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Nilpotent orbits and some small unitary representations of indefinite orthogonal groups ☆

A.W. Knapp[1]

Department of Mathematics, State University of New York, Stony Brook, NY 11794, USA

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Abstract

For $2 \leqslant m \leqslant l/2$, let G be a simply connected Lie group with $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m)$ as Lie algebra, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexification of the usual Cartan decomposition, let K be the analytic subgroup with Lie algebra $\mathfrak{k} \cap \mathfrak{g}_0$, and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. This work examines the unitarity and K spectrum of representations in the "analytic continuation" of discrete series of G, relating these properties to orbits in the nilpotent radical of a certain parabolic subalgebra of $\mathfrak{g}$.

The roots with respect to the usual compact Cartan subalgebra are all $\pm e_i \pm e_j$ with $1 \leqslant i < j \leqslant l$. In the usual positive system of roots, the simple root $e_m - e_{m+1}$ is noncompact and the other simple roots are compact. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the parabolic subalgebra of $\mathfrak{g}$ for which $e_m - e_{m+1}$ contributes to $\mathfrak{u}$ and the other simple roots contribute to $\mathfrak{l}$, let L be the analytic subgroup of G with Lie algebra $\mathfrak{l} \cap \mathfrak{g}_0$, let $L^{\mathbb{C}} = \mathrm{Int}_{\mathfrak{g}}(\mathfrak{l})$, let $2\delta(\mathfrak{u})$ be the sum of the roots contributing to $\mathfrak{u}$, and let $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ be the parabolic subalgebra opposite to $\mathfrak{q}$.

The members of $\mathfrak{u} \cap \mathfrak{p}$ are nilpotent members of $\mathfrak{g}$. The group $L^{\mathbb{C}}$ acts on $\mathfrak{u} \cap \mathfrak{p}$ with finitely many orbits, and the topological closure of each orbit is an irreducible algebraic variety. If Y is one of these varieties, let $R(Y)$ be the dual coordinate ring of Y; this is a quotient of the algebra of symmetric tensors on $\mathfrak{u} \cap \mathfrak{p}$ that carries a fully reducible representation of $L^{\mathbb{C}}$.

For $s \in \mathbb{Z}$, let $\lambda_s = \sum_{k=1}^{m} (-l + \frac{s}{2}) e_k$. Then λ_s defines a one-dimensional $(\mathfrak{l}, L)$ module $\mathbb{C}_{\lambda_s}$. Extend this to a $(\bar{\mathfrak{q}}, L)$ module by having $\bar{\mathfrak{u}}$ act by 0, and define $N(\lambda_s + 2\delta(\mathfrak{u})) = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} \mathbb{C}_{\lambda_s + 2\delta(\mathfrak{u})}$. Let $N'(\lambda_s + 2\delta(\mathfrak{u}))$ be the unique irreducible quotient of $N(\lambda_s + 2\delta(\mathfrak{u}))$. The representations under study are $\pi_s = \Pi_S(N(\lambda_s + 2\delta(\mathfrak{u})))$ and $\pi'_s = \Pi_S(N'(\lambda_s + 2\delta(\mathfrak{u})))$, where $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ and Π_S is the Sth derived Bernstein functor.

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E-mail address: aknapp@math.sunysb.edu.

[1] Current address: 81 Upper Sheep Pasture Road, East Setauket, NY 11733-1729, USA.

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For $s>2l-2$, it is known that $\pi_s=\pi'_s$ and that π'_s is in the discrete series. Enright, Parthsarathy, Wallach, and Wolf showed for $m\leqslant s\leqslant 2l-2$ that $\pi_s=\pi'_s$ and that π'_s is still unitary. The present paper shows that π'_s is unitary for $0\leqslant s\leqslant m-1$ even though $\pi_s\neq\pi'_s$, and it relates the K spectrum of the representations π'_s to the representation of $L^{\mathbb{C}}$ on a suitable $R(Y)$ with Y depending on s. Use of a branching formula of D. E. Littlewood allows one to obtain an explicit multiplicity formula for each K type in π'_s; the variety Y is indispensable in the proof. The chief tools involved are an idea of B. Gross and Wallach, a geometric interpretation of Littlewood's theorem, and some estimates of norms.

It is shown further that the natural invariant Hermitian form on π'_s does not make π'_s unitary for $s<0$ and that the K spectrum of π'_s in these cases is not related in the above way to the representation of $L^{\mathbb{C}}$ on any $R(Y)$.

A final section of the paper treats in similar fashion the simply connected Lie group with Lie algebra $\mathfrak{g}_0=\mathfrak{so}(2m,2l-2m+1)$, $2\leqslant m\leqslant l/2$.

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Let G be a simple Lie group with maximal compact subgroup K and with a compact Cartan subgroup T of G chosen to lie in K, and let $\mathfrak{t}_0$ be the Lie algebra of T. It is known from the work of Harish-Chandra [HC] that the discrete series representations of G, i.e., the irreducible unitary representations of G that are direct summands of $L^2(G)$, occur in finitely many classes. Apart from repetitions, a class is determined by a positive system of roots, and the representations in the class are parametrized by the integral points in a translate of the corresponding dominant cone of the dual $(i\mathfrak{t}_0)^*$ of $i\mathfrak{t}_0$. Wallach [Wa1,Wa2] was the first to raise the question of what properties these representations have when the parameter is moved outside the cone. He studied these "analytically continued" representations initially in the case for G/K Hermitian symmetric of a particularly nice kind of holomorphic discrete series and later with coauthors Enright, B. Gross, Howe, Parthasarathy, and Wolf [EHW,EPWW,GrW1,GrW2] in some other cases and for other groups G. The fundamental problems are

(a) the identification of the continued parameters leading to reasonable unitary representations, and
(b) the determination of the restrictions of the unitary representations to K.

The works [Wa1,Wa2] solved problem (a) completely for all of what will below be called the "line-bundle cases" for G/K Hermitian symmetric when the positive system of roots is the one compatible with the complex structure on G/K. For this same situation, works [EHW,Ja] independently solved problem (a) for all "vector-bundle cases."

For other groups almost all progress has concerned only the "Borel–de Siebenthal positive systems" for the given group. Every simple Lie group with a compact Cartan subgroup has such a positive system [BoS] (cf. [Kn2, Theorem 6.96]). The

work [EPWW] solved problem (a) completely for all Borel–de Siebenthal positive systems in all line-bundle cases for which a certain irreducibility condition is in force; for these cases the answer to problem (b) is fairly easy to see. At about the same time that [EPWW] was written, Vogan [Vo2] discovered a more widely applicable but less sensitive theorem for detecting unitarity. The work [GrW2] went on to address both problems (a) and (b) in line-bundle cases outside the range where the irreducibility condition is in force, but only for G's such that G/K has a quaternionic structure and the positive system is compatible with that structure. More recently the work [Kn3] solved problem (a) in nearly all line-bundle cases and in many vector-bundle cases for all remaining *linear* classical groups, i.e., those linear classical groups that are not of real-rank one and do not have G/K Hermitian symmetric. However, the methods of [Kn3] fail for groups that do not have faithful matrix representations, and they yield no insight into problem (b).

