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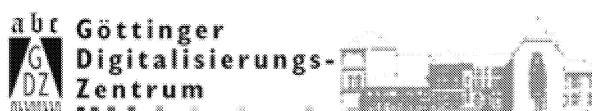
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Closures of Conjugacy Classes of Matrices are Normal

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0. Introduction

0.1. The purpose of this paper is to prove the following theorem: Let A be an $n \times n$ matrix over an algebraically closed field K of characteristic zero, C_A the conjugacy class of A and $\overline{C_A}$ its (Zariski-) closure.

Theorem. $\overline{C_A}$ is normal, Cohen-Macaulay with rational singularities.

If a variety X with a G -action (G reductive) is the closure of an orbit $\mathcal{O}$ and $\dim(X \setminus \mathcal{O}) \leq \dim X - 2$, it is a crucial question for the geometry of X to decide whether the singularity (in $X \setminus \mathcal{O}$) is normal. In fact the normality of X allows to identify the ring $K[X]$ of regular functions on X with the functions on the orbit $\mathcal{O}$ and so, by Frobenius reciprocity, to analyse $K[X]$ as a representation of G (cf. [11], [1]). In our case this is closely related to the “multiplicity conjecture” of Dixmier; we refer the reader to the paper [1] for a detailed description of this connection and some applications.

A different proof of this theorem will appear in [23].

0.2. The theorem has also another interesting application, shown to us by Th. Vust, in the spirit of the classical theory of Schur. If U is a finite dimensional vector space one has the classical relation between the action of $GL(U)$ and of the symmetric group $\mathfrak{S}_m$ on the tensor space $U^{\otimes m}$. If we restrict to the subgroup G_A of $GL(U)$ centralizing a fixed matrix A ($\in \text{End } U$), then one can still compute the centralizer of G_A acting on $U^{\otimes m}$ and one obtains (see Sect. 6): $\text{End}_{G_A}(U^{\otimes m})$ is spanned by the endomorphisms

$$\sigma \cdot A^{h_1} \otimes A^{h_2} \otimes \dots \otimes A^{h_m} \quad (\sigma \in \mathfrak{S}_m, h_1, \dots, h_m \in \mathbb{N}).$$

We remark that the group G_A is not reductive and the commuting algebra is not semisimple in general.

0.3. In many ways the motivation to study this problem came from a fundamental paper of B. Kostant [11] in which he studies in detail the adjoint action on a semisimple Lie algebra $\mathfrak{g}$. In the course of his analysis he shows the normality of

the variety $\overline{C_A}$ in the case in which A is a *regular nilpotent element* of $\mathfrak{g}$ (i.e. $\overline{C_A}$ is the *nilpotent cone* of $\mathfrak{g}$). His method depends on the fact that, in this case, $\overline{C_A}$ can be proved to be a *complete intersection* in $\mathfrak{g}$. This is no more true for the non regular classes in general, nevertheless some particular cases were treated by W. Hesselink [8]; we wish to thank him for his comments on an earlier version of the paper. Our method, on the other hand, consists in constructing an auxiliary variety Z which is a complete intersection and of which $\overline{C_A}$ is a “quotient” (1.4).

0.4. *Remark.* It is known (see [8] proposition 1, or use the method of associated cones [1]) that it is sufficient for the proof of the theorem to treat the case of a *nilpotent matrix* A and so we restrict to this case. Then $\overline{C_A}$ has a resolution of singularities $\pi: X \rightarrow \overline{C_A}$ where X is the cotangent bundle of GL_n/P , P a parabolic subgroup of GL_n ([4], or [1] Anhang). Then the canonical divisor of X is 0 and so by the theorem of Grauert-Riemenschneider (cf. [9] p. 50) it follows that $\overline{C_A}$ has *rational singularities* and the normality of $\overline{C_A}$ is sufficient to insure also the Cohen-Macaulay property. So the main point of the paper is to prove that $\overline{C_A}$ is *normal*. The proof we give should be adaptable also to positive characteristic; it yields at least that the *normalisation of $\overline{C_A}$ is purely inseparable over $\overline{C_A}$* (cf. remark 5.7).

0.5. Let us remark finally that the methods developed here have analogues for all the classical groups. In this case, which will be treated in a subsequent paper, there occur different phenomena which are not yet fully understood. Of course the non connected conjugacy classes have non normal closure, but there are also *infinitely many* connected conjugacy classes C_A for which $\overline{C_A}$ is *not normal*; the simplest known cases are: for the symplectic groups the one in $\mathfrak{sp}_8$ relative to the partition $(3, 3, 1, 1)$, for the orthogonal groups the one in $\mathfrak{so}_{13}$ relative to the partition $(4, 4, 2, 2, 1)$.

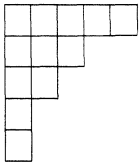
1. Notations, Some Known Results

1.1. Let us fix some notations. Any nilpotent matrix is conjugate to one in normal Jordan block form:

$$\begin{pmatrix} J_{p_1} & 0 & 0 & \dots & 0 \\ 0 & J_{p_2} & 0 & & \vdots \\ 0 & 0 & & & \vdots \\ 0 & 0 & \dots & \dots & J_{p_k} \end{pmatrix}, \quad J_t := \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \quad \text{a } t \times t\text{-block.} \tag{*}$$

We can assume $p_1 \geq p_2 \geq \dots \geq p_k$; this decreasing sequence $\eta = (p_1, p_2, \dots, p_k)$ is a partition of n and it is convenient to represent it geometrically as a *Young-diagram* with rows consisting of $p_1, p_2, \dots, p_k$ boxes respectively:

e.g. the diagram



corresponds to the partition $(5, 3, 2, 1, 1)$.

The *dual partition* $\hat{\eta}=(\hat{p}_1,\hat{p}_2,\dots,\hat{p}_m)$ is defined setting $\hat{p}_i$ equal to the length of the i^{th} column of the diagram η ; more formally $\hat{p}_i:=\#\{j|p_j\geq i\}$. In case of a partition η associated to the normal Jordan block form of a nilpotent matrix A , the dual partition $\hat{\eta}$ has the following interpretation:

$$\dim \text{Ker } A^j = \sum_{i=1}^j \hat{p}_i$$

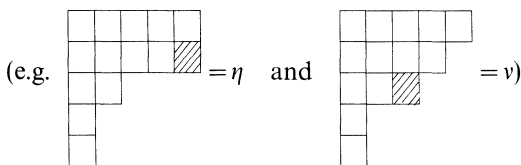
or equivalently

$$\text{rk } A^j = \sum_{i>j} \hat{p}_i.$$

1.2. Given two partitions $\eta=(p_1,\dots,p_s)$ and $v=(q_1,\dots,q_t)$ of n , we say $\eta\geq v$, if we have $\sum_{i=1}^j p_i\geq \sum_{i=1}^j q_i$ for all j . This is equivalent to $\sum_{k>j} \hat{p}_k\geq \sum_{k>j} \hat{q}_k$ for all j .

A simple property of this ordering, which expresses it geometrically, is the following (cf. [7] Proposition 3.9):

Proposition. *If $\eta>v$ and no other partition is in between them (i.e. η and v are adjacent in the ordering), then the diagram of η is obtained from the one of v raising a box from one row to the first allowable position.*



1.3. From now on, if η is a partition of n , we will indicate with C_η the conjugacy class of the matrix $(*)$ in normal Jordan block form with partition η . The following is the basic theorem on degenerations of orbits (cf. [7] Theorem 3.10 and Corollary 3.8 (a)).

