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INVOLUTIONS IN WEYL GROUPS

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ABSTRACT. Let G be a split real group with Weyl group W. Let E be an irreducible representation of W. Let V be the stable Lie algebra version of the coherent continuation representation of W. The main result of this paper is a formula for the multiplicity of E in V. The formula involves the position of E in Lusztig's set $\coprod \mathcal{M}(\mathcal{G})$. The paper treats all quasi-split groups G as well.

1. INTRODUCTION

Let G be a split adjoint group over $\mathbf{R}$ with root system R and Weyl group W. Choose a system R^+ of positive roots in R. By an involution in W we mean an element $\sigma \in W$ such that $\sigma^2 = 1$. Given an involution $\sigma \in W$, we denote by W^σ the centralizer of σ in W, and we denote by R_I the set of imaginary roots relative to σ. There is a sign character ϵ_I on W^σ, defined as follows: for $w \in W^\sigma$ we put $\epsilon_I(w) = (-1)^k$, where k is the number of positive imaginary roots α such that $w\alpha$ is negative.

For any involution $\sigma \in W$ and any finite dimensional representation E of W we define $m(\sigma, E)$ to be the multiplicity of the character ϵ_I in the restriction of E to the subgroup W^σ. For any finite dimensional representation E of W we define $m(E)$ by

$$m(E) := \sum_\sigma m(\sigma, E)$$

where the sum ranges over a set of representatives σ for the conjugacy classes of involutions in W.

Let $\mathfrak{g}$ denote the (real) Lie algebra of G, and let $\mathbf{O}$ denote a nilpotent orbit of $G(\mathbf{C})$ in $\mathfrak{g}_{\mathbf{C}}$. Let E be the Springer representation of W corresponding to $\mathbf{O}$. Then $m(E)$ is equal to the dimension of the space of stably $G(\mathbf{R})$-invariant linear combinations of $G(\mathbf{R})$-invariant measures on the $G(\mathbf{R})$-orbits in $\mathbf{O} \cap \mathfrak{g}$; this assertion (see [Kot98]) is the stable analog of a theorem of Rossmann [Ros90], 3.7. Assuming Assem's conjecture [Ass98] is valid in the real case, then for split classical groups and special $\mathbf{O}$ the number $m(E)$ should be equal to the cardinality of the finite group $\mathcal{G}$ associated by Lusztig to the special representation E. Assem's conjecture also predicts that $m(E)$ should be 0 if $\mathbf{O}$ is not special. See [Kot98] for a detailed discussion of Assem's conjecture, which has been proved by Waldspurger [Wal99] for classical unramified p-adic groups.

Both of these predictions of Assem's conjecture are proved in this paper, which suggests that Assem's conjecture is indeed valid in the real case. More generally,

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this paper calculates (case-by-case) the multiplicities $m(E)$ and $m(\sigma, E)$ for all split classical groups and all irreducible representations E of W, and the last three sections of the paper solve the analogous problem for real groups that are quasi-split but not split. Casselman [Cas98] treats all the split exceptional groups.

The multiplicities $m(E)$ and $m(\sigma, E)$ are stable Lie algebra analogs of multiplicities that have been computed for all classical real groups by Barbasch (see the comment following Theorem 3.5 in [Bar91]) and McGovern [McG98]. I am indebted to Vogan for these references. No ideas beyond those used by Barbasch and McGovern are needed to calculate the stable multiplicities, so this revised version of the paper records the results in the stable case with only brief indications of proofs.

There is a surprisingly simple formula for $m(E)$ (see Theorem 1 below). In order to state it we need to review some results of Lusztig [Lus84], [Lus79]. The set $W^\vee$ of isomorphism classes of irreducible representations of W is a disjoint union of subsets, called *families*. Associated to a family $\mathcal{F}$ is a finite group $\mathcal{G} = \mathcal{G}_{\mathcal{F}}$.

Associated to any finite group $\mathcal{G}$ is a finite set $\mathcal{M}(\mathcal{G})$, defined as follows (see [Lus84], [Lus79]). Consider pairs (x, ρ), where x is an element of $\mathcal{G}$ and ρ is an irreducible (complex) representation of the centralizer $\mathcal{G}_x$ of x in $\mathcal{G}$. There is an obvious conjugation action of $\mathcal{G}$ on this set of pairs, and $\mathcal{M}(\mathcal{G})$ is by definition the set of orbits for this action. The set $\mathcal{M}(\mathcal{G})$ has an obvious basepoint $(1, 1)$.

We now consider the function $\hat{\phi}_0$ on $\mathcal{M}(\mathcal{G})$ whose values are given by

$$\hat{\phi}_0(x, \rho) = \sum_{s \in S} d(s, \rho),$$

where S is a set of representatives for the $\mathcal{G}_x$-conjugacy classes of elements $s \in \mathcal{G}_x$ such that $s^2 = x$, and $d(s, \rho)$ denotes the dimension of the space of vectors in ρ fixed by the centralizer $\mathcal{G}_s$ of s in $\mathcal{G}$. As the notation suggests, $\hat{\phi}_0$ arises naturally as the Fourier-Lusztig transform of a function ϕ_0 on $\mathcal{M}(\mathcal{G})$ (see 2.9 and 2.10). The value of $\hat{\phi}_0$ at the base point in $\mathcal{M}(\mathcal{G})$ is equal to the number of conjugacy classes of involutions in $\mathcal{G}$. If $\mathcal{G}$ is an elementary abelian 2-group, then $\hat{\phi}_0$ is very simple: its value at the basepoint in $\mathcal{M}(\mathcal{G})$ is equal to the order of $\mathcal{G}$ and all remaining values are 0.

Let $\mathcal{F}$ be a family of representations of W, and let $\mathcal{G}$ be the associated finite group. Then Lusztig [Lus84], Ch. 4 defines (case-by-case) an injection

$$\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G}).$$

The image of the unique special representation in $\mathcal{F}$ under this injection is the base point $(1, 1) \in \mathcal{M}(\mathcal{G})$.

We now state a theorem, which in this paper is proved only for the classical root systems. The exceptional root systems are treated by Casselman [Cas98]. In this theorem we assume that the group G is simple, so that the root system R is irreducible. The theorem involves the notion of exceptional representations [BL78] for the Weyl groups of type E_7 and E_8.

Theorem 1. *Let E be an irreducible representation of the Weyl group W, and let $\mathcal{F} \subset W^\vee$ be the unique family containing E. Let $\mathcal{G}$ be the finite group associated to $\mathcal{F}$, and let $x_E \in \mathcal{M}(\mathcal{G})$ denote the image of E under the injection $\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G})$.*

Then there is an equality

$$m(E) = \begin{cases} \hat{\phi}_0(x_E) & \textit{if } E \textit{ is non-exceptional,} \\ 1 & \textit{if } E \textit{ is exceptional.} \end{cases}$$

In particular if E is special and non-exceptional, then $m(E)$ is the number of conjugacy classes of involutions in $\mathcal{G}$. Moreover if R is classical, so that $\mathcal{G}$ is necessarily an elementary abelian 2-group, then $m(E) = 0$ if E is non-special, and $m(E) = |\mathcal{G}|$ if E is special.

The results in this paper are compatible with the following conjecture on left cells. (Left cells in W and left cell representations of W are defined in [KL79].)

Conjecture 1. *Let Γ be a left cell in W, and let $[\Gamma]$ denote the corresponding left cell representation of W. Let σ be an involution in W and let Σ denote its conjugacy class in W. Then the multiplicity $m(\sigma, [\Gamma])$ is equal to the cardinality of the set $\Sigma \cap \Gamma$.*

If the root system R is of type A_n, then left cells can be described in terms of the Robinson-Schensted correspondence [KL79], [BV82a], and the conjecture is true: it follows from 3.1 and a known statement about the Robinson-Schensted correspondence (see Exercise 4 in §4.2 of [Ful97]). Moreover Casselman [Cas98] has verified the conjecture for the root systems F_4 and E_6.

