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Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications

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Summary. Let $G' \subset G$ be real reductive Lie groups and $\mathfrak{q}$ a θ-stable parabolic subalgebra of $\mathrm{Lie}(G) \otimes \mathbb{C}$. This paper offers a sufficient condition on $(G, G', \mathfrak{q})$ that the irreducible unitary representation $\overline{A_{\mathfrak{q}}}$ of G with non-zero continuous cohomology splits into a discrete sum of irreducible unitary representations of a subgroup G', each of finite multiplicity. As an application to purely analytic problems, new results on discrete series are also obtained for some pseudo-Riemannian (non-symmetric) spherical homogeneous spaces, which fit nicely into this framework. Some explicit examples of a decomposition formula are also found in the cases where $A_{\mathfrak{q}}$ is not necessarily a highest weight module.

0 Introduction

Our object of study is the restriction of a unitary representation $\overline{A_{\mathfrak{q}}(\lambda)}$ of a real reductive linear Lie group G with respect to its reductive subgroup G'. Here $\overline{A_{\mathfrak{q}}(\lambda)}$ denotes the Hilbert completion of an irreducible unitary $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ attached to an integral elliptic orbit $\mathrm{Ad}^*(G)\lambda \subset \mathfrak{g}^*$ in the sense of Vogan–Zuckerman, which is a vast generalization of Borel–Weil–Bott's construction of finite dimensional representations of compact Lie groups. It is well-known that the following $(\mathfrak{g}, K)$-modules are described by means of $A_{\mathfrak{q}}(\lambda)$ or its coherent family in the weakly fair range (see [V3, Definition 2.5]; §2):

(0.1)(a) representations with non-zero $(\mathfrak{g}, K)$-cohomology which contributes the de Rham cohomology of locally Riemannian symmetric spaces by Matsushima's formula (see [BoW, VZ]),
(0.1)(b) discrete series for semisimple symmetric spaces (see [FJ, Chap. VIII, §2; V3, §4]), which include Harish–Chandra's discrete series for group manifolds,
(0.1)(c) 'most of' unitary highest weight modules of classical groups [A2].

Suppose $G' \subset G$ are Lie groups, X is a G-space and X' is a G'-space. Then a representation theoretic counterpart of an equivariant morphism $f: X' \to X$ is

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the pullback of function spaces $f^*: \Gamma(X) \to \Gamma(X')$, where the restriction of representations of G with respect to G' naturally arises. If $A_{\mathfrak{q}}(\lambda)$ is realized in a function space $\Gamma(X)$ as in (0.1)(a) and (b), it is natural to ask the restriction formula (*branching rule*) of $\overline{A_{\mathfrak{q}}(\lambda)}_{|G'}$ into irreducible representations of G'. So far, the following special cases of the restriction of $\overline{A_{\mathfrak{q}}(\lambda)}$ with respect to reductive subgroups have been achieved (see also Examples 4.5 and 4.6):

(0.2)(a) G is compact. A classical (but still active) study of branching rules of finite dimensional representations of compact Lie groups ('breaking-symmetry' in physics) is to find explicit restriction formulas with respect to various subgroups.
(0.2)(b) G' is a maximal compact subgroup of G. An explicit decomposition formula is known as a *generalized Blattner formula* (see [HS; V1 Theorem 6.3.12]).
(0.2)(c) $A_{\mathfrak{q}}(\lambda)$ is of a highest (or lowest) weight. An explicit decomposition formula is found (e.g. [M, J, JV]) in the cases where $A_{\mathfrak{q}}(\lambda)$ is holomorphic discrete series with some assumption on G' (see the condition (4.1)(a)').

However, one can observe that the restriction of $\overline{A_{\mathfrak{q}}(\lambda)}$ with respect to G' may have a wild behavior in general, even if G' is a maximal reductive subgroup in G, involving the following cases:

(0.3)(a) The restriction is decomposed into only the continuous spectrum with infinite multiplicity (e.g. a tensor product of principal series of simple complex groups other than $SL(2, \mathbb{C})$; see [GG, Wi]).
(0.3)(b) The restriction is decomposed into the continuous spectrum with finite multiplicity and at most finite many discrete spectrum (possibly no discrete spectrum) (e.g. the tensor product of a holomorphic discrete series and an anti-holomorphic discrete series [R]).
(0.3)(c) The restriction is decomposed into countably many discrete spectrum with finite multiplicity (see §3, §4, §6.1).
(0.3)(d) The restriction is still irreducible (e.g. Theorem 6.4).

For a fruitful study of the restriction of $\overline{A_{\mathfrak{q}}(\lambda)}$ with respect to a reductive subgroup in a general setting, we first want to find *a good framework*, where we can expect to obtain explicit and informative branching rules which are not only interesting from view points of representation theory but also applicable to harmonic analysis as in the situations of (0.1)(a) and (b). For this purpose we focus our attention to the case where the restriction is an 'admissible' representation. Here, we say a unitary representation (π, V) of G is **G-admissible** if (π, V) is decomposed into a discrete Hilbert direct sum with finite multiplicities of irreducible representations of G. Previous examples (0.3)(c) and (d) are the case. Successful theories (0.2)(a) $\sim$ (c) are also the case. One of the advantages of admissibility is to allow one to study *algebraically* the objects in which such representation (π, V) occurs. We also illustrate this in some other familiar results which have laid important foundations on the study of locally symmetric spaces (e.g. [BoW, VZ]), algebraic study of Harish-Chandra modules (e.g. [V1]).

(0.4)(a) (induction) Let Γ be a cocompact discrete subgroup of G. Then the L^2-induced module $L^2\text{-}\mathrm{Ind}_{\Gamma}^{G}(1) = L^2(G/\Gamma)$ is G-admissible (Gelfand and Piatecki-Šapiro, [GGP, Chap. I, §2]).
(0.4)(b) (restriction) Let K be a maximal compact subgroup of a reductive linear Lie group G and (π, V) an irreducible unitary representation of G. Then the restriction $(\pi_{|K}, V)$ is K-admissible (Harish-Chandra).

(0.4)(c) (restriction) The restriction of the Segal–Shale–Weil representation $\widetilde{\mathrm{Sp}}$ with respect to dual reductive pair $G' = G'_1 \times G'_2$ with G'_2 compact is G'-admissible, yielding Howe's correspondence (e.g. [Ho, KV, A1]).

Suppose $G' \subset G$ are real reductive Lie groups and $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$. Now, our main interest is the G'-admissibility of the restriction of $\overline{A_{\mathfrak{q}}(\lambda)}$. Naively, the representation $\overline{A_{\mathfrak{q}}(\lambda)}$ should be 'small' and the subgroup G' should be 'large' for the G'-admissibility of the restriction $\overline{A_{\mathfrak{q}}(\lambda)}_{|G'}$. Though previous examples (0.2) imposed strong assumptions on either G' (compactness) or $\overline{A_{\mathfrak{q}}(\lambda)}$ (unitary highest weight module), it is more natural to treat it as a condition for the triplet $(G, G', \mathfrak{q})$. That is,

Question 0.5 Find a criterion for $(G, G', \mathfrak{q})$ assuring G'-admissibility of the restriction of $\overline{A_{\mathfrak{q}}}_{|G'}$.

Surprisingly, we shall see there is a rich family of a triplet $(G, G', \mathfrak{q})$ such that the restriction of $\overline{A_{\mathfrak{q}}(\lambda)} \in \hat{G}$ with respect to G' is admissible. To be more precise, one of our principal results in the case where (G, G') is a reductive symmetric pair (see Theorem 3.2) asserts:

$$\text{if } \mathbb{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle \cap \sqrt{-1}(\mathfrak{t}^c_{0-})^* = \{0\}, \text{ then the restriction } \overline{A_{\mathfrak{q}}}_{|G} \text{ is } G'\text{-admissible,}$$

where $\mathbb{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle$ is a closed cone determined by a θ-stable parabolic subalgebra $\mathfrak{q}$ and $\sqrt{-1}(\mathfrak{t}^c_{0-})^*$ is a subspace determined by a symmetric pair (G, G'). Such a triplet $(G, G', \mathfrak{q})$ is classified when (G, G') is a semisimple symmetric pair with G classical [Ko4]. For example, if $(G, G') = (U(2,2), \mathrm{Sp}(1,1)) \approx (\mathrm{SO}(4,2), \mathrm{SO}(4,1))$, then Theorem 3.2 says that among inequivalent 18 $A_{\mathfrak{q}}$'s of $U(2,2)$ there exist 12 $A_{\mathfrak{q}}$'s which are $\mathrm{Sp}(1,1)$-admissible, including 7 modules which do not have highest (or lowest) weights (see Example 3.7). In §4 we discuss another case where (G, G') is not necessarily symmetric but satisfies some compatibility condition with $\mathfrak{g} \supset \mathfrak{q}$ generalizing the situation (0.2)(c) (see Theorem 4.1, Corollary 4.4).

Though these are our main results on $A_{\mathfrak{q}}(\lambda)$, another object of the present paper is to study harmonic analysis on spherical homogeneous spaces in connection with the restriction of $A_{\mathfrak{q}}(\lambda)$. Here, a homogeneous space G/H is called **spherical** if $H_{\mathbb{C}}$ has an open orbit on the associated flag variety of $G_{\mathbb{C}}$. The pairs (G, H) have been determined by Krämer and by Brion with G compact [Kr2, Br]. These include the familiar symmetric spaces, but also some others, such as $\mathrm{SO}(2n+1)/U(n)$. A fundamental question in harmonic analysis on homogeneous spaces is to determine discrete series (i.e. to determine which homogeneous spaces admit discrete series and to classify them if exist), which plays an essential role in the Plancherel theorem. This question has been solved for semisimple symmetric spaces by Flensted-Jensen, Oshima and Matsuki (see [FJ] and the references therein), while it remains open for non-symmetric cases except for compact cases [Kr1] and principal bundles over semisimple symmetric spaces with compact fibers [S1, Ko3]. One of the main difficulties in the study of discrete series for non-symmetric homogeneous spaces has been the lack of powerful techniques such as Flensted-Jensen duality for semisimple symmetric spaces. We have no complete answer to this question yet, but our approach here covers various types of pseudo-Riemannian spherical homogeneous spaces, determining which of them admit discrete series. For example, we shall prove

$$\mathrm{Disc}(SO(2p+1, 2q)/U(p, q)) \neq \varnothing \text{ if and only if } (p+1)q \in 2\mathbb{Z}.$$

A very special case $p = 0$ means that there exist (Harish-Chandra's) discrete series for $SO(1, 2n)$ with non-trivial $U(n)$-fixed vector iff $n \in 2\mathbb{Z}$. This is done in §5 (see Corollary 5.6) on the basis of our results on $A_{\mathfrak{q}}(\lambda)$ in §3 and §4 together with deep results on discrete series for symmetric spaces due to Flensted-Jensen, Oshima-Matsuki and Vogan-Zuckerman. In §6 we present explicit decomposition formulas of some family of $\overline{A_{\mathfrak{q}}(\lambda)}$ according to $SO(4p, 4q) \supset U(2p, 2q) \supset Sp(p, q)$ and $SO(4, 3) \supset G_2(\mathbb{R})$, and determine discrete series for indefinite Stiefel manifolds $U(p, q)/U(p-1, q)$, $Sp(p, q)/Sp(p-1, q)$ and non-symmetric spherical homogeneous spaces $G_{2(2)}/SU(2, 1)$ and $G_{2(2)}/SL(3, \mathbb{R})$. This immediately amounts to the Plancherel formula for these spaces since 'continuous series' for these spherical homogeneous spaces are much easier to find thanks to the classical Mackey machine. The first example can be obtained as a reformulation of [HT] (cf. [Ko2] for a special case by a different method), and the second one is a joint work with T. Uzawa. It is remarkable that the irreducible summands in the restriction formula of $\overline{A_{\mathfrak{q}}(\lambda)} = \overline{(\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}})^S(\mathbb{C}_{\lambda+\rho(\mathfrak{u})})}$ (see §2 for notation) in §6 may involve different series of unitary representations (i.e. those attached to different θ-stable parabolic subalgebras $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_m$ of $\mathfrak{g}'$) such as

$$\overline{A_{\mathfrak{q}}(\lambda)}|_{G'} = \bigoplus_{j=1}^{m} \sum_{\nu_i^{(j)} \in \Lambda_j}^{\oplus} \overline{(\mathscr{R}^{\mathfrak{g}'}_{\mathfrak{q}'_j})^{S'_j}(\mathbb{C}_{\nu_i^{(j)}})}.$$

Here $\mathbb{C}_{\nu_i^{(j)}}$ is weakly fair with respect to $\mathfrak{q}'_j$ for any $\nu_i^{(j)} \in \Lambda_j$. It can happen that $\# \Lambda_j$ (the cardinality of the parameter set Λ_j) $= \infty$ for some j and $\# \Lambda_j < \infty$ for the other j (and still each Λ_j meets the good range of parameters). This is a new phenomenon which never occurs in (0.2)(a), (b), and known cases of (c).

Notation. $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

1 The general case

Throughout this section let G be a locally compact group of type I in the sense of von-Neumann algebras. Suppose (π, V) is a unitary representation of G on the (separable) Hilbert space V. A homomorphism between unitary representations of G is a $\mathbb{C}$-linear map which respects both the actions of G and inner products. We denote by Hom_G the totality of such homomorphisms. Given $(\tau, H_\tau) \in \hat{G}$, an irreducible unitary representation of G, we write

$$V(\tau) := \text{the Hilbert completion of the sum of } f(H_\tau),$$
$$f \text{ running through } f \in \mathrm{Hom}_G(H_\tau, V).$$

We define a **discrete part** of the irreducible decomposition of V by

$$V_d := \sum_{(\tau, H_\tau) \in \hat{G}}^{\oplus} V(\tau) \simeq \sum_{(\tau, H_\tau) \in \hat{G}}^{\oplus} m(\tau)(\tau, H_\tau).$$

Here $\sum^{\oplus}$ denotes a direct sum as a Hilbert space and $m(\tau) := \dim_{\mathbb{C}} \mathrm{Hom}_G(H_\tau, V) \in \mathbb{N} \cup \{\infty\}$. Possibly, $V_d = \{0\}$ (e.g. $G = \mathbb{R}$, π is the regular representation on

$V = L^2(\mathbb{R})$). We say (π, V) is **discretely decomposable** if (π, V) is unitarily equivalent to V_d. If G is compact, then (π, V) is always discretely decomposable. We say (π, V) is ***G*-admissible** if the multiplicity $m(\tau) < \infty$ for any $\tau \in \hat{G}$.

