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A CUBIC DIRAC OPERATOR AND THE EMERGENCE OF EULER NUMBER MULTIPLETS OF REPRESENTATIONS FOR EQUAL RANK SUBGROUPS

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CONTENTS

0. Introduction

0.1. The results in this paper arise from two distinct origins. The first, and more recent result, grew out of the joint paper [GKRS]. The second is explained in §0.28.

The paper [GKRS] offered, in very general terms, a mathematical explanation of an interesting phenomenon discovered, empirically, by the physicists Ramond and Pengpan. Presumably motivated by a possible connection with M-theory, they found that there was an infinite set $\mathcal{S}$ of irreducible representations of $\mathrm{Spin}(9,\mathbb{R})$ which partitioned into triplets

$$\mathcal{S} = \bigcup_{i\in I} \{\sigma_1^i, \sigma_2^i, \sigma_3^i\}, \tag{0.2}$$

where the representations in each triplet are related to each other in remarkable ways. For example, the infinitesimal character value of the Casimir operator is constant on the triplet, and there are a number of other infinitesimal character relations on the triplet involving more of the generators of $\mathfrak{Z}(\mathfrak{r})$. Here $\mathfrak{r} = \mathrm{Lie\,Spin}(9)$ and $\mathfrak{Z}(\mathfrak{r})$ is the center of the enveloping algebra $U(\mathfrak{r})$ of $\mathfrak{r}$. Also one has for each $i \in I$,

$$\dim \sigma_1^i + \dim \sigma_2^i = \dim \sigma_3^i. \tag{0.3}$$

The simplest triplet $\{\sigma_1^1, \sigma_2^1, \sigma_3^1\}$ arises from the irreducible 16-dimensional (spin)

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orthogonal representation

$$\nu : \mathfrak{r} \longrightarrow \operatorname{End} \mathbb{C}^{16} \tag{0.4}$$

as follows: The two half-spin representations of Spin 16 are on 128-dimensional vector spaces S_+ and S_-. Composition with ν defines on S_+ and S_- the structure of $\mathfrak{r}$-modules. Curiously, one of these is irreducible, defining σ_3^1, and the other decomposes into a direct sum of σ_1^1 and σ_2^1. The equality (0.3) in this first basic case is

$$44 + 84 = 128.$$

The idea leading to the results in [GKRS] was that the three involved had something to do with the principle of triality, which suggested that the triplets had to do with the embedding $\mathfrak{r} \subset \mathfrak{g}$ where $\mathfrak{g} = F_4$. In fact, if G is the compact group having $\mathfrak{g}$ as its complexified Lie algebra, then $R \subset G$ where $R = \operatorname{Spin}(9, \mathbb{R})$ and

$$3 = \operatorname{Eul}(X) \tag{0.5}$$

where $X = G/R$ is the (16-dimensional) Cayley plane and $\operatorname{Eul}(X)$ is the Euler characteristic of X.

0.6. The setting for the general result in [GKRS] is that $\mathfrak{g}$ is any complex semisimple Lie algebra and $\mathfrak{r} \subset \mathfrak{g}$ is a reductive Lie algebra of $\mathfrak{g}$ having the same rank of $\mathfrak{g}$, so that there exists $\mathfrak{h} \subset \mathfrak{r}$, which is a Cartan subalgebra of both $\mathfrak{r}$ and $\mathfrak{g}$. Let d be the index of $W_{\mathfrak{r}}$ in W, where $W_{\mathfrak{r}} \subset W$ are corresponding respective Weyl groups of $\mathfrak{r}$ and $\mathfrak{g}$. In particular,

$$d = \operatorname{Eul}(X), \tag{0.7}$$

where $X = G/R$, G is the simply connected compact group with $\mathfrak{g}$ as the complexified Lie algebra, and R is the subgroup of G associated to the corresponding compact form of $\mathfrak{r}$. Under these assumptions, X is the most general compact simply connected homogeneous space of positive Euler characteristic.

Let $\Delta_{\mathfrak{r}}^+ \subset \Delta^+$ be positive root systems for $\mathfrak{r}$ and $\mathfrak{g}$, respectively. Correspondingly, let $\rho_{\mathfrak{r}}$ and ρ be one-half the sum of the positive roots. Also let $D_{\mathfrak{r}} \supset D$ be the corresponding Weyl chambers. Let $W^1 = \{\tau \in W \mid \tau(D) \subset D_{\mathfrak{r}}\}$ so that the map

$$W_{\mathfrak{r}} \times W^1 \longrightarrow W, \qquad (w, \tau) \longmapsto w\tau$$

is a bijection and hence $\operatorname{card} W^1 = d$. The set W^1, of coset representatives, itself partitions into two parts

$$W^1 = W_+^1 \cup W_-^1, \tag{0.8}$$

where $W_\pm^1 = \{\tau \in W^1 \mid sg(\tau) = \pm 1\}$. Let $d_\pm = \operatorname{card} W_\pm^1$ so that

$$d = d_+ + d_-.$$

In the example of §0.1, one has setwise $\{d_+, d_-\} = \{2, 1\}$. Now let $\Gamma \subset \mathfrak{h}^*$ be the set of $\mathfrak{g}$-integral linear forms, and let $\Lambda = D \cap \Gamma$. For each $\lambda \in \Lambda$, let $\pi_\lambda : \mathfrak{g} \to \operatorname{End} V_\lambda$ be some fixed irreducible representation with highest weight λ. Let $\Lambda_{\mathfrak{r}} \subset D_{\mathfrak{r}}$ be the set of $\mathfrak{r}$-dominant, and necessarily $\mathfrak{r}$-integral, linear forms in the lattice $\Gamma_{\mathfrak{r}}$ generated by Γ and $\rho_{\mathfrak{r}}$. For each $\mu \in \Lambda_{\mathfrak{r}}$, let $\sigma_\mu : \mathfrak{r} \to \operatorname{End} Z_\mu$ be an irreducible representation with highest weight μ.

If $\mu \in \Lambda_{\mathfrak{r}}$, then of course $\mu + \rho_{\mathfrak{r}}$ is $W_{\mathfrak{r}}$-regular. We pick out an infinite subset by putting $\Lambda_{\mathfrak{r}}^* = \{\mu \in \Lambda_{\mathfrak{r}} \mid \mu + \rho_{\mathfrak{r}} \text{ is } W\text{-regular}\}$. For any $\lambda \in \Lambda$ and $\tau \in W^1$, let $\tau \bullet \lambda = \tau(\lambda + \rho) - \rho_{\mathfrak{r}}$. One readily has that $\tau \bullet \lambda \in \Lambda_{\mathfrak{r}}^*$ and the map

$$W^1 \times \Lambda \longrightarrow \Lambda_{\mathfrak{r}}^*, \qquad (\tau, \lambda) \longmapsto \tau \bullet \lambda$$

is a bijection. Generalizing the triplets in §0.1 (but now using highest weights as parameters), for each $\lambda \in \Lambda$, let $\Lambda_{\mathfrak{r}}^\lambda = \{\tau \bullet \lambda \mid \tau \in W^1\}$. One has $\operatorname{card} \Lambda_{\mathfrak{r}}^\lambda = d$, and we refer to the d-set $\Lambda_{\mathfrak{r}}^\lambda$ of highest weights, or the corresponding set $\{\sigma_\mu\}, \mu \in \Lambda_{\mathfrak{r}}^\lambda$, of irreducible $\mathfrak{r}$-representations, or the corresponding set of $\mathfrak{r}$-modules $\{Z_\mu\}, \mu \in \Lambda_{\mathfrak{r}}^\lambda$, as d-multiplets or just multiplets. In highest weight form, (0.2) generalizes to the partition

$$\Lambda_{\mathfrak{r}}^* = \bigcup_{\lambda \in \Lambda} \Lambda_{\mathfrak{r}}^\lambda. \tag{0.9}$$

The Harish-Chandra homomorphism induces an injection

$$\mathfrak{Z}(\mathfrak{g}) \longrightarrow \mathfrak{Z}(\mathfrak{r}) \tag{0.10}$$

of the center $\mathfrak{Z}(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ into the center $\mathfrak{Z}(\mathfrak{r})$ of the enveloping algebra $U(\mathfrak{r})$ of $\mathfrak{r}$. Let $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{r})$ be the image of (0.10). For any $\mu \in \Lambda_r$, let $\chi_{\mathfrak{r}}^\mu : \mathfrak{Z}(\mathfrak{r}) \to \mathbb{C}$ be the infinitesimal character of σ_μ. Define an equivalence relation in $\Lambda_{\mathfrak{r}}^*$ by defining

$$\mu \sim \mu' \quad \text{if } \chi_{\mathfrak{r}}^\mu \mid \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{r}) = \chi_{\mathfrak{r}}^{\mu'} \mid \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{r}). \tag{0.11}$$

The following generalizes the infinitesimal character relations in $\mathfrak{Z}(\mathfrak{r})$ discovered by Ramond and Pengpan for the triplets in (0.2).

PROPOSITION 0.12. *The equivalence classes for the equivalence relation (0.11) are exactly the d-multiplets $\Lambda_{\mathfrak{r}}^\lambda$ for all $\lambda \in \Lambda$.*

This fact, stated in [GKRS], is proved as Proposition 3.43 in the present paper. Let $\mathfrak{p}$ be the $B_{\mathfrak{g}}$-orthocomplement of $\mathfrak{r}$ in $\mathfrak{g}$, where $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. Then if $\mathrm{SO}(\mathfrak{p})$ is defined with respect to $B_{\mathfrak{p}} = B_{\mathfrak{g}} \mid \mathfrak{p}$, the representation (0.4) generalizes to the representation

$$\nu : \mathfrak{r} \longrightarrow \operatorname{Lie} \mathrm{SO}(\mathfrak{p}), \tag{0.13}$$

defined so that $\nu(x)(y) = [x, y]$ for $x \in \mathfrak{r}$ and $y \in \mathfrak{p}$. Let $C(\mathfrak{p})$ be the ($\mathbb{Z}_2$-graded) Clifford algebra over $\mathfrak{p}$ defined by $B_{\mathfrak{p}}$. Let $n = \dim \mathfrak{p}$. Then $n = 2m$ is even, so that

$C(\mathfrak{p})$ is simple and hence admits a unique irreducible (spin) module S. The module S decomposes into a direct sum $S = S_+ \oplus S_-$ of irreducible $C^{\text{even}}(\mathfrak{p})$ submodules, both of dimension 2^m. The representation ν lifts to a homomorphism

$$\nu_* : \mathfrak{r} \longrightarrow C^{\text{even}}(\mathfrak{p}), \tag{0.14}$$

so that S, S_+, and S_- have the structure of $\mathfrak{r}$-modules. If $\lambda \in \Lambda$, then one has a tensor product representation

$$\zeta_\lambda : \mathfrak{r} \longrightarrow \operatorname{End}(V_\lambda \otimes S),$$

where $\mathfrak{r}$ operates on V_λ by the restriction $\pi_\lambda|\mathfrak{r}$. Clearly, $V_\lambda \otimes S_+$ and $V_\lambda \otimes S_-$ are subrepresentations of ζ_λ. The partition (0.8) clearly induces a partition of the multiplet $\lambda_{\mathfrak{r}}^\lambda = \Lambda_{\mathfrak{r}}^{\lambda,+} \cup \Lambda_{\mathfrak{r}}^{\lambda,-}$, where the two parts have respective cardinalities d_+ and d_- (of course, independent of λ). The main significance of the multiplets in the representation theories of $\mathfrak{g}$ and $\mathfrak{r}$ is the following result.

THEOREM 0.15. *Let $\lambda \in \Lambda$. Then, in the ring of virtual representations of $\mathfrak{r}$, one has*

$$V_\lambda \otimes S_+ - V_\lambda \otimes S_- = \sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} Z_{\mu'} - \sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} Z_{\mu''}. \tag{0.16}$$

Obviously, the left side of (0.16) has zero dimension in the virtual representation ring. The following consequence generalizes the dimensional equality (0.3).

THEOREM 0.17. *Let $\lambda \in \Lambda$. Then*

$$\sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} \dim Z_{\mu'} = \sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} \dim Z_{\mu''}.$$

Theorems 0.2 and 0.3 of [GKRS] are reproved as Theorems 3.51 and 3.56 in the present paper; §3 is devoted to recovering the results of [GKRS].

0.18. Let $\lambda \in \Lambda$. As an immediate consequence of (0.16), one can make several statements as to how $V_\lambda \otimes S$ decomposes under the tensor product representation ζ_λ of $\mathfrak{r}$. One striking consequence of (0.16) is that $V_\lambda \otimes S_+$ and $V_\lambda \otimes S_-$ always differ, independent of λ, by the same number (namely, d) of irreducible representations of $\mathfrak{r}$. More precisely, let $\mu \in \Lambda_{\mathfrak{r}}$. Then if $\mu \notin \Lambda_{\mathfrak{r}}^\lambda$, the multiplicity of σ_μ in $V_\lambda \otimes S$ is even, with half occurring in $V_\lambda \otimes S_+$ and half occurring in $V_\lambda \otimes S_-$. On the other hand, if $\mu \in \Lambda_{\mathfrak{r}}^\lambda$, the multiplicity of σ_μ is odd; it has one more occurrence in $V_\lambda \otimes S_+$ than in $V_\lambda \otimes S_-$ in the case $\mu \in \Lambda_{\mathfrak{r}}^{\lambda,+}$ and vice versa in the case $\mu \in \Lambda_{\mathfrak{r}}^{\lambda,-}$. The question of the actual multiplicity of σ_μ for μ in the multiplet $\Lambda_{\mathfrak{r}}^\lambda$ is settled in the next theorem.

THEOREM 0.19. *Let $\lambda \in \Lambda$ and $\mu \in \Lambda^\lambda$. Then the multiplicity of σ_μ in $V_\lambda \otimes S$ is* 1*, so we can unambiguously regard $Z_\mu \subset V_\lambda \otimes S$. Furthermore, $Z_\mu \subset V_\lambda \otimes S_+$ or $Z_\mu \subset V_\lambda \otimes S_-$ according as $\mu \in \Lambda_{\mathfrak{r}}^{\lambda,+}$ or $\mu \in \Lambda_{\mathfrak{r}}^\lambda$.*

Theorem 0.19 is part of the statement of Theorem 4.17 in the present paper.

Remark 0.20. Theorem 4.17 also explicitly exhibits a highest vector z_μ of Z_μ. The vector z_μ is a decomposable tensor in $V_\lambda \otimes S$.

Given Theorems 0.15, 0.17, and 0.19, experience with Dirac operators leads one to expect the existence of such an operator, $\square_\lambda \in \mathrm{End}(V_\lambda \otimes S)$, commuting with the action of $\mathfrak{r}$, such that if $\square_\lambda^+ = \square_\lambda \mid (V_\lambda \otimes S_+)$, then

$$\square_\lambda^+ : V_\lambda \otimes S_+ \longrightarrow V_\lambda \otimes S_-,$$

where

$$\mathrm{Ker}\,\square_\lambda^+ = \sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} Z_{\mu'}$$

and

$$\mathrm{Coker}\,\square_\lambda^+ = \sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} Z_{\mu''}.$$

In such a case, $\square_\lambda^+$ would induce an explicit quotient $\mathfrak{r}$-isomorphism

$$\widetilde{\square}_\lambda^+ : \frac{V_\lambda \otimes S_+}{\sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} Z_{\mu'}} \longrightarrow \frac{V_\lambda \otimes S_-}{\sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} Z_{\mu''}}. \tag{0.21}$$

The Dirac operator $\square_\lambda$ having the desired properties is constructed in §2. The fact that the constructed $\square_\lambda$ has these properties rests, among other results, on Theorem 0.22. Let $B_{\mathfrak{r}} = B_{\mathfrak{g}} \mid \mathfrak{r}$, and let $\mathrm{Cas}_{\mathfrak{r}} \in \mathfrak{Z}(\mathfrak{r})$ be the Casimir element corresponding to $B_{\mathfrak{r}}$. It is easy to see that $\mathrm{Cas}_{\mathfrak{r}} \in \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{r})$, so $\chi_{\mathfrak{r}}^{\mu}(\mathrm{Cas}_{\mathfrak{r}})$ is constant for all μ in a multiplet by Proposition 0.12. But much more is true.

THEOREM 0.22. *Let $\lambda \in \Lambda$. Then the maximal eigenvalue of $\xi_\lambda(\mathrm{Cas}_{\mathfrak{r}})$ in $V_\lambda \otimes S$ is $(\lambda+\rho, \lambda+\rho) - (\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}})$, and the multiplicity-free $\mathfrak{r}$-submodule $\sum_{\mu \in \Lambda_{\mathfrak{r}}^{\lambda}} Z_\mu$ is the corresponding eigenspace.*

Theorem 0.22 is proved in §4. It is another part of Theorem 4.17.

0.23. The operator $\square_\lambda$ is a specialization of a more general Dirac operator $\square \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$. One has $\square = \square' + \square''$. The operator $\square'$ is linear in $\mathfrak{p}$, a usual expectation in an expression for a Dirac operator. On the other hand, $\square'' = 1 \otimes v$, where v is cubic in $\mathfrak{p}$. As a consequence, we refer to $\square$ or $\square_\lambda$ as a cubic Dirac operator. The main results of §2—Theorems 2.13, 2.16, and 2.21—compute $\square^2$ and $(\square_\lambda)^2$ under more general assumptions about $\mathfrak{r}$ and $\mathfrak{g}$ than those considered above. The following result is the statement of Theorem 2.21 under our present assumptions.

THEOREM 0.24. *Let $\lambda \in \Lambda$. Then if I is the identity on $V_\lambda \otimes S$,*

$$\left(\square_\lambda\right)^2 = \left((\lambda+\rho, \lambda+\rho) - \left(\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}}\right)\right) I - \zeta_\lambda\left(\mathrm{Cas}_{\mathfrak{r}}\right). \tag{0.25}$$

Theorems (0.19), (0.22), and (0.24) immediately yield the desired equality

$$\operatorname{Ker}\Box_\lambda = \sum_{\mu\in\Lambda_{\mathfrak{r}}^{\lambda}} Z_\mu. \tag{0.26}$$

Remark 0.27. Consider the special case where $\mathfrak{r}$ is the Levi factor of a proper parabolic Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{q}$. Then $V_\lambda \otimes S$, as an $\mathfrak{r}$-module, differs from the standard cochain complex $C(\mathfrak{n}, V_\lambda)$ defining the cohomology $H^*(\mathfrak{n}, V_\lambda)$ only by tensoring with a 1-dimensional character χ of $\mathfrak{r}$. The $\mathfrak{r}$-module $H^*(\mathfrak{n}, V_\lambda)$ was determined in [Ko1] by introducing a Laplacian, written here as L_λ, on $C(\mathfrak{n}, V_\lambda)$ and computing the $\mathfrak{r}$-module structure on $\operatorname{Ker} L_\lambda$. Given the expression for L_λ made in [Ko1], Theorems 0.19 and 0.22 easily give that structure. Thus together, Theorems 0.19 and 0.22 yield a very extensive generalization of the main results of [Ko1]. Curiously, it should also be noted that, although no analogue of $\Box_\lambda$ is introduced in [Ko1], Theorem 0.24 asserts that $\Box_\lambda^2$ is very similar to L_λ. We emphasize that the latter is an operator, introduced in [Ko1] only for the purpose of computing $H^*(\mathfrak{n}, V_\lambda)$. In contrast, in the present paper, where $\mathfrak{r}$ is much more general, no such nilpotent cohomology is visible to us.

As one knows, the Bott-Borel-Weil theorem is equivalent to the determination of the $\mathfrak{r}$-module structure on $H(\mathfrak{n}, V_\lambda)$ when r is a Cartan subalgebra. Thus, from this perspective, we can regard the pair, Theorems 0.19 and 0.22, as a very extensive generalization of the Bott-Borel-Weil theorem.

0.28. The theorems about the cubic Dirac operators in §2 depend upon the results in §1. These results, except for the determination of the scalars in (1.82) and (1.92), are about thirty years old (see Remarks 1.52 and 1.63). I believe they are of independent interest; they constitute the second origin of the results in this paper (see the first sentence in §0.1).

The assumption in §1 is that $\mathfrak{r}$ is an arbitrary complex Lie algebra that possesses an ad-invariant, nonsingular, symmetric bilinear form $B_{\mathfrak{r}}$. Next, $\mathfrak{p}$ is an arbitrary finite-dimensional complex vector space also possessing a nonsingular symmetric bilinear form $B_{\mathfrak{p}}$, and we are given a representation $\nu : \mathfrak{r} \to \operatorname{Lie}\operatorname{SO}(\mathfrak{p})$. Let $\mathfrak{g}$ be the linear space $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}$, and let $B_{\mathfrak{g}}$ be the nonsingular symmetric bilinear form on $\mathfrak{g}$ defined so that $B_{\mathfrak{r}} = B_{\mathfrak{g}} \mid \mathfrak{r}$, $B_{\mathfrak{p}} = B_{\mathfrak{g}} \mid \mathfrak{p}$, and $\mathfrak{r}$ and $\mathfrak{p}$ are $B_{\mathfrak{g}}$-orthogonal. We say the pair $(\nu, B_{\mathfrak{g}})$ is of Lie type if there is a Lie algebra structure on $\mathfrak{g}$ satisfying conditions (a) and (b) of (1.3). Using the Clifford algebra $C(\mathfrak{p})$ over $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$, Theorem 1.50 gives a necessary and sufficient condition for $(\nu, B_{\mathfrak{g}})$ to be of Lie type. To state the condition, one notes first that ν lifts to a homomorphism $\nu_* : U(\mathfrak{r}) \to C^{\text{even}}(\mathfrak{p})$. We may identify (à la Chevalley) the underlying vector space of $C(\mathfrak{p})$ with the exterior algebra $\wedge\mathfrak{p}$ and recognize that $\wedge\mathfrak{p}$ has two multiplicative structures. If $\operatorname{Cas}_{\mathfrak{r}} \in \operatorname{Cent} U(\mathfrak{r})$ is the Casimir element with respect to $B_{\mathfrak{r}}$, then one easily has that $\nu_*(\operatorname{Cas}_{\mathfrak{r}}) \in \wedge^4\mathfrak{p} + \mathbb{C}$, where we identify $\wedge^0\mathfrak{p}$ with $\mathbb{C}$. On the other hand, if $v \in \wedge^3\mathfrak{p}$, then one also has (Clifford square) $v^2 \in \wedge^4\mathfrak{p} + \mathbb{C}$. The condition is a canceling of the terms in degree 4. Part of Theorem 1.50 is as follows.

THEOREM 0.29. *The pair* $(\nu, B_{\mathfrak{g}})$ *is of Lie type if and only if there exists an* $\mathfrak{r}$-*invariant* $v \in \wedge^3\mathfrak{p}$ *such that*

$$\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big) + v^2 \in \mathbb{C}. \tag{0.30}$$

Remark 0.31. The v in Theorem 0.29 becomes the cubic term in the Dirac operators $\square$ and $\square_\lambda$ of §2.

In complete generality, the scalar in (0.30) is determined in Theorem 1.81. In the special case where $\mathfrak{r}$ and $\mathfrak{g}$ are reductive (but with no equal rank assumption), using the "strange" formula of Freudenthal and de Vries, one has the following theorem.

THEOREM 0.32. *Assume that an* $\mathfrak{r}$-*invariant* $v \in \wedge^3\mathfrak{p}$ *satisfies (0.30) and that* $\mathfrak{r}$ *and* $\mathfrak{g}$ *are reductive. Then*

$$\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big) + v^2 = (\rho, \rho) - \big(\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}}\big). \tag{0.33}$$

Remark 0.34. The equality (0.33) can be regarded as a complete generalization of a key result of Parthasarathy in [P]. In [P, Lemma 2.2] the scalar in (0.33) is obtained under the assumption that $v = 0$ (i.e., $(\mathfrak{r}, g)$ is a symmetric space pair) and $\operatorname{rank} r = \operatorname{rank} g$. See Theorems 1.59 and 1.61 for the symmetric space case; see also Remark 1.63.

0.35. Returning to the assumptions and notation of §0.6, one readily has that $\mathfrak{Z}(r)$ is a free module of rank d over the image $\mathfrak{Z}_{\mathfrak{g}}(r)$ of $\mathfrak{Z}(\mathfrak{g})$ in $\mathfrak{Z}(r)$. In §5 we define a d-dimensional subspace $\mathfrak{Z}_J(\mathfrak{r})$ of $\mathfrak{Z}(r)$ such that the map

$$\mathfrak{Z}_{\mathfrak{g}}(r) \otimes \mathfrak{Z}_J(\mathfrak{r}) \longrightarrow \mathfrak{Z}(r),$$

defined by multiplication, is a linear isomorphism. The subspace $\mathfrak{Z}_J(\mathfrak{r})$ is chosen so that, among other things, the Harish-Chandra homomorphism defines a grading on $\mathfrak{Z}_J(\mathfrak{r})$. One has

$$\dim \mathfrak{Z}_J^k(\mathfrak{r}) = b_{2k}(X), \tag{0.36}$$

where $b_{2k}(X)$ is the $2k$th Betti number of $X = G/R$. In particular, since $\dim X = 2m$, one has

$$\dim \mathfrak{Z}_J^m(\mathfrak{r}) = 1. \tag{0.37}$$

By Proposition 0.12, the infinitesimal character values of the elements in $\mathfrak{Z}_{\mathfrak{g}}(r)$ on any multiplet is constant. This constant value is explicitly given in (5.19). In §5 there are some results about the infinitesimal character values of $\mathfrak{Z}_J(\mathfrak{r})$ on any multiplet. The most explicit result is obtained for a "Pfaffian-type" element q_d that spans $\mathfrak{Z}_J^m(\mathfrak{r})$. The following is part of the statement of Theorem 5.40.