The present paper uses a completely different approach to solve problems (a) and (b) completely for the line-bundle cases and a Borel–de Siebenthal positive system in the simply connected covering groups of $SO(2m, 2l-2m)_0$ when $4 \leqslant 2m \leqslant 2l-2m$, i.e., $2 \leqslant m \leqslant l/2$. The main positive results for problem (a) are Theorems 8.1 and 8.2, and the main positive results for problem (b) are Theorems 9.4 and 9.5. Modifications necessary for $SO(2m, 2l-2m+1)_0$ with $2 \leqslant m \leqslant l/2$ are noted in the last section, and thus the line-bundle cases are settled for one particular Borel–de Siebenthal ordering for each of these groups. These results, in combination with results in the above cited papers, solve problem (a) completely for the line-bundle cases in at least one Borel–de Siebenthal positive system for each G other than split E_7, split E_8, and the simply connected cover of $SO(2m, 2l-2m+1)_0$ when $2m > l \geqslant 3$.

A tool in the new approach is a theory of a certain kind of nilpotent orbits. There are various theories about certain kinds of nilpotent orbits associated with semisimple groups, whether in the Lie algebra $\mathfrak{g}_0$ of G or in the complexified Lie algebra $\mathfrak{g}$. Noël [No] and others have explained some of these theories and relationships among them. The classification theorem that Noël proves seems to include the orbits that arise here, but the ones here do not seem to be grouped in his classification in a handy way for current purposes. We choose therefore to work directly with the orbits that do arise.

In order to describe our results more quantitatively, let us introduce a minimal amount of notation. More detail will be provided in Section 1. We start from G, K, T, etc., as at the beginning. Let θ be the Cartan involution of $\mathfrak{g}_0$ corresponding to K, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ of Borel–de Siebenthal type relative to $\mathfrak{t}$. The Levi factor $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$, and the analytic subgroup L of G corresponding to $\mathfrak{l}_0$ is a compact subgroup of K. For now, write $L^{\mathbb{C}}$ for $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{l})$. The orbits of interest will be those of $L^{\mathbb{C}}$ on the noncompact part of the nilpotent radical $\mathfrak{u}$ of $\mathfrak{q}$.

The representations of initial interest will be cohomologically induced, in the sense and notation of [KnV], from irreducible (finite-dimensional) $(\mathfrak{l}, L)$ modules. We speak of *line-bundle cases* or *vector-bundle cases* according as the $(\mathfrak{l}, L)$ module is one dimensional or higher dimensional; we shall work exclusively with the line-bundle cases.

To describe cohomological induction, let λ be the differential of a character of L, and write $\mathbb{C}_\lambda$ for its representation space. Let $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ be the parabolic subalgebra opposite to $\mathfrak{q}$, extend the $\mathfrak{l}$ action on $\mathbb{C}_\lambda$ to $\bar{\mathfrak{q}}$ by having $\bar{\mathfrak{u}}$ act by 0, and let $N(\lambda)$ be the upside-down generalized Verma module $N(\lambda) = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} \mathbb{C}_\lambda$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$; $N(\lambda)$ is a $(\mathfrak{g}, L)$ module. If $2\delta(\mathfrak{u})$ denotes the sum of the roots of $\mathfrak{t}$ in $\mathfrak{u}$, the $(\mathfrak{g}, K)$ module of initial interest is $\pi(\lambda) = (\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S(N(\lambda + 2\delta(\mathfrak{u})))$, where $(\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S$ is a functor to be described in Section 1. The $(\mathfrak{g}, L)$ module $N(\lambda)$ has a unique irreducible quotient $N'(\lambda)$, and we let $\pi'(\lambda) = (\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S(N'(\lambda + 2\delta(\mathfrak{u})))$; this is the actual $(\mathfrak{g}, K)$ module of interest. Along the lines of the problems mentioned at the beginning, the specific goals are to determine

(a) when $\pi'(\lambda)$ is unitary, and, in such cases,
(b) what its K decomposition is.

To state results for G equal to a simply connected cover of $SO(2m, 2l-2m)_0$, we need to pin λ down a bit. For this group G, the dual of the complexified Cartan subalgebra can be identified with l tuples, and integrality of such a tuple means that the first m entries are all integers or all half integers and the last $l-m$ entries are all integers or all half integers. Write $\lambda_s = (-l + \frac{s}{2}, \dots, -l + \frac{s}{2}; 0, \dots, 0)$ with s an integer; the semicolon separates the first m entries from the last $l-m$. For reasons that will be given in Section 1, we assume that $s \geqslant 2(2m-l)$. Put $\pi_s = \pi(\lambda_s)$ and $\pi'_s = \pi'(\lambda_s)$. The answer to problem (a) appears in Table 1. Regard s as a decreasing parameter.

The new results in Table 1 concern $s < m$: for $0 \leqslant s \leqslant m-1$, there is unitarity, and for $s < 0$, there is not.[2] The proof of these new results involves an analytic step, which will be carried out in Sections 6 and 7 below, and an algebraic step, which has been largely carried out in [Kn4] and will be completed in Section 8 below. A by-product of these steps is a first approximation to a solution of problem (b) for these representations. A full solution requires additional steps that are taken in Section 9 of this paper. We return to this matter in a moment.

As far as the universal covers of indefinite orthogonal groups $SO(2m, 2l-2m)_0$ with $2 \leqslant m \leqslant l/2$ go, the paper [EPWW] had shown unitarity for $s \geqslant m$, and the general "weakly fair" test in [Vo2][3] comes close to that, handling $s \geqslant m+1$; the Gross–Wallach paper handled $m = 2$, adding to the results of [EPWW] the conclusion of unitarity for $s = 1$ and 0 for the universal cover of $SO(4, 2l-4)_0$ when $l \geqslant 4$. The paper [Kn3] had already proved the unitarity for all *even* $s \geqslant 0$ when $2 \leqslant m \leqslant l/2$ except for $s = 0$ when $m = l/2$.

Authors of some other papers have constructed similar-appearing finite or infinite sequences of small unitary representations of indefinite orthogonal groups. It seems that these sequences often have some representations in common with the ones obtained by analytic continuation of discrete series but are basically just different sequences of representations. Two early papers of this kind are the ones by Strichartz [St] and Vogan [Vo1]. Kostant [Kos1,Kos2] extensively investigated the $s = 0$

[2] See Section 1 for a precise statement.
[3] See p. 35 of [KnV] for the terminology "weakly fair."

Table 1
Unitary line-bundle cases π'_s for universal cover of $SO(2m, 2l-2m)_0$, $2 \leqslant m \leqslant l/2$

Discrete series:	$s > 2l-2$
Limit of discrete series:	$s = 2l-2$
Last [Vo2] unitary point:	$s = m+1$
Last [EPWW] unitary point:	$s = m$
Last unitary point:	$s = 0$
First nonunitary point:[2]	$s = -1$
Last point under study:	$s = -2(l-2m)$

representation of the group $SO(4,4)_0$, which in effect had already been shown to be unitary in [Vo1]. The same representation has been investigated also in [BrK,GrW1,KaS].

Some other papers that have investigated for unitarity certain sequences of small representations of indefinite orthogonal groups are [BiZ,Kob,ZhH]. Sections 8 and 9 of [EPWW], which are sections addressing examples other than analytic continuation of discrete series, construct more sequences of this kind. The papers [Li1,Li2] classify a certain kind of small representation for classical groups, and one may expect that some of the representations shown now to be unitary are small in the sense of Li's papers. See [NOTY], for example.

In connection with problem (b), let $\mathfrak{p}$ be the -1 eigenspace of θ in $\mathfrak{g}$. The nilpotent orbits in question are of $L^{\mathbb{C}}$ on $\mathfrak{u} \cap \mathfrak{p}$; they are finite in number and, for the most part, are parametrized by pairs of integers (p,q) with $0 \leqslant q \leqslant p \leqslant m$. Let $\mathcal{O}(p,q)$ be the orbit indexed by the pair (p,q). The only exception to the parametrization occurs when $l = 2m$; in this case the set that we define as $\mathcal{O}(m,0)$ consists of two orbits, which we can denote $\mathcal{O}(m,0)^+$ and $\mathcal{O}(m,0)^-$.