The following simple lemma is used both in Theorem 1.2 and in Theorem 5.4.

Lemma 1.1 *Let (π, V) be a unitary representation of G. Assume that there exist a closed subgroup K and an irreducible representation $\tau \in \hat{K}$ such that $V(\tau) \neq 0$ and $m(\tau) < \infty$. Then there exists an irreducible closed G-subspace W in V such that $W(\tau) \neq 0$.*

Proof. Select a non-zero subspace E in $V(\tau)$ which is minimal with the property that E is of the form $U(\tau)$ where U is a closed G-subspace in V (this is possible because $m(\tau) < \infty$). We put $W := \cap U_\lambda$, where the intersection is taken over all closed G-subspaces U_λ in V such that $U_\lambda(\tau) = E$. Clearly, W is also a closed G-subspace in V such that $W(\tau) = E$. Let us show W is irreducible. If this were not the case, we would have a direct sum decomposition of two G-invariant closed subspaces $W = W_1 \oplus W_2$. Because $E = W(\tau) = W_1(\tau) \oplus W_2(\tau)$ and because of the minimality of E, we have either $W_1(\tau) = E$ or $W_2(\tau) = E$. In either case, the minimality of W contradicts to $W \supsetneqq W_i$. Hence W is irreducible. □

If G is compact, then it is easy to see that K-admissibility implies G-admissibility. In the following theorem, we show this without any assumption of the compactness. Our idea here parallels a proof of a theorem of Gelfand and Piatecki-Šapiro mentioned in (0.4)(a).

Theorem 1.2 *Let (π, V) be a unitary representation of G, and K a subgroup of G. If $(\pi_{|K}, V)$ is K-admissible, then (π, V) is G-admissible.*

Proof. Using Zorn's lemma, we find a closed G-subspace V' in V which is maximal with the property that V' is discretely decomposable. Let us show $V' = V$, or equivalently, the orthogonally complementary subspace U of V' in V is zero. If this were not the case, there exists $\tau \in \hat{K}$ such that $U(\tau) \neq \{0\}$. Applying Lemma 1.1 to U, we find an irreducible closed G-subspace W in U. Then $V' \oplus W \supsetneqq V'$ is discretely decomposable, which contradicts to the maximality of V'. Hence $V' = V$. The statement of finite multiplicity is obvious from that for K in the assumption. □

In §3 and §4 we shall find a sufficient condition assuring the G'-admissibility of $\overline{A_{\mathfrak{q}}(\lambda)}_{|G'}$ (see §2 for definition), which is independent of λ. As for the independence of the parameter λ we can go one step further by focusing our attention to coherent continuation and representations of Weyl groups on virtual $(\mathfrak{g}, K)$-modules (see [V2]) based on the following simple observation.

Corollary 1.3 *Suppose G is a real reductive Lie group and $K' \subset G'$ are subgroups of G. Let $\pi \in \hat{G}$. Assume $\pi_{|K'}$ is K'-admissible and that $\sigma \in \hat{G}$ appears in a subquotient of a coherent family of π (see* [V1, Definition 7.2.5]*). Then $\sigma_{|G'}$ is G'-admissible.*

Proof. We can find a finite dimensional representation τ of G such that σ appears as a subquotient in $\pi \otimes \tau$ because π is K'-admissible and $\dim \tau < \infty$, the multiplicity of a fixed K'-type occurring in $\pi \otimes \tau$ is finite. Thus σ is also K'-admissible. Hence σ is G'-admissible by Theorem 1.2. □

2 Notation and preliminaries on $A_{\mathfrak{q}}(\lambda)$

Throughout this section we suppose that G is a real reductive linear Lie group. We fix a Cartan involution θ of G. Write $\mathfrak{g}_0$ for the Lie algebra of G, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ for its complexification, $K = G^\theta$ for the fixed point group of θ, and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ for the corresponding Cartan decomposition. Analogous notation is used for other groups. Now a θ-stable parabolic subalgebra $\mathfrak{q}$ is given by an element $X \in \sqrt{-1}\,\mathfrak{k}_0$:

Definition 2.1 *Given an element* $X \in \sqrt{-1}\,\mathfrak{k}_0$. Let
$L \equiv L(X) :=$ *the centralizer of* X *in* G *under the adjoint action.*
$\mathfrak{l} \equiv \mathfrak{l}(X) :=$ *the centralizer of* $\operatorname{ad}(X)$ *in* $\mathfrak{g}$,
$\mathfrak{u} \equiv \mathfrak{u}(X) :=$ *the sum of eigenspaces with positive eigenvalues of* $\operatorname{ad}(X)$,
$\bar{\mathfrak{u}} \equiv \bar{\mathfrak{u}}(X) :=$ *the sum of eigenspaces with negative eigenvalues of* $\operatorname{ad}(X)$,
$\mathfrak{q} \equiv \mathfrak{q}(X) := \mathfrak{l}(X) + \mathfrak{u}(X)$.

Then we have a direct sum $\mathfrak{g} = \bar{\mathfrak{u}}(X) + \mathfrak{l}(X) + \mathfrak{u}(X)$. Choose an $\operatorname{Ad}(G)$-invariant non-degenerate bilinear form on $\mathfrak{g}_0$ which is negative definite on $\mathfrak{k}_0$, and we identify $\mathfrak{k}_0$ and $\mathfrak{k}_0^*$. Via this identification, we also use the notation $\mathfrak{l}(\lambda)$, $\mathfrak{u}(\lambda)$ and so on, if we are given $\lambda \in \sqrt{-1}\,\mathfrak{k}_0^*$.

The elliptic orbit $\operatorname{Ad}(G)X \simeq G/L$ carries a G-invariant complex structure, with the holomorphic tangent bundle $T(G/L)$ given by $G \times_L \bar{\mathfrak{u}}$. An L-module (τ, V_τ) defines an associated holomorphic vector bundle $\mathscr{V}_\tau = G \times_L V_\tau$ over G/L. As an algebraic analogue of a Dolbeault cohomology $H^j(G/L, V_\tau)$ with coefficients in $\mathscr{V}_\tau$, Zuckerman introduced the cohomological parabolic induction $\mathscr{R}^j_{\mathfrak{q}} \equiv (\mathscr{R}^{\mathfrak{g}}_{\mathfrak{q}})^j$ $(j \in \mathbb{N})$, which is a covariant functor from the category of metaplectic $(\mathfrak{l}, (L \cap K)^\sim)$-modules to that of $(\mathfrak{g}, K)$-modules ($\tilde{L}$ is a metaplectic covering of L defined by a character of L acting on $\wedge^{\dim \mathfrak{u}} \mathfrak{u}$) (see [V1, Chap. 6; V2, Chap. 6; Wa1, Chap. 6]). In this paper, we follow the normalization in [V2, Definition 6.20] which is different from the one in [V1] by a '$\rho(\mathfrak{u})$-shift'. To be more precise, we take a fundamental Cartan subalgebra $\mathfrak{h}^c_0$ ($\subset \mathfrak{l}_0$). Then $\mathfrak{h}^c_0$ contains the center $\mathfrak{z}_0$ of $\mathfrak{l}_0$ and $\mathfrak{t}^c_0 := \mathfrak{h}^c_0 \cap \mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. Fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t}^c) \supset \Delta(\mathfrak{k} \cap \mathfrak{u}, \mathfrak{t}^c)$. Suppose W is an $(\mathfrak{l}, (L \cap K)^\sim)$-module with $\mathscr{Z}(\mathfrak{l})$-infinitesimal character $\gamma \in (\mathfrak{h}^c)^*$ in the Harish-Chandra parametrization. Following [V3, Definition 2.5], we say W is in the good range if

$$\text{(2.2)(a)} \qquad \operatorname{Re}\langle \gamma, \alpha \rangle > 0 \quad \text{for any} \quad \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^c).$$

In the case where W is one dimensional, we say W is in the fair range if

$$\text{(2.2)(b)} \qquad \operatorname{Re}\langle \gamma_{|\mathfrak{z}}, \alpha \rangle > 0 \quad \text{for any} \quad \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^c),$$

which is implied by (2.2)(a). It is weakly good (respectively weakly fair) if the weak inequalities hold. Then the $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ with the notation in [VZ, §5] is isomorphic to $\mathscr{R}^S_{\mathfrak{q}}(\mathbb{C}_{\lambda + \rho(\mathfrak{u})})$ with our notation, where $S = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$, $\rho(\mathfrak{u}) = \frac{1}{2}\operatorname{Trace}(\operatorname{ad}_{|\mathfrak{u}})$ and $\mathbb{C}_{\lambda + \rho(\mathfrak{u})}$ is in the good range of parameters. In particular, $A_{\mathfrak{q}} \equiv A_{\mathfrak{q}}(0) \simeq \mathscr{R}^S_{\mathfrak{q}}(\mathbb{C}_{\rho(\mathfrak{u})})$ has the same infinitesimal character as that of the trivial representation. For later convenience, we define a condition (2.2)(c) on W:

(2.2)(c)

W is a finite dimensional metaplectic unitary representation of $\tilde{L}$ in the weakly good range, or in the weakly fair range with $\dim W = 1$.

We recall some important results of Zuckerman and Vogan. See [V2, Theorem 6.8] for the first statement and the proof of [V1, Theorem 6.3.12] for the second.

Fact 2.3 *Retain the notation as above. Suppose W satisfies* (2.2)(c).
(1) *If W is infinitesimally unitary, so is $\mathscr{R}_{\mathfrak{q}}^S(W)$.*
(2) *For K-modules σ_i $(i = 1, 2)$, we write $\sigma_1 \leqq_K \sigma_2$ if $[\sigma_1 : \pi] \leqq [\sigma_2 : \pi]$ for any $\pi \in \hat{K}$, where $[\sigma_i : \pi]$ is the multiplicity of π occurring in σ_i. Then for any $j \in \mathbb{N}$ we have*

$$\mathscr{R}_{\mathfrak{q}}^j(W)_{|K} \underset{K}{\leqq} H^j(K/L, S(\mathfrak{u} \cap \mathfrak{p}) \otimes W \otimes \mathbb{C}_{\rho(\mathfrak{u})}).$$

Hereafter, we write $\overline{\mathscr{R}_{\mathfrak{q}}^S(W)}$ for the completion of the pre-Hilbert space $\mathscr{R}_{\mathfrak{q}}^S(W)$ if W satisfies the condition in Fact 2.3(1). If W is in the good range, then $\mathscr{R}_{\mathfrak{q}}^S(W)$ is essentially the same with $A_{\mathfrak{q}}(\lambda)$ in the sense of [VZ]. In fact, if such W satisfies $\dim W > 1$, then we can find

(2.4) a θ-stable parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ contained in $\mathfrak{q}$, with the property that L/M is compact,

such that $\mathscr{R}_{\mathfrak{q}}^S(W)$ is isomorphic to $\mathscr{R}_{\mathfrak{p}}^{S'}(\mathbb{C}_\lambda)$ with a metaplectic character $\mathbb{C}_\lambda$ of $\tilde{M}$ by using the Borel-Weil-Bott theorem for a compact normal subgroup of a Levi factor L and induction by stages [V1, Proposition 6.3.6] with respect to $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{g}$.

3 Discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ for a symmetric pair (G, G')

In this section we give a sufficient condition that the Hilbert completion of a unitarizable cohomological parabolic induced is discretely decomposable with respect to a symmetric pair (G, G').

Suppose that σ is an involutive automorphism of G and that G' is the connected component of the fixed points of σ. Then (G, G') is called a **reductive symmetric pair**. Choose a Cartan involution θ of G so that $\sigma\theta = \theta\sigma$. Then $\theta G' = G'$, $K' := K \cap G'$ is a maximal compact subgroup of G' and the pair (K, K') forms a compact symmetric pair. We write $\mathfrak{k}_{0\pm} := \{X \in \mathfrak{k}_0 : \sigma(X) = \pm X\}$. Fix a σ-stable Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{k}_0$ such that $\mathfrak{t}_{0-}^c := \mathfrak{t}_0^c \cap \mathfrak{k}_{0-}$ is a maximal abelian subspace in $\mathfrak{k}_{0-}$. Then we have a direct sum $\mathfrak{t}^c = \mathfrak{t}_+^c \oplus \mathfrak{t}_-^c$. Choose a positive system $\Sigma^+(\mathfrak{k}, \mathfrak{t}_-^c)$ of the restricted root system $\Sigma(\mathfrak{k}, \mathfrak{t}_-^c)$ and a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ which is compatible with $\Sigma^+(\mathfrak{k}, \mathfrak{t}_-^c)$ (i.e. if $\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ then $\alpha_{|\mathfrak{t}_-^c} \in \Sigma(\mathfrak{k}, \mathfrak{t}_-^c)$ or $\alpha_{|\mathfrak{t}_-^c} = 0$). Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. After a conjugation by an element of K, we may and do assume that

$$\mathfrak{q} = \mathfrak{q}(X) \qquad \text{(Notation 2.1)},$$

with an element $X \in \sqrt{-1}\,\mathfrak{t}_0$ $(\mathfrak{t}_0 \subset \mathfrak{t}_0^c)$ which is dominant with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. Define a closed cone in $\sqrt{-1}(\mathfrak{t}_0^c)^*$ by

$$\text{(3.1)} \qquad \mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p} \rangle := \left\{ \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}^c)} n_\beta \beta : n_\beta \geqq 0 \right\}.$$

Now we are ready to state one of our main results:

Theorem 3.2 *Suppose that (G, G') is a reductive symmetric pair and that $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ-stable parabolic subalgebra. In the setting as above (possibly after a conjugation of $\mathfrak{q}$ under K), assume*

$$\text{(3.2)(a)} \qquad \mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p}\rangle \cap \sqrt{-1}(\mathfrak{t}^c_{0-})^* = \{0\}.$$

Then $\overline{\mathscr{R}^S_{\mathfrak{q}}(W)}$ is G'-admissible for any W satisfying (2.2)(c).