Conjecture 1 can also be generalized to the quasi-split case (see Conjecture 2 in §4). In the special case of complex groups Conjecture 2 is true: it reduces to a result of Lusztig, namely Proposition 12.15 in [Lus84].

As is clear from the discussion above, this paper owes much to ideas of Assem and Rossmann. I would like to thank two other people for their direct contributions to this paper. Casselman [Cas98] used a computer to prove Theorem 1 for E_6, E_7 and E_8 and in doing so discovered that the exceptional representations of the Weyl groups of type E_7 and E_8 behave differently from the non-exceptional ones. Moreover my confidence in Conjecture 1 was greatly bolstered by Casselman's verification of its truth for F_4 and E_6. Fulton's help with the representation theory of S_n was invaluable, and in fact the work on type A_n in 3.1 was done jointly with him, including the proof of Conjecture 1 for type A_n.

This paper is organized in the following way. Theorem 1 is stated in §2. In §3 Theorem 1 is proved for the classical root systems. At the same time the multiplicities $m(\sigma, E)$ are calculated. The five exceptional root systems are treated by Casselman [Cas98], who tabulates all the individual multiplicities $m(\sigma, E)$. Theorem 2 is stated in §4 and proved in §5 and §6. Again the multiplicities $m(\sigma, E)$ are calculated at the same time. Fortunately Theorem 2 for the quasi-split outer form of E_6 requires no essentially new computation; we need only appeal to Casselman's verification of Theorem 1 for E_6.

2. Statement of Theorem 1 (the split case)

2.1. **The goal.** In this section we introduce notation, review some results of Lusztig, and state Theorem 1 (see 2.13).

2.2. **Involutions.** Let W be the Weyl group of an irreducible reduced root system R. Choose a system R^+ of positive roots in R. By an involution in W we mean an element $\sigma \in W$ such that $\sigma^2 = 1$. Given an involution $\sigma \in W$, we denote by W^σ

the centralizer of σ in W, and we denote by R_I the set of imaginary roots relative to σ (by definition a root α is said to be imaginary if $\sigma(\alpha) = -\alpha$). Of course W^σ acts on R_I by automorphisms of that root system, and therefore there is a sign character ϵ_I on W^σ, defined as follows: for $w \in W^\sigma$ we put $\epsilon_I(w) = (-1)^k$, where k is the number of positive imaginary roots α such that $w\alpha$ is negative. Note that the Weyl group $W(R_I)$ of R_I is a subgroup of W^σ and that the restriction of ϵ_I to $W(R_I)$ is the usual sign character on that Weyl group.

2.3. **Definition of $m(\sigma, E)$.** For any involution $\sigma \in W$ and any finite dimensional representation E of W we define $m(\sigma, E)$ to be the multiplicity of the character ϵ_I in the restriction of E to the subgroup W^σ; clearly $m(\sigma, E)$ depends only on the conjugacy class of σ in W.

2.4. **Definition of $m(E)$.** For any finite dimensional representation E of W we define $m(E)$ by

$$m(E) := \sum_\sigma m(\sigma, E)$$

where the sum ranges over a set of representatives σ for the conjugacy classes of involutions in W.

2.5. **Families of representations of W.** We need to review some results of Lusztig [Lus84], [Lus79]. The set $W^\vee$ of isomorphism classes of irreducible representations of W is a disjoint union of subsets, called *families*. Each family $\mathcal{F} \subset W^\vee$ contains a unique special representation; thus the set of families is in one-to-one correspondence with the set of special representations in $W^\vee$. Associated to a family $\mathcal{F}$ is a finite group $\mathcal{G} = \mathcal{G}_\mathcal{F}$. If R is classical, then $\mathcal{G}$ is an elementary abelian 2-group. If R is exceptional, then $\mathcal{G}$ is one of the symmetric groups S_n $(1 \le n \le 5)$.

2.6. **The set $\mathcal{M}(\mathcal{G})$.** Associated to any finite group $\mathcal{G}$ is a finite set $\mathcal{M}(\mathcal{G})$, defined as follows (see [Lus84], [Lus79]). Consider pairs (x, ρ), where x is an element of $\mathcal{G}$ and ρ is an irreducible (complex) representation of the centralizer $\mathcal{G}_x$ of x in $\mathcal{G}$. There is an obvious conjugation action of $\mathcal{G}$ on this set of pairs, and $\mathcal{M}(\mathcal{G})$ is by definition the set of orbits for this action. The set $\mathcal{M}(\mathcal{G})$ has an obvious basepoint $(1, 1)$, the first entry being the identity element of $\mathcal{G}$ and the second entry being the trivial representation of $\mathcal{G}$.

2.7. **Pairing $\{\cdot,\cdot\}$ on $\mathcal{M}(\mathcal{G})$.** Given elements (x, ρ), (y, τ) in $\mathcal{M}(\mathcal{G})$ one defines (see [Lus79], §4) a complex number $\{(x, \rho), (y, \tau)\}$ by the formula

$$\{(x,\rho),(y,\tau)\} = |\mathcal{G}_x|^{-1}|\mathcal{G}_y|^{-1} \sum_g \chi_\tau(g^{-1}x^{-1}g)\chi_\rho(gyg^{-1}), \tag{2.7.1}$$

where the sum is taken over the set of elements $g \in \mathcal{G}$ such that x commutes with gyg^{-1}, and where we have written χ_τ and χ_ρ for the characters of the representations τ and ρ respectively. Clearly the pairing $\{\cdot,\cdot\}$ has the following property:

$$\{m, n\} = \overline{\{n, m\}} \text{ for all } m, n \in \mathcal{M}(\mathcal{G}). \tag{2.7.2}$$

2.8. **Fourier transform on $\mathcal{M}(\mathcal{G})$.** Let ϕ be a complex-valued function on $\mathcal{M}(\mathcal{G})$. The *Fourier transform* $\hat{\phi}$ of ϕ is the complex-valued function on $\mathcal{M}(\mathcal{G})$ defined by

$$\hat{\phi}(m) = \sum_{n\in\mathcal{M}(\mathcal{G})} \{m,n\}\phi(n).$$

It follows easily from the definitions that

$$\hat{\phi}(x,\rho) = |\mathcal{G}_x|^{-1} \sum_{y\in\mathcal{G}_x} \chi_\rho(y) \sum_{\tau\in\mathcal{G}_y^\vee} \chi_\tau(x^{-1})\cdot\phi(y,\tau). \tag{2.8.1}$$

In [Lus79], §4 it is shown that the square of the Fourier transform operator is the identity (in other words, the square of the matrix $\{m,n\}$ is the identity).

2.9. **Special function ϕ_0 on $\mathcal{M}(\mathcal{G})$.** Let $\mathcal{G}^\vee$ denote the set of isomorphism classes of irreducible representations of $\mathcal{G}$. For any $\rho\in\mathcal{G}^\vee$ we define a number $\varepsilon(\rho)$ in the following way. If ρ is not isomorphic to its contragredient, we put $\varepsilon(\rho)=0$. Otherwise, either ρ admits a non-degenerate $\mathcal{G}$-invariant symmetric bilinear form, in which case we put $\varepsilon(\rho)=1$, or ρ admits a non-degenerate $\mathcal{G}$-invariant alternating bilinear form, in which case we put $\varepsilon(\rho)=-1$. To understand the significance of $\varepsilon(\rho)$, one should recall that the number of involutions in $\mathcal{G}$ is equal to $\sum_{\rho\in\mathcal{G}^\vee}\varepsilon(\rho)\cdot\dim(\rho)$. More generally, for any element $x\in\mathcal{G}$ one has an equality

$$\big|\{g\in\mathcal{G} \;\big|\; g^2=x\}\big| = \sum_{\rho\in\mathcal{G}^\vee}\varepsilon(\rho)\cdot\chi_\rho(x). \tag{2.9.1}$$

Define an involution θ on the group algebra $\mathbf{C}[\mathcal{G}]$ by $\theta(g)=g^{-1}$ $(g\in\mathcal{G})$ and let L_x be the $\mathbf{C}$-linear endomorphism of $\mathbf{C}[\mathcal{G}]$ given by left multiplication by x; then (2.9.1) can be proved by calculating the trace of the endomorphism $L_x\theta$ in two ways.