In Section 1 we follow [GrW2] and define a notion of a $(\mathfrak{g}, K)$ module that is "associated" to a particular orbit; this condition will relate the K types of the given $(\mathfrak{g}, K)$ module to the action of L on the closure of the orbit. Table 2 gives the first part of the answer to problem (b).

The first line of Table 2 follows from [EPWW] for all $m \geqslant 2$. The next two lines of the table are proved in [GrW2] for the case $m = 2$. It is instructive to see the lattice of orbit closures in the Gross–Wallach case. When $m = 2$ and $l > 4$, the lattice is

$$\begin{array}{ccccc} \mathcal{O}(2,2) & & & & \\ \uparrow & & & & \\ \mathcal{O}(2,1) & \leftarrow & \mathcal{O}(1,1) & & \\ \uparrow & & \uparrow & & \\ \mathcal{O}(2,0) & \leftarrow & \mathcal{O}(1,0) & \leftarrow & \mathcal{O}(0,0) \end{array}$$

The arrows indicate inclusions into the closures. The representations π'_2, π'_1, and π'_0 are associated to the members of the left column, and the other orbits play no role in the correspondence. When $m = 2$ and $l = 4$, $\mathcal{O}(2,0)$ splits into two orbits, and π'_0 is associated to $\mathcal{O}(1,0)$; the other orbits outside the first column play no role in the

Table 2
Association of π'_s to orbits for universal cover of $SO(2m, 2l-2m)_0$, $2 \leqslant m \leqslant l/2$

π'_s for $s \geqslant m$ is associated to	$\mathcal{O}(m,m)$
π'_s for $0 \leqslant s < m$ with $(l,s) \neq (2m,0)$ is associated to	$\mathcal{O}(m,s)$
π'_0 for $l = 2m$ is associated to	$\mathcal{O}(m-1,0)$
π'_s for $s < 0$ is associated to	No orbit

correspondence. The dimensions of the orbits when $m = 2$ and $l = 4$ give a little insight into matters; arranged in an array to match the configuration of the lattice, they are $\begin{bmatrix} 8 & & \\ 7 & 5 & \\ 5,5 & 4 & 0 \end{bmatrix}$. The three five-dimensional orbits are conjugate under triality, but none of these orbits plays a role in the correspondence under study of representations to orbits. In the text of [GrW2], these dimensions of orbits are all reduced by one because [GrW2] considers the orbits projectively.

For larger values of m, the lattice of orbit closures has the same triangular appearance. Table 2 is asserting that the correspondence of orbits and representations π'_s for larger values of m is similar to what happens for $m = 2$. When $l > 2m$, the $(\mathfrak{g}, K)$ modules π'_m, $\pi'_{m-1}, \dots, \pi'_0$ are associated to the orbits of the left column; the other orbits play no role. When $l = 2m$, the set $\mathcal{O}(m,0)$ splits into two orbits, and π'_0 is associated to $\mathcal{O}(m-1,0)$. These results will be proved in Section 9. The solution of problem (a) gives a start to a solution to problem (b), and what is left in establishing Table 2 is the proof that certain ideals in the symmetric algebra of $\mathfrak{u} \cap \mathfrak{p}$ are prime. We shall prove that these ideals are prime while addressing a second, ostensibly deeper, aspect of problem (b).

The second aspect of problem (b) is to determine the K types and their multiplicities explicitly when π'_s is unitary. This step was carried out for $m = 2$ in [GrW2] with the aid of "Luna's Slice Theorem" and some other tools. Those results are insufficient for $m > 2$, and we shall make use of a branching theorem of D. E. Littlewood for decomposing irreducible representations of $U(n)$ upon restriction to $SO(n)$. Littlewood's theorem will be stated precisely in Section 1. Its use is the main reason our results are limited to $m \leqslant l/2$. The K spectrum will be identified for general m in Section 9 by using Littlewood's theorem and the details of a construction in [Kn4]; the appropriate orbit $\mathcal{O}(p,q)$ in Table 2 will be indispensable in the proof.

From the formulas for the K types and their multiplicities, one can see that a unitary π'_s has all its K types of multiplicity one if and only if $s = 0$ or $s = 1$. In these cases the number of parameters for the K types is $m + 1$ for $s = 1$, m for $l > 2m$ and $s = 0$, and $m - 1$ for $l = 2m$ and $s = 0$. Among the unitary π'_s's, the only spherical representations are the π'_0's for $l = 2m$, and the only ladder representation is the well-studied case of π'_0 for $l = 4$ and $m = 2$.

The paper is organized as follows. Sections 1–5 contain preliminary material, including statements of what is needed from [EPWW,GrW1,GrW2]. The solution

of problem (a) is in Sections 6–8, and the solution of problem (b) is in Section 9. Section 10 addresses converse results for the two problems, and Section 11 discusses the theory for $SO(2m, 2l-2m+1)_0$ that corresponds to Sections 6–10.

It is a pleasure to acknowledge helpful discussions with David Vogan concerning Section 4 and the structure of the ideal in Section 9. This work was done in part while I was a visitor at the Institute for Advanced Study in Princeton during 2000–02. I am grateful to the Institute for its hospitality.

Note added in proof: Peter Trapa has circulated a preprint in which he proves that the representations π'_s for $s \geqslant 0$ in Tables 1 and 4 are irreducible. The preprint is entitled "Some small unipotent representations of indefinite orthogonal groups."

1. Setting

In this section we establish some notation for general G, as well as whatever special notation we need for $SO(2m, 2l-2m)_0$. Groups will be denoted by upper-case Latin letters, their Lie algebras will be denoted by the corresponding lower-case German letters with a subscript 0, and the complexified Lie algebras (as well as some complex Lie algebras and vector spaces that are not necessarily complexifications) will be denoted by lower-case German letters with no subscript.

Let G, K, and T be as at the start. The group G is to be simple with rank $G =$ rank K, and T is a compact Cartan subgroup of G lying in K. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ corresponding to K, and let θ be the Cartan involution. The complexified Cartan decomposition is written $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{t})$. Each such root is either *compact* or *noncompact* according as its root vectors lie in $\mathfrak{k}$ or $\mathfrak{p}$. A *Borel–de Siebenthal positive system* is a positive system of roots for Δ for which there is exactly one noncompact simple root and that root occurs at most twice in the highest root. Such a positive system always exists, according to [BoS] (cf. [Kn2, Theorem 6.96]). We fix such a system and write Δ^+ for it.

Out of such a Δ^+, we can form a *Borel–de Siebenthal parabolic subalgebra* of $\mathfrak{g}$. This subalgebra will be written $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Its Levi factor $\mathfrak{l}$ is built from $\mathfrak{t}$ and the root vectors for all roots in the span of the compact simple roots. The nilpotent radical $\mathfrak{u}$ is built from the root vectors for all remaining positive roots. The subalgebra $\mathfrak{q}$ is θ stable, with $\mathfrak{l}$ equal to the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$. An important property of a Borel–de Siebenthal parabolic subalgebra is that

$$[\mathfrak{u}, \mathfrak{u} \cap \bar{\mathfrak{u}}] = 0. \tag{1.1}$$

Let L be the analytic subgroup of G with Lie algebra $\mathfrak{l}_0$; L is a compact subgroup of K. The opposite parabolic subalgebra to $\mathfrak{q}$ is $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$, where the "bar" denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. The use of the "bar" symbol is to be distinguished from the related function *bar*, which will be defined below. We write $L^{\mathbb{C}}$ initially for the group $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{l})$ with Lie algebra $\mathfrak{l}$.

