_4}}{(\kappa_4 + \kappa_2)(\kappa_4 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_4}}{(\kappa_0 - \kappa_2)(\kappa_2 + \kappa_4)} \right) c_i c_{i+1} v_{L_5} \\
&+ \left(\frac{\mathcal{Y}_{i,L_4}}{(\kappa_4 + \kappa_2)(\kappa_0 - \kappa_4)} - \frac{\mathcal{Y}_{i,L_4}}{(\kappa_0 + \kappa_2)(\kappa_2 + \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_5} \\
&+ \left(\frac{\mathcal{Y}_{i,L_4}}{(\kappa_4 - \kappa_2)(\kappa_0 + \kappa_4)} + \frac{\mathcal{Y}_{i,L_4}}{(\kappa_0 + \kappa_2)(\kappa_2 - \kappa_4)} \right) c_i c_{i+2} v_{L_5} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_4}}{(\kappa_4 - \kappa_2)(\kappa_0 - \kappa_2)} \right) v_{L_2} + \left(\frac{\mathcal{Y}_{i+1,L_4}}{(\kappa_4 - \kappa_2)(\kappa_0 + \kappa_2)} \right) c_i c_{i+1} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_4}}{(\kappa_4 + \kappa_2)(\kappa_0 - \kappa_2)} \right) c_{i+1} c_{i+2} v_{L_2} + \left(\frac{\mathcal{Y}_{i+1,L_4}}{(\kappa_4 + \kappa_2)(\kappa_0 + \kappa_2)} \right) c_i c_{i+2} v_{L_2} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_4} \mathcal{Y}_{i,L_2}}{\kappa_4 - \kappa_2} \right) v_{L_1} + \left(\frac{\mathcal{Y}_{i+1,L_4} \mathcal{Y}_{i,L_2}}{\kappa_4 + \kappa_2} \right) c_{i+1} c_{i+2} v_{L_1} \\
&+ \left(\frac{\mathcal{Y}_{i,L_4} \mathcal{Y}_{i+1,L_5}}{\kappa_0 - \kappa_4} \right) v_{L_6} + \left(\frac{\mathcal{Y}_{i,L_4} \mathcal{Y}_{i+1,L_5}}{\kappa_4 + \kappa_0} \right) c_i c_{i+1} v_{L_6} + (\mathcal{Y}_{i,L_4} \mathcal{Y}_{i+1,L_5} \mathcal{Y}_{i,L_4}) v_{L_3}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_5} &= s_{i+1} s_i s_{i+1} v_{L_5} = \left(\frac{1}{(\kappa_4 - \kappa_2)^2 (\kappa_0 - \kappa_2)} + \frac{1}{(\kappa_4 + \kappa_2)^2 (\kappa_2 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_5} \mathcal{Y}_{i,L_4}}{\kappa_0 - \kappa_4} \right) v_{L_5} \\
&+ \left(\frac{1}{(\kappa_2^2 - \kappa_4^2) (\kappa_0 - \kappa_2)} - \frac{1}{(\kappa_2^2 - \kappa_4^2) (\kappa_2 + \kappa_0)} \right) c_i c_{i+1} v_{L_5} \\
&+ \left(\frac{-1}{(\kappa_2^2 - \kappa_4^2) (\kappa_2 + \kappa_0)} + \frac{1}{(\kappa_2^2 - \kappa_4^2) (\kappa_0 - \kappa_2)} \right) c_{i+1} c_{i+2} v_{L_5} \\
&+ \left(\frac{1}{(\kappa_4 - \kappa_2)^2 (\kappa_2 + \kappa_0)} + \frac{1}{(\kappa_4 + \kappa_2)^2 (\kappa_0 - \kappa_2)} + \frac{\mathcal{Y}_{i,L_5} \mathcal{Y}_{i,L_4}}{\kappa_0 + \kappa_4} \right) c_i c_{i+2} v_{L_5} \\
&+ \left(\frac{\mathcal{Y}_{i,L_5}}{(\kappa_2 - \kappa_4) (\kappa_0 - \kappa_2)} + \frac{\mathcal{Y}_{i,L_5}}{(\kappa_0 - \kappa_4) (\kappa_4 - \kappa_2)} \right) v_{L_4} \\
&+ \left(\frac{-\mathcal{Y}_{i,L_5}}{(\kappa_4 + \kappa_2) (\kappa_2 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_5}}{(\kappa_0 - \kappa_4) (\kappa_2 + \kappa_4)} \right) c_i c_{i+1} v_{L_4} \\
&+ \left(\frac{\mathcal{Y}_{i,L_5}}{(\kappa_4 + \kappa_2) (\kappa_0 - \kappa_2)} - \frac{\mathcal{Y}_{i,L_5}}{(\kappa_0 + \kappa_4) (\kappa_2 + \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_4} \\
&+ \left(\frac{\mathcal{Y}_{i,L_5}}{(\kappa_2 - \kappa_4) (\kappa_0 + \kappa_2)} + \frac{\mathcal{Y}_{i,L_5}}{(\kappa_0 + \kappa_4) (\kappa_4 - \kappa_2)} \right) c_i c_{i+2} v_{L_4} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_5}}{(\kappa_2 - \kappa_4) (\kappa_0 - \kappa_4)} \right) v_{L_6} + \left(\frac{\mathcal{Y}_{i+1,L_5}}{(\kappa_4 - \kappa_2) (\kappa_0 + \kappa_2)} \right) c_i c_{i+1} v_{L_6} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_5}}{(\kappa_4 + \kappa_2) (\kappa_0 - \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_6} + \left(\frac{\mathcal{Y}_{i+1,L_5}}{(\kappa_4 + \kappa_2) (\kappa_0 + \kappa_4)} \right) c_i c_{i+2} v_{L_6} \\
&+ \left(\frac{(\mathcal{Y}_{i+1,L_5}) (\mathcal{Y}_{i,L_6})}{\kappa_2 - \kappa_4} \right) v_{L_3} + \left(\frac{(\mathcal{Y}_{i+1,L_5}) (\mathcal{Y}_{i,L_6})}{\kappa_4 + \kappa_2} \right) c_{i+1} c_{i+2} v_{L_3} \\
&+ \left(\frac{(\mathcal{Y}_{i,L_5}) (\mathcal{Y}_{i+1,L_4})}{\kappa_0 - \kappa_2} \right) v_{L_2} + \left(\frac{(\mathcal{Y}_{i,L_5}) (\mathcal{Y}_{i+1,L_4})}{\kappa_2 + \kappa_0} \right) c_i c_{i+1} v_{L_6} + (\mathcal{Y}_{i,L_5} \mathcal{Y}_{i+1,L_4} \mathcal{Y}_{i,L_2}) v_{L_1}.
\end{aligned}$$

$$\begin{aligned}
s_i s_{i+1} s_i v_{L_6} &= s_{i+1} s_i s_{i+1} v_{L_6} = \left(\frac{1}{(\kappa_0 - \kappa_4)^2 (\kappa_2 - \kappa_0)} + \frac{1}{(\kappa_0 + \kappa_4)^2 (\kappa_2 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_6} \mathcal{Y}_{i,L_3}}{\kappa_2 - \kappa_4} \right) v_{L_6} \\
&+ \left(\frac{1}{(\kappa_0^2 - \kappa_4^2) (\kappa_2 - \kappa_0)} - \frac{1}{(\kappa_0^2 - \kappa_4^2) (\kappa_2 + \kappa_0)} \right) c_i c_{i+1} v_{L_6} \\
&+ \left(\frac{-1}{(\kappa_0^2 - \kappa_4^2) (\kappa_2 + \kappa_0)} + \frac{1}{(\kappa_0^2 - \kappa_4^2) (\kappa_2 - \kappa_0)} \right) c_{i+1} c_{i+2} v_{L_6} \\
&+ \left(\frac{1}{(\kappa_0 - \kappa_4)^2 (\kappa_2 + \kappa_0)} + \frac{1}{(\kappa_4 + \kappa_0)^2 (\kappa_2 - \kappa_0)} + \frac{\mathcal{Y}_{i,L_6} \mathcal{Y}_{i,L_3}}{\kappa_2 + \kappa_4} \right) c_i c_{i+2} v_{L_6} \\
&+ \left(\frac{\mathcal{Y}_{i,L_6}}{(\kappa_0 - \kappa_4) (\kappa_2 - \kappa_0)} + \frac{\mathcal{Y}_{i,L_6}}{(\kappa_2 - \kappa_4) (\kappa_4 - \kappa_0)} \right) v_{L_3} \\
&+ \left(\frac{-\mathcal{Y}_{i,L_6}}{(\kappa_4 + \kappa_0) (\kappa_2 + \kappa_0)} + \frac{\mathcal{Y}_{i,L_6}}{(\kappa_2 - \kappa_4) (\kappa_0 + \kappa_4)} \right) c_i c_{i+1} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i,L_6}}{(\kappa_0 + \kappa_4) (\kappa_2 - \kappa_0)} - \frac{\mathcal{Y}_{i,L_6}}{(\kappa_2 + \kappa_4) (\kappa_0 + \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i,L_6}}{(\kappa_0 - \kappa_4) (\kappa_0 + \kappa_2)} + \frac{\mathcal{Y}_{i,L_6}}{(\kappa_2 + \kappa_4) (\kappa_4 - \kappa_0)} \right) c_i c_{i+2} v_{L_3} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_6}}{(\kappa_0 - \kappa_4) (\kappa_2 - \kappa_4)} \right) v_{L_5} + \left(\frac{\mathcal{Y}_{i+1,L_6}}{(\kappa_0 - \kappa_4) (\kappa_2 + \kappa_4)} \right) c_i c_{i+1} v_{L_5} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_6}}{(\kappa_0 + \kappa_4) (\kappa_2 - \kappa_4)} \right) c_{i+1} c_{i+2} v_{L_5} + \left(\frac{\mathcal{Y}_{i+1,L_6}}{(\kappa_0 + \kappa_4) (\kappa_2 + \kappa_4)} \right) c_i c_{i+2} v_{L_5} \\
&+ \left(\frac{\mathcal{Y}_{i+1,L_6} \mathcal{Y}_{i,L_5}}{\kappa_0 - \kappa_4} \right) v_{L_4} + \left(\frac{\mathcal{Y}_{i+1,L_6} \mathcal{Y}_{i,L_5}}{\kappa_0 + \kappa_4} \right) c_{i+1} c_{i+2} v_{L_4} \\
&+ \left(\frac{\mathcal{Y}_{i,L_6} \mathcal{Y}_{i+1,L_3}}{\kappa_2 - \kappa_0} \right) v_{L_1} + \left(\frac{(\mathcal{Y}_{i,L_6}) (\mathcal{Y}_{i+1,L_3})}{\kappa_2 + \kappa_0} \right) c_i c_{i+1} v_{L_1} + (\mathcal{Y}_{i,L_6} \mathcal{Y}_{i+1,L_3} \mathcal{Y}_{i,L_6}) v_{L_2}.
\end{aligned}$$

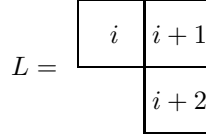


FIGURE 7. Case 6

Case 6: Let L be as in Figure 7. Then

$$s_i s_{i+1} s_i v_L = s_{i+1} s_i s_{i+1} v_L = \frac{1}{\sqrt{2}}(-c_i c_{i+1} v_L + c_i c_{i+2} v_L).$$

□

Now define an $\mathcal{A}(d)$ -module $H^{\lambda/\mu}$ to be $\sum_{w \in S_n} \phi_w \mathcal{L}(c(L))$ where L is a fixed standard filling of the shifted skew shape λ/μ and $\mathcal{L}(c(L)) = \mathcal{L}(c(L_1)) \otimes \cdots \otimes \mathcal{L}(c(L_d))$ is an irreducible $\mathcal{A}(d)$ submodule of $Cl(d)v_L$ introduced in section 3.5.

Proposition 5.1.2. *The $\mathcal{A}(d)$ -module $H^{\lambda/\mu}$ is a $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module.*

Proof. Let $cv_L \in H^{\lambda/\mu}$. Then $(\phi_i - s_i(x_i^2 - x_{i+1}^2))cv_L \in H^{\lambda/\mu}$. Note that $\phi_i cv_L = s_i c \phi_i v_L = k^{s_i} c v_{s_i L}$ where $s_i c = s_i c s_i$ denotes the Clifford element twisted by s_i . This element is in $H^{\lambda/\mu}$ because the twisting of the Clifford element c by s_i is compatible with the permutation of the zero eigenvalues of the x'_j s by s_i . Thus $s_i(x_i^2 - x_{i+1}^2)cv_L = k' s_i cv_L \in H^{\lambda/\mu}$. Since by construction $(x_i^2 - x_{i+1}^2)v_L \neq 0$ by construction, $s_i cv_L \in H^{\lambda/\mu}$. □

Theorem 5.1.3. *For each shifted skew shape λ/μ , $H^{\lambda/\mu}$ is an irreducible $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module. Every irreducible, calibrated $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module is isomorphic to exactly one such $H^{\lambda/\mu}$.*

Proof. First to show that $H^{(\lambda, \mu)}$ is irreducible. Let L be a standard tableaux of shape λ/μ . Let N be a non-zero submodule of $H^{\lambda/\mu}$ and let $v = \sum_Q C_Q v_Q \in N$ be non-zero where $C_Q \in Cl(d)$. Let L be a standard tableaux such that $\mathcal{Y}_L \neq 0$. If $P \neq L$ then there exists an i such that $x_i v_P \neq x_i v_L$. Suppose $\mathcal{Y}_P \neq 0$. Then $\frac{x_i - \kappa_{i,P}}{\kappa_{i,L} - \kappa_{i,P}} v$ no longer has a v_P term but still has a v_L term. This element is also in N . Iterating this process it is clear that $v_L \in N$. The set of tableaux is identified with an interval of S_n under the Bruhat order. The minimal element is the column reading C . Thus there exists a chain $C < s_{i_1} C < \cdots < s_{i_p} \cdots s_{i_1} C = L$. Therefore $\tau_{i_1} \cdots \tau_{i_p} v_L = \kappa v_C$ for some non-zero complex number κ . This implies $v_C \in N$. Now let Q be an arbitrary standard tableaux of λ/μ . There is a chain $C < s_{j_1} C < \cdots < s_{j_p} \cdots s_{j_1} C = Q$. Then $\tau_{j_p} \cdots \tau_{j_1} v_C = \kappa' v_Q$ for some non-zero complex number κ' . Thus $v_Q \in N$ so $N = H^{\lambda/\mu}$.

It is clear by looking at the eigenvalues that if $\lambda/\mu \neq \lambda'/\mu'$, then $H^{\lambda/\mu} \neq H^{\lambda'/\mu'}$.

Next to show that the weight of a calibrated module M is obtained by reading the contents of a shifted skew shape via a standard filling. That is, if $(t_1, \dots, t_d)$ be such a weight, then it is necessary to show that it is equal to $(c(L_1), \dots, c(L_d))$ for some standard tableaux L . It will be shown that if $t_i = t_j$ for some $i < j$, then there exists k, l such that $i < k < l < j$ such that $t_k = t_i \pm 1$ and $t_l = t_i \mp 1$ unless $t_i = t_j = 0$ in which case there is a k with $i < k < j$ such that $t_k = 1$.

Let $j > i$ be such that $t_j = t_i$ and $j - i$ is minimal, let $m_t \in M$ be a nonzero vector of weight $t = (t_1, \dots, t_d)$, and let $\varrho_i = \sqrt{q(t_i)}$. The proof will be by induction on $j - i$.

Case 1: Suppose $j - i = 1$.

First the case that $t_i = 0$. If $t_i = 0$, then $t_{i+1} = 0$ by assumption and then $x_i s_i m_t = -m_t - c_i c_{i+1} m_t$. It is clear that $-m_t - c_i c_{i+1} m_t \neq 0$. Otherwise, $m_t = -c_i c_{i+1} m_t$ which implies after multiplying both sides by $c_i c_{i+1}$ that $m_t = -m_t$ giving $m_t = 0$. Thus $x_i^2 s_i m_t = 0$ but $x_i s_i m_t \neq 0$. Similarly, $x_{i+1}^2 s_i m_t = 0$, but $x_{i+1} s_i m_t \neq 0$. Clearly $(x_j - \varrho_j) s_i m_t = 0$ for $j \neq i, i+1$. Thus if $t_i = 0$, then $s_i m_t \in M_t^{\text{gen}}$, but not in M_t contradicting the assumption that M is calibrated.

Now assume $t_i \neq 0$. Then, $s_i m_t - \frac{1}{2t_i} c_i c_{i+1} m_t \in M_t^{\text{gen}}$ but not in M_t . To see this, calculate:

$$x_i(s_i m_t - \frac{1}{2\varrho_i} c_i c_{i+1} m_t) = t_i s_i m_t - \frac{1}{2} c_i c_{i+1} m_t - m_t.$$

This implies $(x_i - \varrho_i)(s_i m_t - \frac{1}{2\varrho_i} c_i c_{i+1} m_t) = -m_t \neq 0$ and $(x_i - \varrho_i)^2(s_i m_t - \frac{1}{2\varrho_i} c_i c_{i+1} m_t) = 0$. Similarly, $(x_{i+1} - \varrho_{i+1})(s_i m_t - \frac{1}{2\varrho_i} c_i c_{i+1} m_t) = m_t \neq 0$ and $(x_{i+1} - \varrho_{i+1})^2(s_i m_t - \frac{1}{2\varrho_i} c_i c_{i+1} m_t) = 0$. If $j \neq i, i+1$, then $(x_j - \varrho_j)(s_i m_t - \frac{1}{2\varrho_i} c_i c_{i+1} m_t) = 0$. Thus $s_i m_t - \frac{1}{2\varrho_i} c_i c_{i+1} m_t \in M_t^{\text{gen}}$ but not in M_t verifying case 1.

Case 2: Suppose $j - i = 2$.

Since m_t is a weight vector, the vector

$$m_{s_i t} = \phi_i m_t = (\varrho_i - \varrho_{i+1}) s_i m_t - (\varrho_i - \varrho_{i+1}) c_i c_{i+1} m_t + (\varrho_i + \varrho_{i+1}) m_t$$

is a weight vector of weight $t' = s_i t$. Then $t'_{i+1} = t'_{i+2}$. By case 1, this is impossible so $m_{s_i t} = 0$. Note that $\varrho_i + \varrho_{i+1} \neq 0$. If it did, then $m_{s_i t} = 0$ which would imply $c_i c_{i+1} m_t = 0$ which would imply $m_t = 0$. Thus, $s_i m_t = \frac{m_t}{\varrho_{i+1} - \varrho_i} + \frac{c_i c_{i+1} m_t}{\varrho_{i+1} + \varrho_i}$. Since $s_i^2 m_t = m_t$, it follows that $m_t = (\frac{2(\varrho_i + \varrho_{i+1})}{(\varrho_i - \varrho_{i+1})^2}) m_t$. This implies $2(\varrho_i + \varrho_{i+1}) = (\varrho_i - \varrho_{i+1})^2$. The solutions of this equation are

$$\varrho_{i+1} \in \{\pm\sqrt{(t_i + 1)(t_i + 2)}, \pm\sqrt{(t_i - 1)(t_i)}\}.$$

Since it is assumed that the positive square root is taken, there are only two subcases to investigate.