THEOREM 0.38. *Let $\mu \in \Lambda_{\mathfrak{r}}$. Then $\chi_{\mathfrak{r}}^{\mu}(q_d) = 0$ if and only if $\mu \notin \Lambda_{\mathfrak{r}}^{*}$. Assume $\mu \in \Lambda_{\mathfrak{r}}^{*}$ so that there exists a unique $\lambda \in \Lambda$ such that $\mu \in \Lambda_{\mathfrak{r}}^{\lambda}$. Then $\chi_{\mathfrak{r}}^{\mu}(q_d)$ is positive or negative according as $\mu \in \Lambda_{\mathfrak{r}}^{\lambda,+}$ or $\mu \in \Lambda_{\mathfrak{r}}^{\lambda,-}$. In fact, if we write $\mu = \tau \bullet \lambda$ for $\tau \in W^1$, then for an explicitly determined constant k_o, which is independent of μ, one has*

$$\chi_{\mathfrak{r}}^{\mu}(q_d) = sg(\tau)k_o \frac{\dim V_\lambda}{\dim Z_\mu}. \tag{0.39}$$

0.40. The manifold X may or may not be a spin manifold (i.e., the second Stiefel-Whitney class of X might or might not vanish). Both cases abound. In §6 it is shown that X is a spin manifold if and only if $\Gamma = \Gamma_{\mathfrak{r}}$. Assume this to be the case, so that for any $\mu \in \Lambda_{\mathfrak{r}}$ the representation $\sigma_{\mathfrak{r}}$ exponentiates to a representation $\sigma_\mu : R \to \operatorname{Aut} Z_\mu$. The latter then defines a homogeneous vector bundle

$$E_\mu = G \times_R Z_\mu$$

over X. Note that the assumptions are such that X is the most general compact simply connected homogeneous space with positive Euler characteristic having a spin structure. Let $K(X)$ be the topological K-cohomology group for X with complex coefficients. Since, as one knows $b_k(X) = 0$ for odd k, one has

$$\dim K(X) = d.$$

For any $\mu \in \Lambda_{\mathfrak{r}}$, let $[E_\mu] \in K(X)$ be the class defined by E_μ. The following result is a restatement of Theorem 6.30.

THEOREM 0.41. *Let $\lambda \in \Lambda$. Then the classes $[E_\mu]$ over all μ in the d-multiplet $\Lambda_{\mathfrak{r}}^{\lambda}$ are a basis of $K(X)$. In particular, $K(X)$ is spanned by the classes of homogeneous vector bundles.*

0.42. We thank Anton Alekseev, Dick Gross, and Shlomo Sternberg for valuable conversations during the preparation of this paper. We also thank Nolan Wallach for acquainting us with his classification of a family of symmetric spaces X that are spin manifolds. See Remark 6.19.

Note added in proof. Pierre Ramond pointed out to me that a Dirac operator with a cubic term was introduced earlier by physicists in coset models. See [LVW, (5.3)] and [KS, (2.37)]. In [LVW] the authors restrict themselves to the case where τ is a Levi factor of a parabolic subalgebra of $\mathfrak{g}$. The cubic term in [LVW] is not the same as the cubic term introduced in the present paper; however, the authors do concern themselves with the kernel of the Dirac operator (but not its square) and cite my paper [Ko1] for the multiplicity-1 result. See Remark 0.27 in the present paper. The second reference [KS] is in a Kac-Moody setting. The cubic term appears to be a Kac-Moody version of the cubic term in the present paper. No attempt is made to determine the kernel or square of this operator in the general case. More specific results are obtained in the Hermitian-symmetric case where, of course, τ is again a Levi factor.

E. Meinrenken directed me to the papers [S1] and [S2], where geometric Dirac operators are studied on compact homogeneous spaces $X = G/R$ with spin structure and positive Euler number. The elements of a multiplet make an appearance here in a determination of the kernel of these operators. See Theorem 2 in [S2]. No cubic term is explicitly visible (to me) in these operators, and their square is explicitly written in terms of Casimir operators only in the case where X is the flag manifold.

1. A Clifford algebra criterion for $(\nu, B_{\mathfrak{g}})$ to be of Lie type

1.1. We adopt the following notation and conventions throughout the paper. If $\mathfrak{s}$ is a finite-dimensional complex Lie algebra, then $U(\mathfrak{s})$ is the universal enveloping algebra of $\mathfrak{s}$ and $\mathfrak{Z}(\mathfrak{s}) = \operatorname{Cent} U(\mathfrak{s})$. If C is a complex associative algebra (e.g., $C = \operatorname{End} V$, where V is a complex vector space) and $\pi : \mathfrak{s} \to C$ is a Lie algebra homomorphism (e.g., π is a representation of $\mathfrak{s}$ on V), we use the same letter, in this case, π, to denote the homomorphism $U(\mathfrak{s}) \to C$ extending $\pi : \mathfrak{s} \to C$.

Let $\mathfrak{p}$ be a finite-dimensional complex vector space. Assume that (y, y') for $y, y' \in \mathfrak{p}$ is a nonsingular symmetric bilinear form $B_{\mathfrak{p}}$ on $\mathfrak{p}$. Let $\mathrm{SO}(\mathfrak{p})$ be the special orthogonal group on $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$. Let $\mathfrak{r}$ be a complex finite-dimensional Lie algebra, and assume that (x, x') for $x, x' \in \mathfrak{r}$ is a nonsingular ad $\mathfrak{r}$-invariant symmetric bilinear form $B_{\mathfrak{r}}$ on $\mathfrak{r}$.

We assume throughout that

$$\nu : \mathfrak{r} \longrightarrow \operatorname{Lie} \mathrm{SO}(\mathfrak{p})$$

is a $B_{\mathfrak{p}}$-invariant representation of $\mathfrak{r}$ on $\mathfrak{p}$.

Let

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}, \tag{1.2}$$

and let $B_{\mathfrak{g}}$ be the nonsingular symmetric bilinear form on $\mathfrak{g}$ defined so that $B_{\mathfrak{g}} \mid \mathfrak{r} = B_{\mathfrak{r}}$, $B_{\mathfrak{g}} \mid \mathfrak{p} = B_{\mathfrak{p}}$, and $\mathfrak{p}$ is $B_{\mathfrak{g}}$-orthogonal to $\mathfrak{r}$. It is consistent with previous notation to let (z, z') denote the value of $B_{\mathfrak{g}}$ on $z, z' \in \mathfrak{g}$. We say that the representation $(\nu, B_{\mathfrak{g}})$ is of Lie type if there exists a Lie algebra structure $[z, z']$ on $\mathfrak{g}$ such that

(a) $\mathfrak{r}$ is a Lie subalgebra of $\mathfrak{g}$ and

$$[x, y] = \nu(x)y \quad \text{for } x \in \mathfrak{r},\ y \in \mathfrak{p}; \tag{1.3}$$

(b) $B_{\mathfrak{g}}$ is ad $\mathfrak{g}$-invariant.

There are two objectives in §1. The first is to give a necessary and sufficient condition for $(\nu, B_{\mathfrak{g}})$ to be of Lie type. This is Theorem 1.50. The condition is a simple statement involving the Clifford algebra $C(\mathfrak{p})$ over $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$. The second objective is to determine the scalar arising in Theorem 1.50 for the reductive case. This is given in Theorem 1.91.

Remark 1.4. Although our main applications are to the case where $\mathfrak{r}$ and (assuming $(\nu, B_{\mathfrak{g}})$ to be of Lie type) $\mathfrak{g}$ are reductive, no such assumptions are made in Sections 1 and 2, unless specified otherwise. Nonreductive Lie algebras having nonsingular ad invariant symmetric bilinear forms abound. Even nonabelian nilpotent Lie algebras can have nonsingular ad invariant symmetric bilinear forms.

1.5. We recall some of the relations arising from Chevalley's identification of the underlying vector spaces of the exterior algebra $\wedge\mathfrak{p}$ and the Clifford algebra $C(\mathfrak{p})$ over $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$. We think of $\wedge\mathfrak{p}$ as having two multiplicative structures. If $u, w \in \wedge\mathfrak{p}$, then $uw \in \wedge\mathfrak{p}$ denotes the Clifford product of u and w and $u \wedge w \in \wedge\mathfrak{p}$ is the exterior product of u and w. For more details, see §2 in [Ko2]. If $u, w \in \wedge\mathfrak{p}$, the natural extension (see, e.g., §2.1 in [Ko2]) (u, w) of $B_{\mathfrak{p}}$ to $\wedge\mathfrak{p}$ is again denoted by $B_{\mathfrak{p}}$. If $A \in \operatorname{End}\wedge\mathfrak{p}$, then $A^t \in \operatorname{End}\wedge\mathfrak{p}$ denotes the transpose of A with respect to $B_{\mathfrak{p}}$. If $w \in \wedge\mathfrak{p}$, then $\epsilon(w) \in \operatorname{End}\wedge\mathfrak{p}$ denotes the operator of left exterior multiplication by w and $\iota(w) = \epsilon(w)^t$. In particular, $\iota(y)$ for $y \in \mathfrak{p}$ is the antiderivation of degree -1 of the exterior algebra $\wedge\mathfrak{p}$ such that $\iota(y)y' = (y, y')$ if $y' \in \mathfrak{p}$. Clifford multiplication in $\wedge\mathfrak{p}$ is readily determined from the equality

$$yw = \big(\epsilon(y) + \iota(y)\big)w \tag{1.6}$$

for any $y \in \mathfrak{p}$ and $w \in \wedge\mathfrak{p}$.

For any $u \in \wedge^2\mathfrak{p}$, let $\operatorname{ad} u \in \operatorname{End}\wedge\mathfrak{p}$ be the operator defined so that $\operatorname{ad} u(w) = uw - wu$ for any $w \in \wedge\mathfrak{p}$. Then $\operatorname{ad} u$, clearly a derivation of the Clifford algebra structure of $\wedge^2\mathfrak{p}$, is also a derivation of degree zero of the exterior algebra structure on $\wedge\mathfrak{g}$. In particular, $\wedge^2\mathfrak{p}$ is a Lie algebra under Clifford product, and

$$\wedge^2\mathfrak{p} \longrightarrow \operatorname{End}\wedge\mathfrak{p}, \qquad u \longmapsto \operatorname{ad} u$$

is a Lie algebra representation. Next, one has that

$$\tau : \wedge^2\mathfrak{p} \longrightarrow \operatorname{Lie} \mathrm{SO}(\mathfrak{p}) \tag{1.7}$$

is a Lie algebra isomorphism, where for $u \in \wedge^2\mathfrak{p}$ and $y \in \mathfrak{p}$ one has

$$\tau(u)(y) = \operatorname{ad} u(y) = -2\iota(y)u. \tag{1.8}$$

For a proof of the above statements in this paragraph, see Proposition 7 and Theorem 8 in [Ko2].

Remark 1.9. The second equality in (1.8) is a special case of the general fact that if $w \in \wedge^k\mathfrak{p}$ and $y \in \mathfrak{p}$, then

$$yw - (-1)^k wy = 2\iota(y)w.$$

In particular, $\iota(y)$ is an antiderivation of the Clifford algebra structure of $\wedge\mathfrak{p}$ as well as the exterior algebra structure of $\wedge\mathfrak{p}$. See [Ko2, Lemma 5, p. 284].

It clearly follows from (1.8) that there exists a unique Lie algebra homomorphism

$$\nu_* : \mathfrak{r} \longrightarrow \wedge^2\mathfrak{p} \tag{1.10}$$

such that $\tau \circ \nu_* = \nu$ (see [Ko2, Theorem 8]. Putting $u = \nu_*(x)$ in (1.8), one has

$$\nu(x)y = -2\iota(y)\nu_*(x). \tag{1.11}$$

For any $x \in \mathfrak{r}$, let $\theta_\nu(x)$ be the unique derivation of the exterior algebra $\wedge\mathfrak{p}$, which extends $\nu(x)$. Regard $\wedge\mathfrak{p}$ as an $\mathfrak{r}$-module where $x \in \mathfrak{r}$ maps to $\theta_\nu(x)$. One notes that the extended bilinear form $B_\mathfrak{p}$ is invariant under $\theta_\nu(x)$ and recalls that (see [Ko2, (8)] where $u = \nu_*(x)$)

$$\theta_\nu(x) = \operatorname{ad}\nu_*(x). \tag{1.12}$$

Now since $B_\mathfrak{g} \mid \mathfrak{r}$ is nonsingular and $B_\mathfrak{p}$ is defined (and is in fact nonsingular) on $\wedge^2\mathfrak{p}$, the map (1.10) has a well-defined transpose

$$\nu_*^t : \wedge^2\mathfrak{p} \longrightarrow \mathfrak{r}.$$

For $y, y' \in \mathfrak{p}$, let

$$\big[y, y'\big]_\mathfrak{r} = -2\nu_*^t\big(y \wedge y'\big). \tag{1.13}$$

LEMMA 1.14. *Let $x \in \mathfrak{r}$ and $y, y' \in \mathfrak{p}$. Then*

$$\big[x, \big[y, y'\big]_\mathfrak{r}\big] = \big[\nu(x)y, y'\big]_\mathfrak{r} + \big[y, \nu(x)y'\big]_\mathfrak{r} \tag{1.15}$$

and

$$\big(x, \big[y, y'\big]_\mathfrak{r}\big) = \big(\nu(x)y, y'\big). \tag{1.16}$$

Proof. Regarding $\mathfrak{r}$ as an $\mathfrak{r}$-module using the adjoint representation, it follows from (1.12) that ν_* is an $\mathfrak{r}$-map. But then ν_*^t is an $\mathfrak{r}$-map. This proves (1.15). But now by (1.11)

$$\begin{aligned}\big(x, \big[y, y'\big]_\mathfrak{r}\big) &= -2\big(x, \nu_*^t\big(y \wedge y'\big)\big) = -2\big(\nu_*(x), y \wedge y'\big)\\ &= -2\big(\iota(y)\nu_*(x), y'\big) = \big(\nu(x)y, y'\big),\end{aligned}$$

which proves (1.16). □

Let $P_\mathfrak{r} : \mathfrak{g} \to \mathfrak{r}$ and $P_\mathfrak{p} : \mathfrak{g} \to \mathfrak{p}$ be the projections with respect to the decomposition (1.2). An immediate consequence of (1.16) is the following proposition.

PROPOSITION 1.17. *If $[z, z']$ is a Lie algebra structure on $\mathfrak{g}$ such that (a) and (b) of (1.3) are satisfied, one necessarily has*

$$P_\mathfrak{r}\big(\big[y, y'\big]\big) = \big[y, y'\big]_\mathfrak{r}$$

for any $y, y' \in \mathfrak{p}$.

1.18. If $[z, z']$ is a Lie algebra structure on $\mathfrak{g}$ satisfying (a) and (b) of (1.3), then the only bracket relation that has not yet been determined is $P_{\mathfrak{p}}([y, y'])$, where $y, y' \in \mathfrak{p}$. But clearly $P_{\mathfrak{p}}([y, y'])$ is determined by the trilinear form ϕ on $\mathfrak{p}$, defined by putting $\phi(y, y', y'') = (P_{\mathfrak{p}}([y, y']), y'')$. But since $\mathfrak{r}$ and $\mathfrak{p}$ are $B_{\mathfrak{g}}$-orthogonal, one has

$$\phi(y, y', y'') = ([y, y'], y''). \tag{1.19}$$

But by the ad $\mathfrak{g}$-invariance of $B_{\mathfrak{g}}$, it follows that ϕ is alternating and hence there exists a unique $v \in \wedge^3\mathfrak{p}$ such that

$$([y, y'], y'') = -2(v, y \wedge y' \wedge y''). \tag{1.20}$$

Furthermore, since

$$[x, [y, y']] = [[x, y], y'] + [y, [x, y'']] \tag{1.21}$$

for $x \in \mathfrak{r}$, it follows from (1.20) that $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$, where $(\wedge^3\mathfrak{p})^{\mathfrak{r}}$ is the space of $\mathfrak{r}$-invariants in $\wedge^3\mathfrak{p}$. But now $-2(v, y \wedge y' \wedge y'') = -2(\iota(y')\iota(y)v, y'')$. But then, since $y \wedge y' = -y' \wedge y$, it follows from (1.20) that

$$P_{\mathfrak{p}}([y, y']) = 2\iota(y)\iota(y')v, \tag{1.22}$$

explicitly expressing $P_{\mathfrak{p}}([y, y'])$ in terms of $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$. This leads to the following definition. For any $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$ (noting that $(\wedge^3\mathfrak{p})^{\mathfrak{r}}$ is not empty, since it at least contains the zero element in $\wedge^3\mathfrak{p}$), let

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad \{z, z'\} \longmapsto [z, z']^v$$

be the unique alternating bilinear map such that (1) $[x, x']^v = [x, x']$ for $x, x' \in \mathfrak{r}$, (2) $[x, y]^v = \nu(x)y$ for $x \in \mathfrak{r}$, $y \in \mathfrak{p}$, and (3)

$$[y, y']^v = [y, y']_{\mathfrak{r}} + 2\iota(y)\iota(y')v, \tag{1.23}$$

noting that the two components on the right side of (1.23) are $P_{\mathfrak{r}}([y, y']^v)$ and $P_{\mathfrak{p}}([y, y']^v)$, respectively. The statements in the following proposition have either been established above or are immediately verified.

PROPOSITION 1.24. *Let $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$. Then, for any $z, z', z'' \in \mathfrak{g}$, one has*

$$([z, z']^v, z'') = (z, [z', z'']^v). \tag{1.25}$$

Furthermore, the Jacobi identity for z, z', z'' is satisfied if at least one of these three elements is in $\mathfrak{r}$. Finally, if $[z, z']$ is a Lie algebra structure on $\mathfrak{g}$ that satisfies (a) and (b) of (1.3), then for all $z, z' \in \mathfrak{g}$,

$$[z, z'] = [z, z']^v \tag{1.26}$$

for some unique $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$.

Let $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$ and put $[z,z'] = [z,z']^v$. To determine whether $[z,z']$ is a Lie algebra structure on $\mathfrak{g}$ satisfying (a) and (b) of (1.3), we are reduced, by Proposition 1.24, to finding a simple, necessary, and sufficient condition on v so that the Jacobi identity

$$\big[[y,y']^v,y''\big]^v+\big[[y',y'']^v,y\big]^v+\big[[y'',y]^v,y'\big]^v=0$$

is satisfied for all $y,y',y''\in\mathfrak{p}$. Actually, by the following result we have only to prove

$$P_{\mathfrak{p}}\Big(\big[[y,y']^v,y''\big]^v+\big[[y',y'']^v,y\big]^v+\big[[y'',y]^v,y'\big]^v\Big)=0. \tag{1.27}$$

PROPOSITION 1.28. *Let $y,y',y''\in\mathfrak{p}$. Then*

$$P_{\mathfrak{r}}\Big(\big[[y,y']^v,y''\big]^v+\big[[y',y'']^v,y\big]^v+\big[[y'',y]^v,y'\big]^v\Big)=0. \tag{1.29}$$

Proof. Let $x\in\mathfrak{r}$. Since $x\in\mathfrak{r}$ is arbitrary, it suffices to prove

$$\Big(\big[[y,y']^v,y''\big]^v+\big[[y',y'']^v,y\big]^v+\big[[y'',y]^v,y'\big]^v,x\Big)=0. \tag{1.30}$$

But by Proposition 1.24 (especially (1.25)) and the statement of validity of the Jacobi identity, if one of the three elements lies in $\mathfrak{r}$,

$$\begin{aligned}\Big(\big[[y,y']^v,y''\big]^v,x\Big)&=\Big([y,y']^v,[y'',x]^v\Big)=\Big(\big[x,[y,y']^v\big]^v,y''\Big)\\&=\Big(\big[[x,y]^v,y'\big]^v,y''\Big)+\Big(\big[y,[x,y']^v\big]^v,y''\Big)\\&=\Big([x,y]^v,[y',y'']^v\Big)+\Big([x,y']^v,[y'',y]^v\Big)\\&=\Big(x,\big[y,[y',y'']^v\big]^v\Big)+\Big(x,\big[y',[y'',y]^v\big]^v\Big)\\&=-\Big(\big[[y',y'']^v,y\big]^v,x\Big)-\Big(\big[[y'',y]^v,y'\big]^v,x\Big).\end{aligned}$$

But this immediately yields (1.30). □

1.31. Let $r=\dim\mathfrak{r}$, and let $\{x_i\}, i=1,\ldots,r$, be an orthonormal basis of $\mathfrak{r}$ with respect to $B_{\mathfrak{r}}=B_{\mathfrak{g}}\mid\mathfrak{r}$. Then $\mathrm{Cas}_{\mathfrak{r}}\in\mathfrak{Z}(\mathfrak{r})$, where $\mathrm{Cas}_{\mathfrak{r}}=\sum_{i=1}^k x_i^2$. Of course, $\mathrm{Cas}_{\mathfrak{r}}$ is independent of the choice of the orthonormal basis. Let $C^{\mathrm{even}}(\mathfrak{p})=\sum_{i=0}\wedge^{2i}\mathfrak{p}$. Then $C^{\mathrm{even}}(\mathfrak{p})$ is a Clifford subalgebra of $C(\mathfrak{p})$. Now we can regard (see (1.10)) $\nu_*:\mathfrak{r}\to C^{\mathrm{even}}(\mathfrak{p})$ as a Lie algebra homomorphism; hence, by extension,

$$\nu_*:U(\mathfrak{r})\longrightarrow C^{\mathrm{even}}(\mathfrak{p}) \tag{1.32}$$

is a homomorphism of associative algebras. We are particularly interested in the element $\nu_*(\mathrm{Cas}_{\mathfrak{r}})$. Clearly,

$$\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)=\sum_{i=1}^r\nu_*(x_i)^2. \tag{1.33}$$

For any element $w \in \wedge\mathfrak{p}$ and $k \in \mathbb{Z}_+$, let w_k be the homogeneous component of w in $\wedge^k\mathfrak{p}$. We identify $\wedge^0\mathfrak{p}$ with $\mathbb{C}$ so that w_0 is a scalar in $\mathbb{C}$. Let (following Chevalley's notation) $\alpha \in \operatorname{End}\mathfrak{p}$ be the Clifford (and also exterior) algebra antiautomorphism of $\wedge\mathfrak{p}$ such that $\alpha(y) = y$ for $y \in \mathfrak{p}$. Then one easily has that

$$\alpha \mid \wedge^k\mathfrak{p} = (-1)^{k(k-1)/2} \tag{1.34}$$

(see, e.g., [Ch1, §2.1, p. 38, and Theorem III.4.1, p. 90]).

Proposition 1.35. *One has $(\nu_*(\mathrm{Cas}_\mathfrak{r}))_k = 0$ if $k \notin \{0,4\}$ so that there exists a constant $c_\mathfrak{r} \in \mathbb{C}$ such that*

$$\nu_*\big(\mathrm{Cas}_\mathfrak{r}\big) = \big(\nu_*\big(\mathrm{Cas}_\mathfrak{r}\big)\big)_4 + c_\mathfrak{r}. \tag{1.36}$$

Proof. It is immediate from (1.6) and (1.33) that $(\nu_*(\mathrm{Cas}_\mathfrak{r}))_k = 0$ if $k \notin \{0,2,4\}$. It suffices only to show that $(\nu_*(\mathrm{Cas}_\mathfrak{r}))_2 = 0$. By (1.33) it suffices to show that

$$\big(u^2\big)_2 = 0 \tag{1.37}$$

for any $u \in \wedge^2\mathfrak{p}$. But by (1.34), one has $\alpha(u^2) = (-u)^2 = u^2$. Then (1.37) follows, since $\alpha = -1$ on $\wedge^2\mathfrak{p}$. $\square$

If $z, z' \in \mathfrak{p}$, note that, by Remark 1.9,

$$zz' - z'z = 2z \wedge z'. \tag{1.38}$$

Let $y, y' \in \mathfrak{p}$ and let $u \in \wedge^2\mathfrak{p}$. By (1.38) and the second statement in Remark 1.9, one has

$$\begin{aligned}\iota\big(y'\big)\iota(y)u^2 &= 2\big(\iota\big(y'\big)\iota(y)u\big)u + \big(\iota\big(y'\big)u\big)\big(\iota(y)u\big) - \big(\iota(y)u\big)\big(\iota\big(y'\big)u\big)\\ &= 2\big((y\wedge y', u)u + \big(\iota\big(y'\big)u\big)\wedge\big(\iota(y)u\big)\big).\end{aligned}$$

But if $y'' \in \mathfrak{p}$, then

$$\begin{aligned}&\iota\big(y''\big)\iota\big(y'\big)\iota(y)u^2\\ &\quad= 2\big((y\wedge y', u)\iota\big(y''\big)u + \big(\iota\big(y''\big)\iota\big(y'\big)u\big)\wedge\iota(y)u - \big(\iota\big(y'\big)u\big)\wedge\iota\big(y''\big)\iota(y)u\big)\\ &\quad= 2\big((y\wedge y', u)\iota\big(y''\big)u + (y'\wedge y'', u)\iota(y)u + (y''\wedge y, u)\iota\big(y'\big)u\big).\end{aligned}$$

But then, recalling (1.11) and the definition of $[z,z']_\mathfrak{r}$ for $z, z' \in \mathfrak{p}$, if $u = \nu_*(x_i)$, one has

$$\begin{aligned}\iota\big(y''\big)\iota\big(y'\big)\iota(y)\nu_*(x_i)^2 &= 2\big(y\wedge y', \nu_*(x_i)\big)\iota\big(y''\big)\nu_*(x_i) + \big(y'\wedge y'', \nu_*(x_i)\big)\iota(y)\nu_*(x_i)\\ &\quad+ \big(y''\wedge y, \nu_*(x_i)\big)\iota\big(y'\big)\nu_*(x_i)\\ &= \frac{1}{2}\big(\big[y,y'\big]_\mathfrak{r}, x_i\big)\nu(x_i)y'' + \big(\big[y',y''\big]_\mathfrak{r}, x_i\big)\nu(x_i)y''\\ &\quad+ \big(\big[y,y'\big]_\mathfrak{r}, x_i\big)\nu(x_i)y''.\end{aligned}$$

But then summing over $i = 1, \ldots, r$, we have proved the following proposition.

PROPOSITION 1.39. *Let $y, y', y'' \in \mathfrak{p}$. Then*

$$\iota(y'')\iota(y')\iota(y)\nu_*(\mathrm{Cas}_{\mathfrak{r}}) = \frac{1}{2}\big(\nu([y,y']_{\mathfrak{r}})y'' + \nu([y',y'']_{\mathfrak{r}})y + \nu([y'',y]_{\mathfrak{r}})y'\big). \quad (1.40)$$

In particular, for our arbitrary choice $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$, one has

$$\begin{aligned}\iota(y'')\iota(y')\iota(y)\nu_*(\mathrm{Cas}_{\mathfrak{r}})\\ = \frac{1}{2}\big([P_{\mathfrak{r}}[y,y']^v, y'']^v + [P_{\mathfrak{r}}[y',y'']^v, y]^v + [P_{\mathfrak{r}}[y'',y]^v, y']^v\big).\end{aligned} \quad (1.41)$$

1.42. For any $y \in \mathfrak{p}$, let $v^y = \iota(y)v$, so that since $v \in \wedge^3\mathfrak{p}$, $v^y \in \wedge^2\mathfrak{p}$. But then if $y' \in \mathfrak{p}$, putting $u = v^y$ and $y' = y$ in (1.8), it follows from (1.8) and (1.22) that

$$\mathrm{ad}\, v^y(y') = -2\iota(y')v^y = -2\iota(y')\iota(y)v = P_{\mathfrak{p}}([y,y']^v). \quad (1.43)$$

Let $w \in \wedge\mathfrak{p}$. We have observed in the proof of Proposition 1.35 that if $w \in \wedge^2\mathfrak{p}$, then $(w^2)_k = 0$ if $k \notin \{0,4\}$. Curiously, the same statement is true if $w \in \wedge^3\mathfrak{p}$.