To simplify the notation when dealing with the representation theory of compact connected Lie groups, we shall often identify a highest weight with an irreducible representation having that highest weight. More precisely, the expression H *type* is permitted to refer indifferently to a highest weight for the compact connected group H, or to a particular irreducible representation of H with the stated highest weight, or to the equivalence class of irreducible representations of H with the stated highest weight.

The group $L^{\mathbb{C}}$ acts on $\mathfrak{u}\cap\mathfrak{p}$. This action yields an action by contragredient on the dual $(\mathfrak{u}\cap\mathfrak{p})^*$ of $\mathfrak{u}\cap\mathfrak{p}$, and also it extends to the symmetric algebra $S(\mathfrak{u}\cap\mathfrak{p})$. We get an action on the polynomial algebra $P(\mathfrak{u}\cap\mathfrak{p})$, as well, because $P(\mathfrak{u}\cap\mathfrak{p})$ is canonically isomorphic to the symmetric algebra of $(\mathfrak{u}\cap\mathfrak{p})^*$.

We write $S^d(\mathfrak{u}\cap\mathfrak{p})$ and $P^d(\mathfrak{u}\cap\mathfrak{p})$ for the subspaces of elements homogeneous of degree d. The representation of L on $\mathfrak{u}\cap\mathfrak{p}$ is irreducible, and hence the center of L acts by nonzero scalars on $\mathfrak{u}\cap\mathfrak{p}$. Then it follows that the action of the center can be used to isolate the homogeneous components of members of $S(\mathfrak{u}\cap\mathfrak{p})$ and $P(\mathfrak{u}\cap\mathfrak{p})$ and to give their degrees. Consequently a given L type appears in $S(\mathfrak{u}\cap\mathfrak{p})$ or $P(\mathfrak{u}\cap\mathfrak{p})$ only finitely often.

Let C be a symmetric invariant bilinear form on $\mathfrak{g}$ whose restriction to $i\mathfrak{t}_0$ is positive definite. Under C, the vector spaces $\mathfrak{u}\cap\mathfrak{p}$ and $\bar{\mathfrak{u}}\cap\mathfrak{p}$ are nonsingularly paired, and it follows that the mapping

$$bar : \mathfrak{u}\cap\mathfrak{p}\to(\mathfrak{u}\cap\mathfrak{p})^* \quad \text{with } bar(X) = C(\bar{X},\cdot) \tag{1.2}$$

is a canonical conjugate-linear isomorphism. We use the notation *bar* to refer also to the extension $bar : S(\mathfrak{u}\cap\mathfrak{p})\to P(\mathfrak{u}\cap\mathfrak{p})$ of this mapping. For $l\in L^{\mathbb{C}}$, the computation

$$\begin{aligned} C(\overline{\mathrm{Ad}(l)X}, Y) &= C(\mathrm{Ad}(\bar{l})\bar{X}, Y) = C(\bar{X}, \mathrm{Ad}(\bar{l})^{-1}Y) \\ &= bar(X)(\mathrm{Ad}(\bar{l})^{-1}Y) = (\bar{l}(bar(X)))(Y) \end{aligned}$$

shows that

$$bar(\mathrm{Ad}(l)X) = \bar{l}(bar(X)). \tag{1.3}$$

If we specify a subset V of $\mathfrak{u}\cap\mathfrak{p}$, it makes sense to speak of the ideal I_V of all polynomials on $\mathfrak{u}\cap\mathfrak{p}$ that vanish on V. The quotient $P(\mathfrak{u}\cap\mathfrak{p})/I_V$ is called the *coordinate ring* of V. If V is stable under $L^{\mathbb{C}}$, then so is I_V, and it follows that the coordinate ring inherits a fully reducible action by $L^{\mathbb{C}}$.

Using V, we introduce also the *dual coordinate ring* $R(V)$ as a quotient of $S(\mathfrak{u}\cap\mathfrak{p})$. The definition is $R(V) = S(\mathfrak{u}\cap\mathfrak{p})/J_V$, where $J_V = bar^{-1}(I_V)$. This J_V is an ideal even though *bar* is only conjugate linear. If V and thus I_V are stable under $L^{\mathbb{C}}$, then so is J_V, as a consequence of (1.3). In this case the dual coordinate ring inherits a fully reducible action by $L^{\mathbb{C}}$; the multiplicity of an irreducible representation of $L^{\mathbb{C}}$ in the dual coordinate ring equals the multiplicity of the contragredient representation in the coordinate ring.

According to a theorem of Vinberg [Vi], there are only finitely many orbits of $L^{\mathbb{C}}$ in its action on $\mathfrak{u}\cap\mathfrak{p}$, and one of them is open and dense.[4] (See Section 4.1 of [Ru] and Section X.3 of the second edition of [Kn2] for expositions.) Let $\mathcal{O}$ be such an orbit, and let $\mathcal{O}^{\mathrm{cl}}$ be its closure. The holomorphic representation of $L^{\mathbb{C}}$ on the dual coordinate ring $S(\mathfrak{u}\cap\mathfrak{p})/J_{\mathcal{O}^{\mathrm{cl}}}$ will be of great interest to us.

Let π' be a $(\mathfrak{g},K)$ module, and suppose that π' has a unique minimal K type, say Λ. Following [GrW2], we say that π' is *associated* to $\mathcal{O}$ if, for every L type σ, the multiplicity of the K type $\Lambda+\sigma$ in π' equals the multiplicity of the L type σ in $S(\mathfrak{u}\cap\mathfrak{p})/J_{\mathcal{O}^{\mathrm{cl}}}$ and if all K types in π' and L types in $S(\mathfrak{u}\cap\mathfrak{p})/J_{\mathcal{O}^{\mathrm{cl}}}$ are accounted for by this correspondence.

Let us expand upon the definitions given earlier of the $(\mathfrak{g},K)$ modules of interest to us. If E is a complex subspace of $\mathfrak{g}$ spanned by root spaces and a subspace of $\mathfrak{t}$, let $\Delta(E)$ be the set of roots contributing to E, and let $\Delta^+(E)$ be the set of positive roots contributing to E. We write $\delta(E)$ for half the sum of the members of $\Delta^+(E)$, and we abbreviate $\delta(\mathfrak{g})$ as δ. The symmetric invariant bilinear form C allows us to pair members of $i\mathfrak{t}_0$ with members of $i\mathfrak{t}_0^*$, say by $\gamma\mapsto H_\gamma$, and it gives us an inner product $\langle\cdot,\cdot\rangle$ on $i\mathfrak{t}_0^*$. We write $|\gamma|^2$ for $\langle\gamma,\gamma\rangle$.

Corollary 4.69 of [KnV] shows that $iH_{\delta(\mathfrak{u})}$ is in the center $Z_{\mathfrak{l}_0}$ and that $\beta(H_{\delta(\mathfrak{u})})>0$ for every member of $\Delta(\mathfrak{u})$. In fact, all the members β of $\Delta(\mathfrak{u}\cap\mathfrak{p})$ have $\beta(H_{\delta(\mathfrak{u})})$ equal to the same positive constant, and all the members β of $\Delta(\mathfrak{u}\cap\mathfrak{k})$ have $\beta(H_{\delta(\mathfrak{u})})$ equal to twice that positive constant. Let F be an irreducible finite-dimensional $(\mathfrak{l},L)$ module. If ν is its highest weight, we may write $F=F_\nu$, or $F=F_\nu^L$ if there is a need to emphasize the group. If F is one dimensional, i.e., if ν is orthogonal to all members of $\Delta(\mathfrak{l})$, then we may write $\mathbb{C}_\nu$ in place of F_ν.

