19th Century Gems:

$\boxed{\text{Notation: } e(x) = e^{2\pi i x}}$

Following Bump: p. 3-11
p 254-278

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1} \quad Re(s) > 1$$

Dirichlet Series

Euler Product

$$\Lambda(s) = \underbrace{\pi^{-s/2}\,\Gamma(s/2)}_{\substack{\text{Gamma factor} \\ \text{or} \\ \text{Euler factor at } \infty}} \zeta(s)$$

Then Thm. (Riemann)

(1) $\exists$ merom. analytic continuation

(2) Functional Eq. $\Lambda(1-s) = \Lambda(s)$

proof:

let $\Theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi t n^2}$

Recall:

$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt \quad Re\, s > 0$

Mellin transform: $\quad Re\, s > 1$

$$2\Lambda(s) = \int_0^\infty (\Theta(t) - 1)\, t^{\frac{s}{2}-1}\,dt = 2\sum_{n=1}^{\infty}\int_0^\infty e^{-\pi t n^2} t^{\frac{s}{2}-1}\,dt$$

$$= 2\sum_{n=1}^{\infty} (\pi n^2)^{-s/2}\,\Gamma(s/2)$$

$$= 2\pi^{-s/2}\,\Gamma(s/2)\sum_{n=1}^{\infty} n^{-s}$$

Poisson Summation:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

$\therefore \; f(x) = e^{-\pi t x^2}$

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)\, e(-xy)\,dy$$

Calculate: $\hat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi t y^2 - 2\pi i x y}\, dy$

$t^{-1/2} y$ for y

$$= t^{-1/2} \int_{-\infty}^{\infty} e^{-\pi y^2 - 2\pi i (t^{-1/2} x) y}\, dy$$

(*) Recall: $e^{-\pi x^2}$ is self dual for Fourier transform:

ie.

$$\int_{-\infty}^{\infty} e^{-\pi y^2 - 2\pi i x y}\, dy = e^{-\pi x^2}$$

proof: Ex.

by (*)

$$= t^{1/2} e^{-\pi t^{-1} x^2}$$

So applying Poisson sumation:

$$\boxed{\theta(t) = t^{-1/2}\theta(t^{-1})}$$

Not as $t \to 0$ $\int_0^{\infty} (\theta(t) - 1) t^{s/2 - 1}\, dt$ goes like $t^{-1/2}$.

Write:

$$\underbrace{\int_1^{\infty} (\theta(t) - 1) t^{s/2-1}\, dt}_{\text{Entire}} + \int_0^1 (\theta(t) - 1) t^{s/2-1}\, dt.$$

$$= \int_0^1 (t^{-1/2}\theta(t^{-1}) - 1) t^{s/2-1}\, dt. \quad \text{by box Formula.}$$

$$= \int_0^1 \left(t^{-1/2}(\theta(t^{-1}) - 1) + t^{-1/2} - 1\right) t^{s/2-1}\, dt.$$

$t \to t^{-1}$

$$2\Lambda(s) = \underbrace{\int_1^{\infty} (\theta(t) - 1) t^{\frac{s}{2}-1}\, dt + \int_1^{\infty} (\theta(t) - 1) t^{\frac{1-s}{2}-1}\, dt}_{\text{Entire}} + \underbrace{\frac{2}{s-1} - \frac{2}{s}}_{\text{Just Simple poles.}}$$

So part (1) Proved:

Functional Eq is obvious by $s \to 1-s$.

also showed (3) $\Lambda(s)$ has simple poles at $s = 0, 1$ with residues $= -1, 1$.

Dirichlet characters:

if $N \in \mathbb{Z}_{>0}$

Dirichlet char mod N is a fcn: χ;

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{C} \\
\mathbb{Z}/N\mathbb{Z} \arrow[dd, bend right, "\text{if } N_1 \mid N"'] & \\
\cup & \\
(\mathbb{Z}/N\mathbb{Z})^\times \arrow[r] \arrow[d] & \mathbb{C}^\times \\
\mathbb{Z}/N_1\mathbb{Z} & \\
\cup & \\
(\mathbb{Z}/N_1\mathbb{Z})^\times \arrow[uuur] &
\end{tikzcd}$$

$$\chi(n) = \begin{cases} \chi(n) & \text{if } (n, N) = 1 \\ 0 & \text{otherwise} \end{cases}$$

χ: imprimitive if it comes from some char mod N_1, $N_1 \mid N$ ($N_1 < N$)
otherwise primitive

Define: $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p \left(1 - \chi(p) p^{-s}\right)^{-1}$

<u>Note</u>: if $p \mid N$, corresponding Euler factor is 1.

<u>Gauss Sum</u>:

$$\tau(\chi) = \sum_{n \bmod N} \chi(n)\, e\left(\frac{n}{N}\right)$$

<u>Recall</u>:
G = loc cpt. top. group
$\hat{G}$ = pont. dual = $\mathrm{Hom}_{cont}(G, \mathbb{C}^1)$

$R = \mathbb{Z}/N\mathbb{Z}$ a cpt (finite) topological ring.
(<u>discrete</u>)

$\psi : n \longmapsto e\left(\frac{n}{N}\right) \in \mathbb{C}^1$, $\psi \in \hat{R}$

$\cdots \xrightarrow{\sim} \hat{R}$

$m \longmapsto \psi_m$ where $\psi_m(x) = \psi(mx) = e\left(\frac{mn}{N}\right)$ (self dual)

So have Fourier Transform:

$f \in S(R)$ (= any $\mathbb{C}$-val. function) (using counting measure)

but self dual meas is $\dfrac{\text{count. meas}}{\sqrt{N}}$

$$\hat{f}(m) = \sum_{n \in R} f(n)\psi_m(n) = \sum_{n \in R} f(n)\, e\left(\frac{mn}{N}\right)$$

Let $\chi \in S(R)$ (a Dir. char)

<u>lemma</u>: If χ is primitive then

$$\hat{\chi}(m) = \tau(\chi) \cdot \bar{\chi}(m)$$

i.e.

$$\hat{\chi} = \tau(\chi)\,\chi^{-1}$$

<u>pf</u>: $(m,N)=1$. $\exists\ \bar{m} m \equiv 1\ (N)$

$$\hat{\chi}(m) = \sum_{n \in R} \chi(n)\, e\left(\frac{mn}{N}\right)$$

substitute $\bar{m}n$ for n

$$= \sum_{n \in R} \chi(\bar{m} n)\, e\left(\frac{n}{N}\right)$$

$$= \chi(m)^{-1} \underbrace{\sum_{n \in R} \chi(n)\, e\left(\frac{n}{N}\right)}_{\tau(\chi)}$$

If $(m,N) > 1$

write $m = dm_1$, $N = dN_1$

depends on n mod N_1 now. $\equiv 0$ by primitive

$$\hat{\chi}(m) = \sum_{n} \chi(n)\, e\left(\frac{\cancel{d}m_1 n}{\cancel{d}N_1}\right) = \sum_{r \bmod N_1} \Big(\sum_{\substack{n \bmod N \\ n \equiv r (N_1)}} \chi(n) \Big)\, e\left(\frac{m_1 r}{N_1}\right)$$

lemma: $|\tau(\chi)|^2 = N$

pf.

$$|\hat{\chi}(m)|^2 = \left(\sum_{n_1} \chi(n_1)\, e\left(\frac{mn_1}{N}\right)\right)\left(\sum_{n_2} \bar{\chi}(n_2)\, e\left(\frac{-mn_2}{N}\right)\right)$$

$$= \sum_{n_1,n_2} \chi(n_1)\,\bar{\chi}(n_2)\, e\left(\frac{m(n_1-n_2)}{N}\right)$$

want to compute $|\hat{\chi}(m)|^2$

$$|\hat{\chi}(m)|^2 = \underbrace{}_{|\tau(\chi)|^2 \text{ if } (m,N)=1 \quad 0 \text{ if } (m,N)>1}$$

$$\phi(N)\,|\tau(\chi)|^2 = \|\hat{\chi}\|^2 = \sum_{n_1,n_2} \chi(n_1)\,\bar{\chi}(n_2) \underbrace{\sum_m e\left(\frac{m(n_1-n_2)}{N}\right)}_{= \begin{cases} N, & n_1 \equiv n_2\ (N) \\ 0 & \text{otherwise.}\end{cases}}$$

$$= N\,\phi(N) \qquad \boxtimes$$

Note: $\hat{\chi} = \tau(\chi)\bar{\chi} \Rightarrow$

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_m \bar{\chi}(m)\, e\left(\frac{mn}{N}\right)$$

So this lets us define $\chi(x)$ for any real x by the RHS.

let χ = primitive, non-trivial; $\chi(-1) = 1$ (even);

let

$$\theta_\chi(t) = \sum_{n\in\mathbb{Z}} \chi(n)\, e^{-\pi t n^2} = 2\sum_{n=1}^{\infty} \chi(n)\, e^{-\pi t n^2}$$

So

$$\int_0^\infty \theta_\chi(t)\, t^{s/2-1}\, dt = \int_{N^{-1}}^\infty \theta_\chi(t)\, t^{s/2-1}\, dt + \int_0^{N^{-1}} ''$$

by prev. calc. $\longrightarrow$ $\|$

$$2\pi^{-s/2}\,\Gamma(s/2)\, L(s,\chi)$$

to Apply Poisson Summ; $f(x) = \frac{1}{\tau(\bar{\chi})} \sum_m \bar{\chi}(m)\left(e\left(\frac{mx}{N}\right)\cdot e^{-\pi t x^2}\right)$

$$\hat{f}(x) = \text{same sum}\cdots \int_{-\infty}^{\infty} e^{-\pi t y^2}\, e\Big(\underbrace{\frac{my}{N} - xy}_{}\Big)\, dy$$

$$= t^{-1/2} e^{-\pi(m-xN)^2/tN^2}$$

Apply Poiss-Sum to Θ_χ

really goes over $\mathbb{Z}$.

$$\Theta_\chi(t) = \sum_{n\in\mathbb{Z}} \frac{1}{\tau(\chi)} \sum_{\substack{m \\ (\mathrm{mod}\, N)}} \bar{\chi}(m) t^{-1/2} e^{-\pi(m-nN)^2/tN^2}$$

$$\boxed{\Theta_\chi(t) = \frac{1}{\tau(\chi)} t^{-1/2} \Theta_{\bar\chi}\left(\frac{1}{tN^2}\right)}$$

change vars etc. 2^{nd} int. becomes $\int_{N^{-1}}^{\infty} \Theta_{\bar\chi}(t) t^{s/2-1}\,dt + \frac{1}{\tau(\chi)} N^{-s} \int_{N^{-1}}^{\infty} \Theta_\chi(t) t^{\frac{1-s}{2}-1}\,dt.$

so get:

$$\Lambda(1-s, \bar\chi) = \frac{\sqrt{N}}{\tau(\chi)} N^{s-\frac{1}{2}} \Lambda(s,\chi)$$

fixes abs. value

$$N^{\frac{1-s}{2}} \Lambda(1-s, \bar\chi) = \frac{\sqrt{N}}{\tau(\chi)} \left(N^{s/2} \Lambda(s,\chi)\right)$$

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local fields:

(could also consider $\mathbb{F}_q((t))$ but we'll assume char 0)

$$F = \begin{cases} \mathbb{R}, \mathbb{C} & \text{archimedean case} \\ \mathbb{Q}_p, \text{ fin. extension of } \mathbb{Q}_p : & \text{nonarchimedean case.} \end{cases}$$

Suppose F non-archimedean:

$\mathbb{Q}_p \supseteq \mathbb{Q}$: $|x|_p = p^{-\mathrm{ord}_p(x)}$. $|0| = 0$.

$\mathbb{Z}_p \subseteq \mathbb{Q}_p$, $\mathbb{Z}_p = \{x \mid \mathrm{ord}_p(x) \geq 0\}$, $p\mathbb{Z}_p = \{x \mid \mathrm{ord}_p(x) > 0\}$

$$\mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \quad \text{res. field.}$$

Fix:

$|F : \mathbb{Q}_p| = n$, $\exists!$ extension of $|\cdot|_p$ to F and $\exists!$ ext of ord_p to F

$\mathcal{O} = \{x \mid \mathrm{ord}_p x \geq 0\}$

$\cup$

$\mathfrak{p} = \{x \mid \mathrm{ord}_p x > 0\}$. $\mathrm{ord}_p(\varpi) = \frac{1}{e}$ $e =$ ramification index of $F/\mathbb{Q}_p$

$\mathfrak{p} = \varpi\mathcal{O}$

$\uparrow$ uniformizer. (not unique)

So def: $\mathrm{ord}(\varpi) = 1$ then $\mathrm{ord}(p) = e$. , $|\varpi| = \frac{1}{q}$.

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{O}/\mathfrak{p} = \mathbb{F}_q \\ \cup & & | \\ \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p \end{array}$$

where $q = p^f$ and $\underline{n = ef}$

then let $|x| = q^{-\mathrm{ord}(x)}$. $x \in \mathbb{Z}_p \Rightarrow |x| = |x|_p^n$.

F - add. group - nbhd basis of 0 given by $\mathfrak{p}^m$

$F^\times$ = mult. group. Have $U_F = \{x \mid |x| = 1\} = \mathcal{O}^\times$.

$$F^\times \simeq \mathbb{Z} \times U$$

$= GL_1(F)$

$x = \varpi^{\mathrm{ord}(x)} \cdot u$

$u \in U$

Additive char:

$\psi \in \hat{F}$: $\psi: F \longrightarrow \mathbb{C}^1$. Say ψ is non-trivial. Then:

<u>lemma</u>: Fix $a \in F$. Consider $x \longmapsto \psi(ax) = \psi_a(x)$.
also an additive character, ie., $\psi_a \in \hat{F}$.

Then $\begin{array}{ccc} F & \longrightarrow & \hat{F} \\ a & \longmapsto & \psi_a \end{array}$ is an isom. of top. groups.

<u>pf</u>: TATE

So Need a nontrivial ψ:

$F = \mathbb{R}$, $\psi(x) = e(x)$

$F = \mathbb{C}$, $\psi(z) = e(tr(z))$ where $tr: \mathbb{C} \longrightarrow \mathbb{R}$, $z \longmapsto z + \bar{z}$.

F nonarchimedean:

$\mathbb{Q}_p$ define $\psi_0(\ )$ then:

F: $\psi(x) = \psi_0(tr(x))$ where $tr: F \longrightarrow \mathbb{Q}_p$.

if: $x \in \mathbb{Q}_p$ write: $x = \sum_{r >> -\infty} a_r p^r$

$$\lambda(x) = \sum_{r<0} a_r p^r \in \mathbb{Q}$$

then $\psi_0(x) = e(-\lambda(x))$ is a nontrivial add. char

<u>Note</u>: ψ_0 is trivial on $\mathbb{Z}_p$.

Suppose $\psi \in \hat{F}$. Claim: $\exists\ m \in \mathbb{Z}$ s.t. ψ is trivial on $\wp^m$.

<u>pf</u>: $\boxed{\text{NO SMALL SUBGROUPS}}$
$\psi: F \to \mathbb{C}^1$ cont. hom.

Take a small open nbhd. V of 1 in $\mathbb{C}^1$. There are no subgroups of $\mathbb{C}^1$ in this nbhd. But $\psi^{-1}(V)$ is open in F and contains some $\wp^m$ which is a subgroup. Hence $\psi(\wp^m) = 1$.

Defn: For $\psi \in \hat{F}$ non trivial, the conductor of ψ, $c(\psi)$ is the largest integer m s.t. ψ is trivial on $\wp^{-m}$.

ψ_0 above has $c(\psi_0) = 0$.

Note: if $\psi(x) = \psi_0(tr(x))$, ψ is trivial on $\mathfrak{d}^{-1}$.

i.e., $tr: F \to \mathbb{Q}_p$ and ψ is trivial on $\overset{\shortparallel}{tr^{-1}(\mathbb{Z}_p)}$

Fourier Transform:

take dx on F : Haar Measure, unique up to scalar in $\mathbb{R}_+^\times$

$\psi \in \hat{F}$. non. triv.

let ϕ nice function in F, define $\hat{\phi}(x) = \int_F \phi(y)\psi(-xy)\, dy$.

Fourier inversion:

$$\hat{\hat{\phi}}(x) = c \cdot \phi(-x)$$

Say the measure dx is self dual wrt ψ if $c = 1$ in this eq.

So choose the unique self dual meas.

$\boxed{\text{Ex: if } dx \text{ is self dual for } \psi, \text{ what is self dual for } \psi_a\text{?}}$

let $a \in F^\times$, consider $d(ax)$. pull back of dx under $\to$ $F \xrightarrow{a} F$

$$d(ax) = |a|\, dx. \qquad |a| = \begin{cases} |\cdot| & \text{for } F = \mathbb{R} \\ |\cdot|^2 & \text{for } F = \mathbb{C} \\ |a| = q^{-ord(a)} & F \text{ padic} \end{cases}$$

Consider $\int_{\mathcal{O}} d(ax) = |a| \, Vol(\mathcal{O}, dx)$

$\shortparallel$

$\int_{a\mathcal{O}} dx.$

$a = \text{unit} \Rightarrow a\mathcal{O} = \mathcal{O}$ and $|a| = 1$ so $\int_{\wp} dx = |a| \, Vol(\mathcal{O})$. Consider $\mathcal{O} \to \mathcal{O}/\wp = \mathbb{F}_q$.

$a = \varpi \Rightarrow a\mathcal{O} = \wp$

So therefore: $\mathcal{O} = \coprod_{\bar{a} \in \mathbb{F}_q} (\alpha + \wp)$

So $\mathrm{Vol}(\mathcal{O}) = \sum_{\bar{\alpha}} \mathrm{vol}(\alpha + \wp) = q \cdot \mathrm{vol}(\wp)$.

Hence $\mathrm{Vol}(\wp) = |\varpi| q \cdot \mathrm{Vol}(\wp) \Rightarrow |\varpi| = q^{-1}$.

Recall $S(\mathbb{R})$ = Schwartz Space: C^∞ fns s.t

$\sup_{\mathbb{R}} \left| |x|^n f^{(r)}(x) \right| < \infty$

''

$O_{n,r}(f)$

''

Seminorms

All seminorms finite.

topologize s.t. all seminorms are continuous.

Self dual measure: wrt ψ

$F = \mathbb{R}$ dx = Lebesgue

$F = \mathbb{C}$ $dx = 2 \cdot$ Lebesgue on $\mathbb{C} \simeq \mathbb{R}^2$.

F p-adic: Suppose $c(\psi) = 0$.

let $\phi = \mathrm{char}(\mathcal{O})$ and compute:

$\hat{\phi}(x) = \int_F \phi(y) \psi(-xy)\, dy = \int_{\mathcal{O}} \psi(-xy)\, dy$ $\quad \psi_x$ trivial on $\mathcal{O} \Leftrightarrow x \in \mathcal{O}$. $\quad x\mathcal{O} \subseteq \wp^{-c(\psi)}$.

$= \mathrm{vol}(\mathcal{O}) \cdot \phi(x)$.

So $\hat{\hat{\phi}}(x) \overset{\text{by calc.}}{=} \mathrm{Vol}(\mathcal{O})^2 \phi(x)$.

by theory $\to$ '' $c \cdot \phi(-x)$ $\quad$ '' $\phi(x)$)

$\phi(x)$''

dy = arbitrary. so $c = \mathrm{Vol}(\mathcal{O})^2$.

$$\boxed{\text{ex: } \mathrm{Vol}(\mathcal{O}) = q^{\pm \frac{c(\psi)}{2}}}$$

~~Schwartz~~:

F: non-archimedean

Schwartz Space: $S(F) := \{\phi \mid \phi$ locally const and cpt. support$\}$. (this set stable under $\wedge$)

resp. (ϕ const. on $\wp^m$ cosets)

Say ϕ has cpt support. ie. $\mathrm{supp}\, f \subseteq \wp^{-m}$ some m, $c(\psi) = 0$.

(resp. Supp in $\wp^{-m}$ $\Rightarrow$ $\hat{\phi}$ constant on $\wp^m$ cosets.

$\hat{\phi}(x) = \int_F \phi(y) \psi(-xy)\, dy$ (in $\wp^{-m}$) $\quad$ Conclude: $\hat{\phi}$ is constant on $x + \wp^m$.

$\therefore = \hat{\phi}(x)$.

Say $x \mapsto x + \varpi^m t$ then $\hat{\phi}(x + \varpi^m t) = \int_F \phi(y) \psi(-xy - \varpi^m t y)\, dy$ but $y \in \wp^{-m}$ so $\varpi^m t y \in \mathcal{O}$ where ψ is trivial

F

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$$F^\times = GL_1(F).$$

(quasi) - characters :

characters are $\hat{F}^\times$ = cont. homs of $F^\times$ into $\mathbb{C}^1$. i.e. $\mathrm{Hom}_{cont}(F^\times, \mathbb{C}^1)$

$\cap$

quasi-characters = $\mathrm{Hom}_{cont}(F^\times, \mathbb{C}^\times)$

Say $\chi \in \mathrm{Hom}_{cont}(F^\times, \mathbb{C}^\times)$

$$\underbrace{F^\times \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{|\cdot|} \mathbb{R}_+^\times}_{|\chi|}$$

$|\cdot| : F^\times \to \mathbb{R}_+^\times$.
$|\cdot|^s,\ s \in \mathbb{C}$ are quasi-characters.

$\mathbb{R}^\times$, $\mathbb{C}^\times$: easy : $\mathbb{F} = \mathbb{R}$, $\chi(x) = |x|^s$ $s \in \mathbb{C}$ or $\chi(x) = \mathrm{sgn}(x)^a |x|^s$ $s \in \mathbb{C}$, $a = 0, 1$ that's all.

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non-arch. case :

$F^\times \simeq \mathbb{Z} \times U_F$ via choice of ϖ — cpt

$$\hat{F}^\times \simeq \mathbb{C}^1 \times \hat{U}_F$$

then quasi chars : $\simeq \mathbb{C}^\times \times \hat{U}_F \ni (z, \chi_0)$

If $z = q^{-1}$ then $z^{\mathrm{ord}\,x} = q^{-\mathrm{ord}\,x} = |x|$.

So

$|x|^s$ depends only on $s + \frac{2\pi i}{\log(q)}\mathbb{Z}$.

write $x = \varpi^{\mathrm{ord}\,x} \cdot u(x)$

$$\chi(x) = z^{\mathrm{ord}(x)} \cdot \chi_0(u(x)) = |x|^s \chi_0(u(x))$$

$$|\chi(x)| = |z|^{\mathrm{ord}\,x}$$

& Fix χ_0 :

$$|\chi(x)| = |x|^\sigma \quad \sigma = \mathrm{Re}\, s = \sigma(\chi) = \text{exponent of } \chi.$$

χ is unramified $\Leftrightarrow$ trivial on U_F

unramified $\Rightarrow$ $\chi(x) = |x|^s$

otherwise χ is ramified.

$U_F = \mathcal{O}_F^\times$ have $\mathcal{O} \longrightarrow \mathcal{O}/\mathfrak{p} \simeq \mathbb{F}_q$.

$$\begin{array}{ll} u(0) = U & U = \mathcal{O}^\times \longrightarrow \mathbb{F}_q^\times \\ & \cup \\ & U(1) = 1+\mathfrak{p} = \text{kernel.} \\ & \cup \\ & \vdots \quad \text{filtration.} \\ & \cup \\ & u(r) = 1+\mathfrak{p}^m \end{array}$$

$U(r)$ is a nbhd basis of $1 \in F^\times$.

$\exists$ smallest r s.t. $\chi|_{U(r)} \equiv 1$.

$r =$ conductor of χ.

So: $U =$ cpt $\Rightarrow$ $\hat{U}$: discrete $=$ union of $\widehat{U/U(r)}$

so $\hat{U} = \varinjlim_r \widehat{U/U(r)}$

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mult haar measure : $d^\times x$, $\mathrm{Vol}(U_F, d^\times x) = 1$. (good choice for global considerations)

exercise:

$d^\times x = \mu \cdot \frac{dx}{|x|}$ since $d(ax) = |a|\,dx$. Determine μ.

$$\boxed{\begin{array}{ll} \text{lemma:} & \text{If } c = c(\psi),\ dx = \text{self dual for } \psi. \\ & \mathrm{Vol}(\mathcal{O}, dx) = q^{-c/2} \end{array}}$$

Recall: if χ : prim dirichlet char mod N , i.e. $R = \mathbb{Z}/N\mathbb{Z}$

we got: $\hat{\chi} = \tau(\chi)\,\bar{\chi}$ $\qquad \chi\ R^\times$

Using $\uparrow$ motivation:

let $\chi =$ quasicharacter. Fix ψ, dx

extend by 0

Compute: "$\hat{\chi}(x) = \int_{F^\times} \chi(y)\,\psi(-xy)\,dy$". $\quad$ | integrand $| = 1$ so doesn't make sense.

But: View χ as a distribution

But View X as a distribution

F = non-arch.

$S(F)$ = loc. const. cpt. support.

$S(F)'$ = space of all $\mathbb{C}$-linear functionals, ie., $\mathrm{Hom}(S(F), \mathbb{C})$

Consider $\chi \cdot |\,|^s$ where $\chi(x) = |x|^{s_0} \cdot \chi_0(u(x))$.

$\phi \in S(F)$ then define the pairing:

$$\langle \chi|\cdot|^s, \phi \rangle = \int_F \phi(x)\, \chi(x)\, |x|^s\, dx.$$

ϕ loc const so $\exists$ n s.t. $\phi|_{\mathfrak{p}^n}$ = constant, and value is $\phi(0)$.

(only prob near zero)

so $\phi(0) \int_{\mathfrak{p}^n} \chi(x)\, |x|^s\, dx$ taking abs value get

$$\int_{\mathfrak{p}^n} |x|^{\sigma + \sigma(\chi)}\, dx = \sum_{m=n}^{\infty} \int_{\operatorname{ord} x = m} q^{-(\sigma + \sigma(\chi))m}\, dx$$

So what is $\mathrm{vol}(\varpi^m U) = q^{-m}\, \mathrm{vol}(U)$

$$= \sum_{m=n}^{\infty} q^{-(\sigma+\sigma(\chi))m - m}\, \mathrm{vol}(U).$$

So this is <u>ok</u> if $\mathrm{Re}(s) + \sigma(\chi) + 1 > 0$.

lets set $\sigma(\chi) = 0$, only consider characters from now on.

So $\mathrm{Re}(s) + 1 > 0$.

Let

$$\underset{\text{Tate integral}}{\zeta}(\overset{\in\mathbb{C}}{s}, \overset{\in\text{char}}{\chi}, \overset{\in S(F)}{\phi}) = \int_{F^\times} \phi(x)\chi(x)|x|^s d^\times x. \quad \text{ok for } \operatorname{Re} s > 0. \quad \left(\underset{\text{shifts}}{d^\times x = c\frac{dx}{|x|}} \right)$$

For $\operatorname{Re} s > -1$, $\chi|\,|^s$ defines a distribution:

Fourier trans. of a dist:

$$\langle \widehat{\chi|\cdot|^s}, \phi \rangle := \langle \chi|\cdot|^s, \hat{\phi} \rangle.$$

Fact: this distribution is proportional to $\bar{\chi}|\cdot|^{-1}$

We want funct. eqn. ~~of~~ Compute Four. trans of dist

Let $\chi \in \widehat{F^\times}$, ψ add char.

Thm: (i) $\zeta(s,\chi,\phi)$ is absolutely convergent $\forall \phi$ provided $\operatorname{Re} s > 0$.

(ii) There exists a meromorphic analytic continuation of $\zeta(s,\chi,\phi)$.

(iii) (Local Functional equation) There is a meromorphic fn $\gamma(s,\chi,\psi)$, independent of ϕ, s.t.

$$\zeta(1-s, \chi^{-1}, \hat{\phi}) = \gamma(s,\chi,\psi)\,\zeta(s,\chi,\phi).$$

Pf:

(i) proved already for non-arch

functional eqn is well defined in $0 < \operatorname{Re}(s) < 1$.

idea (Tate)
1. prove funct. eqn in the strip.
2. prove $\exists$ merom. cont. of γ.
3. conclude 2.

Wrong proof:

$$\zeta(s,\chi,\hat{\phi}) = \int_{F^\times}\left(\int_F \phi(y)\,\psi(-xy)\,dy\right)\chi(x)\,|x|^s\,d^\times x.$$

$$= \mu\int_F\left(\int_{F^\times} \phi(y)\,\psi(-xy)\,\chi(x)\,|x|^{s-1}\,dx\,dy\right.$$

$$x \to y^{-1}x$$

$$= \mu\int_F \phi(y)\left(\int_F \psi(-x)\,\chi(x)\,|x|^{s-1}\,dx\right)\chi(y^{-1})\,\underbrace{|y^{-1}|^s}_{|y|^{-s}}\,dy$$

$$= \zeta(1-s,\chi^{-1},\phi)\cdot\underbrace{\int_F \psi(-x)\,\chi(x)\,|x|^{s-1}\,dx}_{\gamma}$$

But this integral might not converge for Re s big since ψ is add. char, and there's no ϕ with cpt support to cut things off.

Tate: suffices to prove for $\phi_1,\phi_2\in S(F)$

$$\zeta(1-s,\chi_1^{-1},\hat{\phi}_1)\,\zeta(s,\chi,\phi_2) = \zeta(1-s,\chi^{-1},\hat{\phi}_2)\,\zeta(s,\chi,\phi_1)$$

So:

$$\text{RHS} = \int_{F^\times\times F^\times} \phi_1(x)\,\chi(x)\,|x|^{s-1}\,\hat{\phi}_2(y)\,\chi^{-1}(y)\,|y|^{-s}\,dx\,dy.$$

We showed
$(\phi_1,\phi_2)\mapsto RHS$
same as
$(\hat{\phi}_2,\hat{\phi}_1)\mapsto RHS.$

$$(x,y)\longmapsto(x,xy)$$

$$\int_{F\times F}\phi_1(x)\,\cancel{\chi_1(x)\,|x|^{s-1}}\,\hat{\phi}_2(xy)\,\chi^{-1}(xy)\,|y|^{-s}\,dx\,dy$$

$$= \int_F\underbrace{\left(\int_F \phi_1(x)\,\hat{\phi}_2(xy)\,dx\right)}_{\overset{\text{exercise}}{=}\ \int \phi_2(x)\cdot\hat{\phi}_1(xy)\,dx.}\chi^{-1}(y)\,|y|^{-s}\,dy$$

Kudla:

2/10 χ = character.

Recall: $\zeta(s,\chi,\phi) = \int_{F^\times} \phi(x)\chi(x)|x|^s d^\times x$. is

- absolutely convergent for $\operatorname{Re} s > 0$.
- Functional equation in strip $0 < \operatorname{Re} s < 1$: $\zeta(1-s,\chi^{-1},\hat{\phi}) = \gamma(s,\chi,\psi)\,\zeta(s,\chi,\phi)$.

F = non-arch

lemma: let $a \in F^\times$. Define $(\rho(a)\phi)(x) = \phi(a^{-1}x)$

Then $\zeta(s,\chi,\rho(a)\phi) = \chi(a)|a|^s \zeta(s,\chi,\phi)$.

proof:

$$\int_{F^\times} \phi(a^{-1}x)\chi(x)|x|^s d^\times x \underset{\substack{\text{put } ax\\ \text{for } x}}{=} \uparrow.$$

Observe: Say $\phi(0) = 0$, hence $\phi|_{\mathfrak{p}^n} = 0$ for some n.

Then $\zeta(s,\chi,\phi)$ is entire since $|x|^s$ is bounded away from zero on supp ϕ.

So $\operatorname{supp}(\phi) \subseteq \mathfrak{p}^{-N} - \mathfrak{p}^n$; $-N \le \operatorname{ord} x \le n$.

$$\zeta(s,\chi,\phi) = \sum_{r=-N}^{n} q^{-rs} \int_{\substack{\operatorname{ord} x = r\\ \downarrow\\ \varpi^r U_F}} \phi(x)\chi(x)d^\times x = \sum_{\substack{r\\ \text{finite \# of } r\text{'s}}} (q^{-s})^r \cdot a_r.$$

let $a = \varpi$

Consider

$\zeta(s,\chi,\phi - \rho(\varpi)\phi)$ is entire. But its also, by linearity, and lemma

note: $\phi - \rho(\varpi)\phi(0) = \phi(0) - \phi(0) = 0 \Rightarrow$

$$= (1-\chi(\varpi)q^{-s})\,\zeta(s,\chi,\phi)$$

So $\zeta(s,\chi,\phi) = \dfrac{\zeta(s,\chi,\phi-\rho(\varpi)\phi)}{1-\chi(\varpi)q^{-s}}$ is meromorphic with possible poles for s s.t. $\chi(\varpi)q^{-s} = 1$

Suppose χ is unramified, take $\chi = 1$.