PROPOSITION 1.44. *Let $w \in \wedge^3\mathfrak{p}$. Then $(w^2)_k = 0$ if $k \notin \{0,4\}$. In particular, there exists a constant $c_v \in \mathbb{C}$ such that*

$$v^2 = (v^2)_4 + c_v. \quad (1.45)$$

Proof. By (1.6) it is immediate that $(w^2)_k = 0$ if $k \notin \{0,2,4,6\}$. Recalling (1.34), $\alpha = -1$ on $\wedge^k\mathfrak{p}$ if $k = 2,3,6$. But then $\alpha w^2 = (-w)^2 = w^2$. Hence, $(w^2)_k = 0$ for $k = 2,6$. □

Now let $y, y', y'' \in \mathfrak{p}$. By (1.45), one has $\iota(y'')\iota(y')\iota(y)v^2 \in \mathfrak{p}$. By the Clifford algebra antiderivation properties of $\iota(z)$ for $z \in \mathfrak{p}$ (see Remark 1.9), one readily has, by (1.22) and (1.43),

$$\begin{aligned}&\iota(y'')\iota(y')\iota(y)v^2\\ &= \frac{1}{2}\big(\mathrm{ad}\, v^{y''}(2\iota(y')\iota(y)v) + \mathrm{ad}\, v^{y'}(2\iota(y)\iota(y'')v) + \mathrm{ad}\, v^{y}(2\iota(y'')\iota(y')v)\big)\\ &= -\frac{1}{2}\big(\mathrm{ad}\, v^{y''}P_{\mathfrak{p}}([y,y']^v) + \mathrm{ad}\, v^{y'}P_{\mathfrak{p}}([y'',y]^v) + \mathrm{ad}\, v^{y}P_{\mathfrak{p}}([y',y'']^v)\big)\\ &= -\frac{1}{2}P_{\mathfrak{p}}\big([y'', P_{\mathfrak{p}}[y,y']^v]^v + [y', P_{\mathfrak{p}}[y'',y]^v]^v + [y, P_{\mathfrak{p}}[y',y]^v]\big)\\ &= \frac{1}{2}P_{\mathfrak{p}}\big([P_{\mathfrak{p}}[y,y']^v, y'']^v + [P_{\mathfrak{p}}[y'',y]^v, y']^v + [P_{\mathfrak{p}}[y',y]^v, y'']^v\big).\end{aligned} \quad (1.46)$$

The next proposition is a key result from which one of the main theorems in §1, Theorem 1.50, easily follows.

PROPOSITION 1.47. *Let $y, y', y'' \in \mathfrak{p}$. Then*

$$\iota(y'')\iota(y')\iota(y)\big(\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)+v^2\big) \\ = P_{\mathfrak{p}}\big(\big[\big[y,y'\big]^v, y''\big]^v+\big[\big[y',y''\big]^v, y\big]^v+\big[\big[y'',y\big]^v, y'\big]^v\big). \tag{1.48}$$

Proof. Note that the right-hand side of (1.41) is in the image of $P_{\mathfrak{r}}$. But then (1.48) follows from the addition of (1.41) and (1.46). □

Let $\mathcal{V} = \{v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}} \mid (v^2)_4 = -(\nu_*(\mathrm{Cas}_{\mathfrak{r}}))_4\}$. That is, by Propositions 1.35 and 1.44,

$$\mathcal{V} = \big\{v \in \big(\wedge^3\mathfrak{p}\big)^{\mathfrak{r}} \mid \mathrm{Cas}_{\mathfrak{r}} + v^2 \in \mathbb{C}\big\}. \tag{1.49}$$

THEOREM 1.50. *The pair $(\nu, B_{\mathfrak{g}})$ is of Lie type if and only if $\mathcal{V}$ is not empty. That is, if and only if there exists a constant $c_o \in \mathbb{C}$ such that $\nu_*(\mathrm{Cas}_{\mathfrak{r}})+c_o$ is the square of an element in $(\wedge^3\mathfrak{p})^{\mathfrak{r}}$. Alternatively, that is, if and only if there exists $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$ such that*

$$\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)+v^2 \in \mathbb{C}. \tag{1.51}$$

In fact, putting $[z,z'] = [z,z']^v$ for all $z, z' \in \mathfrak{g}$ and $v \in \mathcal{V}$ sets up a bijection between $\mathcal{V}$ and the set of all Lie algebra structures on $\mathfrak{g}$ that satisfy (a) and (b) of (1.3).

Proof. Let $v \in (\wedge^3\mathfrak{p})^{\mathfrak{r}}$, and put $w = \mathrm{Cas}_{\mathfrak{r}} + v^2$. One has $w \in \wedge^4\mathfrak{p}+\mathbb{C}$ by Propositions 1.35 and 1.44. Since $\iota(y\wedge y'\wedge y'') = \iota(y'')\iota(y')\iota(y)$, for $y, y', y'' \in \mathfrak{p}$, one clearly has $w_4 = 0$ if and only if $\iota(y'')\iota(y')\iota(y)w = 0$ for any $y, y', y'' \in \mathfrak{p}$. But by (1.48) this is the case if and only if $P_{\mathfrak{p}}([[y,y']^v, y'']^v + [[y',y'']^v, y]^v + [[y'',y]^v, y']^v) = 0$ for any $y, y', y'' \in \mathfrak{p}$. But then recalling (1.27) (see Proposition 1.28), this is the case if and only if $[z,z'] = [z,z']^v$ defines a Lie algebra structure on $\mathfrak{g}$ that satisfies (a) and (b) of (1.3). □

Remark 1.52. We proved Theorem 1.50 about thirty years ago. It was unpublished, but a variation of it was properly cited in the 1972 paper by Conlon (see [Co, p. 152]). A major reason for publishing the theorem at this time is the application of (1.51) that we make in §2 in determining the square of the cubic Dirac operator.

1.53. Let $O(\mathfrak{p})$ be the full $B_{\mathfrak{p}}$-orthogonal of $\mathfrak{p}$. If $a \in O(\mathfrak{p})$, let $\Theta(a)$ be the unique exterior algebra automorphism of $\wedge\mathfrak{p}$ that extends a. It is obvious that $\Theta(a)$ is also an automorphism of the Clifford algebra structure and is $B_{\mathfrak{p}}$-orthogonal on $\wedge\mathfrak{p}$. Let $F = \{a \in O(\mathfrak{p}) \mid a\nu(x) = \nu(x)a, \forall x \in \mathfrak{r}\}$ so that F is the orthogonal commuting group of ν. It is obvious that $\nu_*(x)$ is fixed under $\Theta(a)$ for all $a \in F$ and $x \in \mathfrak{r}$. Consequently, $\nu_*(\mathrm{Cas}_{\mathfrak{r}})$ is fixed and $(\wedge^3\mathfrak{p})^{\mathfrak{r}}$ is stabilized by $\Theta(a)$ for any $a \in \mathfrak{r}$. In particular, it is obvious from (1.51) that the variety $\mathcal{V}$ is stabilized by $\Theta(a)$ for any $a \in F$. This defines an equivalence relation in $\mathcal{V}$. Given $v, v' \in \mathcal{V}$ we say that v is equivalent to v', and we write $v \sim v'$ if there exists $a \in F$ such that

$$v' = \Theta(a)v. \tag{1.54}$$

Equivalent elements define isomorphic structures on $\mathfrak{g}$.

PROPOSITION 1.55. *Let $v, \mathfrak{v}' \in \mathcal{V}$, and assume that there exists $a \in F$ satisfying (1.54). Retain the notation $\mathfrak{g}$ for the space $\mathfrak{g}$ with the Lie algebra structure defined by putting $[z, z'] = [z, z']^v$, and let $\mathfrak{g}'$ be the space $\mathfrak{g}$ where the Lie algebra structure is written $[z, z']'$ and $[z, z']' = [z, z']^{v'}$. Let $A_a : \mathfrak{g} \to \mathfrak{g}'$ be the linear isomorphism where $A_a \mid \mathfrak{r}$ is the identity map and $A_a \mid \mathfrak{p} = a$. Then A_a is a Lie algebra isomorphism.*

Proof. It is immediate that one is reduced to showing that $a(P_{\mathfrak{p}}[y, y']) = P_{\mathfrak{p}}([ay, ay']')$ for $y, y' \in \mathfrak{p}$. But clearly, by (1.22),

$$a\big(P_{\mathfrak{p}}\big[y, y'\big]\big) = 2a\big(\iota(y)\iota\big(y'\big)v\big) = 2\iota(ay)\iota\big(ay'\big)v' = P_{\mathfrak{p}}\big(\big[ay, ay'\big]'\big). \qquad \square$$

1.56. The Clifford algebra $C(\mathfrak{p})$ has a unique (up to equivalence) faithful multiplicity-free module S. Let

$$\varepsilon : C(\mathfrak{p}) \longrightarrow \operatorname{End} S \tag{1.57}$$

be the corresponding homomorphism. One refers to S as the spin module for $C(\mathfrak{p})$. The Lie algebra homomorphism $\nu_* : \mathfrak{r} \to \wedge^2\mathfrak{p}$, composed with ε, defines a representation

$$\operatorname{Spin} \nu : \mathfrak{r} \longrightarrow \operatorname{End} S, \tag{1.58}$$

which is referred to as the spin of ν.

If $\mathfrak{u}$ is a Lie subalgebra of a Lie algebra $\mathfrak{v}$, we refer to the pair $(\mathfrak{u}, \mathfrak{v})$ as a symmetric pair if there exists an involutory automorphism θ of $\mathfrak{v}$ such that $\mathfrak{u}$ is a set of θ invariants in $\mathfrak{v}$. The automorphism θ is referred to as a corresponding Cartan involution.

THEOREM 1.59. *The following four conditions are equivalent:*
(1) $0 \in \mathcal{V}$*;*
(2) $\nu_*(\mathrm{Cas}_{\mathfrak{r}}) \in \mathbb{C}$*;*
(3) $\operatorname{Spin}\nu(\mathrm{Cas}_{\mathfrak{r}})$ *is a scalar multiple of the identity operator on S;*
(4) $(\nu, B_{\mathfrak{g}})$ *is of Lie type and $(\mathfrak{r}, \mathfrak{g})$ is a symmetric pair where $\mathfrak{p}$ is the -1 eigenspace for a corresponding Cartan involution.*

Proof. We see that (1) and (2) are clearly equivalent by Theorem 1.50. Also (2) and (3) are obviously equivalent, since $C(\mathfrak{p})$ is faithfully represented on S. If (1) is true, then upon choosing $v = 0$, one has $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{r}$ so that the element $\theta \in \operatorname{End}\mathfrak{g}$, where $\theta = 1$ on $\mathfrak{r}$ and -1 on $\mathfrak{p}$ is an involutory automorphism. Hence (1) implies (4). Conversely, (4) implies (1), since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{r}$ implies that the corresponding element $v \in \mathcal{V}$ must be zero. $\square$

Remark 1.60. E. Cartan was concerned with the set of all symmetric pairs $(\mathfrak{r}, \mathfrak{g})$, where $\mathfrak{g}$ is semisimple and ν is the complexified differential of an orthogonal representation of a compact group on $\mathfrak{p}$. In particular, $\mathfrak{r}$ is reductive and ν is completely reducible. Cartan determined all such ν. One is readily reduced to the case where ν is faithful and $\mathfrak{p}^{\mathfrak{r}} = 0$ where $\mathfrak{p}^{\mathfrak{r}}$ is the space of $\mathfrak{r}$ invariants in $\mathfrak{p}$. In view of the following result, conditions (2) or (3) of Theorem 1.59 characterize all such representations ν.

THEOREM 1.61. *Assume that ν is the complexified differential of a faithful $B_{\mathfrak{p}}$-orthogonal representation of a compact group and $\mathfrak{p}^{\mathfrak{r}} = 0$. Assume that any and hence all of the conditions of Theorem 1.59 are satisfied. Then $\mathfrak{g}$ is semisimple, and $\mathfrak{p}$ is the Killing form orthogonal complement of $\mathfrak{r}$ in $\mathfrak{g}$.*

Proof. Let θ be the Cartan involution defined in (4) of Theorem 1.59. Assume that $\mathfrak{g}$ is not semisimple. Let $\mathfrak{s} \neq 0$ be the radical of $\mathfrak{g}$. Since θ is a $\mathfrak{g}$-automorphism, it follows that $\mathfrak{s}$ is stable under θ. But then each ideal in the commutator series of $\mathfrak{s}$ is stable under θ. Thus there exists a θ-stable abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$ where $\mathfrak{a} \neq 0$. Thus $\mathfrak{a} = \mathfrak{a}_{\mathfrak{r}} \oplus \mathfrak{a}_{\mathfrak{p}}$ where $\mathfrak{a}_{\mathfrak{r}} = \mathfrak{a} \cap \mathfrak{r}$ and $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a} \cap \mathfrak{p}$. Since $\mathfrak{a}$ is an ideal, $[\mathfrak{r}, \mathfrak{a}_{\mathfrak{p}}] \subset \mathfrak{a}_{\mathfrak{p}}$. By the complete reducibility of ν, there exists an ad $\mathfrak{r}$-stable complement $\mathfrak{q}$ of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{p}$. But again, since $\mathfrak{a}$ is an ideal, one has $[\mathfrak{a}_{\mathfrak{r}}, \mathfrak{q}] \subset \mathfrak{q} \cap \mathfrak{a}_{\mathfrak{p}} = 0$. But $[\mathfrak{a}_{\mathfrak{r}}, \mathfrak{a}_{\mathfrak{p}}] = 0$ since $\mathfrak{a}$ is abelian. Thus $\mathfrak{a}_{\mathfrak{r}} \subset \operatorname{Ker} \nu$. Thus $\mathfrak{a}_{\mathfrak{r}} = 0$ since ν is faithful. Hence $\mathfrak{a} = \mathfrak{a}_{\mathfrak{p}}$. But since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{r}$, one has

$$\left[\mathfrak{p}, \mathfrak{a}_{\mathfrak{p}}\right] \subset \mathfrak{a}_{\mathfrak{p}} \cap \mathfrak{r} = 0. \tag{1.62}$$

But if there exists $x \in \mathfrak{r}$ and $y \in \mathfrak{a}_{\mathfrak{p}}$ such that $[x, y] \neq 0$, then by the nonsingularity of $B_{\mathfrak{p}}$ there exists $y' \in \mathfrak{p}$ such that $([x, y], y') \neq 0$. But by (1.62), $([x, y], y') = (x, [y, y']) = 0$. Thus $\mathfrak{a}_{\mathfrak{p}} \subset \mathfrak{p}^{\mathfrak{r}}$. But by assumption $\mathfrak{p}^{\mathfrak{r}} = 0$. Thus $\mathfrak{a} = 0$. This is a contradiction proving that $\mathfrak{g}$ is semisimple. But the Killing form is fixed by the automorphism θ. But since $\mathfrak{r}$ and $\mathfrak{p}$ are eigenspaces of θ for different eigenvalues, it follows that $\mathfrak{r}$ and $\mathfrak{p}$ are orthogonal with respect to the Killing form. □

Remark 1.63. In the special case of Theorem 1.61 where $\operatorname{rank} \mathfrak{r} = \operatorname{rank} \mathfrak{g}$, Parthasarathy computed the scalar $\nu_*(\mathrm{Cas}_{\mathfrak{r}})$ (see Theorem 1.59(3)) in [P, Lemma 2.2]. This computation plays a major role in [P]. Before [P] was written, we informed Parthasarathy that $\nu_*(\mathrm{Cas}_{\mathfrak{r}})$ is indeed a scalar in the symmetric case; however, no such attribution is made in [P] (see the remark following Lemma 2.2 in [P]).

1.64. Returning to the general case, assume $(\nu, B_{\mathfrak{g}})$ is of Lie type. Let $v \in \mathcal{V}$ so that $\mathfrak{g}$ is a Lie algebra where $[z, z'] = [z, z']^v$ (see Theorem 1.50). We generalize Lemma 2.2 in [P] (see Theorem 1.91) by showing that $\nu_*(\mathrm{Cas}_{\mathfrak{r}}) + v^2$ is the same scalar in the very general case where $\mathfrak{r}$ and $\mathfrak{g}$ are reductive. (Hence, of course, $(\mathfrak{r}, \mathfrak{g})$ is not necessarily a symmetric pair and there is no assumption about rank equality.) This generalization is used in connection with the Dirac operator having a cubic term, which is introduced in §2. We first need some computational lemmas. As in §1.31, $\{x_i\}, i = 1, \ldots, r$, is an orthonormal basis of $\mathfrak{r}$ with respect to $B_{\mathfrak{r}}$. Let $n = \dim \mathfrak{p}$ and let $\{y_j\}, j = 1, \ldots, n$, be an orthonormal basis of $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$. Taken together, $\{x_i, y_j\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $B_{\mathfrak{g}}$. Then the Casmir element $\mathrm{Cas}_{\mathfrak{g}} \in \mathcal{Z}(\mathfrak{g})$ corresponding to $B_{\mathfrak{g}}$ is given by

$$\mathrm{Cas}_{\mathfrak{g}} = \sum_{i=1}^{r} x_i^2 + \sum_{j=1}^{n} y_j^2 = \mathrm{Cas}_{\mathfrak{r}} + \sum_{j=1}^{n} y_j^2,$$

where, of course, y_j^2 is taken in $U(\mathfrak{g})$ and not in $C(\mathfrak{p})$. However, with regard to multiplication in $C(\mathfrak{p})$, note that by (1.6) and (1.34) if $w, w' \in \wedge^k \mathfrak{p}$, then

$$\big(ww'\big)_0 = \big(\alpha(w), w'\big) = (-1)^{k(k-1)/2}(w, w).$$

In particular,

$$\big(v^2\big)_0 = -(v, v), \tag{1.65}$$

and for any $u \in \wedge^2 \mathfrak{p}$,

$$\big(u^2\big)_0 = -(u, u). \tag{1.66}$$

Now the set of all elements $y_j \wedge y_k$, where $1 \le j < k \le n$, is an orthonormal basis of $\wedge^2 \mathfrak{p}$. On the other hand, since $\tau(y_j \wedge y_k) y_i = 0$ if $i \notin \{j, k\}$ (see (1.8)) and since

$$\begin{aligned} \tau\big(y_j \wedge y_k\big) y_j &= -2 y_k, \\ \tau\big(y_j \wedge y_k\big) y_k &= 2 y_j, \end{aligned}$$

it follows that $\operatorname{tr} \tau(y_j \wedge y_k)^2 = -8$. In addition, note that the elements $\{\tau(y_j \wedge y_k)\}$, $1 \le j < k \le n$, are an orthogonal basis of Lie SO($\mathfrak{p}$) with respect to the bilinear form defined as $(s, s') = \operatorname{tr} s s'$ for $s, s' \in$ Lie SO($\mathfrak{p}$). This proves the following lemma.

LEMMA 1.67. *For any $u, u' \in \wedge^2 \mathfrak{p}$, one has*

$$-\frac{1}{8} \operatorname{tr} \tau(u) \tau\big(u'\big) = (u, v). \tag{1.68}$$

But as a consequence of (1.66) and Lemma 1.67, one has, in $C(\mathfrak{p})$,

$$\Big(\big(\nu_*(x_i)\big)^2\Big)_0 = -\big(\nu_*(x_i), \nu_*(x_i)\big) = \frac{1}{8} \operatorname{tr} \nu(x_i)^2,$$

and extending ν to a homomorphism $\nu : U(\mathfrak{r}) \to \operatorname{End} \mathfrak{p}$, we have proved the next lemma.

LEMMA 1.69. *One has*

$$\big(\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big)_0 = \frac{1}{8} \operatorname{tr} \nu\big(\mathrm{Cas}_{\mathfrak{r}}\big). \tag{1.70}$$

Explicitly computing the right side of (1.70), one has

$$\begin{aligned} \big(\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big)_0 &= \frac{1}{8} \sum_{i=1,\ j=1}^{r,n} \big(\big[x_i, [x_i, y_j]\big], y_j\big) = \frac{1}{8} \sum_{i=1,\ j=1}^{r,n} \big([x_i, y_j], [y_j, x_i]\big) \\ &= \frac{1}{8} \sum_{i=1,\ j=1}^{r,n} \big(x_i, \big[y_j, [y_j, x_i]\big]\big). \end{aligned} \tag{1.71}$$

To avoid confusion, let $\mathrm{ad}_{\mathfrak{r}}$ (resp., $\mathrm{ad}_{\mathfrak{g}}$) be the adjoint representation of $\mathfrak{r}$ on itself (resp., $\mathfrak{g}$ on itself) extended also to $U(\mathfrak{r})$ (resp., $U(\mathfrak{g})$). Equation (1.71) establishes the following lemma.

LEMMA 1.72. *One has*

$$\big(\nu_*(\mathrm{Cas}_{\mathfrak{r}})\big)_0 = \frac{1}{8}\,\mathrm{tr}\, P_{\mathfrak{r}} \sum_{j=1}^{n} \mathrm{ad}_{\mathfrak{g}}\big(y_j\big)^2 P_{\mathfrak{r}} = \frac{1}{8}\,\mathrm{tr} \sum_{j=1}^{n} \mathrm{ad}_{\mathfrak{g}}\big(y_j\big)^2 P_{\mathfrak{r}}. \tag{1.73}$$

But now $([x_i, y_j], [y_j, x_i]) = \sum_{k=1}^{n}([x_i, y_j], y_k)(y_k, [y_j, x_i])$. Hence, recalling the second to last equation of (1.71), one has

$$\begin{aligned}
\big(\nu_*(\mathrm{Cas}_{\mathfrak{r}})\big)_0 &= \frac{1}{8} \sum_{i=1,\ j=1,\ k=1}^{r,n,n} \big([x_i, y_j], y_k\big)\big(y_k, [y_j, x_i]\big) \\
&= \frac{1}{8} \sum_{i=1,\ j=1,\ k=1}^{r,n,n} \big(x_i, [y_j, y_k]\big)\big([y_k, y_j], x_i\big) \\
&= \frac{1}{8} \sum_{j=1,\ k=1}^{n,n} \big([y_j, y_k]_{\mathfrak{r}}, [y_k, y_j]_{\mathfrak{r}}\big) \\
&= \frac{1}{8} \sum_{j=1,\ k=1}^{n,n} \big([y_j, y_k]_{\mathfrak{r}}, [y_k, y_j]\big) \\
&= \frac{1}{8} \sum_{j=1,\ k=1}^{n,n} \big([y_j, [y_j, y_k]_{\mathfrak{r}}], y_k\big).
\end{aligned}$$

But this proves the next lemma.

LEMMA 1.74. *One has*

$$\begin{aligned}
\big(\nu_*(\mathrm{Cas}_{\mathfrak{r}})\big)_0 &= \frac{1}{8}\,\mathrm{tr}\, P_{\mathfrak{p}} \sum_{j=1}^{n} \mathrm{ad}_{\mathfrak{g}}\big(y_j\big) P_{\mathfrak{r}}\, \mathrm{ad}_{\mathfrak{g}}\big(y_j\big) P_{\mathfrak{p}} \\
&= \frac{1}{8}\,\mathrm{tr} \sum_{j=1}^{n} \mathrm{ad}_{\mathfrak{g}}\big(y_j\big) P_{\mathfrak{r}}\, \mathrm{ad}_{\mathfrak{g}}\big(y_j\big) P_{\mathfrak{p}}.
\end{aligned} \tag{1.75}$$

The set of all elements $y_i \wedge y_j \wedge y_k$, $1 \le i < j < k \le n$, is an orthonormal basis of $\wedge^3\mathfrak{p}$. But then by (1.65)

$$\begin{aligned}
(v^2)_0 = -(v, v) &= -\sum_{1\le i<j<k\le n} \big(v, y_i \wedge y_j \wedge y_k\big)^2 \\
&= -\frac{1}{6} \sum_{i=1,\ j=1,\ k=1}^{n,n,n} \big(v, y_i \wedge y_j \wedge y_k\big)^2
\end{aligned}$$

$$= -\frac{1}{6}\sum_{i=1,\ j=1,\ k=1}^{n,n,n}\big(\iota(y_k)\iota\big(y_j\big)v,\, y_i\big)^2$$
$$= -\frac{1}{24}\sum_{i=1,\ j=1,\ k=1}^{n,n,n}\big(2\iota(y_k)\iota\big(y_j\big)v,\, y_i\big)^2$$
$$= -\frac{1}{24}\sum_{i=1,\ j=1,\ k=1}^{n,n,n}\big(\big[y_k, y_j\big]_{\mathfrak{p}},\, y_i\big)^2$$
$$= -\frac{1}{24}\sum_{j=1,\ k=1}^{n,n}\big(\big[y_k, y_j\big]_{\mathfrak{p}},\big[y_k, y_j\big]_{\mathfrak{p}}\big)$$
$$= -\frac{1}{24}\sum_{j=1,k=1}^{n,n}\big(\big[y_k, y_j\big]_{\mathfrak{p}},\big[y_k, y_j\big]\big)$$
$$= \frac{1}{24}\sum_{j=1,\ k=1}^{n,n}\big(\big[y_j,\big[y_j, y_k\big]_{\mathfrak{p}}\big],\, y_k\big).$$

But this proves the following lemma.

LEMMA 1.76. *One has*

$$\begin{aligned}\big(v^2\big)_0 &= \frac{1}{24}\operatorname{tr} P_{\mathfrak{p}}\sum_{j=1}^{n}\operatorname{ad}_{\mathfrak{g}}\big(y_j\big)P_{\mathfrak{p}}\operatorname{ad}_{\mathfrak{g}}\big(y_j\big)P_{\mathfrak{p}} \\ &= \frac{1}{24}\operatorname{tr}\sum_{j=1}^{n}\operatorname{ad}_{\mathfrak{g}}\big(y_j\big)P_{\mathfrak{p}}\operatorname{ad}_{\mathfrak{g}}\big(y_j\big)P_{\mathfrak{p}}.\end{aligned} \tag{1.77}$$

But now, adding $1/3$ of (1.75) to (1.77), one has

$$\frac{1}{3}\big(\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big)_0 + \big(v^2\big)_0 = \frac{1}{24}\operatorname{tr}\sum_{j=1}^{n}\operatorname{ad}_{\mathfrak{g}}\big(y_j\big)^2 P_{\mathfrak{p}}. \tag{1.78}$$

Then adding $1/3$ of (1.73) to (1.78), one has

$$\frac{2}{3}\big(\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big)_0 + \big(v^2\big)_0 = \frac{1}{24}\operatorname{tr}\sum_{j=1}^{n}\operatorname{ad}_{\mathfrak{g}}\big(y_j\big)^2. \tag{1.79}$$

But if we add and subtract $(1/8)\operatorname{tr}\operatorname{ad}_{\mathfrak{r}}(\mathrm{Cas}_{\mathfrak{r}})$ to the right side of (1.70), one has

$$\big(\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big)_0 = \frac{1}{8}\big(\operatorname{tr}\operatorname{ad}_{\mathfrak{g}}\big(\mathrm{Cas}_{\mathfrak{r}}\big) - \operatorname{tr}\operatorname{ad}_{\mathfrak{r}}\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big). \tag{1.80}$$

We can now determine the constant $\nu_*(\mathrm{Cas}_{\mathfrak{r}}) + v^2$, expressing it in terms of traces of Casimir elements.