Proposition. a) *Given two partitions η and v of n , we have $\eta\geq v$ if and only if $\overline{C_\eta}\supseteq C_v$.*

b) *If $\eta=(p_1,\dots,p_t)$ is a partition of n and $\hat{\eta}=(\hat{p}_1,\dots,\hat{p}_k)$ the dual partition, we have:*

$$\dim C_\eta = n^2 - \sum_{i,j=1}^t \min(p_i, p_j) = n^2 - \sum_{i=1}^k \hat{p}_i^2 = 2 \sum_{i<j} \hat{p}_i \hat{p}_j.$$

1.4. We are working always with affine varieties and we will use the following terminology. If X is an (affine) variety with the action of a reductive group G and $\pi: X\rightarrow Y$ a morphism, we say that π is a *quotient* (under G), if the coordinate ring of Y is identified, via π , with the ring of G -invariant functions on X . We denote this quotient by $\pi: X\rightarrow X/G$. The following properties of quotient maps are well known ([20], Chap. 1, § 2):

a) *Let $Z\subseteq X$ be a G -stable closed subvariety. Then $\pi(Z)\subseteq X/G$ is closed and $\pi|_Z: Z\rightarrow \pi(Z)$ is a quotient.*

b) Consider the following fibre product:

$$\begin{array}{ccc} X' := Y \times_{X/G} X & \xrightarrow{\phi'} & X \\ \downarrow \pi' & & \downarrow \pi \\ Y & \xrightarrow{\phi} & X/G. \end{array}$$

The action of G on X induces an action on the fibre product X' in a natural way and π' is a quotient with respect to this action.

2. The Induction Lemma

2.1. If U, V are vector spaces we will write $L(U, V)$ for the space of linear maps from U to V and $L(U)$ instead of $L(U, U)$. If V is n dimensional and η is a partition of n , we may consider the elements of $L(V)$ as $n \times n$ matrices and so $C_\eta \subseteq L(V)$.

2.2. Let $\eta = (p_1, \dots, p_k)$ be a partition of n . Erasing the first column in the Young diagram η one obtains a partition $\eta' = (p'_1, p'_2, \dots, p'_k)$ of $m := n - \hat{p}_1 = n - k$, formally defined by $p'_i = p_i - 1$ for all i . In terms of dual partitions we have $\hat{\eta}' = (\hat{p}_2, \hat{p}_3, \dots)$.

Fix vector spaces U, V of dimension m, n respectively and consider the two maps

$$\begin{array}{ccc} L(U, V) \times L(V, U) & \xrightarrow{\pi} & L(U) \\ \downarrow \rho & & \\ L(V) & & \end{array}$$

defined by $\pi(A, B) = BA$, $\rho(A, B) = AB$.

Theorem (First fundamental theorem of invariant theory): π and ρ are quotient maps (under $GL(V)$, $GL(U)$ respectively) and the image of ρ is the determinantal variety of matrices of rank $\leq m$. (cf. [22] § 3, Théorème 3 or [18] II.6, Theorem 2.6. A; for a characteristic free proof see [2] § 3.)

2.3. Consider finally the orbits $C_{\eta'} \subseteq L(U)$, $C_\eta \subseteq L(V)$ and the variety $N_\eta := \pi^{-1}(\overline{C_{\eta'}})$.

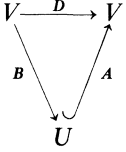
Lemma. $\rho(N_\eta) = \overline{C_\eta}$:

$$\begin{array}{ccc} \pi^{-1}(\overline{C_{\eta'}}) = N_\eta & \xrightarrow{\pi} & \overline{C_{\eta'}} \\ \downarrow \rho & & \\ \overline{C_\eta} & & \end{array}$$

Proof. First of all we show that $\rho(N_\eta) \subseteq \overline{C_\eta}$, using repeatedly Proposition 1.3 a). Let $(A, B) \in N_\eta$, i.e. $BA \in \overline{C_{\eta'}}$. To prove that $AB \in \overline{C_\eta}$ we must verify that, for any

$i \geq 1$, we have $\text{rk}(AB)^i \leq \sum_{j \geq i+1} \hat{p}_j$. Now $(AB)^i = A(BA)^{i-1}B$, so $\text{rk}(AB)^i \leq \text{rk}(BA)^{i-1} \leq \sum_{j \geq i} \hat{p}_j = \sum_{j \geq i+1} \hat{p}_j$ as desired.

To show that $\rho(N_\eta) = C_\eta$ it is sufficient to prove that $C_\eta \subseteq \rho(N_\eta)$ (since ρ is a quotient map and so $\rho(N_\eta)$ is closed, cf. 1.4a). Let us then fix $D \in C_\eta$, $D: V \rightarrow V$. We can clearly identify U with $D(V)$ since $\text{rk } D = m$. It is immediate to verify that $D|_U$ has Young diagram η' and clearly we have a factorization



where A is the inclusion and B coincides with D . On the other hand BA is just $D|_U$ so that the pair (A, B) is in N_η and the claim is proved¹. $\square$

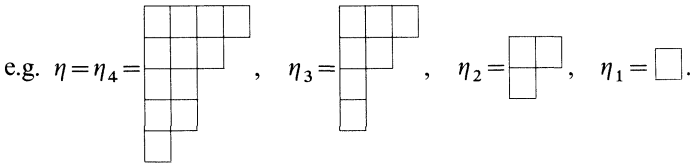
We will use this lemma to present the variety $\overline{C}_\eta$ as a quotient of a suitable variety Z for which we will be able to prove normality (Theorem 3.3).

3. The Variety Z

3.1. Notations being as in section 2 we make the following construction. Starting with a fixed partition $\eta = (p_1, \dots, p_k)$ and dual partition $\hat{\eta} = (\hat{p}_1, \dots, \hat{p}_t)$, $t := p_1$, we define a sequence of partitions

$$\eta_i := \eta, \quad \eta_{i-1}, \quad \eta_{i-2}, \dots, \eta_1$$

by $\eta_{i-1} := \eta'_i$ (i.e. by erasing successively the first column of the corresponding Young diagrams);



Then η_i is a partition of $n_i := \hat{p}_t + \hat{p}_{t-1} + \dots + \hat{p}_{t-i+1}$ with dual partition $\hat{\eta}_i = (\hat{p}_{t-i+1}, \dots, \hat{p}_{t-1}, \hat{p}_t)$.

Construct next vector spaces $U_1, U_2, \dots, U_t$ of dimensions $n_1, n_2, \dots, n_t$ respectively and consider the affine spaces:

$$M := L(U_1, U_2) \times L(U_2, U_1) \times L(U_2, U_3) \\ \times L(U_3, U_2) \times \dots \times L(U_{t-1}, U_t) \times L(U_t, U_{t-1})$$

and

$$N := L(U_1) \times L(U_2) \times \dots \times L(U_{t-1}).$$

¹ This argument due to W. Hesselink replaces a more direct matrix computation we had made.

We will indicate a point α of M by

$$\alpha = (A_1, B_1, A_2, B_2, \dots, A_{t-1}, B_{t-1})$$

where $A_i: U_i \rightarrow U_{i+1}$, $B_i: U_{i+1} \rightarrow U_i$.