For the first subcase, assume $\varrho_{i+1} = \sqrt{q(t_i + 1)}$. A routine calculation gives

$$s_i s_{i+1} s_i m_t = \frac{-s_i m_t}{(\varrho_i - \varrho_{i+1})^2} + \frac{c_i c_{i+2} s_i m_t}{\varrho_{i+1} - \varrho_i} + \frac{c_{i+1} c_{i+2} s_i m_t}{\varrho_i - \varrho_{i+1}} - \frac{c_i c_{i+1} s_i m_t}{(\varrho_i + \varrho_{i+1})^2}.$$

From this it follows that the coefficient of m_t is $\frac{1}{(\varrho_i - \varrho_{i+1})^3} + \frac{1}{(\varrho_i + \varrho_{i+1})^3}$. Similarly, from

$$s_{i+1} s_i s_{i+1} m_t = \frac{-s_{i+1} m_t}{(\varrho_i - \varrho_{i+1})^2} + \frac{c_i c_{i+2} s_{i+1} m_t}{\varrho_i - \varrho_{i+1}} + \frac{c_i c_{i+1} s_{i+1} m_t}{\varrho_{i+1} - \varrho_i} - \frac{c_{i+1} c_{i+2} s_{i+1} m_t}{(\varrho_i + \varrho_{i+1})^2}$$

it follows that the coefficient of m_t is $\frac{-1}{(\varrho_i - \varrho_{i+1})^3} + \frac{-1}{(\varrho_i + \varrho_{i+1})^3}$. Therefore $(\varrho_i - \varrho_{i+1})^3 + (\varrho_i + \varrho_{i+1})^3 = 0$. Recalling that $\varrho_{i+1} = \sqrt{q(t_i + 1)}$ in this subcase, it is clear that $t_i = t_{i+2} = 0$ and $t_{i+1} = 1$. The other subcase is similar.

Now for the induction step. Assume $j - i > 2$. If $t_{j-1} \neq t_j \pm 1$, then the vector $\phi_{j-1} m_t$ is a non-zero weight vector of weight $t' = s_{j-1} t$ by [24, Lemma 14.8.1]. Since $t'_i = t_i = t_j = t'_{j-1}$, the induction hypothesis may be applied to conclude that there exists k and l with $i < k < l < j - 1$ such that $t'_k = t_j \pm 1$ and $t'_l = t_j \mp 1$. (In the case $t_i = t_j = 0$, then there exists $t'_k = 1$.) This implies $t_k = t_j \pm 1$ and $t_l = t_j \mp 1$. (In the case $t_i = t_j = 0$, there exists $t_k = 1$.) Similarly, if $t_{i+1} \neq t_i \pm 1$, consider $\phi_i m_t$ and proceed by induction. Otherwise, $t_{i+1} = t_i \pm 1$ and $t_{j-1} = t_i \pm 1$. Since i and j are chosen such that $t_i = t_j$ and $j - i$ is minimal, $t_{i+1} \neq t_{j-1}$. This then gives the conclusion. (If $t_i = t_j = 0$, then $t_{i+1} = 1$ or $t_{j-1} = 1$.)

Suppose M is an irreducible, calibrated $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -module such that m_t is a weight vector with weight $t = (t_1, \dots, t_d)$ such that $t_{i+1} = t_i \pm 1$. Then $\phi_i m_t = 0$. This follows exactly as in step 5 of [35, Theorem 4.1].

Finally, let m_t be a non-zero weight vector of an irreducible, calibrated module M . By the above, $t = (c(L_1), \dots, c(L_d))$ for L some standard tableaux of shifted skew shape λ/μ . The rest of the proof follows as in step 6 of [35, Theorem 4.1]. Choose a word $w = s_{i_p} \cdots s_{i_1}$ such that w applied to the column reading tableaux of λ/μ gives the tableaux L . Then $m_C = \phi_{i_1} \cdots \phi_{i_p} m_t$ is non-zero. Now to any other standard tableaux Q of λ/μ there is a non-zero weight vector obtained by applying a sequence of intertwiners to m_C . By the above, $\phi_i m_Q = 0$ if $s_i Q$ is not standard. Thus the span of vectors $\{m_Q\}$ over all the standard tableaux of shape λ/μ is a submodule of M . Since M is irreducible, this span must be the entire module. Thus there is an isomorphism $M \cong H^{\lambda/\mu}$ defined by sending $\phi_w m_C$ to $\phi_w v_C$. $\square$

Corollary 5.1.4. *Let λ/μ be a shifted skew shape. Then, $\mathcal{L}(\lambda, \mu) \cong H^{\lambda/\mu}$.*

Proof. Let T be the standard tableaux obtained by filling in the numbers $1, \dots, d$ along rows from top to bottom and left to right. Note that if $s_i \in S_{\lambda-\mu}$, then $v_{s_i T} = 0$ because $s_i T$ is not standard. By Frobenius reciprocity, it follows that there exists a surjective $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -homomorphism $f : \mathcal{M}(\lambda, \mu) \rightarrow H^{\lambda/\mu}$ given by $f(\mathbf{1}_{\lambda-\mu}) = v_T$. $\square$

Furthermore, by construction we have the following result. Note that this agrees with Leclerc's conjectural formula for the calibrated simple modules of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ [27, Proposition 51].

Corollary 5.1.5. *Let λ/μ be a shifted skew shape. Then,*

$$\text{ch } \mathcal{L}(\lambda, \mu) = \sum_L [c(L_1), \dots, c(L_d)],$$

where the sum is over all standard fillings of the shape λ/μ .

6. THE LIE SUPERALGEBRAS $\mathfrak{gl}(n|n)$ AND $\mathfrak{q}(n)$

6.1. The Algebras. Let $I = \{-n, \dots, -1, 1, \dots, n\}$, and $I^+ = \{1, \dots, n\}$. Let $V = \mathbb{C}^{n|n}$ be the $2n$ -dimensional vector superspace with standard basis $\{v_i\}_{i \in I}$. The standard basis for the superalgebra $\text{End}(V)$ is the set of matrix units $\{E_{ij}\}_{i,j \in I}$, and the $\mathbb{Z}_2$ -grading for $\text{End}(V)$ and V are given by

$$p(v_k) = \bar{0}, \quad p(v_{-k}) = \bar{1}, \quad \text{and} \quad p(E_{ij}) = p(v_i) + p(v_j)$$

for $k \in I^+$ and $i, j \in I$.

Let $C = \sum_{i,j \in I^+} (E_{-i,j} - E_{i,-j})$, and let $Q(V) \subset \text{End}(V)$ be the supercentralizer of C . Then, $Q(V)$ has basis given by elements

$$e_{ij} = E_{ij} + E_{-i,-j}, \quad \text{and} \quad f_{ij} = E_{-i,j} + E_{i,-j} \quad i, j \in I^+.$$

When $Q(V)$ and $\text{End}(V)$ are viewed as Lie superalgebras relative to the superbracket:

$$[x, y] = xy - (-1)^{p(x)p(y)}yx,$$

for homogeneous $x, y \in \text{End}(V)$, we denote them $\mathfrak{q}(n)$ and $\mathfrak{gl}(n|n)$ respectively.

We end this section by introducing important elements of $\mathfrak{gl}(n|n)$ that will be needed later. Set

$$\bar{e}_{ij} = E_{ij} - E_{-i,-j}, \quad \text{and} \quad \bar{f}_{ij} = E_{-i,j} - E_{i,-j}, \quad i, j \in I^+. \quad (6.1.1)$$

6.2. Root Data, Category $\mathcal{O}$, and Verma Modules. Fix the triangular decomposition

$$\mathfrak{q}(n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}_0^+$ (resp. $\mathfrak{n}_0^-$) is the subalgebra spanned by the e_{ij} for $1 \leq i < j \leq n$ (resp. $i > j$), $\mathfrak{h}_0$ is spanned by the e_{ii} , $1 \leq i \leq n$, $\mathfrak{n}_1^+$ (resp. $\mathfrak{n}_1^-$) is the subalgebra spanned by the f_{ij} for $1 \leq i < j \leq n$ (resp. $i > j$) and $\mathfrak{h}_1$ is spanned by the f_{ii} , $1 \leq i \leq n$. Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and let $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$.

The isomorphism $\mathfrak{q}(n)_{\bar{0}} \rightarrow \mathfrak{gl}(n)$, $e_{ij} \mapsto E_{ij}$, identifies $\mathfrak{h}_0$ with the standard torus for $\mathfrak{gl}(n)$. Let $\varepsilon_i \in \mathfrak{h}_0^*$ denote the i th coordinate function. For $i \neq j$, define $\alpha_{ij} = \varepsilon_i - \varepsilon_j$, and fix the choice of simple roots $\Delta = \{\alpha_i = \alpha_{i,i+1} | 1 \leq i < n\}$. The corresponding root system is $R = \{\alpha_{ij} | 1 \leq i \neq j \leq n\}$, and the positive roots are $R^+ = \{\alpha_{ij} | 1 \leq i < j \leq n\}$. The root lattice is $Q = \sum_{i=1}^{n-1} \mathbb{Z}\alpha_i$ and weight lattice $P = \sum_{i=1}^n \mathbb{Z}\varepsilon_i$. We can, and will, identify $P = \mathbb{Z}^n$, and $Q = \{\lambda \in P | \lambda_1 + \cdots + \lambda_n = 0\}$. Define the sets of weights P^+ , P^{++} , P_{rat}^+ , P_{poly}^+ and $P_{\geq 0}$ as in §4.2. We call these sets dominant, dominant-typical, rational, polynomial, and positive, respectively. Finally, let $P_{\text{rat}}^{++} = P_{\text{rat}}^+ \cap P^{++}$, and $P_{\text{poly}}^{++} = P_{\text{poly}}^+ \cap P^{++}$.

To begin, let $\mathcal{O} := \mathcal{O}(\mathfrak{q}(n))$ denote the category of all finitely generated $\mathfrak{q}(n)$ -supermodules M that are locally finite dimensional over $\mathfrak{b}$ and satisfy

$$M = \bigoplus_{\lambda \in P} M_\lambda$$

where $M_\lambda = \{v \in M \mid h.v = \lambda(h)v \text{ for all } h \in \mathfrak{h}_0\}$ is the λ -weight space of M .

We now define two classes of *Verma modules*. To this end, given $\lambda \in P$, let $\mathbb{C}_\lambda$ be the 1-dimensional $\mathfrak{h}_0$ -module associated to the weight λ . Let $\theta_\lambda : \mathfrak{h}_1 \rightarrow \mathbb{C}$ be given by $\theta_\lambda(k) = \lambda([k, k])$ for all $k \in \mathfrak{h}_1$. Let $\mathfrak{h}'_1 = \ker \theta$. Let $\overline{\mathcal{U}(\mathfrak{h})} = \mathcal{U}(\mathfrak{h})/\mathfrak{i}$, where $\mathfrak{i}$ is the left ideal of $\mathcal{U}(\mathfrak{h})$ generated by $\{h - \lambda(h) \mid h \in \mathfrak{h}_0\} \cup \mathfrak{h}'_1$. Recall, $\gamma_0(\lambda) = |\{i \mid \lambda_i = 0\}|$. Since $\overline{\mathcal{U}(\mathfrak{h})}$ is isomorphic to a Clifford algebra of rank $n - \gamma_0(\lambda)$, we can define the $\overline{\mathcal{U}(\mathfrak{h})}$ -modules $C(\lambda)$ and $E(\lambda)$ where $C(\lambda)$ is the regular representation of the resulting Clifford algebra and $E(\lambda)$ is its unique irreducible quotient. Both $C(\lambda)$ and $E(\lambda)$ become modules for $\mathcal{U}(\mathfrak{h})$ via inflation through the canonical projection $\mathcal{U}(\mathfrak{h}) \rightarrow \overline{\mathcal{U}(\mathfrak{h})}$. Note that as a $\mathcal{U}(\mathfrak{h})$ -module, $C(\lambda) \cong \text{Ind}_{\mathcal{U}(\mathfrak{h}_0 + \mathfrak{h}'_1)}^{\mathcal{U}(\mathfrak{h})} \mathbb{C}_\lambda$. Extend $C(\lambda)$ and $E(\lambda)$ to representations of $\mathcal{U}(\mathfrak{b}^+)$ by inflation, and define the *Big Verma* $\widehat{M}(\lambda)$ and *Little Verma* $M(\lambda)$ by

$$\widehat{M}(\lambda) = \text{Ind}_{\mathcal{U}(\mathfrak{b}^+)}^{\mathcal{U}(\mathfrak{q}(n))} C(\lambda) \quad \text{and} \quad M(\lambda) = \text{Ind}_{\mathcal{U}(\mathfrak{b}^+)}^{\mathcal{U}(\mathfrak{q}(n))} E(\lambda).$$

The following lemma is obtained from the standard decomposition of the Clifford algebra into irreducible modules:

Lemma 6.2.1. *We have $\widehat{M}(\lambda) \cong M(\lambda) \oplus 2^{\lfloor \frac{n - \gamma_0(\lambda)}{2} \rfloor}$.*

It is known that $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$ (see, for example, [15]). Moreover, it is known $L(\lambda)$ is finite dimensional if, and only if, $\lambda \in P_{\text{rat}}^+$ (see [32]).

The following lemma seems standard, but we cannot find it stated in the literature. See [15, Corollary 7.1, 11.6] for related statements. If M is a $\mathcal{U}(\mathfrak{q})$ -module, then recall that a vector $m \in M$ is called *primitive* if $\mathfrak{n}^+v = 0$.

Lemma 6.2.2. *Let $\lambda \in P$, and assume that for some $\alpha \in R^+$, there exists $r > 0$ such that $s_\alpha \lambda = \lambda - r\alpha$. Then, there exists an injective homomorphism*

$$M(s_\alpha \lambda) \rightarrow M(\lambda).$$

Proof. Let $\alpha = \alpha_{ij}$, and let $v_\lambda \in M(\lambda)_\lambda$ be an odd primitive vector. Then, direct calculation verifies that

$$v_{\lambda-r\alpha} := (e_{ji}^{r-1}(rf_{ji} - e_{ji}(f_{ii} - f_{jj}))) \cdot v_\lambda$$

is a primitive vector of weight $\lambda - r\alpha$ (see, for example [15, Corollary 7.1]). This implies that there is an injective $\mathcal{U}(\mathfrak{b}^+)$ -homomorphism

$$E(s_\alpha \lambda) \rightarrow \mathcal{U}(\mathfrak{h}) \cdot v_{\lambda-r\alpha}.$$

Indeed, clearly every vector in $\mathcal{U}(\mathfrak{h}) \cdot v_{\lambda-r\alpha}$ has weight $\lambda - r\alpha$. Moreover, if $N \in \mathcal{U}(\mathfrak{n}^+)$ and $H \in \mathcal{U}(\mathfrak{h})$, then $[N, H] \in \mathcal{U}(\mathfrak{n}^+)$, so

$$N \cdot (H \cdot v_{\lambda-r\alpha}) = (HN + [N, H]) \cdot v_{\lambda-r\alpha} = 0.$$

The result follows because, by our choice of primitive vector, a standard argument using the filtration of $\mathcal{U}(\mathfrak{q}(n))$ by total degree and a calculation in $U(\mathfrak{q}(2))$ shows that $\mathcal{U}(\mathfrak{b}^-) \cdot v_{\lambda-r\alpha}$ is a free $\mathcal{U}(\mathfrak{n}^-)$ -module. $\square$

6.3. The Shapovalov Form. The Shapovalov map for $\mathfrak{q}(n)$ was constructed in [15]. We review this construction briefly.

Let $\mathcal{D}$ be the category of $Q^- = -Q^+$ -graded $\mathfrak{q}(n)$ -modules with degree 0 with respect to this grading. We regard the big and little Verma's as objects in this category by declaring $\deg M(\lambda)_{\lambda-\nu} = -\nu$ for all $\nu \in Q^+$. Let $\mathcal{C}$ be the category of left $\mathfrak{h}$ -modules.

Let $\Psi_0 : \mathcal{D} \rightarrow \mathcal{C}$ be the functor $\Psi_0(N) = N_0$ (i.e. the degree 0 component). The functor Ψ_0 has a left adjoint $\text{Ind} : \mathcal{C} \rightarrow \mathcal{D}$ given by $\text{Ind } A = \text{Ind}_{\mathfrak{b}^+}^{\mathfrak{q}(n)} A$, where we regard the $\mathfrak{h}$ -module A as a $\mathfrak{b}^+$ -module by inflation. The functor Ψ_0 also has an exact right adjoint Coind (see [15, Proposition 4.3]).

As in [15], let $\Theta(A) : \text{Ind } A \rightarrow \text{Coind } A$ be the morphism corresponding to the identity map $\text{id}_A : A \rightarrow A$. This induces a morphism of functors $\Theta : \text{Ind} \rightarrow \text{Coind}$. The main property we will use is

Theorem 6.3.1. [15, Proposition 4.4] *We have $\ker \Theta(A)$ is the maximal graded submodule of $\text{Ind } A$ which avoids A .*

Define the Shapovalov map $S := \Theta(\mathcal{U}(\mathfrak{h})) : \text{Ind}(\mathcal{U}(\mathfrak{h})) \rightarrow \text{Coind}(\mathcal{U}(\mathfrak{h}))$. Given an object A in $\mathcal{C}$, proposition 4.3 of [15] shows there is a canonical isomorphism $\text{Ind } A \cong \text{Ind } \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{h})} A$ and $\text{Coind } A \cong \text{Coind } \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{h})} A$. In this way, we may identify $\Theta(A)$ with $\Theta(\mathcal{U}(\mathfrak{h})) \otimes_{\mathcal{U}(\mathfrak{h})} \text{id}_A$. It follows that the map $\Theta(A)$ is completely determined by the Shapovalov map.

In order to describe S in more detail, we introduce some auxiliary data. Let $\varsigma : \mathcal{U}(\mathfrak{q}(n)) \rightarrow \mathcal{U}(\mathfrak{q}(n))$ be the antiautomorphism defined by $\varsigma(x) = -x$ for all $x \in \mathfrak{q}(n)$ and extended to $\mathcal{U}(\mathfrak{q}(n))$ by the rule $\varsigma(xy) = (-1)^{p(x)p(y)} \varsigma(y)\varsigma(x)$ for $x, y \in \mathcal{U}(\mathfrak{q}(n))$. Also, define the Harish-Chandra projection $HC : \mathcal{U}(\mathfrak{q}(n)) \rightarrow \mathcal{U}(\mathfrak{h})$ along the decomposition

$$\mathcal{U}(\mathfrak{q}(n)) = \mathcal{U}(\mathfrak{h}) \oplus (\mathcal{U}(\mathfrak{q}(n))\mathfrak{n}^+ + \mathfrak{n}^-\mathcal{U}(\mathfrak{q}(n))).$$

Now, we may naturally identify $\text{Ind}\mathcal{U}(\mathfrak{h}) \cong \mathcal{U}(\mathfrak{b}^-)$ as $(\mathfrak{b}^-, \mathfrak{h})$ -bimodules. The Q^- -grading on $\mathcal{U}(\mathfrak{b}^-)$ is given by

$$\mathcal{U}(\mathfrak{b}^-)_{-\nu} = \{x \in \mathcal{U}(\mathfrak{b}^-) \mid [h, x] = -\nu(h)x \text{ for all } h \in \mathfrak{h}_{\bar{0}}\} \quad (6.3.1)$$

for all $\nu \in Q^+$.

