The Coxeter element and the branching law for the finite subgroups of $SU(2)$

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0. Introduction

0.1. Let Γ be a finite subgroup of $SU(2)$. The question we will deal with in this paper is how an arbitrary (unitary) irreducible representation of $SU(2)$ decomposes under the action of Γ. The theory of McKay assigns to Γ a complex simple Lie algebra $\mathfrak{g}$ of type $A-D-E$. The assignment is such that if $\widetilde{\Gamma}$ is the unitary dual of Γ we may parameterize $\widetilde{\Gamma}$ by the nodes (or vertices) of the extended Coxeter-Dynkin diagram of $\mathfrak{g}$.

Let $\ell = rank\,\mathfrak{g}$ and let $I = \{1,\ldots,\ell\}$. Let $I_{ext} = I \cup \{0\}$. The nodes may be identified with a set of simple roots of the affine Kac-Moody Lie algebra associated to $\mathfrak{g}$ and are indexed by I_{ext}. We can then write $\Gamma = \{\gamma_i\}$, $i \in I_{ext}$. Let $\Pi = \{\alpha_i\}$, $i \in I$, be the set of simple roots of $\mathfrak{g}$ itself. One has γ_0 is the trivial 1 dimensional representation of Γ and, for $i \in I$,

$$dim\,\gamma_i = d_i \tag{0.1}$$

where

$$\psi = \sum_{i\in I} d_i\,\alpha_i \tag{0.2}$$

is the highest root. For proofs and details about the McKay correspondence see e.g. [G-S,V], [M] and [St].

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0.2. The unitary dual of $SU(2)$ is indexed by the set $\mathbb{Z}_+$ of nonnegative integers and will be written as $\{\pi_n\}$, $n \in \mathbb{Z}_+$ where

$$dim\, \pi_n = n + 1 \tag{0.3}$$

Our problem is the determination of $m_{n,i}$ where $n \in \mathbb{Z}_+,\ i \in I_{ext}$ and

$$m_{n,i} = \text{multiplicity of } \gamma_i \text{ in } \pi_n | \Gamma$$

It is resolved with the determination of the formal power series

$$m(t)_i = \sum_{n=0}^{\infty} m_{n,i}\, t^n \tag{0.4}$$

To do this one readily notes that it suffices to consider only the case where $\Gamma = F^*$ and F^* is the pullback to $SU(2)$ of a finite subgroup F of $SO(3)$. This eliminates only the case where Γ is a cyclic group of odd order and $\mathfrak{g}$ is of type A_ℓ where ℓ is even. For the remaining cases the Coxeter number h of $\mathfrak{g}$ is even and we will put

$$g = h/2 \tag{0.5}$$

Also for the remaining cases there is a special index $i_* \in I$. If $\mathfrak{g}$ is of type D or E then α_{i_*} is the branch point of the Coxeter-Dynkin diagram of $\mathfrak{g}$. If $\mathfrak{g}$ is of type A_ℓ then α_{i_*} is the midpoint of the diagram (recalling that ℓ is odd).

If $i = 0$ the determination of $m(t)_0$ is classical and is known from the theory of Kleinian singularities. In fact there exists positive integers $a < b$ such that

$$m(t)_0 = \frac{1 + t^h}{(1 - t^a)(1 - t^b)} \tag{0.6}$$

The numbers a and b in Lie theoretic terms is given in

Theorem 0.1. *One has $a = 2d_{i_*}$ and b is given by the condition that*

$$\begin{aligned} a\, b &= 2\, |F^*| \\ &= 4\, |F| \end{aligned}$$

It remains then to determine $m(t)_i$ for $i \in I$.

Proposition 0.2. *If $i \in I$ there exists a polynomial $z(t)_i$ of degree less than h such that*

$$m(t)_i = \frac{z(t)_i}{(1-t^a)(1-t^b)} \tag{0.7}$$

The problem then is to determine the polynomial $z(t)_i$. This problem was solved in [K] using the orbits of a Coxeter element σ on a set of roots Δ for $\mathfrak{g}$. In the present paper we will put the main result of [K] in a simplified form. See Theorem 1.13 in the present paper. Also the present paper makes explicit some results that are only implicit in [K]. For example introducing $\widetilde{\Pi}$ (see (1.10)) and making the assertions in Remark 10 and Theorems 8, 9, 11 and 12.