Applications are given in Corollary 3.3, Examples 3.6 and 3.7. See also Theorem 6.1 for an example of an explicit decomposition formula in a special case. First, if we apply Theorem 3.2 to a tensor product of two unitary representations $\pi_1, \pi_2 \in \hat{G}$, by regarding it as a restriction of the outer product $\pi_1 \boxtimes \pi_2 \in \widehat{G \times G}$ with respect to the symmetric pair $(G \times G, \operatorname{diag} G)$, then it is easy to see:

Corollary 3.3 *Let G be a real reductive linear Lie group and $\mathfrak{q}_j = \mathfrak{l}_j + \mathfrak{u}_j (j = 1, 2)$ be θ-stable parabolic subalgebras. Fix a Cartan subalgebra $\mathfrak{t}^c_0$ of $\mathfrak{k}_0$ and a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. We may and do assume that $\mathfrak{l}_j \supset \mathfrak{t}^c$ and $\Delta(\mathfrak{u}_j \cap \mathfrak{k}, \mathfrak{t}^c) \subset \Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ (possibly after conjugations of $\mathfrak{q}_j$ by $\mathrm{Ad}(K)$). Denote by w_0 the longest element in the Weyl group of $\Delta(\mathfrak{k}, \mathfrak{t}^c)$. Assume*

$$\text{(3.3)(a)} \qquad \mathbb{R}_+\langle \mathfrak{u}_1 \cap \mathfrak{p}\rangle \cap \mathbb{R}_-\langle w_0(\mathfrak{u}_2 \cap \mathfrak{p})\rangle = \{0\}.$$

Here $\mathbb{R}_-\langle w_0(\mathfrak{u}_2 \cap \mathfrak{p})\rangle$ is defined similarly to (3.1). *Then the tensor product $\overline{\mathscr{R}^{S_1}_{\mathfrak{q}_1}(W_1)} \hat{\otimes} \overline{\mathscr{R}^{S_2}_{\mathfrak{q}_2}(W_2)}$ is G-admissible for any W_j satisfying* (2.2)(c).

Now, let us prove Theorem 3.2. First, we need the following result on finite dimensional representations of compact groups:

Lemma 3.4 *Let (K, K') be a compact symmetric pair. Retain the notation as before. We put $\hat{K}(\tau) := \{\pi \in \hat{K} : [\pi_{|K'} : \tau] \neq 0\}$ for $\tau \in \hat{K}'$ and regard $\hat{K}(\tau) \subset \hat{K} \subset \sqrt{-1}(\mathfrak{t}^c_0)^*$ by means of highest weights with respect to a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ which is compatible with $\Sigma^+(\mathfrak{k}, \mathfrak{t}^c_-)$. We write $P : \sqrt{-1}(\mathfrak{t}^c_0)^* \to \sqrt{-1}(\mathfrak{t}^c_{0+})^*$ for the projection corresponding to a direct sum $\mathfrak{t}^c_0 = \mathfrak{t}^c_{0+} \oplus \mathfrak{t}^c_{0-}$. Then $P(\hat{K}(\tau))$ is a finite set for each $\tau \in \hat{K}'$.*

Remark 3.5 We shall give an explicit upper bound (denoted by $\Xi(\tau)$) of $P(\hat{K}(\tau))$ in the proof of Lemma 3.4. The special case $P(\hat{K}(\mathbf{1})) \subset \Xi(\mathbf{1}) = \{0\}$ is a part of a theorem of Cartan-Helgason (see [War, Theorem 3.3.1.1]).

Proof of Lemma 3.4 We define a reductive Lie group K^d $(\subset K_{\mathbb{C}})$ by a Cartan decomposition $K^d := K' \exp(\sqrt{-1}\,\mathfrak{k}_{0-})$. This definition does not depend on the choice of a complexification $K_{\mathbb{C}}$ because $\exp(\sqrt{-1}\,\mathfrak{k}_{0-}) \simeq \mathbb{R}^{\dim \mathfrak{k}_{0-}}$ is simply connected. Then (K^d, K') is the non-compact dual Riemannian symmetric pair of (K, K') (see [He, Chap. V §2]). A finite dimensional representation π of K defines that of K^d (also denoted by π) by a holomorphic continuation with respect to $K \subset K_{\mathbb{C}} \supset K^d$.

Now let us regard π as a Langlands quotient in the following way (see [Wal, Chap. 5]): Let $P^d = M^d A^d N^d$ be a minimal parabolic subgroup of K^d associated to $\Sigma^+(\mathfrak{k}, \mathfrak{t}^c_-)$. Define a positive system $\Delta^+(\mathfrak{m}^d, \mathfrak{t}^c_+)$ by $\Delta^+(\mathfrak{k}, \mathfrak{t}^c) \cap \Delta(\mathfrak{m}^d, \mathfrak{t}^c_+)$. Suppose $\mu \in (\mathfrak{t}^c)^*$ is the highest weight of $\pi \in \hat{K}(\tau)$. Denote by (σ, V_σ) the irreducible representation of M^d with highest weight $\mu_{|\mathfrak{t}^c_+} \in (\mathfrak{t}^c_+)^*$. We put $\nu = \mu_{|\mathfrak{t}^c_-} \in (\mathfrak{t}^c_-)^*$. Then π is the unique quotient of the (non-unitary) principal series (without ρ-shift):

$$\operatorname{Ind}_{P^d}^{K^d}(\sigma \otimes \nu \otimes 1) = \{f : K^d \xrightarrow{C^\infty} V_\sigma : f(gman) = \sigma(m)^{-1} a^{-\nu} f(g),$$
$$\text{for} \quad man \in M^d A^d N^d, g \in K^d\}.$$

For each $\tau \in \widehat{K'}$ we define

$$\Xi(\tau) \subset \sqrt{-1}(\mathfrak{t}^c_{0+})^*$$

by the set of highest weights for $\Delta^+(\mathfrak{m}^d, \mathfrak{t}^c_+)$ of M^d-types occurring in τ. Then $\Xi(\tau)$ is a finite set because $\dim \tau < \infty$. Finally, let us show $P(\hat{K}(\tau)) \subset \Xi(\tau)$. Because π contains $\tau \in \widehat{K'}$, we have $[\tau_{|M^d} : \sigma] = [\operatorname{Ind}_{M^d}^{K'} \sigma : \tau] \neq 0$ by the Frobenius reciprocity theorem. Thus $P(\pi) = \mu_{|\mathfrak{t}^c_+} \in \Xi(\tau)$ by definition. □

Proof of Theorem 3.2 To apply Theorem 1.2 we shall show $[\mathscr{R}^S_{\mathfrak{q}}(W)_{|K'} : \tau] < \infty$ for each $\tau \in \widehat{K'}$. In view of

$$[\mathscr{R}^S_{\mathfrak{q}}(W)_{|K'} : \tau] = \sum_{\pi \in \hat{K}} [\mathscr{R}^S_{\mathfrak{q}}(W)_{|K} : \pi][\pi_{|K'} : \tau],$$

let us see the sum of the right side is finite. First, it is an easy geometric observation that the condition (3.2)(a) implies the compactness of the set

$$(\mathbb{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle + C_1) \cap \sqrt{-1}((\mathfrak{t}^c_{0-})^* \times C_2) \qquad (\subset \sqrt{-1}(\mathfrak{t}^c_0)^*)$$

for any compact sets $C_1 \subset \sqrt{-1}(\mathfrak{t}^c_0)^*$ and $C_2 \subset \sqrt{-1}(\mathfrak{t}^c_{0+})^*$. In particular, we apply this to the case where C_1 and C_2 are the following finite sets:

$$C_1 := \{\delta + \rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k}) : \delta \text{ is a highest weight of } (L \cap K)^\sim$$
$$\text{type occurring in } W\},$$

$$C_2 := P(\hat{K}(\tau)) (\subset \Xi(\tau)) \text{ (see the proof of Lemma 3.4 for notation).}$$

Assume that μ is a highest weight of a K-type π occurring in $\mathscr{R}^S_{\mathfrak{q}}(W)$ and satisfying $[\pi_{|K} : \tau] \neq 0$. Then it follows from Fact 2.3(2) and Lemma 3.4 that $\mu \in (\mathbb{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle + C_1) \cap \sqrt{-1}((\mathfrak{t}^c_{0-})^* \times C_2)$. Hence there are only finitely many possibilities of such dominant integral weights μ. Thus we have completed the proof. □

We end this section with some Examples 3.6 and 3.7 where (G, G') and $\mathscr{R}^S_{\mathfrak{q}}(W)$ satisfy the assumptions of Theorem 3.2 or Corollary 3.3. Perhaps, the first example is observed by many experts although we could not find it in the literature:

Example 3.6 Suppose (G, K) is an irreducible Hermitian symmetric pair. Letting $\sqrt{-1}Z (\neq 0)$ be a central element in $\mathfrak{k}_0$, we write $\mathfrak{p}_+$ for $\mathfrak{u}(Z)$ with the Notation 2.1.

Then $\mathfrak{q}(Z) = \mathfrak{k} + \mathfrak{p}_+$. According to [A2], we call a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ **holomorphic** if $\mathfrak{u} \cap \mathfrak{p} \subset \mathfrak{p}_+$. Suppose that $\mathfrak{q}$ is holomorphic. Then $A_{\mathfrak{q}}(\lambda)$ is a unitary highest module [A2, Lemma 1.7]. We remark that $\mathfrak{q} \subset \mathfrak{q}(Z)$ iff $A_{\mathfrak{q}}(\lambda)$ is a holomorphic discrete series.

If $\mathfrak{q}_1, \mathfrak{q}_2$ are holomorphic, then the assumption (3.3)(a) is satisfied because $w_0(\mathfrak{u} \cap \mathfrak{p}) \subset w_0\mathfrak{p}_+ = \mathfrak{p}_+$. Thus $\overline{A_{\mathfrak{q}_1}(\lambda_1)} \mathbin{\hat{\otimes}} \overline{A_{\mathfrak{q}_2}(\lambda_2)}$ is G-admissible. In the case where $\mathfrak{q} := \mathfrak{q}_1 = \mathfrak{q}_2 = \mathfrak{q}(Z)$, a decomposition formula of the tensor product of holomorphic discrete series $\overline{A_{\mathfrak{q}}(\lambda_1)} \mathbin{\hat{\otimes}} \overline{A_{\mathfrak{q}}(\lambda_2)}$ is found in [JV, Corollary 2.6]. On the other hand, if we put $\mathfrak{q}_1 = \mathfrak{q}(Z)$ and $\mathfrak{q}_2 = \theta\mathfrak{q}(Z)$, then the assumption (3.3)(a) does not hold. In this case, it is known that $\overline{A_{\mathfrak{q}}(\lambda_1)} \mathbin{\hat{\otimes}} \overline{A_{\theta\mathfrak{q}}(\lambda_2)}$ necessarily contains a continuous spectrum [R, Theorem 2].

In order to see how Theorem 3.2 is applied to $A_{\mathfrak{q}}(\lambda)$ (which does not necessarily have highest weights), let us list all $A_{\mathfrak{q}}$'s of $U(2,2)$ and find which among them are $\mathrm{Sp}(1,1)$-admissible:

Example 3.7 Suppose $(G, G') = (U(2,2), \mathrm{Sp}(1,1)) \approx (\mathrm{SO}(4,2), \mathrm{SO}(4,1))$. First we find K-conjugacy classes of θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. For the study of $A_{\mathfrak{q}}$, we can restrict ourselves to $\mathfrak{q}$, for which there does not exist a proper θ-stable parabolic subalgebra $\mathfrak{p}$ satisfying (2.4). We choose a coordinate in $\sqrt{-1}\,\mathfrak{t}_0^c$ so that $\Delta^+(\mathfrak{k}, \mathfrak{t}^c) = \{e_1 - e_2, e_3 - e_4\}$ and $\sqrt{-1}\,\mathfrak{t}_{0-}^c = \mathbb{R}(H_1 - H_2) + \mathbb{R}(H_3 - H_4)$. By using the above basis, we put $X_1 = (4,3,2,1)$, $X_2 = (4,2,3,1)$, $X_3 = (4,1,3,2)$, $X_4 = (3,2,4,1)$, $X_5 = (3,1,4,2)$, $X_6 = (2,1,4,3)$, $Y_1 = (2,1,1,0)$, $Y_2 = (2,0,1,0)$, $Y_3 = (2,1,2,0)$, $Y_4 = (1,0,2,0)$, $Y_5 = (2,0,2,1)$, $Y_6 = (1,0,2,1)$, $Z_1 = (1,0,0,0)$, $Z_2 = (1,1,1,0)$, $Z_3 = (0,0,1,0)$, $Z_4 = (1,0,1,1)$, $W = (1,0,1,0)$, $U = (0,0,0,0) \in \sqrt{-1}\,\mathfrak{t}_{0-}^c$. Then the set of $(\mathfrak{g}, K)$-modules

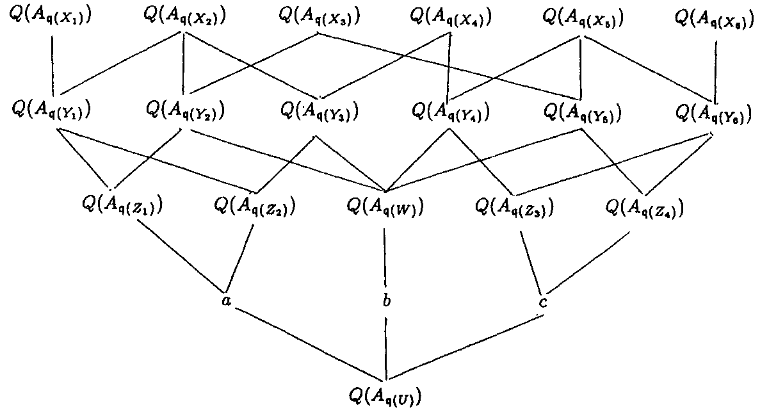
$$\{A_{\mathfrak{q}} : \mathfrak{q} = \mathfrak{q}(X_i), \mathfrak{q}(Y_i) (1 \leqq i \leqq 6), \mathfrak{q}(Z_i) (1 \leqq i \leqq 4), \mathfrak{q}(W), \mathfrak{q}(U)\}$$

is the totality of irreducible, unitary $(\mathfrak{g}, K)$-modules with non-vanishing $(\mathfrak{g}, K)$-cohomology [VZ]. Applying Theorem 3.2 we conclude that if

$$\mathfrak{q} = \mathfrak{q}(X_3), \mathfrak{q}(X_4), \mathfrak{q}(Y_2), \mathfrak{q}(Y_3), \mathfrak{q}(Y_4), \mathfrak{q}(Y_5), \mathfrak{q}(Z_i), (1 \leqq i \leqq 4), \mathfrak{q}(W), \mathfrak{q}(U),$$

then $\overline{A_{\mathfrak{q}}} \in \widehat{G}$ is $G' = \mathrm{Sp}(1,1)$-admissible.