Theorem 1.81. *Assume that $(\nu, B_{\mathfrak{g}})$ is of Lie type. Let $v \in \mathcal{V}$ so that $\mathfrak{g}$ is a Lie algebra where $[z,z'] = [z,z']^v$. Then the constant (see Theorem 1.11) $\nu_*(\mathrm{Cas}_{\mathfrak{r}}) + v^2$ is given by*

$$\nu_*\big(\mathrm{Cas}_{\mathfrak{r}}\big) + v^2 = \frac{1}{24}\big(\operatorname{tr}\operatorname{ad}_{\mathfrak{g}}\big(\mathrm{Cas}_{\mathfrak{g}}\big) - \operatorname{tr}\operatorname{ad}_{\mathfrak{r}}\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big). \tag{1.82}$$

Proof. Obviously,

$$\operatorname{ad}_{\mathfrak{g}}\big(\mathrm{Cas}_{\mathfrak{r}}\big) + \sum_{j=1}^{n}\big(\operatorname{ad}_{\mathfrak{g}} y_j\big)^2 = \operatorname{ad}_{\mathfrak{g}}\big(\mathrm{Cas}_{\mathfrak{g}}\big).$$

But then, recalling (1.51), equation (1.82) follows by adding 1/3 of (1.80) to (1.79). □

1.83. In the case when both $\mathfrak{r}$ and $\mathfrak{g}$ are reductive, Theorem 1.81 simplifies even further. Assume that $\mathfrak{u}$ is a complex finite-dimensional reductive Lie algebra and (z,z') is a nonsingular symmetric ad $\mathfrak{u}$-invariant bilinear form $B_{\mathfrak{u}}$ on $\mathfrak{u}$. Let $\mathrm{Cas}_{\mathfrak{u}} \in \mathfrak{Z}(\mathfrak{u})$ be the Casimir element corresponding to $B_{\mathfrak{u}}$. Let $\mathfrak{h}_{\mathfrak{u}}$ be a Cartan subalgebra of $\mathfrak{u}$. Then, clearly, $B_{\mathfrak{u}} \mid \mathfrak{h}_{\mathfrak{u}}$ is nonsingular, and this restriction induces a nonsingular symmetric bilinear form (γ, δ) on the dual space $\mathfrak{h}_{\mathfrak{u}}^*$ to $\mathfrak{h}_{\mathfrak{u}}$. Let $\Delta_{\mathfrak{u}}^+ \subset \mathfrak{h}_{\mathfrak{u}}^*$ be a choice of a full set of positive roots for the action of $\operatorname{ad}\mathfrak{h}_{\mathfrak{u}}$ on $\mathfrak{u}$. Let $\rho_{\mathfrak{u}} = (1/2)\sum_{\beta\in\Delta_{\mathfrak{u}}^+}\beta$. Obviously, the number $(\rho_{\mathfrak{u}}, \rho_{\mathfrak{u}})$ is independent of the choice of $\mathfrak{h}_{\mathfrak{u}}$ and the choice of $\Delta_{\mathfrak{u}}^+$. The following result is just a more general formulation of the formula of Freudenthal and de Vries.

Proposition 1.84. *With the notation above, one has*

$$\frac{1}{24}\operatorname{tr}\operatorname{ad}\big(\mathrm{Cas}_{\mathfrak{u}}\big) = \big(\rho_{\mathfrak{u}}, \rho_{\mathfrak{u}}\big). \tag{1.85}$$

Proof. First of all, one notes that the decomposition of $\mathfrak{u}$ as a direct sum of its center and the simple components of $[\mathfrak{u},\mathfrak{u}]$ is necessarily a $B_{\mathfrak{u}}$-orthogonal decomposition (since there are no nontrivial intertwining operators for the summands of this decomposition). Furthermore, $\mathfrak{h}_{\mathfrak{u}}$ is a direct sum of its intersection with these summands. It therefore follows that both sides of (1.85) decompose accordingly. The center of $\mathfrak{u}$ cancels out on both sides of (1.85), and hence it suffices to prove (1.85) in the case where $\mathfrak{u}$ is simple. Let $(z,z')'$ be the Killing form $B_{\mathfrak{u}}'$ on $\mathfrak{u}$. Then there exists a nonzero constant c so that $(z,z') = c(z,z')'$ for any $z,z' \in \mathfrak{u}$. If $\{z_i\}$ is an orthonormal basis of $\mathfrak{u}$ with respect to $B_{\mathfrak{u}}'$, then $\{z_i/d\}$ is an orthonormal basis of $\mathfrak{u}$ with respect to $B_{\mathfrak{u}}$, where d is chosen so that $d^2 = c$. Thus if $\mathrm{Cas}_{\mathfrak{u}}'$ is the Casimir element with respect to $B_{\mathfrak{u}}'$, then

$$\mathrm{Cas}_{\mathfrak{u}} = \frac{1}{c}\,\mathrm{Cas}_{\mathfrak{u}}'. \tag{1.86}$$

But clearly, $\operatorname{tr}\operatorname{ad}(\mathrm{Cas}'_{\mathfrak{u}}) = \dim \mathfrak{u}$. Thus

$$\frac{1}{24}\operatorname{tr}\operatorname{ad}\left(\mathrm{Cas}_{\mathfrak{u}}\right) = \frac{1}{c}\frac{\dim \mathfrak{u}}{24}. \tag{1.87}$$

On the other hand, if $(\gamma, \delta)'$ is the bilinear form on $\mathfrak{h}_{\mathfrak{u}}^*$ induced by $B'_{\mathfrak{u}} \mid \mathfrak{h}_{\mathfrak{u}}$, then one easily has that

$$(\gamma, \delta) = \frac{1}{c}(\gamma, \delta)'. \tag{1.88}$$

Thus $(\rho_{\mathfrak{u}}, \rho_{\mathfrak{u}}) = (1/c)(\rho_{\mathfrak{u}}, \rho_{\mathfrak{u}})'$. But $(\rho_{\mathfrak{u}}, \rho_{\mathfrak{u}})' = (1/24)\dim \mathfrak{u}$ is the classic formula of Freudenthal and de Vries (see, e.g., [FD, p. 243]). Thus (1.85) follows from (1.87). □

Remark 1.89. Retain the notation in the proof of Proposition 1.84. Assume that $\eta_\delta : \mathfrak{u} \to \operatorname{End} Y_\delta$ is an irreducible representation with highest (relative to $\Delta_{\mathfrak{u}}^+$) weight $\delta \in \mathfrak{h}_{\mathfrak{u}}^*$. Let I be the identity operator on Y_δ. It is a familiar fact that if $\mathfrak{u}$ is semisimple, then

$$\eta_\delta\left(\mathrm{Cas}'_{\mathfrak{u}}\right) = \left((\delta + \rho_{\mathfrak{u}}, \delta + \rho_{\mathfrak{u}})' - \left(\rho_{\mathfrak{u}}, \rho_{\mathfrak{u}}\right)'\right) I.$$

We wish to point out that the same formula is valid in the reductive case for the arbitrary nonsingular, ad $\mathfrak{u}$-invariant, bilinear symmetric form $B_{\mathfrak{u}}$. That is,

$$\eta_\delta\left(\mathrm{Cas}_{\mathfrak{u}}\right) = \left((\delta + \rho_{\mathfrak{u}}, \delta + \rho_{\mathfrak{u}}) - \left(\rho_{\mathfrak{u}}, \rho_{\mathfrak{u}}\right)\right) I. \tag{1.90}$$

Indeed, as in the proof of Proposition 1.84, one is reduced to the summands. If $\mathfrak{u}$ is abelian, then (1.90) is obvious. If $\mathfrak{u}$ is simple, the result follows from (1.87) and (1.88).

In the reductive cases (for $\mathfrak{g}$ and $\mathfrak{r}$), Theorem 1.81 becomes the following.

THEOREM 1.91. *Assume that $(\nu, B_{\mathfrak{g}})$ is of Lie type. Let $v \in \mathscr{V}$ so that $\mathfrak{g}$ is a Lie algebra where $[z, z'] = [z, z']^v$. Assume both $\mathfrak{r}$ and $\mathfrak{g}$ are reductive Lie algebras. Then the constant (see Theorem 1.50) $\nu_*(\mathrm{Cas}_{\mathfrak{r}}) + v^2$ is given by*

$$\nu_*\left(\mathrm{Cas}_{\mathfrak{r}}\right) + v^2 = (\rho, \rho) - \left(\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}}\right), \tag{1.92}$$

where, using the notation of Proposition 1.84, ρ and (ρ, ρ) are defined as in Proposition 1.84 for the case where $\mathfrak{u} = \mathfrak{g}$ and $B_{\mathfrak{u}} = B_{\mathfrak{g}}$ and where we have written $\rho = \rho_{\mathfrak{g}}$. Similarly, $\rho_{\mathfrak{r}}$ and $(\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}})$ are defined as in Proposition 1.84 for the case where $\mathfrak{u} = \mathfrak{r}$ and $B_{\mathfrak{u}} = B_{\mathfrak{r}}$.

Proof. This is immediate from Theorem 1.81 and Proposition 1.84. □

1.93. Our main interest in this paper (assumed in §3 and all later sections) is in the case where $\mathfrak{r}$ is a reductive equal rank Lie subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Without assuming the symmetric pair condition, we can deduce (assuming that $(\nu, B_{\mathfrak{g}})$ is of Lie type) that $\mathfrak{g}$ is semisimple and $\mathfrak{r}$ is a reductive equal rank Lie subalgebra if we strengthen the assumptions of Theorem 1.61. We say that ν is normal if ν is the complexified differential of a faithful $B_{\mathfrak{p}}$-orthogonal representation of a compact Lie group. If ν is normal, then not only is $\mathfrak{r}$ reductive but ν is completely reducible. In addition, the normality of ν and, in particular, its faithfulness guarantee that we can take $B_{\mathfrak{r}}$ so that

$$\left(x, x'\right) = \operatorname{tr} \operatorname{ad}_{\mathfrak{r}}(x) \operatorname{ad}_{\mathfrak{r}}\left(x'\right) + \operatorname{tr} \nu(x)\nu\left(x'\right). \tag{1.94}$$

If ν is normal and $B_{\mathfrak{r}}$ is given by (1.94), we say that $B_{\mathfrak{r}}$ is normalized.

Remark 1.95. The first term on the right side of (1.94) is just the Killing form of $\mathfrak{r}$. One further notes that if $(\nu, B_{\mathfrak{g}})$ is of Lie type, ν is normal, and $B_{\mathfrak{r}}$ is normalized, then $B_{\mathfrak{r}}$ is just the restriction to $\mathfrak{r}$ of the Killing form of $\mathfrak{g}$.

THEOREM 1.96. *Assume that $(\nu, B_{\mathfrak{g}})$ is of Lie type. Let $v \in \mathcal{V}$ so that $\mathfrak{g}$ is a Lie algebra where $[z, z'] = [z, z']^{v}$. Now assume that* (1) *ν is normal and $B_{\mathfrak{r}}$ is normalized. Furthermore, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{r}$, assume that* (2) $\mathfrak{p}^{\mathfrak{h}} = 0$ *(i.e.,* 0 *is not a weight of ν). Then $\mathfrak{g}$ is a semisimple Lie algebra and $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. Moreover, $\mathfrak{r}$ is a reductive equal rank subalgebra of $\mathfrak{g}$ so that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Finally, v is unique up to equivalence (see (1.54)); that is, $v \sim v'$ for any $v' \in \mathcal{V}$.*

Proof. Let $B'_{\mathfrak{g}}$ be the Killing form for $\mathfrak{g}$ so that

$$B_{\mathfrak{g}} \mid \mathfrak{r} = B'_{\mathfrak{g}} \mid \mathfrak{r}. \tag{1.97}$$

Let $\Gamma(\mathfrak{p}) \subset \mathfrak{h}^*$ be the set of weights for ν. The condition that the zero weight does not occur in $\mathfrak{p}$ as an $\mathfrak{r}$-module implies, as one knows from the representation theory of reductive Lie algebras, that $\Gamma(\mathfrak{p})$ has no intersection with the root lattice of $\mathfrak{r}$. Thus $\operatorname{Hom}_{\mathfrak{h}}(\mathfrak{r}, \mathfrak{p}) = 0$. This immediately implies that $\mathfrak{r}$ is $B'_{\mathfrak{g}}$-orthogonal to $\mathfrak{p}$. But then if $B''_{\mathfrak{g}} = B_{\mathfrak{g}} - B'_{\mathfrak{g}}$, it follows that $B''_{\mathfrak{g}}$ is an ad $\mathfrak{g}$-invariant symmetric bilinear form $(z, z')''$ on $\mathfrak{g}$ and that $\mathfrak{r} \subset \mathfrak{v}$ where $\mathfrak{v} = \{z \in \mathfrak{g} \mid (z, \mathfrak{g})'' = 0\}$. Let $\mathfrak{s} = \mathfrak{r} \cap \mathfrak{v}$ so that $\mathfrak{s}$ is an $\mathfrak{r}$-submodule of $\mathfrak{p}$, and by complete reducibility, let $\mathfrak{q} \subset \mathfrak{p}$ be a complementary $\mathfrak{r}$-submodule. But since $\mathfrak{r} \subset \mathfrak{v}$ and $\mathfrak{v}$ is an ideal of $\mathfrak{g}$, it follows that $[r, \mathfrak{q}] \subset \mathfrak{q} \cap \mathfrak{s} = 0$. In particular, $\mathfrak{q}^{\mathfrak{h}} = 0$. Thus $\mathfrak{q} = 0$ since $\mathfrak{p}^{\mathfrak{h}} = 0$. Hence $\mathfrak{v} = \mathfrak{g}$ so that $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. But then $\mathfrak{g}$ is semisimple by the nonsingularity of the Killing form. Now since $\mathfrak{p}^{\mathfrak{h}} = 0$, it follows also that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. If $\Delta \subset \mathfrak{h}^*$ (resp., $\Delta_{\mathfrak{r}} \subset \mathfrak{h}^*$) is the set of roots of $\mathfrak{g}$ (resp., $\mathfrak{r}$) with respect to $\mathfrak{h}$, one of course has that

$$\Delta = \Delta_{\mathfrak{r}} \cup \Gamma(\mathfrak{p}). \tag{1.98}$$

Now let $v' \in \mathcal{V}$, and let $\mathfrak{g}'$ be the Lie algebra with underlying space $\mathfrak{g}$ and with bracket structure $[z,z']' = [z,z']^{v'}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'$, and by (1.98), both $\mathfrak{g}$ and $\mathfrak{g}'$ are semisimple and have the same set of roots. Thus there exists a Lie algebra isomorphism $C : \mathfrak{g} \to \mathfrak{g}'$ that reduces to the identity on $\mathfrak{h}$. Clearly, C must stabilize $\mathfrak{g}_\varphi$, where $\mathfrak{g}_\varphi$ is the 1-dimensional eigenspace for the weight $\varphi \in \Delta$. But then (see (1.98)) $\mathfrak{r}$ and $\mathfrak{p}$ are stable under C. However, $\mathfrak{r}$ is a Lie subalgebra of both $\mathfrak{g}$ and $\mathfrak{g}'$. Thus $C \mid \mathfrak{r}$ is an automorphism. But since $C \mid \mathfrak{h}$ is the identity, one knows that $C \mid \mathfrak{r}$ is an inner automorphism. Thus there exists $b \in \exp \mathrm{ad}_{\mathfrak{r}} \mathfrak{r}$ such that $C \mid \mathfrak{r} = b$. But then if $G_{\mathbb{C}}$ is the simply connected Lie group corresponding to $\mathfrak{g}$ and $R_{\mathbb{C}} \subset G_{\mathbb{C}}$ is the subgroup corresponding to $\mathfrak{r}$, it follows that there exists $c \in R_{\mathbb{C}}$ such that $\mathrm{Ad}_{\mathfrak{g}} c \mid \mathfrak{r} = b$. But then if $A = C \circ (\mathrm{Ad}_{\mathfrak{g}} c)^{-1}$, it follows that $A : \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism such that $A \mid \mathfrak{r}$ is the identity. But then $a \in F$ where $a = A \mid \mathfrak{p}$. But then for any $y, y' \in \mathfrak{p}$, $a(P_{\mathfrak{p}}[a^{-1}y, a^{-1}y']^v) = P_{\mathfrak{p}}([y,y']^{v'})$. However,

$$a\big(P_{\mathfrak{p}}\big[a^{-1}y, a^{-1}y'\big]^v\big) = 2a\big(\iota\big(a^{-1}y\big)\iota\big(a^{-1}y'\big)v\big) = 2\iota(y)\iota\big(y'\big)\Theta(a)v$$

and

$$P_{\mathfrak{p}}\Big(\big[y,y'\big]^{v'}\Big) = 2\iota(y)\iota\big(y'\big)v'.$$

Hence $\Theta(a)v = v'$. $\square$

2. The cubic Dirac operator $\square$

2.1. Assume that $(\nu, B_{\mathfrak{g}})$ is of Lie type. Let $v \in \mathcal{V}$ so that $\mathfrak{g}$ is a Lie algebra where $[z,z'] = [z,z']^v$. Let $\mathcal{A}$ be the algebra

$$A = U(\mathfrak{g}) \otimes C(\mathfrak{p}). \tag{2.2}$$

To avoid confusion when multiplying elements of $\mathfrak{p}$, the injection map $\mathfrak{g} \to U(\mathfrak{g})$ is tautologically denoted by ξ. Its extension $\xi : U(\mathfrak{g}) \to U(\mathfrak{g})$, of course, is just the identity map. Multiplication of elements in $\mathfrak{p}$ without ξ is multiplication in the Clifford algebra $C(\mathfrak{p})$. Now let $\square' \in \mathcal{A}$ be defined by putting

$$\square' = \sum_{i=1}^{n} \xi\big(y_j\big) \otimes y_j. \tag{2.3}$$

It is clear that $\square'$ is independent of the orthonormal basis $\{y_j\}$ of $\mathfrak{p}$. Now

$$\begin{aligned}\big(\square'\big)^2 &= \sum_{j,k=1}^{n} \xi\big(y_j\big)\xi\big(y_k\big) \otimes y_j y_k = \sum_{q=1}^{n} \xi(y_q)^2 \otimes 1 + \sum_{j<k}^{n} \xi\big(\big[y_j, y_k\big]\big) \otimes y_j y_k \\ &= \sum_{q=1}^{n} \xi(y_q)^2 \otimes 1 + \sum_{j<k}^{n} \xi\big(\big[y_j, y_k\big]_{\mathfrak{r}}\big) \otimes y_j y_k + \sum_{j<k}^{n} \xi\big(P_{\mathfrak{p}}\big(\big[y_j, y_k\big]\big)\big) \otimes y_j y_k.\end{aligned} \tag{2.4}$$

Write the three sums in the last line of (2.4) as I, II, and III, respectively, so that

$$(\square')^2 = I + II + III. \tag{2.5}$$

Now if $j \neq k$, then $y_j y_k = y_j \wedge y_k$ and, over all $j < k$, define an orthonormal basis of $\wedge^2\mathfrak{p}$. Recall that $\{x_i\}, i = 1, \ldots, r$, is an orthonormal basis of $\mathfrak{r}$. One therefore has, by (1.13),

$$\begin{aligned}
II &= \sum_{j<k}^{n} \xi\big([y_j, y_k]_{\mathfrak{r}}\big) \otimes y_j \wedge y_k = \sum_{i=1}^{r}\sum_{j<k}^{n} \big([y_j, y_k]_{\mathfrak{r}}, x_i\big)\xi(x_i) \otimes y_j \wedge y_k \\
&= -2\sum_{i=1}^{r}\sum_{j<k}^{n} \big(\nu_*^t(y_j \wedge y_k), x_i\big)\xi(x_i) \otimes y_j \wedge y_k \\
&= -2\sum_{i=1}^{r}\sum_{j<k}^{n} \big(y_j \wedge y_k, \nu_*(x_i)\big)\xi(x_i) \otimes y_j \wedge y_k \\
&= -2\sum_{i=1}^{r} \xi(x_i) \otimes \nu_*(x_i).
\end{aligned} \tag{2.6}$$

On the other hand, if $\zeta : \mathfrak{r} \to U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is the tensor (diagonal) product Lie algebra homomorphism defined by ξ and ν_*, then

$$\zeta\big(\mathrm{Cas}_{\mathfrak{r}}\big) = \sum_{i=1}^{r} \big(\xi(x_i) \otimes 1 + 1 \otimes \nu_*(x_i)\big)^2.$$

Hence

$$\zeta\big(\mathrm{Cas}_{\mathfrak{r}}\big) = \sum_{i=1}^{r} \xi(x_i)^2 \otimes 1 + 2\sum_{i=1}^{r} \xi(x_i) \otimes \nu_*(x_i) + \sum_{j=1}^{r} 1 \otimes \nu_*(x_i)^2.$$

It follows therefore from (2.5) and (2.6) that

$$I + II + \zeta\big(\mathrm{Cas}_{\mathfrak{r}}\big) = \xi\big(\mathrm{Cas}_{\mathfrak{g}}\big) \otimes 1 + \sum_{i=1}^{r} 1 \otimes \nu_*(x_i)^2. \tag{2.7}$$

Now we introduce a cubic term. Let

$$\square'' = 1 \otimes v. \tag{2.8}$$

By Theorem 1.81,

$$(\square'')^2 + \sum_{i=1}^{r} 1 \otimes \nu_*(x_i)^2 = \frac{1}{24}\big(\mathrm{tr\,ad}_{\mathfrak{g}}\big(\mathrm{Cas}_{\mathfrak{g}}\big) - \mathrm{tr\,ad}_{\mathfrak{r}}\big(\mathrm{Cas}_{\mathfrak{r}}\big)\big)(1 \otimes 1).$$

For notational convenience, in the following calculation put

$$c = \frac{1}{24}\left(\operatorname{tr}\operatorname{ad}_{\mathfrak{g}}\left(\operatorname{Cas}_{\mathfrak{g}}\right) - \operatorname{tr}\operatorname{ad}_{\mathfrak{r}}\left(\operatorname{Cas}_{\mathfrak{r}}\right)\right). \tag{2.9}$$

Thus by (2.7),

$$I + II + \zeta\left(\operatorname{Cas}_{\mathfrak{r}}\right) + \left(\square''\right)^2 = \xi\left(\operatorname{Cas}_{\mathfrak{g}}\right)\otimes 1 + c1\otimes 1;$$

that is,

$$\left(\square'\right)^2 + \left(\square''\right)^2 - III = \xi\left(\operatorname{Cas}_{\mathfrak{g}}\right)\otimes 1 - \zeta\left(\operatorname{Cas}_{\mathfrak{r}}\right) + c1\otimes 1. \tag{2.10}$$

Recall that $\{y_i \wedge y_k\}$, for all $i < k$, is an orthonormal basis of $\wedge^2\mathfrak{p}$. But then by Remark 1.9,

$$\begin{aligned}
\square'\square'' + \square''\square' &= \sum_{j=1}^{n} \xi\left(y_j\right)\otimes\left(y_j v + v y_j\right) = 2\sum_{j=1} \xi\left(y_j\right)\otimes\iota\left(y_j\right)v \\
&= 2\sum_{i<k}^{n}\sum_{j=1}^{n} \xi\left(y_j\right)\otimes\left(\iota\left(y_j\right)v,\, y_i\wedge y_k\right)y_i\wedge y_k \\
&= 2\sum_{i<k}^{n}\sum_{j=1}^{n} \xi\left(y_j\right)\otimes\left(v,\, y_j\wedge y_i\wedge y_k\right)y_i\wedge y_k \\
&= \sum_{i<k}^{n}\sum_{j=1}^{n} \xi\left(y_j\right)\otimes\left(2\iota\left(y_k\right)\iota\left(y_i\right)v,\, y_j\right)y_i\wedge y_k \\
&= \text{(by (1.22))} \sum_{i<k}^{n}\sum_{j=1}^{n} \xi\left(y_j\right)\otimes\left(P_{\mathfrak{p}}\left[y_k, y_i\right],\, y_j\right)y_i\wedge y_k \\
&= \sum_{i<k}^{n} \xi\left(P_{\mathfrak{p}}\left[y_k, y_i\right]\right)\otimes y_i\wedge y_k = -III.
\end{aligned} \tag{2.11}$$

Thus if

$$\square = \square' + \square'', \tag{2.12}$$

then (2.9), (2.10), and (2.11) prove the following result expressing the square of the Dirac operator $\square$, under very general circumstances, in terms of Casimir elements.

THEOREM 2.13. *Assume that $(v, B_{\mathfrak{g}})$ is of Lie type. Let $v \in \mathcal{V}$ so that $\mathfrak{g}$ is a Lie algebra where $[z, z'] = [z, z']^v$. Let $\mathcal{A}$ be the algebra $\mathcal{A} = U(\mathfrak{g})\otimes C(\mathfrak{p})$. Let $\square \in \mathcal{A}$ be defined by (2.3), (2.8), and (2.12). Then*

$$\square^2 = \xi\left(\operatorname{Cas}_{\mathfrak{g}}\right)\otimes 1 - \zeta\left(\operatorname{Cas}_{\mathfrak{r}}\right) + \frac{1}{24}\left(\operatorname{tr}\operatorname{ad}_{\mathfrak{g}}\left(\operatorname{Cas}_{\mathfrak{g}}\right) - \operatorname{tr}\operatorname{ad}_{\mathfrak{r}}\left(\operatorname{Cas}_{\mathfrak{r}}\right)\right)(1\otimes 1). \tag{2.14}$$

2.15. We now specialize to the case where $\mathfrak{g}$ and $\mathfrak{r}$ are reductive. Recalling Theorem 1.91 and (1.82), we have proved the following theorem.