To describe $\text{Coind}\mathcal{U}(\mathfrak{h})$, let $\mathcal{D}_+$ be the category of Q^+ graded submodules and Ind_+ be the left adjoint to the functor $\Psi_0^+ : \mathcal{C} \rightarrow \mathcal{D}_+$. We may naturally identify $\text{Ind}_+\mathcal{U}(\mathfrak{h}) \cong \mathcal{U}(\mathfrak{b}^+)$ as $(\mathfrak{b}^+, \mathfrak{h})$ -bimodules and $\mathcal{U}(\mathfrak{b}^+)$ has a Q^+ -grading analogous to (6.3.1). Now, let $\mathcal{U}(\mathfrak{h})^\varsigma$ be the $(\mathfrak{h}, \mathfrak{h})$ -bimodule obtained by twisting the action of $\mathfrak{h}$ with ς . That is, $h.x = (-1)^{p(h)p(x)}\varsigma(h)x$ and $x.h = (-1)^{p(h)p(x)}x\varsigma(h)$ for all $x \in \mathcal{U}(\mathfrak{h})^\varsigma$ and $h \in \mathfrak{h}$. Then, there is a natural identification of $\text{Coind}\mathcal{U}(\mathfrak{h})$ with the graded dual of $\mathcal{U}(\mathfrak{b}^+)$ as $(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{h})$ -bimodules:

$$\text{Coind}\mathcal{U}(\mathfrak{h}) \cong \mathcal{U}(\mathfrak{b}^+)^\# := \bigoplus_{\nu \in Q^+} \text{Hom}_{\mathcal{C}}(\mathcal{U}(\mathfrak{b}^+)_{\nu}, \mathcal{U}(\mathfrak{h})^\varsigma),$$

see [15, Proposition 4.3(iii)]. Observe that $\mathcal{U}(\mathfrak{b}^+)^\#$ has a Q^- grading given by $\mathcal{U}(\mathfrak{b}^+)^\#_{-\nu} = \text{Hom}_{\mathcal{C}}(\mathcal{U}(\mathfrak{b}^+)_{\nu}, \mathcal{U}(\mathfrak{h})^\varsigma)$.

Using these identifications, we realize the Shapovalov map via the formula:

$$S(x)(y) = (-1)^{p(x)p(y)} HC(\varsigma(y)x),$$

for $x \in \mathcal{U}(\mathfrak{q}(n))$ and $y \in \mathcal{U}(\mathfrak{q}(n))$, [15, §4.2.4, Claim 3].

The Shapovalov map is homogeneous of degree 0. Therefore, $S = \sum_{\nu \in Q^+} S_\nu$, where $S_\nu : \mathcal{U}(\mathfrak{b}^-)_{-\nu} \rightarrow \mathcal{U}(\mathfrak{b}^+)^\#_{-\nu}$ is given by restriction.

For our purposes, it is more convenient to introduce a bilinear form

$$(\cdot, \cdot)_S : \mathcal{U}(\mathfrak{q}(n)) \otimes \mathcal{U}(\mathfrak{q}(n)) \rightarrow \mathcal{U}(\mathfrak{h})$$

with the property that $\text{Rad}(\cdot, \cdot)_S = \ker S$. To do this we introduce the (non-super) *transpose* antiautomorphism $\tau : \mathcal{U}(\mathfrak{q}(n)) \rightarrow \mathcal{U}(\mathfrak{q}(n))$ given by $\tau(x) = x^t$ if $x \in \mathfrak{q}(n)$ and extend to $\mathcal{U}(\mathfrak{q}(n))$ by $\tau(xy) = \tau(y)\tau(x)$. Note that this is the “naive” antiautomorphism introduced in [15]. Define $(\cdot, \cdot)_S$ by

$$(u, v)_S = (-1)^{p(u)p(v)} S(v)(\varsigma\tau(u)) = HC(\tau(u)v)$$

for all $u, v \in \mathcal{U}(\mathfrak{q}(n))$.

Proposition 6.3.2. *The radical of the form may be identified as: $\text{Rad}(\cdot, \cdot)_S = \ker S$.*

Proof. Assume $u \in \ker S$ and $v \in \mathcal{U}(\mathfrak{b}^-)$. Then, $\tau(v) \in \mathcal{U}(\mathfrak{b}^+)$ and

$$(\tau\varsigma(v), u)_S = (-1)^{p(u)p(v)} S(u)(\varsigma\tau\tau\varsigma(v)) = (-1)^{p(u)p(v)} S(u)(v) = 0,$$

showing that $u \in \text{Rad}(\cdot, \cdot)_S$.

Conversely, assume $u \in \text{Rad}(\cdot, \cdot)_S$ and $v \in \mathcal{U}(\mathfrak{b}^+)$. Then, $\tau\varsigma(v) \in \mathcal{U}(\mathfrak{b}^-)$ and

$$0 = (\tau\varsigma(v), u)_S = (-1)^{p(u)p(v)} S(u)(\varsigma\tau\tau\varsigma(v)) = (-1)^{p(u)p(v)} S(u)(v).$$

Hence, $u \in \text{Ker } S$. □

Remark 6.3.3. *We have already defined τ to be an antiautomorphism of the AHCA. We will show the compatibility of the two anti-automorphisms in Proposition 7.4.2.*

7. A LIE-THEORETIC CONSTRUCTION OF $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$

Let X be a $\mathfrak{q}(n)$ -supermodule. In this section we construct a homomorphism of superalgebras

$$\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d) \rightarrow \text{End}_{\mathfrak{q}(n)}(X \otimes V^{\otimes d})$$

along the lines of Arakawa and Suzuki, [1]. The main difficulty is the lack of an even invariant bilinear form, and consequently, a lack of a suitable Casimir element in $\mathfrak{q}(n)^{\otimes 2}$. However, we find inspiration for a suitable substitute in Olshanski's work in the quantum setting [30].

7.1. Lie Bialgebra structures on $\mathfrak{q}(n)$. We begin by reviewing the construction of a Manin triple for $\mathfrak{q}(n)$ from [30] (see also [12]). A Manin triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ consists of a Lie superalgebra $\mathfrak{p}$, a nondegenerate even invariant bilinear symmetric form B and two subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$ which are B -isotropic transversal subspaces of $\mathfrak{p}$. Then, B defines a nondegenerate pairing between $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

Define a cobracket $\Delta : \mathfrak{p}_1 \rightarrow \mathfrak{p}_1^{\otimes 2}$ by dualizing the bracket $\mathfrak{p}_2^{\otimes 2} \rightarrow \mathfrak{p}_2$:

$$B^{\otimes 2}(\Delta(X), Y_1 \otimes Y_2) = B(X, [Y_1, Y_2]), \quad (X \in \mathfrak{p}_1).$$

Then, the pair $(\mathfrak{p}_1, \Delta)$ is called a Lie (super)bialgebra.

Choose a basis $\{X_\alpha\}$ for $\mathfrak{p}_1$ and a basis $\{Y_\alpha\}$ for $\mathfrak{p}_2$ such that $B(X_\alpha, Y_\beta) = \delta_{\alpha\beta}$, and set $s = \sum_\alpha X_\alpha \otimes Y_\alpha$. Then, it turns out that s satisfies the classical Yang-Baxter equation

$$[s^{12}, s^{13}] + [s^{12}, s^{23}] + [s^{13}, s^{23}] = 0$$

and $\Delta(X) = [1 \otimes X + X \otimes 1, s]$, for $X \in \mathfrak{p}_1$.

7.2. The Super Casimir. Note that when $\mathfrak{p} = \mathfrak{g}$ is a simple Lie algebra, $\mathfrak{p}_1 = \mathfrak{b}_+$, $\mathfrak{p}_2 = \mathfrak{b}_-$ are the positive and negative Borel subalgebras and B is the trace form, s becomes the classical r -matrix, which we will denote r^{12} . We can repeat this construction with the roles of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ reversed and obtain another classical r -matrix which we denote r^{21} . Then, the Casimir is simply $\Omega = r^{12} + r^{21}$, see [1] §1.2.

In [30], Olshanski constructs such an element s for $\mathfrak{p} = \mathfrak{gl}(n|n)$, $\mathfrak{p}_1 = \mathfrak{q}(n)$ and some fixed choice of $\mathfrak{p}_2$ analogous to a positive Borel. We will review this construction to obtain an element which we will call s_+ , then replace $\mathfrak{p}_2$ with an analogue of a negative Borel to obtain another element called s_- . Then, we show that the element $\Omega = s_+ + s_-$ performs the role of the Casimir in our setting.

Definition 7.2.1. Let $\mathfrak{p} = \mathfrak{gl}(n|n)$, $B(x, y) = \text{str}(xy)$ (where $\text{str}(E_{ij}) = \delta_{ij} \text{sgn}(i)$ for $i, j \in I$), and $\mathfrak{p}_1 = \mathfrak{q}(n)$.

(1) Let

$$\mathfrak{p}_2^+ = \sum_{i \in I^+} \mathbb{C}(E_{ii} - E_{-i, -i}) + \sum_{\substack{i, j \in I, \\ i < j}} \mathbb{C}E_{ij}.$$

Then the corresponding element s_+ is given by

$$s_+ = \frac{1}{2} \sum_{i \in I^+} e_{ii} \otimes \bar{e}_{ii} + \sum_{\substack{i, j \in I^+ \\ i > j}} e_{ij} \otimes E_{ji} - \sum_{\substack{i, j \in I^+ \\ i < j}} e_{ij} \otimes E_{-j, -i} - \sum_{i, j \in I^+} f_{ij} \otimes E_{-j, i}.$$

(2) *Let*

$$\mathfrak{p}_2^- = \sum_{i \in I^+} \mathbb{C}(E_{ii} - E_{-i, -i}) + \sum_{\substack{i, j \in I, \\ i > j}} \mathbb{C}E_{ij}.$$

Then, the corresponding element s_- is given by

$$s_- = \frac{1}{2} \sum_{i \in I^+} e_{ii} \otimes \bar{e}_{ii} - \sum_{\substack{i, j \in I^+ \\ i > j}} e_{ij} \otimes E_{-j, -i} + \sum_{\substack{i, j \in I^+ \\ i < j}} e_{ij} \otimes E_{j, i} + \sum_{i, j \in I^+} f_{ij} \otimes E_{j, -i}.$$

We now define our substitute Casimir:

$$\Omega = s_+ + s_- = \sum_{i, j \in I^+} e_{ij} \otimes \bar{e}_{ji} - \sum_{i, j \in I^+} f_{ij} \otimes \bar{f}_{ji} \in Q(V) \otimes \text{End}(V), \quad (7.2.1)$$

where $\bar{e}_{ij}$ and $\bar{f}_{ij}$ are given in (6.1.1).

7.3. Classical Sergeev Duality. We now need to recall Sergeev's duality between $\mathcal{S}(d)$ and $\mathfrak{q}(n)$. Recall the matrix $C = \sum_{i \in I^+} \bar{f}_{ii}$ from the previous section, and define the superpermutation operator

$$S = \sum_{i, j \in I} \text{sgn}(j) E_{ij} \otimes E_{ji} \in \text{End}(V)^{\otimes 2},$$

where $\text{sgn}(j)$ is the sign of j . Let $\pi_i : \text{End}(V) \rightarrow \text{End}(V)^{\otimes d}$ be given by $\pi_i(x) = 1^{\otimes i-1} \otimes x \otimes 1^{\otimes d-i}$ for all $x \in \text{End}(V)$ and $i = 1, \dots, d$; similarly, define $\pi_{ij} : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)^{\otimes d}$ by $\pi_{ij}(x \otimes y) = 1^{\otimes i-1} \otimes x \otimes 1^{\otimes j-i-1} \otimes y \otimes 1^{\otimes d-j}$. Set $C_i = \pi_i(C)$ and, for $1 \leq i < j \leq d$, set $S_{ij} = \pi_{ij}(S)$. Then,

Theorem 7.3.1. [41, Theorem 3] *The map which sends $c_i \mapsto C_i$ and $s_i \mapsto S_{i, i+1}$ is an isomorphism of superalgebras*

$$\mathcal{S}(d) \rightarrow \text{End}_{\mathfrak{q}(n)}(V^{\otimes d}).$$

7.4. $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -action. Let M be a $\mathfrak{q}(n)$ -supermodule. In this section we construct an action of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ on $M \otimes V^{\otimes d}$ that commutes with the action of $\mathfrak{q}(n)$. To this end, extend the map π_i from §7.3 to a map $\pi_i : \text{End}(V) \rightarrow \text{End}(V)^{\otimes d+1}$ so that $\pi_i(x) = 1^{\otimes i} \otimes x \otimes 1^{\otimes d-i}$ for $x \in \text{End}(V)$ and $i = 0, \dots, d$ (i.e. add a 0th tensor place); similarly, extend π_{ij} .

Define C_i and S_{ij} as in §7.3. Define

$$\Omega_{ij} = \pi_{ij}(\Omega) \quad 0 \leq i < j \leq d$$

and set $X_i = \Omega_{0i} + \sum_{1 \leq j < i} (1 - C_j C_i) S_{ji}$.

Theorem 7.4.1. *Let M be a $\mathfrak{q}(n)$ -supermodule. Then, the map which sends $c_i \mapsto C_i$, $s_i \mapsto S_{i, i+1}$ and $x_i \mapsto X_i$ defines a homomorphism*

$$\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d) \rightarrow \text{End}_{\mathfrak{q}(n)}(M \otimes V^{\otimes d}).$$

Proof. It is clear from Theorem 7.3.1 that the C_i and $S_{i, i+1}$ form a copy of the Sergeev algebra $\mathcal{S}(d)$ inside $\text{End}_{\mathfrak{q}(n)}(M \otimes V^{\otimes d})$ via the obvious embedding $\text{End}_{\mathfrak{q}(n)}(V^{\otimes d}) \hookrightarrow \text{End}_{\mathfrak{q}(n)}(M \otimes V^{\otimes d})$, $A \mapsto \text{id}_M \otimes A$. Moreover, for $i = 1, \dots, d$, $X_i \in \text{End}(M \otimes V^{\otimes d})$, since $X_i \in Q(n) \otimes \text{End}(V)^{\otimes d}$. Therefore it is enough to check the following properties:

- (a) The X_i satisfy the mixed relations (3.1.4) and (3.1.5),
- (b) $X_i X_j - X_j X_i = 0$, and

(c) the X_i commute with the action of $\mathfrak{q}(n)$ on $M \otimes V^{\otimes d}$.

First, we check that $\Omega(1 \otimes C) = -(1 \otimes C)\Omega$. To do this, a calculation shows that $C\bar{e}_{ji} = -\bar{e}_{ji}$ and $C\bar{f}_{ji} = \bar{f}_{ji}C$. Hence,

$$(1 \otimes C)(e_{ij} \otimes \bar{e}_{ji}) = -(e_{ij} \otimes \bar{e}_{ji})(1 \otimes C)$$

and

$$\begin{aligned} (1 \otimes C)(f_{ij} \otimes \bar{f}_{ji}) &= (-1)^{p(f_{ij})p(C)}(f_{ij} \otimes C\bar{f}_{ji}) \\ &= (-1)^{p(\bar{f}_{ji})p(C)}(f_{ij} \otimes \bar{f}_{ji}C) \\ &= (-1)^{p(\bar{f}_{ji})p(C)+p(1)p(\bar{f}_{ji})}(f_{ij} \otimes \bar{f}_{ji})(1 \otimes C), \end{aligned}$$

so the result follows since $p(1) = \bar{0}$. Next, it is easy to see that $S_i\Omega_{0i}S_i = \Omega_{i+1}$ using (2.0.1). Therefore, (a) follows from the definition of X_i . It is now easy to show that, for $i < j$, (b) is equivalent to

$$\Omega_{0i}\Omega_{0j} - \Omega_{0j}\Omega_{0i} = (\Omega_{0j} - \Omega_{0i})S_{ij} + (\Omega_{0j} + \Omega_{0i})C_iC_jS_{ij}.$$

This equality is then a direct calculation. Finally, to verify (c), it is enough to show that for any $X \in \mathfrak{q}(n)$,

$$[1 \otimes X + X \otimes 1, \Omega] = 0.$$

This is another routine calculation using (2.0.1). $\square$

Now, recall the “naive” antiautomorphism $\tau : \mathcal{U}(\mathfrak{q}(n)) \rightarrow \mathcal{U}(\mathfrak{q}(n))$. This extends to an antiautomorphism of $\mathcal{U}(\mathfrak{gl}(n|n))$. Extend τ to an antiautomorphism of $\mathcal{U}(\mathfrak{gl}(n|n))^{\otimes 2}$ by $\tau(x \otimes y) = (-1)^{p(x)}\tau(x) \otimes \tau(y)$. By induction, extend τ to an antiautomorphism of $\mathcal{U}(\mathfrak{gl}(n|n))^{\otimes k}$ by $\tau(x_1 \otimes \cdots \otimes x_k) = (-1)^{p(x_1)}\tau(x_1) \otimes \tau(x_2 \otimes \cdots \otimes x_k)$. A direct check verifies the following result.

Proposition 7.4.2. *We have that $\tau(C_i) = -C_i$, $\tau(S_{i,i+1}) = S_{i,i+1}$ and $\tau(X_i) = X_i$ for all admissible i ’s. In particular, the antiautomorphism $\tau^{\otimes d+1} : \mathcal{U}(\mathfrak{gl}(n|n))^{\otimes d+1} \rightarrow \mathcal{U}(\mathfrak{gl}(n|n))^{\otimes d+1}$ coincides with the antiautomorphism $\tau : \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d) \rightarrow \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$.*

7.5. The Functor F_λ . In the previous section, we showed that there is a homomorphism from $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ to $\text{End}_{\mathfrak{q}(n)}(M \otimes V^{\otimes d})$. Since the action of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ on $M \otimes V^{\otimes d}$ commutes with the action of $\mathfrak{q}(n)$, it preserves both primitive vectors and weight spaces. By *primitive vector* we mean an element of $M \otimes V^{\otimes d}$ which is annihilated by the subalgebra $\mathfrak{n}^+$ given by the triangular decomposition of $\mathfrak{q}(n)$ as in Section 6.2. Therefore, given a weight $\lambda \in P(M \otimes V^{\otimes d})$ we have an action of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ on

$$F_\lambda M := \{m \in M \otimes V^{\otimes d} \mid \mathfrak{n}^+.m = 0 \text{ and } m \in (M \otimes V^{\otimes d})_\lambda\} \quad (7.5.1)$$

In the case when $\lambda \in P^{++}$ we can provide alternative descriptions of the functor F_λ . First we recall the following key result of Penkov [32]. Given a weight $\lambda \in P$, we write χ_λ for the central character defined by the simple $\mathfrak{q}(n)$ -module of highest weight λ . Then, there is a block decomposition

$$\mathcal{O}(\mathfrak{q}(n)) = \bigoplus_{\chi_\lambda} \mathcal{O}(\mathfrak{q}(n))^{[\lambda]} \quad (7.5.2)$$

where the sum is over all central characters χ_λ and $\mathcal{O}(\mathfrak{q}(n))^{[\lambda]} = \mathcal{O}(\mathfrak{q}(n))^{[\chi_\lambda]}$ denotes the block determined by the central character χ_λ . Given N in $\mathcal{O}(\mathfrak{q}(n))$, let $N^{[\chi_\gamma]} = N^{[\gamma]}$ denote the projection of N onto the direct summand which lies in $\mathcal{O}(\mathfrak{q}(n))^{[\chi_\gamma]}$

The question then becomes to describe when $\chi_\lambda = \chi_\mu$ for $\lambda, \mu \in P$. This is answered in the case when λ is typical by the following result of Penkov [32]. Recall that the symmetric group acts on P by permutation of coordinates.