For the benefit of the reader, we give some explanation of the above description of representations. $\overline{A_{\mathfrak{q}(X_i)}}$ $(1 \leqq i \leqq 6)$ is Harish-Chandra's discrete series for a group manifold G, and $A_{\mathfrak{q}(U)}$ is the trivial representation. If $\mathfrak{q} = \mathfrak{q}(X_1), \mathfrak{q}(X_6), \mathfrak{q}(Y_1), \mathfrak{q}(Y_6)$, $\mathfrak{q}(Z_i)$ $(1 \leqq i \leqq 4)$ or $\mathfrak{q}(U)$, then $A_{\mathfrak{q}}$ is a unitary highest (or lowest) weight module (see [A2]). In the context of the Beilinson-Bernstein correspondence between irreducible Harish-Chandra modules and irreducible K-equivariant sheaves of $\mathscr{D}$-modules on the flag variety X of $G_{\mathbb{C}} \simeq GL(4, \mathbb{C})$, we associate a single $K_{\mathbb{C}}$-orbit $Q(\pi)$ on X to an irreducible $(\mathfrak{g}, K)$-module π so that the closure of $Q(\pi)$ is the support of the corresponding localization. There are 21 $K_{\mathbb{C}}$-orbits on X, with 6 closed orbits $Q(A_{\mathfrak{q}(X_i)})$ $(1 \leqq i \leqq 6)$, and one open orbit $Q(A_{\mathfrak{q}(U)})$. These are described in the following Matsuki-Oshima diagram [MO2]. Here, the k-th column has a complex dimension $k + 2$ $(0 \leqq k \leqq 4)$, and a, b, c are orbits which are not associated to $A_{\mathfrak{q}}$.



We remark that the above position of $K_{\mathbb{C}}$-orbits is slightly different from Fig. 7 in [MO2] so that it is easier to read from the above diagram the duality relations of $(\mathfrak{g}, K)$-modules $A_{\mathfrak{q}}(X_i) \simeq A_{\mathfrak{q}}(X_{6-i})^{\vee}$ $(i = 1, 2)$, $A_{\mathfrak{q}}(Y_i) \simeq A_{\mathfrak{q}}(Y_{6-i})^{\vee}$ $(i = 1, 2, 3)$, $A_{\mathfrak{q}}(Z_i) \simeq A_{\mathfrak{q}}(Z_{4-i})^{\vee}$ $(i = 1, 2)$.

4 Discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ for a holomorphically embedding pair (G, G')

A θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ defines a complex structure on G/L so that the holomorphic tangent bundle is given by $T(G/L) = G \times_L \bar{\mathfrak{u}}$. We also denote by $(G/L)^{\circ}$ a complex manifold G/L endowed with the conjugate complex structure of G/L so that $T((G/L)^{\circ}) = G \times_L \mathfrak{u}$. Let G' be a connected closed subgroup which is reductive in G and fix a Cartan involution θ of G which stabilizes G'. We write $K' = K \cap G'$ as usual. In this section, we consider a discrete decomposability of the restriction $\overline{A_{\mathfrak{q}}(\lambda)}_{|G'}$ in the case where $K'/L \cap K' \subset G/L$ is a holomorphic embedding (see (4.1)(a)).

Theorem 4.1 *In the above setting, assume (possibly after a conjugation by an element of K):*

(4.1)(a) $$\mathfrak{k}' = (\bar{\mathfrak{u}} \cap \mathfrak{k}') \oplus (\mathfrak{l} \cap \mathfrak{k}') \oplus (\mathfrak{u} \cap \mathfrak{k}'),$$

(4.1)(b) $S(\mathfrak{u} \cap \mathfrak{p}) \otimes S(^{\bar{\mathfrak{u}} \cap \mathfrak{k}}/_{\bar{\mathfrak{u}} \cap \mathfrak{k}'})$ *is an admissible* $L \cap K'$ module.

Then $\overline{\mathscr{R}_{\mathfrak{q}}^{S}(W)}$ is *G'-admissible for any W satisfying* (2.2)(c).

Applications are given in Corollary 4.4, Examples 4.5 ~ 4.7. We remark that the compatibility condition (4.1)(a) implies that $K'/L \cap K' \subset G/L$ is a holomorphic embedding, which is a weaker condition than that $G'/L \cap G' \subset G/L$ is a holomorphic embedding, that is,

(4.1)(a)′ $$\mathfrak{g}' = (\bar{\mathfrak{u}} \cap \mathfrak{g}') \oplus (\mathfrak{l} \cap \mathfrak{g}') \oplus (\mathfrak{u} \cap \mathfrak{g}').$$

First, forgetting the above setting for a while, we prepare a result on compact groups.

Lemma 4.2 *Suppose $K \supset K'$ are connected compact Lie groups. Let $X' \in \sqrt{-1}\mathfrak{k}'_0$ and we use the notation* 2.1 *for K', defining $\mathfrak{k}' = \bar{\mathfrak{u}}' + \mathfrak{l}' + \mathfrak{u}'$ and $L' \subset K'$.* Likewise, let $X \in \sqrt{-1}\mathfrak{k}_0$ *and we define $\mathfrak{k} = \bar{\mathfrak{u}} + \mathfrak{l} + \mathfrak{u}$ and $L \subset K$.* Assume that $\mathfrak{l}' \subset \mathfrak{l}$ *and* $\mathfrak{u}' \subset \mathfrak{u}$. (*Such $X \in \sqrt{-1}\mathfrak{k}_0$ exists for each X' (e.g. $X = X'$)). For $(\pi, V) \in \widehat{K}$, we define an L-module $V_\tau := V/\pi(\bar{\mathfrak{u}})V$. Then V_τ is irreducible (the "highest L-type" of V). With the notation $\leqq_K$ defined in Fact* 2.3(2), *we have*

$$\text{(4.2)(a)} \qquad \pi_{|_{K'}} \underset{K'}{\leqq} H^0((K'/L')^\circ, V_{\tau|L'} \otimes S(\bar{\mathfrak{u}}/\bar{\mathfrak{u}}')).$$

The underlying geometric idea of Lemma 4.2 is similar to a proof of the generalized Blattner formla in Fact 2.3(2), but we give an account of it for the sake of completeness.

Proof of Lemma 4.2 By the Borel-Weil-Bott theorem for compact groups we realize V as holomorphic sections: $V \simeq H^0((K/L)^\circ, V_\tau)$.

In view of $\mathfrak{l}' \subset \mathfrak{l}$, $\mathfrak{u}' \subset \mathfrak{u}$ and $\bar{\mathfrak{u}}' \subset \bar{\mathfrak{u}}$, the holomorphic normal bundle $T_{(K'/L')^\circ}((K/L)^\circ)$ of K'/L' in K/L is given by $T_{(K'/L')^\circ}((K/L)^\circ) = \operatorname{Coker}(T((K'/L')^\circ) \to T((K/L)^\circ)_{|(K'/L')^\circ}) = \operatorname{Coker}(K' \times_{L'} \mathfrak{u}' \to K' \times_{L'} \mathfrak{u}) = K' \times_{L'} (\mathfrak{u}/u')$. Then (4.2)(a) follows from the same argument in [JV, §1] based on the technique of differentiation in the direction normal to a submanifold K'/L' due to S. Martens [M]. (We also use an L'-isomorphism $(\mathfrak{u}/\mathfrak{u}')^\vee \simeq \bar{\mathfrak{u}}/\bar{\mathfrak{u}}'$.) □

Proof of Theorem 4.1 Given a metaplectic representation W satisfying (2.2)(c), we write $\tilde{W} := W \otimes \mathbb{C}_{\rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k})}$, which is an $L \cap K$ module. It follows from Fact 2.3(2) and the Poincaré duality $H^0((K/L)^\circ, V_\tau) \simeq H^S(K/L, V_\tau \otimes \wedge^S(\mathfrak{u} \cap \mathfrak{k}))$ that

$$\text{(4.3)} \qquad \mathscr{R}^S_{\mathfrak{q}}(W)|_K \underset{K}{\leqq} H^0((K/L \cap K)^\circ, \tilde{W} \otimes S(\mathfrak{u} \cap \mathfrak{p})).$$

Applying (4.2)(a) to the right side of (4.3), we obtain

$$\mathscr{R}^S_{\mathfrak{q}}(W)|_{K'} \underset{K'}{\leqq} H^0((K'/L \cap K')^\circ, \tilde{W} \otimes S(\mathfrak{u} \cap \mathfrak{p}) \otimes S({}^{\bar{\mathfrak{u}} \cap \mathfrak{k}}/{}_{\bar{\mathfrak{u}} \cap \mathfrak{k}'})).$$

Because $\dim \tilde{W} < \infty$, the assumption (4.1)(b) implies that $\tilde{W} \otimes S(\mathfrak{u} \cap \mathfrak{p}) \otimes S({}^{\bar{\mathfrak{u}} \cap \mathfrak{k}}/{}_{\bar{\mathfrak{u}} \cap \mathfrak{k}'})$ is also admissible as an $L \cap K'$ module. Hence $\mathscr{R}^S_{\mathfrak{q}}(W)$ is admissible as a K'-module. Now Theorem 4.1 is a direct consequence of Theorem 1.2. □

We put $\mathfrak{t} := \mathfrak{k} \cap$ (the center of $\mathfrak{l}$).

Corollary 4.4 *In the setting of Theorem* 4.1, *assume that there exists an ideal $\mathfrak{d}$ of $\mathfrak{k}$ such that $\mathfrak{t} \subset \mathfrak{d} \subset \mathfrak{k}'$. Then $\mathscr{R}^S_{\mathfrak{q}}(W)$ is G'-admissible for any W* satisfying (2.2)(c).

Proof. Let us check the assumptions (4.1)(a) and (b). Since $\mathfrak{d}$ is an ideal of $\mathfrak{k}$, we can find the ideal $\tilde{\mathfrak{d}}$ of $\mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{d} \oplus \tilde{\mathfrak{d}}$. Because $\mathfrak{t} \subset \mathfrak{d}$, we have $\mathfrak{k} = (\bar{\mathfrak{u}} \cap \mathfrak{k}) \oplus (\mathfrak{l} \cap \mathfrak{k}) \oplus (\mathfrak{u} \cap \mathfrak{k}) = (\bar{\mathfrak{u}} \cap \mathfrak{d}) \oplus ((\mathfrak{l} \cap \mathfrak{d}) \oplus \tilde{\mathfrak{d}}) \oplus (\mathfrak{u} \cap \mathfrak{d})$. Because $\mathfrak{d} \triangleleft \mathfrak{k}'$, we have $\mathfrak{k}' = (\bar{\mathfrak{u}} \cap \mathfrak{d}) \oplus (\mathfrak{l} \cap \mathfrak{k}') \oplus (\mathfrak{u} \cap \mathfrak{d}) = (\bar{\mathfrak{u}} \cap \mathfrak{k}) \oplus (\mathfrak{l} \cap \mathfrak{k}') \oplus (\mathfrak{u} \cap \mathfrak{k})$, in particular, $\bar{\mathfrak{u}} \cap \mathfrak{k}' = \bar{\mathfrak{u}} \cap \mathfrak{k}$. Because the symmetric tensor algebra $S(\mathfrak{u} \cap \mathfrak{p})$ is admissible as a T-module, so is it as an $L \cap K' (\supset T)$ module. Hence the assumption (4.1)(b) is also satisfied. □

We begin with two known examples:

Example 4.5 (G' is compact) Assume $G' := K$, a maximal compact subgroup of G. In this case the assumption in Corollary 4.4 is obviously satisfied if we take $\mathfrak{b} := \mathfrak{k}$. It is well-known that any $\pi \in \hat{G}$ is admissible as a K-module (Harish-Chandra). Also a formula of the restriction of $\mathscr{R}_{\mathfrak{q}}^{S}(W)$ to K is known as a generalized Blattner formula [V1, Theorem 6.3.12].

Example 4.6 (holomorphic discrete series) Suppose (G, K) is an irreducible Hermitian symmetric pair. With the notation in Example 3.6 (in particular, Z is a central element in $\sqrt{-1}\mathfrak{k}_0$), we have $\mathfrak{g} = \bar{\mathfrak{u}}(Z) \oplus \mathfrak{l}(Z) \oplus \mathfrak{u}(Z) \equiv \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+$. Assume that $K' = K \cap G'$ satisfies

(4.6)(a) $\qquad S(\mathfrak{p}_+)$ is admissible as K'-module.

Then Theorem 4.1 assures that any holomorphic discrete series of G is G'-admissible because (4.1)(a) is automatically satisfied. For example, if G' contains $\exp(\sqrt{-1}\,\mathbb{R}Z)$, then (4.6)(a) is satisfied. In this case, the result of finite multiplicity of $A_{\mathfrak{q}}(\lambda)_{|G'}$ was obtained in [M], and also in [L], Theorem 4.2 in his study of restrictions of principal series of complex groups to real forms. In [JV] a formula of decomposition of a holomorphic discrete series of G in terms of $\widehat{G'}$ is given under a stronger assumption (4.1) (a)′ (i.e. $\mathfrak{p}'(=\mathfrak{p} \cap \mathfrak{g}') = (\mathfrak{p}' \cap \mathfrak{p}_-) \oplus (\mathfrak{p}' \cap \mathfrak{p}_+)$) but without the assumption (4.6)(a). Under the assumption (4.1)(a)′ in this case, the decomposition is always discrete ([JV]) and (4.6)(a) is a necessary and sufficient condition of the finite multiplicity in the decomposition.