The set Π generates a system, Δ_+, of posiive roots. The highest root $\psi \in \Delta$ defines a certain subset $\Phi \subset \Delta_+$ of cardinality $2h-3$. Because of its connection with a Heisenberg subalgebra of $\mathfrak{g}$ this subset is referred to as the Heisenberg subsystem of Δ_+. The new formulation explicitly shows how the polynomials $z(t)_i$ arise from the intersection

$$(\text{orbits of the Coxeter element } \sigma) \cap (\text{the Heisenberg subsystem } \Phi) \tag{0.8}$$

The polynomials $z(t)_i$ are listed in [K]. The special case where $\mathfrak{g}$ is of type E_8 is also given in the present paper (see Example 1.7.). Unrelated to the Coxeter element the polynomials $z(t)_i$ are also determined in Springer, [Sp]. They also appear in another context in Lusztig, [L1] and [L2]. Recently, in a beautiful result, Rossmann, [R], relates the character of γ_i to the polynomial $z(t)_i$.

1. The main result - Theorem 1.13.

1.1. Proofs of the main results stated here are in [K].

Let F be a finite subgroup of $SO(3)$ and let

$$F^* \subset SU(2) \tag{1.1}$$

be the pullback of the double covering

$$SU(2) \to SO(3)$$

The unitary dual $\widehat{SU(2)}$ of $SU(2)$ is represented by the set $\{\pi_n\}$, $n \in \mathbb{Z}_+$, where if $S(\mathbb{C}^2)$ is the symmetric algebra then

$$\pi_n : SU(2) \to S^n(\mathbb{C}^2)$$

is the $n+1$ dimensional representation defined by the natural action of $SU(2)$ on $\mathbb{C}^2$. We are ultimately interested in determining how the restriction $\pi_n|F^*$ decomposes into irreducible representations of the finite subgroup F^*, for any n, and relating this determination to the structure of the simple Lie algebra corresponding to F^* by the McKay correspondence. We now recall this correspondence.

Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\ell = rank\, \mathfrak{g}$, and if $\mathfrak{h}'$ is the dual space to $\mathfrak{h}$, let $\Delta \subset \mathfrak{h}'$ be the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$. Let W, operating in $\mathfrak{h}'$, be the Weyl group. Let Π be the set of simple positive roots with respect to a choice, Δ_+, of positive roots. If $I = \{1, \ldots, \ell\}$ we will write $\Pi = \{\alpha_i\}$, $i \in I$. We may regard Π as the vertices (or nodes) of the Coxeter-Dynkin diagram associated to $\mathfrak{g}$. The extended Coxeter-Dynkin diagram has an additional node α_0.

The McKay correspondence assigns to F^* a complex simple Lie algebra $\mathfrak{g} = \mu(F^*)$ of type $A - D - E$. The assignment has a number of properties: (1), the unitary dual $\widehat{F^*}$ may be parameterized by indices of the nodes of the extended Coxeter-Dynkin diagram of $\mathfrak{g}$. In particular $card\, \widehat{F^*} = \ell + 1$ and we can write $\widehat{F^*} = \{\gamma_i\}$, $i \in I_{ext} = I \cup \{0\}$. Next (2), γ_0 is the trivial 1-dimensional representation, and if $i \in I$, then

$$dim\, \gamma_i = d_i$$

where

$$\psi = \sum_{i=1}^{\ell} d_i \alpha_i$$

is the highest root in Δ. In addition (3), if γ is the two-dimensional representation defined by (1.1) and A is the $(\ell+1) \times (\ell+1)$ matrix defined so that

$$\gamma_i \otimes \gamma = \sum_{j=0}^{\ell} A_{ij} \gamma_j \tag{1.2}$$

then C is the Cartan matrix of the extended Coxeter-Dynkin of $\mathfrak{g}$ where

$$C_{ij} = 2\delta_{ij} - A_{ij}$$

1.2. Returning to our main problem, for $i \in I_{ext}$ and $n \in \mathbb{Z}_+$, let

$$m_{n,i} = \text{multiplicity of } \gamma_i \text{ in } \pi_n | F^*$$

and introduce the generating formal power series

$$m(t)_i = \sum_{n=0}^{\infty} m_{n,i}\, t^n$$

If $i = 0$, the determination of $m(t)_i$ is classical and is known from the theory of Kleinian singularities. That is, in this case