Proposition 7.5.1. *Let $\lambda \in P^{++}$ be a typical weight and let $\mu \in P$. Then $\chi_\lambda = \chi_\mu$ if and only if $\mu = w(\lambda)$ for some $w \in S_n$.*

For short we call a weight $\lambda \in P$ *atypical* if it is not typical. By the description of the blocks $\mathcal{O}(\mathfrak{q}(n))^{[\lambda]}$, if $L(\mu)$ is an object of $\mathcal{O}(\mathfrak{q}(n))^{[\lambda]}$ then λ is typical if and only if μ is typical (c.f. [34, Proposition 1.1] and the remarks which follow it). We then have the following preparatory lemma.

Lemma 7.5.2. *Let $\lambda, \gamma \in P$. Then the following statements hold:*

- (i) *Assume γ is atypical and λ is typical. If N is an object of $\mathcal{O}^{[\gamma]}$, then $N_\lambda = (\mathfrak{n}^- N)_\lambda$.*
- (ii) *Assume $\lambda, \gamma \in P^{++}$ are typical and dominant and $\lambda \neq \gamma$. If N is an object of $\mathcal{O}^{[\gamma]}$, then $N_\lambda = (\mathfrak{n}^- N)_\lambda$.*

Proof. By [3, Lemma 4.5], every object $\mathcal{O}(\mathfrak{q}(n))$ has a finite Jordan-Hölder series. The proof of (i) is by induction on the length of a composition series of N . The base case is when N has length one (ie. $N \cong L(\nu)$ is a simple module). This case immediately follows from the fact that in order for N_λ to be nontrivial it must be that $\lambda < \nu$. But then it follows from the assumption that ν is atypical (since $L(\nu)$ is an object of $\mathcal{O}^{[\gamma]}$) while λ is typical. Now consider a composition series

$$0 = N_0 \subset N_1 \subset \cdots \subset N_t = N.$$

Let $v \in N_\lambda$ so that $v + N_{t-1} \in N_t/N_{t-1}$ is nonzero. Since N_t/N_{t-1} is a simple module in $\mathcal{O}^{[\gamma]}$, by the base case there exists a $w \in N_t = N$ and $y \in \mathfrak{n}^-$ so that $yw + N_{t-1} = v + N_{t-1}$. Thus, $v - yw \in N_{t-1}$ and is of weight λ . By the inductive assumption, there exists $w' \in N_{t-1} \subset N$ and $y' \in \mathfrak{n}^-$ such that $y'w' = v - yw$. That is, $v = yw + y'w' \in \mathfrak{n}^- N$. This proves the desired result.

Now, (ii) follows by a similar argument by induction on the length of a composition series. If N is simple and $N_\lambda \neq 0$, then λ is not the highest weight of N (as γ is the unique dominant highest weight among the simple modules in $\mathcal{O}^{[\gamma]}$ by Proposition 7.5.1). From this it immediately follows that $N_\lambda = (\mathfrak{n}^- N)_\lambda$. Now proceed by induction as in the previous paragraph. $\square$

Lemma 7.5.3. *Let $\lambda \in P^{++}$ be typical and dominant, and let $M \in \mathcal{O}$. Then*

$$F_\lambda(M) \cong \left((M \otimes V^{\otimes d})^{[\lambda]} \right)_\lambda \cong [M \otimes V^{\otimes d} / \mathfrak{n}_-(M \otimes V^{\otimes d})]_\lambda$$

as $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -modules.

Proof. It should first be remarked that since the action of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ commutes with the action of $\mathfrak{q}(n)$, the action of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ on $M \otimes V^{\otimes d}$ induces an action on each of the vector spaces given in the theorem.

Now, by Proposition 7.5.1 and the assumption that λ is dominant, it follows that for any module $N \in \mathcal{O}^{[\lambda]}$, $N_\nu \neq 0$ only if $\nu \leq \lambda$ in the dominance order. Thus any vector of weight λ in $M \otimes V^{\otimes d}$

is necessarily a primitive vector. On the other hand, if there is a primitive vector of weight λ in $M \otimes V^{\otimes d}$, then it must lie in the image of a nonzero homomorphism $M(\lambda) \rightarrow M \otimes V^{\otimes d}$. But as $M(\lambda)$ is an object in $\mathcal{O}^{[\lambda]}$, it follows that the primitive vector lies in $((M \otimes V^{\otimes d})^{[\lambda]})_\lambda$. Thus, there exists a canonical projection map

$$F_\lambda(M) \rightarrow ((M \otimes V^{\otimes d})^{[\lambda]})_\lambda$$

and this map is necessarily a vector space isomorphism. The fact that it is a $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module homomorphism follows from the fact that the action of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ on both vector spaces is induced by the action of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ on $M \otimes V^{\otimes d}$.

Now consider the block decomposition

$$M \otimes V^{\otimes d} = \oplus_{\chi_\gamma} (M \otimes V^{\otimes d})^{[\chi_\gamma]},$$

where the direct sum runs over dominant $\gamma \in \mathfrak{h}_0^*$ so that different χ_γ are different central characters of $U(\mathfrak{g})$. This then induces the vector space direct sum decomposition

$$(M \otimes V^{\otimes d})/\mathfrak{n}^-(M \otimes V^{\otimes d}) = \oplus_{\chi_\gamma} (M \otimes V^{\otimes d})^{[\chi_\gamma]}/\mathfrak{n}^-(M \otimes V^{\otimes d})^{[\chi_\gamma]},$$

where $(M \otimes V^{\otimes d})^{[\chi_\gamma]}$ denotes the direct summand of $M \otimes V^{\otimes d}$ which lies in the block $\mathcal{O}^{[\gamma]}$.

By the previous lemma, if γ is atypical or if γ is typical and $\gamma \neq \lambda$, then

$$\left[(M \otimes V^{\otimes d})^{[\chi_\gamma]}/\mathfrak{n}^-(M \otimes V^{\otimes d})^{[\chi_\gamma]} \right]_\lambda = 0.$$

Therefore,

$$\left[(M \otimes V^{\otimes d})/\mathfrak{n}^-(M \otimes V^{\otimes d}) \right]_\lambda = \left[(M \otimes V^{\otimes d})^{[\chi_\lambda]}/\mathfrak{n}^-(M \otimes V^{\otimes d})^{[\chi_\lambda]} \right]_\lambda. \quad (7.5.3)$$

Finally, if N is an object of $\mathcal{O}^{[\lambda]}$, then $N_\mu \neq 0$ only if $\mu \leq \lambda$ in the dominance order. Thus weight considerations imply $[\mathfrak{n}^-(M \otimes V^{\otimes d})^{[\chi_\lambda]}]_\lambda = 0$ which, in turn, implies that canonical projection

$$((M \otimes V^{\otimes d})^{[\lambda]})_\lambda \rightarrow [M \otimes V^{\otimes d}/\mathfrak{n}_-(M \otimes V^{\otimes d})]_\lambda$$

is a vector space isomorphism. That is its a $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module homomorphism follows from the fact that in both cases the action is induced from the $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ action on $M \otimes V^{\otimes d}$. $\square$

Corollary 7.5.4. *If $\lambda \in P^{++}$ is dominant and typical, then the functor $F_\lambda : \mathcal{O} \rightarrow \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)\text{-mod}$ is exact.*

Proof. This follows immediately from the first alternative description of F_λ in the above theorem as it is the composition of the exact functors $- \otimes V^{\otimes d}$, projection onto the direct summand lying in the block $\mathcal{O}^{[\lambda]}$, and projection onto the λ weight space. $\square$

In what follows when λ is dominant and typical we use whichever description of F_λ given in lemma 7.5.3 is most convenient.

7.6. Image of the Functor. We can now describe the image of Verma modules under the functor.

Lemma 7.6.1. *Let $M(\mu)$ be a Verma module in $\mathcal{O}$ and let $\lambda \in P^{++}$ be a dominant and typical weight. The natural inclusion*

$$E(\mu) \otimes (V^{\otimes d})_{\lambda-\mu} \hookrightarrow (M(\mu) \otimes V^{\otimes d})_{\lambda}$$

induces an isomorphism of $\mathcal{S}(d)$ -modules $E(\mu) \otimes (V^{\otimes d})_{\lambda-\mu} \cong F_{\lambda}(M(\mu))$. In particular, $F_{\lambda}(M(\mu)) = 0$ unless $\lambda - \mu \in P_{\geq 0}(d)$.

Proof. This is proved exactly as in [1, Lemma 3.3.2], except now the highest weight space of $M(\mu)$ is $E(\mu)$. Namely, by the tensor identity and the PBW theorem,

$$M(\mu) \otimes V^{\otimes d} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} (E(\mu) \otimes V^{\otimes d}) \cong U(\mathfrak{n}^-) \otimes E(\mu) \otimes V^{\otimes d}, \quad (7.6.1)$$

where the first isomorphism is as $\mathfrak{g}$ -modules and the second is as $\mathfrak{h}_0$ -modules. Thus the canonical projection map induces the isomorphism of $\mathfrak{h}_0$ -modules given by

$$1 \otimes E(\mu) \otimes V^{\otimes d} \cong M(\mu) \otimes V^{\otimes d} / \mathfrak{n}^- (M(\mu) \otimes V^{\otimes d}).$$

Taking λ weight spaces on both sides yields the vector space isomorphism

$$1 \otimes E(\mu) \otimes (V^{\otimes d})_{\lambda-\mu} \cong [M(\mu) \otimes V^{\otimes d} / \mathfrak{n}^- (M(\mu) \otimes V^{\otimes d})]_{\lambda}.$$

Now, the composition of the natural inclusion $E(\mu) \otimes (V^{\otimes d})_{\lambda-\mu} \hookrightarrow (M(\mu) \otimes V^{\otimes d})_{\lambda}$ with (7.6.1), and the isomorphism above implies that

$$E(\mu) \otimes (V^{\otimes d})_{\lambda-\mu} \cong 1 \otimes E(\mu) \otimes (V^{\otimes d})_{\lambda-\mu} \cong [M(\mu) \otimes V^{\otimes d} / \mathfrak{n}^- (M(\mu) \otimes V^{\otimes d})]_{\lambda} = F_{\lambda}(M(\mu)),$$

That it is an isomorphism of $\mathcal{S}(d)$ -modules follows from the fact that in each case the action of $\mathcal{S}(d)$ is via the action induced from the action of $\mathcal{S}(d)$ on $M(\mu) \otimes V^{\otimes d}$. $\square$

Corollary 7.6.2. *Let $\lambda \in P^{++}$ be a dominant and typical weight and let $\mu \in P$ with $\lambda - \mu \in P_{\geq 0}(d)$. Set $d_i = \lambda_i - \mu_i$ for $i = 1, \dots, n$.*

(i) *Let $M(\mu)$ be the little Verma module of highest weight μ . Then,*

$$\dim F_{\lambda}(M(\mu)) = 2^{d + \lfloor (n - \gamma_0(\mu) + 1)/2 \rfloor} \frac{d!}{d_1! \cdots d_n!}.$$

(ii) *Let $\widehat{M}(\mu)$ be the big Verma module of highest weight μ . Then,*

$$\dim F_{\lambda}(\widehat{M}(\mu)) = 2^{d + n - \gamma_0(\mu)} \frac{d!}{d_1! \cdots d_n!}.$$

Proof. We have $\dim E(\mu) = 2^{\lfloor (n - \gamma_0(\mu) + 1)/2 \rfloor}$. For each ε_i ($i = 1, \dots, n$), $\dim V_{\varepsilon_i} = 2$. A combinatorial count shows that

$$\dim (V^{\otimes d})_{\lambda-\mu} = \frac{d!}{d_1! \cdots d_n!} 2^d.$$

The statement of (i) then follows by Lemma 7.6.1. The statement of (ii) follows from (i) and Lemma 6.2.1. $\square$

Fix $\lambda, \mu \in P$ such that $\lambda - \mu \in P_{\geq 0}(d)$, and let $d_i = \lambda_i - \mu_i$. Let $\{u_i, u_{\bar{i}}\}_{i=1, \dots, n}$ be the standard basis for V , let $v_\mu \in E(\mu)$, and let $u_{\lambda-\mu} = u_1^{\otimes d_1} \otimes \cdots \otimes u_n^{\otimes d_n} \in (V^{\otimes d})_{\lambda-\mu}$. Finally, let

$$m_k = \sum_{i=1}^k d_i,$$

and define $F_k = \pi_0(f_{kk})$ (see Section 7.4).

Lemma 7.6.3. *Let $v_\mu \in M(\mu)_\mu$ be a primitive vector of weight μ , and let $u = u_{\lambda-\mu} = u_1^{\otimes d_1} \otimes \cdots \otimes u_n^{\otimes d_n}$. For each $1 \leq k \leq n$ and $m_{k-1} < i \leq m_k$,*

$$X_i v_\mu \otimes u_{\lambda-\mu} \equiv \left(\mu_k + i - m_{k-1} - 1 - \sum_{m_{k-1} < l < i} C_l C_i - F_k C_i \right) v_\mu \otimes u_{\lambda-\mu}$$

modulo $\mathfrak{n}_-(M(\mu) \otimes V^{\otimes d})$. As a consequence,

$$X_i^2 v_\mu \otimes u_{\lambda-\mu} \equiv (\mu_k + i - m_{k-1} - 1)(\mu_k + i - m_{k-1}) v_\mu \otimes u_{\lambda-\mu},$$

again modulo $\mathfrak{n}_-(M(\mu) \otimes V^{\otimes d})$.

Proof. We first do some preliminary calculations. Let $1 \leq j < k \leq n$ be fixed, let $m_{k-1} \leq i \leq m_k$ be fixed, and consider the vector

$$v_\mu \otimes u_1^{\otimes d_1} \otimes \cdots \otimes u_k^{\otimes a} \otimes u_j \otimes u_k^{\otimes b} \otimes \cdots \otimes u_n^{\otimes d_n},$$

where the u_j is the i th tensor and $a + b + 1 = d_k$ (i.e. among the u_k 's, the one in the i th position, recalling that v_μ is in the zeroth position, is replaced with u_j). For short, let us write $u = u_1^{\otimes d_1} \otimes \cdots \otimes u_n^{\otimes d_n}$ and $\hat{u} = u_1^{\otimes d_1} \otimes \cdots \otimes u_k^{\otimes a} \otimes u_j \otimes u_k^{\otimes b} \otimes \cdots \otimes u_n^{\otimes d_n}$. Then,

$$\begin{aligned} e_{kj}(v_\mu \otimes \hat{u}) &= (e_{kj}v_\mu) \otimes \hat{u} \\ &\quad + \sum_{r=1}^{d_j} v_\mu \otimes u_1^{\otimes d_1} \otimes \cdots \otimes u_j^{\otimes r-1} \otimes u_k \otimes u_j^{\otimes d_j-r} \otimes \cdots \otimes u_k^{\otimes a} \otimes u_j \otimes u_k^{\otimes b} \otimes \cdots \otimes u_n^{\otimes d_n} + v_\mu \otimes u \\ &= (e_{kj}v_\mu) \otimes \hat{u} + \sum_{r=1}^{d_j} S_{m_{j-1}+r,i}(v_\mu \otimes u) + v_\mu \otimes u. \end{aligned}$$

Similarly, if we write $\check{u} = C_i \hat{u} = u_1^{\otimes d_1} \otimes \cdots \otimes u_k^{\otimes a} \otimes v_{-j} \otimes v_k^{\otimes b} \otimes \cdots \otimes v_n^{\otimes d_n}$, then

$$\begin{aligned} f_{kj}(v_\mu \otimes \check{u}) &= (f_{kj}v_\mu) \otimes \check{u} + (-1)^{p(v_\mu)} \times \\ &\quad \times \sum_{r=1}^{d_j} v_\mu \otimes u_1^{\otimes d_1} \otimes \cdots \otimes u_j^{\otimes r-1} \otimes u_{-k} \otimes u_j^{\otimes d_j-r} \otimes \cdots \otimes u_k^{\otimes a} \otimes u_{-j} \otimes u_k^{\otimes b} \otimes \cdots \otimes u_n^{\otimes d_n} \\ &\quad + (-1)^{p(v_\mu)} v_\mu \otimes u \\ &= (f_{kj}v_\mu) \otimes \check{u} + (-1)^{p(v_\mu)} \sum_{r=1}^{d_j} C_{m_{j-1}+a} C_i S_{m_{j-1}+r,i}(v_\mu \otimes u) + (-1)^{p(v_\mu)} v_\mu \otimes u. \end{aligned}$$

We can now consider the first statement of the lemma. Throughout, we write $\equiv$ for congruence modulo the subspace $\mathfrak{n}_-(M(\mu) \otimes V^{\otimes d})$. Let $1 \leq k \leq n$ be fixed so that $m_{k-1} < i \leq m_k$ (ie. there

is a u_k in the i th position of $v_\mu \otimes u$). Using that v_μ is a primitive vector and the equalities given above, we deduce that

$$\begin{aligned}
X_i(v_\mu \otimes u_{\lambda-\mu}) &= \sum_{\ell,j=1}^n e_{\ell j} v_\mu \otimes u_1^{\otimes d_1} \otimes \cdots \otimes u_k^{\otimes i-m_{k-1}-1} \otimes \bar{e}_{j\ell} u_k \otimes u_k^{\otimes m_k-i} \otimes \cdots \otimes u_n^{\otimes d_n} \\
&\quad - (-1)^{p(v_\mu)} \sum_{\ell,j=1}^n f_{\ell j} v_\mu \otimes u_1^{\otimes d_1} \otimes \cdots \otimes u_k^{\otimes i-m_{k-1}-1} \otimes \bar{f}_{j\ell} u_k \otimes u_k^{\otimes m_k-i} \otimes \cdots \otimes u_n^{\otimes d_n} \\
&\quad + \sum_{\ell < i} (1 - C_\ell C_i) S_{\ell i}(v_\mu \otimes u) \\
&= \sum_{j \leq k} e_{kj} v_\mu \otimes u_1^{\otimes d_1} \otimes \cdots \otimes u_k^{\otimes i-m_{k-1}-1} \otimes u_j \otimes u_k^{\otimes m_k-i} \otimes \cdots \otimes u_n^{\otimes d_n} \\
&\quad - (-1)^{p(v_\mu)} \sum_{j \leq k} f_{kj} v_\mu \otimes u_1^{\otimes d_1} \otimes \cdots \otimes u_k^{\otimes i-m_{k-1}-1} \otimes u_{-j} \otimes u_k^{\otimes m_k-i} \otimes \cdots \otimes u_n^{\otimes d_n} \\
&\quad + \sum_{\ell < i} (1 - C_\ell C_i) S_{\ell i}(v_\mu \otimes u) \\
&\equiv - \sum_{j < k} \left[\sum_{a=1}^{d_j} S_{m_{j-1}+a,i}(v_\mu \otimes u) + v_\mu \otimes u \right] \\
&\quad + \sum_{j < k} \left[\sum_{a=1}^{d_j} C_{m_{j-1}+a} C_i S_{m_{j-1}+a,i}(v_\mu \otimes u) + v_\mu \otimes u \right] \\
&\quad + \mu_k v_\mu \otimes u - C_i((f_{kk} v_\mu) \otimes u) + \sum_{\ell < i} (1 - C_\ell C_i) S_{\ell i}(v_\mu \otimes u) \\
&= - \sum_{l \leq m_{k-1}} S_{l,i} v_\mu \otimes u - (k-1) v_\mu \otimes u + \sum_{l \leq m_{k-1}} C_l C_i S_{l,i} v_\mu \otimes u + (k-1) v_\mu \otimes u \\
&\quad + \mu_k v_\mu \otimes u - C_i((f_{kk} v_\mu) \otimes u) + \sum_{\ell < i} (1 - C_\ell C_i) S_{\ell i}(v_\mu \otimes u) \\
&= \mu_k v_\mu \otimes u_{\lambda-\mu} + C_i((f_{kk} v_\mu) \otimes u_{\lambda-\mu}) + \sum_{m_{k-1} < \ell < i} (1 - C_\ell C_i) S_{\ell i}(v_\mu \otimes u) \\
&= \left(\mu_k + i - m_{k-1} - 1 - \sum_{m_{k-1} < \ell < i} C_\ell C_i S_{\ell i} \right) (v_\mu \otimes u_{\lambda-\mu}) + C_i((f_{kk} v_\mu) \otimes u) \\
&= \left(\mu_k + i - m_{k-1} - 1 - \sum_{m_{k-1} < \ell < i} C_\ell C_i - F_k C_i \right) (v_\mu \otimes u).
\end{aligned}$$

Note the last equality makes use of the fact that $S_{l,i} v_\mu \otimes u = v_\mu \otimes u$ for $m_{k-1} < l < i$ and that as (odd) linear maps $F_k C_i = -C_i F_k$.