Do poles occur as ϕ varies?

Let $\phi = \phi_0 = \mathrm{char}(\mathcal{O})$.

$$\int_{F^\times} \phi_0(x)\,|x|^s d^\times x = \sum_{n=0}^{\infty} q^{-ns} \underbrace{\int_{\mathrm{ord}\,x = n} d^\times x}_{1} = (1-q^{-s})^{-1}.$$

So all poles occur.

$\mathrm{vol}(\varpi^n U_F) = \mathrm{vol}\, U_F = 1$.

χ ramified:

$\phi = \phi_0$

$\zeta(s, \chi, \phi_0) = 0$.

proof:

Take $a \in U_F$ s.t. $\chi(a) \neq 1$.

$\rho(a)\phi_0(x) = \phi_0(a^{-1}x) = \phi_0(x)$

Apply above eqn. ☒

Conclude: χ: ramified, $\zeta(s, \chi, \phi)$: entire $\forall \phi$.

pf: $\zeta(s, \chi, \phi) = \zeta(s, \chi, \phi - \phi(0)\cdot\phi_0)$

Take $\phi_1 = \mathrm{char}(1 + \mathfrak{p}^r)$ $r =$ conductor of χ $r \geq 1$. bc ramified.

$$\zeta(s, \chi, \phi_1) = \int_{1+\mathfrak{p}^r} \chi(x)\,|x|^s d^\times x = \int_{1+\mathfrak{p}^r} d^\times x = \mathrm{vol}(1+\mathfrak{p}^r)$$

So $\zeta(s, \chi, \phi_1) = 1$ if $\phi_1 = \mathrm{char}(1+\mathfrak{p}^r)\,\mathrm{vol}(1+\mathfrak{p}^r)^{-1}$

Theorem continued: F non-arch

$\exists$ an Euler factor $L(s,\chi) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \chi \text{ unramified} \\ 1 & \chi \text{ ramified} \end{cases}$

s.t.

$$\frac{\zeta(s,\chi,\phi)}{L(s,\chi)}$$ is entire and non-vanishing. (minimally cleared poles, i.e., didn't "create" any zeros)

(And in fact choose ϕ to get $\equiv 1$)

We can consider by dividing both sides of fnl eqn.

$$\frac{\zeta(1-s,\chi,\hat{\phi})}{L(1-s,\chi^{-1})} = \varepsilon(s,\chi,\psi)\frac{\zeta(s,\chi,\phi)}{L(s,\chi)}$$

$$\varepsilon(s,\chi,\psi) = \gamma(s,\chi,\psi)\frac{L(s,\chi)}{L(1-s,\chi^{-1})}$$

Claim: $\varepsilon(s,\chi,\psi)$ is entire, no zeros, (note: γ could have poles)

So $\varepsilon(s,\chi,\psi) = Ae^{Bs}$

Pf: take ϕ s.t. $\frac{\zeta(s,\chi,\phi)}{L(s,\chi)} = 1. \Rightarrow \varepsilon(s,\chi,\psi) = \frac{\zeta(1-s,\chi^{-1},\hat{\phi})}{L(1-s,\chi^{-1})}$ = entire.

Take ϕ s.t. $\frac{\zeta(1-s,\chi^{-1},\hat{\phi})}{L(1-s,\chi^{-1})} = 1.$ then $\varepsilon^{-1}(s,\chi,\psi) = \frac{\zeta(s,\chi,\phi)}{L(s,\chi)}$ is entire.

So ε has **no** zeros

we showed

$\zeta(s,\chi,\phi)$'s and $L(s,\chi)$ are rational fns in q^{-s}. So $\varepsilon(s,\chi,\psi) = Ae^{Bs}$.

<u>lets calculate ε's:</u>

χ unramified, say $\chi = 1$

c = conductor of ψ. $= c(\psi)$. $\psi \to dx$ self dual.

If $\phi = \phi_0 = \mathrm{char}(\mathcal{O}) \Rightarrow \varepsilon(s,\chi,\psi) = \dfrac{\zeta(1-s,\chi^{-1},\hat{\phi})}{L(1-s,\chi^{-1})}$: $\hat{\phi} = \mathrm{vol}(\mathcal{O})\,\mathrm{char}(\mathfrak{p}^{-c})$

$$\zeta(1-s,\chi^{-1},\hat{\phi}) = \mathrm{vol}(\mathcal{O})\int_{\mathfrak{p}^{-c}} |x|^{1-s}\,d^\times x \qquad (\mathrm{Re}\, s<1)$$

$$= \mathrm{vol}(\mathcal{O}) \sum_{n=-c}^{\infty} \int_{\mathrm{ord}\, x = n} q^{-n(1-s)}\,d^\times x$$

$$= \mathrm{vol}(\mathcal{O}) \sum_{n=-c}^{\infty} q^{-n(1-s)} = q^{-c/2}\, q^{c(1-s)}\left(1-q^{-(1-s)}\right)^{-1}$$

$$= q^{c(\frac{1}{2}-c)}\, L(1-s,\chi^{-1})$$

So $\boxed{\varepsilon(s,\chi,\psi) = q^{c(\frac{1}{2}-s)}.}$

Take χ ramified, r = conductor of χ i.e. $1+\mathfrak{p}^r$.

Fix ψ, $c = c(\psi)$.

> Note: we could have determined earlier that
> $\varepsilon(s,\chi,\psi_a) = C\,\varepsilon(s,\chi,\psi)$
> where
> $d_0 x = C\,dx$, self dual measures

$$\phi = \frac{\text{char}(1+\mathfrak{p}^r)}{\text{Vol}(1+\mathfrak{p}^r)} \Rightarrow \zeta(s,\chi,\phi) = 1.$$

$$\Rightarrow \varepsilon(s,\chi,\phi) = \zeta(1-s,\chi^{-1},\hat{\phi}).$$

So
$$\hat{\phi}(x) = \int_{1+\mathfrak{p}^r} \psi(-xy)\,dy \underset{\uparrow}{=} \psi(-x)\int_{\mathfrak{p}^r}\psi(-xv)\,dv \underset{\uparrow}{=} \psi(-x)\cdot\begin{cases}\text{Vol}(\mathfrak{p}^r) & \text{if } \underbrace{\operatorname{ord} x + r}_{\operatorname{ord} v} \geq -c \\ 0 & \text{otherwise}\end{cases}$$

write $y = 1+v$, $v \in \mathfrak{p}^r$ $\qquad$ $v \mapsto \psi(-xv)$ is a char. of $\mathfrak{p}^r$

$$= \psi(-x)\,\text{char}(\mathfrak{p}^{-r-c})(x)\cdot\text{Vol}(\mathfrak{p}^r)$$

$$= \psi(-x)\,\text{char}(\mathfrak{p}^{-r-c})(x)\,q^{-r-c/2}$$

So:

$$\zeta(1-s,\chi^{-1},\hat{\phi}) = q^{-r-c/2}\int_{\mathfrak{p}^{-r-c}}\psi(-x)\,\chi(x)^{-1}|x|^{1-s}\,d^\times x = q^{-r-\frac{c}{2}}\sum_{n=-r-c}^{\infty} q^{-n(1-s)}\cdot\int_{\operatorname{ord} x = n}\psi(-x)\chi(x)^{-1}\,d^\times x.$$

<u>Famous Lemma</u>:

$$\int_{\operatorname{ord} x = n}\psi(x)\,\chi(x)\,d^\times x = \begin{cases} 0 & \text{if } n \neq -r-c \\ \mu\cdot q^{-r-c/2}\,\chi(\varpi)^n \sum_{U/U(r)} \chi(u)\,\psi(\varpi^{-r-c}u). \end{cases}$$

and $d^\times x = \mu \dfrac{dx}{|x|}$

proof:

Suppose $n \neq -r-c$:

let $a \in U(1)$ s.t. $\chi(a) \neq 1$.

say $a = 1+b$ put ax for x, get

$$\int_{\mathrm{ord}\, x = n} \psi(ax)\chi(ax)\, d^\times x = \chi(a) \int \psi(x)\psi(bx)\chi(x)\, d^\times x.$$

$\psi(x(1+b))$
$=$
$\psi(x)\psi(bx)$

Suppose b is s.t. $\mathrm{ord}(b)+n \geq -c$, then $\psi(bx)=1$. $\Rightarrow \int = \chi(a)\int \Rightarrow \int = 0$.

(remember that $\chi(a) \neq 1 \Rightarrow \mathrm{ord}(b) < r$.)

So we require $-c-n \leq \mathrm{ord}\, b < r$

$\Rightarrow$ if $n > -r-c$, we're ok.

Kudla
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χ = ramified, r = conductor,

ψ = add. $\quad c$: conductor. $\qquad L(s,\chi) = 1.$

So

$$Z(1-s, \chi^{-1}, \hat{\phi}) = \varepsilon(s,\chi,\psi) \underbrace{Z(s,\chi,\phi)}_{=1} \qquad \phi = \frac{\mathrm{char}(1+\mathfrak{p}^r)}{\mathrm{Vol}(1+\mathfrak{p}^r)\, d^\times x)}$$

then

$$= \frac{1}{\mathrm{Vol}(1+\mathfrak{p}^r)}\, q^{-r-\frac{c}{2}} \sum_{n=-c-r}^{\infty} \int_{\mathrm{ord}\, x = n} \psi(-x)\chi^{-1}(x)\, d^\times x \; q^{-n(1-s)}$$

Lemma:

$$\int_{\mathrm{ord}\,x=n} \chi(x)\psi(x)\, d^\times x = \begin{cases} 0 & \text{if } n \neq -r-c \\ \mu\cdot q^{-\frac{c}{2}-r}\chi(\varpi)^n \sum_{U/U(r)} \chi(u)\psi(\varpi^{-r-c}u) & \text{if } n = -r-c \end{cases}$$

(where $\frac{\mu\, dx}{|x|} = d^\times x$)

pf:

$a = 1+b \in U(1) \qquad 0 < \mathrm{ord}\, b$

$$\int_{\mathrm{ord}\,x=n} \chi(x)\psi(x)\, d^\times x$$

$\|$ if c ⟵ ⟶ $\begin{cases} \psi(ax) = \psi(x)\psi(bx) \\ \text{If } \mathrm{ord}\, b + n \geq -c \Rightarrow \psi(bx) = 1. \end{cases}$

$\chi(a) \int_{\mathrm{ord}\,x=n} \chi(x)\psi(x)\, d^\times x. \Rightarrow \int = 0$ if $\chi(a) \neq 1$.

So we want $r > \mathrm{ord}\, b \geq -n-c$

$\Rightarrow n > -r-c.$

Now, $\chi(a)=1$ if $\operatorname{ord}(b)\geq r$.

if, $\psi(bx)\neq 1$ then $n+\operatorname{ord} b<-c$. so we'd like $r<-c-n$, so $n<-c-r$.

$\{\operatorname{ord} x=n\}=\varpi^n U$. Write $U=\coprod_i u_i\, U(r)$

so

$x=\varpi^n\cdot u_i(1+b)$, $d^\times x=\mu\cdot\frac{dx}{|x|}$ but $|x|=1$ here in U.

$$\int_{\operatorname{ord} x=n}\chi(x)\psi(x)\,d^\times x = \sum_i\int_{\mathfrak{p}^r}\chi(\varpi^n u_i)\psi(\varpi^n u_i)\psi(\varpi^n u_i b)\,\mu\, db$$

ch. vars.

$$=\sum_i \chi(\varpi^n u_i)\psi(\varpi^n u_i)\int_{\mathfrak{p}^r}\psi(\varpi^n \cancel{u_i} b)\,db. = 0.$$

$n<-c-r \Rightarrow \psi$ non triv. char of $\mathfrak{p}^r$ so

(triv. $\Leftrightarrow n+r\geq -c$)

Now Suppose $n=-r-c$.

ψ is trivial on $\mathfrak{p}^r$ so $\int_{\mathfrak{p}^r}\psi(\varpi^n u_i b)\,db = q^{-r}\operatorname{Vol}(U)$

hence we get:

$$\mu\cdot\chi(\varpi^n)\,q^{-r-\frac{c}{2}}\underbrace{\sum_{u\in U/U(r)}\chi(u)\psi(\varpi^n u)}_{\text{Gauss Sum.}}.\quad\text{as claimed}.\ \boxtimes$$

So now compute ε,

first $\operatorname{Vol}(1+\mathfrak{p}^r, d^\times x) = \mu\cdot\operatorname{Vol}(\mathfrak{p}^r)$.

$\mu\frac{dx}{\cancel{|x|}\to 1}$. $\varepsilon = \ldots$

$F = \mathbb{R}$. $\boxed{F = \mathbb{C} : \text{exercise}}$

$\hookrightarrow \quad \chi(x) = \operatorname{sgn}(x)^a |x|^{s_0} \qquad a = 0, 1$ (even or odd:) $\quad s \in \mathbb{C}$.

$$d^\times x = \frac{dx}{|x|}.$$

$$\psi(x) = e(x)$$

$$\underset{\substack{\| \\ 1}}{\zeta(s, \chi, \phi)} = \int_{\mathbb{R}^\times} \phi(x) |x|^{s-1} dx \quad , \operatorname{Re} s > 0.$$

We'll show $\gamma(s, \chi, \psi)$ has a continuation.

choose $\phi(x) = e^{-\pi x^2}$
$\hat{\phi}(x) = e^{-\pi x^2}$

$$\zeta(s, \chi, \phi) = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} dx = 2\pi^{-s/2} \Gamma(s/2) \quad \text{– simple poles at } 0, -2, -4, -6, \dots$$

So

$$\gamma(s, \chi, \psi) = \frac{\pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}$$ gives meromorphic continuation.

let
$$L(s, \chi) = \begin{cases} \pi^{-s/2} \Gamma(s/2) & a = 0 \quad \leftrightarrow \varepsilon = 1 \\ \pi^{-\left(\frac{s+1}{2}\right)} \Gamma\left(\frac{s+1}{2}\right) & a = 1. \quad \leftrightarrow \varepsilon = i \end{cases}$$

take $s_0 = 0$.

then $$\frac{\zeta(s, \chi, \phi)}{L(s, \chi)}$$ is entire $\forall \phi$

idea:

$$2 \int_0^\infty \phi(x) x^{s-1} dx = \underbrace{2 \int_1^\infty}_{\text{entire}} + 2 \int_0^1$$

Global Theory:

$F = \#$ field, $v \in \Sigma_F$ = set of places, F_v = completion of F wrt. $| \, |_v$.

$$\mathbb{A} = \mathbb{A}_F = \prod_v{}' F_v \ni (x_v)_v \text{ has almost all } x_v \in \mathcal{O}_v.$$

topology: nbhd basis for 0:

$$V \times \prod_v \wp_v^{n_v} \quad \text{almost all } n_v = 0.$$

open in $F_\infty = \prod_{\substack{v \in \Sigma_{F,\infty} \\ \text{arch}}} F_v$, $\mathbb{A}_f = \prod_{\substack{v \in \Sigma_F \\ \text{non arch}}}{}' F_v$

$F \hookrightarrow \mathbb{A}$
$\alpha \longmapsto (\ldots, \alpha, \alpha, \ldots)$

$\mathbb{A}/F$ is cpt.

$F_v \hookrightarrow \mathbb{A}$
$x \longmapsto (0, \ldots, 0, \underset{v}{x}, 0, 0, \ldots)$

$\psi : \mathbb{A}/F \longrightarrow \mathbb{C}^1$ additive character, i.e., $\psi : \mathbb{A} \longrightarrow \mathbb{C}^1$ trivial on F.

$\left(\text{ex: } F = \mathbb{Q},\ \underset{\underset{\times}{\cup}}{\mathbb{A}} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p. \right)$

$$\begin{tikzcd} F_v \arrow[r, hook] \arrow[rr, bend right, "\psi_v"'] & \mathbb{A} \arrow[r, "\psi"] & \mathbb{C}^1 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}_f \arrow[r, hook] \arrow[rr, bend right, "\psi_f"'] & \mathbb{A} \arrow[r, "\psi"] & \mathbb{C}^1 \end{tikzcd}$$

$c_v = c(\psi_v)$ v non arch

ψ_f trivial by continuity on some $\prod_v \wp_v^{-c_v}$

turns out $c_v = 0$ almost all v.

$\alpha \in F$

$F \xrightarrow{\sim} \widehat{\mathbb{A}/F}$ by

$\alpha \longmapsto \psi_\alpha$: $\psi_\alpha(x) = \psi(\alpha x)$ This is an isomorphism.

claim: $\psi(x) = \prod_v \psi_v(x_v)$.

let $x = (\ldots, x_v, \ldots)$

S = finite set s.t. $v \notin S \Rightarrow c_v = 0$, $x_v \in \mathcal{O}_v$.

then write.

$$x = \sum_{v \in S} i_v(x_v) + \underset{\substack{\| \\ (\underbrace{\ldots,0,\ldots,0}_{S},\, x_v, \ldots)}}{x^S}$$

So $$\psi(x) = \prod_{v \in S} \psi_v(x_v) \cdot \underbrace{\psi(x^S)}_{=1}.$$

ex: $\mathbb{A}_{\mathbb{Q}}$.

$\psi_\infty(x) = e(-x)$.

$\psi_p(x) = e(\lambda_p(x))$

$$\psi(x) = \psi_\infty(x_\infty) \prod_p \psi_p(x_p)$$ with $x = (x_\infty, x_2, x_3, \ldots)$

$\psi(a) = 1 \quad a \in \mathbb{Q}$.

$\cancel{\psi}\ c_p(\psi) = 0 \quad \forall p$.

$$\begin{array}{ccc} \mathrm{tr}: \mathbb{A}_F & \longrightarrow & \mathbb{A}_{\mathbb{Q}} \\ \cup\!\!| & & \\ F & \longrightarrow & \mathbb{Q} \end{array} \qquad \psi(x) = \psi_0(\mathrm{tr}(x))$$

$\mathbb{A}^\times = \prod_v{}' F_v^\times \ni (x_v)_v$ has $x_v \in U_v$ almost all v.

top nbhd basis of 1.

$V \times \prod_{v \text{ fin.}} U_v(r_v)$ almost all $r_v = 0$. recall $U_v(0) = U_v = \mathcal{O}_v^\times$.

$F_\infty^\times$:

$F^\times \hookrightarrow \mathbb{A}^\times$.
discrete

Def. $|x| = \prod_v |x_v|_v$. $|\alpha|_{\mathbb{A}} = 1 \quad \alpha \in F^\times$.

$$\begin{array}{l} \mathbb{A}^1 \longrightarrow \mathbb{A}^\times \xrightarrow{|\,|_{\mathbb{A}}} \mathbb{R}_+^\times \\ \cup \\ F^\times \end{array}$$

$i_v : F_v^\times \hookrightarrow \mathbb{A}^\times$

$\mathbb{A}^1 / F^\times$ is cpt.

Hecke (quasi) characters:

$\chi : \mathbb{A}^\times / F^\times \longrightarrow \mathbb{C}^\times$.

$\chi(x)|x|^s$ gives any char so restrict to unitary char.

So χ = char.

[Have a notion of conductor of χ.]

Def $\chi_v = \chi \circ i_v$ char of $F^\times$.

Lemma: $\chi(x) = \prod_v \chi_v(x_v)$

Pf exercise.

Ex

Kudla
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F # Field, $\mathbb{A} = \mathbb{A}_F$, $\mathbb{A}^\times = \mathbb{A}_F^\times$ = idèles.

Hecke Character:

$\chi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ continuous character.

χ trivial on some $\prod_v U_v(r_v)$

$\chi = \prod_v \chi_v$

eg's:

$|\cdot|_{\mathbb{A}} : \mathbb{A}^\times \to \mathbb{R}_+^\times$. trivial on $\prod_v U_v$.

$\prod_v |\cdot|_v$

eg:

$$\begin{tikzcd} \mathbb{A}^\times/F^\times F_\infty^\times \prod_v U_v \arrow[r, "\sim"] \arrow[rr, bend left] & Cl(F) \arrow[r, "\chi_0"] & \mathbb{C}^\times \end{tikzcd}$$

$Cl(F)$ = ideal class group of F.

$$x \longmapsto \left[\prod_{v \text{ finite}} \mathfrak{p}_v^{\mathrm{ord}_v(x_v)} \right].$$

eg: $F = \mathbb{Q}$.

$\mathbb{Q}^\times = (\alpha, \alpha, \alpha, \dots)$
$\mathbb{R}_+^\times = (t, 1, 1, \dots)$

$\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_+^\times \prod_p \mathbb{Z}_p^\times$. direct product.

$|\cdot|_{\mathbb{A}} \downarrow$
$\mathbb{R}_+^\times$

Can write $\chi = \chi_0 |\cdot|^s$ where χ_0 is trivial on $\mathbb{R}_+^\times \mathbb{Q}^\times$.

and on $\prod_p U_p(r_p)$ (conductor) $= U(\underline{r})$

$1 + p^{r_p}\mathbb{Z}_p$ $r_p > 0$
$\mathbb{Z}_p^\times$ $r_p = 0$.

Hecke chars of finite order $\overset{\sim}{\hookrightarrow}$ pr. Dirichlet characters

$$\begin{tikzcd} \mathbb{A}^\times/U(\underline{r}) \arrow[d, "\chi_0"] \arrow[r, phantom, "\simeq"] & \prod_{p,\, r_p>0} \mathbb{Z}_p^\times/(1+p^{r_p}\mathbb{Z}_p) \simeq \prod_p (\mathbb{Z}/p^{r_p}\mathbb{Z}_p)^\times \simeq (\mathbb{Z}/N\mathbb{Z})^\times \arrow[dl, bend left] \\ \mathbb{C}^\times & \end{tikzcd}$$

$N = \prod_p p^{r_p}$

Let $F = \mathbb{Q}(\sqrt{d})$. $\square \neq d > 0$. class # $= 1$.

$\mathbb{A}^\times = F^\times \cdot F_\infty^\times \cdot U$ (not direct product), let $\chi = \prod_v \chi_v$, $U(\mathfrak{f})$ = conductor.

$\mathbb{A}^\times / F^\times U(\mathfrak{f})$ $\Big\}$ $\quad \mathbb{A}^\times / F^\times \simeq F_\infty^\times \cdot U / F^\times \cap (F_\infty^\times \cdot U) = F_\infty^\times U / \mathcal{O}^\times$.

$F_\infty^\times \cdot U / U(\mathfrak{f})$ $\qquad$ Consider $\mathcal{O}^\times \cap F_\infty^\times \cdot U(\mathfrak{f})$.

$\chi_\infty \cdot \chi_f$.

$\uparrow$ need trivial on $\mathcal{O}^\times$.

Exercise: Find all Hecke characters of all quad. extensions $F/\mathbb{Q}$.

Global Tate: $\Big($ lit: Need $S(\mathbb{A})$, def $S(\mathbb{A}) = S(F_\infty) \otimes S(\mathbb{A}_F)$

Choose χ = Hecke char.

$S(\mathbb{A}_F) = \bigotimes_v' S(F_v)$

$\phi \in S(\mathbb{A})$.

$=$ finite linear comb's of $\phi = \prod_v \phi_v$ where almost all ϕ_v are char$(\mathcal{O}_v)$.

$$\zeta(s, \chi, \phi) = \int_{\mathbb{A}^\times} \phi(x)\chi(x)|x|^s d^\times x \qquad d^\times x = \prod_v d^\times x_v.$$

$\mathrm{Vol}(U_v) = 1$

suppose $\phi = \prod_v \phi_v$ (cheating a bit, there are factorizable; would need to look at closures of Frechet spaces for arch. places)

rewrite the integral:

$$= \prod_v \int_{F_v^\times} \phi_v(x_v)\,\chi_v(x_v)\,|x_v|_v^s\, d^\times x_v$$

more carefully:

$$\int_{\mathbb{A}^\times} \phi(x)\chi(x)|x|^s d^\times x = \lim_S \int_{\mathbb{A}_S^\times} \phi(x)\chi(x)|x|^s d^\times x.$$

$\mathbb{A}_S^\times = \prod_{v \in S} F^\times \times \underbrace{\prod_{v \notin S} U_v}_{U^S}$

(2)

But:

$$\int_{\mathbb{A}_S^\times} = \int_{\prod_{v\in S} F_v^\times} \phi(x)\chi(x)|x|^s d^\times x \underbrace{\int_{U^S}\left(\prod_{v\notin S}\phi_v(x)\chi_v(x)|x|_v^s\right)d^\times x}_{1.}$$

$\Leftarrow$ S large enough $\Rightarrow$ $v\notin S \Rightarrow \phi_v = \mathrm{char}(\mathcal{O}_v)$
χ_v trivial on U_v

so really have a finite integral here.

Now Need

$$\lim_S \prod_{v\in S}\underbrace{\int_{F_v^\times}\phi_v(x_v)\chi_v(x_v)|x_v|_v^s d^\times x_v}_{(1-\chi_v(\varpi_v)q_v^{-s})^{-1}} \quad \text{exists.}$$

So $\prod_v (1-q_v^{-\sigma})^{-1}$ ok for $\mathrm{Re}(s)>1$
since its $\sim \zeta_F(s)$.

So $\zeta(s,\chi,\phi) = \prod_v \zeta_v(s,\chi_v,\phi_v)$ for $\mathrm{Re}(s)>1$

Write:

$$\mathbb{A}^\times = \underset{\text{abs. val} >1}{\mathbb{A}^\times_{>1}} \cup \underset{\text{abs val} <1}{\mathbb{A}^\times_{<1}}$$

then $\zeta(s,\chi,\phi) = \underset{?}{\zeta_0(s,\chi,\phi)} + \overbrace{\zeta_1(s,\chi,\phi)}^{\text{entire.}}$?

Note:

$$\int_{\mathbb{A}^\times}\phi(x)\chi(x)|x|^s d^\times x = \int_{\mathbb{A}^\times/F^\times}\left(\sum_{\alpha\in F^\times}\phi(\alpha x)\right)\chi(x)|x|^s d^\times x.$$

$$= \int_0^\infty \left(\int_{\mathbb{A}^1/F^\times}\right) t^{s-1}\,dt.$$

1st arch. place
$t = (t,1,1,\ldots)$

$\boxed{\mathbb{A}^1/F^\times = \text{cpt.}}$

$$\zeta_0(s,\chi,\phi) = \int_{\substack{\mathbb{A}^\times/F^\times \\ |x|<1}} \left(\sum_{\alpha\in F^\times} \phi(\alpha x)\right) \chi(x)\,|x|^s\, d^\times x.$$

$$\left(= \sum_{\alpha\in F} \phi(\alpha x) - \phi(0) \right)$$

$$= \phi(0) \int_{\substack{\mathbb{A}^\times/F^\times \\ |x|<1}} \chi(x)|x|^s d^\times x + \int_{\substack{\mathbb{A}^\times/F^\times \\ |x|<1}} \left(\sum_{\alpha\in F} \phi(\alpha x)\right) \chi(x)\,|x|^s\, d^\times x.$$

$$\int_0^1 \left(\int_{\mathbb{A}^1/F^\times} \chi(x)\, d^\times x \right) \chi(t)\, t^{s-1}\, dt \qquad |\cdot| = 1.$$

$$\begin{cases} 0 & \text{if } \chi|_{\mathbb{A}^1} \neq 1 \\ \mathrm{Vol}(\mathbb{A}^1/F^\times) & \text{if } \chi|_{\mathbb{A}^1} = 1 \end{cases}$$

Poisson Summation:

Four. trans. wrt self dual dx on $\mathbb{A}$ for ψ.

$$\phi \in S(\mathbb{A}), \quad \sum_{\alpha\in F} \phi(\alpha x) = |x|^{-1} \sum_{\alpha\in F} \hat{\phi}\left(\frac{\alpha}{x}\right)$$

We get:

using $\mathbb{A} \longrightarrow \hat{\mathbb{A}}$, $a \longmapsto$

$$\int_{\substack{\mathbb{A}^\times/F^\times \\ |x|<1}} \sum_{\alpha\in F} \hat{\phi}(\alpha x^{-1}) \cdot \chi(x)\,|x|^{s-1}\, d^\times x$$

$$\parallel$$

$$\int \left(\sum_{\alpha\in F} \hat{\phi}(\alpha x^{-1}) - \hat{\phi}(0)\right) * + \hat{\phi}(0) \int \chi(x)\,|x|^{s-1}\, d^\times x$$

just like other piece

$x \to x^{-1}$

$$\int_{|x|>1} \sum_{\alpha\in F^\times} \hat{\phi}(\alpha x)\, \chi^{-1}(x)\,|x|^{1-s}\, d^\times x$$

$$\parallel$$

$$\zeta_1(1-s, \chi^{-1}, \hat{\phi})$$

$\exists$ mero. anal cont. and Global functional eqn:

$$\zeta(s,\chi,\phi) = \zeta_1(s,\chi,\phi) + \frac{\mathrm{Vol}(\,)\,\phi(0)}{s} + \frac{\mathrm{vol}(\,)\,\hat{\phi}(0)}{1-s} + \zeta_1(1-s,\chi_1^{-1},\hat{\phi}).$$

$$= \zeta(1-s, \chi^{-1}, \phi)$$

just write it out.

just write it out.

For $Re(s) > 1$.

take S large enough

$$\zeta(s, \chi, \phi) = \prod_v \zeta_v(s, \chi_v, \phi_v) = \left(\prod_{v \in S} \frac{\zeta_v(s, \chi_v, \phi_v)}{L(s, \chi_v)} \right) \cdot \underbrace{\prod_v L(s, \chi_v)}_{v \notin S \Rightarrow \zeta(s,\chi,\phi_v) = L(s,\chi_v)}$$

$L(s, \chi)$. $\overset{!!}{=} \leftarrow Re\, s > 1$

So $L(s, \chi)$ has anal. cont.

Now apply Global funct eqn:

$$\zeta(s, \chi, \phi)$$
$$\|$$
$$\zeta(1-s, \chi^{-1}, \hat{\phi}) = \underbrace{\left(\prod_{v \in S} \frac{\zeta_v(1-s, \chi^{-1}_v, \hat{\phi}_v)}{L(s, \chi_v^{-1})} \right)}_{\downarrow \text{ Local funct. eqn}} \cdot L(1-s, \chi^{-1})$$

$$\varepsilon_\phi(s, \chi_v, \psi_v) \frac{\zeta_v(s, \chi_v, \phi_v)}{L(s, \chi_v)}$$

We get:

$$L(s, \chi) = \underbrace{\left(\prod_v \varepsilon(s, \chi_v, \psi_v) \right)}_{\varepsilon(s, \chi)} \cdot L(1-s, \chi^{-1}).$$

with $\left(\prod_v \varepsilon(s, \chi_v, \psi_v) \right) \overset{!!}{=} \varepsilon(s, \chi)$

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IDEA:

$GL(1) \longrightarrow GL(2)$

Tate. Jacquet Langlands.

$F = \#$ field.

$\chi =$ Hecke char. $= \bigotimes_v \chi_v$

$\mathbb{A}^\times / F^\times = G(F) \backslash G(\mathbb{A})$ $G = GL(1)$

$L(s, \chi)$

$G = GL(2)$.

$\pi =$ irred. automorphic repn. of $G(\mathbb{A}) = GL_2(\mathbb{A})$

$\simeq \bigotimes_v \pi_v$

somehow related to $G(F) \backslash G(\mathbb{A})$ but this isn't even a group...

π_v irred repns of $G(F_v) = GL_2(F_v)$.

$L(s, \pi)$ $L(s, \pi_v)$

ε ε_v

F: non arch. local

$G = GL_2(F)$

$\cup$

$K = GL_2(\mathcal{O}) = \{ g \in M_2(\mathcal{O}) \mid \det g \in \mathcal{O}^\times \}$ is cpt open. subgroup.

Consider $1_2 + \varpi^r M_2(\mathcal{O})$. $r > 0$ gives a nbhd basis for 1 in G.

$\parallel$

$K(r)$ = cpt and open subgroups. all normal in K.

$1 \to K(r) \to K \to GL_2(\mathcal{O}/p^r) \to 1$.

following Bernstein-Zelevinsky:

An <u>l-space</u> X is a Hausdorff top. space where every pt. has a nbhd basis of cpt open sets.

ex: F, $M_2(F)$

An <u>l-group</u> G is a top. group with nbhd basis of e consisting of cpt open subgroups.

ex: $F^\times$, $GL_n(F)$

top. facts:

1. $C \subset X$ a cpt subset

Any open cover has a refinement consisting of a finite number of disjoint cpt. open subsets.

2. $G \supseteq H$ closed subgroup $\Rightarrow$ $H\backslash G$ is an l-space with quotient-top.

and $G \to H\backslash G$ is open and continuous.