To describe cohomological induction in the situation of interest here, let $\mathbb{C}_\lambda$ be a one-dimensional $(\mathfrak{l},L)$ module. Extend the $\mathfrak{l}$ action on $\mathbb{C}_\lambda$ to $\bar{\mathfrak{q}}$ by having $\bar{\mathfrak{u}}$ act by 0, so that $\mathbb{C}_\lambda$ becomes a $(\bar{\mathfrak{q}},L)$ module. Define a $(\mathfrak{g},L)$ module by

$$N(\lambda)=U(\mathfrak{g})\otimes_{\bar{\mathfrak{q}}}\mathbb{C}_\lambda.$$

Let $\Pi_{\mathfrak{g},L}^{\mathfrak{g},K}$ be the Bernstein functor defined on page 30 of [KnV]; this is a covariant right-exact functor sending $(\mathfrak{g},L)$ modules to $(\mathfrak{g},K)$ modules.[5] We shall use its Sth derived functor $(\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S$, where $S=\dim(\mathfrak{u}\cap\mathfrak{k})$. The $(\mathfrak{g},K)$ module of initial interest is

$$\pi(\lambda)=(\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S(N(\lambda+2\delta(\mathfrak{u}))).$$

[4] A linear holomorphic group action on a complex vector space with a dense orbit is called a "prehomogeneous vector space" in the literature. Sato–Kimura [SaK] gives a classification of the irreducible such spaces, up to a certain kind of equivalence.

[5] The better-known Zuckerman functor $I_{\mathfrak{g},L}^{\mathfrak{g},K}$ is defined on page 24 of [KnV] and is the dual of $\Pi_{\mathfrak{g},L}^{\mathfrak{g},K}$ in a sense made precise in Chapter III of [KnV]. The Zuckerman functor is a covariant left-exact functor sending $(\mathfrak{g},L)$ modules to $(\mathfrak{g},K)$ modules.

Any $(\mathfrak{g}, L)$ module of the form $N(\lambda)$ has a unique irreducible quotient $N'(\lambda)$, and we define

$$\pi'(\lambda) = (\Pi^{\mathfrak{g},K}_{\mathfrak{g},L})_S(N'(\lambda + 2\delta(\mathfrak{u}))).$$

This is the actual $(\mathfrak{g}, K)$ module of interest.

The infinitesimal character of both $\pi(\lambda)$ and $\pi'(\lambda)$ is $\lambda + \delta$. From Chapter VI of [KnV], we know that if $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant, then the K type Λ occurs in $\pi(\lambda)$ and $\pi'(\lambda)$ with multiplicity one. Further it is shown in Corollary 8 of [Kn1] that if Λ is $\Delta^+(\mathfrak{k})$ dominant[6] and $\mathfrak{u} \cap \mathfrak{k}$ is not 0, then Λ is the unique minimal K type of $\pi(\lambda)$ and $\pi'(\lambda)$. We

$$\text{assume throughout that } \Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}) \text{ is } \Delta^+(\mathfrak{k}) \text{ dominant.} \tag{1.4}$$

It is not known whether $\pi'(\lambda)$ is irreducible under assumption (1.4). What is known is that the unique irreducible subquotient $\bar{\pi}(\lambda)$ of $\pi(\lambda)$ containing the K type Λ is also a subquotient of $\pi'(\lambda)$. For a proof, see Section 2 of [Kn3].

From [EPWW] the $(\mathfrak{g}, L)$ module $N(\lambda + 2\delta(\mathfrak{u}))$ carries a canonical invariant Hermitian form known as the *Shapovalov form*, and this descends to be nondegenerate on the quotient $N'(\lambda + 2\delta(\mathfrak{u}))$. Under $(\Pi^{\mathfrak{g},K}_{\mathfrak{g},L})_S$, the Shapovalov form is carried to $\pi(\lambda)$ and $\pi'(\lambda)$. When we speak of addressing the unitarity of $\pi'(\lambda)$, we actually are referring to the definiteness or indefiniteness of this particular form. Since $\pi'(\lambda)$ may in principle be reducible, $\bar{\pi}(\lambda)$ or even $\pi'(\lambda)$ could in principle be unitary even though the Shapovalov form on $\pi'(\lambda)$ is indefinite.

Let us record now what our notation looks like for the universal covering group G of $SO(2m, 2l-2m)_0$. The Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}_0$ is the usual one consisting of two-by-two diagonal blocks $\left(\begin{smallmatrix} 0 & ih_j \\ -ih_j & 0 \end{smallmatrix}\right)$, $1 \leqslant j \leqslant l$. The roots are given by $\Delta = \{\pm e_j \pm e_k \,|\, j \neq k\}$, and we take $\Delta^+ = \{e_j \pm e_k \,|\, j < k\}$. A positive root $e_j \pm e_k$ is compact if j and k are both $\leqslant m$ or both $\geqslant m+1$. The only noncompact simple root is $e_m - e_{m+1}$, and hence Δ^+ is a Borel–de Siebenthal positive system. We identify the linear form $\sum_{j=1}^l c_j e_j$ on $\mathfrak{t}$ with the l-tuple $(c_1, \ldots, c_m; c_{m+1}, \ldots, c_l)$, often separating the first m entries and the last $l-m$ by a semicolon. Such a linear form is analytically integral for G if $c_1, \ldots, c_m$ are all integers or all half-integers and if $c_{m+1}, \ldots, c_l$ are all integers or all half-integers. Up to a positive multiplicative constant the quadratic form $|\cdot|^2$ induced on $i\mathfrak{t}_0^*$ by the symmetric invariant bilinear form C is given by

$$|(c_1, \ldots, c_m; c_{m+1}, \ldots, c_l))|^2 = \sum_{j=1}^{l} c_j^2, \tag{1.5}$$

with all the entries c_j real. There will be no harm in treating this positive multiplicative constant as if it is 1.

[6] If $\mathfrak{u} \cap \mathfrak{k} = 0$, then Λ is necessarily $\Delta^+(\mathfrak{k})$ dominant but is not necessarily minimal.

The parameter λ of a one-dimensional $(\mathfrak{l}, L)$ module $\mathbb{C}_\lambda$ has to be of the form

$$\lambda = \lambda_s = \left(-l + \frac{s}{2}, \dots, -l + \frac{s}{2}; 0, \dots, 0\right) \tag{1.6}$$

for some integer s. The $-l$ in (1.6) may be regarded as an additive normalization. We define

$$\pi_s = \pi(\lambda_s) \quad \text{and} \quad \pi'_s = \pi'(\lambda_s).$$

The infinitesimal character of π_s and π'_s is $\lambda_s + \delta$, and we set

$$\Lambda = \Lambda_s = \lambda_s + 2\delta(\mathfrak{u} \cap \mathfrak{p}) = \left(\frac{s}{2} + l - 2m, \dots, \frac{s}{2} + l - 2m; 0, \dots, 0\right).$$

By (1.4) we are assuming that Λ_s is $\Delta^+(\mathfrak{k})$ dominant, i.e., that

$$s \geqslant 2(2m - l). \tag{1.7}$$

Then Λ_s is the highest weight of the unique minimal K type of π_s and π'_s.

Let us identify concretely the action of L on $\mathfrak{u} \cap \mathfrak{p}$. We write $M_{mn} = M_{mn}(\mathbb{C})$ for the vector space of m-by-n complex matrices.