Now we consider the second statement of the lemma. Using the previous calculation, the fact that X_i and the C 's satisfy relation (3.1.4) of the degenerate AHCA, and the fact that $f_{kk} v_\mu \in M(\mu)_\mu$ is again a primitive vector of weight μ ,

$$\begin{aligned}
X_i^2(v_\mu \otimes u_{\lambda-\mu}) &\equiv X_i \left(\mu_k + i - m_{k-1} - 1 - \sum_{m_{k-1} < \ell < i} C_\ell C_i - F_k C_i \right) (v_\mu \otimes u_{\lambda-\mu}) \\
&= \left(\mu_k + i - m_{k-1} - 1 + \sum_{m_{k-1} < \ell < i} C_\ell C_i \right) X_i(v_\mu \otimes u) - C_i X_i((f_{kk} v_\mu) \otimes u_{\lambda-\mu}) \\
&\equiv \left(\mu_k + i - m_{k-1} - 1 + \sum_{m_{k-1} < \ell < i} C_\ell C_i \right) \times \\
&\quad \times \left(\mu_k + i - m_{k-1} - 1 - \sum_{m_{k-1} < \ell < i} C_\ell C_i - F_k C_i \right) (v_\mu \otimes u_{\lambda-\mu}) \\
&\quad - C_i \left(\mu_k + i - m_{k-1} - 1 - \sum_{m_{k-1} < \ell < i} C_\ell C_i - F_k C_i \right) ((f_{kk} v_\mu) \otimes u_{\lambda-\mu}) \\
&= \left(\mu_k + i - m_{k-1} - 1 + \sum_{m_{k-1} < \ell < i} C_\ell C_i \right) \left(\mu_k + i - m_{k-1} - 1 - \sum_{m_{k-1} < \ell < i} C_\ell C_i \right) v_\mu \otimes u \\
&\quad + C_i F_k C_i ((f_{kk} v_\mu) \otimes u) \\
&= \left((\mu_k + i - m_{k-1} - 1)^2 - \left(\sum_{m_{k-1} < \ell < i} C_\ell C_i \right)^2 \right) v_\mu \otimes u + (f_{kk}^2 v_\mu) \otimes u \\
&= ((\mu_k + i - m_{k-1} - 1)^2 + (\mu_k + i - m_{k-1} - 1)) v_\mu \otimes u.
\end{aligned}$$

The last equality follows from the fact that in the Clifford algebra

$$\left(\sum_{m_{k-1} < \ell < i} C_\ell C_i \right)^2 = \sum_{m_{k-1} < \ell < i} (C_\ell C_i)^2 = \sum_{m_{k-1} < \ell < i} -1 = -(i - m_{k-1} + 1)$$

and that, in $\mathfrak{q}(n)$, $f_{kk}^2 = e_{kk}$. $\square$

Corollary 7.6.4. *Let $\lambda \in P^{++}$ be a dominant typical weight, let $\mu \in P$, and let $M(\mu)$ be a Verma module in $\mathcal{O}(\mathfrak{q}(n))$. Then for $i = 1, \dots, d$ the element x_i^2 acts on $F_\lambda(M(\mu))$ with generalized eigenvalues of the form $q(a)$ for various $a \in \mathbb{Z}$. Hence, $F_\lambda(M(\mu))$ is integral.*

As a consequence of the previous corollary we see that for $\lambda \in P^{++}$ we have that $F_\lambda(L(\mu))$ is integral for any simple module $L(\mu)$ in $\mathcal{O}$ and, therefore,

$$F_\lambda : \mathcal{O}(\mathfrak{q}(n)) \rightarrow \text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d).$$

Proposition 7.6.5. *Let $\lambda \in P^{++}$ and $\mu \in \lambda - P_{\geq 0}(d)$. Then, $F_\lambda(\widehat{M}(\mu)) \cong \widehat{\mathcal{M}}(\lambda, \mu)$.*

Proof. Let $v_+ \in \mathbb{C}_\mu$ be a nonzero vector in the 1-dimensional $\mathfrak{h}_{\bar{0}}$ -module $\mathbb{C}_\mu$, let $v_\mu = 1 \otimes v_+ \in C(\mu)_{\bar{0}}$ be its image and let $u_{\lambda-\mu}$ be as in the previous lemma. Then $v_\mu \otimes u_{\lambda-\mu}$ is a cyclic vector for $F_\lambda(\widehat{M}(\mu))$ as a $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module.

Recall the cyclic vector $\hat{\mathbf{1}}_{\lambda, \mu} \in \widehat{\mathcal{M}}(\lambda, \mu)$. For $\delta_1, \dots, \delta_n \in \{0, 1\}$, let $\varphi_1^{\delta_1} \cdots \varphi_n^{\delta_n} \hat{\mathbf{1}}_{\lambda, \mu} = 1 \otimes \varphi_1^{\delta_1} \hat{\mathbf{1}} \otimes \cdots \otimes \varphi_n^{\delta_n} \hat{\mathbf{1}}$, cf. (4.3.3).

Note that $w.(v_\mu \otimes u_{\lambda-\mu}) = v_\mu \otimes u_{\lambda-\mu}$ for all $w \in S_{\lambda-\mu}$. Comparing Lemma 7.6.3 and Proposition 4.1.1, we deduce that, by Frobenius reciprocity, there exists a surjective $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -homomorphism $\widehat{\mathcal{M}}(\lambda, \mu) \rightarrow F_\lambda(\widehat{M}(\mu))$ sending $\varphi_1^{\delta_1} \cdots \varphi_n^{\delta_n} \widehat{1}_{\lambda, \mu} \mapsto F_1^{\delta_1} \cdots F_n^{\delta_n} v_\mu \otimes u_{\lambda-\mu}$. That this is an isomorphism follows by comparing dimensions using Lemmas 4.3.1 and 7.6.2. $\square$

Corollary 7.6.6. *We have*

$$F_\lambda M(\mu) \cong \mathcal{M}(\lambda, \mu)^{\oplus 2^{\varpi(\mu)}}$$

where

$$\varpi(\mu) = \begin{cases} \lfloor \frac{n+1}{2} \rfloor & \text{if } \gamma_0(\mu) \text{ is even,} \\ \lfloor \frac{n}{2} \rfloor & \text{if } \gamma_0(\mu) \text{ is odd.} \end{cases}$$

Proof. Using the additivity of the functor F_λ , the previous proposition, and Lemmas 6.2.1 and 4.3.1 we obtain $F_\lambda M(\mu) = 2^{n - \lfloor \frac{\gamma_0(\mu)+1}{2} \rfloor - \lfloor \frac{n-\gamma_0(\mu)}{2} \rfloor} \mathcal{M}(\lambda, \mu)$. It is just left to observe that

$$n - \lfloor \frac{\gamma_0(\mu)+1}{2} \rfloor - \lfloor \frac{n-\gamma_0(\mu)}{2} \rfloor = \varpi(\mu).$$

$\square$

Lemma 7.6.7. *Assume that $\lambda \in P^{++}$, $\mu \in P^+[\lambda]$, $\lambda - \mu \in P_{\geq 0}(d)$, and $\alpha \in R^+[\lambda]$. Then, $\mathcal{M}(\lambda, \mu) \cong \mathcal{M}(\lambda, s_\alpha \mu)$.*

Proof. By Lemma 6.2.2, there exists an injective homomorphism $M(s_\alpha \mu) \rightarrow M(\mu)$. Since $\varpi(\mu) = \varpi(s_\alpha \mu)$, there exists an injective homomorphism

$$\mathcal{M}(\lambda, s_\alpha \mu)^{\varpi(\mu)} = F_\lambda M(s_\alpha \mu) \rightarrow F_\lambda M(\mu) = \mathcal{M}(\lambda, \mu)^{\varpi(\mu)}.$$

Since $\dim \mathcal{M}(\lambda, s_\alpha \mu) = \dim \mathcal{M}(\lambda, \mu)$ and by Theorem 4.4.10 $\mathcal{M}(\lambda, \mu)$ is indecomposable, it follows that this map is an isomorphism. $\square$

Theorem 7.6.8. *Assume $\lambda \in P^{++}$ and $\mu \in \lambda - P_{\geq 0}(d)$. Then, $\mathcal{M}(\lambda, \mu)$ has a unique maximal submodule $\mathcal{R}(\lambda, \mu)$ and unique irreducible quotient $\mathcal{L}(\lambda, \mu)$.*

Proof. There exists $w \in S_d[\lambda]$ such that $w\mu \in P^+[\lambda]$. By Lemma 7.6.7, $\mathcal{M}(\lambda, w\mu) \cong \mathcal{M}(\lambda, \mu)$. By Theorem 4.4.10, $\mathcal{M}(\lambda, w\mu)$ has a unique maximal submodule and unique irreducible quotient, so the result follows. $\square$

Given $\mu \in P$, the Shapovalov form on $M(\mu)$ induces a non-degenerate $\mathfrak{q}(n)$ -contravariant form on $L(\mu)$, which we will denote $(\cdot, \cdot)_\mu$. In turn we have a non-degenerate $\mathfrak{q}(n)$ -contravariant form on $L(\mu) \otimes V^{\otimes d}$ given by $(\cdot, \cdot)_\mu \otimes (\cdot, \cdot)_{\varepsilon_1}^{\otimes d}$. Observe that different weight spaces are orthogonal with respect to this form and different blocks of $\mathcal{O}(\mathfrak{q}(n))$ given by central characters are also orthogonal. Therefore, when $\lambda \in P^{++}$ is dominant and typical it follows that the bilinear form restricts to a form on $(L(\mu) \otimes V^{\otimes d})_\lambda^{[\lambda]} = F_\lambda(L(\mu))$, which is non-degenerate whenever it is nonzero. By Proposition 7.4.2, this form is $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -contravariant.

Similarly, Proposition 7.4.2 implies that the Shapovalov form on $\widehat{M}(\mu)$ induces an $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -contravariant form on $\widehat{\mathcal{M}}(\lambda, \mu)$. Now, if $\lambda \in P^{++}$ and $\mu \in \lambda - P_{\geq 0}(d)$, then by Theorem 7.6.8, $\widehat{\mathcal{M}}(\lambda, \mu)$ possesses a unique submodule $\widehat{\mathcal{R}}(\lambda, \mu)$ which is maximal among those which avoid the generalized $\zeta_{\lambda, \mu}$ weight space. Indeed,

$$\widehat{\mathcal{R}}(\lambda, \mu) = \mathcal{R}(\lambda, \mu)^{\oplus 2^{n - \lfloor \frac{\gamma_0(\mu)+1}{2} \rfloor}}.$$

Proposition 7.6.9. *Assume that $\lambda \in P^{++}$, $\mu \in \lambda - P_{\geq 0}(d)$, and $\widehat{\mathcal{M}}(\lambda, \mu)$ possesses a nonzero contravariant form $(\cdot, \cdot)$. Let $\mathcal{R}$ denote the radical of this form. Then,*

$$\mathcal{R} \supseteq \widehat{\mathcal{R}}(\lambda, \mu).$$

Proof. First, recall that $\widehat{\mathcal{M}}(\lambda, \mu)$ is cyclically generated by $\hat{\mathbf{1}}_{\lambda, \mu} \in \widehat{\mathcal{M}}(\lambda, \mu)_{\zeta_{\lambda, \mu}}$. Now, assume $v \in \widehat{\mathcal{R}}(\lambda, \mu)$ and $v' \in \widehat{\mathcal{M}}(\lambda, \mu)$. Then, $v' = X \cdot \hat{\mathbf{1}}_{\lambda, \mu}$ for some $X \in \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$. Moreover, $\tau(X) \cdot v \in \widehat{\mathcal{R}}(\lambda, \mu)$. Applying Lemma 3.7.1 and the definition of $\widehat{\mathcal{R}}(\lambda, \mu)$ we deduce that

$$(v', v) = (X \cdot \hat{\mathbf{1}}_{\lambda, \mu}, v) = (\hat{\mathbf{1}}_{\lambda, \mu}, \tau(X) \cdot v) = 0.$$

Hence, $v \in \mathcal{R}$. □

Corollary 7.6.10. *Given $\lambda \in P^{++}$ and $\mu \in \lambda - P_{\geq 0}(d)$,*

$$\mathcal{R} = \mathcal{R}(\lambda, \mu)^{\oplus k} \oplus \mathcal{M}(\lambda, \mu)^{\oplus 2^{n - \lfloor \frac{\gamma_0(\mu) + 1}{2} \rfloor - k}}$$

for some $0 \leq k \leq 2^{n - \lfloor \frac{\gamma_0(\mu) + 1}{2} \rfloor}$.

Theorem 7.6.11. *Assume $\lambda \in P^{++}$, and $\mu \in \lambda - P_{\geq 0}(d)$. If $F_\lambda L(\mu)$ is nonzero, then*

$$F_\lambda L(\mu) \cong \mathcal{L}(\lambda, \mu)^{\oplus \ell}$$

for some $0 < \ell \leq \varpi(\mu)$.

Proof. Let $\widehat{L}(\mu) = L(\mu)^{\oplus 2^{\lfloor \frac{n - \gamma_0(\mu) + 1}{2} \rfloor}}$, so that $\widehat{L}(\mu) = \widehat{\mathcal{M}}(\mu) / \widehat{R}(\mu)$ where $\widehat{R}(\mu)$ is the radical of the Shapovalov form on $\widehat{\mathcal{M}}(\mu)$. Applying the functor, we see that

$$F_\lambda \widehat{L}(\mu) = \widehat{\mathcal{M}}(\lambda, \mu) / F_\lambda \widehat{R}(\mu).$$

Now, $F_\lambda \widehat{R}(\mu) = \mathcal{R}$. Hence, Corollary 7.6.10 and a calculation similar to Corollary 7.6.6 gives the result. □

Proposition 7.6.12. [24, Proposition 18.18.1] *Any finite dimensional irreducible $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -module is a composition factor of $\mathcal{M}(\lambda, \lambda - \varepsilon)$ for some $\lambda \in P^{++}$.*

Theorem 7.6.13. *Any finite dimensional simple module for $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ is isomorphic $\mathcal{L}(\lambda, \mu)$ for some $\mu \in (\lambda - \varepsilon) - Q^+$.*

Proof. The functor F_λ transforms the composition series for $M(\lambda - \varepsilon)$ into the composition series for $\mathcal{M}(\lambda, \lambda - \varepsilon)$. It is now just left to observe that if $L(\mu)$ is a composition factor for $M(\lambda - \varepsilon)$, then $\mu \in (\lambda - \varepsilon) - Q^+$. □

7.7. Calibrated Representations Revisited.

Theorem 7.7.1. *If $\lambda, \mu \in P_{\text{poly}}^+$ satisfy $\lambda - \mu \in P_{\geq 0}(d)$, then $F_\lambda(L(\mu)) \neq 0$ and hence one has a simple module $\mathcal{L}(\lambda, \mu)$.*

Proof. The formal character of $L(\mu)$ when $\mu \in P_{\text{poly}}^+$ is given by the Q -Schur function Q_μ (c.f. [41]). There is a nondegenerate bilinear form, $(\cdot, \cdot)_{P_{\text{poly}}^+}$ on the subring of symmetric functions spanned by Schur's Q -functions given by

$$(Q_\lambda, Q_\mu)_{P_{\text{poly}}^+} = \text{Hom}_{\mathfrak{q}(n)}(L(\lambda), L(\mu)).$$

Furthermore, the basis Q_μ ($\mu \in P_{\text{poly}}^+$) is an orthogonal basis. Within this subring are the skew Q -Schur functions $Q_{\lambda/\mu}$. We refer the reader to [43, 28] for details.

Under the hypotheses of the theorem, λ/μ is a skew shape. Moreover, $F_\lambda L(\mu) = 0$ implies that

$$0 = \text{Hom}_{\mathfrak{q}(n)}(L(\lambda), L(\mu) \otimes V^{\otimes d}) = \bigoplus_{\nu \in P_{\text{poly}}^+(d)} \text{Hom}(L(\lambda), L(\mu) \otimes L(\nu))^{\oplus N_\nu}. \quad (7.7.1)$$

The second equality follows from Sergeev duality which implies that as a $\mathfrak{q}(n)$ -module

$$V^{\otimes d} = \bigoplus_{\nu \in P_{\text{poly}}^+(d)} L(\nu)^{\oplus N_\nu},$$

where N_ν is the dimension of the Specht module of $\mathcal{S}(d)$ corresponding to ν [42].

In terms of the bilinear form on symmetric functions, (7.7.1) implies

$$0 = (Q_\lambda, Q_\mu Q_\nu) \quad (7.7.2)$$

for all $\nu \in P_{\text{poly}}^+(d)$. In fact (7.7.2) holds for all $\nu \in P_{\text{poly}}^+$ since different graded summands of the symmetric function ring are orthogonal. However,

$$(Q_\lambda, Q_\mu Q_\nu) = (Q_\mu^\perp Q_\lambda, Q_\nu) = 2^{\ell(\mu)} (Q_{\lambda/\mu}, Q_\nu),$$

where $Q_\mu^\perp$ denotes the adjoint of Q_μ with respect to the form and the second equality follows from $Q_\mu^\perp Q_\lambda = 2^{-\ell(\mu)} Q_{\lambda/\mu}$ (cf. [28, II.8]). Thus, (7.7.1) implies that

$$(Q_{\lambda/\mu}, Q_\nu) = 0$$

for all $\nu \in P_{\text{poly}}^+$. But the Q -functions form an orthogonal basis for this subring. This implies $Q_{\lambda/\mu} = 0$, which is not true. Hence, $F_\lambda L(\mu) \neq 0$. $\square$

Arguing as in section 7 of [45] using Sergeev duality [41, 42] we obtain the following result.