3. $G \times X \to X$ continuous (G countable at infinity)

with a finite # of orbits $\Rightarrow$ $\exists$ an open orbit X_0

and $\begin{array}{l} G \to X_0 \\ g \mapsto g \cdot x_0 \end{array}$ is open

4. G transitive on X, $H = \text{Stab}$ of $x_0 \in X$ then

$\begin{array}{l} H\backslash G \xrightarrow{\sim} X \\ g \mapsto g^{-1} x_0 \end{array}$ is a homeomorphism

Distributions:

X $l$space. $S(X)$ = locally const. cpt. support fcns ($\mathbb{C}$-valued) on X.

$S^*(X) = \text{Hom}(S(X), \mathbb{C})$ linear functionals on $S(X)$.

= distributions on X.

$\underset{\text{open}}{Y} \subset X \supset \underset{\text{closed}}{Z} = X - Y$

get:

$S(X) \xrightarrow[\text{restriction}]{} S(Z)$

$S(Y) \xrightarrow[\text{extend by } 0]{} S(Z)$

Proposition:

$0 \to S(Y) \xrightarrow{\text{ext}} S(X) \xrightarrow{\text{res}} S(Z) \xrightarrow{\checkmark} 0$

is exact.

Corollary:

$0 \xrightarrow{\checkmark} S^*(Z) \to S^*(X) \to S^*(Y) \to 0$

is exact.

$T \in S^*(X)$ then $\mathrm{Supp}(T) = \{ x \in X \mid$ for all $\underset{x}{\cup} U \subset X$ open $\mathrm{res}_U(T) \neq 0 \}$.

(ie. $x \notin \mathrm{supp}\, T \iff \exists$ open $U \ni x$ s.t. $\mathrm{res}_U(T) = 0$.).

note: $\mathrm{Supp}\, T$ = closed.

ex: $x_0 \in X$ $\delta_{x_0} \in S^*(X)$ $\mathrm{supp}\, \delta_{x_0} = \{x_0\}$

Defn: T is called finite if $\mathrm{supp}\, T$ is cpt.

$S^*_c(X)$ = finite distributions $\leq S^*(X)$.

T a dist, f a fcn write $\langle T, f \rangle$ for evaluation.

$G \times X \longrightarrow X$ $\Rightarrow$ action of G on $S(X)$ and on $S^*(X)$.

take $X = G$

Def $L: G \times X \longrightarrow X$
$g, x \longmapsto gx$.

$R: G \times X \longrightarrow X$
$g, x \longrightarrow xg^{-1}$

get two actions of G on $\underset{f}{S(G)}$, $S^*(G)$

$$(L_g f)(x) = f(g^{-1}x)$$

$$\langle L_g T, f \rangle = \langle T, L_{g^{-1}}(f) \rangle.$$

Prop: (Left Haar Measure)

Up to scalar, $\exists!$ distribution $\mu_G \in S^*(G)$ s.t. $L_g \mu_G = \mu_G \ \forall g \in G$.

normalize μ_G to be positive (ie. $f \geq 0 \Rightarrow \langle \mu_G, f \rangle \geq 0$).

[resp. get ν_G for right Haar Measure]

Pf: Take $K_\alpha \subseteq G$ nbhd basis of c.o. subgroups.

Can assume $\exists\, \alpha_0$ s.t. all $K_\alpha \subseteq K_{\alpha_0}$

let $S_\alpha = \{ f \in S(G) \mid R_g f = f \ \forall g \in K_\alpha \}$

$$S(G) = \bigcup_\alpha S_\alpha.$$

Suppose $f \in S_\alpha$. Claim: $f = \sum'_{g \in G/K_\alpha} f(g) \cdot \mathrm{char}(gK_\alpha)$

Define $\mu_\alpha(f) = \mu_\alpha(K_\alpha) \sum_{g \in G/K_\alpha} f(g)$

take $|K_0 : K_\alpha|^{-1}$

($\Rightarrow \mu_\alpha$ unique up to scalar).

$$\begin{array}{ccc} S_\beta & \xrightarrow{\mu_\beta} & \\ \cup & & \text{consistent!} \\ S_\alpha & \xrightarrow[\mu_\alpha]{} & \end{array}$$

■

$R_g \mu_G = \Delta(g) \mu_G$

So $\Delta : G \to \mathbb{R}^\times_+$ a continuous homomorphism (cont. since $g \in K_\alpha \Rightarrow \Delta(g) = 1$)

G = unimodular $\Leftrightarrow \Delta_G \equiv 1$.

$\Delta_G^{-1} \mu_G = \nu_G$.

Exercise: $\check{\mu}_G = \Delta_G^{-1} \cdot \mu_G$ $\quad \begin{array}{l} G \to G \\ g \mapsto g^{-1}. \end{array}$

$H \subseteq G$ closed.

$X = H\backslash G \qquad R: G \times X \to X.$

let $\Delta = \Delta_G / \Delta_H$ = char. of H.

let $S(G;\Delta) = \left\{ f \in C^\infty(G) \;\middle|\; \begin{array}{l} (i) \; f(hg) = \Delta(h) f(g). \\ (ii) \; \exists\, C = \text{cpt in } G \text{ s.t. } supp(f) \subseteq H \cdot C \end{array} \right\}.$

Thm: up to scalar $\exists !$ functional $\nu_{H\backslash G} : S(G,\Delta) \to \mathbb{C}$

s.t. $R_g \nu_{H\backslash G} = \nu_{H\backslash G}$ can be taken to be positive.

pf:

Def $pr: S(G) \to S(G,\Delta)$
ϕ

$$pr(\phi)(g) = \int_H \phi(hg)\, \Delta_G(h)^{-1} d\mu_H(h). = \langle res_H(R_g f), \Delta_G^{-1} \mu_H \rangle.$$

check: prop. (i) $\quad pr(\phi)(h_0 g) = \int_H \phi(h h_0 g)\, \Delta_G(h)^{-1} d\mu_H(h)$

$$= \int_H \phi(hg)\, \Delta_G(h h_0^{-1})^{-1} d\mu_H(h h_0^{-1})$$

$$= \underbrace{\Delta_G(h_0)\, \Delta_H(h_0^{-1})}_{\Delta(h)} \; pr(\phi)(g). \quad ✓$$

ii) $Supp(pr(\phi)) \subseteq H \cdot Supp(\phi)$. ✓

Lemma: 1. pr is surjective

2. If $pr(\phi) = 0$, then $\langle \nu_G, \phi \rangle = 0$.

Pf: next time.

Using this:

$$\begin{tikzcd} S(G) \arrow[rr] \arrow[dr] & & S(G, \Delta). \\ & S(G)/\ker(pr) \arrow[ur, "\cong"] & \end{tikzcd}$$

write $f = pr(\phi)$.

$$\langle \nu_{H\backslash G}, f \rangle = \langle \nu_G, \phi \rangle \quad \text{where} \quad \phi \mapsto f.$$

this is well defined

$$\int_{H\backslash G} f(x)\, d\nu_{H\backslash G}(x) = \int_G \phi(g)\, d\nu_G(g) \quad \text{where } pr(\phi) = f.$$

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$H \subseteq G \quad \Delta = \Delta_G/\Delta_H : H \to \mathbb{R}_+^\times$

$S(G,\Delta) = \{ f : G \to \mathbb{C} \mid f(hg) = \Delta(h) f(g)$ and $\exists$ C cpt in G $\operatorname{supp} f \subseteq H\cdot C$

$\exists!\ \nu_{H\backslash G} : S(G,\Delta) \to \mathbb{C}$ s.t. $R_g \nu_{H\backslash G} = \nu_{H\backslash G}$.

$pr : S(G) \longrightarrow S(G,\Delta)$

$$pr(\phi)(g) = \int_H \phi(hg)\, \Delta_G(h)^{-1}\, d\mu_H(h)$$

lemma: (i) pr is surjective

(ii) if $pr(\phi) = 0 \Rightarrow \langle \nu_G, \phi \rangle = 0$.

Pf:

$K \subseteq G$ c.o. subgp.

$S(G)^K_g$ (fcns rt. invariant by K) $\phi \in S(G)$ s.t. $\operatorname{supp}(f) \subseteq HgK$ + rt. K inv.

$$\begin{tikzcd} S(G)^K_g \arrow[d, "pr"'] \arrow[dr, dashed] & \\ S(G,\Delta)^K_g \arrow[r, "\sim"] & \mathbb{C} \end{tikzcd}$$

$f \longmapsto f(g)$ then $f(hgk) = \Delta(h) f(g)$.

enough to show the resulting $S(G)^K_g \dashrightarrow \mathbb{C}$ is surjective.

let $\phi = \operatorname{char}(gK)$

$$pr(\phi)(g) = \int_H \phi(hg)\, \Delta_G(h)^{-1}\, d\mu_H(h)$$

$\phi(hg) = 1 \Leftrightarrow hg \in gK \Leftrightarrow h \in gKg^{-1} \cap H$

$$= \int_{gKg^{-1}\cap H} \overbrace{\Delta_G(h)^{-1}}^{=1}\, d\mu_H(h) \neq 0$$

$\phi \in S(G)_g^K$ is determined by values on HgK/K

$\operatorname{supp}\phi = \bigcup_i h_i g K$

~~[illegible]~~ consider $\langle \nu_G , L_{h_0}\cdot \phi \rangle$

$$= \int_G \phi(h_0^{-1} g)\, d\nu_G(g) = \Delta_G(h_0)^{-1} \langle \nu_G , \phi \rangle . \qquad \left(d\nu_G = \Delta_G^{-1} d\mu_G \right).$$

But

$$\operatorname{pr}(L_{h_0}\phi)(g) = \Delta_G(h_0)^{-1} \operatorname{pr}(\phi)(g)$$

If T = any lin. fcnl. transforming by Δ_G^{-1} as above then

if $\phi = \sum_i \phi(h_i g) \underbrace{\text{char}(h_i g K)}_{L_{h_i}\,\text{char}(gK)}$

then

$$T(\phi) = \sum_i \phi(h_i g)\, \Delta_G(h_i)^{-1}\, T(\text{char}(gK)).$$ this is unique up to a scalar, so the dim. of space of such functionals is one

ϕ $\mathbb{R}_+$ valued $\Rightarrow$ $T(\phi) \geq 0$.

So rt. haar measure and pr are proportional.

G

dirac delta of g.

$S(G) \qquad S^*(G) \supseteq S_c^*(G) \ni \varepsilon_g$

X_1, X_2 - ℓ spaces , $S(X_1 \times X_2) \simeq S(X_1) \otimes S(X_2)$

$\phi_1 \cdot \phi_2 \longleftarrow \phi_1 \otimes \phi_2$

So $S^*(X_1 \times X_2) \simeq S^*(X_1) \otimes S^*(X_2)$. $\Rightarrow \operatorname{Supp}(T_1 \otimes T_2) \subseteq \operatorname{Supp} T_1 \times \operatorname{Supp} T_2$

$m: G \times G \to G$

really $C^\infty(G\times G)$

$S(G \times G) \xleftarrow{m^*} S(G)$

$\cong$

$S(G) \otimes S(G)$

$S^*(G) \otimes S^*(G) \to S^*(G)$

$T_1, T_2 \longmapsto T_1 * T_2$. convolution product

$\langle T_1 * T_2, f \rangle = \langle T_1 \otimes T_2, m^*(f) \rangle \qquad m^*(f)(g_1, g_2) = f(g_1 g_2)$.

$\operatorname{Supp}(T_1 * T_2) \subset m(\operatorname{supp} T_1 \times \operatorname{supp} T_2)$

all this makes $S_c^*(G)$ an associative algebra.

$\varepsilon_e, \varepsilon_g$ for $g \in G$ are in $S_c^*(G)$

unit.

$T \in S_c^*(G) \qquad \varepsilon_g * T = L_g T$

$T * \varepsilon_g = R_g T$

$\varepsilon_{g_1} * \varepsilon_{g_2} = \varepsilon_{g_1 g_2}$

$\mathbb{C}[G] \hookrightarrow S_c^*(G)$

$g \longmapsto \varepsilon_g$.

Say $\Gamma \subseteq G$ cpt subgroup

μ_Γ = Haar meas on Γ norm

s.t. $\langle \mu_\Gamma, 1 \rangle = 1$.

$$\Gamma \overset{i_\Gamma}{\hookrightarrow} G$$

$$i_\Gamma^* : S(G) \longrightarrow S(\Gamma)$$

$$(i_\Gamma)_* : S^*(\Gamma) \longrightarrow S^*(G).$$
$$\mu_\Gamma \in S^*(\Gamma)$$

$\varepsilon_\Gamma \in S_c^*(G)$

$= (i_\Gamma)_*(\mu_\Gamma)$

$\phi \mapsto \int_\Gamma \phi(\gamma)\, d\mu_\Gamma(\gamma)$.

facts:

1. $g \in \Gamma$ $\quad \varepsilon_g * \varepsilon_\Gamma = \varepsilon_\Gamma = \varepsilon_\Gamma * \varepsilon_g$

$\Gamma_1, \Gamma_2 \subseteq \Gamma$ closed s.t. $\Gamma = \Gamma_1 \cdot \Gamma_2$. then $\quad \varepsilon_\Gamma = \varepsilon_{\Gamma_1} * \varepsilon_{\Gamma_2} = \varepsilon_{\Gamma_2} * \varepsilon_{\Gamma_1}$

$\Rightarrow \varepsilon_\Gamma^2 = \varepsilon_\Gamma$ idempotent

Lemma: Given Γ_1, Γ_2 closed cpt subgroups of G. fix $g \in G$.

$\exists!\ T \in S_c^*(G)$ s.t.

(1) supp $T \subseteq \Gamma_1 g \Gamma_2$

(2) $L_{g_1} T = T = R_{g_2} T$. $\quad g_1 \in \Gamma_1,\ g_2 \in \Gamma_2$.

(3) $\langle T, 1 \rangle = 1$.

Pf:

let $X = \Gamma_1 g \Gamma_2$

$\Gamma = \Gamma_1 \times \Gamma_2$ acts trans. on X so $X \cong \Gamma / \Gamma_0$ — stab of a pt.

Then $\exists!$ Γ inv. functional ν on $S(X)$ by prev. prop., norm s.t. $\langle \nu, 1 \rangle = 1$

Now: $S^*(\Gamma_1 g \Gamma_2) \longrightarrow S_c^*(G)$ that does it.

Def: $T \in S^*(G)$

is left locally constant $\iff \exists$ c.o. subgroup $K \leq G$ s.t. $\varepsilon_K * T = T$.

$(\text{i.e. } L_g T = T,\ \forall g \in K)$

Note:

locally constant $\not\Rightarrow$ left locally constant.

ex: give an example.

Prop: Say $T \in S^*(G)$ left locally const.

then $\exists\, f \in C^\infty(G)$ s.t. $T = f \cdot \mu_G$.

f left loc. const.

$$S(G) \hookrightarrow S_c^*(G)$$
$$f \longmapsto f \cdot \mu_G$$

depends on choice of Haar measure.

Pf:

Suppose T is left loc const for K.

then

L_{ε_K} is proj to left K-inv. fun.

for $\phi \in S(G)$ $\quad \langle T, \phi \rangle = \langle T, L_g \phi \rangle$

Def: $(L_{\varepsilon_K} \phi)(g) \quad \int_K \phi(k^{-1} g)\, d\mu_K(k)$

$$= \langle T, P_K(\phi) \rangle$$

Consider G/K

$P_K(\phi) = \sum (P_K \phi)(g_i)\, \mathrm{char}(g_i K)$

$$\langle T, P_K(\phi) \rangle = \sum_i (P_K \phi)(g_i) \underbrace{\langle T, \mathrm{char}(g_i K) \rangle}_{f(g_i)}$$

$$= \int_G \phi(g) f(g)\, d\mu_G(g)$$

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DATE______ CLASSROOM INSTR.________________ Sec.#____ GRADING

G = l group.

$S^*(G) \supseteq S_c^*(G) \supset \mathcal{H}(G)$

$S_c^*(G)$: algebra under $*$

$\mathcal{H}(G)$: locally constant

contains: ε_e = identity, ε_g

so $\supseteq \mathbb{C}[G]$.

Fact: 1. $\mathcal{H}(G)$ = 2-sided ideal in $S_c^*(G)$.
2. $S(G) \simeq \mathcal{H}(G)$
$f \longmapsto f \cdot \mu_G$
get usual formula for convolution
$(f * g)\mu_G$

Reps of G:

(π, V) V = $\mathbb{C}$-vector space

$\pi : G \longrightarrow Aut(V)$

smooth: for every $v \in V$ $\{g \in G \mid \pi(g)v = v\}$ = open in G.

π admissible iff smooth and for every cpt open subgp $K \subseteq G$ $V^K = \{v \in V \mid \pi(k)v = v\}$ is finite dimensional.

---

ex: $G = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ an l-group.

Consider $\bar{\mathbb{Q}}$ a v.s. / $\mathbb{Q}$.

Check: repn of G on $\bar{\mathbb{Q}}$ is admissible.

---

let $\mathcal{A}(G)$ = category of smooth reps of G

eg: $S(G)$ with right or left translation action are smooth reps of G, not admissible if $|G/K| = \infty$.

eg: $(R, C^\infty(G))$ not smooth

$g_i K_i$ as $K_i \downarrow e$

choose g_i s.t. $g_i K_i$'s disjoint

$f = \sum_i \text{char}(g_i K_i)$ no cpt open s.t. f is rt. inv.

ie. Stab(f) not open

Given (π, V) smooth, get a repn of $S_c^*(G)$.

take $v \in V$, let $g \overset{f_v}{\longmapsto} \pi(g)v. \quad \in C^\infty(G, V)$

given $T \in S_c^*(G)$, $\quad \langle T, f_v \rangle \in V$. so define $\pi(T)v := \langle T, f_v \rangle = \int_G \pi(g) v \, dT(g)$

eg's $T = \varepsilon_{g_0} \Rightarrow \pi(T) = \pi(g_0)$

$f \in S(G) \simeq \mathcal{H}(G) \qquad \pi(f)v = \int_G \pi(g)v \cdot f(g) \, d\mu_G(g)$

eg: $\Gamma \leq G$ cpt. subgroup.

$\varepsilon_\Gamma \in S_c^*(G). \qquad \varepsilon_\Gamma^2 = \varepsilon_\Gamma.$

$\pi(\varepsilon_\Gamma)V = V^\Gamma$ since $\varepsilon_g * \varepsilon_\Gamma = \varepsilon_\Gamma \quad \forall g \in \Gamma$.

Corollary: If $0 \to V_1 \to V_2 \xrightarrow{p} V_3 \to 0$ exact seq. of smooth repns

then $0 \to V_1^\Gamma \to V_2^\Gamma \xrightarrow{p} V_3^\Gamma \to 0$ is exact.

pf: $v_3 \in V_3^\Gamma$. Know $v_3 = p(v_2)$ some $v_2 \in V_2$.

But $\pi_3(\varepsilon_\Gamma)v_3 = v_3$.

$$\begin{aligned} &\| \\ &\pi_3(\varepsilon_\Gamma)p(v_2) \underset{\text{by setup}}{=} p(\underbrace{\pi_2(\varepsilon_\Gamma)v_2}_{\in V_2^\Gamma}) \checkmark. \end{aligned}$$

Cor: $0 \to \ker(\pi(\varepsilon_\Gamma)) \to V \to V^\Gamma \to 0$

let $V(\Gamma) = \operatorname{span}\{\pi(\gamma)v - v \mid v \in V, \gamma \in \Gamma\}$

then $V(\Gamma) = \ker(\pi(\varepsilon_\Gamma))$

i.e. $V^\Gamma \simeq V/V(\Gamma)$

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Pf:

$$0 \to V(\Gamma) \to V \to V/V(\Gamma) \to 0.$$

apply $\mathcal{E}_\Gamma$ $\downarrow \pi(\mathcal{E}_\Gamma)$

$\bar\pi(\mathcal{E}_\Gamma) \equiv 1$ here.

$$0 \to V(\Gamma)^\Gamma \to V^\Gamma \to (V/V(\Gamma))^\Gamma \to 0$$

$V(\Gamma)^\Gamma = 0.$

$\pi(\mathcal{E}_\Gamma)(\pi(\gamma)v - v)$

$\pi(\underbrace{\mathcal{E}_\Gamma * \mathcal{E}_\gamma}_{e_\Gamma})v - \pi(\mathcal{E}_\Gamma)v$

clearly: $\bar\pi(\gamma) = 1$ for $\gamma \in \Gamma$.

$\bar\pi(\gamma)\bar v = \overline{(\pi(\gamma)v - v + v)} = \bar v$.

$$\bar\pi(\mathcal{E}_\Gamma)\bar v = \overline{\int_\Gamma \pi(\gamma)v\, d\mu_\Gamma(\gamma)\, \tfrac{1}{\mathrm{Vol}(\Gamma)}}$$

let $\Gamma_0 = \mathrm{Stab}$ of $v \in \Gamma$ <u>open</u>.

Γ/Γ_0 finite so $\Gamma = \coprod \gamma_i \Gamma_0$.

$$= \frac{\mathrm{Vol}(\Gamma_0)}{\mathrm{Vol}(\Gamma)} \overline{\sum_i \pi(\gamma_i)v} = \bar v.$$

eg: (π, V) = smooth repn of G

get a repn of $S_c^*(G)$, and a module (rep) for $\mathcal{H}(G)$

Smooth $\Rightarrow$ $(*)$ $V = \bigcup_{K_\alpha} V^{K_\alpha}$ $\quad K_\alpha \downarrow e$ c.o. subgroups.

$\Updownarrow$

$(*)$ $V = \bigcup_{K_\alpha} \pi(\mathcal{E}_{K_\alpha})V$ $\quad$ for $\mathcal{H}(G)$ module.

Prop: $\exists$ equiv. of categories (isomorphism) $\mathcal{A}(G) \simeq$ smooth modules for $\mathcal{H}(G)$.

smooth means $(*)$

Pf: Suppose (π, U) smooth $\mathcal{H}$-module.

For any $T \in S_c^*(G)$. $\pi(\varepsilon_K)v = V$ for some c.o. K.

$\pi(T)v := \pi(\underbrace{T * \varepsilon_K}_{\text{in } \mathcal{H} \text{ (2 sided ideal)}}) v$. ✓. Gives action

Facts: $K \subseteq G$ c.o.

$\mathcal{H}_K = \varepsilon_K * \mathcal{H} * \varepsilon_K$ ε_K is a unit in $\mathcal{H}_K$. (subalg)

if (π, V) a G module, then $(\pi_K, V^K) = \mathcal{H}_K$ module

$V^K = \pi(\varepsilon_K)V$

Prop:

(i) π irreducible $\iff \forall K$ $\pi_K = 0$ or irred for $\mathcal{H}_K$.

(ii) π_1, π_2 irred G-modules s.t. $V_1^K, V_2^K \neq 0$

then $\pi_1 \simeq \pi_2 \iff \pi_1^K \simeq \pi_2^K$

(iii) for every irred $(\tau, \mathcal{H}_K)$ $\exists$ irred π for G s.t. $\pi_K = \tau$

Schur's lemma ✓

Contragredients: (π, V) : smooth

$V^* = \mathrm{Hom}(V, \mathbb{C})$ not nec. smooth. eg: $V = S(G)$, $V^* = S^*(G)$.

$\tilde{V} = (V^*)_{\text{smooth}}$.

$= \bigcup_{K_\alpha} \pi^*(\varepsilon_{K_\alpha})V^*$

lemma: (i) $\Gamma \subseteq G$ cpt subgp., $v^* \in V^*$ which is Γ inv. then $\langle v^*, v\rangle = \langle v^*, \underbrace{\pi(\varepsilon\ \ )}_{\in V^\Gamma} \rangle$

ie. & $V = V(\Gamma) \oplus V^\Gamma$ $\forall v \in V$.

$(V^*)^\Gamma = (V^\Gamma)^*$.

③

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Cor: (i) (π, V) admissible $\iff$ $(\tilde{\pi}, \tilde{V})$ admissible.

(ii) $\tilde{\tilde{\pi}} = \pi$

Induction: $H \leq G$ a closed subgp

(σ, V) smooth rep. of H.

$$\mathrm{Ind}_H^G \sigma = \left\{ f : G \to V \;\middle|\; \begin{array}{l} \text{(i) } f(hg) = \sigma(h) f(g) \\ \text{(ii) } \exists \text{ cpt open } K_f \text{ s.t. } f(gk) = f(g) \ \forall k \in K \end{array} \right\}$$

this gives smooth rep^n of G. Have a functor.

$$\mathcal{A}(H) \longrightarrow \mathcal{A}(G)$$

Prop: Ind_H^G is exact., transitive: ind. in stages.

$\mathrm{Ind}_H^G \sigma \supseteq \mathrm{ind}_H^G(\sigma)$ add cond. (iii) $\exists\, \underset{\text{cpt.}}{C_f} \subseteq G$ s.t. $\mathrm{supp} f \subseteq H \cdot C_f$.

Prop: $\widetilde{\mathrm{ind}_H^G \sigma} \cong \mathrm{Ind}_H^G(\Delta \cdot \hat{\sigma})$ $\quad \Delta = \Delta_G / \Delta_H$.

pf:

$f \in \mathrm{ind}_H^G(\sigma)$, $\tilde{f} \in \mathrm{Ind}_H^G(\Delta \cdot \tilde{\sigma})$

$g \mapsto \langle \tilde{f}(g), f(g) \rangle$ makes sense. Consider $\langle \tilde{f}(hg), f(hg) \rangle = \langle \Delta(h) \tilde{\sigma}(h) \tilde{f}(g), \sigma(h) f(g) \rangle$

this fun^c is in

$\mathcal{S}(G, \Delta)$ ($\leftarrow \exists!$ Hinv. functional here.) $\quad = \Delta(h) \langle \tilde{f}(g), f(g) \rangle$

~~Set~~ Def $[\tilde{f}, f] = \displaystyle\int_{H \backslash G} \langle \tilde{f}(g), f(g) \rangle \, d\nu_{H\backslash G}(g)$ $\quad$ this gives isom.

Frobenius Reciprocity:

let $\sigma \in \mathcal{A}(H)$, $\pi \in \mathcal{A}(G)$

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)) \simeq \mathrm{Hom}_H(\mathrm{res}_H^G(\pi), \sigma).$$

$$\mathrm{Hom}_G(\mathrm{ind}_G^H(\sigma), \pi) =$$

observe:

$$\mathrm{Hom}_G(\pi_1, \pi_2) \simeq \mathrm{Hom}_G(\tilde{\pi}_2, \tilde{\pi}_1)$$

$$\cong \mathrm{Hom}_G(\tilde{\pi}, \widetilde{\mathrm{ind}_H^G(\sigma)}) = \mathrm{Hom}_G(\tilde{\pi}, \mathrm{Ind}_H^G(\Delta\tilde{\sigma})) = \mathrm{Hom}_H(\mathrm{Res}_H^G \tilde{\pi}, \Delta\tilde{\sigma})$$

$$= \mathrm{Hom}_H(\Delta^{-1}\sigma, \widetilde{\mathrm{res}_H^G(\tilde{\pi})})$$

$$= \mathrm{Hom}_H(\sigma, \Delta\, \widetilde{\mathrm{res}_H^G(\tilde{\pi})})$$

Normalized induction:

$$I_H^G(\sigma) = \mathrm{Ind}_H^G(\Delta^{1/2}\sigma)$$

$$i_H^G(\sigma) = \mathrm{ind}_H^G(\Delta^{1/2}\sigma)$$

$$\widetilde{i_H^G(\sigma)} = I_H^G(\tilde{\sigma})$$

$$\mathrm{Hom}_G(\pi, I_H^G(\sigma)) \simeq \mathrm{Hom}_H(\mathrm{res}_H^G(\pi), \Delta^{1/2}\sigma)$$

$$\simeq \mathrm{Hom}_H(r_H^G(\pi), \sigma).$$

where $\Delta^{-1/2}\mathrm{res}_H^G(\pi) = r_H^G(\pi)$.

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(1)

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$G = GL_2(F)$. . Understand irred, adm. reps. of G.

$$\underset{\cup}{B} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} = AN = NA \quad \text{where } N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}_{b \in F}$$

$$A = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \mid a_1, a_2 \in F^\times \right\}.$$

$B \longrightarrow B/N \cong A \simeq (F^\times)^2$ $\mu = (\mu_1, \mu_2)$ pair of characters $\mu_i \in \widehat{F^\times}$ gives a character of B by pulling back. still called μ. (trivial on N)

Form: $I_B^G(\mu) = \{ f : G \to \mathbb{C} \mid f(bg) = \Delta(b)^{1/2} \mu(b) f(g) = \left|\frac{a_1}{a_2}\right|^{1/2} \mu_1(a_1)\mu_2(a_2) f(g) \}$.

: $\Delta_G = 1$ (G is unimodular)

If $K = GL_2(\mathcal{O})$ then $G = BK$ so cpt ind coincides with Ind

But $\Delta_B = ?$

Write $b = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}$

measure $\to$ $du \cdot d^\times a_1 \cdot d^\times a_2$ = right haar measure $d\nu_B(b)$

$$\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} b = \begin{pmatrix} 1 & \alpha_1 \alpha_2^{-1} u_1 \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 a_1 & \\ & \alpha_2 a_2 \end{pmatrix}$$ so left inv. meas is $d\mu_B(b) = |a_1/a_2|^{-1} d\mu_B(b)$

So $\Delta_B(b) = \left|\frac{a_1}{a_2}\right|^{-1}$.

So $\Delta = \dfrac{\Delta_G}{\Delta_b} = \left|\dfrac{a_1}{a_2}\right|$

$I_B^G \mu$ = smooth admissible rep of G

$f \in I(\mu)$ determined by $f|_K$ for any cpt open $K_1 \leq K$

f determined on K/K_1 $\leftarrow$ finite

$Z = \{ \begin{pmatrix} a & \\ & a \end{pmatrix} \mid a \in \mathbb{F}^\times \} =$ center of G.

acts in $I(\mu)$ by $\mu_1\mu_2 =$ central character

$\widetilde{I(\mu)} = I(\tilde{\mu})$ (contragredient)

Note: $\det : G \twoheadrightarrow \mathbb{F}^\times$ so $\chi \in \widehat{\mathbb{F}^\times}$ gives a repn of G irred, adm.

$I_B^G(\mu) \otimes (\chi \circ \det) \simeq I_B^G(\mu\chi)$ by $f \to \chi \cdot f$.

eg: $I(\Delta^{1/2})$, ie., $\mu_1 = |\cdot|^{1/2}$, $\mu_2 = |\cdot|^{-1/2}$.

$f \overset{\in}{\Rightarrow} f(bg) = \Delta(b) f(g)$.

so $I(\Delta^{1/2}) = S(G;\Delta) \xrightarrow{\nu_{B\backslash G}} \mathbb{C} \to 0$

$$f \longmapsto \int_{B\backslash G} f(g)\, d\nu_{B\backslash G}(g).$$

write: $0 \to S \to I(\Delta^{1/2}) \to 1\!\!1 \to 0$

S: ∞-dim, some submod.

$1\!\!1$: trivial rep of G.

---

eg: $I(\Delta^{-1/2}) \simeq S(B\backslash G)$ $\leftarrow$ no inv. Haar meas here

$f(bg) = f(g)$

$S(B\backslash G) \supseteq \mathbb{C}\cdot 1$. (constant fns.)

So we have

$$0 \to 1\!\!1 \to I(\Delta^{-1/2}) \to \tilde{S} \to 0.$$

$\tilde{S}$: contragredient of S above.

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Thm: $I_B^G(\mu)$ is irreducible except when: quotient is the char. $\mu_1 1\cdot 1^{-1/2}$.

(i) $\mu_1\mu_2^{-1} = 1\cdot 1$. Then $I_B^G(\mu)$ has a unique 1dim quotient and an irred. submodule.

(ii) $\mu_1\mu_2^{-1} = 1\cdot 1^{-1}$ then $I_B^G(\mu)$ has a unique 1-dim sub and irred inf. dim. quotient.

Pf

Say π irred adm rep.

Consider $\mathrm{Hom}_G(\pi, I_B^G(\mu)) = \mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \mu\cdot\Delta^{1/2})$ by Frob. Rec.

aside:

suppose (π, V) smooth of G.

let $V(N) = \mathrm{span}\{\pi(n)v - v \mid v\in V, n\in N\}$.

let $V_N = V/V(N)$ (N-coinvariants).

Facts: 1. $N \cong F = \bigcup_n \wp^{-n}$ ← cpt.

consequence: $V \longrightarrow V_N$ is exact.