Lemma 1.1. *For the universal covering G of $SO(2m, 2l-2m)_0$ with $2 \leqslant m \leqslant l/2$, the group $\mathrm{Ad}_{\mathfrak{g}}(L)$ is isomorphic to $(U(m) \times SO(2l-2m))/\{\pm 1\}$ and the space $\mathfrak{u} \cap \mathfrak{p}$ is isomorphic to $M_{m,2l-2m}$ in such a way that the action of L on $\mathfrak{u} \cap \mathfrak{p}$ corresponds to the action of $U(m) \times SO(2l-2m)$ on members X of $M_{m,2l-2m}$ given by*

$$\begin{aligned} u(X) &= uX \quad \text{for } u \in U(m), \\ r(X) &= Xr^{-1} \quad \text{for } r \in SO(2l-2m). \end{aligned} \tag{1.8}$$

The operations on the right-hand sides of (1.8) *are matrix multiplications.*

Proof. We know that $\mathfrak{l}_0$ is compact, that $\mathfrak{l}$ contains the Cartan subalgebra $\mathfrak{t}$, and that $\Delta(\mathfrak{l}) = \{\pm(e_i - e_j) \mid i < j \leqslant m\} \cup \{\pm e_i \pm e_j \mid m < i < j\}$. It is immediate that

$$\mathfrak{l}_0 \cong \mathbb{R} \oplus \mathfrak{su}(m) \oplus \mathfrak{so}(2l-2m) \cong u(m) \oplus \mathfrak{so}(2l-2m).$$

The adjoint action of $\mathrm{Ad}_{\mathfrak{g}}(L)$ on $\mathfrak{u} \cap \mathfrak{p}$ is irreducible, and the highest weight is the largest noncompact root, namely $e_1 + e_{m+1}$. Consequently the representation of $\mathfrak{l}_0$, namely of $\mathfrak{u}(m) \oplus \mathfrak{so}(2l-2m)$, is the outer tensor product of the standard representation of $\mathfrak{u}(m)$ and the standard representation of $\mathfrak{so}(2l-2m)$. Since the standard representation of $\mathfrak{so}(2l-2m)$ is equivalent with its contragredient, we can view the representation of $\mathfrak{l}_0$ as occurring on $M_{m,2l-2m}$, the action being the Lie algebra action corresponding to (1.8).

This representation on the Lie algebra level lifts to $U(m) \times SO(2l-2m)$, acting by (1.8), and the only elements of $U(m) \times SO(2l-2m)$ that act on $M_{m,2l-2m}$ as the identity are ± 1. On the other hand, no nontrivial element of $\mathrm{Ad}_{\mathfrak{g}}(L)$ acts on $\mathfrak{u} \cap \mathfrak{p}$ as

the identity since $\mathfrak{u}\cap\mathfrak{p}$ and $\bar{\mathfrak{u}}\cap\mathfrak{p}$ generate $\mathfrak{g}$. Thus $\mathrm{Ad}_{\mathfrak{g}}(L)$ is isomorphic to the quotient $(U(m)\times SO(2l-2m))/\{\pm 1\}$ in such a way that the action of $\mathrm{Ad}_{\mathfrak{g}}(L)$ on $\mathfrak{u}\cap\mathfrak{p}$ corresponds to the action of $(U(m)\times SO(2l-2m))/\{\pm 1\}$ on $M_{m,2l-2m}$.

Whenever there is a need to be concrete about the action of L on $\mathfrak{u}\cap\mathfrak{p}$, we shall treat L as equal to $U(m)\times SO(2l-2m)$, with action on $\mathfrak{u}\cap\mathfrak{p}$ as in (1.8), and we shall not feel any need to quote Lemma 1.1. The group $L^{\mathbb{C}}$, which was tentatively defined as $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{l})$, may be redefined as $GL(m,\mathbb{C})\times SO(2l-2m,\mathbb{C})$; it acts on $\mathfrak{u}\cap\mathfrak{p}$ by extension of the action (1.8). The actions of L and $L^{\mathbb{C}}$ on $S(\mathfrak{u}\cap\mathfrak{p})$ and $P(\mathfrak{u}\cap\mathfrak{p})$ may similarly be carried over to actions of $U(m)\times SO(2l-2m)$ and $GL(m,\mathbb{C})\times SO(2l-2m,\mathbb{C})$ on $S(M_{m,2l-2m})$ and $P(M_{m,2l-2m})$.

Because of this correspondence, a typical L type occurring in $S(\mathfrak{u}\cap\mathfrak{p})$ will often be written as $\sigma=(\xi,v)$, where

$$\xi=(a_1,\dots,a_m) \tag{1.9a}$$

is a $U(m)$ type and

$$v=(b_1,\dots,b_m,0,\dots,0) \tag{1.9b}$$

is an $SO(2l-2m)$ type.

Let $\xi=\sum_{i=1}^m \xi_i e_i$ be dominant integral for $U(m)$ with $\xi_m\geqslant 0$, regard ξ also as dominant integral for $U(2l-2m)$, and let $v=\sum_{i=1}^{l-m} v_i e_i$ be dominant integral for $SO(2l-2m)$. It is shown in (0.2) of [Kn4] as an application of invariant theory that

$$\begin{aligned}&\text{the } L \text{ type } \sigma=(\xi,v) \text{ occurs in } S(\mathfrak{u}\cap\mathfrak{p}) \text{ as many times as the}\\ &SO(2l-2m) \text{ type } v \text{ occurs in the restriction of the } U(2l-2m)\\ &\text{type } \xi \text{ from } U(2l-2m) \text{ to } SO(2l-2m).\end{aligned} \tag{1.10}$$

Branching from $U(n)$ to $SO(n)$ is described by a 1940 theorem of D. E. Littlewood [Lit] that will apply in (1.10) under the assumption $l\geqslant 2m$ and only then. Other references having some bearing on Littlewood's theorem include [Mal1,Mal2,-Mu,Ne]. The statement of Littlewood's theorem identifies nonnegative linear combinations of the e_i's having different numbers of entries as long as the nonzero entries match. An n-tuple will be said to be *nonnegative* if all its entries are nonnegative. We work largely with nonnegative dominant integral forms, namely n-tuples $\xi=(a_1,\dots,a_n)$ with integer entries $\geqslant 0$ such that $a_1\geqslant\cdots\geqslant a_n$, and we define $||\xi||=\sum_i a_i$. The *depth* of a nonnegative dominant integral form is defined to be the number of nonzero entries.

If ξ, μ, and v are nonnegative integral forms of depth $\leqslant d$, the *Littlewood–Richardson coefficient* $c_{\mu v}^{\xi}$ is the multiplicity of the irreducible representation of $U(d)$ with highest weight ξ in the tensor product of the irreducible $U(d)$ representations with respective highest weights μ and v. If $c_{\mu v}^{\xi}>0$, then it is easy to see that

$$||\xi||=||\mu||+||v||, \tag{1.11}$$

that $\xi-\mu$ and $\xi-\nu$ are both nonnegative, and that μ and ν have depth $\leqslant$ the depth of ξ. Littlewood–Richardson coefficients can be computed by a well-known combinatorial method that will not concern us.[7] It follows from the method of computation that $c_{\mu\nu}^{\xi}$ is independent of d as long as ξ, μ, and ν all have depth $\leqslant d$.

If μ is a nonnegative dominant integral form, we say that μ is *even* if all of its entries are even integers.

Theorem 1.2 (Littlewood). *Fix n, and let ξ and ν be nonnegative dominant integral forms of depth $\leqslant n/2$. Then the multiplicity of the irreducible representation of $SO(n)$ of highest weight ν in the restriction to $SO(n)$ of the irreducible representation of $U(n)$ with highest weight ξ equals the sum over all nonnegative even dominant integral forms μ of the Littlewood–Richardson coefficients $c_{\mu\nu}^{\xi}$.*

The sum in the theorem has only finitely many nonzero terms because of (1.11). In verifying (1.10), one uses $n=2l-2m$. The forms ξ and ν have depth $\leqslant m$, which is $\leqslant n/2$ because $m\leqslant l-m$.