3.2. We are now ready to define the variety Z . It is the subvariety of M defined by the equations

$$\begin{aligned} B_1 A_1 &= 0 \\ B_2 A_2 &= A_1 B_1 \\ B_3 A_3 &= A_2 B_2 \\ &\vdots \\ B_{t-1} A_{t-1} &= A_{t-2} B_{t-2}. \end{aligned} \tag{**}$$

In more suggestive notation we write

$$\alpha: U_0 = 0 \xrightleftharpoons[B_0=0]{A_0=0} U_1 \xrightleftharpoons[B_1]{A_1} U_2 \xrightleftharpoons[B_2]{A_2} U_3 \dots U_{t-1} \xrightleftharpoons[B_{t-1}]{A_{t-1}} U_t$$

for the elements of Z . The equations just require that for each $i=1 \dots t-1$ the two compositions $U_{i-1} \rightrightarrows U_i \rightrightarrows U_{i+1}$ yield the same endomorphism of U_i . (See also [19] 5.3, where this objects occur as representations of a certain Lie algebra.)

In due time we will prove that the equations we have given actually define a reduced variety; for the moment we think of Z as a scheme, possibly not reduced, and we indicate by Z_{red} the reduced variety associated.

The best way to understand the equations is to construct a map $\Phi: M \rightarrow N$ given by the formula:

$$\begin{aligned} \Phi(A_1, B_1, A_2, B_2, \dots, A_{t-1}, B_{t-1}) \\ = (B_1 A_1, B_2 A_2 - A_1 B_1, B_3 A_3 - A_2 B_2, \dots, B_{t-1} A_{t-1} - A_{t-2} B_{t-2}). \end{aligned}$$

Then Z , as a scheme, is the fiber $\Phi^{-1}(0)$ of the 0 point in N .

3.3. Consider the group $G := GL(U_1) \times GL(U_2) \times \dots \times GL(U_t)$ and its normal subgroup $H := GL(U_1) \times GL(U_2) \times \dots \times GL(U_{t-1})$. The group G acts on M and N in a natural way:

On

$$\begin{aligned} M: (g_1, g_2, \dots, g_t)(A_1, B_1, A_2, B_2, \dots, A_{t-1}, B_{t-1}) \\ := (g_2 A_1 g_1^{-1}, g_1 B_1 g_2^{-1}, \dots, g_t A_{t-1} g_{t-1}^{-1}, g_{t-1} B_{t-1} g_t^{-1}) \end{aligned}$$

and on

$$\begin{aligned} N: (g_1, g_2, \dots, g_t)(E_1, E_2, \dots, E_{t-1}) \\ := (g_1 E_1 g_1^{-1}, \dots, g_{t-1} E_{t-1} g_{t-1}^{-1}). \end{aligned}$$

It is easy to verify that the map $\Phi: M \rightarrow N$ is equivariant under G and so Z is invariant under G . Now the main theorem is an immediate consequence of the following more precise result (use the fact that a quotient of a normal variety is also normal):

Theorem. i) Z is a complete intersection in M ; the equations $(**)$ give a regular sequence.

ii) Z is non singular in codimension 1.

iii) Z is reduced, irreducible and normal.

iv) There is an isomorphism $Z/H \xrightarrow{\sim} \overline{C}_\eta$ being compatible with the actions of $GL(U_t) = G/H$.

The rest of the paper is devoted to the proof of this theorem. We first reduce it to a lemma (3.7) whose proof will be given in Sect. 5 (5.4, 5.5, 5.6) using the theory of nilpotent pairs (section 4) and a dimension formula for nilpotent pair orbits (5.3).

3.4. First of all we settle part iv) which gives the connection between Z and $\overline{C}_\eta$. We consider the map

$$\Theta: M \rightarrow L(U_t)$$

given by $(A_1, B_1, \dots, A_{t-1}, B_{t-1}) \mapsto A_{t-1} B_{t-1}$, which is clearly $GL(U_t)$ equivariant.

Proposition. $\Theta(Z_{\text{red}}) = \overline{C}_\eta$ and the induced map $\Theta': Z_{\text{red}} \rightarrow \overline{C}_\eta$ is a quotient map under H (i.e. $Z_{\text{red}}/H \xrightarrow{\sim} \overline{C}_\eta$).

Proof. We use repeatedly lemma 2.3. Since $B_1 A_1 = 0$ the pair (A_1, B_1) is in the variety N_{η_2} and so $A_1 B_1 \in \overline{C}_{\eta_2}$. By induction we may assume $A_{i-1} B_{i-1} \in \overline{C}_{\eta_i}$. Since $B_i A_i = A_{i-1} B_{i-1}$ we have that $(A_i, B_i) = N_{\eta_{i+1}}$ and so again by 2.3, $A_i B_i \in \overline{C}_{\eta_{i+1}}$. Thus finally Θ maps Z_{red} into $\overline{C}_\eta = \overline{C}_{\eta_t}$ and the same lemma 2.3 applied inductively shows that Z_{red} is mapped onto $\overline{C}_\eta$. To see that the map is a quotient under H we perform the quotients in succession. First under $GL(U_1)$, we have the quotient map (Theorem 2.2)

$$\begin{aligned} \Theta_1: M \rightarrow L(U_2) \times L(U_2, U_3) \times L(U_3, U_2) \times \dots \\ \times L(U_{t-1}, U_t) \times L(U_{t-1} U_{t-1}) \end{aligned}$$

given by

$$(A_1, B_1, A_2, B_2, \dots, A_{t-1}, B_{t-1}) \mapsto (A_1 B_1, A_2 B_2, \dots, A_{t-1} B_{t-1}).$$

If we restrict this map to Z_{red} we have again a quotient map (since we are in characteristic zero cf. 1.4a). Now on Z_{red} we have $A_1 B_1 = B_2 A_2$ (if $t > 2$); thus we see that Θ_1 maps Z_{red} into the graph of the map

$$\begin{aligned} \gamma: M_1 := L(U_2, U_3) \times L(U_3, U_2) \times \dots \\ \times L(U_{t-1}, U_t) \times L(U_t, U_{t-1}) \rightarrow L(U_2) \\ (A_2, B_2, A_3, B_3, \dots, A_{t-1}, B_{t-1}) \mapsto B_2 A_2. \end{aligned}$$

Thus we may replace the graph of γ with its domain and drop the first coordinate $A_1 B_1$. Hence on Z_{red} the mapping

$$(A_1, B_1, \dots, A_{t-1}, B_{t-1}) \mapsto (A_2, B_2, \dots, A_{t-1}, B_{t-1}) \in M_1$$

is a quotient under $GL(U_1)$. Its image $Z_1 \subseteq M_1$ is easily seen to be defined by the “equations”:

$$\begin{aligned} B_2 A_2 &\in \overline{C_{\eta_2}} \\ A_2 B_2 &= B_3 A_3 \\ A_3 B_3 &= B_4 A_4 \\ &\vdots \\ A_{t-2} B_{t-2} &= B_{t-1} A_{t-1}. \end{aligned}$$

Similarly (if $t > 3$) $Z_1/GL(U_2) = Z_{\text{red}}/GL(U_1) \times GL(U_2)$ is naturally contained in

$$M_2 := L(U_3, U_4) \times L(U_4, U_3) \times \dots \times L(U_{t-1}, U_t) \times L(U_t, U_{t-1})$$

and given by the “equations”:

$$\begin{aligned} B_3 A_3 &\in \overline{C_{\eta_3}} \\ A_3 B_3 &= B_4 A_4 \\ &\vdots \\ A_{t-2} B_{t-2} &= B_{t-1} A_{t-1}. \end{aligned}$$