2. $\phi : V \longrightarrow V'$ rep where N acts trivially, then ϕ factors through V_N.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d] & V' \\ V_N \arrow[ur, dashed] & \end{tikzcd}$$

$\mu\cdot\Delta^{1/2}$ is trivial on N. Hence

$\mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \mu\cdot\Delta^{1/2}) = \mathrm{Hom}_A(\boxed{(\mathrm{Res}_B^G(\pi))_N}, \mu\Delta^{1/2})$.

let $r_N(\pi)$ denote Jacquet functor applied to π

So

$$\mathrm{Hom}_G(\pi, I_B^G(\mu)) \simeq \mathrm{Hom}_A(r_N(\pi), \mu\Delta^{1/2})$$

if we def $r_N(\pi) = (\mathrm{res}_B^G(\pi))_N \Delta^{-1/2}$. then

$$\mathrm{Hom}_G(\pi, I_B^G(\mu)) \simeq \mathrm{Hom}_A(r_N(\pi), \mu)$$

So $\pi \hookrightarrow I_B^G(\mu) \Leftrightarrow \mu$ is quot. of $r_N(\pi)$.

Consider:

$$\mathrm{Hom}_G(I(\mu), I(\nu)) \simeq \mathrm{Hom}_A(r_N(I(\mu)), \nu)$$

<u>Prop</u>: (1) There is an exact sequence of A-modules

$$0 \to \mathbb{C}\cdot\mu' \to r_N(I(\mu)) \longrightarrow \mathbb{C}_\mu. \longrightarrow 0$$

where $\mu' = (\mu_2, \mu_1)$

(2) This sequence splits $\Leftrightarrow \mu' \neq \mu$

If $\mu' = \mu$ i.e. $\mu_1 = \mu_2$

$$\mu \begin{pmatrix} 1 & \mathrm{ord}(\frac{a_1}{a_2}) \\ & 1 \end{pmatrix}$$

i.e. $\exists$ basis ζ_1, ζ_2 for $r_N(I(\mu))$ s.t.

$$(a_1, a_2) \longmapsto \mu(a_1)\mu(a_2) \begin{pmatrix} 1 & \mathrm{ord}(\frac{a_1}{a_2}) \\ & 1 \end{pmatrix}$$

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①

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Recall: $I(\mu) = I_B^G(\mu)$ $\mu = (\mu_1, \mu_2)$

let $\mu' = (\mu_2, \mu_1)$

Thm: If $\mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1}$ then $I(\mu)$ is irred.

If $\mu_1\mu_2^{-1} = ||$ then ...

$0 \to S \to I(\mu) \to Q \to 0.$

If $\mu_1\mu_2^{-1} = ||^{-1}$ then ...

Jacquet Functor: $\mathcal{A}(G) \to \mathcal{A}(A)$. (exact)

$\mathrm{Hom}_G(\pi, I(\mu)) = \mathrm{Hom}_A(r_N(\pi), \mu)$ where $r_N(\pi) = \left(\mathrm{res}_B^G(\pi)\right)_N \cdot \Delta^{-1/2}$ N-covariants

Suppose π = irred. ad. Then

$\pi \hookrightarrow I(\mu) \iff \mu$ = quotient of $r_N(\pi)$

Prop: (i) Say $\pi = \chi \circ \det$ $r_N(\pi) = (\chi, \chi)\cdot\Delta^{-1/2} = \chi \cdot \Delta^{-1/2}$ $\left(\text{i.e. } \begin{pmatrix} a_1 & \\ & a_2\end{pmatrix} \mapsto \chi(a_1)\chi(a_2)\Delta^{-1/2}\right)$

(ii) $r_N(I(\mu)) \to \mathbb{C}_\mu \to 0$

$0 \to \mathbb{C}_{\mu'}$ Thus: This is split $\iff \mu' \neq \mu$. (not obvious)

Cor: $I(\mu)$ has a unique irred. sub (resp. quotient) since $\widetilde{I(\mu)} = I(\hat{\mu})$.

Pf: If $0 \to \pi_1 \oplus \pi_2 \hookrightarrow I(\mu)$ $\left(\Rightarrow r_N(\pi_i) \to \mu \to 0.\right)$
then

$0 \to r_N(\pi_1) \oplus r_N(\pi_2) \longrightarrow r_N(I(\mu))$ C! by (ii) μ can't appear twice.

Proof of Prop:

(ii) Recall: Bruhat decomp

$G = B \cup BwN$; $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. ($B$-closed $\Rightarrow$ compliment BwN is open)

$\Rightarrow$ ~~As a repn of B, get a~~

gives a filtration of $I(\mu)$.

$$0 \to I^{\circ}(\mu) \to I(\mu) \to \mathbb{C} \to 0$$
$$f \mapsto f(e)$$

$\{f \in I(\mu) \mid \operatorname{supp} f \subseteq BwN\} \simeq S(N)$ by $\phi(n) = f(wn)$ since $f(nawn') = \mu(a)\Delta(a)^{+1/2} f(wn')$

This is a filtration of B-submodules.

Now take N-coinvariants:

$$0 \to S(N)_N \to I^{\circ}(\mu)_N \to \underset{A \text{ acts by } \mu \Delta^{1/2}}{\mathbb{C}_N} \to 0.$$

Claim: $S(N) \to S(N)_N \simeq \mathbb{C}$

$\phi \mapsto \int_N \phi(n)\, d\mu_N(n)$

Have:

$$0 \to \mathbb{C} \to \Gamma_N(I(\mu)) \to \mathbb{C}_{\mu} \to 0.$$

How does A act.

$a = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \Rightarrow a' = \begin{pmatrix} a_2 & \\ & a_1 \end{pmatrix}$, $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Rightarrow n' = \begin{pmatrix} 1 & a_1^{-1}a_2 x \\ 0 & 1 \end{pmatrix}$

$\Delta(a') = \Delta(a)^{-1}$

$$\int_N (R(a)f)(wn)\, dn = \int_N f(wna)\, dn = \int_N f(a'wn')\, dn \qquad x \mapsto a_1 a_2^{-1} x$$

$$= \mu'(a) \underbrace{\Delta(a)^{-1/2} |a_1 a_2^{-1}|}_{\Delta^{1/2}(a)} \int_N f(wn)\, dn$$

$f \mapsto \phi \to \int_N \phi(n)\, d\mu_N(n)$

$\int_N f(wn)\, d\mu_N(n)$

So A acts by μ' once we pull out the $\Delta^{1/2}$.

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Remains to show non-split when $\mu' \neq \mu$.

Suppose $\mu = \mu'$ ie. $\mu_1 = \mu_2$. $0 \to C_\mu \to r_N(I(\mu)) \to C_\mu \to 0$

Step 1: may as well assume that $\mu_1 = \mu_2 = 1$

(since twisting with $\chi \circ \det$ can give $I(1) = I(\mu) \otimes \mu^{-1}$)

$K = GL_2(\mathcal{O}_F)$, $G = BK$ Iwasawa dec.

Consider $I(1)$.

let $f_0 =$ unique fcn in $I(1)$ s.t. $f_0(k) = f_0(e)$ $\forall k \in K$. normalize s.t. $f_0(e) = 1$.

$$I(1) \longrightarrow r_N(I(1))$$
$$f_0 \longrightarrow [f_0] \qquad \text{let } \Delta^{1/2}(a) = \delta(a)$$

Consider:

$$\left[\delta(a)^{-1} R(a) f_0 - f_0\right] = r_N(a) \cdot [f_0] - [f_0]$$

So

$$\left(\delta(a)^{-1} R(a) f_0 - f_0\right)(e)$$
$$\shortparallel$$
$$\delta(a^{-1}) f_0(a) - f_0(e) = \delta(a)^{-1}\delta(a) f_0(e) - f_0(e) = 0. \quad \checkmark$$

So now this fcn is in $I^0(1)$

$$\begin{array}{cc} I^0(1) & f \\ \downarrow & \downarrow \\ \mathbb{C} & \int_N f(wn)\,dn \end{array}$$

So $$\int_{\mathbb{F}} \left(\delta(a)^{-1} R(a) f_0 - f_0\right)\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx = \int_{\mathbb{F}} \left(\delta(a)^{-1} f_0(w n(x) a) - f_0(w n(x))\right) dx.$$

So now: <u>lemma</u>:

$wn(x) = n \cdot a \cdot k$ by Iwasawa.

$$(0,1)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = (0,1)\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}\begin{pmatrix} * & * \\ c & d \end{pmatrix}:$$

$$(1,x) = \underbrace{a_1^{-1}(c,d)}_{\text{bottom row of something in } GL_2(\mathcal{O}_F)}$$

if $\operatorname{ord} x \geq 0$, $a_1 = a_2 = 1$

if $\operatorname{ord} x = -r < 0$ then write:

$$(1,x) = (1, \varpi^{-r} \cdot u) = \overset{a_1^{-1}}{\varpi^{-r}}(\varpi^r, u)$$

So if $w(n(x)) = n \cdot a \cdot k$. then

$a = \begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}$ where $\operatorname{ord} a_1 = 0$ if $\operatorname{ord} x \geq 0$

$= r$ if $\operatorname{ord} x = -r$

So calculating the integral now:

~~So image of the difference in $\mathbb{C}$ is $\operatorname{ord}(a_1/a_2)$~~

$$\int_F \left(\delta(a)^{-2} f_0 (w n(a_1^{-1} a_2 x)) - f(wn(x)) \right) dx$$

$x \to a_1 a_2^{-1} x = \alpha$.

now we can separate the integrals since $\mathfrak{p}^{-r}$ is cpt.

$$\lim_{r \to \infty} \left(\int_{\mathfrak{p}^{-r-\alpha}} f_0(wn(x))\, dx - \int_{\mathfrak{p}^{-r}} f_0(w(n(x))\, dx \right)$$

$$= \lim_{r \to \infty} \int_{\mathfrak{p}^{-r-\alpha} - \mathfrak{p}^{-r}} f_0(w n(x))\, dx.$$

but $f_0(wn(x)) = \begin{cases} 1 & \text{if } \operatorname{ord} x \geq 0. \ (K \cdot \ ) \\ q^{\operatorname{ord} x} & \text{if } \operatorname{ord} x < 0. \end{cases}$

$$= \lim_{r \to \infty} \sum_{i=-r-1}^{\alpha - r} q^i \cdot \underbrace{q^{-i} \operatorname{vol}(U_F)}_{\text{Vol of set unit. on.}} = \operatorname{vol}(U_F) \cdot \alpha$$

So conclude: image of difference above in $\mathbb{C}$ is $\operatorname{ord}(a_1/a_2)$.

Vol of set unit. on.
in $\mathbb{C}$ is $\operatorname{ord}(a_1/a_2)$.

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Recall: $\Delta = \Delta_G / \Delta_B$, $\delta = \Delta^{1/2}$.

$\mathrm{Hom}_G(\pi, I(\mu)) \simeq \mathrm{Hom}_A(r_N(\pi), \mu)$ $\quad \mu = (\mu_1, \mu_2)$, $\mu' = (\mu_2, \mu_1)$

$r_N(\pi) = \delta^{-1}\pi_N$ $\quad$ Jacquet functor: $\mathcal{A}(G) \to \mathcal{A}(A)$

$\pi_N : V_N = V/V(N)$

prop:

$$0 \to \mathbb{C}_{\mu'} \xrightarrow{(2)} r_N(I(\mu)) \xrightarrow{(1)} \mathbb{C}_\mu \to 0$$

this is not split when $\mu' = \mu$, i.e., $\mu_1 = \mu_2$.

Pf. $G = B \,\dot\cup\, BwN$

the map (1) is $f \mapsto f(e)$

$$\begin{array}{ccc} I(\mu) & \longrightarrow & \mathbb{C}_\mu \\ \downarrow & \nearrow & \\ r_N(I(\mu)) & & \end{array}$$

$\{ f \in I(\mu) \mid \mathrm{supp}(f) \subseteq BwN \} \simeq S(N)$

$f \longleftrightarrow \phi$

$f(wn) = \phi(n)$

$f \mapsto \int_N f(wn)\,dn.$

If $\mu' = \mu$ then we may as well consider $I(\mu) \simeq \mu \otimes I(1)$.

Take f_0 = unique K-invariant vector with $f_0(e) = 1$.

$f_1 \leftrightarrow \mathrm{char}(\mathcal{O})$ under the isom above.

Let $[f_0]$, $[f_1]$ be images in $r_N(\mathcal{I}(\mu))$, they give a basis.

Calculate action:

$$r(a)[f_1] = [\delta^{-1}(a) R(a) f_1] = \mu^{\cancel{1}}(a)[f_1] \quad 1$$

$$r(a)[f_0] = [\delta^{-1}(a) R(a) f_0] \xrightarrow{\text{evaluate}} 1.$$

$$= [\delta^{-1}(a) R(a) f_0 - f_0] + [f_0]$$

$$= \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(K)} \cdot \underset{\substack{| \\ \mathrm{ord}\left(\frac{a_1}{a_2}\right)}}{\mathrm{ord}(a)} [f_1] + [f_0].$$

$\downarrow$

renormalize so that

$$r(a) = \begin{pmatrix} 1 & \mathrm{ord}(a) \\ 0 & 1 \end{pmatrix}$$

So $r_N(a) = \mu(a) \begin{pmatrix} 1 & \mathrm{ord}(a) \\ 0 & 1 \end{pmatrix}$. w.r.t basis $\begin{bmatrix} f_1^* \\ f_0 \end{bmatrix}$.

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If π irred $\pi \hookrightarrow I(\mu) \Leftrightarrow \mu$ a quot. of $r_N(\pi)$

Cor: $I(\mu)$ has a unique irred sub (quotient)

pf: $0 \to V_1 \oplus V_2 \to I(\mu)$

$0 \to r_N(V_1) \oplus r_N(V_2) \to r_N(I(\mu))$ c!

So we have $0 \to Sub \to I(\mu) \to Q \to 0$.

so comp. series:

$\underbrace{0 \subseteq S}_{\text{irred}} \subset \underbrace{\cdots}_{?} \subset \underbrace{S' \subset I(\mu)}_{\text{irred.}}$

Whittaker Models:

ψ non triv.

$W_\psi = Ind_N^G(\psi)$: Wh. Space

(π, V) irr. rep. A Wh. model is $\pi \hookrightarrow W_\psi$.

$V(N,\psi) = \{v \in V \mid \int_{\wp^{-n}} \psi(x)\pi(n(x))v\,dx = 0$ for $n >> 0\}$.

$Hom_G(\pi, W_\psi) \cong Hom_N(Res_N^G(\pi), \psi)$. $\ni \lambda$ is a lin functional $\lambda: V_\pi \to \mathbb{C}$ s.t. $\lambda(\pi(n)v) = \psi(n)$

$V_{N,\psi} := V/V(N,\psi)$ | Space of Wh. functionals is $(V/V(N,\psi))^*$ | this is a ψ-Wh. functional.

Theorem:

(1) $r_{N,\psi}(\pi) = 0 \Leftrightarrow \pi = 1$-dim. (Existence)

(2) If π is inf. dim. then $\dim V_{N,\psi} = 1$. (uniqueness)

remarks:

(1) True only for $GL(2)$, false for $GL(n)$ $n \geq 3$.

π is generic if $W(\pi,\psi)$ exists.

Prop: let ψ be non-triv.

Then $\dim \Gamma_{N,\psi}(I(\mu)) = 1.$

proof:

$$0 \to I^\circ(\mu) \to I(\mu) \to \mathbb{C} \to 0 \qquad f \mapsto f(e).$$
ex. seq. of N-modules.

$$0 \to \Gamma_{N,\psi}(I^\circ(\mu)) \to \Gamma_{N,\psi}(I(\mu)) \to \underset{\substack{\| \\ 0}}{\Gamma_{N,\psi}(\mathbb{C})} \to 0$$

$$I^\circ(\mu) \overset{N}{\simeq} S(N) \simeq S(F)$$
$$f \to \phi$$

Define:
$$\lambda: \phi \mapsto \int_F \psi(-x)\,\phi(x)\,dx$$
So $\dim > 0$, it's also $= 1$

λ is a ψ-Wh. functional. ⊠

Back to

$$0 \to S \to I(\mu) \to Q \to 0$$

$$0 \to \Gamma_{N,\psi}(S) \to \underset{\substack{\| \\ \text{1-dim.}}}{\Gamma_{N,\psi}(I(\mu))} \to \Gamma_{N,\psi}(Q) \to 0$$

$\Rightarrow \Gamma_{N,\psi}(S) = 0$ or $\Gamma_{N,\psi}(Q) = 0.$

Suppose $\Gamma_{N,\psi}(S) = 0$.

Cor: $\exists$ precisely one ~~irred~~ inf. ~~dim~~. dim. constituent of $I(\mu)$

Pf: Thm

also: $2 = \dim \Gamma_N(I(\mu)) > \#$ of 1-dim S.

Suppose $I(\mu)$ not irred.

If sub $= \infty$-dim then quot. is one-dim. apply contragredient to get $\tilde{Q} = 1$ dim sub.
Back to

Case 1: $S =$ irred sub is 1-dim.

$$0 \to \underset{\substack{| \\ \chi\circ\det}}{\mathbb{C}_\chi} \to I(\mu) \quad \Leftrightarrow \Gamma_N(\mathbb{C}_\chi) \text{ has } \mu \text{ as a quot.}$$
i.e.
$$\delta^{-1}\chi = \mu \Rightarrow \mu = (\mu_1,\mu_2) \leftarrow (|\;|^{-1/2}\chi, |\;|^{1/2}\chi).$$

3/30

Kudla:

L. time: If $I(\mu)$ has a 1 dim. sub (resp. quot.) then $\mu_1\mu_2^{-1} = |\,|^{-1}$ (resp. $|\cdot|$) since $\widetilde{I(\mu)} = I(\hat{\mu})$

Pf:

$\pi \hookrightarrow I(\mu) \iff r_N(\pi) \to \mu$

But $r_N(\chi) = \delta^{-1}\chi = \mu \Rightarrow \mu = (\chi|\cdot|^{-1/2}, \chi|\cdot|^{1/2})$

, ie., $\mu_1\mu_2^{-1} = |\cdot|^{-1}$

Know: $\exists!$ ∞-dim. constituent. since $r_{N,\psi}(I(\mu)) = $ 1-dim.

Now

Suppose: $\mu_1\mu_2^{-1} = |\cdot|^{-1}$ or $|\cdot|$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}_\chi \arrow[r] & I(\mu) \arrow[r] \arrow[dr] & Q \arrow[r] \arrow[d] & 0 \\ & & & \square & \end{tikzcd}$$

claim: Q is irred.

Since $I(\mu)$ has unique irred quot, Q has un. irred. quot.

Suppose its 1-dim. then $\mu_1\mu_2^{-1} = |\cdot|$ C! since $\mu_1\mu_2^{-1} = |\cdot|^{-1}$.

Suppose Q has ∞-dim. quot.

If Q not irred, ie., $\exists$ 1-dim sub in Q, then $r_N(\square) = 0$.

by contragredients:

$0 \to \tilde{\square} \to I(\hat{\mu}) \to$ 1-dim $\to 0$.

$r_N(\tilde{\square}) \to \mu$

$\Rightarrow r_N(\tilde{\square}) \neq 0$.

Case (0): $I(\mu)$ irred $\iff \mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1}$; $\mathrm{Hom}_G(I(\mu), I(\nu)) = \mathrm{Hom}_A(r_N(I(\mu)), \nu) = \begin{cases} 1 & \text{if } \nu = \mu \text{ or } \mu \\ 0 & \text{other} \end{cases}$

Case (1):

$\mu_1\mu_2^{-1} = |\cdot|^{-1}$

$0 \to \mathbb{C}_\chi \to I(\mu) \to Q \to 0$

Case (2):

$\mu_1\mu_2^{-1} = |\cdot|$

$0 \to \sigma \to I(\mu) \to \mathbb{C}_\chi \to 0$

$I(\mu) \longrightarrow I(\mu')$ $\qquad\qquad\qquad\qquad (\chi|\cdot|^{-1/2}, \chi|\cdot|^{1/2})$.

Say $\mu_1\mu_2^{-1} = |\cdot|^{-1}$ $\quad$ consider $0 \to \mathbb{C}_\chi \to I(\mu) \to Q \to 0$ $\qquad \chi = \mu_1|\cdot|^{-1/2}$.

$\Downarrow$

~~$r_N(I(\mu)) \to r_N(I(\mu'))$~~ $\quad 0 \to \chi\delta^{-1} \to r_N(I(\mu)) \to r_N(Q) \to 0.$

$\chi\delta^{-1} = \mu'$, $\quad r_N(Q) = \mu'$

$I(\mu) \to I(\mu')$

(quot. of $I(\mu)$) $= Q \nearrow$ $\quad$ but Hom_G is 1-dim so this is the only one.

$r_N(Q) = \mu'$ so $\exists$

So Must have $\quad Q(\mu) \simeq S(\mu')$.

---

irreducibles of G

- $\chi \circ \det$.
- $I(\mu)$ where $\mu = (\mu_1, \mu_2)$ $\quad \mu_1\mu_2^{-1} \neq |\cdot|^{-1}, |\cdot|$

  $= \pi(\mu)$ $\qquad \pi(\mu) \simeq \pi(\mu')$.
- If $\mu_1\mu_2^{-1} = |\cdot|^{-1}$

  let $\sigma(\mu) = Q(\mu) =$ inf. dim. quotient.

  $(= S(\mu'))$
- π supercuspidal, i.e., π s.t. $r_N(\pi) = 0$.

  so that $\mathrm{Hom}_G(\pi, I(\mu)) = 0 \quad \forall \mu$.

Have proved everything except:

Thm (ii): ψ non-triv., $r_{N,\psi}(\pi) = 0 \iff \pi =$ 1-dim.

Bump: p. 441-447

let X = ℓ-space $\qquad U$ cpt. opens.

$S(X)$ = loc. const. cpt. support. fcns.

$\mathcal{S}$ = associated sheaf on X by $S(U) \overset{\text{not.}}{\hookrightarrow} S(X) = \Gamma_c(X, \mathcal{S})$

$\qquad\qquad 1_U \in S(U)$ = char fcn of U if U = cpt. open.

If V = sheaf of $\mathbb{C}$-v.s. on X

$\Rightarrow$ it is an $\mathcal{S}$-module.

If $M = S(X)$-module then $\mathcal{M}$ = corr. sheaf of $\mathcal{S}$-modules ; $\mathcal{M}(U) = 1_U \cdot M \subset M$ = pre-sheaf

Prop: There are equivalences of categories

Sheaves of $\mathbb{C}$ v.s. on X $\longleftrightarrow$ sheaf of $\mathcal{S}$-modules. $\mathcal{M}$

$\longleftrightarrow$ cosmooth $S(X)$-Module. M $\qquad$ $\mathcal{M} \mapsto \Gamma_c(X, \mathcal{M})$ Global Sections.

$\supset \forall m \in M \ \exists\, U$ c.o. in X s.t. $1_U \cdot m = m$.

Prop: M = cosmooth $S(X)$-module

$\mathcal{M}$ = assoc. sheaf.

For $x \in X$, $\mathcal{M}_x = \varinjlim_{U \ni x} \mathcal{M}(U) \simeq M/M(x)$ where $M(x) = \{ m \in M \mid \exists\, U \ni x \text{ c.o. with } 1_U \cdot m = 0 \}$

the "stalk"

(π, V)

Suppose $r_{N,\psi}(\pi) = 0 \iff V(N,\psi) = V \iff V(N, \psi_a) = V \quad \forall a \in F^\times$

since

let $a \in F^\times$. Consider $\pi(a)\, V(N,\psi) = \pi(a)\pi(n)v - \psi(n)\pi(a)v$

$a = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$

$= \pi(n')\,\pi(a)v - \psi(n)\pi(a)v. \qquad n' = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$.

$= \pi(n')\,\pi(a)v - \psi_{a^{-1}}(n')\,\pi(a)v.$

$= V(N, \psi_{a^{-1}})$

Let V = repn of N

get a repn of $S(F)$ by $\pi(f)v = \int_{\mathbb{F}} f(x)\pi(x)v\,dx$.

$(S(\mathbb{F}), *)$ is an alg.

Hence we get an $(S(\mathbb{F}), \underset{\text{mult.}}{\cdot})$ module by $\hat{\pi}(f)v = \pi(\hat{f})v$. Call this module M.

Now make $\mathcal{M}$ the assoc. sheaf on $X = \mathbb{F}$ as above.

Compute the stalks: let $a \in \mathbb{F}$

$\mathcal{M}_a = M/M(a)$ $\quad M(a) = \{m \in M \mid f_U \cdot m = 0$ for some U o. $U \ni a\}$

turns out $\mathcal{M}_a \cong \begin{cases} V_{N,\psi_a} & a \neq 0 \\ V_{N,1} & a = 0 \end{cases}$

Kudla
4/4.

(π, V), ψ n.t.

$$r_{N,\psi}(\pi) = V/V(N,\psi)$$

$\hat{\subset}$: $\langle \pi(n)v - \psi(n)v \rangle$ or $\{v \mid \int_{\mathfrak{p}^{-r}} \psi(-x)\,\pi(n(x))v\,dx = 0$ for r large $\}$.

Thm: (1) $r_{N,\psi}(\pi) = 0 \iff \pi = 1$-dim.

Recall: $X = \ell$-space, $S(X) \rightsquigarrow S$ sheaf ; $S(X) = \Gamma_c(X, S)$

l.c. cpt spt. fcns. $\quad \cap \quad C^\infty(X) = \Gamma(X, S)$.

If $M = S(X)$-module (cosmooth.)

$\mathcal{M}$ = assoc. sheaf of S-mod.

Then $M = \Gamma_c(X, \mathcal{M})$ $\qquad \Gamma_c(X,\mathcal{M}) \subset \qquad \subset \Gamma(X,\mathcal{M})$

$\qquad \qquad \overset{\shortparallel}{M}$

Exercise: Find modules that live strictly in between

ex: $X = F \times F$

Prop: $\forall x \in X$

$\mathcal{M}_x = M/M(x)$; $M(x) = \{m \in M \mid \exists$ c.o. $U \ni x$ s.t. $1_U \cdot m = 0\}$

Proof: let $a \in F^\times$

note: $\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix} V(N,\psi) = V(N, \psi_{a^{-1}})$.

$\Rightarrow$ We want to show: $V = V(N, \psi_a)$ $\forall a \in F^\times$. iff $r_{N,\psi}(\pi) = 0$.

V = repn of N, hence a repn of $(S(N), *)$

$\qquad \uparrow \wedge$ — Fourier tr. wrt ψ.

$(S(N), \cdot)$

let $\mathcal{V}$ = assoc. sheaf to the module V.

Claim: (1) V is cosmooth, thus, $V = \Gamma_c(N, \mathcal{V})$

(2) For $x \in F \simeq N$.

$$V(x) = V(N, \psi_x)$$

Thus: $V_x = \begin{cases} 0 & \text{if } x \neq 0 \\ r_N(\pi) = V/V(N,1) & \text{if } x = 0 \end{cases}$

(1), (2) gives us:

$V \cong V/V(N,1)$, ie., $V(N,1) = 0$, ie, $\pi(n)v - v = 0 \ \forall v, n$

so N acts trivially in V.

Since conjugates of N generate $SL_2(\mathbb{F}) \subseteq G \Rightarrow SL_2(\mathbb{F})$ act trivially

$\Rightarrow \pi = \chi \circ \det$

check (1), (2):

Let $v \in V$, $\pi(n(x))v = v$ if ord x sufficiently large, say $\operatorname{ord}(x) \geq r$ some $r > 0$.
Let $\phi_r = \text{char } \wp^r / \operatorname{Vol}(\wp^r) \in S(N)$

So $\pi(\phi_r)v = v$. since $\pi(\phi_r)v = \int_N \phi_r(n)\pi(n)v \, dn = (\cdot)v = v$

Let $c = \text{cond. of } \psi_0 = c(\psi)$

$(S(N), \cdot) \ni 1_{\wp^{-r-c}}$, $\pi(\widehat{1_{\wp^{-r-c}}})v = 1_{\wp^{-r-c}} \cdot v$

$$\widehat{1}_{\wp^{-r-c}}(x) = \int_{\wp^{-r+c}} \psi(-xy)\,dy = \begin{cases} 0 & \text{if } \operatorname{ord}(y) < r \\ \operatorname{Vol}(\wp^{-r-c}) & \operatorname{ord}(y) \geq r \end{cases}$$

so this four. trans is exactly ϕ_r.

But $\operatorname{Vol}(\wp^r)\operatorname{Vol}(\wp^{-r+c}) = 1$

This proves cosmooth.

check (2):

Claim: $v \in V(\alpha) \iff \exists \ U$ c.o. $U \ni \alpha$ s.t. $1_U \cdot v = 0$.

$\begin{cases} 0 & \operatorname{ord} x + t < c \\ \operatorname{Vol}(\wp^t) & \text{if } \operatorname{ord} x + t \geq c. \end{cases}$

Take $1_U = \text{char}(\alpha + \wp^t)$

so $\hat{1}_U(X) = \int_{a+\wp^t} \psi(-xy)\,dy = \psi(-ax)\int_{\wp^t} \psi(-xy)\,dy$

s.t. $1_U \cdot v = 0$

$\longrightarrow = \pi(\hat{1}_U) \cdot v$

$= \psi(-ax)\operatorname{Vol}(\wp^t) \cdot \text{char}(\wp^{c-t}).(x).$

So $\pi(\hat{1}_u)v = \frac{1}{\mathrm{vol}(\mathfrak{p}^t)} \int_{\mathfrak{p}^{c-t}} \psi(-ax)\,\pi(u(x))v\,dx. = 0$ for large t.

But this is exactly $V(N, \psi_a)$. ✓.

Df. (π, V) irred. supercuspidal $:= r_N(\pi) = 0$.

$\Leftrightarrow \pi \not\hookrightarrow I(\mu)$ any μ.

π is ∞-dim., hence $\exists$ a Whittaker Model, and its unique.

characterization: π supercuspidal $\Leftrightarrow$ matrix coeffs. of π are cptly supported Modulo center.

recall: $v \in \pi$ Matrix coeff. $\phi(g) = \langle \pi(g)v, \tilde{v} \rangle$

$\tilde{v} \in \tilde{\pi}$ = contragred.

Let ω_π = central char. of π. then $\phi(zg) = \omega_\pi(z)\phi(g)$

Note: "S.C.'s are projective in the category $\mathcal{A}(G)$ sm. reps of G"

Unramified reps (spherical):

$K = GL_2(\mathcal{O}) \leq G$ max cpt.

$G = BK$ (Iwasawa)

$= KAK$ (Cartan)

Df: (π, V) is irred adm rep. of G is unramified $\Leftrightarrow V^K \neq 0$

recall: (Assume $\mathrm{Vol}(K, \mu_G) = 1$.) ($\Rightarrow \varepsilon_K = \mathrm{char}_K$)

We have $\mathcal{H} \overset{\mu_G}{\simeq} S(G)$.

$\mathcal{H}_K = \varepsilon_K * \mathcal{H} * \varepsilon_K$.

$$\begin{matrix}\text{irred adm. } (\pi, V) \\ \text{s.t. } V^K \neq 0\end{matrix} \overset{\text{bij}}{\rightleftarrows} \text{irred reps of } \mathcal{H}_K.$$

$(\pi, V) \longrightarrow (\pi^K, V^K)$

Cor:

Prop: $\mathcal{H}_K$ = commutative (with ε_K as a unit).

Pf: $G \longrightarrow G$
$g \longrightarrow {}^t g$ Note: $K \xrightarrow{\sim} K$.

For $T \in S(G)$ let $T'(g) = T({}^t g)$.

$\mathcal{H}_K \longrightarrow \mathcal{H}_K$ under prime
$\shortparallel$
$S(G//K)$

Recall: $T_1, T_2 \in \mathcal{H}$ then

$$(T_1 * T_2)(g) = \int_G T_1(g_1) T_2(g_1^{-1} g)\, dg_1$$

as a fcn in $S(G)$.
Vcn $\simeq$

Note: $(T_1 * T_2)' = T_2' * T_1'$. check this: uses $\mu_G' = \mu_G$.

In fact: $KgK = K \cdot \begin{pmatrix} \varpi^{r_1} & \\ & \varpi^{r_2} \end{pmatrix} \cdot K$ where we can take $r_1 \geq r_2$.

$\phi_{r_1, r_2} = $ char $K(\ ')K$. span $\mathcal{H}_K$.

$\phi_{r_1, r_2}' = \phi_{r_1, r_2}$.

So $(T_1 * T_2)' = T_2' * T_1' = T_2 * T_1$

$T_2 * T_1$ is commutative.

Fact: See Bump ~~(p.~~ (p. 494-495).

$\mathcal{H}_K$ is generated by $T(p) = \phi_{1,0} = $ char $K\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}K$.

$R(p) = \phi_{1,1}$

$R(p^{-1}) = \phi_{-1,-1}$.

Cor of Prop: irred. reps of $\mathcal{H}_K$ are 1-dim.

Consider: $I(\mu)$, $\mu = (\mu_1, \mu_2)$ unramified
i.e., $\mu_1(U_F) = 1$, $\mu_2(U_F) = 1$.