We will need the following function ϕ_0 on $\mathcal{M}(\mathcal{G})$. Let $m=(x,\rho)\in\mathcal{M}(\mathcal{G})$. We put $\phi_0(m)=\varepsilon(\rho)$ (of course the function ε occurring here is the one for the finite group $\mathcal{G}_x$).

2.10. **Fourier transform of ϕ_0.** We are interested in the Fourier transform of the function ϕ_0 defined in 2.9. The value of this Fourier transform at the base point in $\mathcal{M}(\mathcal{G})$ is equal to the number of conjugacy classes of involutions in $\mathcal{G}$. In fact the following more general statement holds:

$$\hat{\phi}_0(x,\rho) = \sum_{s\in S} d(s,\rho), \tag{2.10.1}$$

where S is a set of representatives for the $\mathcal{G}_x$-conjugacy classes of elements $s\in\mathcal{G}_x$ such that $s^2=x$, and $d(s,\rho)$ denotes the dimension of the space of vectors in ρ fixed by $\mathcal{G}_x\cap\mathcal{G}_s$ (here $\mathcal{G}_x$, $\mathcal{G}_s$ denote the centralizers of x, s in $\mathcal{G}$). In particular $\hat{\phi}_0(x,\rho)$ is always a non-negative integer.

Indeed, it follows from (2.8.1) and (2.9.1) that

$$\hat{\phi}_0(x,\rho) = |\mathcal{G}_x|^{-1}\sum_{y\in\mathcal{G}_x}\chi_\rho(y)\cdot|\{s\in\mathcal{G}_y \;\big|\; s^2=x\}| \tag{2.10.2}$$

(note that $\{s\in\mathcal{G}_y \;\big|\; s^2=x^{-1}\}$ and $\{s\in\mathcal{G}_y \;\big|\; s^2=x\}$ have the same cardinality). The right side of (2.10.2) is equal to

$$\sum_{\{s\in\mathcal{G}\,|\,s^2=x\}} |\mathcal{G}_x|^{-1}\sum_{y\in\mathcal{G}_x\cap\mathcal{G}_s}\chi_\rho(y),$$

which is in turn equal to the right side of (2.10.1). (Note that any element $s \in \mathcal{G}$ such that $s^2 = x$ automatically lies in $\mathcal{G}_x$.)

If $\mathcal{G}$ is an elementary abelian 2-group, then the Fourier transform of ϕ_0 is very simple; its value at the basepoint in $\mathcal{M}(\mathcal{G})$ is equal to the order of $\mathcal{G}$ and all remaining values are 0.

To handle the exceptional root systems one also needs to compute the Fourier transform of ϕ_0 for the symmetric groups S_n ($3 \leq n \leq 5$). We give the results in the form of lists. We adopt (without further explanation) the notation used by Lusztig [Lus84], 4.3 to describe the elements in $\mathcal{M}(\mathcal{G})$ for these three symmetric groups. We do not give all the values of $\hat{\phi}_0$, just the ones needed for our purposes.

We begin with the case $\mathcal{G} = S_3$. Then $\mathcal{M}(\mathcal{G})$ has 8 elements, but we need only consider 5 of them: $(1,1)$, $(g_2,1)$, $(1,r)$, $(g_3,1)$, $(1,\varepsilon)$. The values of $\hat{\phi}_0$ on these 5 elements are 2, 0, 1, 1, 0 (in the order given).

Next we consider the case $\mathcal{G} = S_4$. Then $\mathcal{M}(\mathcal{G})$ has 21 elements, but we need only consider 11 of them: $(1,1)$, $(1,\lambda^1)$, $(1,\lambda^2)$, $(1,\sigma)$, $(g_2,1)$, (g_2,ε''), $(g_2',1)$, (g_2',ε''), (g_2',ε'), $(g_3,1)$, $(g_4,1)$. The values of $\hat{\phi}_0$ on these 11 elements are 3, 1, 0, 2, 0, 0, 1, 0, 0, 1, 0 (in the order given).

Finally we consider the case $\mathcal{G} = S_5$. Then $\mathcal{M}(\mathcal{G})$ has 39 elements, but we need only consider 17 of them: $(1,1)$, $(g_3,1)$, $(g_2',1)$, $(1,\nu)$, $(1,\lambda^1)$, $(g_5,1)$, (g_3,ε), $(1,\nu')$, (g_2',ε''), $(1,\lambda^2)$, (g_2',ε'), $(1,\lambda^3)$, $(g_2,1)$, $(g_4,1)$, $(g_6,1)$, (g_2,r), (g_2,ε). The values of $\hat{\phi}_0$ on these 17 elements are 3, 2, 1, 2, 2, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 (in the order given).

2.11. **Injection $\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G})$.** Let $\mathcal{F}$ be a family of representations of W, and let $\mathcal{G}$ be the associated finite group (see 2.5). Then Lusztig [Lus84], Ch. 4 defines (case-by-case) an injection

$$\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G}). \tag{2.11.1}$$

The image of the unique special representation in $\mathcal{F}$ under the injection (2.11.1) is the base point $(1,1) \in \mathcal{M}(\mathcal{G})$.

2.12. **Exceptional representations of W.** Let E be an irreducible representation of the Weyl group W. Recall that E is said to be exceptional (see [BL78]) if R is of type E_7 and $\dim(E) = 512$ (there are two of these) or if R is of type E_8 and $\dim(E) = 4096$ (there are four of these). Recall also the involution $i : W^\vee \to W^\vee$ introduced in [BL78]. If E is non-exceptional, then $i(E) = E$. If E is exceptional, then $i(E) \neq E$ and the unique family $\mathcal{F} \subset W^\vee$ containing E has two elements, namely E and $i(E)$.

2.13. **Statement of Theorem 1.** Let E be an irreducible representation of the Weyl group W, and let $\mathcal{F} \subset W^\vee$ be the unique family containing E. Let $\mathcal{G}$ be the finite group associated to $\mathcal{F}$ (see 2.5), and let $x_E \in \mathcal{M}(\mathcal{G})$ denote the image of E under the injection (2.11.1). Theorem 1 states that

$$m(E) = \begin{cases} \hat{\phi}_0(x_E) & \text{if } E \text{ is non-exceptional,} \\ 1 & \text{if } E \text{ is exceptional.} \end{cases} \tag{2.13.1}$$

In particular (see 2.10) if E is special and non-exceptional, then $m(E)$ is the number of conjugacy classes of involutions in $\mathcal{G}$. Moreover (again see 2.10) if R is classical, so that $\mathcal{G}$ is necessarily an elementary abelian 2-group, then $m(E) = 0$ if E is non-special, and $m(E) = |\mathcal{G}|$ if E is special.

If E is exceptional, then $\mathcal{F} = \{E, i(E)\}$ (see 2.12), the group $\mathcal{G}$ is cyclic of order 2, and the function $E' \mapsto \hat{\phi}_0(x_{E'})$ takes the value 2 on the unique special member of $\mathcal{F}$ and 0 on the unique non-special member of $\mathcal{F}$. Therefore Theorem 1 is equivalent to the statement that

$$m(E) = [\hat{\phi}_0(x_E) + \hat{\phi}_0(x_{i(E)})]/2 \tag{2.13.2}$$

for all E, exceptional or not.

3. Proof of Theorem 1 for classical split groups

3.1. **Root system A_n.** Let $\sigma \in S_n$ be an involution and let s be the number of fixed points of σ on $\{1, \dots, n\}$. Let E be an irreducible representation of S_n, and let t be the number of columns of odd length in the Young diagram corresponding to E. Then the multiplicity $m(\sigma, E)$ is equal to $\delta_{s,t}$ (Kronecker delta). This follows from the Littlewood-Richardson rule and 4.1(b) in [BV82b] (see also [Tan85] and [Tho80]).