The next example is remarkable which treats more general cases where $A_q(\lambda)$ is not necessarily a unitary highest module and where G' is not necessarily compact.

Example 4.7 Let $G = U(p, q; \mathbb{F})$, an indefinite orthogonal group over an Archimedean field $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (a quaternionic number field). We write $K = K_1 \times K_2 = U(p; \mathbb{F}) \times U(q; \mathbb{F})$. Assume that a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is defined by an element of $\mathfrak{k}_1 = \mathfrak{u}(p; \mathbb{F})$ ($\subset \mathfrak{k}$) (see Definition 2.1). (The representations $A_{\mathfrak{q}}(\lambda)$ here have been intensively studied, for instance, in [EPWW, S1, Ko3].) Then $\overline{A_{\mathfrak{q}}(\lambda)}$ is $G' = U(p, r; \mathbb{F}) \times U(q - r; \mathbb{F})$-admissible for any $0 \leqq r \leqq q$. Actually, it is $U(p, r; \mathbb{F})$-admissible. See also [Ko3, Proposition 4.1.3].

5 Harmonic analysis on spherical homogeneous spaces and restrictions of $A_{\mathfrak{q}}(\lambda)$

In this section we study discrete series for spherical homogeneous spaces (see Theorem 5.4 and Corollary 5.6) on the basis of discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ studied in previous sections.

Suppose H, G' are closed subgroups of G. In general, the existence of an open orbit of H on G/G' is merely necessary for the transitivity of the H action (e.g. Bruhat decomposition, Matsuki decomposition). However, it is also sufficient in the case of reductive groups:

Lemma 5.1 *Suppose that G is a connected real reductive linear Lie group and that H and G' are closed subgroups reductive in G with finitely many connected* components. Then the following three conditions are equivalent:

(1) *The natural immersion* $H/H \cap G' \hookrightarrow G/G'$ *is surjective.*
(2) *The natural immersion* $G'/H \cap G' \hookrightarrow G/H$ *is surjective*
(3) $\dim H + \dim G' = \dim G + \dim (H \cap G')$.

Proof. The non-trivial part is the implication (3) ⇒ (1) (or (3) ⇒ (2)), or equivalently, to show that an open H-orbit on G/G' is necessarily a closed orbit. We may and do assume that a Cartan involution θ of G stabilizes both H and G' (possibly after taking conjugates of H and G'). We write $K' := K \cap G'$. According to the vector bundle structure $G/G' \simeq K \times_{K'} (\mathfrak{p}_0/\mathfrak{p}_0 \cap \mathfrak{g}_0')$, we have an $H \cap K$ equivariant vector bundle map

$$(H \cap K) \underset{H \cap K'}{\times} (\mathfrak{p}_0 \cap \mathfrak{h}_0 / \mathfrak{p}_0 \cap \mathfrak{h}_0 \cap \mathfrak{g}_0') \to K \underset{K'}{\times} (\mathfrak{p}_0 / \mathfrak{p}_0 \cap \mathfrak{g}_0'),$$

corresponding to the immersion $H/H \cap G' \hookrightarrow G/G'$. An open orbit of a compact group is automatically closed, and so we have an isomorphism between base spaces $H \cap K/H \cap K' \xrightarrow{\sim} K/K'$. On the other hand, an open image of a homomorphism between vector spaces is obviously closed, and so we have an isomorphism between fibers $\mathfrak{p}_0 \cap \mathfrak{h}_0/\mathfrak{p}_0 \cap \mathfrak{h}_0 \cap \mathfrak{g}_0' \xrightarrow{\sim} \mathfrak{p}_0/\mathfrak{p}_0 \cap \mathfrak{g}_0'$. Therefore an open H-orbit on G/G' is also closed. Hence (3) ⇒ (1). □

We should remark that the third condition in Lemma 5.1 depends only on their complexifications (or their compact real forms). For instance, we have

Example 5.2 The natural inclusions $\mathrm{Sp}(m) \subset U(2m)$, $U(m) \subset \mathrm{SO}(2m)$ and $\mathrm{SU}(3) \subset G_2 \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8)$ induce (see [Bo]):

$$\mathrm{Sp}(m)/\mathrm{Sp}(m-1) \xrightarrow{\sim} U(2m)/U(2m-1) \xrightarrow{\sim} S^{4m-1}, \quad U(m)/U(m-1) \xrightarrow{\sim} \mathrm{SO}(2m)/\mathrm{SO}(2m-1) \xrightarrow{\sim} S^{2m-1},$$

$$\mathrm{Spin}(7)/G_2 \xrightarrow{\sim} \mathrm{Spin}(8)/\mathrm{Spin}(7) \xrightarrow{\sim} S^7, \quad G_2/SU(3) \xrightarrow{\sim} \mathrm{Spin}(7)/\mathrm{Spin}(6) \xrightarrow{\sim} S^6 .$$

Hence we have also isomorphisms of different real forms such as

$$\mathrm{Sp}(p,q)/\mathrm{Sp}(p-1,q) \xrightarrow{\sim} U(2p,2q)/U(2p-1,2q), \quad U(p,q)/U(p-1,q) \xrightarrow{\sim} \mathrm{SO}(2p,2q)/\mathrm{SO}(2p-1,2q),$$

$$U(m,m)/U(m-1,m) \xrightarrow{\sim} \mathrm{Sp}(m,\mathbb{R})/\mathrm{Sp}(m-1,\mathbb{R}) \xrightarrow{\sim} \mathrm{GL}(2m,\mathbb{R})/\mathrm{GL}(2m-1,\mathbb{R}),$$

$$\mathrm{GL}(m,\mathbb{R})/\mathrm{GL}(m-1,\mathbb{R}) \xrightarrow{\sim} \mathrm{SO}(m,m)/\mathrm{SO}(m-1,m),$$

$$G_2(\mathbb{R})/\mathrm{SL}(3,\mathbb{R}) \xrightarrow{\sim} \mathrm{SO}_0(4,3)/\mathrm{SO}_0(3,3), \quad G_2(\mathbb{R})/\mathrm{SU}(2,1) \xrightarrow{\sim} \mathrm{SO}_0(4,3)/\mathrm{SO}_0(4,2)$$

and those obtained by (1) ⇔ (2) in Lemma (5.1) such as

$$\mathrm{SU}(2p-1,2q)/\mathrm{Sp}(p-1,q) \xrightarrow{\sim} \mathrm{SU}(2p,2q)/\mathrm{Sp}(p,q), \quad \mathrm{SO}(2p-1,2q)/U(p-1,q) \xrightarrow{\sim} \mathrm{SO}(2p,2q)/U(p,q),$$

$$\mathrm{SU}(m-1,m)/\mathrm{Sp}(m-1,\mathbb{R}) \xrightarrow{\sim} \mathrm{SU}(m,m)/\mathrm{Sp}(m,\mathbb{R}), \quad \mathrm{SL}(2m-1,\mathbb{R})/\mathrm{Sp}(m-1,\mathbb{R}) \xrightarrow{\sim} \mathrm{SL}(2m,\mathbb{R})/\mathrm{Sp}(m,\mathbb{R}),$$

$$\mathrm{SO}(m-1,m)/\mathrm{GL}(m-1,\mathbb{R}) \xrightarrow{\sim} \mathrm{SO}(m,m)/\mathrm{GL}(m,\mathbb{R}),$$

$$\mathrm{SO}_0(3,3)/\mathrm{SL}(3,\mathbb{R}) \xrightarrow{\sim} \mathrm{SO}_0(4,3)/G_2(\mathbb{R}) \xleftarrow{\sim} \mathrm{SO}_0(4,2)/\mathrm{SU}(2,1) .$$

In the setting of Lemma 5.1, G/H carries a G-invariant (pseudo-) Riemannian metric and then a G-invariant measure. Let $L^2(G/H)$ denote the space of square integrable functions on G/H with respect to this invariant measure. Then naturally we have a unitary representation of G on the Hilbert space $L^2(G/H)$. An irreducible unitary representation $\pi \in \hat{G}$ is called **discrete series** for $L^2(G/H)$ if π can be realized as a closed subspace of $L^2(G/H)$.

Definition 5.3 *The totality of discrete series for $L^2(G/H)$ is denoted by* $\mathrm{Disc}(G/H)$ $(\subset \hat{G})$. *We also write* $\mathbb{D}\mathrm{isc}(G/H)$ *for the multiset of* $\mathrm{Disc}(G/H)$ *counted with multiplicity occurring in $L^2(G/H)$. Suppose G' is also a subgroup reductive in G with finitely many connected components. For $(\pi, V) \in \hat{G}$, we write* $\mathrm{Disc}(\pi_{|G'})(\subset \widehat{G'})$ *for the irreducible discrete summands of the restriction $\pi_{|G'}$, and* $\mathbb{D}\mathrm{isc}(\pi_{|G'})$ *for the corresponding multiset counted with multiplicity. We note that* $(\pi_{|G'}, V)_d \simeq \sum^{\oplus}_{\sigma \in \mathbb{D}\mathrm{isc}(\pi_{|G'})} \sigma$ *with the notation in §1.*

The isomorphism of G'-manifolds $\iota\colon G'/H' \xrightarrow{\sim} G/H$ induces that of Hilbert spaces $\iota^*\colon L^2(G/H) \xrightarrow{\sim} L^2(G'/H')$, on which the regular representations of G and G' are compatible with the restriction with respect to $G \supset G'$. Then the following theorem relates $\mathbb{D}\mathrm{isc}(G/H)$ and $\mathbb{D}\mathrm{isc}(G'/H')$.

Theorem 5.4 *Suppose that G is a connected real reductive linear Lie group and that H and G' are closed subgroups reductive in G with finitely many connected components. We assume*

$$\text{(5.4)(a)} \qquad \dim H + \dim G' = \dim G + \dim (H \cap G')\,,$$

and that there exists a minimal parabolic subgroup P' of G' such that

$$\text{(5.4)(b)} \qquad \dim H' + \dim P' = \dim G' + \dim (H' \cap P')\,.$$

Then there exists a surjective map between multisets

$$\mathscr{T}\colon \mathbb{D}\mathrm{isc}(G'/H') \to \mathbb{D}\mathrm{isc}(G/H)\,,$$

such that if $\pi \in \mathrm{Disc}(G/H)(\subset \hat{G})$ *then the fiber* $\mathscr{T}^{-1}(\pi) = \mathbb{D}\mathrm{isc}(\pi_{|G'})$ *(see Definition 5.3). This means that we have a bijection between multisets:*

$$\text{(5.4)(c)} \qquad \bigcup_{\pi \in \mathbb{D}\mathrm{isc}(G/H)} \mathbb{D}\mathrm{isc}(\pi_{|G'}) = \mathbb{D}\mathrm{isc}(G'/H')\,.$$

In particular, $\mathrm{Disc}(G'/H') = \emptyset$ *if and only if either* $\mathrm{Disc}(G/H) = \emptyset$ *or the discrete part $(\pi_{|G'})_d = \{0\}$ for any* $\pi \in \mathrm{Disc}(G/H)$. *Moreover, if discrete series for G'/H' is multiplicity free, then the discrete part of the restriction $\pi_{|G'}$ is multiplicity free for any* $\pi \in \mathrm{Disc}(G/H) \subset \hat{G}$.

Proof. For $\pi \in \mathrm{Disc}(G/H)(\subset \hat{G})$, we write $m(\pi)$ for the multiplicity of π occurring in $L^2(G/H)$ and fix a base $\{T_1^\pi, \ldots, T_{m(\pi)}^\pi\}$ of the $\mathbb{C}$-vector space $\mathrm{Hom}_G(\pi, L^2(G/H)))$ so that the image $T_j^\pi(\pi)$ is mutually orthogonal. Likewise, if $\tau \in \widehat{G'}$ occurs in the decomposition of $\pi_{|G'}$ as a discrete summand with multiplicity $m(\tau, \pi)$, we fix a base $\{S_1^{\tau,\pi}, \ldots, S_{m(\tau,\pi)}^{\tau,\pi}\}$ of the $\mathbb{C}$-vector space $\mathrm{Hom}_{G'}(\tau, \pi)$. Then $\{T_j^\pi S_i^{\tau,\pi}(\tau) : \pi \in \mathrm{Disc}(G/H), 1 \leq i \leq m(\tau, \pi), 1 \leq j \leq m(\pi)\}$ forms G'-irreducible, mutually orthogonal closed subspaces of $L^2(G'/H') \simeq L^2(G/H)$ which are isotypic to τ. This gives rise to discrete series for $L^2(G'/H')$, and what we want to prove now is the exhaustion of discrete series by this construction. Suppose U is a G'-irreducible closed subspace of $L^2(G'/H')$, which is isotypic to $\tau \in \widehat{G'}$. Because the multiplicity of τ in $L^2(G'/H')$ is finite from the assumption (5.4)(b) by a recent result of Bien, Oshima and Yamashita (cf. [Os, Y]), we can apply Lemma 1.1 so that we find a G-irreducible closed subspace W in $L^2(G/H) \simeq L^2(G'/H')$ such that the τ-isotypic subspace $W(\tau)$ contains U. From our definition of T_j^π and $S_i^{\tau,\pi}$, this means that $\{T_j^\pi S_i^{\tau,\pi} : \pi \in \mathrm{Disc}(G/H),\ 1 \leqq i \leqq m(\tau,\pi),\ 1 \leqq j \leqq m(\pi)\}$ spans the $\mathbb{C}$-vector space $\mathrm{Hom}_{G'}(\tau, L^2(G'/H'))$. In other words, the multiset $\mathbb{D}\mathrm{isc}(G'/H')$ contains $\tau \in \widehat{G'}$ with multiplicity

$$\sum_{\pi \in \mathrm{Disc}(G/H)} m(\tau,\pi) m(\pi) \qquad \text{(finite sum)}\,.$$

Now the map $\mathscr{T}$ can be well-defined on the multiset $\mathbb{D}\mathrm{isc}(G'/H')$ via the above basis. The remaining part of the theorem is clear. $\square$

If one knows $\mathbb{D}\mathrm{isc}(G/H)$ and the restriction formula $\pi_{|G'}$ for $\pi \in \mathrm{Disc}(G/H)$, then Theorem 5.4 gives a construction and exhaustion of discrete series for G'/H'. Conversely, on the knowledge of $\mathbb{D}\mathrm{isc}(G'/H')$ and $\mathbb{D}\mathrm{isc}(G/H)$, one can establish an explicit decomposition formula in some cases, which was the approach taken in [Ko2]. In the scheme of Theorem 5.4, we shall present a classification of discrete series for some non-symmetric spherical homogeneous spaces in §6 in the cases where the restriction of any $\pi \in \mathrm{Disc}(G/H)$ is G'-admissible.