THEOREM 2.16. *Assume that $(\nu, B_{\mathfrak{g}})$ is of Lie type. Let $v \in \mathcal{V}$ so that $\mathfrak{g}$ is a Lie algebra where $[z, z'] = [z, z']^v$. Now assume, in addition, that $\mathfrak{g}$ and $\mathfrak{r}$ are reductive Lie algebras. Then*

$$\Box^2 = \xi\big(\mathrm{Cas}_{\mathfrak{g}}\big) \otimes 1 - \zeta\big(\mathrm{Cas}_{\mathfrak{r}}\big) + \big((\rho, \rho) - \big(\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}}\big)\big)(1 \otimes 1), \tag{2.17}$$

where the scalars in (2.17) are defined in the paragraph preceding Proposition 1.84.

Let the notation and assumptions be as in Theorem 2.16. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Delta^+ \subset \mathfrak{h}^*$ be a choice of a positive root system. We may then take $\rho = (1/2)\sum_{\varphi \in \Delta^+} \varphi$. Let $\lambda \in \mathfrak{h}^*$ be the highest weight (relative to Δ^+) of an irreducible finite-dimensional representation $\pi_\lambda : \mathfrak{g} \to \mathrm{End}\, V_\lambda$. Let

$$\mathcal{A}_\lambda = \mathrm{End}\, V_\lambda \otimes C(\mathfrak{p}). \tag{2.18}$$

Clearly, $\pi_\lambda \otimes 1 : \mathcal{A} \to \mathcal{A}_\lambda$ is an epimorphism of algebras. Let

$$\Box_\lambda = \big(\pi_\lambda \otimes 1\big)(\Box). \tag{2.19}$$

Let $\zeta_\lambda = (\pi_\lambda \otimes 1) \circ \zeta$ so that $\zeta_\lambda : \mathfrak{r} \to \mathcal{A}_\lambda$ is a Lie algebra homomorphism, where explicitly

$$\zeta_\lambda(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes \nu_*(x) \tag{2.20}$$

for any $x \in \mathfrak{r}$. Again, by abuse of notation, denote the identity element of $\mathcal{A}_\lambda$ by $1 \otimes 1$.

THEOREM 2.21. *Let the notation and assumptions be as in Theorem 2.2. Then*

$$\big(\Box_\lambda\big)^2 = \big((\lambda + \rho, \lambda + \rho) - \big(\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}}\big)\big)(1 \otimes 1) - \zeta_\lambda\big(\mathrm{Cas}_{\mathfrak{r}}\big), \tag{2.22}$$

where the scalars in (2.22) are defined in the paragraph preceding Proposition 1.84.

Proof. By (1.90) one has $\pi_\lambda(\mathrm{Cas}_{\mathfrak{g}}) = ((\lambda + \rho, \lambda + \rho) - (\rho, \rho))I$; here I is the identity operator on V_λ. The equation (2.22) then follows from Theorem 2.16. $\Box$

Remark 2.23. Originally, I introduced the operator $\Box_\lambda$ and established (2.22) in the case where $\mathfrak{r}$ is a reductive equal rank reductive Lie subalgebra of a semisimple Lie algebra. It was used to solve a problem that arose from the results in the paper [GKRS] and from the multiplicity-1 statement of Theorem 0.19 in the present paper. The problem (using notation in [GKRS] and also later in this paper) was to find a Dirac-type operator $V_\lambda \otimes S_+ \to V_\lambda \otimes S_-$ that would have, as kernel, the span of the positive multiplets and, as cokernel, the span of the negative multiplets. The successful use of (2.22) to accomplish this was the first application of my thirty-year-old result, Theorem 1.50. My conversation with Anton Alekseev about a joint work of his and

Eckhard Meinreinken [AM] made it apparent that they had independently obtained (2.17) in the very special case where $\mathfrak{r}=0$, $\mathfrak{p}=\mathfrak{g}$ is semisimple, and $v\in\wedge^3\mathfrak{g}$ is given by the classic 3-form $\phi(z,z',z'')=(z,[z',z''])$. What was illuminating about this for me was that zero is not an equal rank subalgebra of $\mathfrak{g}$ (assuming $\mathfrak{g}\neq 0$). Subsequently upon examining my proof of (2.17), I realized that I never needed to assume $\mathfrak{r}$ had the same rank as $\mathfrak{g}$. In particular, the pair $(0,\mathfrak{g})$ is a special case of $(\mathfrak{r},\mathfrak{g})$ in Theorems 2.16 and 2.21.

3. Tensoring with the spin representation and the emergence of d-multiplets

3.1. Henceforth, assume that $\mathfrak{g}$ is a complex semisimple Lie algebra. Let $l=\operatorname{rank}\mathfrak{g}$. Let $\mathfrak{r}$ be a reductive Lie subalgebra of $\mathfrak{g}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{r}$. Assume $\operatorname{rank}\mathfrak{r}=\operatorname{rank}\mathfrak{g}$ so that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$ and $\dim\mathfrak{h}=l$. Let $\mathfrak{h}^*$ be the dual space to $\mathfrak{h}$, and let $\Delta\subset\mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\Delta_{\mathfrak{r}}\subset\Delta$ be the set of roots of $\mathfrak{r}$ with respect to $\mathfrak{h}$. Let (z,z') for $z,z'\in\mathfrak{g}$ be the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$, and let $B_{\mathfrak{r}}=B_{\mathfrak{g}}\mid\mathfrak{r}$. For any $\varphi\in\Delta$, let $e_\varphi\in\mathfrak{g}$ be a corresponding root vector where the choice is normalized so that

$$(e_\varphi,e_{-\varphi})=1 \tag{3.2}$$

for any $\varphi\in\Delta$. Let $\mathfrak{h}^*_{\mathbb{R}}$ be the real span of Δ so that $\mathfrak{h}^*_{\mathbb{R}}$ is a real form of $\mathfrak{h}^*$. The restriction $B_{\mathfrak{g}}\mid\mathfrak{h}$ is nonsingular and induces a nonsingular symmetric bilinear form (δ,γ) on $\mathfrak{h}^*$, which is positive-definite on $\mathfrak{h}^*_{\mathbb{R}}$. Let $\Gamma\subset\mathfrak{h}^*_{\mathbb{R}}$ be the lattice of $\mathfrak{g}$-integral linear forms on $\mathfrak{h}$; that is, $\Gamma=\{\mu\in\mathfrak{h}^*\mid 2(\mu,\varphi)/(\varphi,\varphi)\in\mathbb{Z},\forall\varphi\in\Delta\}$. For any subspace $\mathfrak{m}\subset\mathfrak{g}$ that is stable under $\operatorname{ad}\mathfrak{h}$, let $\Delta(\mathfrak{m})=\{\varphi\in\Delta\mid e_\varphi\in\mathfrak{m}\}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$. Then $\mathfrak{b}_{\mathfrak{r}}=\mathfrak{b}\cap\mathfrak{r}$ is a Borel subalgebra of $\mathfrak{r}$. Positive root systems for $\mathfrak{g}$ and $\mathfrak{r}$ can then be given by letting $\Delta^+=\Delta(\mathfrak{b})$ and $\Delta^+_{\mathfrak{r}}=\Delta(\mathfrak{b}_{\mathfrak{r}})$, respectively. Let $\rho=(1/2)\sum_{\varphi\in\Delta^+}\varphi$ and $\rho_{\mathfrak{r}}=(1/2)\sum_{\varphi\in\Delta^+_{\mathfrak{r}}}\varphi$. One has $\rho\in\Gamma$, but $\rho_{\mathfrak{r}}$ is not necessarily in Γ. Of course, $2\rho_{\mathfrak{r}}\in\Gamma$. Let $\Gamma_{\mathfrak{r}}\subset\mathfrak{h}^*_{\mathbb{R}}$ be the lattice generated by Γ and $\rho_{\mathfrak{r}}$. One, of course, has

$$\frac{2(\mu,\varphi)}{(\varphi,\varphi)}\in\mathbb{Z},\quad \forall\varphi\in\Delta_{\mathfrak{r}}\quad\text{and}\quad\forall\mu\in\Gamma_{\mathfrak{r}}.$$

Let D be the $\mathfrak{g}$-Weyl chamber in $\mathfrak{h}^*_{\mathbb{R}}$ defined by $D=\{\gamma\in\mathfrak{h}^*_{\mathbb{R}}\mid(\gamma,\varphi)\geq 0,\forall\varphi\in\Delta^+\}$, and let $D_{\mathfrak{r}}$ be the $\mathfrak{r}$-Weyl chamber in $\mathfrak{h}^*$ defined by $D_{\mathfrak{r}}=\{\gamma\in\mathfrak{h}^*_{\mathbb{R}}\mid(\gamma,\varphi)\geq 0,\forall\varphi\in\Delta^+_{\mathfrak{r}}\}$. Now let $\Lambda=D\cap\Gamma$, and let $\Lambda_{\mathfrak{r}}=D_{\mathfrak{r}}\cap\Gamma_{\mathfrak{r}}$. For each $\lambda\in\Lambda$, let

$$\pi_\lambda:\mathfrak{g}\longrightarrow\operatorname{End}V_\lambda$$

be an irreducible representation with highest weight λ, and for any $\mu\in\Lambda_{\mathfrak{r}}$, let

$$\sigma_\mu:\mathfrak{r}\longrightarrow\operatorname{End}Z_\mu$$

be an irreducible representation with highest weight μ.

Let $\mathfrak{p}$ be the $B_{\mathfrak{g}}$-orthocomplement of $\mathfrak{r}$ in $\mathfrak{g}$, and let $B_{\mathfrak{p}} = B_{\mathfrak{g}} \mid \mathfrak{p}$. Let $\nu : \mathfrak{r} \to \operatorname{Lie} \mathrm{SO}(\mathfrak{p})$ be the representation defined so that $\nu(x)y = [x, y]$ for $x \in \mathfrak{r}$ and $y \in \mathfrak{p}$. We are therefore in the case of §1, where $(\nu, \mathfrak{g})$ is of Lie type, $[z, z'] = [z, z']^{v}$ for $v \in \mathcal{V}$, and v is defined so that

$$P_{\mathfrak{p}}\big([y, y']\big) = 2\iota(y)\iota\big(y'\big)v \tag{3.3}$$

for $y, y' \in \mathfrak{p}$. See (1.23).

Of course, $\Delta(\mathfrak{p})$ is the complement of $\Delta(\mathfrak{r})$ in Δ, and one readily has card $\Delta(\mathfrak{p}) = n$, where, as in §1.64, $n = \dim \mathfrak{p}$. Let $\Delta^{+}(\mathfrak{p}) = \Delta^{+} \cap \Delta(\mathfrak{p})$ so that $\Delta^{+}(\mathfrak{p})$ is the complement of $\Delta^{+}_{\mathfrak{r}}$ in Δ^{+}. Put $m = \operatorname{card} \Delta^{+}(\mathfrak{p})$; since one readily has the partition

$$\Delta(\mathfrak{p}) = \Delta^{+}(\mathfrak{p}) \cup -\Delta^{+}(\mathfrak{p}), \tag{3.4}$$

it follows that

$$n = 2m. \tag{3.5}$$

In particular, n is even, so the Clifford algebra $C(\mathfrak{p})$ is simple and the spin module S is irreducible. In fact, $\dim S = 2^{m}$ and the map (1.57) defines an isomorphism

$$C(\mathfrak{p}) \cong \operatorname{End} S. \tag{3.6}$$

For computational purposes, it is convenient to take S to be a minimal left ideal in $C(\mathfrak{p})$, and the map ε (see (1.57)) defining the action of $C(\mathfrak{p})$ on S is left multiplication. Let $\mathfrak{p}_{+}$ (resp., $\mathfrak{p}_{-}$) be the span of e_{φ} (resp., $e_{-\varphi}$) for $\varphi \in \Delta^{+}(\mathfrak{p})$ so that, as linear spaces,

$$\mathfrak{p} = \mathfrak{p}_{+} \oplus \mathfrak{p}_{-}. \tag{3.7}$$

Clearly, $B_{\mathfrak{p}} \mid \mathfrak{p}_{+} = 0$ and $B_{\mathfrak{p}} \mid \mathfrak{p}_{-} = 0$ so that Clifford multiplication and exterior multiplication are the same on both $\wedge\mathfrak{p}_{+}$ and $\wedge\mathfrak{p}_{-}$. On the other hand, if $w \in \wedge^{k}\mathfrak{p}$ and $w' \in \wedge^{k'}\mathfrak{p}$, it follows from (1.6) that

$$ww' - w \wedge w' \in \sum_{j=1}^{[(k+k')/2]} \wedge^{k+k'-2j}\mathfrak{p}. \tag{3.8}$$

This readily implies that the map

$$\wedge\mathfrak{p}_{-} \otimes \wedge\mathfrak{p}_{+} \longrightarrow C(\mathfrak{p}), \qquad w \otimes w' \longmapsto ww' \tag{3.9}$$

is a linear bijection. It therefore follows that if $\Delta^{+}(\mathfrak{p}) = \{\beta_1, \ldots, \beta_m\}$ and we put $e_{\mathfrak{p}_{+}} = e_{\beta_1} \cdots e_{\beta_m}$, then since clearly $\wedge\mathfrak{p}_{+} e_{\mathfrak{p}_{+}} = \mathbb{C} e_{\mathfrak{p}_{+}}$, one has that

$$C(\mathfrak{p}) e_{\mathfrak{p}_{+}} = \wedge\mathfrak{p}_{-} e_{\mathfrak{p}_{+}}; \tag{3.10}$$

thus the left ideal $C(\mathfrak{p})e_{\mathfrak{p}_+}$ has dimension 2^m and consequently is a minimal ideal. We therefore can take

$$S = C(\mathfrak{p})e_{\mathfrak{p}_+} = \wedge\mathfrak{p}_- e_{\mathfrak{p}_+}. \tag{3.11}$$

Let $\wedge^{\text{even}}\mathfrak{p}_- = \sum_{j=0}\wedge^{2j}\mathfrak{p}_-$ and $\wedge^{\text{odd}}\mathfrak{p}_- = \sum_{j=0}\wedge^{2j+1}\mathfrak{p}_-$. Put

$$S_+ = \wedge^{\text{even}}\mathfrak{p}_- e_{\mathfrak{p}_+}, \qquad S_- = \wedge^{\text{odd}}\mathfrak{p}_- e_{\mathfrak{p}_+}. \tag{3.12}$$

It then follows from (3.8) that

$$S = S_+ \oplus S_- \tag{3.13}$$

is a decomposition of S as a module for $C^{\text{even}}(\mathfrak{p})$ into two submodules (half-spin) for $C^{\text{even}}(\mathfrak{p})$. Then it also follows (see (1.58)) that S, as an $\mathfrak{r}$-module with respect to $\operatorname{Spin}\nu$, has S_+ and S_- as submodules; hence $\operatorname{Spin}\nu$ is a direct sum of its two subrepresentations

$$\operatorname{Spin}_+\nu : \mathfrak{r} \longrightarrow \operatorname{End} S_+, \qquad \operatorname{Spin}_-\nu : \mathfrak{r} \longrightarrow \operatorname{End} S_-. \tag{3.14}$$

For any subset $\Phi \subset \Delta$, let $\langle\Phi\rangle = \sum_{\varphi\in\Phi}\varphi$. Obviously, $\langle\Phi\rangle \in \Gamma$. Let $\Phi \subset \Delta_{\mathfrak{p}}^+$ and $k = \operatorname{card}\Phi$. Write $\Phi = \{\beta_{i_1}, \ldots, \beta_{i_k}\}$ where $i_1 < \cdots < i_k$ so that $\langle\Phi\rangle = \sum_{j=1}^k \beta_{i_j}$. Also let $e_{-\Phi} = e_{-\beta_{i_1}}\cdots e_{-\beta_{i_k}}$ and put $s_\Phi = e_{-\Phi}e_{\mathfrak{p}_+}$ so that

$$\{s_\Phi\}, \Phi \subset \Delta_{\mathfrak{p}}^+, \text{ is a basis of } S \tag{3.15}$$

and

$$\begin{aligned} &\{s_\Phi\}, \Phi \subset \Delta_{\mathfrak{p}}^+, \text{ card } \Phi \text{ is even, is a basis of } S_+, \\ &\{s_\Phi\}, \Phi \subset \Delta_{\mathfrak{p}}^+, \text{ card } \Phi \text{ is odd, is a basis of } S_-. \end{aligned} \tag{3.16}$$

Let $\rho_{\mathfrak{p}} = (1/2)\sum_{i=1}^m \beta_i$. Note that $\rho_{\mathfrak{p}} = \rho - \rho_{\mathfrak{r}}$ so that $\rho_{\mathfrak{p}} \in \Gamma_{\mathfrak{r}}$.

PROPOSITION 3.17. *For any $x \in \mathfrak{h}$, one has*

$$\nu_*(x) = \frac{1}{2}\sum_{\beta\in\Delta_{\mathfrak{p}}^+}\beta(x)e_\beta \wedge e_{-\beta} = \frac{1}{2}\sum_{\beta\in\Delta_{\mathfrak{p}}^+}\beta(x)(1 - e_{-\beta}e_\beta). \tag{3.18}$$

Furthermore, for any $\Phi \subset \Delta_{\mathfrak{p}}^+$, the basal element $s_\Phi \in S$ is a weight vector for $\operatorname{Spin}\nu$ *(see (1.58)) with weight $\rho_{\mathfrak{p}} - \langle\Phi\rangle \in \Gamma_{\mathfrak{r}}$.*

Proof. If $y, y' \in \mathfrak{p}$, then $yy' = 2(y, y') - y'y$. Let $\beta \in \Delta_{\mathfrak{p}}^+$. Then, by (3.2), $e_\beta e_{-\beta} = 2 - e_{-\beta}e_\beta$. But $e_\beta e_{-\beta} = e_\beta \wedge e_{-\beta} + 1$ by (1.6). Thus

$$e_\beta \wedge e_{-\beta} = 1 - e_{-\beta}e_\beta. \tag{3.19}$$

Hence to prove (3.18), it suffices only to prove the left inequality. Let u be the middle sum in (3.18). Clearly, $-2\iota(e_\beta)u = \beta(x)e_\beta$ and $-2\iota(e_{-\beta})u = -\beta(x)e_{-\beta}$. Hence $u = \nu_*(x)$ by (1.11). This establishes (3.18).

Clearly $e_\beta e_{\mathfrak{p}_+} = 0$ for any $\beta \in \Delta_{\mathfrak{p}}^+$. Hence from the right side in (3.18), one has

$$\nu_*(x)e_{\mathfrak{p}_+} = \rho_{\mathfrak{p}}(x)e_{\mathfrak{p}_+} \tag{3.20}$$

for any $x \in \mathfrak{h}$. On the other hand, if $\Phi \subset \Delta_{\mathfrak{p}}^+$, then, by (1.12), clearly

$$\operatorname{ad}\nu_*(x)(e_{-\Phi}) = -\langle\Phi\rangle(x)e_{-\Phi}. \tag{3.21}$$

Thus

$$\operatorname{Spin}\nu(x)s_\Phi = \nu_*(x)s_\Phi = \operatorname{ad}\nu_*(x)(e_{-\Phi})e_{\mathfrak{p}_+} + e_{-\Phi}\nu_*(x)e_{\mathfrak{p}_+} = \big(\rho_{\mathfrak{p}} - \langle\Phi\rangle\big)(x)s_\Phi,$$

by (3.20) and (3.21). □

3.22. Let $R_{\mathbb{C}}^{\mathrm{sc}}$ be a simply connected Lie group corresponding to $\mathfrak{r}$, and let $\exp^{\mathrm{sc}} : \mathfrak{r} \to R_{\mathbb{C}}^{\mathrm{sc}}$ be the exponential map. If $\sigma : \mathfrak{r} \to \operatorname{End} Z$ is a finite-dimensional representation of $\mathfrak{r}$, let $\sigma^{\mathrm{sc}} : R_{\mathbb{C}}^{\mathrm{sc}} \to \operatorname{Aut} Z$ be the corresponding Lie group representation. Now let $\operatorname{ch}\sigma$ be the function on $\mathfrak{r}$ defined by putting $\operatorname{ch}\sigma(x) = \operatorname{tr}\sigma^{\mathrm{sc}}(\exp^{\mathrm{sc}} x)$ for $x \in \mathfrak{r}$. For convenience we write $\operatorname{ch} Z$ for $\operatorname{ch}\sigma$ if the representation σ is understood. The following well-known fact is an immediate consequence of (3.16) and Proposition 3.17.

PROPOSITION 3.23. *One has*

$$(\operatorname{ch} S_+ - \operatorname{ch} S_-) \mid \mathfrak{h} = \prod_{\beta\in\Delta_{\mathfrak{p}}^+} \big(e^{\beta/2} - e^{-\beta/2}\big) = e^{\rho_{\mathfrak{p}}} \prod_{\beta\in\Delta_{\mathfrak{p}}^+} \big(1 - e^{-\beta}\big).$$

We now recall and reprove (mainly to adapt to the present notation) results in the paper [GKRS]. Let W (resp., $W_{\mathfrak{r}}$) be the Weyl group of $\mathfrak{g}$ (resp., $\mathfrak{r}$) operating as usual in both $\mathfrak{h}$ and $\mathfrak{h}^*$. Of course, $W_{\mathfrak{r}}$ is a subgroup of W. Let d be the index of $W_{\mathfrak{r}}$ in W. A cross section W^1 of the set of right $W_{\mathfrak{r}}$ cosets is defined (using the Weyl chambers $D \subset D_{\mathfrak{r}}$; see §3.1) by putting

$$W^1 = \{\tau \in W \mid \tau(D) \subset D_{\mathfrak{r}}\}. \tag{3.24}$$

That is, the map

$$W_{\mathfrak{r}} \times W^1 \to W, \qquad (w,\tau) \longmapsto w\tau \tag{3.25}$$

is clearly a bijection. In particular,

$$\operatorname{card} W^1 = d. \tag{3.26}$$

Let Γ^{reg} be the set of W-regular elements in the lattice $\Gamma \subset \mathfrak{h}^*_{\mathbb{R}}$. One has the partition

$$\Gamma^{\text{reg}} = \bigcup_{w \in W} w(\Lambda + \rho). \tag{3.27}$$

The partition (3.27) clearly defines a partition of the subset $\Gamma^{\text{reg}} \cap D_{\mathfrak{r}}$ of elements in Γ^{reg} that are $\Delta^+_{\mathfrak{r}}$-dominant:

$$\Gamma^{\text{reg}} \cap D_{\mathfrak{r}} = \bigcup_{\tau \in W^1} \tau(\Lambda + \rho). \tag{3.28}$$

Obviously, $\Gamma^{\text{reg}} \cap D_{\mathfrak{r}} \subset \Lambda_{\mathfrak{r}}$. Furthermore, it is also obvious that the elements in $\Gamma^{\text{reg}} \cap D_{\mathfrak{r}}$ are $W_{\mathfrak{r}}$-regular. For any $\tau \in W^1$ and $\lambda \in \Lambda$, let

$$\tau \bullet \lambda = \tau(\lambda + \lambda) - \rho_{\mathfrak{r}}. \tag{3.29}$$

It then follows that $\tau \bullet \lambda \in \Lambda_{\mathfrak{r}}$. In fact, this isolates a distinguished subset $\Lambda^*_{\mathfrak{r}}$ of $\Lambda_{\mathfrak{r}}$. Let

$$\Lambda^*_{\mathfrak{r}} = \left\{\mu \in \Lambda_{\mathfrak{r}} \mid \mu + \rho_{\mathfrak{r}} \in \Gamma^{\text{reg}}\right\}. \tag{3.30}$$

The following proposition is an immediate consequence of the partition (3.28).

Proposition 3.31. *For any $\tau \in W^1$ and $\lambda \in \Lambda$, one has $\tau \bullet \lambda \in \Lambda^*_{\mathfrak{r}}$. Moreover, the map*

$$W^1 \times \Lambda \longrightarrow \Lambda^*_{\mathfrak{r}} \tag{3.32}$$

is a bijection.

Now observe that the elements of Λ parameterize subsets of $\Lambda^*_{\mathfrak{r}}$. For each $\lambda \in \Lambda$, let

$$\Lambda^{\lambda}_{\mathfrak{r}} = \left\{\tau \bullet \lambda \mid \tau \in W^1\right\}. \tag{3.33}$$

From Proposition 3.31, one has

$$\text{card}\, \Lambda^{\lambda}_{\mathfrak{r}} = d, \tag{3.34}$$

so that all the subsets have the same cardinality and

$$\Lambda^*_{\mathfrak{r}} = \bigcup_{\lambda \in \Lambda} \Lambda^{\lambda}_{\mathfrak{r}} \tag{3.35}$$

is a partition. By abuse of notation, we use the word *multiplet* (or *d-multiplet*) to refer to (1) any subset of $\Lambda^*_{\mathfrak{r}}$ of the form $\Lambda^{\lambda}_{\mathfrak{r}}$, (2) the corresponding d-set $\{\sigma_\mu\}$, $\mu \in \Lambda^{\lambda}_{\mathfrak{r}}$, of irreducible $\mathfrak{r}$-representations or, again, (3) the corresponding d-set $\{Z_\mu\}$, $\mu \in \Lambda^{\lambda}_{\mathfrak{r}}$, of irreducible $\mathfrak{r}$-modules. We can characterize the multiplets using infinitesimal characters.

3.36. Let $S(\mathfrak{h})$ be the symmetric algebra over $\mathfrak{h}$. We can regard $S(\mathfrak{h})$ as the ring of polynomial functions on $\mathfrak{h}^*$. Let $S(\mathfrak{h})^W$ (resp., $S(\mathfrak{h})^{W_\mathfrak{r}}$) be the algebra of W-invariants (resp., $W_\mathfrak{r}$-invariants) in $S(\mathfrak{h})$. Obviously,

$$S(\mathfrak{h})^W \subset S(\mathfrak{h})^{W_\mathfrak{r}}. \tag{3.37}$$

Now for any $\lambda \in \Lambda$ (resp., $\mu \in \Lambda_\mathfrak{r}$), let $\chi^\lambda : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ (resp., $\chi_\mathfrak{r}^\mu : \mathfrak{Z}(\mathfrak{r}) \to \mathbb{C}$) be the infinitesimal character of π_λ (resp., σ_μ). The theorem of Harish-Chandra on the determination of infinitesimal characters, in the present cases, asserts that there exists algebra isomorphisms

$$\eta : \mathfrak{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W, \qquad \eta_\mathfrak{r} : \mathfrak{Z}(\mathfrak{r}) \longrightarrow S(\mathfrak{h})^{W_\mathfrak{r}} \tag{3.38}$$

such that for any $p \in \mathfrak{Z}(\mathfrak{g})$ and $\lambda \in \Lambda$ (resp., $q \in \mathfrak{Z}(\mathfrak{r})$ and $\mu \in \Lambda_\mathfrak{r}$), one has

$$\chi^\lambda(p) = \eta(p)(\lambda+\rho) \tag{3.39}$$

and

$$\chi_\mathfrak{r}^\mu(q) = \eta_\mathfrak{r}(q)\big(\mu+\rho_\mathfrak{r}\big). \tag{3.40}$$

But now by (3.37) and (3.38), one has an injection

$$\eta_\mathfrak{r}^{-1} \circ \eta : \mathfrak{Z}(\mathfrak{g}) \longrightarrow \mathfrak{Z}(\mathfrak{r}). \tag{3.41}$$

Let $\mathfrak{Z}_\mathfrak{g}(\mathfrak{r}) \subset \mathfrak{Z}(\mathfrak{r})$ be the image of the map (3.41).