Corollary 7.7.2. *Let $\lambda, \mu \in P_{\text{poly}}^+$ such that $\lambda - \mu \in P_{\geq 0}(d)$. Then the group character of $\mathcal{L}(\lambda, \mu) \downarrow_{\mathcal{S}(d)}$ is a power of 2 multiple of the skew Q -Schur function $Q_{\lambda/\mu}$.*

8. A CLASSIFICATION OF SIMPLE MODULES

In [6, 24], it was shown that the Grothendieck group of finite dimensional integral representations of $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ is a module for the Kostant-Tits $\mathbb{Z}$ -form of the Kac-Moody Lie algebra $\mathfrak{b}_\infty$. Indeed, let $\mathfrak{n}_\infty$ be a maximal nilpotent subalgebra of $\mathfrak{b}_\infty$, and let $\mathcal{U}_{\mathbb{Z}}^*(\mathfrak{n}_\infty)$ be the *minimal* admissible lattice inside the universal envelope of $\mathfrak{n}_\infty$. This lattice is spanned by Lusztig's dual canonical basis,

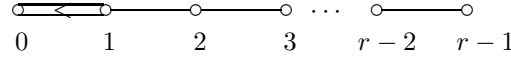
Theorem 8.0.3. [24, Theorem 20.5.2] *There is an isomorphism of graded Hopf algebras*

$$\mathcal{U}_{\mathbb{Z}}^*(\mathfrak{n}_\infty) \cong \bigoplus_{d \geq 0} K(\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)).$$

and,

Theorem 8.0.4. [24, Theorem 21.0.4] *The set $B(\infty)$ of isomorphism classes of simple $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$ -modules, for all d , can be given the structure of a crystal (in the sense of Kashiwara). Moreover, this crystal is isomorphic to Kashiwara's crystal associated to the crystal base of $\mathcal{U}_{\mathbb{Q}}(\mathfrak{n}_\infty)$.*

8.1. Quantum Groups and Shuffle Algebras. Let $\mathfrak{b}_r$ be the simple finite dimensional Lie algebra of type B_r over $\mathbb{C}$, and $\mathcal{U}_q(\mathfrak{b}_r)$ the associated quantum group with Chevalley generators e_i, f_i ($i = 0, \dots, r-1$) corresponding to the labeling of the Dynkin diagram:



Fix a triangular decomposition $\mathfrak{b}_r = \mathfrak{n}_r^+ \oplus \mathfrak{h}_r \oplus \mathfrak{n}_r^-$. Let Δ be the root system of $\mathfrak{b}_r$ relative to this decomposition, Δ^+ the positive roots, and $\Pi = \{\beta_0, \dots, \beta_{r-1}\}$ the simple roots. Let $\mathcal{Q}$ be the root lattice and $\mathcal{Q}^+ = \sum_{i=0}^{r-1} \mathbb{Z}_{\geq 0} \beta_i$. Finally, let $(\cdot, \cdot)$ denote the trace form on $\mathfrak{h}^*$. The Cartan matrix of $\mathfrak{b}_r$ is then $A = (a_{ij})_{i,j=0}^{r-1}$, where

$$a_{ij} = \frac{2(\beta_i, \beta_j)}{(\beta_i, \beta_i)}, \quad d_i = \frac{(\beta_i, \beta_i)}{2} \in \{1, 2\}.$$

Let $q_i = q^{d_i}$. To avoid confusion with notation we will use later, we adopt the following non-standard notation for q -integers and q -binomial coefficients:

$$(k)_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}.$$

The algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{n}_r^+)$ is naturally $\mathcal{Q}^+$ -graded by assigning to e_i the degree β_i . Let $|u|$ be the $\mathcal{Q}^+$ -degree of a homogeneous element $u \in \mathcal{U}_q(\mathfrak{n}_b^+)$.

There exist q -derivations e'_i , $i = 0, \dots, r-1$ given by

$$e'_i(e_j) = \delta_{ij} \quad \text{and} \quad e'_i(uv) = e'_i(u)v + q^{(\beta_i, |u|)} u e'_i(v)$$

for all homogeneous $u, v \in \mathcal{U}_q^+$.

Now, let $\mathcal{F}$ be the free associative algebra over $\mathbb{Q}(q)$ generated by the set of letters $\{[0], \dots, [r-1]\}$. Write $[i_1, \dots, i_k] := [i_1] \cdot [i_2] \cdots [i_k]$, and let $[]$ denote the empty word. The algebra $\mathcal{F}$ is $\mathcal{Q}^+$ graded by assigning the degree β_i to $[i]$ (as before, let $|f|$ denote the $\mathcal{Q}^+$ -degree of a homogeneous $f \in \mathcal{F}$). Notice that $\mathcal{F}$ also has a *principal grading* obtained by setting the degree of a letter $[i]$ to be 1; let $\mathcal{F}_d$ be the d th graded component in this grading.

Now, define the (quantum) shuffle product, $*$, on $\mathcal{F}$ inductively by

$$(x \cdot [i]) * (y \cdot [j]) = (x * (y \cdot [j])) \cdot [i] + q^{-(|x| + \beta_i, \beta_j)} ((x \cdot [i]) * y) \cdot [j], \quad x * [] = [] * x = x. \quad (8.1.1)$$

Iterating this formula yields

$$[i_1, \dots, i_\ell] * [i_{\ell+1}, \dots, i_{\ell+k}] = \sum_{w \in D(\ell, k)} q^{-e(w)} [i_{w(1)}, \dots, i_{w(\ell+k)}]$$

where

$$e(w) = \sum_{\substack{s \leq \ell < t \\ w(s) < w(t)}} (\beta_{i_{w(s)}}, \beta_{i_{w(t)}}),$$

see [27, §2.5] for details. The product $*$ is associative and, [27, Proposition 1],

$$x * y = q^{-(|x|, |y|)} y \bar{*} x \quad (8.1.2)$$

where $\bar{*}$ is obtained by replacing q with q^{-1} in the definition of $*$.

Now, to $f = [i_1, \dots, i_k] \in \mathcal{F}$, associate $\partial_f = e'_{i_1} \cdots e'_{i_k} \in \text{End } \mathcal{U}_q$, and $\partial_{[]} = \text{Id}_{\mathcal{U}_q}$. Then,

Proposition 8.1.1. [37, 38, 16] *There exists an injective $\mathbb{Q}(q)$ -linear homomorphism*

$$\Psi : \mathcal{U}_q \rightarrow (\mathcal{F}, *)$$

defined on homogeneous $u \in \mathcal{U}_q$ by the formula $\Psi(u) = \sum \partial_f(u)f$, where the sum is over all monomials $f \in \mathcal{F}$ such that $|f| = |u|$.

Therefore $\mathcal{U}_q^+$ is isomorphic to the subalgebra $\mathcal{W} \subseteq (\mathcal{F}, *)$ generated by the letters $[i]$, $0 \leq i < r$.

Let $\mathcal{A} = \mathbb{Q}[q, q^{-1}]$, and let $\mathcal{U}_{\mathcal{A}}$ denote the $\mathcal{A}$ -subalgebra of $\mathcal{U}_q$ generated by the divided powers $e_i^k/(k)_i!$ ($0 \leq i < r$, $k \in \mathbb{Z}_{\geq 0}$). Let $(\cdot, \cdot)_K : \mathcal{U}_q \times \mathcal{U}_q \rightarrow \mathbb{Q}(q)$ denote the unique symmetric bilinear form satisfying

$$(1, 1)_K = 1 \quad \text{and} \quad (e'_i(u), v)_K = (u, e_i v)_K$$

for all $0 \leq i < r$, and $u, v \in \mathcal{U}_q$. Let

$$\mathcal{U}_{\mathcal{A}}^* = \{ u \in \mathcal{U}_q \mid (u, v)_K \in \mathcal{A} \text{ for all } v \in \mathcal{U}_{\mathcal{A}} \} \quad (8.1.3)$$

and let $u^* \in \mathcal{U}_{\mathcal{A}}^*$ denote the dual to $u \in \mathcal{U}_{\mathcal{A}}$ relative to $(\cdot, \cdot)_K$.

Now, given a monomial

$$[i_1^{a_1}, i_2^{a_2}, \dots, i_k^{a_k}] = [\underbrace{i_1, \dots, i_1}_{a_1}, \underbrace{i_2, \dots, i_2}_{a_2}, \dots, \underbrace{i_k, \dots, i_k}_{a_k}]$$

with $i_j \neq i_{j+1}$ for $1 \leq j < k$, let $c_{i_1, \dots, i_k}^{a_1, \dots, a_k} = (a_1)_{i_1}! \cdots (a_k)_{i_k}!$, so that $(c_{i_1, \dots, i_k}^{a_1, \dots, a_k})^{-1} e_{i_1}^{a_1} \cdots e_{i_k}^{a_k}$ is a product of divided powers. Let

$$\mathcal{F}_{\mathcal{A}} = \bigoplus \mathcal{A} c_{i_1, \dots, i_k}^{a_1, \dots, a_k} [i_1^{a_1}, i_2^{a_2}, \dots, i_k^{a_k}]$$

and $\mathcal{W}_{\mathcal{A}}^* = \mathcal{W} \cap \mathcal{F}_{\mathcal{A}}$. It is known that $\mathcal{W}_{\mathcal{A}}^* = \Psi(\mathcal{U}_{\mathcal{A}}^*)$, [27, Lemma 8].

Define

$$\mathcal{F}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathcal{A}} \mathcal{F}_{\mathcal{A}}, \quad \text{and} \quad \mathcal{W}_{\mathbb{C}}^* = \mathbb{C} \otimes_{\mathcal{A}} \mathcal{W}_{\mathcal{A}}^*$$

where $\mathbb{C}$ is an $\mathcal{A}$ -module via $q \rightarrow 1$. Given an element $E \in \mathcal{W}_{\mathcal{A}}$ (resp. $\mathcal{F}_{\mathcal{A}}$) let $\underline{E}$ denote its image in $\mathcal{W}_{\mathbb{C}}$ (resp. $\mathcal{F}_{\mathbb{C}}$).

Observe that $(\mathcal{F}_{\mathbb{C}}, *)$ is the classical shuffle algebra and the shuffle product coincides with the formula for the characters associated to parabolic induction of $\mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d)$ -modules (see Lemma 3.5.3).

We close this section by describing the bar involution on $\mathcal{F}$:

Definition 8.1.2. [27, Proposition 6] *Let $- : \mathcal{F} \rightarrow \mathcal{F}$ be the $\mathbb{Q}$ -linear automorphism of $(\mathcal{F}, *)$ defined by $\bar{q} = q^{-1}$ and*

$$\overline{[i_1, \dots, i_k]} = q^{-\sum_{1 \leq s < t \leq k} (\beta_{i_s}, \beta_{i_t})} [i_k, \dots, i_1].$$

8.2. Good Words and Lyndon Words. In what follows, it is convenient to differ from the conventions in [27]. In particular, it is natural from our point of view to order monomial in $\mathcal{F}$ lexicographically reading from *right to left*. Unlike the type A case, this convention leads to some significant differences in the good Lyndon words that appear. This section contains a careful explanation of all the changes that occur.

Fix the ordering on the set of letters in $\mathcal{F}$ (resp. Π): $[0] < [1] < \dots < [r-1] < []$ (resp. $\beta_0 < \beta_1 < \dots < \beta_{r-1}$). Give the set of monomials in $\mathcal{F}$ the associated lexicographic order read from right to left. That is,

$$[i_1, \dots, i_k] < [j_1, \dots, j_\ell] \text{ if } i_k < j_\ell, \text{ or for some } m, i_{k-m} < j_{\ell-m} \text{ and } i_{k-s} < j_{\ell-s} \text{ for all } s < m.$$

Note that since the empty word is larger than any letter, every word is smaller than all of its right factors:

$$[i_1, \dots, i_k] < [i_j, \dots, i_k], \text{ for all } 1 < j \leq k. \quad (8.2.1)$$

(For those familiar with the theory, this definition is needed to ensure that the induced Lyndon ordering on positive roots is convex, cf. §8.3 below.)

For a homogeneous element $f \in \mathcal{F}$, let $\min(f)$ be the smallest monomial occurring in the expansion of f . A monomial $[i_1, \dots, i_k]$ is called a *good word* if there exists a homogeneous $w \in \mathcal{W}$ such that $[i_1, \dots, i_k] = \min(w)$, and is called a *Lyndon word* if it is larger than any of its proper left factors:

$$[i_1, \dots, i_j] < [i_1, \dots, i_k], \text{ for any } 1 \leq j < k.$$

Let $\mathcal{G}$ denote the set of good words, $\mathcal{L}$ the set of Lyndon words, and $\mathcal{GL} = \mathcal{L} \cap \mathcal{G} \subset \mathcal{G}$ the set of good Lyndon words.

Lemma 8.2.1. [27, Lemma 13] *Every factor of a good word is good.*

Because of our ordering conventions, [27, Lemma 15, Proposition 16] become

Lemma 8.2.2. [27, Lemma 15] *Let $l \in \mathcal{L}$, w a monomial such that $w \geq l$. Then, $\min(w * l) = wl$.*

and

Proposition 8.2.3. [27, Proposition 16] *Let $l \in \mathcal{GL}$, and $g \in \mathcal{G}$ with $g \geq l$. Then $gl \in \mathcal{G}$.*

Hence, we deduce from Lemma 8.2.1 and Proposition 8.2.3 [27, Proposition 17]:

Proposition 8.2.4. [26, 27] *A monomial g is a good word if, and only if, there exist good Lyndon words $l_1 \geq \dots \geq l_k$ such that*

$$g = l_1 l_2 \dots l_k.$$

As in [27], we have

Proposition 8.2.5. [26, 27] *The map $l \rightarrow |l|$ is a bijection $\mathcal{GL} \rightarrow \Delta^+$.*

Given $\gamma \in \Delta^+$, let $\gamma \rightarrow l(\gamma)$ be the inverse of the above bijection (called the Lyndon covering of Δ^+).

We now define the *bracketing* of Lyndon words, that gives rise to the *Lyndon basis* of $\mathcal{W}$. To this end, given $l \in \mathcal{L}$ such that l is not a letter, define the standard factorization of l to be $l = l_1 l_2$ where $l_2 \in \mathcal{L}$ is a proper left factor of maximal length. Define the q -bracket

$$[f_1, f_2]_q = f_1 f_2 - q^{(|f_1|, |f_2|)} f_2 f_1 \quad (8.2.2)$$

for homogeneous $f_1, f_2 \in \mathcal{F}$ in the $\mathcal{Q}^+$ -grading. Then, the bracketing $\langle l \rangle$ of $l \in \mathcal{L}$ is defined inductively by $\langle l \rangle = l$ if l is a letter, and

$$\langle l \rangle = [\langle l_1 \rangle, \langle l_2 \rangle]_q \quad (8.2.3)$$

if $l = l_1 l_2$ is the standard factorization of l .

Example 8.2.6. (1) $\langle [0] \rangle = [0]$;

(2) $\langle [12] \rangle = [[1], [2]]_q = [12] - q^{-1}[21]$;

(3) $\langle [012] \rangle = [[0], [12] - q^{-1}[21]]_q = [012] - q^{-1}[021] - q^{-2}[120] + q^{-3}[210]$.

As is suggested in this example, we have

Proposition 8.2.7. [27, Proposition 19] *For $l \in \mathcal{L}$, $\langle l \rangle = l + r$ where r is a linear combination of words w such that $|w| = |l|$ and $w < l$.*

Any word $w \in \mathcal{F}$ has a canonical factorization $w = l_1 \cdots l_k$ such that $l_1, \dots, l_k \in \mathcal{L}$ and $l_1 \geq \cdots \geq l_k$. We define the bracketing of an arbitrary word w in terms of this factorization: $\langle w \rangle = \langle l_1 \rangle \cdots \langle l_k \rangle$. Define a homomorphism $\Xi : (\mathcal{F}, \cdot) \rightarrow (\mathcal{F}, *)$ by $\Xi([i]) = [i]$. Then, $\Xi([i_1, \dots, i_k]) = [i_1] * \cdots * [i_k] = \Psi(e_{i_1} \cdots e_{i_k})$. In particular, $\Xi(\mathcal{F}) = \mathcal{W}$. We have the following characterization of good words:

Lemma 8.2.8. [27, Lemma 21] *The word w is good if and only if it cannot be expressed modulo $\ker \Xi$ as a linear combination of words $v < w$.*

For $g \in \mathcal{G}$, set $r_g = \Xi(\langle g \rangle)$. Then, we have

Theorem 8.2.9. [27, Proposition 22, Theorem 23] *Let $g \in \mathcal{G}$ and $g = l_1 \cdots l_k$ be the canonical factorization of g as a nonincreasing product of good Lyndon words. Then*

- (1) $r_g = r_{l_1} * \cdots * r_{l_k}$,
- (2) $r_g = \Psi(e_g) + \sum_{w < g} x_{gw} \Psi(e_w)$ where, for a word $v = [i_1, \dots, i_k]$, $e_v = e_{i_1} \cdots e_{i_k}$, and
- (3) $\{r_g | g \in \mathcal{G}\}$ is a basis for $\mathcal{W}$.