It follows that $m(E) = 1$ for every irreducible representation E of W. This proves Theorem 1 in this case, since for type A_n the families $\mathcal{F}$ are singletons and the groups $\mathcal{G}_{\mathcal{F}}$ are trivial (see [Lus84], 4.4). For type A_n Theorem 1 is not new; it is a special case of a theorem of Rossmann [Ros90], 3.7.

3.2. **Root systems B_n and C_n.** In this section we assume that the root system R is of type B_n or C_n. In fact these two cases can be treated together: the Weyl groups are the same, the characters ϵ_I are the same, and the families $\mathcal{F}$ and groups $\mathcal{G}_{\mathcal{F}}$ are essentially the same [Lus84], 4.5.

The Weyl group W_n is the semidirect product $S_n \ltimes \{\pm 1\}^n$. For any ordered triple (j, k, l) of non-negative integers satisfying

$$2j + k + l = n \tag{3.2.1}$$

we define an involution $\sigma = \sigma_{j,k,l}$ in W_n by

$$\sigma_{j,k,l} := \theta_j \ltimes (1, \dots, 1, -1, \dots, -1),$$

where 1 is repeated $2j+k$ times in $(1, \dots, 1, -1, \dots, -1)$ and -1 is repeated l times, and where $\theta_j \in S_n$ is the product of the j disjoint transpositions $(i, 2j+1-i)$ $(1 \le i \le j)$. These elements $\sigma_{j,k,l}$ form a set of representatives for the conjugacy classes of involutions in W_n.

The isomorphism classes of irreducible representations of W_n are parametrized by ordered pairs (α, β) of partitions such that $|\alpha| + |\beta| = n$; we write $E_{\alpha,\beta}$ for the representation corresponding to (α, β). The multiplicity $m(\sigma_{j,k,l}, E_{\alpha,\beta})$ is equal to 0 unless $j + k = r$ and $j + l = s$, in which case it is equal to the number of partitions γ of j such that $\gamma \xrightarrow{\text{c}} \alpha$ and $\gamma \xrightarrow{\text{r}} \beta$. Here we are using the notation $\gamma \xrightarrow{\text{c}} \alpha$ (respectively, $\gamma \xrightarrow{\text{r}} \beta$) to indicate that the Young diagram for α can be obtained from the Young diagram for β by adding boxes in such a way that at most one box is added to each column (respectively, row). This follows easily from the Littlewood-Richardson rule (and Mackey's formula for the restriction of an induced representation).

We need to recall some results from [Lus84], 4.5. By adding some parts equal to 0 to α, β, we may assume that α has one more part than β, say $\alpha = (\alpha l_1 \le \alpha_2 \le \dots \le \alpha_{m+1})$ and $\beta = (\beta_1 \le \beta_2 \le \dots \le \beta_m)$. Put $\lambda_i := \alpha_i + i - 1$ (for $1 \le i \le m+1$) and $\mu_i := \beta_i + i - 1$ (for $1 \le i \le m$), and note that both $\lambda_1, \lambda_2, \dots, \lambda_{m+1}$ and

$\mu_1, \mu_2, \dots, \mu_m$ are strictly increasing sequences of non-negative numbers. Then the irreducible representation $E_{\alpha,\beta}$ of W_n is special if and only if

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \mu_m \le \lambda_{m+1}, \tag{3.2.2}$$

and in case it is special, then the corresponding group $\mathcal{G}_{\mathcal{F}}$ is an elementary abelian 2-group of order 2^d, where d is the number of integers in the sequence $\mu_1, \mu_2, \dots, \mu_m$ that do not appear in the sequence $\lambda_1, \lambda_2, \dots, \lambda_{m+1}$.

An easy combinatorial argument shows that the multiplicities $m(\sigma, E_{\alpha,\beta})$ and $m(E_{\alpha,\beta})$ are 0 unless $E_{\alpha,\beta}$ is special, and that in case $E_{\alpha,\beta}$ is special one has

$$m(E_{\alpha,\beta}) = 2^d = |\mathcal{G}_{\mathcal{F}}|. \tag{3.2.3}$$

This proves Theorem 1 in this case. Furthermore, the combinatorial argument also shows that for any non-negative integers j, k, l satisfying $j+k=r$ and $j+l=s$,

$$m(\sigma_{j,k,l}, E_{\alpha,\beta}) = \binom{d}{j_0 - j} \quad \text{(binomial coefficient)}, \tag{3.2.4}$$

where $j_0 := \sum_{i=1}^{m} \inf\{\alpha_{i+1}, \beta_i\}$. In particular $m(\sigma_{j,k,l}, E_{\alpha,\beta}) = 0$ unless $j_0 - d \le j \le j_0$.

3.3. **Root system D_n.** In this section we assume that the root system R is of type D_n; thus $W(D_n)$ is the semidirect product $S_n \ltimes \{\pm 1\}^n_{\text{even}}$, where $\{\pm 1\}^n_{\text{even}}$ is the subgroup of $\{\pm 1\}^n$ consisting of elements $(a_1, \dots, a_n)$ such that $a_1 \dots a_n = 1$. We identify $W(D_n)$ with a subgroup of index 2 in W_n in the usual way.

Recall the involutions $\sigma_{j,k,l} \in W_n$. Clearly $\sigma_{j,k,l}$ lies in the subgroup $W(D_n)$ if and only if l is even. In this case the intersection of $W(D_n)$ with the conjugacy class of $\sigma_{j,k,l}$ in W_n is a single conjugacy class in $W(D_n)$, except when $k = l = 0$, in which case n must be even and the intersection consists of two conjugacy classes in $W(D_n)$; for each even n we now fix an element $\sigma'_{n/2,0,0}$ that is conjugate to $\sigma_{n/2,0,0}$ in W_n but not in $W(D_n)$. Then the elements $\sigma_{j,k,l}$ (for ordered triples of non-negative integers (j,k,l) satisfying (3.2.1) and such that l is even), together with the element $\sigma'_{n/2,0,0}$ if n is even, form a set of representatives for the conjugacy classes of involutions in $W(D_n)$.

Let (α, β) be an ordered pair of partitions such that $|\alpha| + |\beta| = n$. Then the irreducible representation $E_{\alpha,\beta}$ of W_n restricts to an irreducible representation of $W(D_n)$ unless $\alpha = \beta$ (this can happen only if n is even), in which case it decomposes as the direct sum of two irreducible representations $E'_{\alpha,\alpha}$ and $E''_{\alpha,\alpha}$. Obviously $E'_{\alpha,\alpha}$ and $E''_{\alpha,\alpha}$ are conjugate by the non-trivial element of the group $W_n/W(D_n)$; soon we will see how to distinguish them. Every irreducible $W(D_n)$-module is isomorphic to one of the modules above. Note that $E_{\alpha,\beta}$ and $E_{\beta,\alpha}$ are isomorphic as $W(D_n)$-modules, but this is the only redundancy in this description of the irreducible $W(D_n)$-modules.

Let α, β be as above. By adding some parts equal to 0 to α, β, we may assume that α and β have the same number of parts, say $\alpha = (\alpha_1 \le \alpha_2 \le \cdots \le \alpha_m)$ and $\beta = (\beta_1 \le \beta_2 \le \cdots \le \beta_m)$. Put $\lambda_i := \alpha_i + i - 1$ and $\mu_i := \beta_i + i - 1$ for $1 \le i \le m$.

We need to recall some results from [Lus84], 4.6. Suppose first that $\alpha = \beta$. Then the representations $E'_{\alpha,\alpha}$, $E''_{\alpha,\alpha}$ are both special and the associated groups $\mathcal{G}$ are both trivial. Now assume that $\alpha \ne \beta$, and by switching α, β if necessary, assume that $|\beta| \le |\alpha|$. Then $E_{\alpha,\beta}$ is special if and only if

$$\mu_1 \le \lambda_1 \le \mu_2 \le \lambda_2 \le \cdots \le \mu_m \le \lambda_m. \tag{3.3.1}$$

If $E_{\alpha,\beta}$ is special, then the associated group $\mathcal{G}$ is an elementary abelian 2-group of order 2^{c-1}, where c is the number of integers in the sequence $\mu_1, \mu_2, \dots, \mu_m$ that do not appear in the sequence $\lambda_1, \lambda_2, \dots, \lambda_m$. Note that $c \geq 1$ (since $\alpha \neq \beta$).