$$m_{n,0} = dim\, (S^n(\mathbb{C}))^{F^*}$$

In fact let h be the Coxeter number of $\mathfrak{g}$ so that

$$\ell(h+1) = dim\, \mathfrak{g}$$

Then there exists positive integers $a < b$ such that

$$m(t)_0 = \frac{1 + t^h}{(1-t^a)(1-t^b)} \tag{1.3}$$

To define the numbers a and b in Lie theoretic terms one notes that $\mu(F^*)$ is of type D, E or A_ℓ where ℓ is odd. In any of these case there is a special index $i_* \in I$. If $\mu(F^*)$ is of type D or E, then α_{i_*} is the branch point of the Coxeter-Dynkin diagram of $\mathfrak{g}$. If $\mu(F^*)$ is of type A_ℓ, then α_{i_*} is the midpoint of the diagram (recall that ℓ is odd in this case).

Theorem 1.1. *One has $a = 2d_{i_*}$ and b is given by the condition that*

$$\begin{aligned} a\,b &= 2\,|F^*| \\ &= 4\,|F| \end{aligned} \tag{1.4}$$

See Lemma 5.14 in [K]. The cases under consideration are characterized by the condition that h is even. We put $g = h/2$. The parity of g will play a later role.

Remark 1.2. One proves (see Lemma 5.7 in [K]) that b may also be given by

$$b = h + 2 - a \tag{1.5}$$

so that b, as well as a, is even.

The following table lists the various cases under consideration. In the table Δ_n is the dihedral group of order $2n$.

F	$\mathfrak{g}$	a	b	h	g
$\mathbb{Z}_n$	A_{2n-1}	2	$2n$	$2n$	n
Δ_n	D_{n+2}	4	$2n$	$2n+2$	$n+1$
Alt_4	E_6	6	8	12	6
Sym_4	E_7	8	12	18	9
Alt_5	E_8	12	20	30	15

Proposition 1.3. *There exists a unique partition*

$$\Pi = \Pi_1 \cup \Pi_2 \tag{1.6}$$

such that if $k = 1,2$ and $\alpha_i, \alpha_j, \in \Pi_k$ where $i \neq j$ then α_i is orthogonal to α_j. Furthermore all the roots in Π_2 are orthogonal to the highest root ψ, or equivalently the root α_0 is orthogonal to all the roots in Π_2.

One has the disjoint union $I = I_1 \cup I_2$ where, if $k \in \{1,2\}$, $\Pi_k = \{\alpha_i \mid i \in I_k\}$.

Remark 1.4. It is immediate from (1.2) that if $A_{ij} \neq 0$ and $i \in I_k$ then j is in the complement of I_k in I. It then follows that γ_i descends to a representation of F (i.e., $\gamma_i(-1) = 1$) if and only if $k = 2$. In particular

$$m_{n,i} = 0 \text{ if } n \text{ and } k \text{ have opposite parities where } \alpha_i \in \Pi_k. \tag{1.7}$$

If $i \in I$ let $s_i \in W$ be the reflection defined by α_i so that s_i commutes with s_j if $i,j \in I_k$, $k \in \{1,2\}$. Put $\tau_k = \prod_{i \in I_k} s_i$. Then

$$\begin{aligned}\tau_1^2 &= \tau_2^2 \\ &= identity\end{aligned}$$

One defines a Coxeter element $\sigma \in W$ by putting

$$\sigma = \tau_2 \tau_1 \tag{1.8}$$

Remark 1.5. Every element in W is contained in a dihedral subgroup of W. Since, as one knows, the centralizer of a Coxeter element is the cyclic group (necessarily of order h) generated by the Coxeter element, a dihedral group containing the Coxeter element is unique. It is clear that τ_1 and τ_2 are in the dihedral group containing σ and, in fact, are in the complementary coset of the cyclic group generated by σ.

As a extension of (1.3) one knows (see (5.7.2) in [K]) that for any $i \in I$ there exists a polynomial $z(t)_i$ of degree less than h such that

$$m(t)_i = \frac{z(t)_i}{(1-t^a)(1-t^b)} \tag{1.9}$$

so that $m(t)_i$ is known as soon as one knows the polynomial $z(t)_i$.