Then finally by induction $Z_{\text{red}}/GL(U_1) \times GL(U_2) \times \dots \times GL(U_{t-2})$ is given by

$$\{(A_{t-1}, B_{t-1}) | B_{t-1} A_{t-1} \in \overline{C_{\eta_{t-1}}}\} \subseteq L(U_{t-1}, U_t) \times L(U_t, U_{t-1}),$$

i.e. it is the variety $N_{\eta} = N_{\eta_t}$, and hence

$$Z_{\text{red}}/H \cong N_{\eta}/GL(U_{t-1}) \cong \overline{C_{\eta}}$$

(the isomorphism being induced by Θ). $\square$

This reasoning can be also displayed in a more suggestive way constructing a diagram, e.g. $t = 5$:

$$\begin{array}{ccccccc} Z(1, 5) & \rightarrow & Z(1, 4) & \rightarrow & Z(1, 3) & \rightarrow & N_{\eta_1} \rightarrow \overline{C_{\eta_1}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z(2, 5) & \rightarrow & Z(2, 4) & \rightarrow & N_{\eta_2} & \rightarrow & \overline{C_{\eta_2}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z(3, 5) & \rightarrow & N_{\eta_3} & \rightarrow & \overline{C_{\eta_3}} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ N_{\eta_4} & \rightarrow & \overline{C_{\eta_4}} & & & & \\ \downarrow & & & & & & \\ \overline{C_{\eta_5}} & & & & & & \end{array}$$

where each $Z(i, j)$ is constructed inductively forming a fiber product. If we proceed on a column we see that

$$Z(k, n) \leftarrow Z(1, n)/GL(U_1) \times GL(U_2) \times \dots \times GL(U_{k-1}) \quad \text{for } k < n-1$$

(cf. 1.4b).

3.5. We now make a simple remark on the basic map $\Phi: M \rightarrow N$ (3.2) for which $Z = \Phi^{-1}(0)$.

Let M^0 be the open subset of M of those elements $(A_1, B_1, A_2, B_2, \dots, A_{t-1}, B_{t-1})$ such that for each $i=1, 2, \dots, t-1$ either A_i or B_i has maximal rank. Then we have:

Proposition. *The differential $d\Phi$ of Φ is onto at every point α of M^0 .*

Proof. Let $\alpha = (A_1, B_1, A_2, B_2, \dots, A_{t-1}, B_{t-1}) \in M^0$. We can identify the tangent space of M in α with M itself and take a point $T = (X_1, Y_1, X_2, Y_2, \dots, X_{t-1}, Y_{t-1})$ in it. Then the tangent map gives

$$\begin{aligned} d\Phi_\alpha(T) = & (Y_1 A_1 + B_1 X_1, Y_2 A_2 + B_2 X_2 - X_1 B_1 - A_1 Y_1, \dots, Y_{t-1} A_{t-1} \\ & + B_{t-1} X_{t-1} - X_{t-2} B_{t-2} - A_{t-2} Y_{t-2}). \end{aligned}$$

If $W = (W_1, W_2, \dots, W_{t-1})$ is any tangent vector in $\Phi(\alpha) \in N$ we can solve inductively the equations

$$\begin{aligned} Y_1 A_1 + B_1 X_1 &= W_1 \\ Y_2 A_2 + B_2 X_2 - X_1 B_1 - A_1 Y_1 &= W_2 \\ &\vdots \\ Y_{t-1} A_{t-1} + B_{t-1} X_{t-1} - X_{t-2} B_{t-2} - A_{t-2} Y_{t-2} &= W_{t-1} \end{aligned}$$

provided that for each i either A_i or B_i has maximal rank. In fact if A_i has maximal rank then there is an $\overline{A}_i: U_{i+1} \rightarrow U_i$ with $\overline{A}_i A_i = Id_{U_i}$, so the equation $Y_i A_i = R_i$ is solved by $Y_i := R_i \overline{A}_i$. Similarly if B_i has maximal rank then there is an element $\overline{B}_i: U_i \rightarrow U_{i+1}$ with $B_i \overline{B}_i = Id_{U_i}$ and the equation $B_i X_i = S_i$ is solved by $X_i := \overline{B}_i S_i$. qed.

3.6. The net result of this proposition is this:

Corollary. *The open subvariety $Z^0 := Z \cap M^0$ of Z is smooth and of codimension $\sum_{i=1}^{t-1} n_i^2$ in M .*

Proof. The only thing to prove is that $Z^0 \neq \emptyset$ since then the statement is a consequence of 3.5 ($\dim N = \sum_{i=1}^{t-1} n_i^2$ by definition 3.1). Now if we recall the proof of 3.4 we see that we have constructed an element in Z such that for all i both A_i and B_i have maximal rank (see also the construction in 3.1). More precisely if $D: U \rightarrow U$ is any element of C_η we may assume $U_t = U$, $U_{t-1} = D(U)$, $U_{t-2} = D^2(U)$, ..., $U_1 = D^{t-1}(U)$, $U_0 = 0$. Setting $A_i: U_i \rightarrow U_{i+1}$ the inclusion, and $B_i: U_{i+1} \rightarrow U_i$ the map D itself, we have the required element. qed.

Remark. If we insist that for each i both A_i and B_i have maximal rank we still get a non empty open set Z' of Z . One can easily show that Z' is an orbit under G (cf. proof above). It will be proved in fact that this is the unique open orbit of G in Z (5.4).

3.7. To complete the proof of the theorem it is enough to show the following result:

Lemma. $\dim(Z \setminus Z^0) \leq \dim Z - 2$.

In fact using 3.6 this lemma implies $\dim Z = \dim Z^0$ and that Z is non singular in codimension one. Thus again by 3.6 we have that the codimension of Z in M is exactly the number $\sum_{i=1}^{t-1} n_i^2$ of equations defining Z (3.2). This implies that these equations form a regular sequence and hence Z is a complete intersection. Since Z is a cone it is also connected. But then by Serre’s criterion ([6] IV, Théorème 5.8.6) Z is normal reduced and so also irreducible. This completes the proof of the theorem 3.3 modulo the lemma above.

For this basic statement we will need to stratify the complement of Z^0 in Z with strata of which we can compute the dimension (5.1, 5.3) and this will lead us to the theory of nilpotent pairs (cf. the following section).

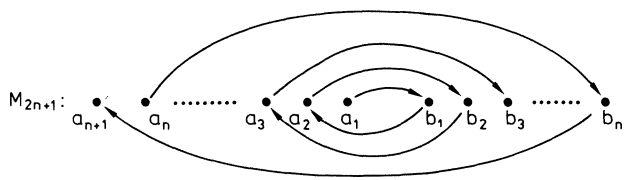
4. Nilpotent Pairs

Given two vector spaces U, V we consider the space $L := L(U, V) \times L(V, U)$ of pairs of maps $U \overset{A}{\underset{B}{\rightleftarrows}} V$ as a representation of $GL(U) \times GL(V)$ in a canonical way:

$$(X, Y)(A, B) = (YAX^{-1}, XBY^{-1}).$$

The theory of orbits for this representation is known (cf. [3], [14], or [5]) and it is in fact a special case of the theory of vector space crowns. One can naturally think of such pairs as of a category of modules, and the classification is (like in the case of Jordan blocks) through indecomposable modules. Also the “invariant theory” of this representation is well known (see [12] and also [17]). In our case we are interested in a special class of pairs, those $U \overset{A}{\underset{B}{\rightleftarrows}} V$ for which BA (or equivalently AB) is *nilpotent*. We will call such pairs “*nilpotent pairs*”. They can be easily seen to be exactly the unstable vectors (in the sense of geometric invariant theory) of the representation L .