The basis $\{r_g | g \in \mathcal{G}\}$ is called the Lyndon basis of $\mathcal{W}$. An immediate consequence of Proposition 8.2.7 and Theorem 8.2.9 is the following:

Proposition 8.2.10. [27, Proposition 24] *Assume $\gamma_1, \gamma_2 \in \Delta^+$, $\gamma_1 + \gamma_2 = \gamma \in \Delta^+$, and $l(\gamma_1) < l(\gamma_2)$. Then, $l(\gamma_1)l(\gamma_2) \geq l(\gamma)$.*

This gives an inductive algorithm to determine $l(\gamma)$ for $\gamma \in \Delta^+$ (cf. [27, §4.3]):

For $\beta_i \in \Pi \subset \Delta^+$, $l(\beta_i) = [i]$. If γ is not a simple root, then there exists a factorization $l(\gamma) = l_1 l_2$ with l_1, l_2 Lyndon words. By Lemma 8.2.1, l_1 and l_2 are good, so $l_1 = l(\gamma_1)$ and $l_2 = l(\gamma_2)$ for some $\gamma_1, \gamma_2 \in \Delta^+$ with $\gamma_1 + \gamma_2 = \gamma$. Assume that we know $l(\gamma_0)$ for all $\gamma_0 \in \Delta^+$ satisfying $\text{ht}(\gamma_0) < \text{ht}(\gamma)$. Define

$$C(\gamma) = \{ (\gamma_1, \gamma_2) \in \Delta^+ \times \Delta^+ \mid \gamma = \gamma_1 + \gamma_2, \text{ and } l(\gamma_1) < l(\gamma_2) \}.$$

Then, Proposition 8.2.10 implies

Proposition 8.2.11. [27, Proposition 25] *We have*

$$l(\gamma) = \min \{ l(\gamma_1)l(\gamma_2) \mid (\gamma_1, \gamma_2) \in C(\gamma) \}$$

In our situation,

$$\Delta^+ = \{\beta_i + \beta_{i+1} + \cdots + \beta_j | 0 \leq i \leq j < r\} \cup \{2\beta_0 + \cdots + 2\beta_j + \beta_{j+1} + \cdots + \beta_k | 0 \leq j < k < r\}.$$

A straightforward inductive argument shows that

$$l(\beta_i + \beta_{i+1} + \cdots + \beta_j) = [i, i+1, \dots, j] \quad \text{and} \quad l(2\beta_0 + \cdots + 2\beta_j + \beta_{j+1} + \cdots + \beta_k) = [j, j-1, \dots, 0, 0, \dots, k-1, k].$$

Remarkably,

Proposition 8.2.12. *In the notation of Lemma 3.5.3 we have*

$$l(\beta_i + \cdots + \beta_j) = \text{ch } \Phi_{[i,j]}$$

and

$$2l(2\beta_0 + \cdots + 2\beta_j + \beta_{j+1} + \cdots + \beta_k) = \text{ch } \Phi_{[-j-1,k]}.$$

Observe that we may write any good Lyndon word uniquely in the form $l = [i, i+1, \dots, j]$ where $i, j \in \mathbb{Z}$ and $0 \leq |i| \leq j < r$. For example,

$$l(2\beta_0 + \cdots + 2\beta_j + \beta_{j+1} + \cdots + \beta_k) = [-j-1, \dots, k]. \quad (8.2.4)$$

In the following definition, we mean for n to vary. Given $\lambda \in P_{>0}^{++}$, let

$$\mathcal{B}_d(\lambda) = \{ \mu \in P^+[\lambda] \mid \lambda - \mu \in P_{\geq 0}(d) \text{ and } |\mu_i| < \lambda_i \text{ for all } i \} \quad (8.2.5)$$

and let

$$\mathcal{B}_d = \{ (\lambda, \mu) \mid \lambda \in P_{>0}^{++} \text{ and } \mu \in \mathcal{B}_d(\lambda) \}. \quad (8.2.6)$$

Let $\mathcal{G}_d = \mathcal{G} \cap \mathcal{F}_d$ be the set of good words of principal degree d . We have

Lemma 8.2.13. *The map $(\lambda, \mu) \mapsto [\lambda - \mu] = [\mu_1, \dots, \lambda_1 - 1, \dots, \mu_n, \dots, \lambda_n - 1]$ induces a bijection $\mathcal{B}_d \rightarrow \mathcal{G}_d$.*

Proof: By (8.2.4), $[\lambda - \mu]$ is a well-defined element of $\mathcal{F}_d$. Since $\lambda \in P_{>0}^{++}$ and $\mu \in P^+[\lambda]$, the ordering convention and (8.2.1) imply that $[\lambda - \mu] \in \mathcal{G}_d$. This map is clearly bijective. ■

8.3. PBW and Canonical Bases. The lexicographic ordering on $\mathcal{GL}$ induces a total ordering on Δ^+ , which is *convex*, meaning that if $\gamma_1, \gamma_2 \in \Delta^+$ with $\gamma_1 < \gamma_2$, and $\gamma = \gamma_1 + \gamma_2 \in \Delta^+$, then $\gamma_1 < \gamma < \gamma_2$ (cf. [39, 27]).

Indeed, assume $\gamma_1, \gamma_2, \gamma = \gamma_1 + \gamma_2 \in \Delta^+$ and $\gamma_1 < \gamma_2$. Proposition 8.2.10 and (8.2.1) imply that $l(\gamma) \leq l(\gamma_1)l(\gamma_2) < l(\gamma_2)$. If $l(\gamma) = l(\gamma_1)l(\gamma_2)$, then the definition of Lyndon words implies $l(\gamma_1) < l(\gamma)$. We are therefore left to prove that $l(\gamma_1) < l(\gamma)$ even if $l(\gamma) < l(\gamma_1)l(\gamma_2)$. This cannot happen if $\gamma = \beta_i + \cdots + \beta_j$. In the case $\gamma = 2\beta_0 + \cdots + 2\beta_j + \beta_{j+1} + \cdots + \beta_k$, the possibilities for $\gamma_1 < \gamma_2$ are $\gamma_1 = \beta_i + \cdots + \beta_j$ and $\gamma_2 = 2\beta_0 + \cdots + 2\beta_{i-1} + \beta_i + \cdots + \beta_k$ for $0 \leq i \leq j$. In any of these cases, $[i, \dots, j] < [j, \dots, 0, 0, \dots, k]$. That is, $l(\gamma_1) < l(\gamma) < l(\gamma_2)$.

Each convex ordering, $\gamma_1 < \cdots < \gamma_N$, on Δ^+ arises from a unique decomposition $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ of the longest element of the Weyl group of type B_r via

$$\gamma_1 = \beta_{i_1}, \gamma_2 = s_{i_1} \beta_{i_2}, \dots, \gamma_N = s_{i_1} \cdots s_{i_{N-1}} \beta_{i_N}.$$

Lusztig associates to this data a PBW basis of $\mathcal{U}_{\mathcal{A}}$ denoted

$$E^{(a_1)}(\gamma_1) \cdots E^{(a_n)}(\gamma_N), \quad (a_1, \dots, a_N) \in \mathbb{Z}_{\geq 0}^N.$$

Leclerc [27, §4.5] describes the image in $\mathcal{W}$ of this basis for the convex Lyndon ordering. We use the same braid group action as Leclerc and the results of [27, §4.5, 4.6] carry over, making changes in the same manner indicated in the previous section. We describe the relevant facts below.

For $g = l(\gamma_1)^{a_1} \cdots l(\gamma_k)^{a_k}$, where $\gamma_1 > \cdots > \gamma_k$ and $a_1, \dots, a_k \in \mathbb{Z}_{>0}$ set

$$E_g = \Psi(E^{(a_k)}(\gamma_k) \cdots E^{(a_1)}(\gamma_1)) \in \mathcal{W}_{\mathcal{A}}$$

and let $E_g^* \in \mathcal{W}_{\mathcal{A}}^*$ be the image of $(E^{(a_k)}(\gamma_k) \cdots E^{(a_1)}(\gamma_1))^* \in \mathcal{U}_{\mathcal{A}}^*$. Observe that the order of the factors in the definition of E_g above are increasing with respect to the Lyndon ordering. Leclerc shows that if $\gamma \in \Delta^+$, then

$$\kappa_{l(\gamma)} E_{l(\gamma)} = r_{l(\gamma)}, \quad (8.3.1)$$

For some $\kappa_{l(\gamma)} \in \mathbb{Q}(q)$, [27, Theorem 28] (the proof of this theorem in our case is obtained by reversing all the inequalities and using the standard factorization as opposed to the costandard factorization).

More generally, let $f \mapsto f^t$ be the linear map defined by $[i_1, \dots, i_k]^t = [i_k, \dots, i_1]$ and $(x * y)^t = y^t * x^t$. Then, E_g is proportional to $\overline{r_g^t}$ (cf. [27, §4.6, §5.5.2 – 5.5.3]).

As in [27, §5.5.3], we see that there exists an explicit $c_g \in \mathbb{Z}$ such that

$$E_g^* = q^{c_g} (E_{l_m}^*) * \cdots * (E_{l_1}^*)$$

if $g = l_1 \cdots l_m$ with $l_1 > \cdots > l_m$. Using (8.1.2) we deduce that

$$E_g^* = q^{C_g} (E_{l_1}^*) \bar{*} \cdots \bar{*} (E_{l_m}^*),$$

where $C_g = c_g - \sum_{1 \leq i < j \leq m} (\beta_i, \beta_j)$. In particular,

$$\underline{E_g^*} = \underline{(E_{l_1}^*) * \cdots * (E_{l_m}^*)}. \quad (8.3.2)$$

Using the bar involution (Definition 8.1.2), Leclerc constructs the canonical basis, $\{b_g \mid g \in \mathcal{G}\}$ for $\mathcal{W}_{\mathcal{A}}$ via the PBW basis $\{E_g \mid g \in \mathcal{G}\}$. It has the form

$$b_g = E_g + \sum_{\substack{h \in \mathcal{G} \\ h < g}} \chi_{gh} E_h.$$

The dual canonical basis then has the form

$$b_g^* = E_g^* + \sum_{\substack{h \in \mathcal{G} \\ h > g}} \chi_{gh}^* E_h^*.$$

In particular, for good Lyndon words, [27, Corollary 41], $b_l^* = E_l^*$ for every $l \in \mathcal{GL}$. As in [27, Lemma 8.2], we see that $b_{[i, \dots, j]}^* = [i, \dots, j]$ for $0 \leq i < j < r$. We now prove

Lemma 8.3.1. *For $0 \leq j < k < r$, one has*

$$b_{[j, \dots, 0, 0, \dots, k]}^* = (2)_0 [j, \dots, 0, 0, \dots, k].$$

Proof: We prove this by induction on j and k with $j < k$, using (8.1.1), (8.2.2), and (8.2.3) for the computations.

Observe that for $k \geq 1$, $r_{[0,1,\dots,k]} = (q^2 - q^{-2})^k [0, 1, \dots, k]$, which can be proved easily by downward induction on j , $0 \leq j < k$, using (8.1.1) and

$$r_{[j,\dots,k]} = \Xi(\langle [j, \dots, k] \rangle) = \Xi([j], \langle [j+1, \dots, k] \rangle_q) = [j] * r_{[j+1,\dots,k]} - q^{-2} r_{[j+1,\dots,k]} * [j].$$

By (8.1.1), we have

$$\begin{aligned} [0] * [0, 1] - [0, 1] * [0] &= [0, 1, 0] + q^2([0] * [0])[1] - ([0] * [0])[1] - [0, 1, 0] \\ &= (q^2 - 1)([0, 0] + q^{-2}[0, 0])[1] = (q^2 - q^{-2})[0, 0, 1] \end{aligned}$$

Therefore, applying (8.2.3) and the relevant definitions, we deduce that

$$\begin{aligned} r_{[0,0,1]} &= \Xi(\langle [0, 0, 1] \rangle) \\ &= \Xi([0], \langle [0, 1] \rangle_q^2) \\ &= [0] * r_{[0,1]} - r_{[0,1]} * [0] \\ &= (q^2 - q^{-2})([0] * [0, 1] - [0, 1] * [0]) \\ &= (q^2 - q^{-2})^2 [0, 0, 1] \end{aligned}$$

Once again, using (8.1.1), we deduce that for all $k \geq 2$,

$$[0] * [0, \dots, k] - [0, \dots, k] * [0] = ([0] * [0, \dots, k-1] - [0, \dots, k-1] * [0])[k]. \quad (8.3.3)$$

Assume $k \geq 2$. Then, $(\beta_0, \beta_0 + \dots + \beta_k) = 0$, so iterated applications of (8.3.3) yields

$$\begin{aligned} r_{[0,0,\dots,k]} &= [0] * r_{[0,\dots,k]} - r_{[0,\dots,k]} * [0] \\ &= (q^2 - q^{-2})^k ([0] * [0, \dots, k] - [0, \dots, k] * [0]) \\ &= (q^2 - q^{-2})^k ([0] * [0, 1] - [0, 1] * [0])[2, \dots, k] \\ &= (q^2 - q^{-2})^{k+1} [0, 0, \dots, k] \end{aligned}$$

Now, assume that $k \geq 2$, and $0 < j < k$. To compute $r_{[j,\dots,0,0,\dots,k]}$, we need the following. For $|j - k| > 1$,

$$\begin{aligned} [j] * [j-1, \dots, k] - q^{-2}[j-1, \dots, k] * [j] \\ = ([j] * [j-1, \dots, k-1] - q^{-2}[j-1, \dots, k-1] * [j])[k]. \end{aligned} \quad (8.3.4)$$

For $j = k-1$,

$$\begin{aligned} [j] * [j-1, \dots, 0, 0, \dots, j+1] - q^{-2}[j-1, \dots, 0, 0, \dots, j+1] * [j] \\ = (q^2[j] * [j-1, \dots, 0, 0, \dots, j] - q^{-2}[j-1, \dots, 0, 0, \dots, j] * [j])[j+1]. \end{aligned} \quad (8.3.5)$$

Finally,

$$\begin{aligned} q^2[j] * [j-1, \dots, 0, 0, \dots, j] - q^{-2}[j-1, \dots, 0, 0, \dots, j] * [j] \\ = ([j] * [j-1, \dots, 0, 0, \dots, j-2] - q^{-2}[j-1, \dots, 0, 0, \dots, j-2] * [j])[j, j+1]. \end{aligned} \quad (8.3.6)$$

Indeed, (8.3.4) and (8.3.5) are straightforward applications of (8.1.1). Equation (8.3.6) involves a little more calculation:

$$\begin{aligned}
q^2[j] * [j-1, \dots, 0, 0, \dots, j] - q^{-2}[j-1, \dots, 0, 0, \dots, j] * [j] \\
&= q^2[j-1, \dots, 0, 0, \dots, j, j] + q^{-2}([j] * [j-1, \dots, 0, 0, \dots, j-1] \\
&\quad - [j-1, \dots, 0, 0, \dots, j-1] * [j])[j] - q^{-2}[j-1, \dots, 0, 0, \dots, j, j] \\
&= (q^2 - q^{-2})[j-1, \dots, 0, 0, \dots, j, j] + q^{-2}([j-1, \dots, 0, 0, \dots, j] \\
&\quad + q^2([j] * [j-1, \dots, 0, 0, \dots, j-2])[j-1] - ([j-1, \dots, 0, 0, \dots, j-2] * [j])[j-1] \\
&\quad - q^4[j-1, \dots, 0, 0, \dots, j])[j] \\
&= ([j] * [j-1, \dots, 0, 0, \dots, j-2] - q^{-2}[j-1, \dots, 0, 0, \dots, j-2] * [j])[j, j+1],
\end{aligned}$$

Note that (8.3.4) holds for both $[j-1, j, \dots, k]$ and $[j-1, \dots, 0, 0, \dots, k]$.

Now, assume that we have shown that $r_{[j-1, \dots, 0, 0, \dots, k]} = (q^2 - q^{-2})^{j+k}[j-1, \dots, 0, 0, \dots, k]$. Then, since $(\beta_j, 2\beta_0 + \dots + 2\beta_{j-1} + \beta_j + \dots + \beta_k) = -2$,

$$\begin{aligned}
r_{[j, \dots, 0, 0, \dots, k]} &= [j] * r_{[j-1, \dots, 0, 0, \dots, k]} - r_{[j-1, \dots, 0, 0, \dots, k]} * [j] \\
&= (q^2 - q^{-2})^{j+k}[j] * [j-1, \dots, 0, 0, \dots, k] - q^{-2}[j-1, \dots, 0, 0, \dots, k] * [j] \\
&= (q^2 - q^{-2})^{j+k}([j] * [j-1, \dots, 0, 0, \dots, j+1] \\
&\quad - q^{-2}[j-1, \dots, 0, 0, \dots, j+1] * [j])[j+2, \dots, k] \quad \text{by (8.3.4)} \\
&= (q^2 - q^{-2})^{j+k}(q^2[j] * [j-1, \dots, 0, 0, \dots, j] \\
&\quad - q^{-2}[j-1, \dots, 0, 0, \dots, j] * [j])[j+1, \dots, k] \quad \text{by (8.3.5)} \\
&= (q^2 - q^{-2})^{j+k}([j] * [j-1, \dots, 0, 0, \dots, j-2] \\
&\quad - q^{-2}[j-1, \dots, 0, 0, \dots, j-2] * [j])[j, \dots, k] \quad \text{by (8.3.6)} \\
&= (q^2 - q^{-2})^{j+k}([j] * [j-1] - q^{-2}[j-1] * [j])[j-2, \dots, 0, 0, \dots, k] \quad \text{by (8.3.4)} \\
&= (q^2 - q^{-2})^{j+k+1}[j, \dots, 0, 0, \dots, k].
\end{aligned}$$

Finally, the result follows after computing the normalizing coefficient (8.3.1) using [27, Equation (28)]. We leave the details to the reader. ■

8.4. In section we give a representation theoretic interpretation of the good Lyndon words associated to the root vectors $2\beta_0 + \dots + 2\beta_j + \beta_{j+1} + \dots + \beta_k$ ($0 \leq j < k < r$) which appear in [27, Lemma 53]. The corresponding dual canonical basis vectors are given by the formula

$$[0] \cdot ([1, \dots, j] * [0, \dots, k]).$$

Lemma 8.4.1. *Let $0 \leq a < b$, $d = b + a + 2$, $\lambda = (b + 1, a + 1)$ and $\alpha = (1, -1)$. Then, for $1 \leq k \leq a$,*

$$\text{ch } \mathcal{L}(\lambda, -k\alpha) = 2[k-1] \cdot \underline{([k-2, k-3, \dots, 1, 0, 0, 1, \dots, b] * [k, \dots, a])}$$

where if $k = 1$, we interpret

$$[k-2, k-3, \dots, 1, 0, 0, 1, \dots, b] = [0, 1, \dots, b]$$

Proof. By [15, Proposition 11.4], for each $k \in \mathbb{Z}_{\geq 0}$, there exists a short exact sequence

$$0 \longrightarrow L(-(k+1)\alpha) \longrightarrow M(-k\alpha) \longrightarrow L(-k\alpha) \longrightarrow 0.$$

For $k \leq a+1$, applying the functor F_λ yields the exact sequence

$$0 \longrightarrow F_\lambda L(-(k+1)\alpha) \longrightarrow 2\mathcal{M}(\lambda, -k\alpha) \longrightarrow F_\lambda L(-k\alpha) \longrightarrow 0. \quad (8.4.1)$$

Therefore,

$$\text{ch } F_\lambda L(-k\alpha) = 4[k-1, \dots, 1, 0, 0, 1, \dots, b] * [k, \dots, a] - \text{ch } F_\lambda L(-(k+1)\alpha).$$

Note that when $k = a+1$, $F_\lambda L(-(k+1)\alpha) = 0$ since $\mathcal{M}(\lambda, -(a+2)\alpha) = 0$. Therefore the sequence (8.4.1) implies $F_\lambda L(-k\alpha) = 2\mathcal{L}(\lambda, -(a+1)\alpha) \cong 2\mathcal{M}(\lambda, -(a+1)\alpha) \cong 2\Phi_{[-a-1, b]}$, and

$$\text{ch } \Phi_{[-a-1, b]} = 2[a, a-1, \dots, 1, 0, 0, 1, \dots, b].$$

We now prove the lemma by downward induction on $k \leq a$. We have

$$\begin{aligned} \text{ch } F_\lambda L(-a\alpha) &= 4[a-1, \dots, 1, 0, 0, 1, \dots, b] * [a] - 4[a, \dots, 1, 0, 0, 1, \dots, b] \\ &= 4[a-1] \cdot ([a-2, \dots, 1, 0, 0, 1, \dots, b] * [a]). \end{aligned}$$

Hence, $F_\lambda L(-a\alpha) = 2\mathcal{L}(\lambda, -a\alpha)$ and the lemma holds for $k = a$. Now, assume $k < a$, $F_\lambda L(-(k+1)\alpha) = 2\mathcal{L}(\lambda, -(k+1)\alpha)$, and

$$\text{ch } \mathcal{L}(\lambda, -(k+1)\alpha) = 2[k] \cdot ([k-1, \dots, 1, 0, 0, 1, \dots, b] * [k+1, \dots, a]).$$

Then,

$$\begin{aligned} \text{ch } F_\lambda L(-k\alpha) &= 4[k-1, \dots, 1, 0, 0, 1, \dots, b] * [k, \dots, a] - 4[k] \cdot ([k-1, \dots, 1, 0, 0, 1, \dots, b] * [k+1, \dots, a]) \\ &= 4[k-1] \cdot ([k-2, \dots, 1, 0, 0, 1, \dots, b] * [k, \dots, a]). \end{aligned}$$

Hence, $F_\lambda L(-k\alpha) \neq 0$, so $F_\lambda L(-k\alpha) = 2\mathcal{L}(\lambda, -k\alpha)$ and the lemma holds. $\square$

Corollary 8.4.2. *Let $0 \leq a < b$, $d = b + a + 2$, $\lambda = (b+1, a+1)$ and $\mu = -\alpha = (-1, 1)$. Then,*

$$\text{ch } \mathcal{L}(\lambda, -\alpha) = 2[0] \cdot [0, \dots, b] * [1, \dots, a].$$

8.5. A Basis for the Grothendieck Group $K(\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d))$.

Theorem 8.5.1. *The set*

$$\{[\mathcal{M}(\lambda, \mu)] \mid (\lambda, \mu) \in \mathcal{B}_d\}$$

forms a basis for $K(\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d))$.