$\mu_1(\varpi) = z_1$; $\mu_1(x) = z_1^{\operatorname{ord} x}$

$\mu_2(\varpi) = z_2$; $\mu_2(x) = z_2^{\operatorname{ord} x}$

$z_1, z_2 \in \mathbb{C}^\times$.

Claim: $\dim I(\mu)^K = 1$.

Say $f \in I(\mu)^K$

then $f(nak) = \mu(a)\delta(a) f(k) = \mu(a)\delta(a) f(1)$.

Set $f_\mu^\circ(e) = 1$.

Then $I(\mu)^K = \mathbb{C} \cdot f_\mu^\circ$.

ex:

Suppose $\mu_1\mu_2^{-1} \neq |\,|^{\pm 1}$ so $I(\mu)$ irred, $\pi(\mu)$ = unramified rep.

eg: $\mathbb{C}_\chi$ const. of $I(\mu)$

$\chi = \mu_1 |\,|^{\pm 1/2}$ = unramified.

So $\mathbb{C}_\chi^K = \mathbb{C}_\chi$.

$\sigma(\mu)$ = special (Steinberg) = other piece in $I(\mu)$. this is not spherical, i.e. $\sigma(\mu)^K = 0$.

Th: The $\pi(\mu)$ irred. sph. pr. series
and $\chi \circ \det$; for χ unramified give all unramified reps of G.

Satake Isom:

$$\left\{ \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \right\} \in GL_2(\mathbb{C}) = G^\vee$$

$t = t(z)$; $\{\ldots\} = T^\vee \subset G^\vee$

$$N_{G^\vee}(T^\vee)/T^\vee = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle = W$$

$t^w = t(z')$

$T^\vee/W$ = semi-simple conjugacy classes in $G^\vee$.

Kudla:

$G^\vee = GL_2(\mathbb{C})$, $T^\vee = \{ t(z) = \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \}$

$T^\vee / W \simeq$ semisimple conj. classes in $G^\vee$.

Theorem:

The isom. classes of unramified irred. adm. reps are in bijection with $T^\vee / W$.

Note: Take $\xi \in \mathcal{H}_K$, $f_\mu^0 \in I(\mu)$

$$r(\xi) f_\mu^0 = \underbrace{\phi_\mu(\xi)}_{\text{a constant}} f_\mu^0$$

For fixed μ:

$\phi_\mu : \mathcal{H} \longrightarrow \mathbb{C}$ alg hom.

Recall: ring of regular fcns on $T^\vee$ is $\mathbb{C}[z_1, z_2, z_1^{-1}, z_2^{-1}]$.

Thm ("Satake isom")

The map

$$\xi \longmapsto \phi_\bullet(\xi)$$

gives an isomorphism of $\mathcal{H}_K \xrightarrow{\sim} \mathbb{C}[z_1, z_2, z_1^{-1}, z_2^{-1}]^W$ ← reg. fcns invariant under W.

"pf" sketch: check $\phi_\bullet(\xi) \in \mathbb{C}[\cdots]^W$

$$r(\xi) f_\mu^0 = \phi_\mu(\xi) \cdot f_\mu^0.$$

assume $\xi = \mathrm{char}(K\alpha K)$. , Assume $\mathrm{Vol}(K) = 1$

$$\Rightarrow \phi_\mu(\xi) = r(\xi) f_\mu^0(e) = \int_{K\alpha K} f_\mu^0(g)\, dg = \sum_i \int_K f_\mu^0(\alpha_i k)\, dk.$$

Write $K\alpha K = \coprod_i \alpha_i K$.

$$= \sum_i f_\mu^0(\alpha_i) \quad \text{write } \alpha_i = n_i a_i k_i$$

Just a poly in $\mathbb{C}[\cdots]$ $\quad = \sum \mu(a_i)\delta(a_i) = \sum z_1^{\mathrm{ord}\, a_{i_1}} z_2^{\mathrm{ord}\, a_{i_2}} \delta(a_i)$.

Recall:

$\mathcal{H}_K$ is generated by $T(p) :$ char$\left(K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K\right) = \phi(1,0)$.

$R(p) = \phi(1,1) :$ char $K\begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}K$

$R(p^{-1}) = \phi(-1,-1)$

consider $K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K = \begin{pmatrix}1 & \\ & \varpi\end{pmatrix}K \cup \bigcup\limits_{\substack{b \in \mathcal{O} \\ \text{mod } p}} \begin{pmatrix}\varpi & b \\ 0 & 1\end{pmatrix}K$

Toral part is $\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}$

run through calculation above:

$$\text{get} \quad \mu_2(\varpi)\, q^{1/2} + q\, \mu_1(\varpi)\, q^{-1/2} = q^{1/2}(z_1 + z_2)$$

$\delta\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}$

So $\phi_\mu(T(p)) = q^{1/2}(z_1 + z_2)$

For $K\begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}K = \begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}K$ get $\mu_1\mu_2(\varpi)$ (central character)

$\phi_\mu(R(p)) = z_1 z_2$.

$\phi_\mu(R(p^{-1})) = z_1^{-1} z_2^{-1}$

these generate all symmetric polys. so surjective

Need More to show isom.

---

we proved the Thm provided we've got all spherical reps.

Say (π, V) spherical

π^K = 1-dim'l of $\mathcal{H}_K$, $\pi^K(T(p)) = \alpha$

$\pi^K(R(p^{\pm 1})) = \beta^{\pm 1}$

Take (z_1, z_2) : roots of $X^2 - q^{-1/2}\alpha X + \beta$.

This means π came from one of our known spherical reps.

Remark:

Say $\xi = z_1 z_2^2 + z_1^2 z_2$ under the isom.

then

$r(\xi) f_\mu^0 = (z_1 z_2^2 + z_2 z_1^2) f_\mu^0$, i.e., get the eigenvalues.

Recall: μ unram.

Thm: $T(\mu)$ has a unique Wh. model provided $\mu_1\mu_2 \neq |\cdot|^{\pm 1}$

i.e.

$$\pi(\mu) \hookrightarrow W_\psi.$$

$$f_\mu^0 \longmapsto W_\mu^0$$ Let's get a formula for W_μ^0.

Recall: $G = NAK$

$$W_\mu^0(nak) = \psi(n) W_\mu^0(a).$$

and

$z = \begin{pmatrix} z & \\ & z \end{pmatrix}$ $$W_\mu^0(z \cdot g) = \underbrace{\omega_\pi(z)}_{\substack{\text{centr char} \\ (=\mu(z))}} W_\mu^0(g)$$

enough to consider $\begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix} = a_m$.

Thm: (Assume $c(\psi) = 0$)

$$W_\mu^0(a_m) = \begin{cases} 0 & \text{if } m < 0 \\ \left(q^{-\frac{m}{2}}\right) \dfrac{z_1^{m+1} - z_2^{m+1}}{z_1 - z_2} W_\mu^0(e) & \text{otherwise.} \end{cases}$$

Pf: $m<0$:

$$W_\mu^0(a_m) = W_\mu^0\left(\begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = W_\mu^0\left(\begin{pmatrix} 1 & \varpi^m x \\ & 1 \end{pmatrix} a_m \right)$$

ord $x > 0$

$= \psi(\varpi^m x) W_\mu^0(a_m)$ Now, $m<0 \Rightarrow$ choose x s.t. $\psi(\varpi^m x) \neq 0$

So $W_\mu^0(a_m) = 0$.

Next :

$$(T(p) W_\mu^0)(a_m) = W_\mu^0\left(a_m \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}\right) + \sum_b W_\mu^0\left(a_m \begin{pmatrix} \varpi & b \\ & 1 \end{pmatrix}\right).$$

$$= z_1 z_2 W_\mu^0(a_{m-1}) + \sum_b \psi(\cancel{\varpi^m b}) q W_\mu^0(a_{m+1})$$

$\to 1$ since $\varpi^m b \in \mathcal{O}$, cond$(\psi) = 0$.

But using other Model we know

$$q^{1/2}(z_1+z_2) W_\mu^0(a_m) = T(p) W_\mu^0(a_m)$$

So get recursive formula.

$m = -1 \Rightarrow 0$

$m = 0 \Rightarrow W_\mu^0(e)$ Now solve to get formula.

Finally: Unitary $\pi(\mu)$'s ?

unitarizable

Need a pos. def. Herm inner product:

Need:

1. $(\,,\,) : \pi \times \pi \longrightarrow \mathbb{C}$ $\mathbb{C}$-anti-linear in 2nd arg. (sesquilinear pairing.)

2. Hermitian: $(x, y) = \overline{(y, x)}$.

3. Pos. Def.

Unitary:

I' $\mu_1\mu_2^{-1} \neq |\,|^{\pm 1}$

μ_1, μ_2 unramified

π unitarizable
if
- $\exists$ $(\cdot,\cdot)$ sesquilinear
- Hermitian
- pos. def

---

we have:

$$I(\mu) \times I(\tilde{\mu}) \longrightarrow \mathbb{C}$$
$$I(\tilde{\mu}) = \widetilde{I(\mu)}$$

f_1, f_2 Consider $f_1(g) f_2(g)$

$$f_1(\tilde{n}\overset{b}{\tilde{a}} g) f_2(\tilde{n}\overset{b}{\tilde{a}} g) = \mu(b)\delta(b) \cdot \hat{\mu}(b)\delta(b) f_1(g) f_2(g)$$

$$= \delta^2(b) f_1(g) f_2(g) = \Delta(b) f_1(g) f_2(g)$$

so $f = f_1 f_2 \in S(G, \Delta)$ and $\exists!$ lin. functional $S(G,\Delta) \longrightarrow \mathbb{C}$ from before.

It is $\displaystyle\int_{B\backslash G} f_1(g) f_2(g)\, dg = \int_K f_1(k) f_2(k)\, dk$

$G = BK$

Now replace $I(\tilde{\mu})$ with $I(\bar{\tilde{\mu}})$ and def $(f_1, f_2) = \displaystyle\int_K f_1(k)\overline{f_2(k)}\, dk$.

this is a sesquilinear pairing on these 2 spaces

Want a pairing on $I(\mu) \times I(\mu)$ so need an int operator:

$$I(\mu) \longrightarrow I(\bar{\tilde{\mu}})$$

G-inv. sesquilinear pairings on $I(\mu) \iff \mathrm{Hom}_G(I(\mu), I(\bar{\tilde{\mu}}))$.

But:

$$\dim \mathrm{Hom}_G(I(\mu), I(\bar{\tilde{\mu}})) = \begin{cases} 1 & \text{if } \bar{\tilde{\mu}} = \mu \text{ or } \mu' \\ 0 & \text{otherwise.} \end{cases}$$

So If $\bar{\tilde{\mu}} = \mu$ $\iff$ μ_1, μ_2 are unitary characters.

$\mu_1^{-1} = \bar{\mu}_1$ i.e. $\mu_1 \bar{\mu}_1 = 1$

$\mu_2^{-1} = \bar{\mu}_2$

<u>Then</u>: $I(\mu)$ is unitarizable

The pairing is just:

$$(f_1, f_2) = \int_K f_1(k) \overline{f_2(k)} \quad \text{which is symm + pos. def} \checkmark$$

other case:

$\bar{\tilde{\mu}} = \mu'$ $\Rightarrow$ $\mu_1^{-1} = \bar{\mu}_2$ i.e. $\mu = (\mu_1, \bar{\mu}_1^{-1})$

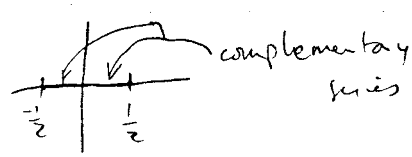
μ_1 = unramified character of $F^\times$ (actually don't need unramified)

$\mu_1(x) = \mu_0(x)|x|^s$ where $s \in \mathbb{R}$, μ_0 unitary.

So $(\mu_1, \bar{\mu}_1^{-1}) = \mu_0(|\,|^s, |\,|^{-s})$

So $I(\mu) = \mu_0 \otimes I(|\,|^s, |\,|^{-s})$. ; $I(s) := I(|\,|^s, |\,|^{-s})$

<u>Thm</u>: $I(s)$ is unitarizable $\iff$ $s \in (-\frac{1}{2}, \frac{1}{2})$



Consider:

$I(s) \times I(-s) \longrightarrow \mathbb{C}$ Have sesq. pairing here by above.

we need $\uparrow M(s)$ = intertwining op.

$I(s)$

$M(s): I(s) \longrightarrow I(-s)$.

$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

Let $f \in I(s)$, then $f(w^{-1}g)$ is s.t.: $f(w^{-1}ag) = f(a'w^{-1}g) = \mu(a')\delta(a')f(w^{-1}g)$.
$= \mu^{-1}(a)\delta^{-1}(a)f(w^{-1}g)$.

So def $M(s)f(g) = \int_N f(w^{-1}ng)\,dn$. ok for $Re(s) >> 0$.

$\exists$ analytic continuation to all s.

So Pairing is defined to be:

Note: $I(s)|_K \cong S(B\cap K \backslash K)$

$(B\cap K) = (N\cap K)(A\cap K)$
$\mu(a)\delta(a) = 1$ by unramified, or,
$\mu = (|\,|^s, |\,|^{-s})$

$$\begin{array}{ccc} I(s)|_K & \simeq & S(B\cap K\backslash K) \\ s \downarrow & \swarrow f & \\ I(s) & & \end{array}$$

$f_s(bk) = \mu_s(b)\delta(b)f(k)$.

D

$$(f_1, f_2)_s = \int_K M(s)f_{1,s}(k)\overline{f_2(k)}\,dk.$$

Furthermore $\overline{(y,x)} = c_s(x,y)_s$. by 1-dim. of the Hom space.

So enough to calc for one fcn to get c_s

$$\begin{array}{ccc} I(s) & \xrightarrow{M(s)} & I(-s) \\ \wr\wr & & \\ C(B\cap K\backslash K) & & \\ \rotatebox{90}{$\in$} & & \\ f^\circ \equiv 1 & f^\circ \rightsquigarrow f_s^\circ & \end{array}$$

$$(f^\circ, f^\circ)_s = \frac{1 - q^{-1-2s}}{1 - q^{-2s}}$$

$s \neq 0, -\frac{1}{2} \in \mathbb{R}$.
$\Rightarrow$ Hermitian (gives $c_s = 1$)
$s = 0$ unitarizable already (from before.)

Let $K_0(p) = \left\{ \begin{pmatrix} * & * \\ p & * \end{pmatrix} \right\} \subseteq K$ Iwahori subgp.

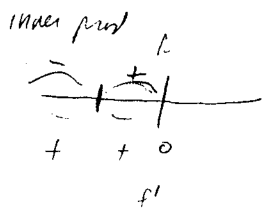
let $f' = \mathrm{char}(K_0(p)) \in C({}_{B\cap K}\backslash {}^{K})$

then:

$$(f', f')_s = \frac{1-q^{-1}}{1+q} \, \frac{q^{-2s}}{1-q^{-2s}}$$

$\Rightarrow s = -1/2$ is ok. , get non zero real #.

What about pos. def ?



only get them pos def for $(-\frac{1}{2}, 0)$

Same for $(0, \frac{1}{2})$

Kolla: 4/11

$I\chi \simeq \mu_0 \otimes \underbrace{I(|\,|^s, |\,|^{-s})}_{\underset{I(s)\ s\in\mathbb{R}}{\shortparallel}}$

μ_0: unitary char

$M(s) : I(s) \longrightarrow I(-s)$ intertwining operator.

$I(s) \simeq C(K\cap B\backslash K)$

$f_s \longleftarrow\!\!\shortmid f$

$$(f_1, f_2)_s = \int_K M(s) f_{1,s}(k)\, \overline{f_2(k)}\, dk.$$

We calculated $(f^0, f^0)_s$, $(f^1, f^1)_s$

For suitable f_0, f_1, found that non-unitarizable for $s < -1/2$, $s > 1/2$.

since $(f^0, f^0)_s < 0$

$(f^1, f^1)_s > 0$.

Let $(f_1, f_2)_s^* = (1 - q^{-2s})(f_1, f_2)_s$

We know $(\,,\,)_0^* > 0$ on $I(0)$

$V = C(K\cap B\backslash K)$

$\shortparallel$

$V(\rho) = \rho$ isotypic, fin. dim'l. $(\rho \in \hat{K})$

$(\,,\,)_0^* |_{V(\rho)} > 0$

if $v_1, \dots, v_N$ = basis. Get hermitian matrix $h_s = {}^t\overline{h_s}$.

Know: $h_0 > 0$

Let $\lambda_1(s), \dots, \lambda_N(s) \in \mathbb{R}$ = eigenvalues of h_s.

$\lambda_i(0) > 0$

$h_s > 0 \iff \lambda_i(s) > 0\ \forall i$.

[...]ce for some $s_0 \in (-1/2, 0)$, $\lambda_i(s_0) = 0$. Thus, $\exists\ v \in V(\rho) \subseteq V$ s.t. $(v, v')_{s_0} = 0$. $\forall\ v' \in V$. (we this by diagonaliz[...]

~~Let $f \in \mathcal{I}$~~ $f_{s_0} \in I(s_0)$ corresp. to v,

~~then $(f_{s_0}, f_{s_0})^* = 0$~~ so $0 \neq \operatorname{rad}(\,,\,)_{s_0}^* \subseteq I(s_0)$

~~So thus $(\pi(g) f, \pi(g) f(s))^*$~~

(G-inv. sub but $I(s_0)$ irred for $s_0 \in (-1/2, 0)$, [...] $|_{V(\rho)} > 0\ \forall$ [...]

Return to Satake Isom. (Reference: Cartier in Corvallis)

$G \supseteq K = GL_2(\mathcal{O})$

(π, V) s.t. $V^K \neq 0 \leftrightarrow \mathcal{H}_K$.

$\mathcal{H}_K \simeq S(G//K)$ (Satake's Proof)

Integral Formulas:

$G = BK$, $\int_K dg = \int_K dk = 1$

$B = NA$ $\quad \int_{N \cap K} dn = 1, \quad \int_{A \cap K} da = 1$

$b = na$

$\Downarrow$

$d\mu(b) = \delta(a)^{-2}\, dn\, da$ where $\delta(a) = |a_1/a_2|^{1/2}$.

left Haar meas.

$d\nu(b) = dn\, da$.

rt. Haar. meas

$$\int_G f(g)\,dg = \int_K\int_N\int_A f(nak)\,\delta(a)^{-2}\,dn\,da\,dk.$$

$$= \int_K\int_N\int_A f(kna)\,dk\,dn\,da.$$

$$\int_N f(ana^{-1})\,dn = \delta(a)^{-2}\int_N f(n)\,dn$$

Lemma:

let $\Delta(a) = |\det(Ad_{\mathfrak{n}}(a) - 1)|$

$= |a_1a_2^{-1} - 1| = \left|\frac{a_1 - a_2}{a_2}\right|$

N, $\mathfrak{n}$ = Lie alg of N.

For us $= (\text{span}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$

So $Ad_{\mathfrak{n}}(a)\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1a_2^{-1} \\ 0 & 1 \end{pmatrix}$

So

Then: if $\Delta(a) \neq 0$

$$\int_N f(n)\,dn = \Delta(a)\int_N f(nan^{-1}a^{-1})\,dn$$

proof:

$$n = n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Rightarrow n a n^{-1} a^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a_1 a_2^{-1} x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (1-a_1 a_2^{-1})x \\ 0 & 1 \end{pmatrix}$$

so everything follows.

If $f \in S(G)$, $a \in A$, $\Delta(a) \neq 0$.

$$F_f(a) = \Delta(a) \int_{G/A} f(gag^{-1})\, d\bar{g}$$

"orbital integral"

Fix a.

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/C_G(a) \cong \text{conj. class of } a = C(a) \\ G/A \arrow[ur] & \end{tikzcd}$$

$C_G(a)$: centralizer

Lemma:

If $f \in \mathcal{H}_K = S(G//K)$ and $\Delta(a) \neq 0$ then

$$\int_N f(na)\, dn = F_f(a).$$

Pf: Say $\phi \in S(G)$ left K-inv

$$\int_G \phi(g)\, dg = \int_N \int_A \phi(na)\, dn\, da \quad \text{(by int. formula above)}$$

(one of)

$$= \int_N \Big(\underbrace{\int_A \phi(na)\, da}_{\bar{\phi}_0(\bar{n})} \Big)\, dn.$$

$\bar{\phi}_0(\bar{n}) = f(nan^{-1})$

take $\phi_0(g) = f(gag^{-1})$

so $\Delta(a) \int_{G/A} f(gag^{-1})\, d\bar{g}$

$= \Delta(a) \int_N f(nan^{-1})\, dn$ by

$= \Delta(a) \int_N f(nan^{-1}a^{-1}a)\, dn. = \int_N f(na)\, dn$

by lemma.

∎

Defn: (Satake Transform)

Note: $A/_{A\cap K} \simeq \mathbb{Z}\times\mathbb{Z}$.
$(\varpi^r\ \varpi^s) \mapsto (r,s)$

$$S : S(G/\!/K) \longrightarrow S(A/\!/A\cap K)$$

let $\Lambda = A/_{A\cap K} \simeq \mathbb{Z}$

$$f \longmapsto \delta(a)^{-1}\int_N f(na)\,dn = \delta(a)\int_N f(an)\,dn.$$

Theorem:

S gives an isom. of algebras:

$$\mathcal{H}_K = S(G/\!/K) \xrightarrow{\sim} S(A/_{A\cap K})^W \simeq \mathbb{C}[\Lambda]^W.$$

proof: ⊙ $Sf(a)$ is invariant under $A\cap K$. clear. ✓

⊙ in $S(A)$. ? f is in $S(G)$ at least.

$$na = \begin{pmatrix} a_1 & x a_2 \\ 0 & a_2 \end{pmatrix} \quad \text{so done ✓.}$$

⊙ algebra homomorphism

S is a composition:

$$\begin{array}{c} S(G) \\ \cup \\ S(G/\!/K) \end{array} \xrightarrow[\text{res.}]{(1)} S(B) \xrightarrow[\int dn]{(2)} S(A) \xrightarrow[\text{Mult. by } \delta^{1/2}]{} S(A)$$

If all these are alg homs, so is S.

(3). let χ = char. T_1, T_2 distributions

$$((\chi T_1) * (\chi T_2))(g) = \chi\,(T_1 * T_2)$$

$$\chi(g)\int_G \chi T_1(g_1)\, \chi T_2(g_1^{-1}g)\,dg_1 \ ✓.$$

(1) let $f_1, f_2 \in S(G)$ Consider $(f_1 * f_2)(b) = \int_G f_1(g_1) f_2(g_1^{-1}b)\,dg_1$

$$= \int_K\int_B f_1(b_1k) f_2(k^{-1}b_1^{-1}b)\,dk\,d\mu(b). \quad \text{(Formula before)}$$

$$= \int_B f_1(b) f_2(b_1^{-1}b)\,d\mu(b) \quad \text{by } K \text{ invariance of fcns.}$$

②
$$\int_N (f_1 * f_2)(na)\,dn = \int_N \Big(\int_N \int_A f_1(n_1 a_1)\, f_2(a_1^{-1} n_1^{-1} n a)\, \underbrace{\delta(a_1)^{-2}\, dn_1\, da_1}_{d\mu(b_1)} \Big) dn.$$

$$\| \qquad \int f_1 * f_2$$

$$= \int_N \int_N \int_A f_1(n_1 a_1)\, f_2(a_1^{-1} n a)\, \delta(a_1)^{-2}\, dn\, dn_1\, da_1 \qquad n \mapsto a_1 n a_1^{-1}$$

$$= \int_N \int_N \int_A f_1(n_1 a_1)\, f_2(n a_1^{-1} a)\, dn_1\, dn\, da_1$$

$$\|$$

$$= \int f_1 * \int f_2. \quad \checkmark.$$

Weyl Invariance :

$$Sf(a) = \delta(a)^{-1} \Delta(a) \int_{G/A} f(gag^{-1})\, d\bar{g} \qquad \text{(by Lemma)}$$

let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K$

$Sf(waw^{-1})$

$$\left|\frac{a_2}{a_1}\right|^{1/2} \left| \frac{a_1 - a_2}{a_2} \right| = \frac{|a_1 - a_2|}{|a_1 a_2|^{1/2}} \quad \text{is } W\text{-inv.}$$

and

$$\int_{G/A} f(gwaw^{-1}g^{-1})\, d\bar{g} = \int_{G/A} f(w^{-1}gwaw^{-1}g^{-1})\, d\bar{g} \quad \text{by } K\text{-inv.}$$

$$\begin{aligned} G/A &\xrightarrow{\sim} G/A \\ g &\longmapsto wgw^{-1} \end{aligned} \qquad \text{so integral is invariant also.}$$

Must check iso.

Kudla

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①

$\underline{4}$. $S: \mathcal{S}(G/\!/K) \xrightarrow{\sim} \mathcal{S}(A/\!/A\cap K)^W \simeq \mathbb{C}(\Lambda)^W$, $\Lambda \simeq \mathbb{Z}\times\mathbb{Z}$

$$\begin{pmatrix} \varpi^r & \\ & \varpi^s \end{pmatrix} \longleftrightarrow (r,s)=\lambda \qquad \Lambda^+ : (r,s),\ r\geq s$$

typical basis elt: order basis.

$\mathrm{char}\left(K\begin{pmatrix} \varpi^r & \\ & \varpi^s \end{pmatrix}K\right)$ $\qquad$ $\mathrm{char}\left((A\cap K)\begin{pmatrix} \varpi^r & \\ & \varpi^s \end{pmatrix}\right) + \mathrm{char}\left((A\cap K)\begin{pmatrix} \varpi^s & \\ & \varpi^r \end{pmatrix}\right)$

ϕ_λ "

basis on this side.

$\lambda = (r,s)\in\Lambda^+$ $\qquad$ let $a_{\lambda'} = \begin{pmatrix} \varpi^{r'} & \\ & \varpi^{s'} \end{pmatrix}$ for $\lambda' = (r',s')\in\Lambda^+$.

Need to calculate:

$S(\phi_\lambda)(a_{\lambda'})$ to get coeffs for change of basis matrix.

Note: $S_0(f)(a) = \delta(a)^{-1}\int_N f(na)\,dn$

$$S\phi_\lambda(a_{\lambda'}) = \delta(a_{\lambda'})^{-1}\int_N \phi_\lambda(na_{\lambda'})\,dn$$

$$= q^{\frac{1}{2}(r'-s')}\ \mathrm{Vol}\left(N\cap Ka_\lambda K a_{\lambda'}^{-1}\right). \qquad \geq 0.$$

Consider $Na_{\lambda'}\cap Ka_\lambda K$.

$\left\{ g = \begin{pmatrix} \varpi^{r'} & x\varpi^{s'} \\ 0 & \varpi^{s'} \end{pmatrix} \right\}$, $x\in F$

$\det g = \varpi^{r'+s'}$

$\Rightarrow \mathrm{ord}\det g = r'+s'$

$pr(g) \leq s'$

$g = k_1\begin{pmatrix} \varpi^r & \\ & \varpi^s \end{pmatrix}k_2 \in M_2(F)\supseteq M_2(\mathcal{O})$

$\mathrm{ord}\det g = r+s.$

$pr(g) = s$ (biggest power of ϖ I can factor out to get something in $M_2(\mathcal{O})$) "primitive".

$\Rightarrow s\leq s'$

if intersection $\neq\emptyset$.

So $r+s = r'+s'$

$\Rightarrow r-r' = s'-s \geq 0.$

$\Rightarrow \lambda' = \lambda + \ldots$

So we get $\lambda = \lambda' + n\alpha$, $n \geq 0$

$\vdots$

get $r - s \geq 2n \geq 0$

since $-/$

$\lambda' \in \Lambda^+$

So $S\phi_\lambda = \sum_{\lambda' \leq \lambda} c(\lambda, \lambda') \psi_{\lambda'}$ and $c(\lambda,\lambda) > 0$.

So change of basis matrix is upper Δ with pos diag entries so we have an isom. ⊠

Archimedean Theory:

$F = \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$ $\quad G = GL_2(\mathbb{R})$.

$K = O(2)$. Let $\mathfrak{g}_0 = Lie(G) = M_2(\mathbb{R})$.

$\mathfrak{k}_0 \subseteq \mathfrak{g}_0$

$\shortparallel$

$Lie(K) = \mathbb{R} \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$; $\mathfrak{g} = (\mathfrak{g}_0)_\mathbb{C} \cong M_2(\mathbb{C})$

$U(\mathfrak{g})$: univ. env. alg.

$\cup$

$\mathfrak{z}(\mathfrak{g})$ = center.

Def: $(\mathfrak{g}, K)$-module V

V: $\mathbb{C}$-v.s. (no top) with

1. action of $\mathfrak{g}$, ie, $\mathfrak{g} \xrightarrow{\pi} End(V)$. Lie alg. hom
$[X,Y] \longmapsto \pi(X)\pi(Y) - \pi(Y)\pi(X)$

2. rep. of K. such that every $v \in V$ lies in a f.d. K-invariant subspace.

Compatibilities:

Say $v \in W \subseteq V$
fin. dim
K-inv.

then

$k \mapsto \pi(k)v$ is a map $K \longrightarrow W$ which is analytic (C^∞)
so

(3) $X \in \mathfrak{k}_0$

$$\frac{d}{dt}\left\{\pi(\exp(tX))v\right\}\Big|_{t=0} = \pi(X)v.$$

(4) $X \in \mathfrak{g}$, $k \in K$ (← Need this if K not connected)

$$\pi(k)\pi(X)v = \pi(Ad(k)X)\pi(k)v$$

(observe: $V = \bigoplus_{\sigma \in \hat{K}} V(\sigma)$).

Def: A $(\mathfrak{g}, K)$ module is admissible $\iff$ K-isotypics $V(\sigma)$ are finite dimensional.

Its irreducible $\Leftrightarrow$ no proper $(\mathfrak{g}, K)$ submodules

---

Shift notation, let $K = SO(2)$ (really using connected components G^+ pos. dets in G, $SO(2)$)

Find irred $(\mathfrak{g}, K)$ modules.

basis for $\mathfrak{g}$:

$H = -i\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ basis for $\mathfrak{k}$.

$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ — basis for $sl(2)$

$L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$

$Z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ basis for center.

Have: $[H, R] = 2R$

$[H, L] = -2L$

$[R, L] = H.$

$\mathfrak{Z}(\mathfrak{g})$ has basis Z, $C = H^2 + 2RL + 2LR.$

Casimir operator

Say $V = (\mathfrak{g}, K)$-module. $\qquad k_\theta = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$V = \bigoplus V(n) \qquad v \in V(n), \quad k_\theta \cdot v = e^{in\theta} v.$

$H \cdot v = n \cdot v.$

(Be careful
~~Note~~ $\pi(H) v = \frac{d}{dt} \{ \pi(\exp(tH)) v \} |_{t=0}$ is not correct.
can't exponentiate from $\mathfrak{k}$ to K
only from $\mathfrak{k}_0$ to K
$H \notin \mathfrak{k}_0$.)

Correct:

$\pi(H) v = -i \frac{d}{dt} \{ \pi(\exp(t \begin{pmatrix} & 1 \\ -1 & \end{pmatrix})) v \}_{t=0}$

Note: $R\, V(n) \subseteq V(n+2)$

since for $v \in V(n)$

$HRv = (HR - RH + RH) v$

$= [H, R] v + RHv$

$= [H, R] v + nRv \quad v \in V(n).$

$= (n+2) Rv.$

$L\, V(n) \subseteq V(n-2).$

Schur's lemma $\Rightarrow$ $\mathfrak{z}$ acts by scalars

Say $V =$ irred, adm. $(\mathfrak{g}, K)$ mod.

$V = \bigoplus_n V(n).$

Let $x_0 \in V(n) \quad x_0 \neq 0.$ Consider $U(\mathfrak{g}) x_0 \subset V.$

So know: $\pi(Z) = \mu, \quad \pi(C) = \lambda.$

Claim:

$V = \mathbb{C} \cdot x_0 \oplus \bigoplus_{m>0} \mathbb{C} \cdot R^m x_0 \oplus \bigoplus_{m>0} \mathbb{C} \cdot L^m x_0$ this is H-stable.

what about R, L?

3

Well,

$\boxed{m>0}$

$$^{m} x_0 \in V(n+2m-2)$$

$LR(R^{m-1})x_0$

Note: $C = H^2 + 2RL + 2LR$. ($= \lambda$. in rep^n)

$$= H^2 + 2(RL - LR) + 4LR$$

$$= H^2 + 2H + 4LR.$$

So if $x \in V(n)$ any n.

$$4LRx = (\lambda - n^2 - 2n)x.$$

$$4RLx = (\lambda - n^2 + 2n)x.$$

$\Longrightarrow LR(R^{m-1})x_0$ is a multiple of $R^{m-1}x_0$

So shows Stability under of hence the claim.

Lem: $\dim V(n) \leq 1$.