4.2. The classification of the *indecomposable nilpotent pairs* is rather simple and resembles the theory of Jordan blocks. The indecomposables are of the following types:



i.e. the space U is spanned by the basis $a_1, a_2, \dots, a_{n+1}$, V has basis $b_1, b_2, \dots, b_n$ and

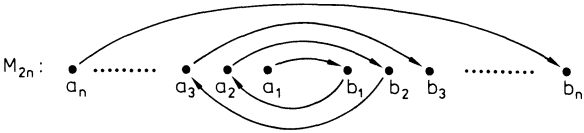
$$A a_i = b_i, \quad B b_j = a_{j+1}.$$

This type will be in short indicated by a string

$$abababab \dots ba$$

with $n+1$ a 's and n b 's.

The type

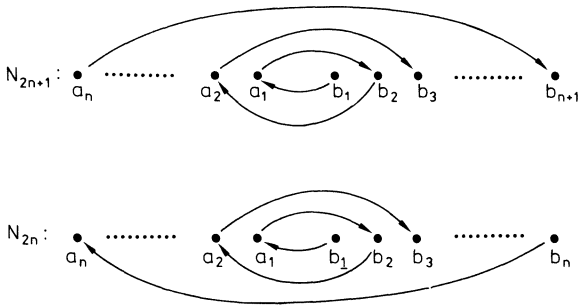


is defined in a similar way and is shortly indicated by the string

$$abab \dots ab$$

with n a 's and n b 's.

We have two other types starting with b instead of a :



shortly indicated by $baba \dots bab$ and $baba \dots ba$ respectively.

4.3. In general a nilpotent pair $U \xrightleftharpoons[B]{A} V$ is a direct sum of indecomposables and so it will be determined by a finite set of such strings (*ab-strings*). We will refer to such a set of strings as the *ab-diagram of the pair*. It is easy to see that two distinct *ab-diagrams* give rise to non isomorphic pairs, since one can easily recover the *ab-diagram* from the knowledge of the ranks of all the compositions $BABA \dots$.

Given a nilpotent pair $U \xrightleftharpoons[B]{A} V$ through its *ab-diagram* δ , it is simple to recognise the Young diagrams of the nilpotent matrices $BA: U \rightarrow U$ and $AB: V \rightarrow V$: For the diagram of BA suppress all the b 's in the *ab-strings* of δ . In this way every *ab-string* gives rise to a string of a 's which can be interpreted as a

row in a Young diagram. Similarly for AB one has to suppress all the a 's. We call these diagrams the *associated a -diagram* and the *associated b -diagram* of δ and denote them as in 2.2 by $\pi(\delta)$ and $\rho(\delta)$.

$ababab$

$babab$

e.g. $\delta :=$

$abab$

$abab$

ba

b

then

aaa

aa

aa

aa

a

$\pi(\delta) =$

$= (3, 2, 2, 2, 1)$

is the Young diagram of $BA \in L(U)$ and

bbb

bb

b

b

b

$\rho(\delta) =$

$= (3, 2, 1, 1, 1, 1)$

is that of $AB \in L(V)$.

4.4. One should make three remarks:

Remark 1. Not all pairs of Young diagrams describing a nilpotent orbit in $L(U)$ and one in $L(V)$ are associated to some nilpotent pair. Furthermore, there can be different nilpotent pairs giving rise to the same pair of Young diagrams.

Remark 2. For a nilpotent pair (A, B) with ab -diagram δ one can immediately verify the following:

- i)

A is injective if and only if every ab -string in δ ends with b .
- ii)

A is surjective if and only if every ab -string in δ starts with a .
- iii)

B is injective if and only if every ab -string in δ ends with a .
- iv)

B is surjective if and only if every ab -string in δ starts with b .

Remark 3. For any ab -diagram δ we denote by X_δ its orbit in L (under the group $GL(U) \times GL(V)$). We have the two maps

X_δ

$\xrightarrow{\pi'}$

$C_{\pi(\delta)}$

ρ'

$\downarrow$

$C_{\rho(\delta)}$

induced by π and ρ (2.2) which are *fibrations* (being of the form $G/H \rightarrow G/H'$ with closed subgroups $H \subseteq H' \subseteq G := GL(U) \times GL(V)$). In particular π' and ρ' are *smooth*.

5. Nilpotent Strings, Proof of Lemma 3.7

5.1. We want to go back to the basic variety Z (3.2) formed by strings

$$\alpha: U_0 = 0 \xleftarrow[A_0=0]{B_0=0} U_1 \xleftarrow[B_1]{A_1} U_2 \xleftarrow[B_2]{A_2} U_3 \dots U_{t-1} \xleftarrow[B_{t-1}]{A_{t-1}} U_t$$

with the conditions $B_{i+1}A_{i+1} = A_iB_i$ for $i=0,1,\dots,t-2$. Let us indicate by $Y_0 = \{\emptyset\}$, $Y_1, Y_2, \dots, Y_t$ the (finite) sets of diagrams indexing nilpotent conjugacy classes in $L(U_0), L(U_1), \dots, L(U_t)$ respectively and by $\Psi_0, \Psi_1, \dots, \Psi_{t-1}$ the (finite) sets of *ab*-diagrams indexing conjugacy classes of nilpotent pairs in

$$L(U_0, U_1) \times L(U_1, U_0), L(U_1, U_2) \times L(U_2, U_1), \dots, \\ L(U_{t-1}, U_t) \times L(U_t, U_{t-1})$$

respectively. We have the already described maps associated to the two compositions (cf. 4.3):

$$Y_0 \xleftarrow{\pi_0} \Psi_0 \xrightarrow{\rho_0} Y_1 \xleftarrow{\pi_1} \Psi_1 \xrightarrow{\rho_1} Y_2 \\ \xleftarrow{\pi_2} \Psi_2 \xrightarrow{\rho_2} Y_3 \leftarrow \dots \rightarrow \Psi_{t-1} \xrightarrow{\rho_{t-1}} Y_t.$$

We can form the iterated fiber products and construct the finite set $\mathcal{A}$ of strings $\lambda = (\delta_0, \delta_1, \delta_2, \dots, \delta_{t-1})$ of *ab*-diagrams $\delta_i \in \Psi_i$ with

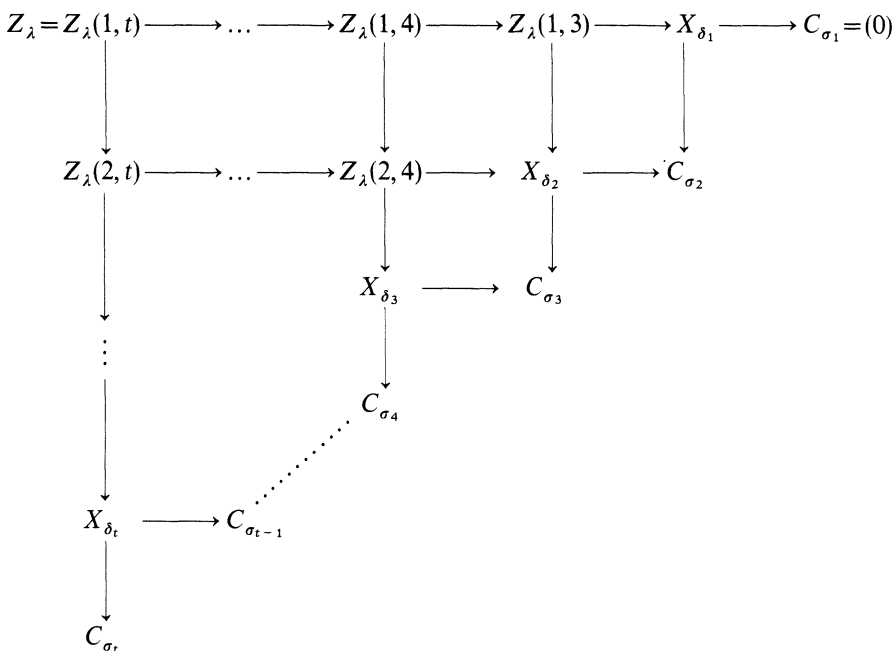
$$\rho_i(\delta_i) = \pi_{i+1}(\delta_{i+1}), \quad i=0,1,\dots,t-2.$$