We start by calculating the multiplicities $m(\sigma, E_{\alpha,\beta})$, even though in case $\alpha = \beta$ this number is the sum of the two multiplicities $m(\sigma, E'_{\alpha,\alpha})$ and $m(\sigma, E''_{\alpha,\alpha})$ that we really want. In case n is even, so that $\sigma'_{n/2,0,0}$ is defined, we note that

$$m(\sigma'_{n/2,0,0}, E_{\alpha,\beta}) = m(\sigma_{n/2,0,0}, E_{\alpha,\beta}),$$

so that we may as well assume that $\sigma = \sigma_{j,k,l}$ with l even.

The calculation of $m(\sigma, E_{\alpha,\beta})$ is very similar to the one made for types B and C, and we just state the result. As before we assume that $|\beta| \leq |\alpha|$. Let $Q(\alpha, \beta)$ denote the set of partitions γ such that $\beta \xrightarrow{\mathrm{r}} \gamma$ and $\gamma \xrightarrow{\mathrm{c}} \alpha$. For any non-negative integer d let $Q_d(\alpha, \beta)$ be the set of partitions $\gamma \in Q(\alpha, \beta)$ such that $|\gamma| = d$. Then $m(\sigma_{j,k,l}, E_{\alpha,\beta})$ is 0 unless $|\alpha| = j + k + l$ and $|\beta| = j$, in which case it is equal to the cardinality of the set $Q_{j+l}(\alpha, \beta)$.

Assume for the moment that $\alpha = \beta$. It follows from the discussion above that $m(\sigma_{j,k,l}, E_{\alpha,\alpha}) = 0$ unless $k = l = 0$, and that $m(\sigma_{n/2,0,0}, E_{\alpha,\alpha}) = 1$. Thus we may choose the labeling of $E'_{\alpha,\alpha}$ and $E''_{\alpha,\alpha}$ so that both $m(\sigma_{n/2,0,0}, E'_{\alpha,\alpha})$ and $m(\sigma'_{n/2,0,0}, E''_{\alpha,\alpha})$ are 1, and both $m(\sigma_{n/2,0,0}, E''_{\alpha,\alpha})$ and $m(\sigma'_{n/2,0,0}, E'_{\alpha,\alpha})$ are 0. Therefore Theorem 1 holds for $E'_{\alpha,\alpha}$ and $E''_{\alpha,\alpha}$.

Now we assume that $\alpha \neq \beta$ (and that $|\beta| \leq |\alpha|$). As above we assume that α, β both have m parts, and we use α, β to define integers λ_i, μ_i. In the discussion below it will be convenient to make the convention that $\alpha_0 = \beta_0 = 0$, so that α_{i-1}, β_{i-1} are defined even for $i = 1$.

One sees easily that $Q(\alpha, \beta)$ is the set of partitions $\gamma = (\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_m)$ such that

$$\sup\{\alpha_{i-1}, \beta_i\} \leq \gamma_i \leq \inf\{\alpha_i, \beta_i + 1\}.$$

In particular $Q(\alpha, \beta)$ is non-empty if and only if

$$\alpha_{i-1} - 1 \leq \beta_i \leq \alpha_i \qquad \text{for } i = 1, \dots, m,$$

or, equivalently, if and only if (3.3.1) holds.

Now assume that $Q(\alpha, \beta)$ is non-empty. Then for each value of the index i, the difference of $\inf\{\alpha_i, \beta_i + 1\}$ and $\sup\{\alpha_{i-1}, \beta_i\}$ is 0 or 1, and we define C to be the set of indices i for which the difference is 1; it is easy to see that $i \in C$ if and only if $\alpha_{i-1} - 1 < \beta_i < \alpha_i$, or, equivalently, if and only if μ_i does not belong to the sequence $\lambda_1, \lambda_2, \dots, \lambda_m$. Therefore the cardinality of C is the number c defined above. Now put

$$d_0 = \sum_{i=1}^{m} \sup\{\alpha_{i-1}, \beta_i\}.$$

Then $|Q_d(\alpha, \beta)|$ is 0 unless $d_0 \leq d \leq d_0 + c$, in which case it is the binomial coefficient $\binom{c}{d-d_0}$. Moreover we have $m(E) = 2^{c-1}$, which proves Theorem 1.

4. Statement of Theorem 2 (the quasi-split case)

4.1. **The group G.** Let G be a connected reductive group over $\mathbf{R}$ that is quasi-split but not split. We assume further that G is adjoint and $\mathbf{R}$-simple. Thus G is either of the form $R_{\mathbf{C}/\mathbf{R}}H$ for a simple group H over $\mathbf{C}$, or G is an outer form of A_n, D_n or E_6.

We fix a Borel subgroup B of G over $\mathbf{R}$ and a maximal $\mathbf{R}$-torus T contained in B. Let R denote the root system of $T_{\mathbf{C}}$ in $G_{\mathbf{C}}$, and let W denote the Weyl group of $T_{\mathbf{C}}$ in $G_{\mathbf{C}}$. Then complex conjugation, which we denote by θ, acts on R and W, preserving the subset R^+ of roots that are positive relative to B. We denote by $\tilde{W}$ the semidirect product $W \rtimes \mathrm{Gal}(\mathbf{C}/\mathbf{R})$. Of course $\tilde{W}$ acts in an obvious way on R.

4.2. **Involutions.** Let $W' = W \cdot \theta$ be the non-trivial coset of W in $\tilde{W}$. By an involution in W' we mean an element $\sigma \in W'$ such that $\sigma^2 = 1$. Given an involution $\sigma \in W'$, we denote by W^σ the centralizer of σ in W, and we denote by R_I the set of imaginary roots in R relative to σ (by definition a root α is said to be imaginary if $\sigma(\alpha) = -\alpha$). Of course W^σ acts on R_I by automorphisms of that root system, and therefore there is a sign character ϵ_I on W^σ, defined as follows: for $w \in W^\sigma$ we put $\epsilon_I(w) = (-1)^k$, where k is the number of positive imaginary roots α such that $w\alpha$ is negative.

4.3. **Definition of $m(\sigma, E)$.** The group $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{1, \theta\}$ acts on the set $W^\vee$ of isomorphism classes of irreducible representations of W. We denote by $W^\vee_{\mathrm{ex}}$ the subset of $W^\vee$ consisting of all isomorphism classes of irreducible representations that are fixed by θ. Any representation $E \in W^\vee_{\mathrm{ex}}$ can be extended (in two ways) to an irreducible representation $\tilde{E}$ of $\tilde{W}$. For any involution $\sigma \in W'$ and any $E \in W^\vee_{\mathrm{ex}}$ we define $m(\sigma, E)$ to be the multiplicity of the character ϵ_I in the restriction of E to the subgroup W^σ; clearly $m(\sigma, E)$ depends only on the W-conjugacy class of σ in W'.

4.4. **Definition of $m_\theta(E)$.** For any $E \in W^\vee_{\mathrm{ex}}$ we define $m_\theta(E)$ by

$$m_\theta(E) := \sum_\sigma m(\sigma, E),$$

where the sum ranges over a set of representatives σ for the W-conjugacy classes of involutions in W'.