For applications of Theorem 5.4, we use a well-known description of $\mathrm{Disc}(G/H)$ for symmetric cases by means of Zuckerman's derived functor modules. Let σ be an involution of G and H be an open subgroup of the fixed points of σ. A homogeneous space G/H is called a **semisimple symmetric space**. Take a Cartan involution θ of G commuting with σ. Then we have

Fact 5.5 (Flensted-Jensen, Matsuki and Oshima, see [MO1; FJ, Chap. VIII, §2; V3, §4]) *Suppose G/H is a semisimple symmetric space. With the notation as above,* $\mathrm{Disc}(G/H) \neq \emptyset$ *if and only if* $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$. *Moreover, if the rank condition is satisfied, we fix a maximal abelian subspace $\mathfrak{t}_0$ in $\{X \in \mathfrak{k}_0 : \sigma(X) = -X\}$. Then any discrete series $\pi \in \mathrm{Disc}(G/H)$ is of the form $\mathscr{R}_\mathfrak{q}^S(\mathbb{C}_\lambda)$, where $\mathfrak{q}$ is defined by a generic element in $\sqrt{-1}\mathfrak{t}_0$ and $\mathbb{C}_\lambda$ is a metaplectic character of $Z_G(\mathfrak{t}_0)^\sim$ in the fair range satisfying some integral conditions determined by (G, H).*

Here is an application of results of §3, §4 and Theorem 5.4 to determine the existence of discrete series for some non-symmetric spherical homogeneous spaces.

Corollary 5.6
(5.6)(a) *real forms of* $SO(2n+1,\mathbb{C})/GL(n,\mathbb{C})$:
$\mathrm{Disc}(SO(2p-1,2q)/U(p-1,q)) \neq \emptyset$ if and only if $pq \in 2\mathbb{Z}$.
(5.6)(b) *real forms of* $SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$:
$\mathrm{Disc}(SU(2p-1,2q)/Sp(p-1,q)) \neq \emptyset$ for any p, q,
$\mathrm{Disc}(SU(n,n+1)/Sp(n,\mathbb{R})) \neq \emptyset$,
$\mathrm{Disc}(SL(2n+1,\mathbb{R})/Sp(n,\mathbb{R})) = \emptyset$.
(5.6)(c) *real forms of* $GL(n+1,\mathbb{C})/GL(n,\mathbb{C})$:
$\mathrm{Disc}(U(p,q)/U(p-1,q)) \neq \emptyset$ for any p, q,
$\mathrm{Disc}(GL(n+1,\mathbb{R})/GL(n,\mathbb{R})) = \emptyset$.
(5.6)(d) *real forms of* $G_2(\mathbb{C})/SL(3,\mathbb{C})$:
$\mathrm{Disc}(G_2(\mathbb{R})/SU(2,1)) \neq \emptyset$,
$\mathrm{Disc}(G_2(\mathbb{R})/SL(3,\mathbb{R})) \neq \emptyset$.

(5.6)(e) *real form of* $SO(7, \mathbb{C})/G_2(\mathbb{C})$:
$\mathrm{Disc}(SO(4, 3)/G_2(\mathbb{R})) \neq \emptyset$.
(5.6)(f) *real forms of* $Sp(1, \mathbb{C}) \times Sp(n+1, \mathbb{C})/\mathrm{diag}(Sp(1, \mathbb{C})) \times Sp(n, \mathbb{C})$:
$\mathrm{Disc}(Sp(1) \times Sp(p, q)/\mathrm{diag}(Sp(1)) \times Sp(p-1, q)) \neq \emptyset$ for any p, q.

Proof. First we note that the above homogeneous spaces are spherical and of course satisfy (5.4)(b). Therefore the multiplicity of discrete series is always finite if exist. The proof of the corollary is divided into four cases:

(i) If G/H is a symmetric space $SO(2p, 2q)/U(p, q)$ with $pq \in 2\mathbb{Z}+1$ or $SL(2m, \mathbb{R})/Sp(m, \mathbb{R})$, then $\mathrm{rank}\, G/H > \mathrm{rank}\, K/H \cap K$ and so $\mathrm{Disc}(G/H) = \emptyset$ by Fact 5.5. Applying Theorem 5.4 to equivariant diffeomorphisms

$$SO(2p-1, 2q)/U(p-1, q) \xrightarrow{\sim} SO(2p, 2q)/U(p, q)$$

and

$$SL(2m-1, \mathbb{R})/Sp(m-1, \mathbb{R}) \xrightarrow{\sim} SL(2m, \mathbb{R})/Sp(m, \mathbb{R})$$

in Example 5.2, we conclude that $\mathrm{Disc}(SO(2p-1, 2q)/U(p-1, q)) = \emptyset$ ($pq \in 2\mathbb{Z}+1$) and $\mathrm{Disc}(SL(2m-1, \mathbb{R})/Sp(m-1, \mathbb{R})) = \emptyset$.
(ii) Because $SO(4, 2)/U(2, 1)$ is a symmetric space with the rank condition, Fact 5.5 says $\mathrm{Disc}(SO(4, 2)/SU(2, 1)) \supset \mathrm{Disc}(SO(4, 2)/U(2, 1)) \neq \emptyset$. From Theorem 5.4, we conclude that $\mathrm{Disc}(SO(4, 3)/G_2(\mathbb{R})) \neq \emptyset$.
(iii) This case is the main part of this corollary. Since the computation is fairly similar for other cases, we shall give the details only in the case $G'/H' = SO(2p-1, 2q)/U(p-1, q)$. Let $G = SO(2p, 2q) \supset K = S(O(2p) \times O(2q))$. We represent the root system of $\mathfrak{g}$ and $\mathfrak{k}$ as $\Delta(\mathfrak{g}, \mathfrak{t}^c) = \{\pm(f_i \pm f_j)\colon 1 \leqq i < j \leqq p+q\}$, $\Delta(\mathfrak{k}, \mathfrak{t}^c) = \{\pm(f_i \pm f_j)\colon 1 \leqq i < j \leqq p$ or $p+1 \leqq i < j \leqq p+q\}$. Suppose $pq \in 2\mathbb{Z}$. To describe discrete series for a symmetric space $G/H = SO(2p, 2q)/U(p, q)$, we take $\sqrt{-1}\mathfrak{t}_0^* = \mathbb{R}(f_1 + f_2) + \cdots + \mathbb{R}(f_{2[\frac{p}{2}]-1} + f_{2[\frac{p}{2}]}) + \mathbb{R}(f_{p+1} + f_{p+2}) + \cdots + \mathbb{R}(f_{p+2[\frac{q}{2}]-1} + f_{p+2[\frac{q}{2}]})$ (see notation in Fact 5.5). We define $\mu := ([\frac{p}{2}], [\frac{p}{2}], \ldots, 1, 1, (0), [\frac{p}{2}] + [\frac{q}{2}], [\frac{p}{2}] + [\frac{q}{2}], \ldots, [\frac{p}{2}] + 1, [\frac{p}{2}] + 1, (0)) \in \sqrt{-1}\mathfrak{t}_0^*$ with the above coordinate and define a θ-stable parabolic subalgebra $\mathfrak{q} \equiv \mathfrak{q}(\mu) = \mathfrak{l} + \mathfrak{u}$ (see Definition 2.1). Then, since $\Delta(\mathfrak{u} \cap \mathfrak{p}) \subset \{f_i \pm f_j\colon 1 \leqq j \leqq p, p+1 \leqq i \leqq p+q\}$, it is easy to see $\mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p} \rangle \cap \mathbb{R}\langle f_1, \ldots, f_p \rangle = \{0\}$. With the notation in Theorem 3.2, $\sqrt{-1}(\mathfrak{t}_{0-}^c)^* = \mathbb{R}f_1$ for a symmetric pair $(SO(2p, 2q), SO(2p-1, 2q))$. Hence the condition (3.2)(a) is satisfied and so Theorem 3.2 assures that $\mathscr{R}_{\mathfrak{q}}^S(W)$ is $SO(2p-1, 2q)$-admissible for any metaplectic representation W satisfying (2.2)(c). Now we have found a discrete series for $SO(2p, 2q)/U(p, q)$ whose restriction to $SO(2p-1, 2q)$ is admissible. Thanks to Theorem 5.4, we now conclude $\mathrm{Disc}(SO(2p-1, 2q)/U(p-1, q)) \neq \emptyset$ if $pq \in 2\mathbb{Z}$. This argument applies to other cases where $G'/H' = SU(2p-1, 2q)/Sp(p-1, q)$, $SU(n, n+1)/Sp(n, \mathbb{R})$, $G_2(\mathbb{R})/SU(2, 1)$, $G_2(\mathbb{R})/SL(3, \mathbb{R})$, $U(p, q)/U(p-1, q)$, $Sp(1) \times Sp(p, q)/\mathrm{diag}(Sp(1)) \times Sp(p-1, q)$.
(iv) It is deduced from the following lemma with $H_1 = \mathbb{R}$ that $\mathrm{Disc}(GL(n+1, \mathbb{R})/GL(n, \mathbb{R})) = \emptyset$. □

Lemma 5.7 *Suppose that* $G \supset H$ *are reductive Lie groups and that* H *has a direct decomposition* $H_1 \times H_2$ *with* H_1 *noncompact. Then either* $\mathrm{Disc}(G/H_2) = \emptyset$ *or any discrete series for* G/H_2 *occurs in* $L^2(G/H_2)$ *with infinite multiplicity.*

Proof. Correspondingly to the G-equivariant H_1-principal bundle $H_1 \to G/H_2 \to G/H$, we regard $G/H_2 \simeq H_1 \times G/\mathrm{diag}(H_1) \times H_2$. Applying Theorem 5.4, if $\tau \in \mathrm{Disc}(G/H_2)$ and if the multiplicity of τ occurring in $L^2(G/H_2)$ is finite, then we can find $\pi \in \mathrm{Disc}(H_1 \times G/\mathrm{diag}(H_1) \times H_2)$. We shall see that this leads to a contradiction. In view of

$$L^2(G/H_2) = L^2(G/H, L^2(H/H_2)) = \int_{\widehat{H_1}}^{\oplus} L^2(G/H, \tau \boxtimes 1) d\mu(\tau) ,$$

where $d\mu(\tau)$ is the Plancherel measure on H_1, π must be of the form $\pi = \sigma \boxtimes \tau$ with some $\sigma \in \mathrm{Disc}(H_1) \subset \widehat{H_1}$. Because H_1 is non-compact, $\dim \sigma = \infty$. On the other hand, the multiplicity of τ in $L^2(G/H_2)$ must be at least $\dim \sigma$. This contradicts to the fact the assumption that the multiplicity of τ in $L^2(G/H_2)$ is finite. $\square$

We remark that we are not intending to make a complete list in Corollary 5.6. Actually, we have omitted several (easy) cases where either [S1; Ko3, Proposition 0.4] or Lemma 5.7 can be applied. We end this section with some remarks suggesting further study.

Remark 5.8 (1) Discrete series for non-symmetric spherical homogeneous spaces in (5.6)(c), (d) and (f) are classified in §6. It would be interesting to find the explicit restriction formula and to classify discrete series for some other non-symmetric spherical homogeneous spaces in the scheme of Theorem 5.4.
(2) There are some other spherical homogeneous spaces such as $SO(m, m+1)/GL(m, \mathbb{R})$ and $Sp(1, \mathbb{R}) \times Sp(n+1, \mathbb{R})/\mathrm{diag}(Sp(1, \mathbb{R})) \times Sp(n, \mathbb{R})$, for which we do not tell the existence of discrete series by this method.
(3) The result in (5.6)(b) suggests a geometric construction of 'holomorphic discrete series' in the sense of 'Olafsson and Ørsted [OO] for a non-symmetric spherical homogeneous space $SU(2p-1, 2q)/Sp(p-1, q)$.
(4) Our approach also presents examples where the restrictions of $\pi \in \hat{G}$ with respect to a reductive subgroup G' is decomposed only by the continuous spectrum (an opposite direction to theorems in §3 and 4). It is the case when $G = SO(4, 3) \supset G' = SO(3, 3)$ and $\pi \in \mathrm{Disc}(SO(4, 3)/G_2(\mathbb{R}))$.

6 Examples of an explicit decomposition formula

In this section, in the framework of Theorem 5.4, we present examples of an explicit decomposition formula of $\overline{A_{\mathfrak{q}}(\lambda)}|_{G'}$ for $\overline{A_{\mathfrak{q}}(\lambda)} \in \hat{G}$ together with the classification of discrete series for non-symmetric spherical homogeneous spaces G'/H'.

In contrast to Harish-Chandra's discrete series, we have to deal with $\mathscr{R}_{\mathfrak{q}}^S(\mathbb{C}_\lambda)$ where the range of parameters of $\mathbb{C}_\lambda$ is not necessarily in the good range but in the weakly fair range in the sense of [V3] (see (2.2)(a), (b)). By this reason, we use the notation $\mathscr{R}_{\mathfrak{q}}^S(\mathbb{C}_\lambda)$ instead of $A_{\mathfrak{q}}(\lambda)$ $(\simeq \mathscr{R}_{\mathfrak{q}}^S(\mathbb{C}_{\lambda+\rho(\mathfrak{u})}))$. For simplicity, we omit the notation for obvious Hilbert completions in this section.

6.1 $Sp(p, q) \subset SU(2p, 2q)$, $U(p, q) \subset SO_0(2p, 2q)$.