Remark 3.42. Note that the algebra $\mathfrak{Z}(\mathfrak{r})$ and the subalgebra $\mathfrak{Z}_\mathfrak{g}(\mathfrak{r})$ are both polynomial rings in l-variables.

The partition (3.35) defines an equivalence relation $\mu \sim \mu'$ in $\Lambda_\mathfrak{r}^*$ where the equivalence classes are just the d-multiplets $\Lambda_\mathfrak{r}^\lambda, \lambda \in \Lambda$. The following result was stated as "Proposition" in [GKRS].

PROPOSITION 3.43. *Let $\mu, \mu' \in \Lambda_\mathfrak{r}^*$. Then $\mu \sim \mu'$ if and only if*

$$\chi_\mathfrak{r}^\mu \mid \mathfrak{Z}_\mathfrak{g}(\mathfrak{r}) = \chi_\mathfrak{r}^{\mu'} \mid \mathfrak{Z}_\mathfrak{g}(\mathfrak{r}). \tag{3.44}$$

Proof. Let $q \in \mathfrak{Z}_\mathfrak{g}(\mathfrak{r})$ so that there exists $p \in \mathfrak{Z}(\mathfrak{g})$ such that $\eta_\mathfrak{r}^{-1}(\eta(p)) = q$. Let $(\tau,\lambda), (\tau',\lambda') \in W^1 \times \Lambda$ be such that $\tau \bullet \lambda = \mu$ and $\tau' \bullet \lambda' = \mu'$. Thus $\tau(\lambda+\rho) = \mu + \rho_\mathfrak{k}$. Consequently, by (3.40), one has

$$\begin{aligned} \chi_\mathfrak{r}^\mu(q) &= \eta_\mathfrak{r}(q)\big(\mu+\rho_\mathfrak{k}\big) = \eta_\mathfrak{r}(q)\big(\tau(\lambda+\rho)\big) \\ &= \eta(p)\big(\tau(\lambda+\rho)\big) = \eta(p)(\lambda+\rho). \end{aligned} \tag{3.45}$$

Similarly, $\chi_\mathfrak{r}^{\mu'}(q) = \eta(p)(\lambda'+\rho)$. But now, by definition, $\mu \sim \mu'$ if and only if $\lambda' = \lambda$. But then if $\mu \sim \mu'$, one has $\chi_\mathfrak{r}^{\mu'}(q) = \chi_\mathfrak{r}^\mu(q)$ by (3.45). Conversely, if $\chi_\mathfrak{r}^{\mu'}(q) = \chi_\mathfrak{r}^\mu(q)$

for all $q \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{r})$, then one has $\eta(p)(\lambda+\rho) = \eta(p)(\lambda'+\rho)$ for all $p \in \mathcal{Z}(\mathfrak{g})$. Hence by the left isomorphism in (3.38), it follows that $\lambda+\rho$ and $\lambda'+\rho$ must be W-conjugate. But since both lie in D, this implies $\lambda+\rho = \lambda'+\rho$. Thus $\lambda = \lambda'$ and hence $\mu \sim \mu'$. □

Remark 3.46. Recall that $\mathrm{Cas}_{\mathfrak{r}} \in \mathcal{Z}(\mathfrak{r})$ is the Casimir element corresponding to $B_{\mathfrak{r}}$. In the case that $\mathfrak{r}$ is simple, it is immediate that $\mathrm{Cas}_{\mathfrak{r}} \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{r})$. By Proposition 3.43, this means that $\chi_{\mathfrak{r}}^{\mu}(\mathrm{Cas}_{\mathfrak{r}})$ is constant over all μ in a multiplet. In the special case where $(\mathfrak{r},\mathfrak{g}) = (\text{Lie Spin } 9, F_4)$, one has $d=3$, so that in this case the multiplets are triplets. Without knowledge of the connection with F_4, the triplets of Spin 9 representations (see §0.1) were empirically discovered by physicists Ramond and Pengpan. A number of properties of these triplets, including the constancy of $\chi_{\mathfrak{r}}^{\mu}(\mathrm{Cas}_{\mathfrak{r}})$ for μ in a triplet, were also empirically observed. The motivation of [GKRS] was to explain mathematically and generalize those empirical observations.

3.47. The d-set W^1 has a natural partition into two parts. Let $W^1_+ = \{\tau \in W^1 \mid sg\tau = 1\}$. Let W^1_- be defined similarly, except that $sg\tau = -1$. Obviously,

$$W^1 = W^1_+ \cup W^1_-. \tag{3.48}$$

Let $d_+ = \operatorname{card} W^1_+$ and $d_- = \operatorname{card} W^1_-$ so that

$$d = d_+ + d_-. \tag{3.49}$$

Correspondingly, any multiplet $\Lambda_{\mathfrak{r}}^{\lambda}$, $\lambda \in \Lambda$, partitions into two parts

$$\Lambda_{\mathfrak{r}}^{\lambda} = \Lambda_{\mathfrak{r}}^{\lambda,+} \cup \Lambda_{\mathfrak{r}}^{\lambda,-} \tag{3.50}$$

of respective cardinalities d_+ and d_-, where $\Lambda_{\mathfrak{r}}^{\lambda,+} = \{\tau \bullet \lambda \mid \tau \in W^1_+\}$ and $\Lambda_{\mathfrak{r}}^{\lambda,-} = \{\tau \bullet \lambda \mid \tau \in W^1_-\}$.

Let $\mathcal{R}$ be the ring of virtual reductive representations of $\mathfrak{r}$. The role of the d-multiplets enters in the following result of [GKRS].

THEOREM 3.51. *Let $\lambda \in \Lambda$ so that V_λ is an $\mathfrak{r}$-module with respect to $\pi_\lambda \mid \mathfrak{r}$. Then in $\mathcal{R}$,*

$$V \otimes S_+ - V_\lambda \otimes S_- = \sum_{\tau \in W^1} sg(\tau) Z_{\tau \bullet \lambda} = \sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} Z_{\mu'} - \sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} Z_{\mu''}. \tag{3.52}$$

Proof. As a function on $\mathfrak{h}$, let F (resp., $F_{\mathfrak{r}}$) be the denominator of the Weyl character formula for $\mathfrak{g}$ (resp., $\mathfrak{r}$). Then, by Proposition 3.23,

$$\begin{aligned} F &= \prod_{\varphi \in \Delta^+} \left(e^{\varphi/2} - e^{-\varphi/2}\right) = \prod_{\varphi \in \Delta_{\mathfrak{r}}^+} \left(e^{\varphi/2} - e^{-\varphi/2}\right) \prod_{\beta \in \Delta_{\mathfrak{p}}^+} \left(e^{\beta/2} - e^{-\beta/2}\right) \\ &= F_{\mathfrak{r}}(\operatorname{ch} S_+ - \operatorname{ch} S_-) \mid \mathfrak{h}. \end{aligned} \tag{3.53}$$

For $\lambda \in \Lambda$ (resp., $\mu \in \Lambda_{\mathfrak{r}}$), let F^{λ} (resp., $F_{\mathfrak{r}}^{\mu}$) be the numerator of Weyl's character formula for $\operatorname{ch} V_{\lambda} \mid \mathfrak{h}$ (resp., $\operatorname{ch} Z_{\mu} \mid \mathfrak{h}$). If $\tau \in W^1$, we recall by definition that $\tau(\lambda+\rho) = \tau \bullet \lambda + \rho_{\mathfrak{r}}$. Then

$$\begin{aligned} F^{\lambda} &= \sum_{w' \in W} sg(w') e^{w'(\lambda+\rho)} = \sum_{\tau \in W^1} sg(\tau) \left(\sum_{w \in W_{\mathfrak{r}}} sg(w) e^{w\tau(\lambda+\rho)} \right) \\ &= \sum_{\tau \in W^1} sg(\tau) \left(\sum_{w \in W_{\mathfrak{r}}} sg(w) e^{w\left(\tau \bullet \lambda + \rho_{\mathfrak{r}}\right)} \right) = \sum_{\tau \in W^1} sg(\tau) F_{\mathfrak{r}}^{\tau \bullet \lambda}. \end{aligned} \tag{3.54}$$

But now $\operatorname{ch} V_{\lambda} \mid \mathfrak{h} = F^{\lambda}/F$. But then, by (3.53) and (3.54),

$$\begin{aligned} &\Big(\operatorname{ch}\left(V_{\lambda} \otimes S_{+}\right) - \operatorname{ch}\left(V_{\lambda} \otimes S_{+}\right)\Big) \mid \mathfrak{h} \\ &\qquad = \operatorname{ch} V_{\lambda} (\operatorname{ch} S_{+} - \operatorname{ch} S_{-}) \mid \mathfrak{h} = \left(\frac{F^{\lambda}}{F} \right) (\operatorname{ch} S_{+} - \operatorname{ch} S_{-}) \mid \mathfrak{h} \\ &\qquad = \sum_{\tau \in W^1} sg(\tau) \frac{F_{\mathfrak{r}}^{\tau \bullet \lambda}}{F_{\mathfrak{r}}} = \sum_{\tau \in W^1} sg(\tau) \operatorname{ch} Z_{\tau \bullet \lambda} \mid \mathfrak{h}. \end{aligned}$$

Thus the character of the extreme left side of (3.52) equals the character of the middle and right side of (3.52). This proves (3.52). □

Remark 3.55. Although the proof is quite simple, Theorem 3.51 and some of its rather striking consequences to follow seem not to have been in the mathematics literature. This curious fact was mentioned in [GKRS]. Subsequently, the authors of [GKRS] have learned that at least the proof of Theorem 3.51 has appeared in the physics literature (see [LVW]). However, the conclusions drawn from the proof in [LVW] appear to be considerably weaker than those in [GKRS]. In addition, as mentioned in [GKRS], Wilfred Schmid informed us that he was aware of Theorem 3.51.

The correspondence $Z \mapsto \dim Z$, for reductive $\mathfrak{r}$-modules Z extends to an additive map on $\mathcal{R}$. The value taken by the extreme left side of (3.52) is clearly zero. Hence one immediate consequence of Theorem 3.51, drawn in [GKRS], is the following dimension equality for multiplets.

THEOREM 3.56. *Let $\lambda \in \Lambda$. Then*

$$\sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} \dim Z_{\mu'} = \sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} \dim Z_{\mu''}. \tag{3.57}$$

Remark 3.58. Recall Remark 3.46, where the motivating case $(\mathfrak{r}, \mathfrak{g}) = (\text{Lie Spin}\,9, F_4)$ was discussed. In that case $d = 3$ and setwise $\{d_{+}, d_{-}\} = \{2, 1\}$. Theorem 3.56 explains and generalizes the empirical observation made by Ramond and Pengpan that the sum of the dimensions of two of the members of a triplet equals the dimension of the third.

Another immediate consequence of Theorem 3.51 is the following multiplicity statement.

THEOREM 3.59. *Let $\lambda \in \Lambda$ and $\mu \in \Lambda_{\mathfrak{r}}$. Then if $\mu \notin \Lambda_{\mathfrak{r}}^{\lambda}$, the multiplicity of σ_μ in $V_\lambda \otimes S$ is even, with half occurring in $V_\lambda \otimes S_+$ and half occurring in $V_\lambda \otimes S_-$. If $\mu \in \Lambda_{\mathfrak{r}}^{\lambda}$, then the multiplicity of σ_μ in $V_\lambda \otimes S$ is odd, with one more occurrence in $V_\lambda \otimes S_+$ (resp., $V_\lambda \otimes S_-$) than in $V_\lambda \otimes S_-$ (resp., $V_\lambda \otimes S_+$) if $\mu \in \Lambda_{\mathfrak{r}}^{\lambda,+}$ (resp., $\mu \in \Lambda_{\mathfrak{r}}^{\lambda,-}$).*

Remark 3.60. Later in this paper (see Theorem 4.17), we show that the multiplicity σ_μ in $V_\lambda \otimes S$ is in fact 1 when μ is in the multiplet $\Lambda_{\mathfrak{r}}^{\lambda}$.

Applying Theorem 3.51 to the case where $\lambda = 0$, as observed in [GKRS], yields the following.

THEOREM 3.61. *In $\mathcal{R}$ one has*

$$S_+ - S_- = \sum_{\tau \in W^1} sg(\tau) Z_{\tau \bullet 0} = \sum_{\tau \in W^1} sg(\tau) Z_{\tau(\rho)-\rho_{\mathfrak{r}}}. \tag{3.62}$$

Finally, Theorem 3.61 was combined in [GKRS] with Theorem 3.51 to immediately yield the following equal rank generalization of Weyl's character formula. It reduces to Weyl's formula when $\mathfrak{r} = \mathfrak{h}$.

THEOREM 3.63. *Let $\lambda \in \Lambda$. Then*

$$\operatorname{ch} V_\lambda = \frac{\sum_{\tau \in W^1} sg(\tau) \operatorname{ch} Z_{\tau \bullet \lambda}}{\sum_{\tau \in W^1} sg(\tau) \operatorname{ch} Z_{\tau \bullet 0}}. \tag{3.64}$$

4. Multiplets and the kernel of the Dirac operator $\square$

4.1. If Z is a (finite-dimensional) reductive $\mathfrak{r}$-module, let $\Gamma(Z) \subset \mathfrak{h}^*$ be the corresponding set of weights. We consider only Z-modules where $\Gamma(Z) \subset \Gamma_{\mathfrak{r}}$. Given such an $\mathfrak{r}$-module, let $m_Z = \max_{\gamma \in \Gamma(Z)} (\gamma + \rho_{\mathfrak{r}}, \gamma + \rho_{\mathfrak{r}})$ and let $\Gamma_{\max}(Z) = \{\mu \in \Gamma(Z) \mid (\mu + \rho_{\mathfrak{k}}, \mu + \rho_{\mathfrak{k}}) = m_Z\}$. The following proposition is known but is proved here for completeness.

PROPOSITION 4.2. *If $\mu \in \Gamma_{\max}(Z)$, then $\mu \in \Lambda_{\mathfrak{r}}$ and any weight vector $0 \neq z \in Z$ with weight μ is a highest weight vector. In particular, $U(\mathfrak{r})z$ is an irreducible $\mathfrak{r}$ module and is equivalent to Z_μ.*

Proof. Let $\mu \in \Gamma_{\max}(Z)$. We first show that $\mu + \rho_{\mathfrak{r}} \in \Lambda_{\mathfrak{r}}$. Assuming it is not true, there then exists $1 \neq w \in W_{\mathfrak{r}}$ such that $w\mu + w\rho_{\mathfrak{r}} \in \Lambda_{\mathfrak{r}}$. But $\rho_{\mathfrak{r}} - w\rho_{\mathfrak{r}}$ is a nontrivial sum of positive $\mathfrak{r}$-roots and $w\mu + \rho_{\mathfrak{r}} = (w\mu + w\rho_{\mathfrak{r}}) + (\rho_{\mathfrak{r}} - w\rho_{\mathfrak{r}})$. Hence

$$\big(w\mu + \rho_{\mathfrak{r}}, w\mu + \rho_{\mathfrak{r}}\big) > \big(w\mu + w\rho_{\mathfrak{r}}, w\mu + w\rho_{\mathfrak{r}}\big) = m_Z.$$

This is a contradiction since $w\mu \in \Gamma(Z)$. Now if $z \in Z$ is a weight vector with weight μ and if z is not a highest weight vector, then by decomposing Z into a sum of irreducible components, it follows that there exists an irreducible component Z' with some highest weight μ' such that $\mu \neq \mu'$ but that μ is a weight of Z'. But then $\mu' - \mu$ is a nontrivial sum of positive $\mathfrak{r}$-roots. Furthermore, $\mu' + \rho_{\mathfrak{r}} = (\mu' - \mu) + (\mu + \rho_{\mathfrak{r}})$. But then

$$\left(\mu' + \rho_{\mathfrak{r}}, \mu' + \rho_{\mathfrak{r}}\right) > \left(\mu + \rho_{\mathfrak{r}}, \mu + \rho_{\mathfrak{r}}\right) = m_Z.$$

This again is a contradiction. Hence z is a highest weight vector. This, of course, implies $\mu \in \Lambda_{\mathfrak{r}}$. □

Remark 4.3. In the notation of Proposition 4.2, note that the eigenvalue of the Casimir operator $\mathrm{Cas}_{\mathfrak{r}}$ on $U(\mathfrak{r})z$ is $m_Z - (\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}})$ and that, by the definition of m_Z, this is the maximal eigenvalue of $\mathrm{Cas}_{\mathfrak{r}}$ on Z. Proposition 4.2 then readily implies the following.

PROPOSITION 4.4. *Let the notation be as in Proposition 4.2. For any $\mu \in \Gamma_{\max}(Z)$, let $Z(\mu)$ be the corresponding weight space and put $Y_{\max} = \sum_{\mu \in \Gamma_{\max}(Z)} Z(\mu)$. Let Y be the $U(\mathfrak{r})$ submodule of Z generated by $Y_{\max}$. Then $m_Z - (\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}})$ is the maximal eigenvalue of* $\mathrm{Cas}_{\mathfrak{r}}$ *on Z, and Y is the corresponding eigenspace.*

4.5. If V is a finite-dimensional $\mathfrak{g}$-module, it is, in particular, an $\mathfrak{r}$-module where $\Gamma(V) \subset \Gamma$. One knows (see [Ko1, Lemma 5.9]) that

$$\Gamma(V_\rho) = \left\{\rho - \langle\Phi\rangle \mid \Phi \subset \Delta^+\right\} \tag{4.6}$$

and that the multiplicity of $\rho - \langle\Phi\rangle$ equals the number of subsets $\Phi' \subset \Delta^+$ such that $\langle\Phi\rangle = \langle\Phi'\rangle$. If $\gamma \in \Gamma(V_\rho)$, then obviously γ is an extremal weight (with respect to the $\mathfrak{g}$-module structure) if and only if there exists $w \in W$, necessarily unique, such that $\gamma = w\rho$. But for any $w \in W$, let $\Phi_w = w(-\Delta^+) \cap \Delta^+$. Then by (5.10.1) and (5.10.2) in [Ko1], for $\Phi \subset \Delta_+$ one has

$$w\rho = \rho - \langle\Phi\rangle \iff \Phi = \Phi_w. \tag{4.7}$$

Thus for $\Phi \subset \Delta_+$ one has

$$(\rho, \rho) \geq \left(\rho - \langle\Phi\rangle, \rho - \langle\Phi\rangle\right), \tag{4.8}$$

where equality occurs if and only if $\Phi = \Phi_w$ for some $w \in W$.

Now let $\lambda \in \Lambda$. The extremal weights of $V_{\lambda+\rho}$, as a $\mathfrak{g}$-module, are uniquely of the form $w(\lambda + \rho)$ for $w \in W$.

PROPOSITION 4.9. *Let $\gamma \in \Gamma(V_\lambda)$ and let $\Phi \subset \Delta^+$. Then*

$$(\lambda + \rho, \lambda + \rho) \geq \left(\gamma + \rho - \langle\Phi\rangle, \gamma + \rho - \langle\Phi\rangle\right), \tag{4.10}$$

and equality occurs if and only if there exists $w \in W$ such that (a) $\gamma = w\lambda$ and (b) $\Phi = \Phi_w$. Furthermore, $w \in W$ satisfying (a) and (b) is unique.

Proof. Let $w \in W$ be such that $w^{-1}(\gamma + \rho - \langle\Phi\rangle) \in \Lambda$ (i.e., is dominant). But if $A = \lambda - w^{-1}(\gamma)$ and $B = \rho - w^{-1}(\rho - \langle\Phi\rangle)$, then A and B are both sums (possibly empty) of positive roots by (4.6). But $\lambda + \rho = A + B + w^{-1}(\gamma + \rho - \langle\Phi\rangle)$. This proves the inequality (4.10) where equality occurs if and only if $A = B = 0$. That is, if and only if $\gamma = w\lambda$ and $\Phi = \Phi_w$ by (4.7). The uniqueness of w follows from (5.10.2) in [Ko1]. □

Let $\lambda \in \Lambda$. We now apply Propositions 4.2 and 4.4 and the notation therein to the case where Z is the tensor product $\mathfrak{r}$-module $Z = V_\lambda \otimes S$. In the notation of (2.20), the action by $\mathfrak{r}$ is given by the representation ζ_λ. Clearly, by Proposition 3.17, $\Gamma(Z)$ is the set of all elements $\mu \in \Gamma_{\mathfrak{r}}$ of the form

$$\mu = \gamma + \rho_{\mathfrak{p}} - \langle\Phi\rangle, \tag{4.11}$$

where $\gamma \in \Gamma(V_\lambda)$ and $\Phi \subset \Delta_{\mathfrak{p}}^+$. But in such a case,

$$\mu + \rho_{\mathfrak{r}} = \gamma + \rho - \langle\Phi\rangle. \tag{4.12}$$

But then, by Proposition 4.9, one has

$$(\lambda + \rho, \lambda + \rho) \geq m_Z. \tag{4.13}$$

However, if we choose $\gamma = \lambda$ and $\Phi = \emptyset$, we get $\lambda + \rho = \mu + \rho_{\mathfrak{r}}$ in (4.12). But this proves

$$m_Z = (\lambda + \rho, \lambda + \rho) \tag{4.14}$$

by (4.13). We now determine $\Gamma_{\max}(Z)$. If $\mu \in \Gamma(Z)$, then using the notation of (4.12) and Proposition 4.9, one has $\mu \in \Gamma_{\max}(Z)$ if and only if there exists $w \in W$, necessarily unique, such that $\gamma = w\lambda$ and $\Phi = \Phi_w$. But $\rho - \langle\Phi_w\rangle = w\rho$. Thus $\mu \in \Gamma_{\max}(Z)$ if and only if

$$\mu + \rho_{\mathfrak{r}} = w(\lambda + \rho), \tag{4.15}$$

where $\Phi_w \subset \Delta_{\mathfrak{p}}^+$. But now if $w \in W$, then $w \in W^1$ if and only if $(\varphi, w\rho) > 0$ for all $\varphi \in \Delta_{\mathfrak{r}}^+$. On the other hand, clearly $(\varphi, w\rho) > 0$ if and only if $\varphi \in w(\Delta^+)$. Thus $w \in W^1$ if and only if $\Phi_w \subset \Delta_{\mathfrak{p}}^+$. Hence an element $w \in W$ satisfies (4.15) with $\Phi_w \subset \Delta_{\mathfrak{p}}^+$ if and only if $w \in W^1$. It follows that $\Gamma_{\max}(Z)$ is a multiplet. In fact, clearly,

$$\Gamma_{\max}(Z) = \Lambda_{\mathfrak{r}}^{\lambda}. \tag{4.16}$$

Also, now writing τ for w, if $\mu = \tau \bullet \lambda \in \Gamma_{\max}(Z)$, then we have shown that the weight components (see (4.11)) $\gamma = \tau(\lambda)$ in $\Gamma(V_\lambda)$ and $\rho_{\mathfrak{p}} - \langle\Phi_\tau\rangle$ in $\Gamma(S)$ are unique. But $\tau\lambda$ has multiplicity 1 in V_λ. Let $0 \neq v_{\tau\lambda}$ be a corresponding (extremal) weight vector. Also $\rho_{\mathfrak{p}} - \langle \Phi_\tau \rangle$ has multiplicity 1 in S, by Proposition 7.2 and (5.10.2) in [Ko1]. By Proposition 3.17 the element $s_{\Phi_\tau} \in S$ is the corresponding (unique up to scalar multiplication) weight vector. It follows that any $\mu \in \Lambda^\lambda_{\mathfrak{r}}$ has multiplicity 1 in $Z = V_\lambda \otimes S$. In fact, the corresponding (unique up to scalar multiplication) weight vector is the decomposable tensor $z_\mu = v_{\tau\lambda} \otimes s_{\Phi_\tau}$. We have proved almost all of the following theorem.

THEOREM 4.17. *Let $\lambda \in \Lambda$. Then any element μ in the d-multiplet $\Lambda^\lambda_{\mathfrak{r}}$ has multiplicity* 1 *as a weight in $V_\lambda \otimes S$. Moreover, if we write $\mu = \tau \bullet \lambda$ for $\tau \in W^1$, the corresponding (unique up to scalar multiplication) weight vector is the decomposable tensor*

$$z_\mu = v_{\tau\lambda} \otimes s_{\Phi_\tau},$$

where $0 \neq v_{\tau\lambda}$ is an extremal weight vector in V_λ for the extremal weight $\tau\lambda$ and s_{Φ_τ} is the basal vector in S defined as in Proposition 3.17. Moreover, z_μ is a highest weight vector with respect to $\mathfrak{b}_{\mathfrak{r}}$, so we can take

$$Z_\mu = U(\mathfrak{r}) z_\mu. \tag{4.18}$$

In particular, improving on a statement in Theorem 3.59, the multiplicity of σ_μ in $V_\lambda \otimes S$ is 1. *In addition,*

$$\begin{aligned} Z_\mu &\subset V_\lambda \otimes S_+ \quad \text{if } sg(\tau) = 1,\\ Z_\mu &\subset V_\lambda \otimes S_- \quad \text{if } sg(\tau) = -1. \end{aligned} \tag{4.19}$$

(We recall that $\sigma_{\mu'}$ has even multiplicity if $\mu' \notin \Lambda^\lambda_{\mathfrak{r}}$.)

Finally, the maximal eigenvalue of $\mathrm{Cas}_{\mathfrak{r}}$ *in $V_\lambda \otimes S$ is*

$$(\lambda+\rho, \lambda+\rho) - \big(\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}}\big), \tag{4.20}$$

and $\sum_{\mu \in \Lambda^\lambda_{\mathfrak{r}}} Z_\mu$ is the corresponding eigenspace.

Proof. The statement that the unique weight vector (up to scalar multiplication) z_μ, for $\mu \in \Lambda^\lambda_{\mathfrak{r}}$, is a highest vector weight follows from (4.16) and Proposition 4.2. This justifies the definition (4.18). We only have to prove (4.19) and the final statement of Theorem 4.17.