For each $\lambda \in \mathcal{A}$ we have a stratum Z_λ of the variety Z : Z_λ is the set of all points

$$\alpha: 0 = U_0 \xleftarrow[B_0]{A_0} U_1 \xleftarrow[B_1]{A_1} U_2 \rightleftarrows \dots \rightleftarrows U_{t-1} \xleftarrow[B_{t-1}]{A_{t-1}} U_t$$

of Z such that for each i the nilpotent pair (A_i, B_i) has *ab*-diagram δ_i .

Put $\sigma_i := \rho_{i-1}(\delta_{i-1})$, $i=1,2,\dots,t$ and let us indicate as usual by C_{σ_i} the conjugacy class of diagram σ_i and by X_{δ_i} the nilpotent pair orbit of diagram δ_i . The definition of $\mathcal{A}$ implies that we have a fiber product diagram subordinate to the basic diagram constructing Z :



in which each map is smooth (4.4 remark 3). Hence we get the following proposition:

- Proposition.** (i) Z_λ Is a locally closed, G -stable, smooth and irreducible subvariety of Z .
 (ii) The set A indexes a stratification of Z into smooth G -stable strata.

5.2. The following result now clearly implies lemma 3.7.

Lemma. For all $\lambda \in A$ either $Z_\lambda \subseteq Z^0$ or $\dim Z_\lambda \leq \dim Z - 2$.

The proof will be given in 5.4, 5.5, 5.6 using the following dimension formula for nilpotent pair orbits.

5.3. Proposition. Let $X = X_\delta \subset L(U, V) \times L(V, U)$ be a nilpotent pair orbit projecting to the nilpotent conjugacy classes $C_1 \subset L(U)$ and $C_2 \subset L(V)$. Then

$$\dim X_\delta = \frac{1}{2}(\dim C_1 + \dim C_2) + \dim U \cdot \dim V - \Delta,$$

$$\Delta := \sum_{i \text{ odd}} a_i b_i$$

where a_i (resp. b_i) denotes the number of ab -strings of length i starting with a (resp. with b).

Proof. The representation of $GL(U) \times GL(V)$ on $L := L(U, V) \times L(V, U)$ is a Θ -group in the sence of Vinberg [17] (cf. also [12]): Consider the automorphism Θ of $\text{End}(U \oplus V)$ (and of $GL(U \oplus V)$) given by conjugation with $J = \begin{pmatrix} Id_U & 0 \\ 0 & -Id_V \end{pmatrix}$; then $GL(U) \times GL(V)$ is the fixed point group and $L \subset \text{End}(U \oplus V)$ the (-1) -

eigenspace of Θ . Furthermore we have the following relation between the dimension of X and the dimension of the conjugacy class $C \subset \text{End}(U \oplus V)$ generated by X :

$$\dim X = \frac{1}{2} \dim C$$

(cf. [17] §2.5 Proposition 5, or [12] 1.3 Proposition 5). In order to calculate $\dim C$ denote by r_i resp. s_i the number of a 's resp. b 's in the i^{th} row of the ab -diagram δ associated to X . Then the partition of the nilpotent conjugacy class C is given by $(p_1, p_2, \dots)$, $p_i := r_i + s_i$, hence

$$\dim C = (n+m)^2 - \sum_{i,j} \min(p_i, p_j)$$

(1.3 Proposition b), $n := \dim V$, $m := \dim U$). By definition we have $|r_i - s_i| \leq 1$ and therefore $\min(p_i, p_j) = \min(r_i, r_j) + \min(s_i, s_j)$ except in the case $p_i = p_j$ odd, $r_i = s_j$ and $r_j = s_i$, where $\min(p_i, p_j) = \min(r_i, r_j) + \min(s_i, s_j) + 1$. This implies

$$\begin{aligned} \dim C &= (n+m)^2 - \sum_{i,j} \min(r_i, r_j) + \sum_{i,j} \min(s_i, s_j) + 2 \cdot \sum_{i \text{ odd}} a_i b_i \\ &= \dim C_1 + \dim C_2 + 2nm - 2\Delta, \end{aligned}$$

hence the required dimension formula.² qed.

Now let $\lambda \in \Delta$, $\lambda = (\delta_0, \dots, \delta_{t-1})$ and $\lambda' = (\delta_0, \dots, \delta_{t-2})$. Set $\sigma = \rho_{t-1}(\delta_{t-1})$, $\sigma' = \rho_{t-2}(\delta_{t-2})$, and $\Delta_\lambda = \sum_{i=0}^{t-1} \Delta_i$, Δ_i the Δ associated to δ_i .

Corollary. $\dim Z_\lambda = \sum_{i=1}^{t-1} n_i n_{i+1} + \frac{1}{2} \dim C_\sigma - \Delta_\lambda$.

Proof. We have the fibre product diagram

$$\begin{array}{ccc} Z_\lambda & \longrightarrow & Z_{\lambda'} \\ \downarrow & & \downarrow \\ X_{\delta_{t-1}} & \longrightarrow & C_{\sigma'} \\ \downarrow & & \\ C_\sigma & & \end{array}$$

Now the proposition implies

$$\dim X_{\delta_{t-1}} = \frac{1}{2} (\dim C_\sigma + \dim C_{\sigma'}) + \dim U_t \cdot \dim U_{t-1} - \Delta_{t-1},$$

so we have by induction:

$$\begin{aligned} \dim Z_\lambda &= \dim Z_{\lambda'} + \dim X_{\delta_{t-1}} - \dim C_{\sigma'} \\ &= \sum_{i=1}^{t-2} n_i \cdot n_{i+1} + \frac{1}{2} \dim C_{\sigma'} + \frac{1}{2} (\dim C_{\sigma'} + \dim C_\sigma) \end{aligned}$$

² This proof was suggested by G. Kempken; it replaces a explicit but lengthy calculation of stabilizers we have made.

$$\begin{aligned} &+n_{t-1}n_t-\dim C_{\sigma'}-\Delta_{t-1} \\ &= \sum_{i=1}^{t-1} n_i n_{i+1} + \tfrac{1}{2} \dim C_{\sigma} - \Delta_{\lambda}. \quad \text{qed.} \end{aligned}$$

5.4. We now look, in view of corollary 5.3, at the projection $\Theta: Z \rightarrow \overline{C_{\eta}}$ (3.4) and try to study the various strata Z_{λ} which lie on top of a given orbit C_{σ} in $\overline{C_{\eta}}$. First of all analyze the open orbit C_{η} : we have to describe the strings $\lambda = (\delta_0, \delta_1, \dots, \delta_{t-1})$ which lead to $\rho_{t-1}(\delta_{t-1}) = \eta$. We claim that *there is only one such string*.