Kudla.
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Weig Notes:

$\cdot l$ $(\mathfrak{g}, K)$-Modules where $\mathfrak{g} = \mathrm{Lie}(GL_2(\mathbb{R}))_{\mathbb{C}}$, $K = SO(2)$, $\mathfrak{k} = \mathbb{C}\cdot H$.

Suppose (π, V) is irred $(\mathfrak{g}, K)$ mod.

$V(n) = \{ v \in V \mid H\cdot v = n\cdot v \}$

We showed that for any $x_0 \in V(n)$, $x_0 \neq 0$, $V = \mathbb{C}\cdot x_0 \oplus \bigoplus_{m>0} \mathbb{C}\cdot R^m x_0 \oplus \bigoplus_{m>0} \mathbb{C} L^m x_0$

$\dim V(n) \leq 1$. So $V = \oplus V(n)$, $\dim V(n) = 0, 1$

$C = H^2 + 2RL + 2LR = H^2 + 2H + 4LR = H^2 - 2H + 4RL$

$\exists\ \mu, \lambda \in \mathbb{C}$ s.t.

$\pi(Z) = \mu$
$\pi(C) = \lambda$ by Schur's lemma

Suppose $x \in V(n)$.

$4LRx = (\lambda - n^2 + 2n)\,x$
$4RLx = (\lambda - n^2 - 2n)\,x$ from above formulas for C.

So

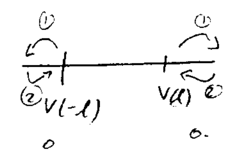
$-\frac{1}{4}\lambda = \frac{n}{2}\left(1 - \frac{n}{2}\right)$ gives vanishing in 1st formula.

$-\frac{1}{4}\lambda = -\frac{n}{2}\left(1 + \frac{n}{2}\right)$ for 2nd formula.

① Say $-\frac{1}{4}\lambda \neq \frac{n}{2}\left(1 - \frac{n}{2}\right)$ for any $n \in \mathbb{Z}$. $\Rightarrow$ K-types give an infinite string.
$\downarrow$
of the form $2\mathbb{Z}$, or, $1 + 2\mathbb{Z}$.

② $-\frac{1}{4}\lambda = \frac{l}{2}\left(1 - \frac{l}{2}\right)$ for some $l \in \mathbb{Z}$ at $V(l)$ Raising then lowering is 0.

at $V(-l)$ lowering then raising is zero



possibilities (take $l \geq 0$)

(i) $V(l+2) \neq 0 \Rightarrow l + 2 + 2\mathbb{Z}_+ = l + 2\mathbb{Z}_{>0}$.

K spectrums

(ii) $V(-l-2) \neq 0 \Rightarrow -l - 2\mathbb{Z}_{>0}$.

Possibility (iii), Have a f.d. repn.

this is the repn of $GL_2(\mathbb{R})^+$

$$g \cdot P(x,y) = P((x,y)g) \quad \text{on homogeneous Polys of degree } \ell.$$

$\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ — $(x,y)k_\theta \begin{pmatrix} 1 \\ i \end{pmatrix} = (x,y)\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\theta}$

---

if we go to $K = O(2)$, only question is how $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ acts.

$Ad\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} : R \leftrightarrow L.$ — this kind of gives a (reflection) symmetry about zero.

leaves H, Z invariant — switches raising + lowering and multiplies by -1.

$G = GL_2(\mathbb{R})$

$B = NA$.

$\mu = (\mu_1, \mu_2)$ $\quad \mu_i = $ quasi-char of $\mathbb{R}^\times \Rightarrow \mu_i(x) = \mathrm{sgn}(x)^{a_i} |x|^{s_i} \quad s_i \in \mathbb{C}, a_i = 0, 1$

$$I(\mu) = I_B^G(\mu) = \{ f : G \to \mathbb{C} \mid f(nag) = \mu(a) \left|\frac{a_1}{a_2}\right|^{1/2} f(g) \}$$

where also, f is (1) smooth i.e. $C^\infty(G)$

(2) K-finite

a fn is det. by res to $SO(2)$ $\left(G = BK, K = O(2) \text{ but } \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in A \cap K \right)$

K-spectrum:

Say $f \in I(\mu)$ consider $f|_{SO(2)}$.

$$f\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} k_\theta\right) = (-1)^{a_1+a_2} f(k_\theta) \qquad f_n(k_\theta) = e^{in\theta}.$$

So K-spectrum is $a_1 + a_2 + 2\mathbb{Z}$.

multiplicity free.

Work out action of R, L, H

$f_n \in I(\mu)$ any $n \in a_1 + a_2 + 2\mathbb{Z}$

$H \cdot f_n = n \cdot f_n$.

$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $R \cdot f_n = * \cdot f_{n+2}$

so $Rf_n(e) = *$ since $f_{n+2}(e) = 1$

$$= \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} + \frac{i}{2}\underbrace{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}_{= \begin{pmatrix} & 2 \\ 0 & \end{pmatrix} - \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}$$

so

$$\left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} f_n\right)(e) = \frac{d}{dt} f_n\left(\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}\right)\Big|_{t=0} = \left\{ e^{t(s_1 + 1/2)} e^{-t(s_2 - 1/2)} \right\}_{t=0} = s_1 - s_2 + 1$$

$\begin{pmatrix} 0 & 2 \\ & \end{pmatrix} f_n(e)$

so $\begin{pmatrix} & 2 \\ 0 & \end{pmatrix}$ acts by 0

$$f_n\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} = f_n(e)$$

so finally:

$$R f_n = \left(\frac{1}{2}(s_1 - s_2 + 1) + \frac{n}{2}\right) f_{n+2} = \frac{1}{2}(s_1 - s_2 + n + 1) f_{n+2}$$

$$L f_n = \frac{1}{2}(s_1 - s_2 - n + 1) f_{n-2}$$

$$C = (s_1 - s_2)^2 - 1.$$

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$G = \underline{GL_2(\mathbb{R})}$

Summary:

$G^+ = GL_2^+(\mathbb{R})$, $\det g > 0$ connected, $K^+ = SO(2)$.

Found $(\mathfrak{g}, K^+)$-modules

$\mathfrak{Z} = $ Center $U(\mathfrak{g}) = \mathbb{C}[Z, C]$

if $(\pi, V) = $ irred $(\mathfrak{g}, K^+)$ mod

then $\pi(Z) = \mu$ $\quad \mu, \lambda \in \mathbb{C}$. $\quad$ We can normalize to get $\mu = 0$

$\pi(C) = \lambda \quad \pi\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = (-1)^a \quad a = 0, 1.$ $\quad$ K-spectrum $\subseteq a + 2\mathbb{Z}$.

If $-\frac{1}{4}\lambda \neq \frac{l}{2}\left(1 - \frac{l}{2}\right)$; $l \equiv a \pmod 2$

then

$\exists !$ irred $(\mathfrak{g}, K^+)$-module, K-spectrum $= a + 2\mathbb{Z}$.

If $-\frac{1}{4}\lambda = \frac{l}{2}\left(1 - \frac{l}{2}\right)$; $l \in a + 2\mathbb{Z}$.

then:

if $l > 1$ $\exists$ 3 possibilities:

D_l^+ $\quad$ K spectrum $l, l+2, \ldots$

D_l^0 $\quad$ K spectrum $2-l, \ldots, l-2$

D_l^- $\quad$ K spectrum $-l, -l-2, \ldots$

This gives everything up to tensoring with characters o det.

If $l = 1$.

D_l^+ $\quad$ K spect. $1, 3, \ldots$

D_l^- $\quad$ K spect. $-1, -3, \ldots$

Now move up to $K = O(2)$, so add element $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

This takes $V(n) \to V(-n)$, and $Ad\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \cdot R = L$.

So for case 1, as for D_ℓ^0, there exist 2 extensions from $(\mathfrak{g}, K^+)$ to $(\mathfrak{g}, K)$

eg: $D_\ell^0 \otimes \text{sgn det} \not\cong D_\ell^0$.

But, just 1 ext of D_ℓ^+ (resp D_ℓ^-) $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ carries basis for D_ℓ^+ to D_ℓ^- vice versa

So there exists a unique irred ext

of $D_\ell^+ \oplus D_\ell^-$ to D_ℓ an irred $(\mathfrak{g}, K)$ mod. and $D_\ell \otimes \text{sgn det} \simeq D_\ell$.

(Clifford Theory on Lie Alg)

Back to $I(\mu)$ $\mu_i(x) = \text{sgn}(x)^{a_i} |x|^{s_i}$

$\omega(z) = \text{sgn}(z)^{a_1+a_2} |z|^{s_1+s_2}$ so $\mu = s_1 + s_2 = \pi(Z)$

$\lambda = (s_1 - s_2)^2 - 1$.

take $I(\mu) = |\det|^{\frac{s_1+s_2}{2}} I(\mu^0)$ where $\mu_1^0(x) = \text{sgn}(x)^{a_1} |x|^{\frac{s_1-s_2}{2}}$

$\mu_2^0(x) = \text{sgn}(x)^{a_2} |x|^{-\frac{s_1-s_2}{2}}$

let $s = \frac{s_1 - s_2}{2}$. Know: $\pi\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = (-1)^{a_1+a_2} = a$.

So

K-spectrum of $I(\mu) = a + 2\mathbb{Z}$

recall: basis for $V(n)$, f_n

$V = I(\mu)$ $f_n(e) = 1$.

(2)

$\boxed{n \in a + 2\mathbb{Z}}$

Here $Rf_n = \left(s + \frac{n+1}{2}\right) f_{n+2}$

$Lf_n = \left(s - \frac{n-1}{2}\right) f_{n-2}$

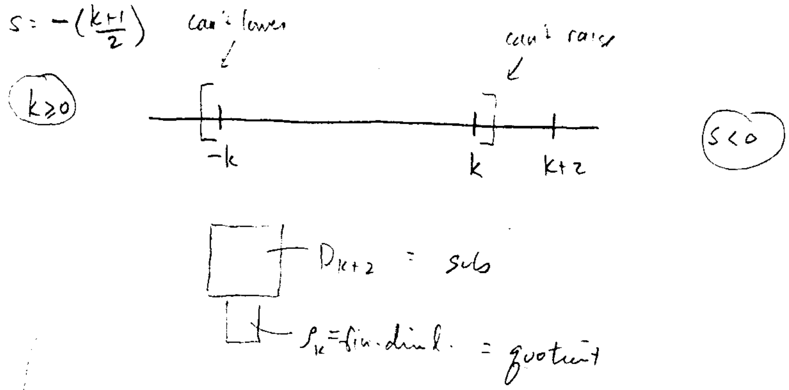
$\Rightarrow$ get a break for R. at $s = -\left(\frac{n+1}{2}\right)$

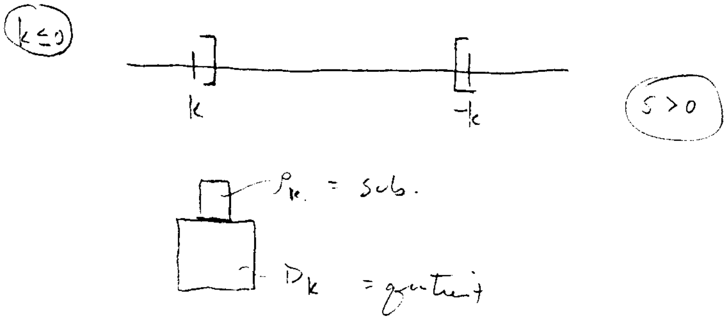
break for L at $s = \left(\frac{n-1}{2}\right)$.

Picture:

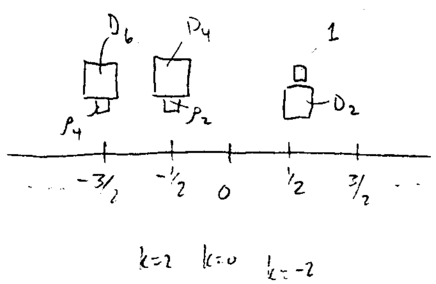
$a = 0$. K spect. is $2\mathbb{Z}$.

$s = -\left(\frac{k+1}{2}\right)$

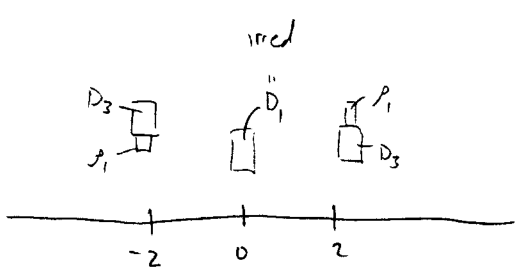




s-plane $a = 0$.



$a = 1$



f.d. ρ_{k-2} live with discrete series D_k..

Whittaker function:

N, ψ, eg: $\psi(x) = e(x)$

$W_\psi = \mathrm{Ind}_N^G(\psi)$, smooth, K-finite. ($\rightsquigarrow$ $(\mathfrak{g}, K)$-module).

let (π, V) = irred. admissible

$V \hookrightarrow W_\psi$ $(\mathfrak{g}, K)$-intertwining gives "weak" Whittaker Model.
(not unique.)

Say $f_n \in V(n)$

W_n = associated Wh. function.

$$W_n(z\,n(x)\,g\,k_\theta) = z^\mu \psi(x) e^{in\theta} W_n(g)$$

suffices to know W_n on $\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}$ for $y > 0$.

let $w_n(4\pi y) = W_n\left(\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right)$

(Normalize s.t. $-\frac{1}{4}\pi(C) = \nu$)

then we also know that

$$-\tfrac{1}{4}\pi(C) W_n = \nu W_n.$$

so for $g \in GL_2^+(\mathbb{R})$ we write:

$$g = z \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k_\theta$$

Consider $C^\infty(G^+)$ have $R, L, \ldots$ acting:

$$r(R) = e^{2i\theta}\left(2iy\,\partial + \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

$$r(L) = e^{-2i\theta}\left(-2iy\,\bar\partial - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

$$-\frac{1}{4}r(C) = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x\,\partial\theta}$$

where $\overset{\tau}{\overset{\shortparallel}{x+iy}} \in \mathfrak{h}$, $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & \\ & y^{-1/2}\end{pmatrix}$

$$\Rightarrow \partial = \frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\bar\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

take $\psi(x) = e^{\frac{i\pi x}{2}}$ to get d.e.

$$w_n'' + \left\{-\frac{1}{4} - \frac{n}{2y} + \frac{\nu}{y^2}\right\}w_n = 0$$

Solutions: (Whittaker-Watson 1903)

take: $(\sigma - \frac{1}{2})^2 = \frac{1}{4} - \nu$

Sol:

$$W_{\frac{n}{2},\sigma-\frac{1}{2}}(y) \xrightarrow[\text{conditions}]{\text{Growth}} e^{-y/2}y^{n/2}$$

as $y \to \infty$.

$$W_{-\frac{n}{2},\sigma-\frac{1}{2}}(-y) \dashrightarrow e^{y/2}(-y)^{-n/2}$$

So imposing growth condition on Wh. model, i.e., slowly increasing as $y \to \infty$.

this gives unique choice $W_{\frac{n}{2},\sigma-\frac{1}{2}}(y)$.

So Thm:

$\exists\,!$ Wh. Model.

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1

$G = GL_2(\mathbb{R}) \supset GL_2^+(\mathbb{R})$, ψ = fixed add. char $= e(x)$

$W_\psi = \mathrm{Ind}_N^G(\psi)$ $\quad (\mathfrak{g}, K^+)$

(π, V) irred adm. $\quad$ (?) $\pi \hookrightarrow W_\psi$.

Central char:

$\mu = \pi(Z)$

$\nu = -\frac{1}{4}\pi(C)$

$n = K^+$-type

Last time: let $W_\psi(\mu,\nu,n) = \{ \phi \in W_\psi \mid \phi = \text{smooth},\ \mu = r(Z)\phi,\ \nu = -\frac{1}{4}r(C)\phi,\ \phi(gk_\theta) = e^{in\theta}\phi(g),\ \phi \text{ has moderate growth} \}$

define

$\|\cdot\|$ on G "height".

$G \longrightarrow SL_4(\mathbb{R})$

$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & {}^tg^{-1} \end{pmatrix} = (g_{ij})$, $1 \le i,j \le 4$

$\|g\| = \max_{i,j} |g_{ij}|$

idea: G = affine alg. group, $G \subseteq \mathbb{A}^N$ (affine space) = Zariski closed subset of affine space.

$GL_2 \subseteq M_2 = \mathbb{A}^4$

def by $\det g \neq 0$ not closed so put GL_2 in a bigger space...

$\|g_1 g_2\| \le 2\|g_1\|\,\|g_2\|$

$\|g^{-1}\| = \|g\|$

let $g = z\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k_\theta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^a$

$\phi \in C(G)$ is of moderate growth $\Leftrightarrow \exists N, C$ s.t. $|\phi(g)| \le C\|g\|^N \ \forall g \in G$.

Fact: $W_\psi(\mu,\nu,n)$ has dim. 1.

Theorem (Harish-Chandra)

$G = SL_2(\mathbb{R})$ $\quad \mathfrak{Z} = \mathbb{C}[C] \supset \mathfrak{a}$ (ideal $\neq 0$) so $\mathfrak{Z}/\mathfrak{a}$ is fin.-diml.

$K = SO(2)$.

Suppose $f \in C^\infty(G)$. s.t. f is K-finite and $\mathfrak{Z}$-finite. $\longleftrightarrow r(\mathfrak{a})\cdot f = 0$ or equiv. $r(\mathfrak{Z}) f$ = fin.-diml.

Then:

(so here $\mathfrak{a} = (\prod_j (C-\lambda_j)^{n_j})$.)

1. f is analytic.
2. $U(\mathfrak{g}) f$ is a admissible $(\mathfrak{g}, K)$-module.
3. $\exists\ \alpha \in C_c^\infty(G)$ s.t. $\alpha(kxk^{-1}) = \alpha(x)$ and $f * \alpha = f$.
4. if $|f(g)| \le C\|g\|^N$ then $\forall\ D \in U(\mathfrak{g})$ $\ |Df(g)| \le C'\|g\|^N$ (uniform moderate growth)

Corollary: π = irred admissible, has at most one Whittaker Model with Moderate growth.

pf: part 4.

raise and lower ϕ above (in W_ψ) stays of Moderate growth, so generates a Wh. model.

In fact π has a unique Wh. Model $\Leftrightarrow$ π is ∞-diml.

proof:

(π, V) = irred adm. we get (μ, ν, a) and K-spectrum

Take $n \in K$-spectrum

take $\phi_n \in W_\psi(\mu, \nu, n)$, $\phi_n \neq 0$. Consider submodule $\mathbb{C}\cdot\phi_n \oplus \left(\bigoplus_{i>0} \mathbb{C}R^i\phi_n\right) \oplus \left(\bigoplus_{i>0} \mathbb{C}L^i\phi_n\right)$

Case 1 $\nu \neq \frac{\ell}{2}(1-\frac{\ell}{2})$ for $\ell \in K$-spectrum.

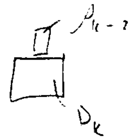
So π = irred pr. series, K spectrum $= a + 2\mathbb{Z}$

then the subbundle we wrote down is π (raising & lowering are never zero)

Now take D_k - discrete series (or limit of if $k=1$)

$$D_k = D_k^- \oplus D_k^+$$

take $n = k > 0$

exercise
~~pf~~ conjecture : get copy of  pieces in W_+. So don't have a wh model for ρ_{k-2}

<u>Global</u> : $G = GL(2)/F$ = # field

$G(\mathbb{A}) = \prod_v{}' G(F_v)$: $G(F_v) = K_v$ a.e. v where $K_v = \begin{cases} O(2) & v \text{ real} \\ GL_2(\mathcal{O}_v) & v \text{ non-arch} \end{cases}$

$G_\infty = \prod_{v \in S_\infty} G(F_v)$, $G(\mathbb{A}_f) = \prod_{v \in S_f}{}' G(F_v)$

$(\mathfrak{g}_\infty, K_\infty)$

prod of $O(2)$'s

$= Lie(G_\infty)_{\mathbb{C}}$

(π, V) $\mathbb{C}$-v.s.

An irred admissible "repn" of $G(\mathbb{A})$ means

$V = (\mathfrak{g}_\infty, K_\infty)$ - module, (every vector is K_∞-finite) a $G(\mathbb{A}_f)$ - module (actions commute with each other)

<u>smooth</u> : as a repn of the l-group $G(\mathbb{A}_f)$

<u>admissible</u>: K- isotypics are finite dim. ($K = K_\infty \cdot K_f$)

<u>irred</u>: No proper submodules.

<u>eg</u>: (π_v, V_v) - irred adm repns of G_v $v \in S_f$.

$(\mathfrak{g}_v, K_v)$ if $v \in S_\infty$

s.t. for almost all $v \in S_f$ (π_v, V_v) is unramified, i.e., $V_v^{K_v}$ = 1-dim.

Fix $0 \neq \xi_v^0 \in V_v^{K_v}$. Let $V = \bigotimes_v{}' V_v$ w.r.t. $\{\xi_v^0\}$

i.e.,
$$V = \varinjlim_{\substack{T \subset S \\ |T| < \infty}} \left(\bigotimes_{v \in T} V_v \right)$$

(Call $\left(\bigotimes_{v \in T} V_v\right) = V_T$ and assume $T \supset S_0 = S_\infty \cup v$ finite and there is no ξ_v^0)

where if $T' \supset T$

$$V_T \longrightarrow V_{T'}$$
$$u \longmapsto u \otimes \left(\bigotimes_{v \in T' - T} \xi_v^0 \right)$$

There is an irred adm. repn of $G(\mathbb{A})$.

eg: Let $\mu = (\mu_1, \mu_2)$ chars of $\mathbb{A}^\times / F^\times$ - gives a char of $B(\mathbb{A})$

define $I(\mu) = \left\{ f : G(\mathbb{A}) \to \mathbb{C} \;\middle|\; f(nag) = \mu(a) \left| \frac{a_1}{a_2} \right|_{\mathbb{A}}^{1/2} f(g) \right\}$

f is smooth, K-finite

recall from Tate: $\mu_i = \bigotimes_v' \mu_{iv}$

Claim: $I(\mu) \simeq \bigotimes_v' I(\mu_v)$ $\qquad \xi_v^0$ are K_v-invariant fns taking value 1 at e.

exercise: prove ↗

and suppose $\mu_1 \mu_2^{-1} = |\,|^{\pm 1}$

What is composition series of $I(\mu)$?

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2

Theorem: $G = SL_2(\mathbb{R})$, $K = SO(2)$. $f \in C^\infty(G)$ s.t. f is K-finite and $\mathfrak{z}$-finite ($U(\mathfrak{g}) \supset \mathfrak{z} = \mathbb{C}[C]$).

Then:

(i) f is analytic

(ii) $U(\mathfrak{g}) f$ is admissible $(\mathfrak{g}, K)$-module.

(iii) $\exists\, \alpha \in C_c^\infty(G)$, Ad K invariant, s.t. $f * \alpha = f$.

(iv) if $|f(g)| \le c\|g\|^N$ then $\forall\, D \in U(\mathfrak{g})$, $|Df(g)| \le c'\|g\|^N$.

proof:

$H \in \mathfrak{g}$ ($\mathfrak{k} = \mathbb{C}\cdot H$)

let $\mathcal{R} = \mathbb{C}[H, C]$. Then $\mathcal{R}\cdot f$ is finite diml is equivalent to f being K, $\mathfrak{z}$-finite.

PBW: (Poincaré, Birkhoff, Witt)
says monomials $H^i R^j L^k$ span $U(\mathfrak{g})$

R deg $= +1$ so deg $R^j L^k H^i = j - k$.
L deg $= -1$.

Lemma: $\mathcal{R} = U(\mathfrak{g})_0$ ← deg 0 elts.

Pf:
Consider $R^j L^j H^i$ (typical deg 0 elt).

Show its in $\mathcal{R}$:

$$C = H^2 + RL + LR = H^2 - H + 2RL.$$

$$\Rightarrow RL = \tfrac{1}{2}(C - H^2 + H)$$

$$\Rightarrow (RL)^j \in \mathcal{R}.$$

$(RL)^j = R^j L^j$ + lower degree. (done by induction)

Thus: $U(\mathfrak{g}) = \mathcal{R} \oplus \left(\bigoplus_{i>0} L^i \mathcal{R}\right) \oplus \left(\bigoplus_{i>0} R^i \mathcal{R}\right)$.

(e.g. $R^j L^k H^i = R^{j-k} \underbrace{R^k L^k H^i}_{\in \mathcal{R}}$)

So now

$$U(\mathfrak{g})f = \mathcal{R}f + \left(\sum_i L^i \mathcal{R}f\right) + \left(\sum_i R^i \mathcal{R}f\right)$$

$\mathcal{R}f$ = fin. dim., take $f_1, \dots f_\ell$ basis.

$$= \bigoplus_{i=1}^{\ell} \underset{[-M,M]}{\mathbb{C}\cdot f_i} + \left(\sum_j \sum_i \mathbb{C} L^j f_i\right) + \left(\sum_j \sum_i \mathbb{C} R^j f_i\right).$$

$[-M,M]$, $[-M,M]-2$, ⋮ $\qquad$ $[-M,M]$, $[-M,M]+2$, ⋮

take $n \in \mathbb{Z}$.

f_i are all K-finite so $\exists M$ so all K-types for f_i are in $[-M,M]$.

For fixed $n \in \mathbb{Z}$.

at some point n is not in the translates of the intervals $[-M,M]+2 \dots$, $-2 \dots$.

So get admissibility.

$C^\infty(G)$ with seminorms:

take $\phi \in C^\infty(G)$, $|(H^i R^j L^k \phi)(g)|$

Consider $\omega \subseteq G$ cpt set.

Define $\nu_{\omega,N}(\phi) = \max\limits_{\substack{i+j+k \le N \\ g \in \omega}} |(H^i R^j L^k \phi)(g)|$

a seminorm $\rightarrow$ gives a Frechet topology. (See Rudin: Funct. Anal.)

G acts continuously.

$U(\mathfrak{g})f$.

$V_0 \subseteq V$ = smallest closed G-inv. subspace containing f.

$E_n : V \longrightarrow V(n)$ (Projection maps)

$$\phi \longmapsto (E_n\phi)(g) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\phi(gk_\theta)\,d\theta$$

②

f K-finite $\Rightarrow \exists N$ s.t. $f = \sum_{n=-N}^{N} E_n f$.

recall
$$-\tfrac{1}{4} C = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y \frac{\partial^2}{\partial x \partial \theta}.$$

$E_n f$ is killed by $\prod_i (C - \lambda_i)^{r_i} = 0$ (since E_n commutes with C)

but on $E_n f$
$$-\tfrac{1}{4} C = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iny \frac{\partial}{\partial x}.$$

a power of this ? is elliptic.

So $\exists$ a thm. if $Cf = 0$, C elliptic then f analytic.

theorem
closure of V_0 : $cl(V_0) = V$

lemma: $cl(V_0) = V$

Pf: If not: $cl(V_0) \subsetneq V$, Then $\exists$ cont. lin. f.l. $\Lambda \neq 0 : V \to \mathbb{C}$

s.t. $V_0 \subseteq \mathrm{Ker}(\Lambda)$.

Say $F \in V \subset C^\infty(G)$

let $\phi_F(g) = \Lambda(r(g)F)$. If $D \in U(\mathfrak{g})$ then $D\phi_F = \phi_{DF}$.

So if $f = \eta$ finite K-finite, so is ϕ_f.

ϕ_f is analytic

so $(D\phi_f)(e) = \phi_{Df}(e) = \Lambda(\underset{\substack{\uparrow \\ V_0}}{Df}) = 0. \Rightarrow \phi_f \equiv 0$ (all derivs vanish at e!)

So $0 = r(g)\phi_f(e) = \phi_{r(g)f}(e) = \Lambda(r(g)f)$

(iii) Let $\mathcal{E} = \{ \alpha \in C_c^\infty(G) \mid \alpha = Ad(K) \text{ invariant} \}$.

$$r(k)(f * \alpha) = (r(k)f) * \alpha.$$

$$(f*\alpha)(g) = \int_G f(g_1)\alpha(g_1^{-1}g)\,dg_1 = \int_G f(gg_1)\,\alpha(g_1^{-1})\,dg_1$$

$$= \int_G \alpha(g_1)\big(r(g_1^{-1})f(g)\big)\,dg_1 \in V$$

Now: $V \supseteq \bigoplus_n V(n) = V_0$

$$\cup$$

Say $f \in \bigoplus_{n=-M}^{M} V(n) = U$ f.d.

above shows $f*\alpha$ in U also.

$\{\alpha_m\}$: Dirac Sequence.

Consider $f*\alpha_m \to f$ $\underset{\text{wrt } \|\cdot\|_\infty}{\underline{\text{uniformly}}}$ on any cpt set $\bar{\omega} \supseteq \omega$ = open.

$$\begin{tikzcd}
C^\infty(G) \arrow[r, "\text{res}"] & C^\infty(\omega) \text{ with } \|\cdot\|_\infty \\
V \arrow[u, phantom, "\cup"] & \\
U \arrow[u, phantom, "\cup"] \arrow[uur, dashed] &
\end{tikzcd}$$

(this is continuous)

can take ω s.t. $\exists$ injection

f, $f*\alpha_m$ $\in U$ = f.d. so gets usual topology.

Now $(f*\alpha_m)|_\omega \xrightarrow{u} f$ in $\|\cdot\|_\infty$.

Now $f*\mathcal{E} \subseteq U$ so $f \in cl(f*\mathcal{E}) \Rightarrow f \in f*\mathcal{E}$

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Recall: $G = SL_2(\mathbb{R})$, $K = SO(2)$, $f \in C^\infty(G)$, $\mathfrak{Z}$-finite, K-finite.

We proved $\exists$ $\alpha \in C_c^\infty(G)$, Ad K-invariant s.t. $f * \alpha = f$.

(iv) If $|f(g)| \leq C\|g\|^N$ $\forall$ $g \in G$ then for all $D \in U(\mathfrak{g})$, $|Df(g)| \leq C'\|g\|^N$.

Pf:

Let $X \in \mathfrak{g}_0$. $(Xf)(g) = \big(X(f*\alpha)\big)(g)$

$$= \frac{d}{dt}\left\{ \int_G f(g_1)\,\alpha(g_1^{-1} g \exp(tX))\, dg_1 \right\}_{t=0}.$$

$$gg_1 \longmapsto g_1$$

$$= \frac{d}{dt}\left\{ \int_G f(gg_1)\,\alpha(g_1^{-1}\exp(tX))\, dg_1 \right\}_{t=0}$$

$$= \int_G f(gg_1)\,(X\alpha)(g_1^{-1})\, dg_1$$

$$= f * X\alpha.$$

So $\underset{\displaystyle Df}{\underset{\|}{D(f*\alpha)}} = f * D\alpha$.

So $|Df(g)| = |f * D\alpha(g)| = \left| \int_G f(gg_1)(D\alpha)(g_1^{-1})\,dg_1 \right| \leq \int_G C\|gg_1\|^N |D\alpha(g_1^{-1})|\,dg_1$

$$\underset{(\|gg_1\| \leq 2\|g\|\|g_1\|)}{=} C'\|g\|^N \int_G \|g_1\|^N |D\alpha(g_1^{-1})|\, dg_1$$

$$\leq \underset{\text{depends only on } \alpha,\, D}{C''}\|g\|^N \quad \checkmark.$$

Remark: $SL_2(\mathbb{R})\, Z(\mathbb{R}) = GL_2^+(\mathbb{R})$

add $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ so extend $\mathfrak{z}$-finiteness, K-fin. to $GL_2(\mathbb{R})$ similarly.

F, $G(\mathbb{A})$, irred adm. "repn" — $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ module.

Thm: Given any π irred adm. "repn" of $G(\mathbb{A})$, then there exists a collection of (π_v, V_v) $\left(\begin{array}{l}\text{irred adm of } G_v \ v \text{ finite}, \ (\mathfrak{g}_v, K_v)\text{- for } v \text{ arch} \\ \text{for almost all } v,\ \pi_v \text{ unramified}, \ \xi_v^0 \in V_v^{K_v}\end{array}\right)$ s.t. $\pi \simeq \bigotimes_v{}' \pi_v$. Moreover, local components are unique up to iso.

---

recall:

$G = GL(1)$ $\quad G(\mathbb{A}) = \mathbb{A}^\times$

ω = char of $\mathbb{A}^\times$

$\omega = \bigotimes_v \omega_v$ But interesting characters were Hecke chars: $\omega : \mathbb{A}^\times / F^\times \longrightarrow \mathbb{C}^\times$

Def: $\mathcal{A}(G)$ = space of automorphic forms on $G(\mathbb{A})$

$f \in \mathcal{A}(G)$ satisfies

(i) $f : G(\mathbb{A}) \longrightarrow \mathbb{C}$ smooth $\begin{cases} g_\infty \longmapsto f(g_\infty g_f) & \text{Fix } g_f \text{; smooth in arch. sense} \\ g_f \overset{\text{Fix } g_\infty}{\longmapsto} f(g_\infty g_f) & \text{smooth in } p\text{-adic sense.} \end{cases}$

(ii) $\forall \gamma \in G(F)$, $g \in G(\mathbb{A})$, $f(\gamma g) = f(g)$.