Let $\tau \in W^1$. Then, as one knows, $sg(\tau) = (-1)^{\mathrm{card}\, \Phi_\tau}$ (since τ is a product, in some order, of the reflections by the roots in Φ_τ). But then (4.19) follows from (3.16). But now if $Z = V_\lambda \otimes S$, then, using the notation of Proposition 4.4, $Y_{\max} = \sum_{\mu \in \Lambda^\lambda_{\mathfrak{r}}} \mathbb{C} z_\mu$ by (4.16). By definition $Y = \sum_{\mu \in \Lambda^\lambda_{\mathfrak{r}}} Z_\mu$. Then (4.20) and the final statement of Theorem 4.17 follow from Proposition 4.4 and (4.14). □

4.21. Let $\lambda \in \Lambda$. We recall the notation of §2.1. By definition (see (2.18)) $\mathcal{A}_\lambda = \mathrm{End}\, V_\lambda \otimes C(\mathfrak{p})$. Using the isomorphism (3.6), identify $\mathcal{A}_\lambda$ with $\mathrm{End}(V_\lambda \otimes S)$.

In particular (see (2.19)), $\square_\lambda \in \mathrm{End}(V_\lambda \otimes S)$. In fact, it is clear from the definitions (2.3), (2.8), and (2.12) that $\square_\lambda$ commutes with the action of $\mathfrak{r}$ on $V_\lambda \otimes S$ so that

$$\square_\lambda \in \mathrm{End}_{\mathfrak{r}}\big(V_\lambda \otimes S\big). \tag{4.22}$$

Also from these definitions, one has $\square_\lambda \in \mathrm{End}\, V_\lambda \otimes \wedge^{\mathrm{odd}}\mathfrak{p}$ so that if $\square_\lambda^+ = \square_\lambda \mid (V_\lambda \otimes S_+)$ and $\square_\lambda^- = \square_\lambda \mid (V_\lambda \otimes S_-)$, then

$$\begin{aligned} \square_\lambda^+ &: V_\lambda \otimes S_+ \longrightarrow V_\lambda \otimes S_-,\\ \square_\lambda^- &: V_\lambda \otimes S_- \longrightarrow V_\lambda \otimes S_+. \end{aligned} \tag{4.23}$$

We can now relate the cubic Dirac operator $\square_\lambda$ with the set of d-multiplets $\Lambda_{\mathfrak{r}}^{\lambda}$.

THEOREM 4.24. *Let $\lambda \in \Lambda$. The operator $\square_\lambda \in \mathrm{End}(V_\lambda \otimes S)$ has the same kernel as its square $(\square_\lambda)^2$ and is given by*

$$\mathrm{Ker}\,\square_\lambda = \sum_{\mu \in \Lambda_{\mathfrak{r}}^{\lambda}} Z_\mu. \tag{4.25}$$

In more detail,

$$\mathrm{Ker}\,\square_\lambda^+ = \sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} Z_{\mu'} \tag{4.26}$$

and

$$\mathrm{Ker}\,\square_\lambda^- = \sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} Z_{\mu''}. \tag{4.27}$$

Proof. The kernel of $(\square_\lambda)^2$, by Theorem 2.21, is the eigenspace for $\zeta_\lambda(\mathrm{Cas}_{\mathfrak{r}})$ corresponding to the eigenvalue $(\lambda+\rho, \lambda+\rho) - (\rho_{\mathfrak{r}}, \rho_{\mathfrak{r}})$. But then

$$\mathrm{Ker}\,\big(\square_\lambda\big)^2 = \sum_{\mu \in \Lambda_{\mathfrak{r}}^{\lambda}} Z_\mu \tag{4.28}$$

by Theorem 4.17. But if $\mu \in \Lambda_{\mathfrak{r}}^{\lambda}$, then Z_μ is contained in either $V_\lambda \otimes S_+$ or $V_\lambda \otimes S_-$ by (4.19). But then $Z_\mu \subset \mathrm{Ker}\,\square_\lambda$ by (4.22) and (4.23), since σ_μ has multiplicity 1 in $V_\lambda \otimes S$. Thus $\mathrm{Ker}(\square_\lambda)^2 \subset \mathrm{Ker}\,\square_\lambda$ by (4.28). But of course $\mathrm{Ker}\,\square_\lambda \subset \mathrm{Ker}(\square_\lambda)^2$. Thus the theorem follows from (4.23) and (4.28). $\square$

Let $\lambda \in \Lambda$. It follows from Theorem 3.51 and Theorem 4.17 that there exists some equivalence

$$\frac{V_\lambda \otimes S_+}{\sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} Z_{\mu'}} \longrightarrow \frac{V_\lambda \otimes S_-}{\sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} Z_{\mu''}}$$

of $\mathfrak{r}$-modules. The question as to how to exhibit an explicit such $\mathfrak{r}$-isomorphism is solved by the cubic Dirac operator $\square_\lambda$. By Theorem 4.24, (4.23), and (4.26), one has

$$\operatorname{Ker}\square_\lambda^- \cong \operatorname{Coker}\square_\lambda^+ \tag{4.29}$$

so that $\square_\lambda^+$ induces an $\mathfrak{r}$-map

$$\widetilde{\square}_\lambda^+ : \frac{V_\lambda \otimes S_+}{\sum_{\mu' \in \Lambda_{\mathfrak{r}}^{\lambda,+}} Z_{\mu'}} \longrightarrow \frac{V_\lambda \otimes S_-}{\sum_{\mu'' \in \Lambda_{\mathfrak{r}}^{\lambda,-}} Z_{\mu''}} \tag{4.30}$$

of quotient spaces.

THEOREM 4.31. *The map $\widetilde{\square}_\lambda^+$ is an isomorphism of $\mathfrak{r}$-modules.*

Proof. This is immediate from Theorem 4.24. □

5. Infinitesimal character values on multiplets

5.1. Let $\mathfrak{g}^*$ be the dual space to $\mathfrak{g}$, and let $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ be the linear isomorphism induced by the Killing form $B_{\mathfrak{g}}$. We may regard $\mathfrak{h}^* \subset \mathfrak{g}^*$ by putting $\langle \gamma, e_\varphi \rangle = 0$ for any $\varphi \in \Delta$. Clearly, $\kappa(\mathfrak{h}) = \mathfrak{h}^*$. The isomorphism κ extends to an algebra isomorphism (also denoted by κ) $S(\mathfrak{g}) \to S(\mathfrak{g}^*)$ of symmetric algebras. If $S(\mathfrak{h}^*)$ is the symmetric algebra over $\mathfrak{h}^*$, then $S(\mathfrak{h}^*)$ is a W-module (and hence also a $W_{\mathfrak{r}}$-module) and clearly $\kappa \mid S(\mathfrak{h})$ is a W-module isomorphism $S(\mathfrak{h}) \to S(\mathfrak{h}^*)$. Let $S(\mathfrak{h}^*)^W$ (resp., $S(\mathfrak{h}^*)^{W_{\mathfrak{r}}}$) be the algebra of W (resp., $W_{\mathfrak{r}}$) invariants in $S(\mathfrak{h}^*)$. Also let $I(\mathfrak{h})$ (resp., $I(\mathfrak{h}^*)$) be the ideal in $S(\mathfrak{h})$ (resp., $S(\mathfrak{h}^*)$) generated by the set of homogeneous elements of positive degree in $S(\mathfrak{h})^W$ (resp., $S(\mathfrak{h}^*)^W$). The duality between $\mathfrak{h}$ and $\mathfrak{h}^*$ extends to a natural duality between $S(\mathfrak{h})$ and $S(\mathfrak{h}^*)$ where, for $\gamma \in \mathfrak{h}^*$ and $x \in \mathfrak{h}$, $\langle \gamma^i, x^j \rangle = j!\langle \gamma, x\rangle^j$ if $i = j$ and 0 otherwise. The graded W-module, $\operatorname{Harm}(\mathfrak{h}) = \sum_{\mathfrak{k}=0} \operatorname{Harm}^k(\mathfrak{h})$, (resp., $\operatorname{Harm}(\mathfrak{h}^*) = \sum_{\mathfrak{k}=0} \operatorname{Harm}^k(\mathfrak{h}^*)$) of harmonic elements in $S(\mathfrak{h})$ (resp., $S(\mathfrak{h}^*)$) is defined to be the orthocomplement of $I(\mathfrak{h}^*)$ (resp., $I(\mathfrak{h})$) in $S(\mathfrak{h})$ (resp., $S(\mathfrak{h}^*)$). One readily has $\kappa(S(\mathfrak{h})^W) = S(\mathfrak{h}^*)^W$ and $\kappa(\operatorname{Harm}(\mathfrak{h})) = \operatorname{Harm}(\mathfrak{h}^*)$ and also one knows that the maps

$$\begin{aligned} S(\mathfrak{h})^W \otimes \operatorname{Harm}(\mathfrak{h}) &\longrightarrow S(\mathfrak{h}), \\ S(\mathfrak{h}^*)^W \otimes \operatorname{Harm}(\mathfrak{h}^*) &\longrightarrow S(\mathfrak{h}^*), \end{aligned} \tag{5.2}$$

defined by multiplication, are W-isomorphisms (see, e.g., [He, Chapter 3, Theorem 3.4]). By (5.2), injection composed with the quotient map clearly induces W-isomorphisms

$$\operatorname{Harm}(\mathfrak{h}) \longrightarrow \frac{S(\mathfrak{h})}{I(\mathfrak{h})}, \qquad \operatorname{Harm}(\mathfrak{h}^*) \longrightarrow \frac{S(\mathfrak{h}^*)}{I(\mathfrak{h}^*)}. \tag{5.3}$$

But then, as one knows (see (B) in [Ch2]), as W-modules,

$$\operatorname{Harm}(\mathfrak{h}) \text{ and } \operatorname{Harm}(\mathfrak{h}^*) \text{ afford the regular representation of } W. \tag{5.4}$$

For any $\varphi \in \Delta$, let $\varphi^\vee \in \mathfrak{h}$ be defined by putting $\varphi^\vee = \kappa^{-1}(\varphi)$. Let $\omega^\vee = \prod_{\varphi\in\Delta^+}\varphi^\vee$. Let $\omega_\mathfrak{r}^\vee$ and $\omega_\mathfrak{p}^\vee$ be defined similarly, where Δ^+ is replaced by $\Delta_\mathfrak{r}^+$ and $\Delta_\mathfrak{p}^+$, respectively. Let $\omega = \kappa(\omega^\vee)$. Let $\omega_\mathfrak{r}$ and $\omega_\mathfrak{p}$ be defined similarly, where $\omega^\vee$ is replaced by $\omega_\mathfrak{r}^\vee$ and $\omega_\mathfrak{p}^\vee$, respectively. Clearly,

$$\omega = \omega_\mathfrak{r}\omega_\mathfrak{p}, \qquad \omega^\vee = \omega_\mathfrak{r}^\vee\omega_\mathfrak{p}^\vee. \tag{5.5}$$

Let $sg : W \to \{1,-1\} \subset \operatorname{End}\mathbb{C}$ be the sign representation on W, and let $sg_\mathfrak{r} = sg \mid W_\mathfrak{r}$. If M is any W-module (resp., $W_\mathfrak{r}$-module), let M^{sg} (resp., $M^{sg_\mathfrak{r}}$) be the sg (resp., $sg_\mathfrak{r}$) primary component of M. The following proposition is well known (see, e.g., [He, Chapter 3, Theorem 3.6 and Corollary 3.8]).

Proposition 5.6. *One has*
(1) $S(\mathfrak{h}^*)^{sg} = \omega S(\mathfrak{h}^*)^W$,
(2) $S(\mathfrak{h}^*)^{sg_\mathfrak{r}} = \omega_\mathfrak{r} S(\mathfrak{h}^*)^{W_\mathfrak{r}}$,
(3) $\operatorname{Harm}(\mathfrak{h}^*)^{sg} = \mathbb{C}\omega$.
Also the same statements hold when $\mathfrak{h}^*, \omega, \omega_\mathfrak{r}$ *are replaced by* $\mathfrak{h}, \omega^\vee, \omega_\mathfrak{r}^\vee$, *respectively.*

5.7. Let
$$J(\mathfrak{h}) = \left\{ f \in S(\mathfrak{h})^{W_\mathfrak{r}} \mid \omega_\mathfrak{r}^\vee f \in \operatorname{Harm}(\mathfrak{h}) \right\}.$$
Define $J(\mathfrak{h}^*)$ similarly, where $\mathfrak{h}$ and $\omega_\mathfrak{r}^\vee$ are replaced by $\mathfrak{h}^*$ and $\omega_\mathfrak{r}$, respectively. It is clear that $J(\mathfrak{h})$ (resp., $J(\mathfrak{h}^*)$) is a graded subspace of $S(\mathfrak{h})^{W_\mathfrak{r}}$ (resp., $S(\mathfrak{h}^*)^{W_\mathfrak{r}}$).

Proposition 5.8. *One has*
(a) $\dim J(\mathfrak{h}^*) = d$,
(b) $\omega_\mathfrak{r} J(\mathfrak{h}^*) = \operatorname{Harm}(\mathfrak{h}^*)^{sg_\mathfrak{r}}$.
The same statements also hold when $\mathfrak{h}^*$ *and* $\omega_\mathfrak{r}$ *are replaced by* $\mathfrak{h}$ *and* $\omega_\mathfrak{r}^\vee$, *respectively.*

Proof. Since $\operatorname{Harm}(\mathfrak{h}^*)$ transforms under W according to the regular representation (see (5.4)), it follows easily that $\operatorname{Harm}(\mathfrak{h}^*)$, as a $W_\mathfrak{r}$-module, is equivalent to d-copies of the regular representation of $W_\mathfrak{r}$. Since $sg_\mathfrak{r}$ has multiplicity 1 in the regular representation of $W_\mathfrak{r}$, it follows then that

$$\dim \operatorname{Harm}(\mathfrak{h}^*)^{sg_\mathfrak{r}} = d. \tag{5.9}$$

But since $S(\mathfrak{h}^*)$ is an integral domain, (a) and (b) of Proposition 5.8 follow from (1) and (2) of Proposition 5.6. A similar argument, of course, yields the analogous last statement of Proposition 5.8. □

If one takes the $sg_\mathfrak{r}$-primary component of both sides of (5.2), the maps (5.2) clearly restrict to linear isomorphisms

$$\begin{aligned} S(\mathfrak{h})^W \otimes \operatorname{Harm}(\mathfrak{h})^{sg_\mathfrak{r}} &\longrightarrow S(\mathfrak{h})^{sg_\mathfrak{r}}, \\ S(\mathfrak{h}^*)^W \otimes \operatorname{Harm}(\mathfrak{h}^*)^{sg_\mathfrak{r}} &\longrightarrow S(\mathfrak{h}^*)^{sg_\mathfrak{r}}. \end{aligned} \tag{5.10}$$

Proposition 5.11. *The maps*

$$S(\mathfrak{h})^W \otimes J(\mathfrak{h}) \longrightarrow S(\mathfrak{h})^{W_\mathfrak{r}},$$
$$S(\mathfrak{h}^*)^W \otimes J(\mathfrak{h}^*) \longrightarrow S(\mathfrak{h}^*)^{W_\mathfrak{r}}, \tag{5.12}$$

defined by multiplication, are linear isomorphisms.

Proof. Using Propositions 5.6 and 5.8, Proposition 5.11 follows from (5.10) upon division by $\omega_r^\vee$ in the first line of (5.12) and division by ω_r in the second line. □

5.13. Recalling (3.38), let $\mathfrak{Z}_J(\mathfrak{r})$ be the inverse image of $J(\mathfrak{h})$ with respect to the Harish-Chandra isomorphism $\eta_\mathfrak{r}$. As an immediate consequence of the first line of (5.12), one has the following proposition.

Proposition 5.14. *The center $\mathfrak{Z}(\mathfrak{r})$ of $U(\mathfrak{r})$ is a free module of rank d over the subalgebra $\mathfrak{Z}_\mathfrak{g}(\mathfrak{r})$. In fact, the map*

$$\mathfrak{Z}_\mathfrak{g}(\mathfrak{r}) \otimes \mathfrak{Z}_J(\mathfrak{r}) \longrightarrow \mathfrak{Z}(\mathfrak{r}), \tag{5.15}$$

defined by multiplication, is a linear isomorphism.

Let $\gamma \in \mathfrak{h}^*$ be regular so that $\operatorname{card} W(\gamma) = \operatorname{card} W$, where $W(\gamma)$ is the W-orbit of γ. Let $\operatorname{Maps}(W(\gamma), \mathbb{C})$ be the $\operatorname{card} W$-dimensional space of all $\mathbb{C}$-valued functions on $W(\gamma)$. Now we may regard $S(\mathfrak{h})$ as the ring of all polynomial functions on $\mathfrak{h}^*$. Let

$$t_\gamma : \operatorname{Harm}(\mathfrak{h}) \longrightarrow \operatorname{Maps}\big(W(\gamma), \mathbb{C}\big) \tag{5.16}$$

be defined so that if $f \in \operatorname{Harm}(\mathfrak{h})$ and $w \in W$, then $t_\gamma(f)(w(\gamma)) = f(w(\gamma))$.

Lemma 5.17. *Let $\gamma \in \mathfrak{h}^*$ be W-regular. Then the map t_γ (see (5.16)) is a linear isomorphism.*

Proof. Let $h \in \operatorname{Maps}(W(\gamma), \mathbb{C})$. Since $W(\gamma)$ is obviously a Zariski closed subset of $\mathfrak{h}^*$, there exists $f' \in S(\mathfrak{h})$ such that $f' \mid W(\gamma) = h$. However, $S(\mathfrak{h})^W \mid W(\gamma)$ reduces to constant functions on $W(\gamma)$. It therefore follows from (5.2) that there exists $f \in \operatorname{Harm}(\mathfrak{h})$ such that $f \mid W(\gamma) = h$. Hence t_γ is surjective. But then t_γ is a linear isomorphism by dimension. □

Now for any $q \in \mathfrak{Z}_\mathfrak{g}(\mathfrak{r})$, let q' be the unique element in $\mathfrak{Z}(\mathfrak{g})$ such that

$$\eta_\mathfrak{r}(q) = \eta(q'). \tag{5.18}$$

Now if $\lambda \in \Lambda$, then Proposition 3.43 asserts that for $q \in \mathfrak{Z}_\mathfrak{g}(\mathfrak{r})$, the infinitesimal character value $\chi_\mathfrak{r}^\mu(q)$ is constant over all μ in the multiplet $\Lambda_\mathfrak{r}^\lambda$. In fact, by (3.39) and (3.45), one has

$$\chi_\mathfrak{r}^\mu(q) = \chi^\lambda(q') \quad \forall \mu \in \Lambda_\mathfrak{r}^\lambda. \tag{5.19}$$

An entirely different behavior occurs for $\mathfrak{Z}_J(\mathfrak{r})$.

PROPOSITION 5.20. *Let $\{q_i\}, i = 1, \dots, d$, be a basis of $\mathfrak{X}_J(\mathfrak{r})$. Let $\lambda \in \Lambda$, and let $\{\mu_j\}, j = 1, \dots, d$, be a basis of the multiplet $\Lambda_{\mathfrak{r}}^{\lambda}$. Then the $d \times d$ matrix $\chi_{\mathfrak{r}}^{\mu_j}(q_i)$ is nonsingular. In particular, if $c = \{c_1, \dots, c_d\}$ is an arbitrary point in $\mathbb{C}$, there exists a unique element $q_c \in \mathfrak{X}_J(\mathfrak{r})$ such that*

$$\chi_{\mathfrak{r}}^{\mu_j}(q_c) = c_j. \tag{5.20}$$

Proof. Assume that the $d \times d$ matrix $\chi_{\mathfrak{r}}^{\mu_i}(q_j)$ is singular. Then there exists a nonzero vector $\{b_1, \dots, b_d\}$ in $\mathbb{C}$ such that if $p = \sum_{i=1}^{d} b_i q_i$, then $\chi_{\mathfrak{r}}^{\mu}(p) = 0$ for all $\mu \in \Lambda_{\mathfrak{r}}^{\lambda}$. But if $f = \eta_{\mathfrak{r}}(p)$, then $0 \neq f \in J(\mathfrak{h})$. Furthermore, $f(\mu + \rho_{\mathfrak{r}}) = 0$ for all $\mu \in \Lambda_{\mathfrak{r}}^{\lambda}$ by (3.40). But then, by the definition of $\Lambda_{\mathfrak{r}}^{\lambda}$, one has $f(\tau(\lambda + \rho)) = 0$ for all $\tau \in W^1$. But f is $W_{\mathfrak{r}}$-invariant. Consequently, $f(w\tau(\lambda+\rho)) = 0$ for all $w \in W_{\mathfrak{r}}$ and $\tau \in W^1$. Thus f vanishes on the W-orbit $W(\lambda+\rho)$. But if $g = \omega_{\mathfrak{r}}^{\vee} f$, then g also vanishes on the orbit $W(\lambda+\rho)$. But $0 \neq g \in \operatorname{Harm}(\mathfrak{h})$. This contradicts Lemma 5.17 since $\lambda + \rho$ is W-regular. □

5.21. Let $\mathfrak{h}_{\mathbb{R}} = \kappa^{-1}(\mathfrak{h}_{\mathbb{R}}^*)$ and $\mathfrak{t} = i\mathfrak{h}_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}}$ be a compact form of $\mathfrak{g}$ containing $\mathfrak{t}$, and let $\mathfrak{r}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{r}$. Let G be a connected and simply connected compact Lie group corresponding to $\mathfrak{g}_{\mathbb{R}}$ so that one can regard $\mathfrak{g}_{\mathbb{R}} = \operatorname{Lie} G$. Since $\mathfrak{t} \subset \mathfrak{r}_{\mathbb{R}}$, it is clear that $\mathfrak{r}_{\mathbb{R}}$ equals its own normalizer in $\mathfrak{g}_{\mathbb{R}}$; hence if R is the subgroup of G corresponding to $\mathfrak{r}_{\mathbb{R}}$, then R is closed and hence is compact. We write X for the connected and simply connected compact homogeneous space G/R. Note that

$$\dim X = n = 2m. \tag{5.22}$$

Let $I_{\mathfrak{r}}(\mathfrak{h}^*)$ be the ideal in $S(\mathfrak{h}^*)^{W_{\mathfrak{r}}}$ generated by all homogeneous elements of positive degree in $S(\mathfrak{h}^*)^W$. It is clear from the second line in (5.12) that a quotient map composed with injection induces a linear isomorphism

$$J(\mathfrak{h}^*) \longrightarrow \frac{S(\mathfrak{h}^*)^{W_{\mathfrak{r}}}}{I_{\mathfrak{r}}(\mathfrak{h}^*)}. \tag{5.23}$$

Furthermore, it is clear that both sides of (5.23) are graded and the restriction of (5.23) to homogeneous components is a linear isomorphism

$$J^k(\mathfrak{h}^*) \longrightarrow \left(\frac{S(\mathfrak{h}^*)^{W_{\mathfrak{r}}}}{I_{\mathfrak{r}}(\mathfrak{h}^*)}\right)^k \tag{5.24}$$

for all $k \in \mathbb{Z}_+$. Let $H^*(X)$ denote the cohomology ring of X with coefficients in $\mathbb{C}$, and let $\operatorname{Eul}(X)$ be the Euler characteristic of X. We recall in the following proposition some well-known facts about homogeneous spaces of the form G/R (where we recall $\operatorname{rank} \mathfrak{r} = \operatorname{rank} \mathfrak{g}$). For proofs see, for example, [GHV, Chapter 10, §6, Theorem 11, p. 442].

PROPOSITION 5.25. *One has* $\mathrm{Eul}(X) > 0$*, and every compact, connected, and simply connected homogeneous space with positive Euler characteristic is of the form* G/R *with our present assumptions about* G *and* R*. Furthermore,* $H^k(X) = 0$ *if* k *is odd so that*

$$H^*(X) = \sum_{k=0}^{m} H^{2k}(X) \tag{5.26}$$

and

$$\mathrm{Eul}(X) = \dim H^*(X). \tag{5.27}$$

In addition, there exists an algebra isomorphism

$$\frac{S(\mathfrak{h}^*)^{W_{\mathfrak{r}}}}{I_{\mathfrak{r}}(\mathfrak{h}^*)} \longrightarrow H^*(X) \tag{5.28}$$

whose restriction to homogeneous components is a linear isomorphism

$$\left(\frac{S(\mathfrak{h}^*)^{W_{\mathfrak{r}}}}{I_{\mathfrak{r}}(\mathfrak{h}^*)}\right)^k \longrightarrow H^{2k}(X) \tag{5.29}$$

for all $k \in \mathbb{Z}_+$.

The composition of the maps (5.28) and (5.23) defines a linear isomorphism

$$J(\mathfrak{h}^*) \longrightarrow H^*(X) \tag{5.30}$$

whose restriction to homogeneous components (composition of (5.29) and (5.24)) is a linear isomorphism

$$J^k(\mathfrak{h}^*) \longrightarrow H^{2k}(X) \tag{5.31}$$

for all k. Also note that (5.27) and (5.30) imply the well-known fact that

$$\mathrm{Eul}(X) = d. \tag{5.32}$$

The grading on $J(\mathfrak{h})$ induces a grading on $Z_J(\mathfrak{r})$ so that $\eta_{\mathfrak{r}}(Z_J^k(\mathfrak{r})) = J^k(\mathfrak{h})$. Now clearly both $\omega_r^\vee$ and $\omega^\vee$ are in $S(\mathfrak{h})^{sg_{\mathfrak{r}}}$ so that $\omega_{\mathfrak{p}}^\vee \in S(\mathfrak{h})^{W_{\mathfrak{r}}}$ by (5.5). On the other hand, $\omega^\vee \in \mathrm{Harm}(\mathfrak{h})$ by Proposition 5.6. Thus $\omega_{\mathfrak{p}}^\vee \in J(\mathfrak{h})$ by (5.5) and the definition of $J(\mathfrak{h})$. More specifically, since $\omega_{\mathfrak{p}}^\vee$ is homogeneous of degree m, one has

$$\omega_{\mathfrak{p}}^\vee \in J^m(\mathfrak{h}). \tag{5.33}$$

We choose the basis $\{q_i\}, i = 1, \ldots, d$, so that (see Proposition 5.14)

$$\eta_{\mathfrak{r}}(q_d) = \omega_{\mathfrak{p}}^\vee. \tag{5.34}$$

Let $b_k(X)$ be the kth Betti number of X so that $b_k(X) = 0$ if k is odd.