Remark 1.6. Note that by (1.6) and evenness of a and b (Remark 1.2) one must have that the only powers of t which have a nonzero coefficient are odd if $i \in I_1$ and even if $i \in I_2$.

Example 1.7. Consider the case where F is the icosahedral group so that $\mu(F^*) = E_8$. In the listing of $z(t)_i$ below we will replace the arbitrary index i by the more informative $\{d_i\}$. Since there exists in certain cases two distinct $i, j \in I$ such that $dim\, \gamma_i = dim\, \gamma_j$ we will write $\underline{\{d_j\}}$ for j when the "distance" of α_j to α_0 is greater than the "distance" of α_i to α_0. Note that $d_{i_*} = 6$.

$$
\begin{aligned}
z(t)_{\{2\}} &= t + t^{11} + t^{19} + t^{29} \\
z(t)_{\{3\}} &= t^2 + t^{10} + t^{12} + t^{18} + t^{20} + t^{28} \\
z(t)_{\{4\}} &= t^3 + t^9 + t^{11} + t^{13} + t^{17} + t^{19} + t^{21} + t^{27} \\
z(t)_{\{5\}} &= t^4 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{22} + t^{26} \\
z(t)_{\{6\}} &= t^5 + t^7 + t^9 + t^{11} + t^{13} + 2\,t^{15} + t^{17} + t^{19} + t^{21} + t^{23} + t^{25} \\
z(t)_{\underline{\{4\}}} &= t^6 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + t^{22} + t^{24} \\
z(t)_{\underline{\{2\}}} &= t^7 + t^{13} + t^{17} + t^{23} \\
z(t)_{\underline{\{3\}}} &= t^6 + t^{10} + t^{14} + t^{16} + t^{20} + t^{24}
\end{aligned}
$$

We now modify Π by defining

$$\widetilde{\Pi} = \{\beta_i \mid i \in I\} \tag{1.10}$$

where $\beta_i = \alpha_i$ if $i \in I_1$ and $\beta_i = -\alpha_i$ if $i \in I_2$. Let $Z \subset W$ be the cyclic group generated by the Coxeter element σ. Recall $(h+1)\,\ell$ so that

$$card\,\Delta = h\,\ell \tag{1.11}$$

We have shown that σ has ℓ orbits in Δ, each with h-elements, and that each orbit contains a unique element of $\widetilde{\Pi}$. That is, one has

Theorem 1.8. *For any $i \in I$ the σ-orbit $Z \cdot \beta_i$ has h elements and one has the disjoint union*

$$\Delta = \sqcup_{i=1}^{\ell} \, Z \cdot \beta_i \tag{1.12}$$

This result is readily proved using (6.9.2) in [K].

For any $i \in I$ let $(Z \cdot \beta_i)_+ = \Delta_+ \cap Z \cdot \beta_i$. One has (see (0.5))

$$\Delta_+ = g\,\ell \tag{1.13}$$

Theorem 1.9. *For any $i \in I$ one has card $(Z \cdot \beta_i)_+ = g$ and the disjoint union*

$$\Delta_+ = \sqcup_{i \in I} \, (Z \cdot \beta_i)_+ \tag{1.14}$$

It follows from (5.6.2) in [K] that (see (0.5))

$$\alpha_{i_*} \in \Pi_2 \text{ if } g \text{ is even and } \alpha_{i_*} \in \Pi_1 \text{ if } g \text{ is odd.} \tag{1.15}$$

Let κ be the long element of the Weyl group. One has (see Lemma 4.9 in [K]) the following result of Steinberg:

$$\sigma^g = \kappa \tag{1.16}$$

so that $\kappa \in Z$.