Let $G = SO_0(2p, 2q)$ $(p \geqq 1,\ q \geqq 0)$. We represent the root system of $\mathfrak{k}$ as $\Delta(\mathfrak{k}, \mathfrak{t}^c) = \{\pm(f_i \pm f_j) : 1 \leqq i < j \leqq p \text{ or } p+1 \leqq i < j \leqq p+q\}$ and define a θ-stable parabolic subalgebra by $\mathfrak{q} := \mathfrak{q}(f_1)$. For $\lambda \in \mathbb{Z}$, we write $\mathbb{C}_{\lambda f_1}$ for the metaplectic representation of $\tilde{L}$ corresponding to $\lambda f_1 \in \sqrt{-1}(\mathfrak{t}_0^c)^*$. We define a $(\mathfrak{g}, K)$-module by

$$U(\lambda) \equiv U^{SO_0(2p,2q)}(\lambda) := (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^{2p-2}(\mathbb{C}_{\lambda f_1}) .$$

If we write $H = SO_0(2p-1, 2q)$, then it is well-known (see also Fact 5.5) that the totality of discrete series for a semisimple symmetric space G/H is given by (multiplicity free)

$$\mathrm{Disc}(G/H) = \begin{cases} \{U(\lambda) : \lambda \in \mathbb{N}_+\} & (p \geqq 2,\ q \geqq 1) , \\ \{U(\lambda),\ U(\lambda)^\vee : \lambda \in \mathbb{N}_+\} & (p = 1, q \geqq 1) , \\ \{U(\lambda) : \lambda \in \mathbb{N}_+,\ \lambda \geqq p-1\} & (p \geqq 2,\ q = 0) , \\ \{U(\lambda),\ U(\lambda)^\vee,\ U(0) : \lambda \in \mathbb{N}_+\} & (p = 1,\ q = 0) . \end{cases}$$

Next, let $G' = U(p,q)$, and represent the root system of $\mathfrak{k}'$ as $\Delta(\mathfrak{k}', \mathfrak{t}'^c) = \{\pm(e_i - e_j) \colon 1 \leqq i < j \leqq p \text{ or } p+1 \leqq i < j \leqq p+q\}$. Fix $A \gg B \gg 0$ and we define θ-stable parabolic subalgebras of $\mathfrak{g}'$ by $\mathfrak{q}'_+ := \mathfrak{q}(Ae_1 + Be_{p+1})$, $\mathfrak{q}'_0 := \mathfrak{q}(e_1 - e_p)$, and $\mathfrak{q}'_- := \mathfrak{q}(-Ae_p - Be_{p+q})$. For $\lambda \in \mathbb{N}_+$, $l \in \mathbb{Z}$ such that $l \equiv \lambda + p + q + 1 \bmod 2$, we define $(\mathfrak{g}', K')$-modules by:

$$V_+(\lambda, l) \equiv V_+^{U(p,q)}(\lambda, l) := (\mathscr{R}_{\mathfrak{q}'_+}^{\mathfrak{g}'})^{p+q-2}(\mathbb{C}_{\frac{\lambda+l}{2}e_1 + \frac{-\lambda+l}{2}e_{p+1}}) \qquad \text{if } l > \lambda > 0,\ pq \geqq 1 ,$$

$$V_0(\lambda, l) \equiv V_0^{U(p,q)}(\lambda, l) := (\mathscr{R}_{\mathfrak{q}'_0}^{\mathfrak{g}'})^{2p-4}(\mathbb{C}_{\frac{\lambda+l}{2}e_1 + \frac{-\lambda+l}{2}e_p}) \qquad \text{if } \lambda \geqq |l|,\ p \geqq 2 ,$$

$$V_-(\lambda, l) \equiv V_-^{U(p,q)}(\lambda, l) := (\mathscr{R}_{\mathfrak{q}'_-}^{\mathfrak{g}'})^{p+q-2}(\mathbb{C}_{\frac{-\lambda+l}{2}e_p + \frac{\lambda+l}{2}e_{p+q}}) \qquad \text{if } -l > \lambda > 0,\ pq \geqq 1 .$$

Third, let $G'' = Sp(p,q)$, and represent the root system of $\mathfrak{k}''$ as $\Delta(\mathfrak{k}'', \mathfrak{t}''^c) = \{\pm(h_i - h_j),\ \pm 2h_l \colon 1 \leqq i < j \leqq p \text{ or } p+1 \leqq i < j \leqq p+q,\ 1 \leqq l \leqq p+q\}$. Fix $A \gg B \gg 0$ and we define θ-stable parabolic subalgebras of $\mathfrak{g}''$ by $\mathfrak{q}''_+ := \mathfrak{q}(Ah_1 + Bh_{p+1})$, $\mathfrak{q}''_0 := \mathfrak{q}(Ah_1 + Bh_2)$. For $\lambda \in \mathbb{N}_+, j \in \mathbb{N}$ such that $j \equiv \lambda + 1 \bmod 2$, we define $(\mathfrak{g}'', K'')$-modules by:

$$W_+(\lambda, j) \equiv W_+^{Sp(p,q)}(\lambda, j) := (\mathscr{R}_{\mathfrak{q}''_+}^{\mathfrak{g}''})^{2p+2q-2}(\mathbb{C}_{\frac{\lambda+j+1}{2}h_1 + \frac{-\lambda+j+1}{2}h_{p+1}})$$

$$\text{if } j+1 > \lambda > 0,\ pq \geqq 1 ,$$

$$W_0(\lambda, j) \equiv W_+^{Sp(p,q)}(\lambda, j) := (\mathscr{R}_{\mathfrak{q}''_0}^{\mathfrak{g}''})^{4p-4}(\mathbb{C}_{\frac{\lambda+j+1}{2}h_1 - \frac{-\lambda+j+1}{2}h_2})$$

$$\text{if } \lambda \geqq j+1 > 0,\ p \geqq 2.$$

Here is a reformulation of [HT] (cf. [Ko2] for a special case) in the scheme of Theorem 5.4:

Theorem 6.1. (1) *Let* $p \geqq 2$ *and* $q \geqq 1$. *First we fix* $\lambda \in \mathbb{N}_+$. *According to* $U(p,q) \subset SO_0(2p, 2q)$, *we have*

$$U^{SO_0(2p,2q)}(\lambda)|_{U(p,q)} = \bigoplus_{l > \lambda} V_+(\lambda, l) + \bigoplus_{|l| \leqq \lambda} V_0(\lambda, l) + \bigoplus_{l < -\lambda} V_-(\lambda, l) ,$$

where the sum is taken over $l \in 2\mathbb{Z} + \lambda + p + q + 1$.

Next, we fix $\lambda \in \mathbb{N}_+$, $l \in \mathbb{Z}$ *such that* $l \equiv \lambda + 1 \bmod 2$. *According to* $\mathrm{Sp}(p,q) \subset U(2p,2q) \subset SO_0(4p,4q)$, *we have*

$$V_+^{U(2p,2q)}(\lambda, l)_{|\mathrm{Sp}(p,q)} = V_-^{U(2p,2q)}(\lambda, -l)_{|\mathrm{Sp}(p,q)} = \bigoplus_{j \geqq l} W_+(\lambda, j) \qquad \text{if } l > \lambda ,$$

$$V_0^{U(2p,2q)}(\lambda, l)_{|\mathrm{Sp}(p,q)} = \bigoplus_{|l| \leqq j \leqq \lambda - 1} W_0(\lambda, j) + \bigoplus_{\lambda + 1 \leqq j} W_+(\lambda, j) \qquad \text{if } |l| \leqq \lambda ,$$

$$U^{SO_0(4p,4q)}(\lambda)_{|\mathrm{Sp}(p,q)} = \bigoplus_{0 \leqq j \leqq \lambda - 1} (j+1) W_0(\lambda, j) + \bigoplus_{\lambda + 1 \leqq j} (j+1) W_+(\lambda, j) ,$$

where the sum is taken over $j \in 2\mathbb{Z} + \lambda + 1$. *On the other hand, if we fix* $l \in \mathbb{Z}$ *and* $j \in \mathbb{N}$, *then we have a classification of discrete series for associated vector bundles*:

$$\mathbb{D}\mathrm{isc}(U(p,q)/U(1) \times U(p-1,q); \chi_l)$$
$$= \begin{cases} \{V_+(\lambda, l) : l > \lambda > 0\} \cup \{V_0(\lambda, l) : \lambda \geqq l\} & (l > 0) , \\ \{V_0(\lambda, l) : \lambda > 0\} & (l = 0) , \\ \{V_-(\lambda, l) : -l > \lambda > 0\} \cup \{V_0(\lambda, l) : \lambda \geqq -l\} & (l < 0) , \end{cases}$$

$$\mathbb{D}\mathrm{isc}(\mathrm{Sp}(p,q)/\mathrm{Sp}(1) \times \mathrm{Sp}(p-1,q); \sigma_j)$$
$$= \{(j+1) W_0(\lambda, j) : \lambda > j\} \cup \{(j+1) W_+(\lambda, j) : j > \lambda > 0\} .$$

Here χ_l *is a character of* $U(1)$ *and* σ_j *is the* $j+1$ *dimensional representation of* $\mathrm{Sp}(1)$. *The parameter* λ *runs over* $\lambda \in 2\mathbb{Z} + l + p + q + 1$, $\lambda \in 2\mathbb{Z} + j + 1$, *respectively. The multiplicity of discrete series is uniformly equal to* $j+1$ *for the* $\mathrm{Sp}(p,q)$ *case.*
(2) *Let* $p = 1$ *and* $q \geqq 1$. *First we fix* $\lambda \in \mathbb{N}_+$.

$$U^{SO_0(2,2q)}(\lambda)_{|U(1,q)} \simeq \bigoplus_{\substack{l \in 2\mathbb{Z} + \lambda + q \\ l \geqq \lambda + q}} V_+^{U(1,q)}(\lambda, l),$$

$$U^{SO_0(2,2q)}(\lambda)^{\vee}{}_{|U(1,q)} \simeq \bigoplus_{\substack{l \in 2\mathbb{Z} + \lambda + q \\ l \leqq -\lambda - q}} V_-^{U(1,q)}(\lambda, l) .$$

Next, we fix $\lambda \in \mathbb{N}_+$, $l \in \mathbb{Z}$ *such that* $l \equiv \lambda + 1 \bmod 2$.

$$V_+^{U(2,2q)}(\lambda, l)_{|\mathrm{Sp}(1,q)} = V_-^{U(2,2q)}(\lambda, -l)_{|\mathrm{Sp}(1,q)} = \bigoplus_{j \geqq \max(l, \lambda + 2q - 1)} W_+(\lambda, j) \text{ if } l > \lambda ,$$

$$V_0^{U(2,2q)}(\lambda, l)_{|\mathrm{Sp}(1,q)} = \bigoplus_{\lambda + 2q - 1 \leqq j} W_+(\lambda, j) \qquad \text{if } |l| \leqq \lambda ,$$

$$U^{SO_0(4,4q)}(\lambda)_{|\mathrm{Sp}(1,q)} = \bigoplus_{\lambda + 2q - 1 \leqq j} (j+1) W_+(\lambda, j) ,$$

where the sum is taken over $j \in 2\mathbb{Z} + \lambda + 1$. *On the other hand, if we fix* $l \in \mathbb{Z}$ *and* $j \in \mathbb{N}$, *then we have a classification of discrete series for associated vector bundles*:

$$\mathbb{D}\mathrm{isc}(U(1,q)/U(1) \times U(q); \chi_l) = \begin{cases} \{V_+(\lambda, l) : l - q \geqq \lambda > 0\} & (l > q) , \\ \emptyset & (|l| \leqq q) , \\ \{V_-(\lambda, l) : -l - q \geqq \lambda > 0\} & (-q > l) , \end{cases}$$

$$\mathbb{D}\mathrm{isc}(\mathrm{Sp}(1,q)/\mathrm{Sp}(1) \times \mathrm{Sp}(q); \sigma_j) = \begin{cases} \{(j+1) W_+(\lambda, j) : j - 2q + 1 \geqq \lambda > 0\}, & (j \geqq 2q), \\ \emptyset, & (j < 2q) . \end{cases}$$

Sketch of Proof. We explain a proof which consists of the following two steps.

Step 1 Knowledge of some degenerate principal series of $U(p, q; \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) such as irreducible constituents and the branching rule with respect to $\mathrm{Sp}(p, q) \subset U(2p, 2q)$ and $U(p, q) \subset SO_0(2p, 2q)$.
Step 2 Identification of some irreducible constituents as derived functor modules.