PROPOSITION 5.35. *For any $k \in \mathbb{Z}_+$, one has*

$$\dim Z_J^k(\mathfrak{r}) = b_{2k}(X). \tag{5.36}$$

In particular, $Z_J^k(\mathfrak{r}) = 0$ for $k > m$ and $\dim Z_J^m(\mathfrak{r}) = 1$. In fact,

$$Z_J^m(\mathfrak{r}) = \mathbb{C}q_d. \tag{5.37}$$

Proof. Obviously, $\kappa(J^k(\mathfrak{h})) = J^k(\mathfrak{h}^*)$. But then (5.36) follows from (5.30), which implies that $\dim Z_J^m(\mathfrak{r}) = 1$. But then (5.37) follows from (5.33). $\square$

5.38. Next we see that the element $q_d \in \mathbb{Z}(\mathfrak{r})$ has remarkable Pfaffian-like properties and that all its infinitesimal character values can be determined. Recall that the $\mathfrak{r}$-highest weight subset $\Lambda_\mathfrak{r}^*$ of $\Lambda_\mathfrak{r}$ (see (3.30)) was distinguished by a W-regular condition. This subset will now be distinguished by q_d in a simpler way. Furthermore, besides the partition of $\lambda_\mathfrak{r}^*$ by d-multiplets (see (3.35)), the subset $\lambda_\mathfrak{r}^*$ has a $\pm$ partition

$$\Lambda_\mathfrak{r}^* = \Lambda_\mathfrak{r}^+ \cup \Lambda_\mathfrak{r}^-, \tag{5.39}$$

defined by putting $\Lambda_\mathfrak{r}^+ = \cup_{\lambda\in\Lambda} \Lambda_\mathfrak{r}^{\lambda,+}$ and $\Lambda_\mathfrak{r}^- = \cup_{\lambda\in\Lambda} \Lambda_\mathfrak{r}^{\lambda,-}$ (see (3.50)).

THEOREM 5.40. *Let $\mu \in \Lambda_\mathfrak{r}$. Then $\mu \in \Lambda_\mathfrak{r}^*$ if and only if $\chi_\mathfrak{r}^\mu(q_d) \neq 0$. Furthermore, if $\mu \in \Lambda_\mathfrak{r}^*$, then $\chi_\mathfrak{r}^\mu(q_d)$ is real and is greater than zero if $\mu \in \Lambda_\mathfrak{r}^+$ and is less than zero if $\mu \in \Lambda_\mathfrak{r}^-$. More explicitly, assume $\mu \in \Lambda_\mathfrak{r}^*$ (see (3.35)) so that there exists $\lambda \in \Lambda$ and $\tau \in W^1$ such that $\mu = \tau \bullet \lambda$. Then*

$$\chi_\mathfrak{r}^\mu(q_d) = sg(\tau)k_o \frac{\dim V_\lambda}{\dim Z_\mu}, \tag{5.41}$$

where k_o is a positive constant independent of μ. In fact, k_o is the quotient of the denominator of Weyl's dimension formula for $\mathfrak{g}$ by the denominator of Weyl's dimension formula for $\mathfrak{r}$.

Proof. By the definition of q_d (see (5.34)), one has

$$\chi_\mathfrak{r}^\mu(q_d) = \omega_\mathfrak{p}^\vee\big(\mu + \rho_\mathfrak{r}\big) \tag{5.42}$$

by (3.40). But $\mu + \rho_\mathfrak{r}$ is clearly $W_\mathfrak{r}$-regular. Thus $\omega_\mathfrak{r}^\vee(\mu + \rho_\mathfrak{r}) \neq 0$. In fact, $\omega_\mathfrak{r}^\vee(\mu + \rho_\mathfrak{r}) = k_o' \dim Z_\mu$, where k_o' is the denominator of Weyl's dimension formula for $\mathfrak{r}$. But then multiplying and dividing the right side of (5.42) by $k_o' \dim Z_\mu$, one has

$$\chi_\mathfrak{r}^\mu(q_d) = \frac{\omega^\vee\big(\mu + \rho_\mathfrak{r}\big)}{k_o' \dim Z_\mu}. \tag{5.43}$$

But the numerator $\omega^\vee(\mu + \rho_\mathfrak{r})$ of the right side of (5.43) is zero or not zero according as $\mu + \rho_\mathfrak{r}$ is W-regular or not. That is, $\chi_\mathfrak{r}^\mu(q_d)$ is zero or not zero according as μ is in

$\Lambda_{\mathfrak{r}}^*$ or not. This proves the first statement of the theorem. Now assume $\mu \in \Lambda_{\mathfrak{r}}^*$, and let $\lambda \in \Lambda$ and $\tau \in W^1$ be such that $\mu = \tau \bullet \lambda$. Then $\mu + \rho_{\mathfrak{r}} = \tau(\lambda + \rho)$, and hence (5.43) can be written as

$$\chi_{\mathfrak{r}}^{\mu}(q_d) = \frac{\omega^{\vee}\big(\tau(\lambda+\rho)\big)}{k_o' \dim Z_{\mu}}. \tag{5.44}$$

But $\omega^{\vee} \in S(\mathfrak{h})^{sg}$. Thus the numerator of the right side of (5.44) equals $sg(\tau)\omega^{\vee}(\lambda+\rho)$. But then, by Weyl's dimension formula for $\mathfrak{g}$, this numerator equals $sg(\tau)k_o'' \dim V_{\lambda}$, where k_o'' is the denominator of Weyl's dimension formula for $\mathfrak{g}$. This proves the theorem, and in particular (5.41), where $k_o = k_o''/k_o'$. □

6. Multiplets and topological K-theory

6.1. Let T be the subgroup of R that corresponds to $\mathfrak{t} = i\mathfrak{h}_{\mathbb{R}}$, so that T is a maximal torus of both R and G. Let Z be a finite-dimensional reductive r-module with respect to a representation

$$\sigma : \mathfrak{r} \longrightarrow \operatorname{End} Z \tag{6.2}$$

such that (recalling the notation of §4.1) $\Gamma(Z) \subset \Gamma_{\mathfrak{r}}$. We say that σ exponentiates to R if there is a representation of R on Z, also denoted by σ, for which (6.2) is the complexified differential. One readily has the following lemma.

Lemma 6.3. *A representation (6.2) of $\mathfrak{r}$ exponentiates to R if and only if $\Gamma(Z) \subset \Gamma$.*

Proof. Since G is simply connected, Γ is just the set of complexified differentials of the set of characters of T. Thus if σ exponentiates to R, one has $\Gamma(Z) \subset \Gamma$. Now let R^{sc} be a simply connected Lie group with Lie algebra $\mathfrak{r}_{\mathbb{R}}$, and let $K \subset R^{\mathrm{sc}}$ be the kernel of the covering homomorphism $h : R^{\mathrm{sc}} \to R$. One knows and easily has that $K \subset T^{\mathrm{sc}}$ where T^{sc} is the subgroup of R^{sc} corresponding to $\mathfrak{t}$. In particular, K is the kernel of the restricted epimorphism $h : T^{\mathrm{sc}} \to T$. Now given σ, let σ^{sc} be the representation of R^{sc} on Z having (6.2) as its complexified differential. But then if $\Gamma(Z) \subset \Gamma$, it is immediate that K is in the kernel of σ^{sc}. Hence σ^{sc} descends to R. □

One of course has $\Gamma \subset \Gamma_{\mathfrak{r}}$. Now consider the assumption

$$\Gamma = \Gamma_{\mathfrak{r}}. \tag{6.4}$$

We refer to (6.4) as the *equal weight assumption*.

Proposition 6.5. *Under the equal weight assumption, all the irreducible representations*

$$\sigma_{\mu} : \mathfrak{r} \longrightarrow \operatorname{End} Z_{\mu},$$

$\mu \in \Lambda_{\mathfrak{r}}$, exponentiate to R.

Proof. This is immediate from Lemma 6.3. □

Since $\rho_{\mathfrak{p}} = \rho - \rho_{\mathfrak{r}}$, the equal weight assumption is equivalent to the assumption

$$\rho_{\mathfrak{p}} \in \Gamma$$

or

$$\rho_{\mathfrak{r}} \in \Gamma. \tag{6.6}$$

PROPOSITION 6.7. *The equal weight assumption is satisfied if and only if the representation (see (1.58))* $\operatorname{Spin}\nu : \mathfrak{r} \to \operatorname{End} S$ *exponentiates to* R. *Moreover, in such a case (see (3.14)),* $\operatorname{Spin}_+ \nu$ *and* $\operatorname{Spin}_- \nu$ *exponentiate to* R *so that* $V_\lambda \otimes S_+$ *and* $V_\lambda \otimes S_-$*, for any* $\lambda \in \Lambda$*, have the structure of* R*-modules arising from their structure as* $\mathfrak{r}$*-modules*.

Proof. If the equal weight assumption is satisfied, then $\operatorname{Spin}\nu$ exponentiates to R by Lemma 6.3 and the last statement in Proposition 3.17. The latter asserts that $\rho_{\mathfrak{p}}$ is a weight of $\operatorname{Spin}\nu$ so that, conversely, if $\operatorname{Spin}\nu$ exponentiates to R, then the equal weight assumption is satisfied by (6.6) and Lemma 6.3. The remaining statements of Proposition 6.7 are obvious because $\operatorname{Spin}_+ \nu$ and $\operatorname{Spin}_- \nu$ are subrepresentations of $\operatorname{Spin}\nu$. □

Remark 6.8. Obviously, the equal weight assumption is satisfied if R is simply connected. An example is the case when $(\mathfrak{r}, \mathfrak{g}) = (\text{Lie Spin } 9, F_4)$. Another condition guaranteeing the equal weight assumption is

$$\rho_{\mathfrak{r}} \text{ is in the root lattice of } \mathfrak{r}. \tag{6.9}$$

Let $\mathfrak{r}' = [\mathfrak{r}, \mathfrak{r}]$ so that $\mathfrak{r}'$ is semisimple. If C is the center of the simply connected Lie group with Lie algebra $\mathfrak{r}'$, then a sufficient (but not necessary) condition for (6.9) is when C has no elements of order 2 (e.g., when the simple components of $\mathfrak{r}'$ are of type A_N with N even or any of the exceptional Lie algebras except E_7). This follows easily since the character induced on C by the irreducible representation of $\mathfrak{r}$ with highest weight $\rho_{\mathfrak{r}}$ has order 2 and, of course, C is isomorphic to its character group. The question as to whether (6.9) is satisfied reduces to considering the simple components of $\mathfrak{r}'$. For a simple Lie algebra $\mathfrak{s}$, one can show that the corresponding $\rho_{\mathfrak{s}}$ is in the root lattice if and only if all the complex Hermitian symmetric spaces associated to a compact form of $\mathfrak{s}$ have even complex dimension.

6.10. Let $T(X)$ be the tangent bundle to $X = G/R$, and let $o \in X$ correspond to the coset R. Let $\mathfrak{p}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{p}$ so that one has a canonical isomorphism

$$\mathfrak{p}_{\mathbb{R}} \cong T_o(X). \tag{6.11}$$

Let M be the G-invariant Riemannian metric on X such that M_o corresponds to $-B_{\mathfrak{g}} \mid \mathfrak{p}_{\mathfrak{r}}$ with respect to the isomorphism (6.11). Since X is simply connected, it is orientable. Choose an orientation, and let $\mathcal{B}$ be the principal $\mathrm{SO}(n,\mathbb{R})$ bundle over X of oriented orthonormal frames in $T(X)$. Let $t:\mathrm{Spin}(n,\mathbb{R})\to\mathrm{SO}(n,\mathbb{R})$ be the simply connected double covering of $\mathrm{SO}(n,\mathbb{R})$. A spin structure on X is a principal $\mathrm{Spin}(n,\mathbb{R})$-bundle $\mathcal{B}'$ over X, whose projection onto X factors through a double covering $\delta:\mathcal{B}'\to\mathcal{B}$ such that $\delta(ga)=\delta(g)t(a)$ for any $a\in\mathrm{Spin}(n,\mathbb{R})$ and $g\in\mathcal{B}'$. One knows that X has a spin structure if and only if the second Stiefel-Whitney class of X vanishes. Since X is simply connected, this structure is unique if it exists.

Recall (1.10). Note that $\wedge^2\mathfrak{p}_{\mathbb{R}}$ is a Lie algebra real form of $\wedge^2\mathfrak{p}$ under Clifford multiplication and $\wedge^2\mathfrak{p}_{\mathbb{R}}$ maps to Lie $\mathrm{SO}(\mathfrak{p}_{\mathbb{R}})$ under the isomorphism (1.7). Consequently, (1.10) restricts to a Lie algebra homomorphism

$$\nu_*:\mathfrak{r}_{\mathbb{R}}\longrightarrow\wedge^2\mathfrak{p}_{\mathbb{R}}. \tag{6.12}$$

Let $\mathrm{Spin}(\mathfrak{p}_{\mathbb{R}})$ be the (real) Lie group in $C(\mathfrak{p})$ generated by $\exp\wedge^2\mathfrak{p}_{\mathbb{R}}$. Then one knows (see, e.g., [Ch1, p. 68]) that

$$t_o:\mathrm{Spin}(\mathfrak{p}_{\mathbb{R}})\longrightarrow\mathrm{SO}(\mathfrak{p}_{\mathbb{R}}) \tag{6.13}$$

is a double covering, where if $a\in\mathrm{Spin}(\mathfrak{p}_{\mathbb{R}})$ and $y\in\mathfrak{p}_{\mathbb{R}}$, then $t_o(a)(y)=aya^{-1}$. Note that the map (1.7) is the complexified differential of t_o. Recalling the isomorphism (3.6), the following is just a restatement of the first statement of Proposition 6.7.

PROPOSITION 6.14. *The equal weight assumption is satisfied if and only if there is a homomorphism*

$$\nu_o:R\longrightarrow\mathrm{Spin}(\mathfrak{p}_{\mathbb{R}}) \tag{6.15}$$

having (6.12) as its differential.

The following is a topological necessary and sufficient condition for the equal weight assumption to be satisfied.

THEOREM 6.16. *The equal weight assumption (see (6.4)) is satisfied if and only if $X=G/R$ is a spin manifold (i.e., the second Stiefel-Whitney class of X vanishes).*

Proof. The fiber $\mathcal{B}_o$ of $\mathcal{B}$ at o is a left principal homogeneous space for $\mathrm{SO}(\mathfrak{p}_{\mathbb{R}})$ and a right principal homogeneous space for $\mathrm{SO}(n,\mathbb{R})$ where the two actions commute. Obviously, $\nu:\mathfrak{r}\to$ Lie $\mathrm{SO}(\mathfrak{p}_{\mathbb{R}})$ exponentiates to R so that one has

$$\nu:R\longrightarrow\mathrm{SO}(\mathfrak{p}_{\mathbb{R}}). \tag{6.17}$$

Clearly, we may take

$$\mathcal{B}=G\times_R\mathcal{B}_o, \tag{6.18}$$

where R operates on $\mathcal{B}_o$ via the composition of (6.17) and the left action of $\mathrm{SO}(\mathfrak{p}_{\mathbb{R}})$. Now let $\delta_o:\widetilde{\mathcal{B}}_o\to\mathcal{B}_o$ be a simply connected double covering map. Then the actions of $\mathrm{SO}(\mathfrak{p}_{\mathbb{R}})$ and $\mathrm{SO}(\mathfrak{p}_{\mathbb{R}})$ on $\mathcal{B}_o$ canonically lift so that $\widetilde{\mathcal{B}}_o$ has the structure of a left principal homogeneous space for $\mathrm{Spin}(\mathfrak{p}_{\mathbb{R}})$ and a right principal homogeneous space for $\mathrm{Spin}(n, \mathbb{R})$, where again the two actions commute. Now assume that the equal weight assumption is satisfied, and let ν_o be as in (6.15). Then we may put

$$\mathcal{B}' = G \times_R \widetilde{\mathcal{B}}_o,$$

where R operates on $\widetilde{\mathcal{B}}_o$ via the composition of ν_o and the left action of $\mathrm{Spin}(\mathfrak{p}_{\mathbb{R}})$ on $\widetilde{\mathcal{B}}_o$. It is immediate that $\mathcal{B}'$ defines a spin structure on X.

Conversely, assume that $\delta : \mathcal{B}' \to \mathcal{B}$ is a double covering defining a spin structure on X. Then the fiber $\mathcal{B}'_o$ of $\mathcal{B}$ at o has a lifted structure of a left principal homogeneous space for $\mathrm{Spin}(\mathfrak{p}_{\mathbb{R}})$ and a right principal homogeneous space for $\mathrm{Spin}(n, \mathbb{R})$ where the two actions commute. In particular, these actions are each others' centralizer in the permutation group of $\mathcal{B}'_o$. But since G is simply connected, the action of G on $\mathcal{B}$ lifts naturally to an action of G on $\mathcal{B}'$. Clearly, R stabilizes $\mathcal{B}'_o$ and commutes with the right action of $\mathrm{Spin}(n, \mathbb{R})$. But then one has a homomorphism $\nu_o : R \to \mathrm{Spin}(\mathfrak{p}_{\mathbb{R}})$, which is clearly a lift of (6.17). But then the equal weight assumption is satisfied by Proposition 6.14. □

Remark 6.19. If $\mathfrak{g}$ is simple and X is a non-Hermitian symmetric space, then Nolan Wallach [W] has determined all cases where the second Stiefel-Whitney class vanishes. (He assumes, as we do, that $\operatorname{rank} \mathfrak{g} = \operatorname{rank} \mathfrak{r}$.) Under these assumptions, his determination includes a proof that this is always the case if $\mathfrak{g}$ is laced simply.

6.20. Now let $K_{\mathbb{C}}(X)$ be the topological K-cohomology group of the compact manifold $X = G/R$ defined by complex vector bundles over X. For details, see [Ka]. For simplicity of notation, put $K(X) = K_{\mathbb{C}}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ so that $K(X)$ has the structure of a complex vector space. If $E = E(X)$ is a complex vector bundle over X, let $[E] \in K(X)$ denote the corresponding cohomology class. If Z is a finite-dimensional $\mathfrak{r}$-module with respect to a representation σ, which exponentiates to R, let $E_Z = E_Z(X)$ be the homogeneous vector bundle over X defined by putting

$$E_Z = G \times_R Z,$$

where, of course, R operates on Z via σ. If $\mu \in \Lambda_{\mathfrak{r}}$ and $\sigma_\mu : \mathfrak{r} \to \operatorname{End} Z_\mu$ exponentiates to R, we write $E_\mu = E_\mu(X)$ for E_{Z_μ}. One knows that there exists a linear map $\mathrm{Ch} : K(X) \to H^*(X)$ defined by the Chern character of a complex vector bundle over X. (See, e.g., [Ka, Chapter 5, §3, especially 3.24]; see also equation (3) of §10 in [Hi].) Furthermore, since $H^k(X) = 0$, if k is odd the map

$$\mathrm{Ch} : K(X) \longrightarrow H^*(X) \tag{6.21}$$

is a linear isomorphism. See [Ka, Chapter 5, Theorem 3.25]. In particular,

$$\dim K(X) = d. \tag{6.22}$$

Let $Y = G/T$ so that Y is the G-flag manifold. Of course, Y is a special case of X (so that the notation above applies) where $R = T$. In general, the embedding $T \to R$ induces a bundle map

$$\psi : Y \longrightarrow X \tag{6.23}$$

with fiber equal to the R-flag manifold R/T. Next note that for Y, (5.28) becomes the (Borel) isomorphism

$$\frac{S(\mathfrak{h}^*)}{I(\mathfrak{h}^*)} \longrightarrow H^*(Y). \tag{6.24}$$

Let $S[[\mathfrak{h}^*]]$ be the completion of $S(\mathfrak{h}^*)$ defined as the direct product of $S^k(\mathfrak{h}^*)$ over all $k \in \mathbb{Z}_+$. One has $e^\gamma \in S[[\mathfrak{h}^*]]$ for any $\gamma \in \mathfrak{h}^*$. The span of elements in $S[[\mathfrak{h}^*]]$ of the form e^γ can be regarded, of course, as functions on $\mathfrak{h}$. Recalling the duality between $S(\mathfrak{h})$ and $S(\mathfrak{h}^*)$ (see §5.1), we may also regard $S[[\mathfrak{h}^*]]$ as the algebraic dual of $S(\mathfrak{h})$. Note that for $\gamma \in \mathfrak{h}^*$ and $f \in S(\mathfrak{h})$,

$$\langle e^\gamma, f \rangle = f(\gamma). \tag{6.25}$$

Let $I[[\mathfrak{h}^*]]$ be the ideal in $S[[\mathfrak{h}^*]]$ generated by $I(\mathfrak{h}^*)$. Since clearly $S^k(\mathfrak{h}^*) = I^k(\mathfrak{h}^*)$ for all $k > \dim Y$, the isomorphism (6.24) induces a surjection

$$g : S[[\mathfrak{h}^*]] \longrightarrow H^*(Y), \tag{6.26}$$

where clearly

$$I[[\mathfrak{h}^*]] = \operatorname{Ker} g. \tag{6.27}$$

Recalling the definition of $\operatorname{Harm}(\mathfrak{h}) \subset S(\mathfrak{h})$, note that if $u \in S[[\mathfrak{h}^*]]$, then

$$u \in I[[\mathfrak{h}^*]] \iff \langle u, f \rangle = 0 \quad \forall f \in \operatorname{Harm}(\mathfrak{h}). \tag{6.28}$$

Let $\gamma \in \mathfrak{h}^*$. We may regard $\gamma : \mathfrak{h} \to \operatorname{End} \mathbb{C}$ as a 1-dimensional representation. Now assume $\gamma \in \Gamma$. Then the representation exponentiates to T. Let $\mathbb{C}_\gamma$ denote $\mathbb{C}$ with the structure of a T-module defined by γ, and let L_γ be the line bundle over Y defined by putting $L_\gamma = G \times_T \mathbb{C}_\gamma$. One has $[L_\gamma] \in K(Y)$ and $\operatorname{Ch}([L_\gamma]) \in H^*(Y)$. A key property of the map (6.26) is

$$g(e^\gamma) = \operatorname{Ch}\big([L_\gamma]\big). \tag{6.29}$$

See, e.g., [CG, Lemmas 6.4.7 and 6.4.18 and Proposition 6.4.19]. Assuming that X is a spin manifold, we now find that the vector bundle classes corresponding to any d-multiplet Λ^λ define a basis of $K(X)$.

THEOREM 6.30. *Assume that $X = G/R$ is a spin manifold so that σ_μ, for any $\mu \in \Lambda_\mu$, exponentiates to a representation of R (see Proposition 6.7 and Theorem 6.16) and hence defines a homogeneous vector bundle E_μ over X. We note that the assumptions are such that X is the most general simply connected compact homogeneous space of positive Euler characteristic that has a vanishing second Stiefel-Whitney class. Then for any $\lambda \in \Lambda$, the classes $[E_\mu] \in K(X), \mu \in \Lambda^\lambda$, are a basis of $K(X)$. In particular, $K(X)$ is generated by the classes of homogeneous vector bundles.*

Proof. Let $\lambda \in \Lambda$. By (6.22) it suffices to prove that the classes $[E_\mu] \in K(X), \mu \in \Lambda^\lambda$, are linearly independent. Assume that $b_\mu \in \mathbb{C}$ and $\mu \in \Lambda^\lambda$ are constants such that

$$\sum_{\mu \in \Lambda^\lambda} b_\mu [E_\mu] = 0.$$

Then by the map (6.21), one has

$$\sum_{\mu \in \Lambda^\lambda} b_\mu \operatorname{Ch}\big([E_\mu]\big) = 0.$$

But then, recalling (6.23), one has

$$\sum_{\mu \in \Lambda^\lambda} b_\mu \psi^*\big(\operatorname{Ch}\big([E_\mu]\big)\big) = 0.$$

By the functorial properties of the Chern character (an immediate consequence of [Hi, Chapter 1, §4.1, Axiom 2, and §10, equation (3)]) one has

$$\sum_{\mu \in \Lambda^\lambda} b_\mu \big(\operatorname{Ch}\big(\big[\psi^*(E_\mu)\big]\big)\big) = 0. \tag{6.31}$$

However, clearly $\psi^*(E_\mu)$ is the vector bundle on the flag manifold $Y = G/T$ given by

$$\psi^*(E_\mu) = G \times_T Z_\mu, \tag{6.32}$$

where Z_μ is a T-module by the restriction $\sigma_\mu \mid T$. But then $\psi^*(E_\mu)$ is a sum of homogeneous line bundles corresponding to the weights, with multiplicities, of σ_μ. Thus recalling the notation of §3.22 and (6.29), one has

$$g\big(\operatorname{ch} Z_\mu \mid \mathfrak{h}\big) = \operatorname{Ch}\big(\big[\psi^*(E_\mu)\big]\big). \tag{6.33}$$

Thus, by (6.31),

$$g\left(\sum_{\mu \in \Lambda^\lambda} b_\mu \operatorname{ch} Z_\mu \mid \mathfrak{h}\right) = 0.$$

Hence,

$$\sum_{\mu\in\Lambda^{\lambda}} b_{\mu}\,\mathrm{ch}\,Z_{\mu}\mid \mathfrak{h}\in I[[\mathfrak{h}^*]] \tag{6.34}$$

by (6.27). Recall Weyl's formula for $\mathrm{ch}\,Z_{\mu}\mid\mathfrak{h}$. Then, since $I[[\mathfrak{h}^*]]$ is an ideal, if we multiply the left side of (6.34) by the denominator of Weyl's formula, we get

$$\sum_{\mu\in\Lambda^{\lambda}} b_{\mu}\left(\sum_{w'\in W_{\mathfrak{r}}} sg\big(w'\big)e^{w'(\mu+\rho_{\mathfrak{r}})}\right)\in I[[\mathfrak{h}^*]]. \tag{6.35}$$

But $\Lambda^{\lambda}=\{\tau\bullet\lambda\}, \tau\in W^1$, where we recall $\tau\bullet\lambda=\tau(\lambda+\rho)-\rho_{\mathfrak{r}}$. Thus we may rewrite (6.35) as

$$\sum_{w'\in W_{\mathfrak{r}},\tau\in W^1} b_{\tau\bullet\lambda}sg\big(w'\big)e^{w'\tau(\lambda+\rho)}\in I[[\mathfrak{h}^*]]. \tag{6.36}$$

But any element $w\in W$ is uniquely of the form $w=w'\tau$ for $w'\in W_{\mathfrak{r}}$ and $\tau\in W^1$. Thus if we define $b_w=b_{\tau\bullet\lambda}sg(w')$, one has

$$\sum_{w\in W} b_w e^{w(\lambda+\rho)}\in I[[\mathfrak{h}^*]]. \tag{6.37}$$

But then for any $f\in\mathrm{Harm}(\mathfrak{h})$, one has

$$\sum_{w\in W} b_w f\big(w(\lambda+\rho)\big)=0$$

by (6.25) and (6.28). But then $b_w=0$ for any $w\in W$ by Lemma 5.17, since $\lambda+\rho$ is W-regular. Thus $b_{\mu}=0$ for any $\mu\in\Lambda^{\lambda}$ so that the set of classes $\{[E_{\mu}]\}, \mu\in\Lambda^{\lambda}$, are linearly independent. □

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