Let us take in general a Young diagram σ and let σ' be the diagram obtained from σ erasing the first column. We want to find an ab -diagram such that the a -diagram and b -diagram associated are σ' and σ .

Given σ (as b -diagram) and a certain number m of a 's, to construct an ab -diagram over σ one has to proceed as follows: First of all every b -string of σ has to be filled internally with a 's. This requires altogether as many a 's as the number of boxes m' in σ' . If m is equal to m' the ab -diagram δ over σ is unique, its associated a -diagram is σ' , and every ab -string of δ starts and ends with b . If $m < m'$ there is no ab -diagram over σ . If $m > m'$, after having used the m' a 's, one can utilize the remaining $m - m'$ a 's in many ways: add an a at the beginning or the end of an ab -string or create a row with single a .

$$\begin{array}{c} b b b b b \\ b b b \\ \text{Example: } \sigma = b b b \quad = (5, 3, 3, 2, 1), m' = 9; \\ b b \\ b \end{array}$$

the unique ab -diagram with $m = 9$ a 's is

$$\begin{array}{c} b a b a b a b a b \\ b a b a b \\ \delta = b a b a b \\ b a b \\ b \end{array}$$

associated to σ and

$$\begin{array}{c} a a a a \\ \pi(\delta) = \frac{a a}{a a} = \sigma' \\ a \end{array}$$

If we give $m = 10$ a 's (for instance) one easily sees that it is possible to construct 11 ab -diagrams over σ , if $m = 11$, we can construct 56 ab -diagrams etc.

Summing up the result we see by induction that *there is a unique string $\lambda^0 = (\delta_1^0, \delta_2^0, \dots, \delta_{t-1}^0)$ such that $\rho_{t-1}(\delta_{t-1}^0) = \eta = \eta_t$* . For this string we have $\rho_i(\delta_i^0) = \eta_{i+1}$ for all i . Since every ab -string in each δ_i^0 starts and ends with b , it follows

from 4.4 remark 2 that $Z_{\lambda^0} \subseteq Z^0$ ($Z_{\lambda^0} = Z'$ with the notations of remark 3.6). Hence we get the following result:

Lemma. *There is a unique string $\lambda^0 = (\delta_1^0, \delta_2^0, \dots, \delta_{t-1}^0)$ such that $\rho_{t-1}(\delta_{t-1}^0) = \eta = \eta_t$. For this string we have $Z_{\lambda^0} \subseteq Z^0$.*

5.5. We now make a further simple remark. By 3.6 and 3.1 we have

$$\begin{aligned} \dim Z^0 &= 2 \sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-1} n_i^2 = \sum_{i=1}^{t-1} n_i n_{i+1} + \sum_{i=1}^{t-1} n_i (n_{i+1} - n_i) \\ &= \sum_{i=1}^{t-1} n_i n_{i+1} + \sum_{i < j} \hat{p}_i \hat{p}_j = \sum_{i=1}^{t-1} n_i n_{i+1} + \frac{1}{2} \dim C_\eta. \end{aligned}$$

Now if $\lambda \neq \lambda^0$, Z_λ projects to some orbit C_σ with $\sigma < \eta$ and thus $\dim Z_\lambda < \dim Z^0$ (Corollary 5.3). This implies (as one can also verify directly) that $\dim Z_{\lambda^0} = \dim Z^0$ and Z_{λ^0} is the unique open orbit of G acting on Z . The same estimate shows that, if $\dim C_\sigma \leq \dim C_\eta - 4$, then $\dim Z_\lambda \leq \dim Z^0 - 2$.

To complete the proof of 5.2 we are thus restricted to analyze the strings λ such that Z_λ projects to some C_σ with $\dim C_\sigma = \dim C_\eta - 2$. In each degeneration the dimension of a nilpotent orbit decreases by at least 2 (since the orbits are even dimensional, see proposition 1.3b). Thus the only case in which we may have $\dim C_\sigma = \dim C_\eta - 2$ is if the diagram σ is obtained from η moving down a single box (Proposition 1.2). The explicit dimension formula (Proposition 1.3b)) shows, in fact, that the only case is *to move a box down to the next row*. Given such a σ we must study which strings $(\delta_1, \delta_2, \dots, \delta_{t-1})$ lead to $\rho_{t-1}(\delta_{t-1}) = \sigma$.

5.6. We analyze inductively this problem as before and claim that the analysis restricts to the following problem:

Given a diagram η , let η' be obtained from η removing the first column, and let σ be a one step degeneration of η (obtained by moving down one box to the next row). We must study the ab -diagrams δ such that $\rho(\delta) = \sigma$ and $\pi(\delta) \leq \eta'$ ($\pi(\delta)$ and η' with the same number of boxes).

We follow the same analysis as before, letting m be the number of boxes of η' . Let σ' be obtained from σ erasing the first column and m' its number of boxes. We have clearly the two possibilities $m = m'$ and $m = m' + 1$.

Case I: $m = m'$; this case is obtained when the box moved in the degeneration of η to σ is attached to a non empty row.

In this case the previous analysis (5.4) shows that i) there is a unique ab -diagram δ over σ , ii) δ is associated to σ, σ' and σ' is a one step degeneration of η' , and iii) every ab -string of δ starts and ends with b .

Case II: $m = m' + 1$; this case is obtained when σ is gotten from η splitting off one box from the last row (to form a new "one box" row). In this case the previous analysis shows that, to form an ab -diagram δ over σ , we are forced to place m' a 's; the remaining single a can be placed only in the last two rows or by itself since we must preserve the condition $\pi(\delta) \leq \eta'$.

Let us consider thus the last two rows; after filling with m' of the a 's they are:

$$\begin{array}{c} b a b a b \dots a b \\ b \end{array}$$

For the remaining a we have 5 choices:

(α) and (α'): *We attach a to the row $b a b \dots a b$ either to the right or to the left.* In this case the ab -diagram δ represents a pair in which one of the two maps has maximal rank. The associated a -diagram is just η' .

(β) and (β'): *We attach a to the row b left or right.* In this case the ab -diagram δ represents also a pair in which one of the two maps has maximal rank. The associated a -diagram is a one step degeneration of η' .

(γ): *We create a new row consisting of the remaining a .* In this case the associated a -diagram is a one step degeneration of η' but neither map in the nilpotent pair has maximal rank. On the other hand for this ab -diagram we have, in the notations of 5.3 $a_1 = b_1 = 1$ and hence $\Delta = 1$. For the corresponding nilpotent pair orbit we have thus the dimension

$$\dim X_\delta = \frac{1}{2}(\dim C_\sigma + \dim C_\sigma) + \dim U \cdot \dim V - 1.$$

Summing up all this analysis we see by an easy induction, that we have proved:

If Z_λ projects to C_σ , σ a one step degeneration of η , then either

$$Z_\lambda \subseteq Z^0$$

or

$$\dim Z_\lambda \leq \sum_{i=1}^{t-1} n_i n_{i+1} + \frac{1}{2} \dim C_\sigma - 1 \leq \dim Z^0 - 2.$$

This completes the proof of 5.2.

5.7. Remark. The only place in which we have used characteristic zero, apart from the implication normal $\Rightarrow$ Cohen Macaulay, was in the proof of proposition 3.4, where we used the following fact: If V is a affine variety on which a reductive group G acts and W a closed subvariety invariant under G , then the induced map $W/G \rightarrow V/G$ is a *closed immersion* (cf. 1.4a)). In characteristic $p > 0$ one can only say that this map is finite and injective, i.e. W/G is purely inseparable over its image (cf. [21] §4).