4.5. **Families in $W^\vee_{\mathrm{ex}}$.** We need to review some results in [Lus84]. If $\mathcal{F} \subset W^\vee$ is a family, then $\theta(\mathcal{F})$ is also a family. Now assume that $\theta(\mathcal{F}) = \mathcal{F}$. Then $\mathcal{F}$ is fixed pointwise by θ, so that $\mathcal{F} \subset W^\vee_{\mathrm{ex}}$ (see [Lus84], 4.17). Let $\tilde{\mathcal{F}}$ be the set of isomorphism classes of irreducible representations of $\tilde{W}$ whose restrictions to W are irreducible and belong to $\mathcal{F}$. The restriction map $\tilde{\mathcal{F}} \to \mathcal{F}$ is surjective, and its fibers have two elements, obtained from one another by tensoring with the non-trivial 1-dimensional character χ of the quotient $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ of $\tilde{W}$.

Associated to $\mathcal{F}$, θ is a finite group $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_{\mathcal{F}}$ containing the group $\mathcal{G}_{\mathcal{F}}$ of 2.5 as a subgroup of index 2 (see 4.18–4.20 in [Lus84]). We identify the quotient group $\tilde{\mathcal{G}}/\mathcal{G}$ with $\{1, \theta\}$ and denote by $\mathcal{G}'$ the non-trivial coset of $\mathcal{G}$ in $\tilde{\mathcal{G}}$.

4.6. **The sets $\mathcal{M}$, $\mathcal{M}'$, $\overline{\mathcal{M}}$.** Let $\tilde{\mathcal{G}}$ be a finite group, and let $\mathcal{G}$ be a subgroup of index 2 in $\tilde{\mathcal{G}}$. Denote by $\mathcal{G}'$ the non-trivial coset of $\mathcal{G}$ in $\tilde{\mathcal{G}}$. Associated to $(\mathcal{G}, \tilde{\mathcal{G}})$ are three finite sets $\mathcal{M}$, $\mathcal{M}'$, $\overline{\mathcal{M}}$, defined as follows (see [Lus84], 4.16). We begin with $\mathcal{M}$ and $\mathcal{M}'$, both of which are by definition subsets of $\mathcal{M}(\tilde{\mathcal{G}})$, the finite set associated to $\tilde{\mathcal{G}}$ (see 2.6). The set $\mathcal{M}$ consists of elements $(x, \rho) \in \mathcal{M}(\tilde{\mathcal{G}})$ such that $x \in \mathcal{G}$, the centralizer $\tilde{\mathcal{G}}_x$ meets $\mathcal{G}'$, and the restriction of ρ to $\mathcal{G}_x$ remains irreducible. The set $\mathcal{M}'$ consists of pairs $(y, \tau) \in \mathcal{M}(\tilde{\mathcal{G}})$ such that $y \in \mathcal{G}'$. Finally, $\overline{\mathcal{M}}$ is defined to be the set of equivalence classes of pairs $(y, \bar{\tau})$ consisting of an element $y \in \mathcal{G}'$

and an element $\bar\tau \in \mathcal{G}_y^\vee$, two such pairs being equivalent if they lie in the same orbit of the conjugation action of $\tilde{\mathcal{G}}$ (or, equivalently, the conjugation action of $\mathcal{G}$).

There is a surjective map $\mathcal{M}' \to \overline{\mathcal{M}}$ defined by $(y,\tau) \mapsto (y,\bar\tau)$, where $\bar\tau$ denotes the restriction of τ to $\mathcal{G}_y$; note that $\bar\tau$ is automatically irreducible, since the non-trivial coset of $\mathcal{G}_y$ in $\tilde{\mathcal{G}}_y$ contains an element of the center of $\tilde{\mathcal{G}}_y$, namely y. Every fiber of this map $\mathcal{M}' \to \overline{\mathcal{M}}$ has two elements.

There is a map $\mathcal{M} \to \mathcal{M}(\mathcal{G})$ defined by $(x,\rho) \mapsto (x,\bar\rho)$, where $\bar\rho$ denotes the restriction of ρ to $\mathcal{G}_x$. Any non-empty fiber of this map has two elements, obtained from one another by tensoring with the non-trivial 1-dimensional character χ of $\tilde{\mathcal{G}}/\mathcal{G}$, which we also view as the non-trivial character on $\tilde{\mathcal{G}}_x/\mathcal{G}_x$ for any $x \in \tilde{\mathcal{G}}$ such that $\tilde{\mathcal{G}}_x$ meets $\mathcal{G}'$.

4.7. **Pairing $\{\cdot,\cdot\}$ between $\mathcal{M}$ and $\mathcal{M}'$.** We restrict the pairing $\{\cdot,\cdot\}$ on $\mathcal{M}(\tilde{\mathcal{G}})$ to the subset $\mathcal{M}\times\mathcal{M}'$ of $\mathcal{M}(\tilde{\mathcal{G}})\times\mathcal{M}(\tilde{\mathcal{G}})$, obtaining a pairing $\mathcal{M}\times\mathcal{M}' \to \mathbf{C}$, as in [Lus84], 4.16.

4.8. **Fourier transform.** Recall from 2.8 the Fourier transform on $\mathcal{M}(\tilde{\mathcal{G}})$. This Fourier transform interchanges (see [Lus84], 4.16) the following two subspaces of the space of complex-valued functions on $\mathcal{M}(\tilde{\mathcal{G}})$: the space $\mathcal{P}$ of functions ϕ supported on $\mathcal{M}$ and satisfying $\phi(x,\rho\otimes\chi) = -\phi(x,\rho)$ for all $(x,\rho)\in\mathcal{M}$, and the space $\mathcal{P}'$ of functions ϕ supported on $\mathcal{M}'$ and satisfying $\phi(y,\tau\otimes\chi) = \phi(y,\tau)$ for all $(y,\tau)\in\mathcal{M}'$. Obviously $\mathcal{P}'$ can be identified with the space of complex-valued functions on $\overline{\mathcal{M}}$, since the fiber of $\mathcal{M}' \to \overline{\mathcal{M}}$ through (y,τ) consists of (y,τ) and $(y,\tau\otimes\chi)$.

4.9. **Special function ϕ_0 on $\overline{\mathcal{M}}$.** We will need the following function ϕ_0 on $\overline{\mathcal{M}}$. Let $(y,\bar\tau)\in\overline{\mathcal{M}}$. Let $\omega_{\bar\tau}$ denote the central character of $\bar\tau$. We put

$$\phi_0(y,\bar\tau) = \varepsilon(\bar\tau)\cdot\omega_{\bar\tau}(y^2),$$

with $\varepsilon(\bar\tau)$ as in 2.9. (Note that y^2 lies in the center of $\mathcal{G}_y$.)

We will need the following analog of (2.9.1). Suppose that $\mathcal{G}'$ contains an element y belonging to the center of $\tilde{\mathcal{G}}$. Then for all $x\in\mathcal{G}$ there is an equality

$$\bigl|\{g\in\mathcal{G}' \bigm| g^2 = x\}\bigr| = \sum_{\rho\in\mathcal{G}^\vee} \varepsilon(\rho)\cdot\omega_\rho(y^2)\cdot\chi_\rho(x). \tag{4.9.1}$$

This follows from (2.9.1), applied to both $\tilde{\mathcal{G}}$ and $\mathcal{G}$.

4.10. **Fourier transform of ϕ_0.** We are interested in the Fourier transform of the function ϕ_0 defined in 4.9. We regard $\hat\phi_0$ as a function on $\mathcal{M}$ satisfying $\hat\phi_0(x,\rho\otimes\chi) = -\hat\phi_0(x,\rho)$ for all $(x,\rho)\in\mathcal{M}$. In fact, just as in 2.10 one can use (4.9.1) to show that for $(x,\rho)\in\mathcal{M}$

$$\hat\phi_0(x,\rho) = \sum_{s\in S}[d(s,\rho) - d(s,\rho\otimes\chi)], \tag{4.10.1}$$

where S is a set of representatives for the $\tilde{\mathcal{G}}_x$-conjugacy classes of elements $s\in\mathcal{G}'$ such that $s^2 = x$, and $d(s,\rho)$ denotes the dimension of the space of vectors in ρ fixed by $\tilde{\mathcal{G}}_x\cap\tilde{\mathcal{G}}_s$. In particular, the value of $\hat\phi_0$ at the base point $(1,1)$ in $\mathcal{M}$ is equal to the number of $\tilde{\mathcal{G}}$-conjugacy classes of involutions in $\mathcal{G}'$ (which is the same as the number of $\mathcal{G}$-conjugacy classes of involutions in $\mathcal{G}'$).