It follows from Theorem 1.2 that if the multiplicity in question is >0, then $||\xi||-||\nu||$ is an even integer $\geqslant 0$. We shall use this fact many times, writing $2t$ for the even integer.

Not every $SO(n)$ highest weight ν occurring in the restriction of an irreducible representation of $U(n)$ with nonnegative highest weight ξ need be nonnegative if n is even. These exceptional ν's can be handled by a simple trick, and we return to them in Section 8.

2. $SU(m,\, l-m)$ as a prototype

Before turning to the details for $SO(2m,2l-2m)$, it may be helpful to consider briefly the linear isometry groups $G=SU(m,l-m)$ of indefinite Hermitian forms. For these groups the theory of this paper reduces in many spots to results in linear algebra and representation theory that are well known and fairly old. Let us assume that $2\leqslant m\leqslant l/2$.

The group $SU(m,l-m)$ has G/K Hermitian symmetric, and we take a compatible positive system of roots. Specifically the group K is $S(U(m)\times U(l-m))$, and we take $K^{\mathbb{C}}$ to be the determinant-one subgroup of $GL(m,\mathbb{C})\times GL(l-m,\mathbb{C})$. The Cartan subalgebra $\mathfrak{t}$ is the diagonal subalgebra, the roots are the usual $\pm(e_a-e_b)$ with $a<b$, the positive roots are the e_a-e_b with $a<b$, and the simple roots are the e_a-e_{a+1}. The simple root e_m-e_{m+1} is noncompact, and the other simple roots are compact. Since G/K is Hermitian symmetric, the space $\mathfrak{p}$ splits into the sum $\mathfrak{p}=\mathfrak{p}^+\oplus\mathfrak{p}^-$ of two abelian subalgebras stable under $K^{\mathbb{C}}$. The parabolic subalgebra $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ is $\mathfrak{k}\oplus\mathfrak{p}^+$ for this example, and $\mathfrak{u}\cap\mathfrak{k}$ is 0. Thus $L=K$, $S=\dim(\mathfrak{u}\cap\mathfrak{k})=0$, and $(\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S$ is the identity.

[7] See [Mac] for the combinatorial method and a proof of its validity.

The Harish–Chandra decomposition (cf. [Kn2, Section VII.9]) shows that $\mathfrak{p}^+$ may be identified with the vector space of m-by-$(l-m)$ complex matrices, and $K^{\mathbb{C}}$ acts on $\mathfrak{p}^+$ by $(k_1,k_2)(X)=k_1Xk_2^{-1}$. This is the analog of Lemma 1.1. The action on the left allows for arbitrary row operations on X, and the action on the right allows for arbitrary column operations. A familiar canonical-form theorem from linear algebra says that any two X's of the same rank are in the same orbit,[8] and consequently the orbits are parametrized by the rank: $\mathcal{O}(m),\mathcal{O}(m-1),\dots,\mathcal{O}(0)$.

The parameter λ of the one-dimensional $(\mathfrak{l},L)=(\mathfrak{k},K)$ module is given by an l-tuple, and we write

$$\lambda=\lambda_s=(-l+s,\dots,-l+s;\ 0,\dots,0),$$

with the semicolon occurring after the mth entry. Our interest is in $\pi_s=\pi(\lambda_s)=N(\lambda_s+2\delta(\mathfrak{u}))$ and $\pi_s'=\pi'(\lambda_s)=N'(\lambda_s+2\delta(\mathfrak{u}))$. The infinitesimal character of π_s and π_s' is $\lambda_s+\delta$, and we set

$$\Lambda=\Lambda_s=\lambda_s+2\delta(\mathfrak{p}^+)=(-m+s,\dots,-m+s;\ \ -m,\dots,-m).$$

This is always $\Delta^+(\mathfrak{k})$ dominant, but it is not always the unique minimal K type parameter of π_s; for example, when $m=l-m=2$ and $s=-1$, Λ and $\Lambda+\sigma$ are both K types if $\sigma=(1,1;-1,-1)$, but $|\Lambda+\sigma+2\delta(\mathfrak{k})|^2=|\Lambda+2\delta(\mathfrak{k})|^2$. However, a brief computation shows that Λ is the unique minimal K type parameter of π_s and π_s' if $s\geqslant 0$.

The representations π_s and π_s' are what are studied as the analytic continuation of the holomorphic discrete series in the line-bundle cases of this G. The unitarity of the representations π_s' for $s\geqslant 0$ was proved by Wallach [Wa1,Wa2].

The methods of Sections 6–8 of this paper, which have [GrW1,GrW2] as their starting point rather than [Wa1,Wa2], will reproduce Wallach's results on unitarity when applied to $SU(m,l-m)$. They will also provide some preliminary information about the K types occurring in π_s' in the unitary cases. The methods of Section 9, which give more detailed information about K types, simplify in the case of $SU(m,l-m)$ to an exercise in classical invariant theory whose solution will be written out below.

The theory in [GrW1,GrW2], as will be explained further in Section 5, looks for a division of the L types $\sigma\neq 0$ in $S(\mathfrak{u}\cap\mathfrak{p})$ into two kinds, one for which the difference

$$c_s(\sigma)=|\lambda_s+\sigma+\delta|^2-|\lambda_s+\delta|^2 \tag{2.1}$$

is >0 and one for which the subspace V^σ of $S(\mathfrak{u}\cap\mathfrak{p})$ transforming according to σ lies completely in an ideal $V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p})$ of $S(\mathfrak{u}\cap\mathfrak{p})$, σ_s being a certain particular L type depending on s such that $c_s(\sigma_s)=0$. Some L types will fit both descriptions, but the method requires having a way of deciding which class to put each L type in. For

[8] The simple Lie groups $Sp(n,\mathbb{R})$ and $SO^*(2n)$ are other groups with G/K Hermitian symmetric, and familiar canonical-form theorems yield the orbits for them, as well. For $Sp(n,\mathbb{R})$ the relevant theorem is the principal-axis theorem for complex symmetric matrices, and for $SO^*(2n)$ the relevant theorem is the corresponding result for complex skew-symmetric matrices.

$SO(2m, 2l-2m)$, this step is carried out in Theorem 7.1, which is fairly complicated to prove.

For $SU(m, l-m)$, things are much simpler. We are considering the K types in $S(\mathfrak{p}^+)$. Since this is the same as the decomposition of the space of m-by-$(l-m)$ matrices under $U(m)$ on the left and $U(l-m)$ on the right, classical invariant theory, particularly Corollary 4.5.19 and Theorem 5.27 of [GoW], tells us the answer immediately. The K types in $S(\mathfrak{p}^+)$ correspond to all nonnegative dominant integral forms $(a_1, \dots, a_m)$ for $U(m)$, the corresponding K type having highest weight

$$\sigma = (a_1, \dots, a_m;\ 0, \dots, 0, -a_m, \dots, -a_1). \tag{2.2}$$

All multiplicities[9] are 1.