Proof. By Lemma 8.3.1 and (8.3.2), it follows that $\text{ch } \mathcal{M}(\lambda, \mu) = E_{[\lambda-\mu]}^*$. The result now follows from Lemma 8.2.13 and the fact that the character map is injective. $\square$

We will now describe a basis for $K(\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d))$ in terms of the simple modules $\mathcal{L}(\lambda, \mu)$.

Proposition 8.5.2. *Let $b \geq 0$, $\lambda = (b+1, b+1)$ and $\alpha = (1, -1)$. Then,*

$$\Phi_{[-b-1, b]} \cong \mathcal{L}(\lambda, b\alpha).$$

Proof. There is a surjective homomorphism $\mathcal{M}(\lambda, b\alpha) \rightarrow \Phi_{[-b-1, b]}$. The result follows since $\Phi_{[-b-1, b]}$ is simple. $\square$

Corollary 8.5.3. *Assume that $\lambda \in \mathcal{P}_{>0}^{++}$, $\mu \in P^+[\lambda]$, $\lambda - \mu \in P_{\geq 0}(d)$, and $|\mu_i| \leq \lambda_i$ for all i . Then, there exists $(\eta, \nu) \in \mathcal{B}_d$ such that*

$$\mathcal{L}(\lambda, \mu) \cong \mathcal{L}(\eta, \nu),$$

and $[\lambda - \mu] \leq [\eta - \nu]$.

Proof. First, we may assume $\mu_i < \lambda_i$ for all i , since the terms for which $\lambda_i = \mu_i$ do not contribute to $\mathcal{L}(\lambda, \mu)$. Proceed by induction on $N(\lambda, \mu) = |\{i = 1, \dots, n \mid \mu_i = -\lambda_i\}|$. If $N(\lambda, \mu) = 0$, then $(\lambda, \mu) \in \mathcal{B}_d^+$ so there is nothing to do. If $N(\lambda, \mu) > 0$, let j be the smallest index such that $\mu_j = -\lambda_j$. Set $\lambda^{(1)} = (\lambda_1, \dots, \lambda_{j-1}, \lambda_j, \lambda_j, \lambda_{j+1}, \dots, \lambda_n)$ and $\mu^{(1)} = (\mu_1, \dots, \mu_{j-1}, \lambda_j - 1, \mu_j + 1, \mu_{j+1}, \dots, \lambda_n)$. Clearly, $\lambda^{(1)} \in \mathcal{P}_{>0}^{++}$ and $\mu^{(1)} \in \lambda^{(1)} - P_{\geq 0}(d)$. We now show $\mu^{(1)} \in P^+[\lambda]$. Indeed, $\lambda_j > 0$, so $\lambda_j - 1 > 1 - \lambda_j = \mu_j + 1$; and, $\mu_j \geq \mu_{j+1}$, so $\mu_j + 1 > \mu_{j+1}$. Since $\mu_j < \lambda_j - 1$, the j th twisted good Lyndon word in $[\lambda^{(1)} - \mu^{(1)}]$ is greater than the j th twisted good Lyndon word in $[\lambda - \mu]$. Hence, $[\lambda - \mu] \leq [\lambda^{(1)} - \mu^{(1)}]$.

Now, there exists a surjective homomorphism

$$\begin{aligned} \Phi_{[\mu_1, \lambda_1 - 1]} \otimes \cdots \otimes \mathcal{M}((\lambda_j, \lambda_j), (\lambda_j - 1, \mu_j + 1)) \otimes \cdots \otimes \Phi_{[\mu_n, \lambda_n - 1]} \\ \rightarrow \Phi_{[\mu_1, \lambda_1 - 1]} \otimes \cdots \otimes \Phi_{[\mu_j, \lambda_j - 1]} \otimes \cdots \otimes \Phi_{[\mu_n, \lambda_n - 1]} \end{aligned}$$

Hence, a surjective homomorphism $\mathcal{M}(\lambda^{(1)}, \mu^{(1)}) \rightarrow \mathcal{L}(\lambda, \mu)$. It follows that $\mathcal{L}(\lambda^{(1)}, \mu^{(1)}) \cong \mathcal{L}(\lambda, \mu)$.

Since $N(\lambda^{(1)}, \mu^{(1)}) < N(\lambda, \mu)$ the result follows. $\square$

Recall that given $\mu \in \lambda - P_{\geq 0}(d)$ there exists a unique $w \in S_d[\lambda]$ such that $w\mu \in P^+[\lambda]$. Let μ^+ denote this element. Also, given $\lambda \in P^{++}$, and $\mu \in \lambda - P_{\geq 0}(d)$, let $[\lambda - \mu]^+ = [\lambda - \mu^+] \in \mathcal{TG}$ be the associated twisted good word. The following lemma is straightforward.

Lemma 8.5.4. *Assume that $\lambda \in P^{++}$, $\lambda - \mu \in P_{\geq 0}(d)$ and $\gamma \in Q^+$. Then, $[\lambda - \mu] \leq [\lambda - (\mu - \gamma)^+]$.*

Theorem 8.5.5. *The following is a complete list of pairwise non-isomorphic simple modules for $\mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d)$:*

$$\{ \mathcal{L}(\lambda, \mu) \mid (\lambda, \mu) \in \mathcal{B}_d^+ \}.$$

Proof. Every composition factor of $M(\mu)$ is of the form $L(\mu - \gamma)$ for some $\gamma \in Q^+$. Applying the functor, we deduce that every composition factor of $\mathcal{M}(\lambda, \mu)$ is of the form $\mathcal{L}(\lambda, \mu - \gamma) \cong \mathcal{L}(\lambda, (\mu - \gamma)^+)$. Now, putting together Corollary 8.5.3 and Lemma 8.5.4, we deduce that in the Grothendieck group

$$[\mathcal{M}(\lambda, \mu)] = \sum_{\substack{\nu \in \mathcal{B}_d(\eta) \\ \eta \in \mathcal{P}_{>0}^{++} \\ [\lambda - \mu] \leq [\eta - \nu]}} c_{\lambda, \mu, \eta, \nu} [\mathcal{L}(\eta, \nu)],$$

where the $c_{\lambda, \mu, \eta, \nu}$ are integers and where $c_{\lambda, \mu, \lambda, \mu} \neq 0$. Therefore, the transition matrix between the basis for $K(\text{Rep } \mathcal{H}_{\mathcal{C}_\ell}^{\text{aff}}(d))$ given by standard modules and that given by simples is triangular. $\square$

9. TABLE OF NOTATION

For the convenience of the reader we provide a table of notation with a reference to where the notation is first defined.

Notation	First Defined
$\mathcal{S}(d), \mathcal{H}_{\mathcal{C}\ell}^{\text{aff}}(d), \mathcal{P}_d[x], \mathcal{A}(d)$	Section 3.1
$q(a)$	Section 3.3, (3.3.1)
$\mathcal{P}_d[x^2]$	Section 3.3
Ind_{μ}^d	Section 3.4
$D_{\nu}, D_{(m,k)}$	Section 3.4
$\gamma_0 = \gamma_0(a_1, \dots, a_d)$	Section 3.5, (3.5.1)
$[a_1, \dots, a_d]$	Section 3.5
$\mathcal{C}\ell_d$	Section 4.1, (4.1.1)
$\mathcal{L}_i, s_{ij}$	Section 4.1, (4.1.4)
$[a, b]$	Section 4.1
$\hat{\Phi}_{[a,b]}, \hat{\Phi}_{[a,b]}^+, \hat{\Phi}_{[a,b]}^-$	Section 4.1
$\Phi_{[a,b]}$	Section 4.1, Definition 4.1.9
$\hat{\mathbf{1}}_{[a,b]}, \varphi \hat{\mathbf{1}}_{[a,b]}$	Section 4.1
$\mathbf{1}_{a,b,n}$	Section 4.4
R, R^+, Q, Q^+	Section 4.2
$P, P_{\geq 0}, P^+, P^{++}, P_{\text{rat}}^+, P_{\text{poly}}^+$	Section 4.2
$P(d), P_{\geq 0}(d), P^+(d), P^{++}(d), P_{\text{rat}}^+(d), P_{\text{poly}}^+(d)$	Section 4.2
$S_n[\lambda], R[\lambda], P^+[\lambda], P^-[\lambda]$	Section 4.2
$\hat{\Phi}(\lambda, \mu), \Phi(\lambda, \mu)$	Section 4.3
$\widehat{\mathcal{M}}(\lambda, \mu), \mathcal{M}(\lambda, \mu)$	Section 4.3, (4.3.1), (4.3.2)
$\mathcal{M}_{a,b,n}$	Section 4.4
$S_n[\zeta]$	Section 4.3
$\mathcal{R}(\lambda, \mu)$	Section 4.4
$L(\lambda, \mu)$	Section 4.4, Theorem 4.4.10
λ/μ	Section 5
$\mathcal{Y}_{i,L}$	Section 5
$H^{\lambda/\mu}$	Section 5
$e_{i,j}, f_{i,j}, \bar{e}_{i,j}, \bar{f}_{i,j}$	Section 6.1
$\mathcal{O}, \mathcal{O}(\mathfrak{q}(n))$	Section 6.2
$\widehat{M}(\lambda), M(\lambda)$	Section 6.2
$(\cdot, \cdot)_S$	Section 6.3

Notation	First Defined
$C_i, S_{i,j}, F_i$	Section 7.3
$\Omega_{i,j}$	Section 7.4
F_λ	Section 7.5, (7.5.1)
$(\cdot, \cdot)_\mu$	Section 7.6
$\varpi(\mu)$	Section 7.6
$\Delta^+, \Pi, \mathcal{Q}, \mathcal{Q}^+$	Section 8.1
$(\mathcal{F}, *), \mathcal{W}$	Section 8.1
$\mathcal{F}_\mathcal{A}, \mathcal{F}_\mathbb{C}, \mathcal{W}_\mathcal{A}, \mathcal{W}_\mathbb{C}$	Section 8.1
$\underline{E} \in \mathcal{W}_\mathbb{C}$	Section 8.1
$\mathcal{GL}, \mathcal{G}$	Section 8.2
$\mathcal{B}_d[\lambda], \mathcal{B}_d$	Section 8.2
$[\cdot, \cdot]_q, \Xi, r_g$	Section 8.2
E_g, E_g^*, b_g, b_g^*	Section 8.3

REFERENCES

- [1] T. Arakawa and T. Suzuki, Duality between $\mathfrak{sl}_n(\mathbb{C})$ and the degenerate affine Hecke algebra of type A , *J. Algebra* **209** (1998), 288–304.
- [2] S. Ariki, On the decomposition numbers of the Hecke algebra of type $G(m, 1, n)$, *J. Math. Kyoto Univ.* **36** (1996), 789–808.
- [3] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$, *Adv. in Math.* **182** (2004), 28–77.
- [4] ———, Centers of degenerate cyclotomic Hecke algebras and parabolic category $\mathcal{O}$. *Represent. Theory* **12** (2008), 236–259.
- [5] J. Brundan and A. Kleshchev, Projective Representations of the Symmetric Group via Sergeev Duality, *Math. Z.* **239** (2002) no. 1, 27–68.
- [6] ———, Hecke-Clifford Superalgebras, Crystals of Type $A_{2l}^{(2)}$ and Modular Branching Rules for $\widehat{S}_n$, *Represent. Theory* **5** (2001), 317–403.
- [7] ———, Schur-Weyl duality for higher levels, *Selecta Math.* **14** (2008), 1–57.
- [8] ———, Representations of shifted Yangians and finite W -algebras, *Mem. Amer. Math. Soc.* **196** (2008), no. 918.
- [9] I. Cherednik, Special bases of irreducible representations of a degenerate affine Hecke algebra. *Functional Anal. Appl.* **20** (1986), no. 1, 76–78.
- [10] ———, Double affine Hecke algebras. London Mathematical Society Lecture Note Series, 319. Cambridge University Press, Cambridge, 2005.
- [11] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and sl_2 -categorification. *Ann. of Math.* (2) **167** (2008), no. 1, 245–298.
- [12] V. G. Drinfeld, *Proc. Intern. Cong. Math., Berkeley*, vol. 1, Academic Press, New York, 1987, pp. 798–820.
- [13] A. Frisk, Typical blocks of the category $\mathcal{O}$ for the queer Lie superalgebra. *J. Algebra Appl.* **6** (2007), no. 5, 731–778.
- [14] V. A. Ginzburg, Proof of the Deligne-Langlands conjecture, *Soviet. Math. Dokl.* **35** (2) (1987), 304–308.
- [15] M. Gorelik, Shapovalov determinants of Q -type Lie superalgebras. *Int. Math. Res. Pap.* 2006, Art. ID 96895, 71 pp.
- [16] J. A. Green, Quantum groups, Hall algebras and quantum shuffles. In: Finite reductive groups (Luminy 1994), 273–290, Birkhuser Prog. Math. 141, 1997.
- [17] I. Grojnowski, Affine sl_p controls the representation theory of the symmetric group and related Hecke algebras. math.RT/9907129.
- [18] A. De Sole and V. Kac, Finite vs affine W -algebras. (English summary) *Jpn. J. Math.* **1** (2006), no. 1, 137–261.
- [19] M. Kashiwara, On crystal bases of the q -analogue of universal enveloping algebras. *Duke Math. J.* **63** (1991), 465–516.
- [20] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I. arXiv:0803.4121
- [21] ———, A diagrammatic approach to categorification of quantum groups II. arXiv:0804.2080
- [22] ———, A diagrammatic approach to categorification of quantum groups III. arXiv:0807.3250
- [23] S. Khoroshkin and M. Nazarov, Yangians and Mickelsson algebras. I. *Transform. Groups* **11** (2006), no. 4, 625–658.
- [24] A. Kleshchev, Linear and Projective Representations of Symmetric Groups, Cambridge University Press, 2005.
- [25] A. Kleshchev and A. Ram, Homogeneous Representations of Khovanov-Lauda Algebras. arXiv:0809.0557
- [26] P. Lalonde and A. Ram, Standard Lyndon bases of Lie algebras and enveloping algebras. *Trans. Am. Math. Soc.* **347** (1995), 1821–1830.
- [27] B. Leclerc, Dual canonical bases, quantum shuffles and q -characters. *Math. Z.* **246** (2004), no. 4, 691–732.
- [28] I. G. Macdonald, Symmetric Functions and Hall Polynomials, second edition, Oxford University Press, Oxford, 1995.
- [29] M. Nazarov, Young’s symmetrizers for projective representations of the symmetric group. *Adv. Math.* **127** (1997), no. 2, 190–257.

- [30] G. I. Olshanski, Quantized Universal Enveloping Superalgebra of type Q and a Super-Extension of the Hecke Algebra, *Letters in Mathematical Physics* **24** (1992), 93–102.
- [31] R. Orellana and A. Ram, Affine braids, Markov traces and the category $\mathcal{O}$. Algebraic groups and homogeneous spaces, *Tata Inst. Fund. Res. Stud. Math.*, Mumbai, (2007), 423–473.
- [32] I. Penkov, Characters of typical irreducible finite-dimensional $\mathfrak{q}(n)$ -modules. (Russian) *Funktsional. Anal. i Prilozhen.* **20** (1986), no. 1, 37–45, 96.
- [33] I. Penkov and Serganova, V., Characters of finite-dimensional irreducible $\mathfrak{q}(n)$ -modules. *Lett. Math. Phys.* **40** (1997), no. 2, 147–158.
- [34] ———, Characters of irreducible G -modules and cohomology of G/P for the Lie supergroup $G = Q(N)$. Algebraic geometry, 7. *J. Math. Sci. (New York)* **84** (1997), no. 5, 1382–1412.
- [35] A. Ram, Skew shape representations are irreducible. Combinatorial and geometric representation theory (Seoul, 2001), *Contemp. Math.*, **325**, Amer. Math. Soc., Providence, RI, (2003), 161–189
- [36] J. D. Rogawski, On modules over the Hecke algebra of a p -adic group, *Invent. Math.* **79** (1985), no. 3, 443–465.
- [37] M. Rosso, Groupes quantiques et algèbres de battage quantiques. *C. R. Acad. Sci. Paris* **320**, (1995), 145–148.
- [38] ———, Quantum groups and quantum shuffles. *Invent. Math.* **133**, (1998), 399–416.
- [39] ———, Lyndon bases and the multiplicative formula for R -matrices. Preprint, 2002.
- [40] R. Rouquier, 2-Kac-Moody Algebras. arXiv:0812.5023
- [41] A. N. Sergeev, Tensor algebra of the identity representation as a module over the Lie superalgebras $GL(n, m)$ and $Q(n)$, *Math. USSR Sbornik* **51** (1985), 419–427.
- [42] ———, The Howe Duality and the Projective Representations of Symmetric Groups, *Represent. Theory* **3** (1999), 416–434.
- [43] J. R. Stembridge, Shifted tableaux and the projective representations of symmetric groups. *Adv. Math.* **74** (1989), no. 1, 87–134.
- [44] T. Suzuki, Rogawski’s Conjecture on the Jantzen Filtration for the Degenerate Affine Hecke Algebra of Type A , *Represent. Theory* (Electronic Jour. of AMS) **2** (1998), 393–409.
- [45] ———, Representations of degenerate affine Hecke algebra and $\mathfrak{gl}_n$. *Combinatorial methods in representation theory* Adv. Stud. Pure Math., 28, Kinokuniya, Tokyo, (2000), 343–372.
- [46] T. Suzuki and M. Vazirani, Tableaux on periodic skew diagrams and irreducible representations of the double affine Hecke algebra of type A . *Int. Math. Res. Not.* (2005), no. 27, 1621–1656.
- [47] M. Vazirani, Irreducible Modules over the Affine Hecke Algebra: A Strong Multiplicity One Result, Ph.D. Thesis, UC Berkeley, 1999.
- [48] J. Wan, Completely Splittable Representations of Affine Hecke-Clifford Algebras. In preparation.
- [49] W. Wang, Spin Hecke algebras of finite and affine types. *Advances in Math.* **212** (2007), 723–748.
- [50] W. Wang and L. Zhao, Representations of Lie Superalgebras in Prime Characteristic II: The Queer Series. preprint.
- [51] A. Zelevinsky, Induced representations of reductive p -adic groups II. *Ann. Sci. E.N.S.* **13** (1980), 165–210.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, BERKELEY, CA 94720-3840

E-mail address: dhill1@math.berkeley.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OK 73019

E-mail address: kujawa@math.ou.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, BERKELEY, CA 94720-3840

E-mail address: sussan@math.berkeley.edu