We need to know more about $\hat\phi_0$ in three special cases. First suppose that $\tilde{\mathcal{G}}$ is the direct product of $\mathcal{G}$ and $\{\pm1\}$. Then the canonical map $\mathcal{M}\to\mathcal{M}(\mathcal{G})$ has a section

$t : \mathcal{M}(\mathcal{G}) \to \mathcal{M}$, defined by $t(x, \bar{\rho}) = (x, \rho)$, where ρ is the irreducible representation of $\tilde{\mathcal{G}}_x$ obtained from $\bar{\rho}$ by means of the canonical surjection $\tilde{\mathcal{G}}_x = \mathcal{G}_x \times \{\pm 1\} \to \mathcal{G}_x$. Moreover $\mathcal{M}$ is the disjoint union of the image of t and the image of t', where t' is the composition of t and the map $\mathcal{M} \to \mathcal{M}$ defined by $(x, \rho) \mapsto (x, \rho \otimes \chi)$. Pulling back the function $\hat{\phi}_0$ to $\mathcal{M}(\mathcal{G})$ by means of t, we get a function on $\mathcal{M}(\mathcal{G})$ which is easily seen (using the description of the pairing on $\mathcal{M} \times \mathcal{M}'$ given in [Lus84], 4.19) to coincide with the one in 2.10 (defined using the Fourier transform on $\mathcal{M}(\mathcal{G})$).

Next suppose that $\tilde{\mathcal{G}}$ is an elementary abelian 2-group. Of course $\mathcal{G}$ is then also an elementary abelian 2-group. Moreover, $\tilde{\mathcal{G}}$ is non-canonically the direct product of $\mathcal{G}$ and $\{\pm 1\}$. Therefore it follows from the discussion above (together with 2.10) that $\hat{\phi}_0(1,1) = |\mathcal{G}|$, $\hat{\phi}_0(1,\chi) = -|\mathcal{G}|$, and the remaining values of $\hat{\phi}_0$ are all 0.

Finally, suppose that $\mathcal{G} = \mathcal{H} \times \mathcal{H}$ for some finite group $\mathcal{H}$, and that $\tilde{\mathcal{G}}$ is the semidirect product $\mathcal{G} \rtimes \{1, \theta\}$, where θ acts on $\mathcal{H} \times \mathcal{H}$ by $\theta(a, b) = (b, a)$. There is a canonical embedding $u : \mathcal{M}(\mathcal{H}) \to \mathcal{M}$, defined by $(x, \rho) \mapsto ((x, x), \rho \otimes \rho)$, where the element θ of $\tilde{\mathcal{G}}_{(x,x)}$ acts on $\rho \otimes \rho$ by $\theta(v \otimes w) = w \otimes v$. Moreover $\mathcal{M}$ is the disjoint union of the image of u and the image of u', where u' is the composition of u and the map $\mathcal{M} \to \mathcal{M}$ defined by $(x, \rho) \mapsto (x, \rho \otimes \chi)$. Therefore $\hat{\phi}_0$ is uniquely determined by the function ψ on $\mathcal{M}(\mathcal{H})$ obtained by composing $\hat{\phi}_0$ with the embedding u. It is easy to check (using the description of the pairing on $\mathcal{M} \times \mathcal{M}'$ given in [Lus84], 4.20) that ψ is equal to the Fourier transform (on $\mathcal{M}(\mathcal{H})$) of the function ϕ_1 on $\mathcal{M}(\mathcal{H})$ defined by $\phi_1(x, \rho) = \varepsilon(\rho) \cdot \omega_\rho(x)$, where ω_ρ denotes the central character of ρ.

We claim that ϕ_1 is its own Fourier transform and hence that $\psi = \phi_1$. Indeed, using (2.8.1) we see that for $(x, \rho) \in \mathcal{M}(\mathcal{H})$

$$\hat{\phi}_1(x, \rho) = |\mathcal{H}_x|^{-1} \sum_{y \in \mathcal{H}_x} \chi_\rho(y) \sum_{\tau \in \mathcal{H}_y^\vee} \chi_\tau(x^{-1}) \cdot \varepsilon(\tau) \cdot \omega_\tau(y). \tag{4.10.2}$$

It follows from (2.9.1) that the second sum in (4.10.2) is equal to

$$\big|\{s \in \mathcal{H}_y \;\big|\; s^2 = x^{-1} y\}\big|.$$

Therefore the right side of (4.10.2) can be rewritten as

$$\omega_\rho(x) \cdot |\mathcal{H}_x|^{-1} \sum_{s \in \mathcal{H}_x} \chi_\rho(s^2),$$

which is in turn equal to $\omega_\rho(x) \cdot \varepsilon(\rho)$ (in other words, equal to $\phi_1(x, \rho)$), since $\chi_\rho(s^2)$ is equal to $\chi_{\mathrm{Sym}^2(\rho)}(s) - \chi_{\wedge^2(\rho)}(s)$.

4.11. **Injection $\tilde{\mathcal{F}} \hookrightarrow \mathcal{M}$.** Let $\mathcal{F}$ be a θ-stable family in $W^\vee$, and define $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}} \to \mathcal{F}$ as in 4.5. Let $\mathcal{G} = \mathcal{G}_{\mathcal{F}}$ be the finite group associated to $\mathcal{F}$ as in 2.5, and let $\tilde{\mathcal{G}}$ be the finite group associated to $\mathcal{F}$, θ in 4.5. Recall that $\mathcal{G}$ is a subgroup of index 2 in $\tilde{\mathcal{G}}$, so that we have finite sets $\mathcal{M}$, $\mathcal{M}'$, $\overline{\mathcal{M}}$ as in 4.6. Lusztig [Lus84], Ch. 4 defines (case-by-case) an injection

$$i : \tilde{\mathcal{F}} \to \mathcal{M} \tag{4.11.1}$$

such that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{F}} & \xrightarrow{i} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{M}(\mathcal{G}) \end{array}$$

commutes, where $\mathcal{F} \to \mathcal{M}(\mathcal{G})$ is the injection in 2.11 and $\mathcal{M} \to \mathcal{M}(\mathcal{G})$ is the map defined in 4.6. Moreover, if the image of $\tilde{E} \in \tilde{\mathcal{F}}$ under the injection i is (x, ρ), then the image of $\tilde{E} \otimes \chi$ under i is $(x, \rho \otimes \chi)$.

4.12. **Statements of Theorem 2 and Conjecture 2.** Let $E \in W^\vee_{\mathrm{ex}}$ and let $\mathcal{F} \subset W^\vee$ be the unique family containing E. Then $\mathcal{F}$ is θ-stable, and we let $\tilde{\mathcal{F}}$, $\mathcal{G}$, $\tilde{\mathcal{G}}$, $\mathcal{M}$ and $\tilde{\mathcal{F}} \to \mathcal{M}$ be as in 4.11. Let $\tilde{E}$ be one of the two extensions of E to a representation of $\tilde{W}$ (the other extension being $\tilde{E} \otimes \chi$), and let $x_{\tilde{E}} \in \mathcal{M}$ be the image of $\tilde{E}$ under the injection $\tilde{\mathcal{F}} \to \mathcal{M}$. Then $\hat{\phi}_0(x_{\tilde{E}})$ is an integer, and $\hat{\phi}_0(x_{\tilde{E}\otimes\chi}) = -\hat{\phi}_0(x_{\tilde{E}})$. Therefore the absolute value $|\hat{\phi}_0(x_{\tilde{E}})|$ is independent of the choice of extension $\tilde{E}$.

Theorem 2. *There is an equality*

$$m_\theta(E) = |\hat{\phi}_0(x_{\tilde{E}})|, \tag{4.12.1}$$

where $m_\theta(E)$ is the multiplicity defined in 4.4.