Step 1 is done in [HT]. We set up some necessary notation. With the notation in §2.1 l.c. (p, q be even, $a \in \mathbb{Z}$, $a \geqq 2 - p - q$), we write $S_{+-}^{O(p,q)}(a)$ for the Harish-Chandra module of the irreducible quotient of the degenerate principal series $S^{a,(-1)^a}(X^o)$ of $O(p, q)$ whose K-type region is in the right wing of the barrier A^{+-} (see p. 17 (i), (ii) l.c.). Second, with the notation in §4 l.c. ($p \geqq 2$, $q \geqq 1$, $\alpha, \beta \in \mathbb{Z}$, $\alpha + \beta \geqq 2 - p - q$), we write $S_{+-}^{U(p,q)}(\alpha, \beta)$ for the Harish-Chandra module of the irreducible quotient of the degenerate principal series $S^{\alpha,\beta}(X^o)$ of $U(p, q)$ whose K-type region is in the right wing of the barrier A^{+-} (see p. 39 (i), (ii) for $q \geqq 2$, §4.5 for $q = 1$). If $p = 1$, the K-type region of $U(1, q)$-module $S^{\alpha,\beta}(X^o)$ meets the right wing of A^{+-} iff $\min(\alpha, \beta) \leqq -q$ (cf. Diagram 4.26 l.c. for $U(p, 1)$). Thus, $S_{+-}^{U(1,q)}(\alpha, \beta)$ is defined only for $\alpha, \beta \in \mathbb{Z}$ such that $\alpha + \beta \geqq 1 - q$ and $\min(\alpha, \beta) \leqq -q$. Third, with the notation in §5 l.c. ($p \geqq 2$, $q \geqq 1$, $a \in \mathbb{Z}$, $j \in \mathbb{N}$, $a \geqq 2 - 2p - 2q$), we write $S_{+-}^{\mathrm{Sp}(p,q)}(a, j)$ for the Harish-Chandra module of the irreducible quotient of the degenerate principal series $S_j^a(X^o)$ of $Sp(p, q)$ whose K-type region is in the right wing of the barrier A^{+-}. If $p = 1$, $S_{+-}^{\mathrm{Sp}(1,q)}(a, j)$ is defined only for $a \in \mathbb{Z}$, $j \in \mathbb{N}$, $a \geqq -2q$ and $a \leqq j - 4q$. It can be read from [HT] that if $a \in \mathbb{Z}$ satisfies $a \geqq 2 - p - q$ (if $\alpha, \beta \in \mathbb{Z}$ satisfies $\alpha + \beta \geqq 2 - 2p - 2q$, if $a \in \mathbb{Z}$ satisfies $a \geqq 2 - 2p - 2q$, respectively), then for $p \geqq 2$ and $q \geqq 1$ we have

$$S_{+-}^{O(2p,2q)}(a) = \bigoplus_{\substack{\alpha, \beta \in \mathbb{Z} \\ \alpha + \beta = a}} S_{+-}^{U(p,q)}(\alpha, \beta), \qquad S_{+-}^{O(2,2q)}(a) = \bigoplus_{\substack{\alpha, \beta \in \mathbb{Z} \\ |\alpha - \beta| \geqq a + 2q \\ \alpha + \beta = a}} S_{+-}^{U(1,q)}(\alpha, \beta),$$

$$S_{+-}^{U(2p,2q)}(\alpha, \beta) = \bigoplus_{\substack{j \in 2\mathbb{Z} + \alpha + \beta \\ j \geqq |\alpha - \beta|}} S_{+-}^{\mathrm{Sp}(p,q)}(\alpha + \beta, j), \quad S_{+-}^{U(2,2q)}(\alpha, \beta) = \bigoplus_{\substack{j \in 2\mathbb{Z} + \alpha + \beta \\ j \geqq |\alpha - \beta| \\ j \geqq \alpha + \beta + 4q}} S_{+-}^{\mathrm{Sp}(1,q)}(\alpha + \beta, j),$$

$$S_{+-}^{O(4p,4q)}(a) = \bigoplus_{\substack{j \in 2\mathbb{Z} + a \\ j \geqq 0}} (j + 1) S_{+-}^{\mathrm{Sp}(p,q)}(a, j), \quad S_{+-}^{O(4,4q)}(a) = \bigoplus_{\substack{j \in 2\mathbb{Z} + a \\ j \geqq a + 4q}} (j + 1) S_{+-}^{\mathrm{Sp}(1,q)}(a, j).$$

Suppose $p, q \geqq 1$, $\lambda \in \mathbb{N}_+$, $l \in 2\mathbb{Z} + \lambda + p + q + 1$, $j \in 2\mathbb{Z} + \lambda + 1$. We assume moreover $\lambda \geqq |l|$ and $j \geqq \lambda + 2q - 1$ if $p = 1$. Step 2 is to prove the following isomorphisms:

(6.2)(a)

$$S_{+-}^{O(2p,2q)}(\lambda - p - q + 1)_{|SO_0(2p,2q)} \simeq \begin{cases} U^{SO_0(2p,2q)}(\lambda) \oplus U^{SO_0(2p,2q)}(\lambda)^\vee & (p = 1), \\ U^{SO_0(2p,2q)}(\lambda) & (p \geqq 2), \end{cases}$$

(6.2)(b) $$S_{+-}^{U(p,q)}\left(\frac{\lambda + l - p - q + 1}{2}, \frac{\lambda - l - p - q + 1}{2}\right) \simeq V_\delta^{U(p,q)}(\lambda, l),$$

(6.2)(c) $$S_{+-}^{\mathrm{Sp}(p,q)}(\lambda - 2p - 2q + 1, j) \simeq W_\varepsilon^{\mathrm{Sp}(p,q)}(\lambda, j).$$

Here $\delta = +$ if $l > \lambda$, $\delta = 0$ if $|l| \leqq \lambda$, $\delta = -$ if $-l > \lambda$ in (6.2)(b), and $\varepsilon = +$ if $j + 1 > \lambda$, $\varepsilon = 0$ if $j + 1 \leqq \lambda$ in (6.2)(c).

Step 2 is divided into three substeps:

Step 2(a) In the good range of parameters, we can apply a characterization of $A_{\mathfrak{q}}(\lambda)$ given in [VZ] Proposition 6.1 to infinitesimally unitarizable representations $S_{+-}^{O(2p,2q)}(a)$ (or their irreducible direct components as $SO_0(2p, 2q)$-modules if $p = 1$), $S_{+-}^{U(p,q)}(\alpha, \beta)$ and $S_{+-}^{Sp(p,q)}(a, j)$ whose K-types we know explicitly.
Step 2(b) We can prove the isomorphism (6.2) as K-modules by finding explicit K-types of $U(\lambda)$, $V_\delta(\lambda, l)$ ($\delta = +, 0, -$) and $W_\varepsilon(\lambda, j)$ ($\varepsilon = +, 0$). Explicit K-type formulas of $U(\lambda)$, $V_0(\lambda, l)$ and $W_0(\lambda, j)$ can be obtained as a special case of [Ko3, §4]. A technical feature of calculations in the case $V_\pm(\lambda, l)$ and $W_+(\lambda, j)$ is slightly different, which we omit here.
Step 2(c) In order to deal with the weakly fair range which does not belong to the good range, we apply translation functors to both degenerate principal series and derived functor modules in a special direction of translations ("from the good parameter $\mathbb{C}_{\nu + 2k\rho(\mathfrak{u})}$ ($k \gg 0$) to the weakly fair parameter $\mathbb{C}_\nu$") based on [V3, Lemma 4.8] (see [V3, Proposition 4.7] for the case of derived functor modules). On the other hand, because we have already established the isomorphism (6.2) as K-modules in Step 2(b), and because irreducible constituents of the degenerate principal series here are characterized only by the K-types, we can now conclude that the above translation functor sends the irreducible quotient "$S_{+-}(\nu + 2k\rho(\mathfrak{u}))$" to "$S_{+-}(\nu)$" (with an abuse of notation). Hence the isomorphisms (6.2) of $(\mathfrak{g}, K)$-modules follow. □

Remark 6.3 In [Ko2] ($p = 1$), we have taken an alternative approach, that is, to find all discrete series first and then to obtain the branching formula. The latter part is based on explicit K-type formulas of corresponding discrete series. Here we use a special property that the restriction of these discrete series to K forms linearly independent K-modules.

Finally, we mention the relation of these discrete series and the degenerate principal series. We note that indefinite Stiefel manifolds $U(p,q)/U(p-1,q)$ and $Sp(p,q)/Sp(p-1,q)$ are principal bundles over semisimple symmetric spaces of rank one. Like the base spaces which are symmetric, we can define the boundary map of discrete series for the above Stiefel manifolds (one of the terms of the asymptotic expansion vanishes because of square integrability). This leads to an embedding of discrete series to the degenerate principal series of $U(p,q)$ or $Sp(p,q)$ which are dual to those considered in the above proof.

6.2 $G_2(\mathbb{R}) \subset SO_0(4,3)$.

Let $G = SO_0(4,3) \supset K = K_1 \times K_2 = SO(4) \times SO(3)$ and represent the root system of $\mathfrak{k}$ as $\Delta(\mathfrak{k}, \mathfrak{t}^c) = \{\pm(f_1 \pm f_2), \pm f_3\}$. We define θ-stable parabolic subalgebras by $\mathfrak{q}_1 := \mathfrak{q}(f_3)$ and $\mathfrak{q}_2 := \mathfrak{q}(f_1)$ (see Definition 2.1). Let $H_1 := SO_0(4,2)$, $H_2 := SO_0(3,3)$ be subgroups (naturally embedded) in G. Then it is well-known (see also Fact 5.5)

$$\mathbb{D}\mathrm{isc}(G/H_1) = \{\mathcal{R}^1_{\mathfrak{q}_1}(\mathbb{C}_{\lambda f_3}) : \lambda \in \mathbb{N} + \tfrac{1}{2}\}, \quad \mathbb{D}\mathrm{isc}(G/H_2) = \{\mathcal{R}^2_{\mathfrak{q}_2}(\mathbb{C}_{\lambda f_1}) : \lambda \in \mathbb{N} + \tfrac{1}{2}\}.$$

Denote by X the generalized flag variety of $G_{\mathbb{C}}$ containing $\mathfrak{q}_1$. There are 4 $K_{\mathbb{C}}$-orbits on X, say, $\mathcal{Q}_1$ (compact), $\mathcal{Q}_2$ (compact), $\mathcal{Q}_4$, and $\mathcal{Q}_5$ (open) such that $\dim_{\mathbb{C}} \mathcal{Q}_i = i$ ($i = 1, 2, 4, 5$), and that $\mathcal{Q}_j \ni \mathfrak{q}_j$ and K_j acts trivially on $\mathcal{Q}_j$ ($j = 1, 2$).

Next, let $G' = G_2(\mathbb{R})$ be a non-compact real form of an exceptional simple Lie group $G'_{\mathbb{C}} = G_2(\mathbb{C})$ with $K' \simeq SO(4)$. We choose a compact short root α and a non-compact short root β. We define θ-stable parabolic subalgebras by $\mathfrak{q}'_1 := \mathfrak{q}(\alpha)$ and $\mathfrak{q}'_2 := \mathfrak{q}(\beta)$. Denote by Y the generalized flag variety of $G'_{\mathbb{C}}$ containing $\mathfrak{q}'_1$. Then $Y \ni \mathfrak{q}'_2$. Let $\mathcal{Q}'_j$ be the $K'_{\mathbb{C}}$ orbit through $\mathfrak{q}'_j$ $(j = 1, 2)$. We embed $\iota\colon G' \subsetneq G$ so that $\iota(K') \subset K$. We also write ι for its complexification and its differential. Then the map $\iota^*\colon X \to Y$, $\mathfrak{q} \mapsto \iota^{-1}(\mathfrak{q})$ is well-defined, giving a $G'_{\mathbb{C}}$-equivariant isomorphism between X and Y. It is easy to see $\iota^*(\mathcal{Q}_j) \ni \mathfrak{q}'_j$ $(j = 1, 2)$. Because

$$\iota\colon \mathfrak{k}'_{\mathbb{C}} \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{k}_{\mathbb{C}} \simeq (\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})) \oplus \mathfrak{sl}(2, \mathbb{C})$$

is given by $(X, Y) \mapsto (X, Y, Y)$ up to $\mathrm{Ad}(K_{\mathbb{C}})$, we have $\iota(\mathfrak{k}'_{\mathbb{C}}) + \mathfrak{k}_{j\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}$ $(j = 1, 2)$. Because the action of $K_{j\mathbb{C}}$ on $\mathcal{Q}_j$ is trivial and that of $K_{\mathbb{C}}$ is transitive, $\iota(K'_{\mathbb{C}})$ acts transitively on $\mathcal{Q}_j$. This means $\iota^*(\mathcal{Q}_j) = \mathcal{Q}'_j$ $(j = 1, 2)$. Thus the local cohomology on X with support $\mathcal{Q}_j$ with coefficients in an invertible sheaf of $\mathcal{O}$-modules is isomorphic to that on Y with support $\mathcal{Q}'_j$ via ι^* $(j = 1, 2)$. By the duality of Harish-Chandra modules between $\mathcal{D}$-module construction and cohomological parabolic induction [HMSW, C]), we have the first half of the following theorem, while the latter half is a direct consequence of Example 5.2 and Theorem 5.4.

Theorem 6.4 (joint work with T. Uzawa)

$$(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}_1})^1(\mathbb{C}_{\lambda f_3})|_{G_2(\mathbb{R})} = (\mathcal{R}^{\mathfrak{g}'}_{\mathfrak{q}'_1})^1(\mathbb{C}_{\lambda\alpha}), \qquad (\lambda \in \mathbb{N} + \tfrac{1}{2}),$$

$$(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}_2})^2(\mathbb{C}_{\lambda f_1})|_{G_2(\mathbb{R})} = (\mathcal{R}^{\mathfrak{g}'}_{\mathfrak{q}'_2})^2(\mathbb{C}_{\lambda\beta}), \qquad (\lambda \in \mathbb{N} + \tfrac{1}{2}),$$

$$\mathrm{Disc}(G_2(\mathbb{R})/SU(2,1)) = \{(\mathcal{R}^{\mathfrak{g}'}_{\mathfrak{q}'_1})^1(\mathbb{C}_{\lambda\alpha}) : \lambda \in \mathbb{N} + \tfrac{1}{2}\},$$

$$\mathrm{Disc}(G_2(\mathbb{R})/SL(3,\mathbb{R})) = \{(\mathcal{R}^{\mathfrak{g}'}_{\mathfrak{q}'_2})^2(\mathbb{C}_{\lambda\beta}) : \lambda \in \mathbb{N} + \tfrac{1}{2}\}.$$

Remark 6.5 There are 5 $K'_{\mathbb{C}}$ orbits $\mathcal{Q}'_i$ on Y with $\dim_{\mathbb{C}} \mathcal{Q}'_i = i$ $(i = 1, 2, 3, 4, 5)$ such that $\iota^*(\mathcal{Q}_i) = \mathcal{Q}'_i$ $(i = 1, 2, 5)$, $\iota^*(\mathcal{Q}_4) = \mathcal{Q}'_3 \cup \mathcal{Q}'_4$ and the boundary $\partial\mathcal{Q}'_3 = \mathcal{Q}'_1 \cup \mathcal{Q}'_2$.

The irreducibility of the $(\mathfrak{g}', K')$-modules $(\mathcal{R}^{\mathfrak{g}'}_{\mathfrak{q}'_1})^1(\mathbb{C}_{\lambda\alpha})$ and $(\mathcal{R}^{\mathfrak{g}'}_{\mathfrak{q}'_2})^2(\mathbb{C}_{\lambda\beta})$ at $\lambda = \frac{1}{2}$ in Theorem 3.6 is not known to the author. In this sense there is a little abuse of notation in the latter part of the above theorem.

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