(iii) $\mathfrak{z}$-finite

(iv) K-finite $(K \subseteq G(\mathbb{A}) : K = \prod_v K_v)$

(v) Moderate Growth $|f(g)| \le C \|g\|^N$ where $\|g\| = \prod_v \max\{|g_{rs}|_v\}$ (this is a finite prod.)

get: $\|g_1 g_2\| \le C \|g_1\| \|g_2\|$

recall: $g \mapsto \begin{pmatrix} g & \\ & {}^t g^{-1} \end{pmatrix} - (a_{rs})$
$\qquad \cap$
$\qquad M(4)$.

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②

Also get:

$\|\cdot\| > c \quad \forall g \in G(\mathbb{A})$, (See Moeglin-Waldspurger).

<u>Claim</u>: $\mathcal{A}(G)$ is a "representation" of $G(\mathbb{A})$.

Pf:

Let $g \in G(\mathbb{A}_f)$, $f \in \mathcal{A}(G)$.

$$r(g)f(x_\infty x_f) = f(x_\infty x_f g) \quad \text{so (i) is immediate.}$$

because $x_\infty \longmapsto f(x_\infty x_f g)$ is smooth.

Also:

$x_f \longmapsto r(g)f(x_\infty x_f) = f(x_\infty x_f g)$ is loc. constant.

(ii) Act on right, so left invariance is fine.

(iii) $\mathfrak{z} \supset \mathfrak{a}$, $r(\mathfrak{a})f \equiv 0$, not affected by rt. trans. by $g \in G(\mathbb{A}_f)$.

(iv) obvious

(v) $|r(g)f(x)| = |f(gx)| \le C\|gx\|^N \le \underbrace{C'\|g\|^N}_{C''}\|x\|^N$ ✓.

---

for $(\mathfrak{g}, K_\infty)$ action:

(i)–(iii) easy.

(iv), (v) follow from Harish-Chandra's Thm for $GL_2(\mathbb{R})$

Thm: (i) $\mathcal{A}(G)$ is a rep. of $G(\mathbb{A})$.

(ii) $f \in \mathcal{A}(G)$ $f * \mathcal{H} = r(\mathcal{H}) f$. = admissible $\mathcal{H}$-module.

$$\left(\begin{array}{l} \mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f \ , \ \mathcal{H}_f = S(G(\mathbb{A}_f)). \\ \mathcal{H}_\infty = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathcal{E}_{K_\infty} \text{ where } \mathcal{E}_{K_\infty} \text{ is spanned} \\ \text{by } \dfrac{\dim(\sigma)^{-1} tr(\sigma)}{Vol(K_\infty)} = \mathcal{E}_\sigma \text{ where } \sigma \in \hat{K_\infty} \end{array} \right)$$

(iii) Given an ideal $\mathfrak{a} \subseteq \mathfrak{z}$ finite codim.

let $\mathcal{A}(G, \mathfrak{a}) := \{ f \in \mathcal{A}(G) \mid r(\mathfrak{a}) f \equiv 0 \}$

then (H-Ch.) $\mathcal{A}(G, \mathfrak{a})$ = admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ module.

, ie., Given any σ = irred repn of $K = K_\infty K_f$ then

$\mathcal{A}(G, \mathfrak{a}, \sigma)$ is finite dim'l,

(σ isotypic)

Another reference: Borel - Jacquet in Corvallis

Def: (π, V) an irred. rep. of $G(\mathbb{A})$ is automorphic if it occurs as a constituent of $\mathcal{A}(G)$

ie. $\exists$ invariant subspaces $U_2 \subseteq U_1 \subseteq \mathcal{A}(G)$

s.t. $V \simeq U_1 / U_2$.

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Corollary: Such a (π, V) is admissible.

pf:

$$\begin{array}{c} \mathcal{H}f \\ \downarrow \end{array}$$

Let $f \in U_1 - U_2$. Then $V \simeq U_1/U_2 \longleftarrow \mathcal{H}\cdot f + U_2/U_2 \simeq \mathcal{H}f/U_2 \cap \mathcal{H}f$.

these are all intertwining maps and $\mathcal{H}\cdot f$ is admissible so done.

Cor: Any automorphic rep. (π, V) has a factorization $\pi \simeq \bigotimes_v{}' \pi_v$

Thm (very special case of Langlands (in Corvallis))

$\mu = (\mu_1, \mu_2)$ Hecke chars, i.e., on $\mathbb{A}^\times/F^\times$.

We have $I(\mu)$ the global induced rep. $\left(I(\mu) = \bigotimes_v{}' I(\mu_v) \right)$

then every constituent of $I(\mu)$ is automorphic. (!)

Consider: $G = GL(1)$.

What is $\mathcal{A}(G)$?

$f : \mathbb{A}^\times \longrightarrow G$ smooth

take: $F = \mathbb{Q}$. ex. Determine $\mathcal{A}(G)$ for $G = GL(1)$, $F = \mathbb{Q}$

Hint: What does $\mathcal{Z}$-finite mean?

$\mathcal{Z} = \mathfrak{g} = \mathbb{C}[x\partial]$ ← Euler operator E

since:

$G(\mathbb{R}) = \mathbb{R}^\times$

$$(Xf)(x) = \frac{d}{dt}\left(f(xe^t)\right)\Big|_{t=0} = x \cdot f'(x)$$

Say:

$(E - \lambda) f = 0$

$\Rightarrow xf' - \lambda f = 0$

$\Rightarrow f = x^\lambda$ looks like arch. component of Hecke char.

But:

$\prod_j (E - \lambda_j)^{n_j} = \mathfrak{a}$ so annihilated by $\mathfrak{a}$ means sol. space ha

Say $f(x) = x^\lambda \ln(x)$

$$f(xg) = g^\lambda x^\lambda (\ln(x) + \ln(g))$$

$$= g^\lambda [x^\lambda \ln(x) + x^\lambda \ln(g)]$$

group action: $\mu \cdot \begin{pmatrix} 1 & \ln|g| \\ & 1 \end{pmatrix}$

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$\mathcal{A}$) , (π, V) = irred auto. rep. of $G(\mathbb{A})$. $\Leftrightarrow$ irred. constituent of $\mathcal{A}(G)$. (occurs as sub quotient.)

Cor:
(π, V) irred. automorphic $\Rightarrow$ admissible., hence factorizable, ie., $\pi \simeq \otimes_v' \pi_v$. π_v local components.

Constant term:

$$f \in \mathcal{A}(G). \quad f_N(g) = \int_{N(F)\backslash N(\mathbb{A})} f(ng)\, du.$$

Fourier expansion:

$f \in \mathcal{A}(G) \Rightarrow f: G(F)\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$

Fix $g \in G(\mathbb{A})$ then get a fcn

$\mathbb{A}/F \simeq N(F)\backslash N(\mathbb{A}) \longrightarrow \mathbb{C}$ and $\widehat{\mathbb{A}/F} \simeq F$ Fix ψ of $\mathbb{A}/F$.

$n \longmapsto f(ng)$ $\qquad \psi_\alpha \longleftarrow \alpha$.

So get fourier exp. of f with fourier coeffs:

$$f_\alpha(g) = \int_{N(F)\backslash N(\mathbb{A})} \psi(-\alpha x)\, f(n(x) g)\, dx.$$

So expansion is

$$f(n(x)g) = \sum_\alpha f_\alpha(g)\, \psi_\alpha(x)$$

Set $x = 0$;

$$f(g) = \sum_\alpha f_\alpha(g)$$

So const. term = 0^{th} fourier coeff. = $f_N(g)$

Key idea: Growth of f is controlled by growth of $f_N(g)$ = const. term
(Recall const. term of ⊕ from beginning)

$A(G)$
$\cup$
$A_0(G) = \{ f \in A(G) \mid f_N \equiv 0 \}$ = cuspidal automorphic forms.
$\uparrow$
$G(\mathbb{A})$ invariant

(π, V) = cuspidal irred. aut. repn if it occurs in $A_0(G)$.

Fix ω a (quasi) character of $\mathbb{A}^\times / F^\times$

let $A(G, \omega) = \{ f \in A(G) \mid f(zg) = \omega(z) f(g) \ \forall z \in Z(\mathbb{A}) \}$.
$Z(\mathbb{A}) \cong \mathbb{A}^\times$

if χ = quasi-char of $\mathbb{A}^\times / F^\times$, $f \in A(G)$ then $\chi \cdot f \in A(G)$ ($\chi = \chi \circ \det$ here)

if ω = central character of f, ie, $f \in A(G, \omega)$

then $\chi \cdot f \in A(G, \omega \chi^2)$

So $|\omega|^{-1/2} \cdot f$ has central character $\frac{\omega}{|\omega|}$ which is unitary.

So we can assume that ω is unitary.

Fact: $Z(\mathbb{A}) G(F) \backslash G(\mathbb{A})$ has finite volume wrt dg

Consider $L^2(G(F) \backslash G(\mathbb{A}), \omega) = \{ \phi : G(F) \backslash G(\mathbb{A}) \to \mathbb{C}$ measurable,
$\phi(zg) = \omega(z) \phi(g)$
$\|\phi\|^2 = \int_{Z(\mathbb{A}) G(F) \backslash G(\mathbb{A})} |\phi(g)|^2 dg < \infty \}$

This space is a unitary representation of $G(\mathbb{A})$ by rt. translation.

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in general

$$L^2(G(F)\backslash G(\mathbb{A}), \omega) = L^2_{\text{discrete}}(\cdot) \oplus L^2_{\text{cont.}}(\cdot)$$

Define $L^2_0(G(F)\backslash G(\mathbb{A}), \omega) = \{\phi \mid \phi_N(g) \equiv 0 \text{ almost everywhere}\}$

Theorem: (ω = unitary) (1) $A_0(G,\omega) \subseteq L^2_0(G(F)\backslash G(\mathbb{A}), \omega)$

(2) $L^2_0(\cdot)$ is a Hilbert direct sum of irred. unitary reps of G. $= \hat{\bigoplus}_\pi V_\pi$. with finite multiplicities,

(3) Given a closed irred $V \subseteq L^2_0(\cdot)$

$V_{(K)}$ = K-finite vectors $= V \cap A_0(G,\omega)$ are dense in V.

picture:

$$\begin{tikzcd}
A(G) & \\
\cup & \\
A(G,\omega) & L^2(G,\omega) \\
\cup & \cup \\
A_0(G,\omega) \arrow[r, phantom, "\subseteq"] & L^2_0(G,\omega) \\
 & \| \arrow[r, leftarrow] & \text{big Theorem} \\
 & \hat{\oplus} V \\
 & V_K \arrow[uuull]
\end{tikzcd}$$

and $A_0(G,\omega) = \oplus V_K$ alg. dir. sum of K-finite vectors.

Say $f \in \mathcal{A}_0(G,\omega)$

write $f(g) = \sum_{\alpha \in F^\times} f_\alpha(g)$ Fourier expansion. (cuspidal $\Rightarrow$ no constant term).

let $\alpha \in F^\times$.

Consider
$$f_1\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g\right) = \int_{N(F)\backslash N(\mathbb{A})} \psi(-x)\, f\Big(\underbrace{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}}_{=\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \alpha^{-1}x\\ & 1\end{pmatrix}} g\Big)\, dx.$$

But then f automorphic means $f\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \alpha^{-1}x\\ & 1\end{pmatrix}\right) = f\begin{pmatrix}1 & \alpha^{-1}x\\ & 1\end{pmatrix}$

So change vars:

$$= \int_{N(F)\backslash N(\mathbb{A})} \psi(-\alpha x)\, f\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right) dx.$$

$$= f_\alpha(g).$$

Let
$$W_\psi(g,f) = \int_{N(F)\backslash N(\mathbb{A})} \psi(-x)\, f(n(x)g)\, dx \qquad (= f_1(g))$$

Then

$$f(g) = \sum_{\alpha \in F^\times}{}' W_\psi\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g, f\right)$$
<u>Note</u>: $W_\psi(\cdot, f)$ determines f.

<u>Theorem</u>: Given (π, V) irred. adm. repn. of $G(\mathbb{A})$ with central char ω, then
$$\dim \operatorname{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}_0(G,\omega)) \le 1.$$

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pf:

$$(\pi, V) \subseteq \mathcal{A}_0(G,\omega) \longrightarrow W_\psi$$

$$f \longmapsto W_\psi(\cdot, f)$$

where $W_\psi = \{W : G(\mathbb{A}) \to \mathbb{C} :$ (W_ψ = Global Wh. space).

$W(n(x)g) = \psi(x) W(g)$, and W has moderate growth.

So

$$\begin{tikzcd} V \arrow[d, phantom, "\cap"] \arrow[dr] & \\ \mathcal{A}_0(G,\omega) \arrow[r] & W_\psi \end{tikzcd}$$

$$f \longmapsto W_\psi(\cdot, f).$$

uniqueness of Global Wh. Model and fact that $W_\psi(\cdot, f)$ determines f implies uniqueness of V in $\mathcal{A}_0(G,\omega)$, i.e., multiplicity one.

Just Need Global existence/uniqueness of Wh. Model:

$$\pi \simeq \otimes_v' \pi_v \longrightarrow W_\psi.$$

$$\phi \longmapsto \prod_{v \in S} i_v(\phi_v) \prod_{v \notin S} W_v^\circ$$

well defined, gives unique global Wh. Model.

$\pi_v \xrightarrow[i_v]{\sim} W_{\psi_v}(\pi_v) \hookrightarrow W_{\psi_v}$

$\xi_v^\circ \longmapsto W_v^\circ$ s.t. $W_v^\circ(e) = 1$.

then, $\phi \in \pi$ write $\phi = \left(\otimes_{v \in S} \phi_v\right) \otimes \left(\otimes_{v \notin S} \xi_v^\circ\right)$

FACTS: SL_2 : Thm Mult. one holds for $G = SL_2$. (proved in last 2 years)

Thm. (Blasius) : Mult. one fails for $G = SL_n$, $n \geq 3$.

Thm. (Shalika) : Mult. one holds for GL_n

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5/2 Edinborough Conference Articles: Knapp + Rogowski (?)

Recall:

$F = \#$ field, $G = GL_2/_F$

$\mathcal{A}(G) \supseteq \mathcal{A}_0(G,\omega)$ ω = unitary Hecke char of $\mathbb{A}^\times/F^\times$.

cusp forms

$\mathcal{A}_0(G,\omega) \subseteq L^2_0(G,\omega)$

$$\begin{array}{ccc} \| & & \| \\ \bigoplus_\pi V_\pi & \xrightarrow{\text{dense in}} & \hat{\bigoplus_\pi} V_\pi \end{array}$$

alg. sum. irred adm. "reps" of $G(\mathbb{A})$

Hilbert space direct sum. irred unitary rep of $G(\mathbb{A})$

π's occur with multiplicity 1.

Note: $\pi \cong \otimes'_v \pi_v$ factorizable, almost all π_v unramified

(π = irred. cusp. auto repn)

Artin L-functions:

$\Gamma = Gal(\bar{F}/F) = Gal(\bar{\mathbb{Q}}/F) = \varprojlim_{E/F \text{ finite Galois}} Gal(E/F)$ = profinite group, l-group, compact.

(Nbhd basis of e given by $Gal(\bar{\mathbb{Q}}/E)$).

let (ρ, U) = fin dim. $\mathbb{C}$-repn of Γ (continuous)

Hence, $\ker(\rho)$ is open in Γ

let E = fixed field of $\ker(\rho)$ then $\Gamma \longrightarrow \Gamma_{E/F} \xrightarrow{\sigma} GL(U)$

$\Gamma_{E/F} = Gal(E/F)$.

Suppose with $\Gamma_{E/F}$: E / F, $\mathcal{O}_E / \mathcal{O}_F$, $\mathfrak{p}\mathcal{O}_E = \prod_{i=1}^r \mathfrak{P}_i^{e_i}$, $\mathfrak{P} \cap \mathcal{O}_F = \mathfrak{p}$

(Say v = finite place, $\mathfrak{p}_v = \mathfrak{p}$ = prime ideal in $\mathcal{O}_F$.)

so $\Gamma_{E/F}$ acts transitively on primes $\mathfrak{P}_i$ over $\mathfrak{p}$.

$$\begin{array}{ccccccc}
\mathfrak{p} & \to & \mathcal{O}_F & \to & \mathcal{O}_F/\mathfrak{p} & = & \mathbb{F}_q = k_\mathfrak{p} \\
\uparrow & & \uparrow & & | & & | \\
\wp & \to & \mathcal{O}_E & \to & \mathcal{O}_E/\wp & = & \mathbb{F}_{q^f} = k_\wp
\end{array}$$

If $\begin{array}{c} E \\ | \\ F \end{array}$ has degree n get $n = r_\mathfrak{p} e_\mathfrak{p} f_\mathfrak{p}$

def $D_\wp$: stab of $\wp$ in $\Gamma_{E/F}$ = decomposition group

$\sigma \in D_\wp$, then

$\boxed{\sigma \text{ gives aut. of the seq } \wp \to \mathcal{O}_E \to \mathcal{O}_E/\wp \text{ which fixes } \mathfrak{p}, \mathcal{O}_F, \mathcal{O}_F/\mathfrak{p}}$

$$1 \to I_\wp \to D_\wp \to \mathrm{Gal}(k_\wp/k_\mathfrak{p}) \to 1$$

$I_\wp$ = Inertia grp.

$\mathrm{Gal}(k_\wp/k_\mathfrak{p})$ = cyclic = $\langle Fr_\wp \rangle$, $x \mapsto x^q$

$|D_\wp| = e_\mathfrak{p} f_\mathfrak{p}$

$|I_\wp| = e_\mathfrak{p}$ for almost all v, $e_\mathfrak{p} = 1$.

If $e_\mathfrak{p} = 1$ then $D_\wp \simeq \mathrm{Gal}(k_\wp/k_\mathfrak{p})$

$\quad$ so $\exists$ $Fr_\wp$ - canonical generator.

Fact: For each $\mathfrak{p}$ unramified in E/F, $\Delta_{E/F}$ = rel. discriminant. ie. $\mathfrak{p} \nmid \Delta_{E/F}$.

$\exists$ a unique conjugacy class

$\langle Fr_\mathfrak{p} \rangle \subset \Gamma_{E/F}$.

Now take $E \supseteq$ Fixed field of $\ker(\rho)$, still $\Gamma \to \Gamma_{E/F} \to GL(V)$ (not nec. faithful)

Artin L-function:

- unramified: $\rho(Fr_p) \in GL(U)$ let S = finite set of ramified primes

$$\det(1_U - \rho(Fr_p)T) = P_{\rho,p}(T) = \text{char. poly}.$$

$$L^S(s,\rho) = \prod_{p \notin S} \det(1_U - \rho(Fr_p)Np^{-s})^{-1}$$

recall: $|k_p| = Np$.

= Euler product.

putting in bad primes:

$$L(s,\rho) = L^S(s,\rho) \cdot \prod_{p \in S} \det\left(1 - \left(\rho(Fr_p)\big|_{U^{I_p}}\right) \cdot Np^{-s}\right)^{-1}.$$

$U \supset U^{I_p}$, I_p invariants. choose $\wp \mid p$. then $\langle Fr_\wp \rangle \simeq D_\wp / I_\wp$ which acts on U^{I_p}.

This product converges for $\mathrm{Re}\, s$ sufficiently large.

ex: ρ = trivial repn $1\!\!1_F$ (take $E = F$)

$$L(s, 1\!\!1_F) = \prod_p (1 - Np^{-s})^{-1} = \zeta_F(s)$$

Dedekind zeta fcn.

Fact:

$\rho \simeq \rho_1 \oplus \rho_2$.

$$L(s,\rho) = L(s,\rho_1)L(s,\rho_2).$$

Fact: L-function is "inductive" i.e.

F_1 | F

ρ_1 = rep of $\Gamma_{F_1} = \mathrm{Gal}(\bar{F_1}/F_1)$

$\Gamma_{F_1} \subset \Gamma_F$, Form $\mathrm{Ind}_{F_1}^F(\rho_1)$ = repn of Γ_F.

Then $L(s,\rho_1) = L(s, \mathrm{Ind}_{F_1}^F(\rho_1))$

eg:

E | F finite Galois ext. $\Gamma_E \subset \Gamma_F \to \Gamma_{E/F}$.

$$\zeta_E(s) = L(s, \mathbb{1}_E) = L(s, \mathrm{Ind}_E^F(\mathbb{1}_E)) = \zeta_F(s) \cdot \prod_{\rho \neq \mathbb{1}} L(s,\rho)^{\dim(\rho)}.$$

$\mathrm{Ind}_E^F(\mathbb{1}_E) \cong$ reg. repn. of $\Gamma_{E/F}$. So $\simeq \mathbb{1}_F \oplus \sum_{\rho \neq \mathbb{1}} \dim(\rho)\cdot\rho$

Cor: $\zeta_F(s) \,\big|\, \zeta_E(s)$.

(conjectured but not known for non-Galois extension.)

<u>Thm of Brauer</u>: $\Gamma_{E/F}$ = finite grp.

Then every character is a $\mathbb{Z}$-linear comb. of chars of reps induced from one-dim. reps ~~[illegible]~~ of subgroups.

So... write: $\theta_\rho = \sum n_i\, \theta_{\mathrm{Ind}_{\Gamma_i}^{\Gamma}(\chi_i)}$, then

$$L(s,\rho) = \prod_i L(s,\chi_i)^{n_i} \qquad (n_i \in \mathbb{Z}).$$

$$\begin{tikzcd} E \arrow[d, dash] \arrow[dr, dash] & \\ & E_i \arrow[dl, dash] \\ F & \end{tikzcd}$$

$\Gamma_{E/E_i} \subset \Gamma_{E/F}$

$\chi_i : \Gamma_{E/E_i} \to \mathbb{C}^\times$.

$$\begin{tikzcd} \text{if } \chi : \Gamma_F \arrow[r] \arrow[d] & \mathbb{C}^\times & & \\ \Gamma_F^{ab} \arrow[ur] & \mathbb{A}^\times / \overline{F^\times \cdot F_\infty^{\times}}^{+} \arrow[l, "\sim"'] & \mathbb{A}^\times/F^\times \arrow[l] \arrow[ul, bend right, "\omega = \omega(\chi)"'] & \end{tikzcd}$$

connected component

Global class field theory

So a χ corresponds to a Hecke character, call it ω.

So $L(s,\chi) = L(s,\omega)$

Artin $\qquad$ Tate/Hecke

If $\omega \neq 1\!\!1$ then $L(s,\omega)$ is entire

So using Brauer get

$$L(s,\rho) = \prod_i L(s,\chi_i)^{n_i} = \prod_i L(s,\omega(\chi_i))^{n_i}$$

$\Rightarrow L(s,\rho)$ has a meromorphic analytic continuation

Conjecture: ρ irred $\neq 1\!\!1$ then $L(s,\rho)$ is entire.

Back to

$\pi \cong \otimes_v' \pi_v$, S = finite set s.t. $v \notin S \Rightarrow \pi_v$ unramified.

There is a Satake parameter $t(\pi_v) \in G^\vee = Gl_2(\mathbb{C})$ (actually its a conjugacy class of elements in $Gl_2\,\mathbb{C}$)

So **Now**: π irred, auto, cusp

$\pi \longrightarrow S$	**Artin**:
$v \notin S$	$\Gamma_{E/F}$, ρ
$t(\pi_v)$ conj. class in $G^\vee$ $\longleftrightarrow$	$v \notin S$
	Fr_v – conj. class in $\Gamma_{E/F}$.

So

Def: $r: G^\vee \longrightarrow GL(U)$

f.d. $\mathbb{C}$-repn of $G^\vee$.

then

$$L^S(s,\pi,r) = \prod_{v \notin S} \det(1_U - r(t(\pi_v)) \cdot N_{\mathfrak{p}_v}^{-s})^{-1} = \text{Langlands L-function}$$

eg:

r = standard rep. on $\mathbb{C}^2$.

get degree 2 Euler product, $L(s,\pi)$ "standard L-function"

Questions: (1) Convergence?

(2) bad factors: $v \in S$?

(3) Analytic Continuation?

functional eq, poles?

Note: picture of $A_0(G,\omega) \subset L_0^2(G,\omega)$

$\Rightarrow$ π_v are all unitarizable.

Fix v: finite place

~~Say~~ π = unramified rep of $GL_2(F)$

$\exists$ $I(\mu)$, μ unramified s.t. π is the unramified constituent of $I(\mu)$.

But really π_v's are inf. dim'l. So assume $I(\mu) = \pi$.

So

$$I(\mu) = \mu_0 \otimes I(|\,|^s, |\,|^{-s}) \quad (\text{by unitarizable})$$

unitary $\qquad -\tfrac{1}{2} < s < \tfrac{1}{2}$.

What is $t(\pi)$?, $t(\pi) = \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix}$: $z_1 = \mu_1(\varpi)$

$z_2 = \mu_2(\varpi)$

Case 1: μ_1, μ_2 unitary:

$\Rightarrow |z_1| = |z_2| = 1$ $\Rightarrow$ Euler factor $(1 - z_1 q^{-s})(1 - z_2 q^{-s})$ since $r(t(\pi_v)) = \begin{pmatrix} 1 - z_1 \ldots & \\ & 1 - z_2 q^{-s} \end{pmatrix}$

So

$$\prod_v (1 - z_{1,v} q_v^{-s})^{-1} (1 - z_{2,v} q_v^{-s})^{-1} \text{ has same conv as } \prod_v (1 - q_v^{-\mathrm{Re}\, s})^{-1}$$

which converges for $\mathrm{Re}(s) > 1$ (Dedekind zeta fn)

Case 2: $q^{-1/2} < |z_i| < q^{1/2}$

get convergence for

$\mathrm{Re}(s) > 3/2$

$\mathrm{Re}(s) > 3/2$ which converges for $\mathrm{Re}(s) > 1$ (Dedekind zeta fn)

Kudla
5/4.

(π ... = irred. cusp. adm. rep for $G(\mathbb{A})$)

$\pi \simeq \otimes'_v \pi_v$, S = arch. places + finite v s.t. π_v unramified.

$\forall\, v \notin S$, $t(\pi_v) \in G^\vee = GL_2(\mathbb{C})$.

$r: G^\vee \longrightarrow GL(U)$ f.d. $\mathbb{C}$-repn , form

(k_v = res. field at v, $|k_v| = q_v$.)

$$L^S(s,\pi,r) = \prod_{v \notin S} \det(1_U - r(t(\pi_v))\, q_v^{-s})^{-1} \quad \text{Langlands } L\text{-fcn.}$$

conv. for $\mathrm{Re}\, s > 3/2$.

Questions:

1. bad Euler factors, i.e., $v \in S$.
2. meromorphic continuation
3. poles

Back to Artin L-fns:

$\Gamma_F = \mathrm{Gal}(\bar F / F)$, v = non-arch. place.

$$\begin{array}{cccc} \Gamma_F & \bar F & \wp & \bar k_v \\ \cup & | & & | \\ D_\wp & F & \mathfrak{p}_v & k_v \end{array}$$

Stab $\wp$ — depends on choice of $\wp$

$$1 \to I_\wp \to D_\wp \to \mathrm{Gal}(\bar k_v / k_v) \to 1.$$

$$\mathrm{Gal}(\bar k_v / k_v) = \overline{\langle \mathrm{Fr}_v \rangle} \cong \hat{\mathbb{Z}}$$

So get a coset $\mathrm{Fr}_\wp (I_\wp)$

(ρ, U)

$\rho: \Gamma_F \to GL(U)$ f.d. $\mathbb{C}$-repn.

then

$\rho(I_\wp) = 1$ almost all $\mathfrak{p}$. , all places $v \notin S$ get $\{\rho(\mathrm{Fr}_v)\}$ and Artin L-fcn

$$L^S(s,\rho) = \prod_{v \notin S} \det(1_U - \rho(\mathrm{Fr}_v)\, q_v^{-s})^{-1}.$$

Say r = std. repn of $\mathbb{C}^2$ and suppose ρ = 2-dim'l so $U \simeq \mathbb{C}^2$.

<u>Langlands side</u>: <u>Artin side</u>.

$\{t(\pi_v)\}_{v \notin S}$ $\{\rho(Fr_v)\}_{v \notin S}$.

<u>Conjecture</u>: Given ρ irred 2-dim : $\rho : \Gamma_F \to GL_2(\mathbb{C}) = G^\vee$

there is an aut. cusp repn $\pi = \pi(\rho)$ for $G(\mathbb{A})$

s.t. $t(\pi_v) = \rho(Fr_v)$ $\forall v \notin S$. ($v \notin S \Rightarrow$ 1st really, π_v unramified)

<u>eg</u>: Say ρ is 1-dim., $\rho : \Gamma_F \to \mathbb{C}^\times = GL_1(\mathbb{C}) = H^\vee$, $H = GL(1)/_F$.

$\rho(I_\wp) = 1$ $\forall \wp_v$, $v \notin S$.

Could ask for $\omega : \mathbb{A}^\times/_{F^\times} \to \mathbb{C}^\times$

s.t. (0): $\forall v \notin S$, ω_v is unramified ; ie, $\omega_v|_{U_{F_v}} = 1$

(1): $\omega_v(\varpi_v) = \rho(Fr_v)$

Satake parameter

<u>Thm</u>: $\exists$ such an $\omega = \omega(\rho)$

$$\begin{tikzcd}
\Gamma_F \arrow[r] \arrow[rr, bend left, "\rho"] & \Gamma_F^{ab} \arrow[r] & \mathbb{C}^\times \\
 & \mathbb{A}^\times/\overline{F^\times \cdot F^\times_{\infty +}} \arrow[u, "\wr"] & \\
 & \mathbb{A}^\times/F^\times \arrow[u] \arrow[uur, bend right, "\omega(\rho)"'] &
\end{tikzcd}$$

$\leftarrow$ (Consequence of global class field theory)

(Certainly not every Hecke character factors through $\mathbb{A}^\times/\overline{F^\times F^\times_{\infty+}}$.)

If conjecture were true, then $L^S(s, \pi, \text{std.}) = L^S(s, \rho)$

What about if given irred 3-dim repn

$\rho : \Gamma_F \longrightarrow GL_3(\mathbb{C}) = G^\vee$

For $GL(n) = G$, still have

$\mathcal{A}(G) \supseteq \mathcal{A}_0(G, \omega)$ all defined in exactly same way, still have strong Mult. one.

Now $G^\vee = GL_n(\mathbb{C})$

and locally:

$I(\mu)$ is given by $\mu = (\mu_1, \ldots, \mu_n)$ μ_i a char of $F^\times$.

π unramified $\overset{\text{corresponds to}}{\longleftrightarrow}$ unique $I(\mu)$ (π = unique spherical piece)

Still get

$\pi \longmapsto t(\pi) = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$ s.t. $z_i = \mu_i(\varpi)$.

$\cap$
$GL_n(\mathbb{C})$.

Back to conjecture, generalizing:

ρ = n-dim. $V = \mathbb{C}^n$ get $\{\rho(Fr_v)\}, v \notin S$

$\cap$
$GL_n(\mathbb{C})$.

r = std. on $\mathbb{C}^n$.

$\{t(\pi_v)\}\ v \notin S$ — same conjecture —

$\cap$
$GL_n(\mathbb{C})$

Suppose $\pi = \bigotimes_v' \pi_v$ irred auto. cusp. for $GL_2(\mathbb{A})$.

$$\underset{v \notin S}{t(\pi_v)} \in G^\vee = GL_2(\mathbb{C}) \xrightarrow{r_3} GL_3(\mathbb{C}) \ni \underset{v \notin S}{r(t(\pi_v))}.$$

<u>Conjecture</u>: $\exists$ auto. repn Π of $GL_3(\mathbb{A})$.

s.t. (0) Π_v unram. for $v \notin S$

(1) $r(t(\pi_v)) = t(\Pi_v)$ $\forall v \notin S$

Π is a Langlands Functorial lift from $GL(2)$ to $GL(3)$.

then
Note: $L^S(s, \pi, r_3) = L^S(s, \Pi, \text{std.})$

Would conjecture:

π for $GL_2(\mathbb{A})$

r_n = irred n dim of GL_2.

there should be Π for $GL_n(\mathbb{A})$ s.t.

$$L^S(s, \pi, r_n) = L^S(s, \Pi, \text{std.})$$

Say G = connected red. gp. $/F$. — $\mathcal{A}(G)$ auto. cusp. repns

H = ... $\mathcal{A}(H)$

get dual groups $G^\vee$, $H^\vee$.