6. An Application to Tensor Representation

6.1. We present here the application due to Th. Vust announced in the introduction. Let $A \in \text{End}(U)$ be a matrix, G_A the centralizer of A in $GL(U)$. We consider the action of G_A on the tensor space $U^{\otimes m}$ and wish to compute $\text{End}_{G_A}(U^{\otimes m})$. One knows that $\text{End}_{GL(U)}(U^{\otimes m})$ is spanned by the symmetric group $\mathfrak{S}_m$ acting on $U^{\otimes m}$ in the obvious way (cf. [2], [18]). Now clearly the endomorphisms $A^{h_1} \otimes A^{h_2} \otimes \dots \otimes A^{h_m} \in \text{End}(U^{\otimes m})$ also commute with G_A and we have:

Theorem (Th. Vust). *The algebra $\text{End}_{G_A}(U^{\otimes m})$ is spanned by the elements*

$$\sigma \cdot A^{h_1} \otimes \dots \otimes A^{h_m}, \quad \sigma \in \mathfrak{S}_m, \quad h_1, \dots, h_m \in \mathbb{N}.$$

The proof will require some lemmas (mostly well known).

6.2. Let V be an affine variety, G a reductive group acting on V , $W \subseteq V$ a G -stable closed subvariety, M a linear representation of G and $\varphi: W \rightarrow M$ a G -equivariant morphism.

Lemma 1. *There exists a G -equivariant morphism $\Phi: V \rightarrow M$ extending φ .*

Proof. Let $K[U]$, $K[W]$ be the coordinate rings of V , W . A G -equivariant morphism φ from W to M is given by an element $u \in (K[W] \otimes M)^G$. To extend φ to V is equivalent to lift u to $(K[V] \otimes M)^G$, and this is a simple consequence of linear reductivity. $\square$

6.3. Let V be as before, $G := GL(V)$. We take now W to be the closure $\overline{GA}$ of an orbit GA for an element $A \in V$. We assume:

- i) $\dim(\overline{GA} \setminus GA) \leq \dim GA - 2$,
- ii) $\overline{GA}$ is a normal variety.

Let G_A denote the stabilizer of A in G and M be again a linear representation of G .

Lemma 2. *If $B \in M$ is invariant under G_A , then under the condition i) and ii) there exists a G -equivariant morphism $\Phi: V \rightarrow M$ such that $\Phi(A) = B$.*

Proof. First of all we construct a morphism $\varphi: GA \rightarrow M$ given by $gA \mapsto gB$; this is well defined since $B \in M^{G_A}$. The two hypotheses i) and ii) on $\overline{GA}$ imply that φ extends (uniquely) to a G -equivariant map $\varphi': \overline{GA} \rightarrow M$. Finally taking $W = \overline{GA}$ and applying lemma 1 we have the required conclusion. $\square$

6.4. We now want to apply these lemmas to the following set up: $M := \text{End}(U^{\otimes m}) = \text{End}(U)^{\otimes m}$, $A \in \text{End}(U)$ the given matrix, $B \in \text{End}_{G_A}(U^{\otimes m})$, $V := \text{End}(U)$. To finish our proof it only remains to explicit the set of G -equivariant maps $\Phi: \text{End}(U) \rightarrow \text{End}(U)^{\otimes m}$. This set can be easily computed (cf. [15]). We need two lemmas for which we refer to the literature.

Lemma 3 ([10] Lemma 4.9, [15] Theorem 1.2). *Let $\sigma \in \mathfrak{S}_m$ be decomposed into cycles: $\sigma = (i_1 i_2 \dots i_k) (j_1 j_2 \dots j_e) \dots (t_1 t_2 \dots t_s)$ (including cycles of length one), and $Y = X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_k} \in \text{End}(U)^{\otimes m}$. Then*

$$\text{Tr}(\sigma \cdot Y) = \text{Tr}(X_{i_1} X_{i_2} \dots X_{i_k}) \cdot \text{Tr}(X_{j_1} X_{j_2} \dots X_{j_e}) \dots \text{Tr}(X_{t_1} X_{t_2} \dots X_{t_s}).$$

Lemma 4 ([16] Theorem 1, [15] Theorem 1.3). *The ring of invariants of the space of m -tuples of matrices $(X_1, X_2, \dots, X_m)$ under simultaneous conjugation under $GL(U)$ is generated by the invariants*

$$\text{Tr}(X_{v_1} X_{v_2} \dots X_{v_k}), \quad k \in \mathbb{N}, \quad v_1, \dots, v_k \in \{1, 2, \dots, m\}.$$

6.5. Let us now look at the space L of G -equivariant maps $\Phi: \text{End}(U) \rightarrow \text{End}(U)^{\otimes m}$. Clearly L is a module over the ring R of invariants of $\text{End}(U)$.

Proposition. L is spanned, as an R -module, by the maps of type:

$$X \mapsto \sigma \cdot X^{h_1} \otimes X^{h_2} \otimes \dots \otimes X^{h_m}, \quad \sigma \in \mathfrak{S}_m, \quad h_i \in \mathbb{N}.$$

Proof. Let $\Phi: \text{End}(U) \rightarrow \text{End}(U)^{\otimes m}$ be a G -equivariant map. We introduce m new variables $Y_1, Y_2, \dots, Y_m$ in $\text{End}(U)$ and construct a function Ψ on $\text{End}(U)^{m+1}$ by setting

$$\Psi(X, Y_1, Y_2, \dots, Y_m) := \text{Tr}(\Phi(X) \cdot Y_1 \otimes Y_2 \otimes \dots \otimes Y_m).$$

By the non degeneracy of the trace form the mapping $\Phi \mapsto \Psi$ is an injection from L to the space of invariants of $X, Y_1, Y_2, \dots, Y_m$ which are linear in $Y_1, Y_2, \dots, Y_m$. Now by lemma 4 such invariants are of type

$$\begin{aligned} & \sum t(X) \cdot \text{Tr}(X^{h_1} Y_{i_1} X^{h_2} Y_{i_2} \dots X^{h_k} Y_{i_k}) \cdot \text{Tr}(X^{p_1} Y_{j_1} X^{p_2} Y_{j_2} \dots) \dots \\ & \dots \text{Tr}(X^{s_1} Y_{t_1} X^{s_2} Y_{t_2} \dots) \quad (t(X) \in R). \end{aligned}$$

The previous lemma 3 shows then that any such invariant is of type $\text{Tr}(\Phi(X) \cdot Y_1 \otimes Y_2 \otimes \dots \otimes Y_m)$, where $\Phi(X)$ is a linear combination with coefficients in R of the special maps $X \mapsto \sigma \cdot X^{h_1} \otimes X^{h_2} \otimes \dots \otimes X^{h_m}$. The previous remark about the injectivity of $\Phi \mapsto \Psi$ completes the proof. qed .

6.6. We now sum all our work and prove the main theorem 6.1:

Let $B \in \text{End}_{G_A}(U^{\otimes m}) = (\text{End}(U)^{\otimes m})^{G_A}$; we have seen that there exists a G -equivariant map $\Phi: \text{End}(U) \rightarrow \text{End}(U^{\otimes m})$ such that $\Phi(A) = B$. By the previous proposition 6.5 we know all equivariant maps. The very formula given by the proposition implies immediately the theorem. qed .

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