Remark 1.10. Recall that ψ is the highest root. It is a consequence of (5.6.2) in [K] that one has ψ and β_{i_*} are in the same σ orbit. In fact if g is odd then

$$\begin{aligned} \sigma^{\frac{g-1}{2}}(\psi) &= \beta_{i_*} \\ &= \alpha_{i_*} \end{aligned} \tag{1.17}$$

and if g is even then

$$\begin{aligned}\sigma^{\frac{g}{2}}(\psi) &= \beta_{i_*} \\ &= -\alpha_{i_*}\end{aligned} \tag{1.18}$$

One easily has that σ^g commutes with τ_1 and τ_2 so that, for $k \in \{1,2\}$,

$$\sigma^g(\Pi_k) = -(\Pi_k) \tag{1.19}$$

Furthermore since $\kappa(\psi) = -\psi$ one has that

$$\sigma^g(\alpha_{i_*}) = -\alpha_{i_*} \tag{1.20}$$

so that in any case

$$\psi \text{ and } \alpha_{i_*} \text{ lie in the same } \sigma\text{-orbit} \tag{1.21}$$

1.3. We come now to the main result—the determination of $z(t)_i$ in terms of the orbit structure of σ on Δ. For any $\varphi \in \Delta_+$ let $i_\varphi \in I$ be defined so that (by Theorem 1.9)

$$\varphi \in (Z \cdot \beta_{i_\varphi})_+ \tag{1.22}$$

But then there exists $k_\varphi \in \{1,2\}$ such that

$$i_\varphi \in I_{k_\varphi} \tag{1.23}$$

The following result follows from (6.9.2) in [K].

Theorem 1.11. *Let $\varphi \in \Delta_+$. Then there exists a unique positive integer $n(\varphi)$ where $1 \le n(\varphi) \le h$ with the same parity as k_φ such that if $k_\varphi = 1$ then*

$$\sigma^{\frac{n(\varphi)-1}{2}}(\varphi) = \beta_{i_\varphi} \tag{1.24}$$

If $k_\varphi = 2$ then

$$\sigma^{\frac{n(\varphi)}{2}}(\varphi) = \beta_{i_\varphi} \tag{1.25}$$

One also has (see Remark 6.10 in [K])

Theorem 1.12. *For any $i \in I_1$ the map*

$$(Z \cdot \beta_i)_+ \to \{0, 1, \ldots, g-1\}, \qquad \varphi \mapsto \frac{n(\varphi)-1}{2} \tag{1.26}$$

is a bijection and for any $i \in I_2$ the map

$$(Z \cdot \beta_i)_+ \to \{1, \ldots, g\}, \qquad \varphi \mapsto \frac{n(\varphi)}{2} \tag{1.27}$$

is a bijection.

Let (φ, φ') be the restriction to Δ of the W-invariant bilinear form on $\mathfrak{h}'$ induced by the Killing form on $\mathfrak{g}$. Let $\Phi = \{\varphi \in \Delta \mid (\psi, \varphi) > 0\}$. One easily has that $\Phi \subset \Delta_+$. Obviously $\psi \in \Phi$. One has

$$card\, \Phi = 2\,h - 3 \tag{1.28}$$

Because of its connection with a Heisenberg subalgebra of $\mathfrak{g}$ we refer to Φ as the Heisenberg subsystem of Δ_+. For $i \in I$ let $\Phi^i = \Phi \cap (Z \cdot \beta_i)_+$. Our main result is

Theorem 1.13. *Let $i \in I - \{i_*\}$. Then*

$$z(t)_i = \sum_{\varphi \in \Phi^i} t^{n(\varphi)} \tag{1.29}$$

Furthermore

$$card\, \Phi^i = 2 d_i \tag{1.30}$$

In addition all the coefficients of $z(t)_i$ are either 1 or 0 so that

$$z(1)_i = 2\, d_i \tag{1.31}$$

For $i = i_*$ one has

$$z(t)_{i_*} = 2\, t^g + \sum_{\varphi \in \Phi^{i_*},\, \varphi \neq \psi} t^{n(\varphi)} \tag{1.32}$$

In addition the coefficient of t^g is 2 and all the other coefficients of $z(t)_{i_*}$ are either 0 *or* 1*. One also has*

$$\begin{aligned} z(1)_{i_*} &= 2\, d_{i_*} \\ &= a \end{aligned} \tag{1.33}$$

Finally

$$z(t)_{i_*} = t^{g-a+2} + t^{g-a+4} + \cdots + t^{g-2} + 2\, t^g + t^{g+2} + \cdots + t^{g+a-4} + t^{g+a-2} \tag{1.34}$$

Theorem 1.13 combines Theorem 6.6 and Lemma 6.14 in [K]. We note also that the expression (1.32) for $z(t)_{i_*}$ in Theorem 1.13 follows from the proof of Theorem 6.6 in [K] (see especially (5.8.1) in [K]).

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