$v \notin S \Rightarrow \Pi_v$ unram., $\exists$ $t(\Pi_v) \in G^\vee$ or $H^\vee$.

Suppose: given $\phi: G^\vee \longrightarrow H^\vee$

(eg: $G = GL_2$, $H = GL_n$
$\phi = r_n$.)

then <u>conj</u>:

$\exists$ map from auto. cusp. repns for $G(\mathbb{A})$ $\underline{\text{to}}$ auto. repns for $H(\mathbb{A})$. (no cuspidal here.)

drop cuspidal for $H(\mathbb{A})$ since:

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π auto. cusp. for $GL_n(\mathbb{A}) \Rightarrow L(s,\pi,\text{std.})$ = entire.

$$GL_2(\mathbb{C}) \xrightarrow{r_3} GL_3(\mathbb{C})$$

$$\begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \longmapsto \begin{pmatrix} z_1^2 & & \\ & z_1 z_2 & \\ & & z_2^2 \end{pmatrix}$$

(Note: for $I(\mu)$
$z_1 z_2 = \mu_1\mu_2(\varpi I)$.
= central char.)

Say $z_1 z_2 = 1$

$$\begin{pmatrix} z_1^2 & & \\ & 1 & \\ & & z_2^2 \end{pmatrix}$$

$1 \longrightarrow$ give rise to $\zeta_F^S(s)$

So $L^S(s,\pi,r_3) = \zeta_F^S(s)\,\boxed{\text{stuff from } z_1, z_2}$

has pole so can't give L-fcn for auto cuspidal of GL_3.

So Π not cuspidal.

Back to Artin $\longrightarrow$ Auto.

$\rho: \Gamma_F \longrightarrow GL_2(\mathbb{C})$

$\rho(\mathrm{Fr}_v) \quad v \notin S$

$\downarrow$

get π_v s.t. $t(\pi_v) = \rho(\mathrm{Fr}_v)$.

So can start to build

$\bigotimes'_{v \notin S} \pi_v$ what about S?

Now: $D_F \subset \Gamma_F$
$\cup$
$I_F \hookrightarrow \longrightarrow \mathrm{Gal}(\bar{k}_v/k_v)$ so we only really need $\rho|_{D_F}$.

$\mathrm{Gal}(\bar{F}_v/F_v)$
$\cup$
$W_{F_v} \longrightarrow GL_2(\mathbb{C})$

local Langlands: reps of $GL_2(F_v)$ correspond to reps of W_{F_v}.

Kudla 5/9

F = #field, $\Gamma_F = Gal(\bar{F}/F)$

$\rho: \Gamma_F \to GL_2(\mathbb{C}) = G^\vee$ irred.

$\{\rho(Fr_v)\}_{v \notin S}$

Conj: $\exists$ irred cusp. auto. repn π $\underset{A\ s.t.}{\text{fn } G(\mathbb{A})}$

1. $v \notin S \Rightarrow \pi_v$ unramified

2. $v \notin S \Rightarrow t(\pi_v) = \rho(Fr_v)$

To Local Langlands:

$$\begin{array}{ccl} \bar{F} & \wp & \qquad\qquad \Gamma_F \\ | & | & \qquad\qquad \cup \\ | & | & 1 \to I_\wp \to D_\wp \longrightarrow Gal(\bar{k}_v/k_v) \to 1 \\ F & p = p_v & \qquad\qquad\qquad\qquad \langle \widehat{Fr_v} \rangle \simeq \hat{\mathbb{Z}} \end{array}$$

Above we said: ρ gives us π_v for $v \notin S$

$v \notin S \Rightarrow \pi_v$ is determined by $\rho|_{D_\wp}$ Know: $\rho(\Gamma_F)$ = finite grp in $GL_2(\mathbb{C})$

Note:

$$\begin{array}{cl} \bar{F}_v & D_\wp \simeq \Gamma_{F_v} = Gal(\bar{F_v}/F_v) \\ | & \\ | & 1 \to I_{F_v} \to \Gamma_{F_v} \longrightarrow \hat{\mathbb{Z}} \to 0 \\ F_v & \end{array}$$

Suppose we have $\rho_v : \Gamma_{F_v} \to GL_2(\mathbb{C}) = G^\vee$.

We Want to have $\pi_v = \pi_v(\rho_v)$

Ex If $\rho_v(I_v) = 1$.

so ρ_v is unramified and is determined by $\rho_v(Fr_v)$ in $GL_2(\mathbb{C})$, a semi simple conj. class of finite order.

Q's:

1. Suppose ρ_v is ramified, ie., $\rho(I_v) \neq 1$. Is there $\pi_v = \pi_v(\rho_v)$?

2. Is every π_v a $\pi_v(\rho_v)$ for some ρ_v ? (Obviously not but ...)

Start to fix 2):

$$\begin{tikzcd} 1 \arrow[r] & I_{F_v} \arrow[r] & \Gamma_{F_v} \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 0 \\ 1 \arrow[r] & I_{F_v} \arrow[r] \arrow[u, equal] & W_{F_v} \arrow[r] \arrow[u, phantom, "\cup"] & \mathbb{Z} \arrow[r] \arrow[u, phantom, "\cup"] & 0 \end{tikzcd}$$

local Weil group (pullback of $\mathbb{Z}$ without top.)

Forgetting the topology

Observation: The irred. adm unramified repns of $GL_2(F_v)$ $\rightleftharpoons$ semi simple conj. classes in $G^\vee = GL_2(\mathbb{C})$ $\rightleftharpoons$ Semi-simple unramified repns of W_F

(s.s. means $\rho(W_F)$ = s.s. elts)

$G = GL_1$, $G^\vee = \mathbb{C}^\times$, $G(F_v) = F_v^\times$

Say $\rho : W_F \to \mathbb{C}^\times$

$$\begin{tikzcd} W_F \arrow[r, "\rho"] \arrow[d, dashed] & \mathbb{C}^\times \\ W_F^{ab} \arrow[ur] & \\ F^\times \arrow[u, phantom, "\wr|"] & \end{tikzcd}$$

$\wr|$ ← local class field theory.

Moreover:

image of I_F in W^{ab}

$$\begin{tikzcd} 1 \arrow[r] & \overline{I}_F \arrow[r] & W_F^{ab} \arrow[r] \arrow[rrr, bend left, "\|\cdot\|"] & \mathbb{Z} \arrow[r] & 0 & \mathbb{R}_+^\times \\ 1 \arrow[r] & U_F \arrow[r] \arrow[u, "\wr|"] & F^\times \arrow[r, "\mathrm{ord}"] \arrow[u, "r"] \arrow[urrr, bend right, "|\cdot|"'] & \mathbb{Z} \arrow[r] \arrow[u, equal] & 0 & \end{tikzcd}$$

So $\|\cdot\|$ on W_F is def. by comp with $|\cdot|$ on $F^\times$.

So now, ρ gives rise to $\rho \circ r = \mu(\rho)$ a character.

Theorem: let $F = F_v$ (non-arch)

irred adm repns of $GL_2(F)$ $\rightleftharpoons$ s.s. repns of W_F' into $GL_2(\mathbb{C}) = G^\vee$.

Weil-Deligne gp ...

(This was proved recently for GL_n.)

take $I(\mu)$, $\mu = (\mu_1, \mu_2)$ char of $F^\times$, irred if $\mu_1 \mu_2^{-1} \neq |\,|^{\pm 1}$.

$\mu_1 \longrightarrow \rho_1$ char of W_F

$\mu_2 \longrightarrow \rho_2$ char of W_F via reciprocity r above.

let $\rho = \rho_1 \oplus \rho_2$: a repn $W_F \longrightarrow GL_2\,\mathbb{C}$.

If $\mu_1 \mu_2^{-1} = |\cdot|$ $<$ $\sigma(\mu)$ = special (∞-dim) / 1-dim.

A s.s. repn of W_F' is a pair (ρ, N) where

$\rho : W_F \longrightarrow GL(V)$ V = f.d. $\mathbb{C}$-v.s.

ρ is s.s.

$N : V \longrightarrow V$ is a nilpotent endomorphism s.t. $\rho(w) N \rho(w)^{-1} = \|w\| N$.

so $\rho_1 \rho_2^{-1} = \|\cdot\|$

Can take: $\rho(w) = \begin{pmatrix} \rho_1(w) & \\ & \rho_2(w) \end{pmatrix}$, $N = 0$ $\longleftrightarrow$ corresponds to 1-dim'l piece of $I(\mu)$

or

$\rho(w) = \begin{pmatrix} \rho_1(w) & \\ & \rho_2(w) \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\longleftrightarrow$ corresp. to $\sigma(\mu)$.

If we have an irred rep of W_F with $N = 0$

then get a supercuspidal of $GL_n(F)$ under correspondence of the Thm

Suppose $F = \mathbb{R}$:

$$\begin{array}{ccccccccc} & & & & \mathbb{C}^\times \cup \mathbb{C}^\times j & & & & \\ & & & & \| & & & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & W_{\mathbb{R}} & \longrightarrow & \Gamma_{\mathbb{R}} & \longrightarrow & 1 \\ & & \overset{\cup}{z} & & & & \| & & \\ & & & & & & \langle \sigma \rangle & & \end{array}$$

$$j \longmapsto \sigma$$

$$j z j^{-1} = \bar{z}$$

$$j^2 = -1.$$

Thm: irred adm repns of $GL_n(\mathbb{R})$ $\rightleftarrows$ s.s. repns of $W_{\mathbb{R}}$ into $G^\vee = GL_n(\mathbb{C})$.

For $GL_2(\mathbb{R})$:

1st: L_1 consider $\rho : W_{\mathbb{R}} \to \mathbb{C}^\times$.

$$\begin{array}{c} W_{\mathbb{R}} \\ \cup \\ \mathbb{C}^\times \end{array}$$

$$\rho(z) = z^\alpha \bar{z}^\beta = |z|^{\frac{\alpha+\beta}{2}} \left(\frac{z}{\bar{z}}\right)^{\frac{\alpha-\beta}{2}}$$ so require $\alpha, \beta \in \mathbb{C}$, $\alpha - \beta = l \in \mathbb{Z}$.

$$\rho(j)^2 = \rho(j^2) = \rho(-1) = 1.$$

$$\begin{array}{l} \rho(jzj^{-1}) = \rho(\bar{z}) = \bar{z}^\beta \bar{z}^\alpha \\ \quad \| \\ \rho(j)\rho(z)\rho(j)^{-1} = z^\alpha \bar{z}^\beta \end{array} \Rightarrow \alpha = \beta.$$

So $\rho(z) = |z|^s$ $\quad s = \alpha = \beta \in \mathbb{C}$

$\rho(j) = (-1)^a$, $a = 0, 1$

$\to$ Corresponds to

$x \longmapsto |x|^s \operatorname{sgn}(x)^a$ char of $\mathbb{R}^\times$.

New
$GL_2(\mathbb{R})$:

$$\rho: \begin{array}{c} W_{\mathbb{R}} \\ \cup \\ \mathbb{C}^\times \end{array} \longrightarrow GL_2(\mathbb{C}) \qquad \rho(z) = \begin{pmatrix} \rho_1(z) & \\ & \rho_2(z) \end{pmatrix} \qquad \rho_i \text{ char of } \mathbb{C}^\times$$

$\rho(j)$ either preserves or switches eigenspaces

So: preserved by $\rho(j) \Rightarrow$ get pair $((s_1, a_1), (s_2, a_2))$.

otherwise:
Say e_1, e_2 are eigenvectors for above:

$$\rho(j) e_1 = e_2$$
$$\rho(j) e_2 = \rho(j^2) e_1 = \rho(-1) e_1 = \rho_1(-1) e_1.$$

$$\begin{pmatrix} & \rho_1(-1) \\ 1 & \end{pmatrix}$$

But: $\rho_2(z) = \rho_1(\bar{z})$ from $)$.

So really choosing one character here.

So we get:

$$\rho(z) = |z|^s \begin{pmatrix} \left(\frac{z}{\bar{z}}\right)^{\ell/2} & \\ & \left(\frac{z}{\bar{z}}\right)^{-\ell/2} \end{pmatrix}$$

(∞-dim piece in the reducible case) $\longleftrightarrow$

(this rep is irred.)

$$\rho(j) = \begin{pmatrix} & (-1)^\ell \\ 1 & \end{pmatrix}$$

$G = GL_2 / F$

(π, V) = auto. cusp. repn.

$V \subset \mathcal{A}_0(G, \omega)$, ω unitary (for simplicity)

S = ramified set, i.e., $v \notin S \Rightarrow \pi_v$ unramified with $t(\pi_v) \in GL_2(\mathbb{C})$.

$$L^S(s, \pi, \mathrm{std}) = L^S(s,\pi) = \prod_{v \notin S} \det\left(1 - t(\pi_v) \cdot q_v^{-s}\right)^{-1}$$

std = 2-dim. repn of $GL_2(\mathbb{C})$

Say χ = Hecke character, substitute $\chi_v(\varpi_v) q_v^{-s}$ for q_v^{-s}

Define $$L^S(s, \pi, \chi) = \prod_{v \notin S'} \det\left(1 - t(\pi_v)\chi_v(\varpi_v) q_v^{-s}\right)^{-1} \qquad \left(S' = S \cup \{ v \notin S,\ \chi_v \text{ ramified} \}\right)$$

Take $f \in V$. $\quad f(g) = \sum_{\alpha \in F^\times} W\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g\right)$

ψ add. char.

$$W(g) = \int_{N(F)\backslash N(A)} \psi(-x) f(n(x) g)\, dx.$$

π has a Global Wh. Model $W(\pi)$

so $V \xrightarrow{\sim} W(\pi) \simeq \bigotimes_v{}' W(\pi_v)$

$f \longmapsto W$

↳ wrt. $W_v^\circ \in W(\pi_v)$ s.t. $W_v^\circ(e) = 1$.

If $f \mapsto W = \bigotimes_v{}' W_v$ a pure tensor, we say f is "factorizable".

Let $$Z(s, f, \chi) = \int_{F^\times \backslash \mathbb{A}^\times} f\left(\begin{pmatrix}a & \\ & 1\end{pmatrix}\right) \cdot \chi(a) |a|^{s - 1/2}\, d^\times a.$$

Global zeta "Hecke" integral.

Convergence: recall:

$$\mathbb{A}^\times / F^\times \longrightarrow \mathbb{R}_+^\times \longrightarrow 1$$

$(t, 1, \ldots) \longleftarrow\!\shortmid\, t$

$1 \longrightarrow \mathbb{A}^1 / F^\times = \text{cpt.} = \Omega \ni a_0$

So need to consider $\left| f\left(\begin{pmatrix} t \cdot a_0 & \\ & 1 \end{pmatrix}\right) \right|$. a_0 won't play a role

as $t \to \infty$, $W_\infty\left(\begin{pmatrix}t & \\ & 1\end{pmatrix}\right) = O(e^{-ct})$ as $t \to \infty$.

Thus:

$$\left| f\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)\right| = O(e^{-c|a|}) \text{ as } |a| \longrightarrow \infty.$$

for any $f \in V$.

what if $a \to 0$? $\quad w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in G(F)$

$$f\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) = f\left(\underset{\substack{\uparrow \\ \text{goes away by} \\ \text{left inv. by} \\ G(F)}}{w}\begin{pmatrix} 1 & \\ & a\end{pmatrix} w\right) = r(w) f\begin{pmatrix} 1 & \\ & a\end{pmatrix} = \omega(a)(r(w)f)\begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}.$$

So $\left| f\left(\begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}\right)\right| = \left|(r(w)f)\begin{pmatrix} a & \\ & 1\end{pmatrix}\right| = O(e^{-c|a|})$ as $|a| \longrightarrow \infty$.

Hence $Z(s,f,\chi)$ is abs. conv. $\forall s$, hence defines an entire fcn of s.

<u>Global functional</u> Eqn.

say $\mathbb{A}^\times / F^\times = D_{\geq 1} \cup D_{\leq 1}$.

$$Z(s,f,\chi) = \int_{D_{\leq 1}} f\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) \chi(a) |a|^{s-1/2} d^\times a + \int_{D \geq 1} (\text{''})$$

$$= \int_{D \leq 1} (r(w)f)\left(\begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}\right) \omega(a) \chi(a) |a|^{s-1/2} d^\times a = \int_{D \geq 1} (r(w)f)\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) \omega(a)^{-1} \chi(a)^{-1} |a|^{1-s-1/2} d^\times a$$

So

$$Z(s,f,\chi) = Z(1-s, r(w)f, \omega^{-1}\chi^{-1})$$

(2)

$$Z(s,f,\chi) = \int_{F^\times\backslash\mathbb{A}^\times} \left(\sum_{\alpha\in F^\times}' W\left(\begin{pmatrix} \alpha a & \\ & 1\end{pmatrix}\right)\right) \chi(a)\, |a|^{s-1/2}\, d^\times a.$$

Assume f is "factorizable"

$$= \int_{\mathbb{A}^\times} W\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) \chi(a)\, |a|^{s-1/2}\, d^\times a$$

$$= \prod_v \int_{F_v^\times} W_v\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) \chi_v(a) |a|^{s-1/2}\, d_v^\times a \qquad Re(s) > 3/2.$$

(Global integral was abs. conv, but unfolding loses some convergence)

---

let $\mathbb{A}_T^\times = \prod_{v\in T} F_v^\times \times \prod_{v\notin T} U_{F_v} \hookrightarrow \mathbb{A}^\times = \varinjlim_{T} \mathbb{A}_T^\times$

if ϕ a fcn on $\mathbb{A}^\times$, then ~~by~~ def:

$$\int_{\mathbb{A}^\times} \phi(a)\, d^\times a := \lim_T \int_{\mathbb{A}_T^\times} \phi(a)\, d^\times a$$

assuming $\phi = \otimes_v' \phi_v$ wrt ϕ_v° s.t. $\phi_v^\circ|_{U_{F_v}} \equiv 1$.

then get

$$= \prod_{v\in T} \int_{F_v^\times} \phi_v(a)\, d_v^\times a.$$

[since for T big enough $\int_{U_{F_v}} d_v^\times a = \int_{U_{F_v}} \phi_v(a)\, d_v^\times a = 1$.

since $\phi_v(a)|_{U_{F_v}} \equiv 1$.]

Take $v\in T$ (T big, say $v\notin S''$); $c(\psi_v)=0$

then

$$W_v = W_v^\circ \quad\text{and}\quad W_v^\circ\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) = \begin{cases} 0 & \text{if } m = \operatorname{ord} a < 0 \\ q_v^{-\frac{m}{2}} \dfrac{z_1^{m+1} - z_2^{m+1}}{z_1 - z_2}, & m \geq 0 \end{cases}, \quad t(\pi_v) = \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix}$$

$\chi_v(\varpi) = z$

so local integral:

$$\int_{F^\times} W^\circ\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) \chi(a) |a|^{s-1/2} d^\times a = \sum_{m=0}^{\infty} q^{-m/2} \cdot \frac{z_1^{m+1} - z_2^{m+1}}{z_1 - z_2} z^m q^{-(s+1/2)m}.$$

$$= \frac{1}{z_1 - z_2}\left[\frac{z_1}{1 - z z_1 q^{-s}} - \frac{z_2}{1 - z z_2 q^{-s}}\right]$$

$$= \frac{1}{(1 - z z_1 q^{-s})(1 - z z_2 q^{-s})}$$

$$\prod_{v \notin S''} \underbrace{\frac{1}{(1 - z z_1 q_v^{-s})(1 - z z_2 q_v^{-s})}}_{= L(s, \pi_v, \chi_v)}$$

Assume $|z| = 1$, $q^{-1/2} \le |z_i| \le q^{1/2}$. then ah. for $Re(s) > 3/2$

So

$$Z(s, f, \chi) = \prod_{v \in S''} Z(s, W_v, \chi_v) \cdot L^{S''}(s, \pi, \chi). \quad \text{for } Re\, s > 3/2.$$

$$\|$$

$$Z(1-s, r(\omega) f, \omega^{-1}\chi^{-1}) = \prod_{v \in S''} Z(1-s, W_v', \omega_v^{-1}\chi_v^{-1}) \cdot L^{S''}(1-s, \pi, \omega^{-1}\chi^{-1}) \quad \text{for } Re(1-s) > 3/2, \text{ i.e., } Re\, s < -1/2.$$

Theorem: Let F be local, π irred adm., $W(\pi)$ = Wh. Model,

(i) $Z(s, W, \chi)$ have meromorphic analytic cont. $\forall W \in W(\pi)$.

(ii) $\exists$ Euler factor $L(s, \pi, \chi)$ s.t. $\dfrac{Z(s, W, \chi)}{L(s, \pi, \chi)}$ is entire.

and $\exists\, W^\circ \in W(\pi)$ s.t. $\dfrac{Z(s, W^\circ, \chi)}{L(s, \pi, \chi)} \equiv 1$.

1-al functional eqn:

written on wrong side!

- $Z(s, W, \chi) = \boxed{\gamma(s, \pi, \chi, \psi)}\, Z(1-s, W', \omega^{-1}\chi^{-1})$

$\Downarrow$

- $$\frac{Z(s, W, \chi)}{L(s, \pi, \chi)} = \boxed{\varepsilon(s, \pi, \chi, \psi)}\, \frac{Z(1-s, W', \omega^{-1}\chi^{-1})}{L(1-s, \pi, \omega^{-1}\chi^{-1})}$$

So: Finally:
Globally:

$$L(s, \pi, \chi) = \varepsilon(s, \pi, \chi)\, L(1-s, \pi, \omega^{-1}\chi^{-1})$$

"
entire.

(Can also replace $\omega^{-1} \otimes \pi = \tilde{\pi}$ to get another version of func. eqn).

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Kudla

ictionary :

cuspidal auto rep. (π, V) $\rightleftharpoons$ normalized newform

Case when $F = \mathbb{Q}$

(π, V) = irred. auto. cusp. repn.

$V = V(\pi) \subset A_0(G, \omega)$

WLOG assume: ① ω = unitary

Also: $Z \subset G$ = center

$Z(\mathbb{R})^+ \subseteq Z(\mathbb{R})$

$\{\begin{pmatrix} t & \\ & t\end{pmatrix} \mid t \in \mathbb{R}_+^\times\}$ $\quad$ $\{\begin{pmatrix} t & \\ & t\end{pmatrix} \mid t \in \mathbb{R}^\times\}$.

② Assume $\omega|_{Z(\mathbb{R}^\times)^+} \equiv 1$.

since if $t \mapsto t^{ir}$

twist by $|\det(g)|_{\mathbb{A}}^{-\frac{ir}{2}}$ is trivial on $Z(\mathbb{R})^+$.

Now:

$\omega : \mathbb{A}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$

$\mathbb{A}^\times = \mathbb{Q}^\times \cdot \mathbb{R}_+^\times \cdot \hat{\mathbb{Z}}^\times \leftarrow \prod_p \mathbb{Z}_p^\times$.

direct prod

by right, we get.

$$\begin{array}{ccc} \omega: \hat{\mathbb{Z}}^\times & \longrightarrow & \mathbb{C}^\times \\ \| & & \uparrow \chi \\ \prod_\ell \mathbb{Z}_\ell^\times & \longrightarrow & (\mathbb{Z}/N\mathbb{Z})^\times \end{array}$$

$\exists N$ s.t.

Not : let $t_p = (1, \dots, p, 1, \dots) = p \cdot (p^{-1}, \dots, p^{-1}, 1, p^{-1}, \dots)$

p^{th} place.

Consider $\omega(t_p) = \chi(p^{-1})$

$\omega(t_p) = \omega_p(p)$

$\boxed{\text{Fix } \psi,\ c(\psi)=0,\ \psi_\infty(x)=e^{\pi i x}.}$

$\pi \simeq \bigotimes_v{}' \pi_v$, $V(\pi) \xrightarrow{\sim} W(\pi)$: Wh. Model. $= \bigotimes_v{}' W(\pi_v)$

$S = \{ p \mid \pi_p = \text{ramified} \}$ $\qquad f \longmapsto W = W_\infty \otimes \left(\bigotimes_{p \in S} W_p \right) \otimes \left(\bigotimes_{p \notin S} W_p^\circ \right)$

looking for a "distinguished" vector.

(obvious choice here.)

$W_p^\circ(e) = 1$.

let $G' = SL_2$

$K'_\infty = SO(2)$

<u>Case $p = \infty$</u>:

Assume $\pi_\infty \simeq D_\ell$ (limit of) Discrete Series.

(if $\ell = 1$)

$(\mathfrak{g}, K_\infty)$-Module. $(\ell \geq 1)$

K'_∞-spectrum is $\pm\left(\ell + 2\mathbb{Z}_{\geq 0}\right)$

<u>Take</u>:

$W_\infty = W_\infty^\ell$: lowest weight vector $(L W_\infty^\ell = 0)$

<u>In fact</u>:

write $G(\mathbb{R}) = \begin{pmatrix} t & \\ & t \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k_\theta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^a$

$k_\theta = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

$t \in \mathbb{R}_+^\times$, $x \in \mathbb{R}$, $y \in \mathbb{R}_+^\times$ $\qquad a = 0, 1.$

Solve:

$0 = L W_\infty^\ell$

<u>Fact</u>:

$W_\infty^\ell \big|_{G(\mathbb{R})^-} \equiv 0$

So on $G(\mathbb{R})^+$, ~~the~~ W_∞^ℓ is independent of $\begin{pmatrix} t & \\ & t \end{pmatrix}$

So its just a fcn of x, y, θ.

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also:

$$W_\infty^l(gk_\theta) = e^{il\theta} W_\infty^l(g)$$

Further in these coordinates

$$L = e^{-2i\theta}\left((-2iy)\bar{\partial} - \frac{1}{2i}\frac{\partial}{\partial\theta} \right) \quad \text{where} \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$(z = x+iy \in \mathfrak{h})$$

Write:

$$W_\infty^l(g) = y^{l/2} \underbrace{e(x)}_{\text{action of } \left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)} e^{il\theta} w(y)$$

and so

$$0 = L W_\infty^l(g) = e^{-2i\theta}\left((-iy)\left(2\pi i + i\left(\frac{l}{2}y^{-1}\right) w + iw'\right) - \frac{1}{2i} ilw \right) y^{l/2} e(x) e^{il\theta}$$

So

$$0 = w' + 2\pi w \qquad 1^{\text{st}} \text{ order ODE.}$$

Solution: $w(y) = e^{-2\pi y}$

Hence

$$\boxed{W_\infty^l(g) = y^{l/2} \underbrace{e(z)}_{x+iy} e^{il\theta}}$$

Remaining case is $p \in S$:

Thm: (Casselman [Math. Ann. 1973], Miyake, Novodvorsky)

F = nonarch., (π, V) = inf. dim. irred. adm. rep of $G = GL_2(F)$

Let $N = \mathfrak{p}^r$

Let $K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_F) \;\middle|\; c \in \mathfrak{p}^r \text{ i.e. } c \equiv 0 (N) \right\}$.

let $K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_F) \;\middle|\; \begin{matrix} d-1 \in \mathfrak{p}^r \\ c \in \mathfrak{p}^r \end{matrix} \right\}$. i.e. $d \in 1+\mathfrak{p}^r$

let $N(\pi) = \mathfrak{p}^{r(\pi)}$, $r(\pi)$ smallest ≥ 0 s.t. $V^{K_1(N)} \neq 0$

then $r(\pi) < \infty$ and $\dim V^{K_1(N(\pi))} = 1$.

> Moreover: Can take s.t.
>
> $W_p(e) = ?$.

Note: if ω_π = cent. char of π

$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N)$ $(N = N(\pi))$

$v \in V^{K_1(N)}$

then $\pi(k)v = \omega(d)v$.

Thm: (SAT) (Strong Approx. Thm):

$(G' = SL_2)$

$G'(\mathbb{Q}) \cdot G'(\mathbb{R})$ is dense in $G'(\mathbb{A})$

Hence: For any cpt open subgp $K' \leq G'(\mathbb{A}_f)$

$$\underbrace{G'(\mathbb{Q})\, G'(\mathbb{R}) \cdot K'}_{\text{since open; containing } G'(\mathbb{Q})\cdot G'(\mathbb{R})} = G'(\mathbb{A})$$

Note: $G' = [G, G]$

Cor: let $K \leq G(\mathbb{A}_f)$ cpt. open

$$G(\mathbb{Q})G(\mathbb{R})^+ \backslash G(\mathbb{A}) / K \xrightarrow{\sim} \mathbb{Q}^\times \mathbb{R}_+^\times \backslash \mathbb{A}^\times / \det(K) \quad (\text{via } \det)$$

$$\Big\downarrow \wr \quad \leftarrow \text{if } K \leq GL_2(\hat{\mathbb{Z}})$$

$$\hat{\mathbb{Z}}^\times / \det K$$

pf:

$$G(\mathbb{Q})G(\mathbb{R})^+ g K = \underbrace{G(\mathbb{Q})G(\mathbb{R})^+ (\overset{K_1}{g K g^{-1} \cap K})}_{\cup}\, gK.$$

$$G'(\mathbb{Q})\, G'(\mathbb{R})\, K_1' = G'(\mathbb{A})$$

③

Thus any such coset is left invariant under $G'(\mathbb{A})$.

Cor: $G(\mathbb{A}) = G(\mathbb{Q})\, G(\mathbb{R})^+ K_0(N)$.

Recall: $f: G(\mathbb{Q})\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$

$k \in K_0(N) \subseteq G(\mathbb{A}_f)$

$f(gk) = \omega(k) f(g)$

$f(g k_\theta) = e^{il\theta} f(g)$

$G(\mathbb{R}^+)$ acts on $\mathfrak{h}$

$$\underset{\Psi}{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \underset{\Psi}{z} \longmapsto \frac{az+b}{cz+d} = g(z)$$

$$g\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{pmatrix} g(z) \\ 1 \end{pmatrix} \underset{j(g,z)}{\underbrace{(cz+d)}}$$

automorphy factor:

$j(g_1 g_2, z) = j(g_1, g_2 z)\, j(g_2, z)$

$j(k_\theta, i) = e^{-i\theta}$

Define: for $z = x+iy \in \mathfrak{h}$

$$\phi(z) := j(g_\infty, i)^l f(g_\infty) \cdot \det(g_\infty)^{-l/2} \quad \left[\text{where } g_\infty(i) = z\right]$$

(actually independent of choice of g_∞)

plug in Fourier exp:

$$\phi(z) = j(g_\infty, i)^l \det(g_\infty)^{-l/2} \sum_{\alpha \in \mathbb{Q}^\times} W\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_\infty\right)$$

But
$$W\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_\infty\right) = \underbrace{W_\infty^l\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_\infty\right)}_{\text{in } G(\mathbb{R})^{\mathrm{sgn}(\alpha)}} \cdot \overbrace{\prod_{p \in S} W_p\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\right) \cdot \prod_{p \notin S} W_p^\circ\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\right)}^{=0 \text{ unless } \mathrm{ord}_p(\alpha) \geq 0\ \forall p \;\Rightarrow\; \alpha = n \in \mathbb{Z}}$$

$= 0$ if $\alpha < 0 \Rightarrow \alpha = \ldots$

Now, take for $z = x+iy$.

$$g_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}, \quad \text{so} \quad j(g_\infty, i) = y^{-1/2}.$$

Also: $$W_\infty^\ell\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right) = W_\infty^\ell\left(\begin{pmatrix} 1 & nx \\ & 1 \end{pmatrix}\begin{pmatrix} (ny)^{1/2} & \\ & (ny)^{-1/2} \end{pmatrix}\right)$$

$$= (ny)^{\ell/2} \underbrace{e(nz)}_{= q^n}$$

Thus:

$$\phi(z) = y^{-\frac{\ell}{2}} \sum_{n=1}^{\infty} y^{\ell/2} n^{\ell/2} q^n \prod_p W_p\left(\begin{pmatrix} n & \\ & 1 \end{pmatrix}\right)$$

$$= \sum_{n=1}^{\infty} a_n q^n \quad \text{where} \quad a_n = n^{\ell/2} \prod_p W_p\begin{pmatrix} n & \\ & 1 \end{pmatrix}$$

Also: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \overset{\subseteq SL_2(\mathbb{Z})}{=} G(\mathbb{Q}) \cap G(\mathbb{R}^+) K_0(N)$.

$$\phi(\gamma z) = j(\gamma, z)^\ell f(\gamma g_\infty)$$

$\gamma \mapsto \gamma_\infty$ in $G(\mathbb{R}^+)$ or $\gamma_0 \in K_0(N)$

can take γg_∞

rewrite $f(\gamma_\infty g_\infty) = f(\gamma \gamma_0^{-1} g_\infty)$

$$= \omega(\gamma_0^{-1}) f(g_\infty)$$

So

$$\phi(\gamma z) = j(\gamma, z)^\ell \overset{\chi(a)}{\omega(\gamma_0)^{-1}} \phi(z)$$

So ϕ is classical modular form.