Conjecture 2. *Let σ be an involution in W' and let Σ be its conjugacy class in $\tilde{W}$. Let Γ be a left cell in W, let $[\Gamma]$ be the corresponding left cell representation of W, and let $\Gamma \cdot \theta$ be the subset of W' consisting of all products $w\theta$ where $w \in \Gamma$. Then the multiplicity $m(\sigma, [\Gamma])$ is equal to the cardinality of the set $\Sigma \cap (\Gamma \cdot \theta)$.*

This generalizes Conjecture 1 of the introduction to the quasi-split case.

4.13. **Organization.** In §5 we will calculate the multiplicities $m(\sigma, E)$ and $m_\theta(E)$. In §6 we will prove Theorem 2.

5. Multiplicities $m(\sigma, E)$ and $m_\theta(E)$ (quasi-split case)

5.1. **The goal.** We retain the notation and assumptions of §4. For each involution $\sigma \in W'$ and each $E \in W^\vee_{\mathrm{ex}}$ we are going to calculate the multiplicities $m(\sigma, E)$ and $m_\theta(E)$ defined in 4.3 and 4.4. This will be done case-by-case.

5.2. **Complex groups.** Assume that $G = R_{\mathbf{C}/\mathbf{R}}(H)$ for some connected reductive group H over $\mathbf{C}$, and let W_H denote the Weyl group of H. Then $W = W_H \times W_H$ and $\tilde{W}$ is the semidirect product $(W_H \times W_H) \rtimes \{1, \theta\}$, where θ acts on $W_H \times W_H$ by $\theta(a, b) = (b, a)$. The map $E_H \mapsto E_H \otimes E_H$ from $W_H^\vee$ to $W^\vee_{\mathrm{ex}}$ is bijective. Every involution in W' is conjugate to the involution θ. The centralizer W^θ of the involution θ is W_H, embedded diagonally in W. There are no imaginary roots, and therefore the sign character ϵ_I is trivial. Thus, for any $E \in W^\vee_{\mathrm{ex}}$ we have

$$m_\theta(E) = m(\theta, E) = 1. \tag{5.2.1}$$

(We used that every $E_H \in W_H^\vee$ is self-contragredient.)

5.3. **Types A_n and E_6.** Assume that G is of type A_n ($n \geq 2$) or E_6. Then we have

$$\tilde{W} = \mathrm{Aut}(R) = W \times \{\pm 1\},$$

where $-1 \in \tilde{W}$ denotes the automorphism of R sending each root to its negative. Therefore $W^\vee_{\mathrm{ex}} = W^\vee$ and every involution $\sigma \in W'$ is of the form $\sigma = (\tau, -1)$, where τ is an involution in W. Clearly $(\tau, -1)$ is conjugate to $(\tau', -1)$ if and only if τ is conjugate to τ'. The centralizer W^σ is equal to the centralizer W^τ. A root $\alpha \in R$ is imaginary for σ if and only if it is real for τ. Therefore ϵ_I^σ is equal to $\epsilon_W \otimes \epsilon_I^\tau$, where ϵ_I^σ (respectively, ϵ_I^τ) denotes the sign character ϵ_I for σ (respectively, τ), and where ϵ_W denotes the usual sign character for W.

Let $E \in W^\vee$. We conclude from the discussion above that

$$m(\sigma, E) = m(\tau, E \otimes \epsilon_W). \tag{5.3.1}$$

The multiplicities on the right side of (5.3.1) were calculated in 3.1 for type A_n and by Casselman [Cas98] for type E_6. It follows from (5.3.1) that $m_\theta(E) = m(E \otimes \epsilon_W)$. But in fact $m(E \otimes \epsilon_W)$ is equal to $m(E)$; this is obvious for type A_n since $m(E)$ is then always 1, and it is true for type E_6 by Casselman's calculation. Therefore we conclude that

$$m_\theta(E) = m(E). \tag{5.3.2}$$

5.4. **Type D_n.** Assume that G is of type D_n ($n \geq 2$). Then we may identify $\tilde{W}$ with W_n (see 3.3), and $W^\vee_{\text{ex}}$ consists of the representations $E_{\alpha,\beta}$ of 3.3 with $\alpha \neq \beta$. Every involution in W' is conjugate under W to one of the involutions $\sigma_{j,k,l}$ of 3.2 with l odd. Thus we need to calculate the multiplicities $m(\sigma_{j,k,l}, E_{\alpha,\beta})$ for $\alpha \neq \beta$ and l odd. But in fact this calculation is exactly the same as the one made in 3.3, since the fact that l was assumed to be even played no role. Therefore the results of 3.3 remain true for l odd (in particular the non-zero multiplicities are binomial coefficients).

It follows from the discussion above (together with results from 3.3) that $m_\theta(E_{\alpha,\beta})$ is 0 unless $E_{\alpha,\beta}$ is special, in which case $m_\theta(E_{\alpha,\beta}) = 2^{c-1}$, which can be rewritten as (see 3.3)

$$m_\theta(E_{\alpha,\beta}) = |\mathcal{G}|, \tag{5.4.1}$$

where $\mathcal{G}$ is the finite group associated to the unique family $\mathcal{F} \subset W^\vee$ containing $E_{\alpha,\beta}$.

6. Proof of Theorem 2

6.1. **The goal.** We retain the notation and assumptions of §4. Let $E \in W^\vee_{\text{ex}}$ and let $\mathcal{F}$, $\mathcal{G}$, $\tilde{\mathcal{G}}$, $\mathcal{M}$, $\hat{\phi}_0$ be as in 4.12. In this section we will verify case-by-case that Theorem 2 (see (4.12.1)) is true.

6.2. **Complex groups.** As in 5.2 we assume that $G = R_{\mathbf{C}/\mathbf{R}}(H)$. Then $E = E_H \otimes E_H$ for some $E_H \in W^\vee_H$, and $\mathcal{F} = \mathcal{F}_H \times \mathcal{F}_H$, where $\mathcal{F}_H$ is the unique family in $W^\vee_H$ containing E_H. Moreover $\mathcal{G} = \mathcal{H} \times \mathcal{H}$, where $\mathcal{H} = \mathcal{G}_{\mathcal{F}_H}$, and $\tilde{\mathcal{G}} = (\mathcal{H} \times \mathcal{H}) \rtimes \{1, \theta\}$, where θ acts on $\mathcal{H} \times \mathcal{H}$ by interchanging the two factors.

It follows from 4.10 that the function $|\hat{\phi}_0|$ on $\mathcal{M}$ is identically equal to 1. It follows from (5.2.1) that the function m_θ on $W^\vee_{\text{ex}}$ is also identically equal to 1. Therefore Theorem 2 is true in this case.

6.3. **Types A_n and E_6.** Assume that G is of type A_n ($n \geq 2$) or E_6. In this case we have $\tilde{\mathcal{G}} = \mathcal{G} \times \{\pm 1\}$ (see [Lus84], 4.19). It follows from 4.10 that the function $|\hat{\phi}_0|$ on $\mathcal{M}$ is the composition of the map $\mathcal{M} \to \mathcal{M}(\mathcal{G})$ (defined in 4.6) with the function on $\mathcal{M}(\mathcal{G})$ defined in 2.10. Using Theorem 1 and the commutativity of the diagram in 4.11, we see that Theorem 2 is equivalent to the assertion that $m_\theta(E) = m(E)$, which we have already proved (see (5.3.2)).

6.4. **Type D_n.** Assume that G is of type D_n ($n \geq 2$). Then $\tilde{\mathcal{G}}$ is an elementary abelian 2-group (see [Lus84], 4.18). It follows from 4.10 that the function $|\hat{\phi}_0|$ on $\mathcal{M}$ is 0 except at the two points $(1,1)$ and $(1,\chi)$, where it is equal to $|\mathcal{G}|$. Therefore Theorem 2 is equivalent to the assertion that $m_\theta(E) = 0$ unless E is special, in which case $m_\theta(E) = |\mathcal{G}|$. We have already proved this assertion (see (5.4.1)).

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