With σ as in (2.2), we compute (2.1) and find that

$$\begin{aligned} c_s(\sigma) &= 2a_1(a_1 + s - 1) + \cdots + 2a_j(a_j + s - 2j + 1) \\ &+ \cdots + 2a_m(a_m + s - 2m + 1). \end{aligned} \tag{2.3}$$

For $s \geqslant m$, only the first kind of K type occurs in $S(\mathfrak{p}^+)$. There is no σ_s, and (2.3) shows that

$$\begin{aligned} \frac{1}{2} c_s(\sigma) &= \sum_{j=1}^{m} (a_j^2 + a_j(s-m)) + \sum_{j=1}^{m} a_j(m+1-2j) \\ &= \sum_{j=1}^{m} (a_j^2 + a_j(s-m)) + \sum_{1 \leqslant u < v \leqslant m} (a_u - a_v). \end{aligned}$$

The first term on the right-hand side is $\geqslant 0$ for $s \geqslant m$ with equality only if $\sigma = 0$, and the second term is $\geqslant 0$ because of the dominance condition. Thus $c_s(\sigma) > 0$ for $\sigma \neq 0$. It follows from the theory of [GrW1,GrW2], specifically from Theorem 5.2 below, that $\pi'_s = \pi_s$ and that π'_s is unitary for $s \geqslant m$. The comparable estimates for $SO(2m, 2l-2m)$ are in Section 6 below.

For s with $0 \leqslant s \leqslant m-1$, the special σ_s has $a_1 = \cdots = a_{s+1} = 1$ and $a_{s+2} = \cdots = a_m = 0$. Computation shows that it indeed has $c_s(\sigma_s) = 0$. We divide the K types in $S(\mathfrak{p}^+)$ into two classes, those with $a_{s+1} > 0$ and those with $a_{s+1} = 0$. For any σ' and σ'' occurring in $S(\mathfrak{p}^+)$, the product of their highest weight vectors (as symmetric tensors) is a highest weight vector of $V^{\sigma'+\sigma''}$; since $\sigma' + \sigma''$ has multiplicity 1, we see that $V^{\sigma'+\sigma''} \subseteq V^{\sigma'} V^{\sigma''}$. Use of this fact and the expansion-by-cofactors formula for determinants shows that if σ has $a_{s+1} > 0$, then $V^{\sigma} \subseteq V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$. On the other hand, if σ has $a_{s+1} = 0$, then (2.3) gives

$$\frac{1}{2} c_s(\sigma) = \sum_{j=1}^{s} a_j^2 + \sum_{j=1}^{s} a_j(s+1-2j) = \sum_{j=1}^{s} a_j^2 + \sum_{1 \leqslant u < v \leqslant s} (a_u - a_v),$$

[9] For the other cases with G/K Hermitian symmetric, the decomposition still is multiplicity free and is given by a formula in [Sc].

and this is $\geqslant 0$ with equality only if $\sigma = 0$. It follows from the Gross–Wallach theory [GrW1,GrW2], specifically from Theorem 5.3 below, that π'_s is unitary for $0 \leqslant s \leqslant m-1$ and that its K spectrum is the sum of Λ_s and the K spectrum of $S(\mathfrak{p}^+)/V^{\sigma_s}S(\mathfrak{p}^+)$. The comparable results for $SO(2m, 2l-2m)$ are in Sections 7 and 8 below.

One might quickly guess that the K spectrum of $S(\mathfrak{p}^+)/V^{\sigma_s}S(\mathfrak{p}^+)$ consists of all K types with $a_{s+1} = 0$, each with multiplicity one, but some argument is needed. This argument is where the orbits play a critical role. It is apparent that the polynomials $p(X)$ vanishing on $\mathcal{O}(s)$ include all minors of X of size $s+1$, hence the ideal in $P(\mathfrak{p}^+)$ generated by those minors. But it is not so apparent that no polynomials outside this ideal vanish on $\mathcal{O}(s)$. The question is whether the ideal in $P(\mathfrak{p}^+)$ generated by those minors is prime. This is a question answered affirmatively by a version of the Second Fundamental Theorem of classical invariant theory.

The ideal of all polynomials vanishing on $\mathcal{O}(s)$ is stable under K and hence is a sum of its K types, each with multiplicity one. These K types are the contragredients of K types occurring in $S(\mathfrak{p}^+)$. The members of such a contragredient K type σ^c in $P(\mathfrak{p}^+)$ vanish on $\mathcal{O}(s)$ if and only if a nonzero lowest-weight polynomial vanishes on $\mathcal{O}(s)$. The theorems quoted above from [GoW] give a formula for such a polynomial, and inspection of the formula shows that the polynomial vanishes on $\mathcal{O}(s)$ only if $a_{s+1} = 0$. It follows that the ideal is indeed prime and that the K spectrum of $S(\mathfrak{p}^+)/V^{\sigma_s}S(\mathfrak{p}^+)$ indeed is as asserted above. In the terminology of Section 1, π'_s is associated to the orbit $\mathcal{O}(s)$. For further information about the role of invariant theory, see [DeP,Ho]. The comparable argument for $SO(2m, 2l-2m)$ will be carried out in Section 9.

Finally let us consider $s<0$. In this case we form $\sigma_0 = (1, 0, \dots, 0;\ 0, \dots, 0, -1)$. Direct computation gives $c_s(\sigma_0) = 2s<0$, and it follows from the theory of [GrW1,GrW2] that the Shapovalov form for π'_s does not exhibit π'_s as unitary for $s<0$. A comparable result for $SO(2m, 2l-2m)$ will be given in Section 10.

3. Orbits

This section concerns the orbits of $L^{\mathbb{C}}$ in its action on $\mathfrak{u} \cap \mathfrak{p}$ when $\mathfrak{g}_0$ is the Lie algebra $\mathfrak{so}(2m, 2l-2m)$ with $2 \leqslant m \leqslant l/2$. We use the identification in Lemma 1.1 of $\mathfrak{u} \cap \mathfrak{p}$ with $M_{m,2l-2m}$. The action of $L^{\mathbb{C}}$ of $\mathfrak{u} \cap \mathfrak{p}$ is transformed into the action of $GL(m, \mathbb{C}) \times SO(2l-2m, \mathbb{C})$ on $M_{m,2l-2m}$ as in (1.8).

A typical element of $M_{m,2l-2m}$ will be denoted X. By XX^{tr}, we mean the m-by-m matrix product of X and its transpose; the (a,b)th entry of XX^{tr} is the ordinary dot product of the ath and bth rows of X. Here "dot product" refers to the complex bilinear form on $\mathbb{C}^{2l-2m}$ given by $v \cdot w = \sum_{a=1}^{2l-2m} v_a w_a$.

When it is necessary to number the rows and columns of members of $M_{m,2l-2m}$, we number the rows as $1, \dots, m$ and the columns as $1, \dots, l-m, 1', \dots, (l-m)'$. The first occurrence of this numbering will in effect be with the example that follows the statement of Proposition 3.1.

Proposition 3.1. *In the action* (1.8) *on* $M_{m,2m-2l}$ *when* $2 \leqslant m \leqslant l/2$,

(a) *the orbits of* $GL(m,\mathbb{C}) \times O(2l-2m,\mathbb{C})$ *are exactly the sets*

$$\mathcal{O}(p,q) = \{X \in M_{m,2l-2m} \mid \text{rank } X = p \text{ and } \text{rank } XX^{\text{tr}} = q\}$$

for $0 \leqslant q \leqslant p \leqslant m$,

(b) *the dimension of* $\mathcal{O}(p,q)$ *is* $p(2l-m-p) - \frac{1}{2}(p-q)(p-q+1)$,

(c) *all the sets* $\mathcal{O}(p,q)$ *with one exception remain single orbits when the action is restricted to* $GL(m,\mathbb{C}) \times SO(2l-2m,\mathbb{C})$, *the one exception being* $\mathcal{O}(m,0)$ *in the case that* $l = 2m$,

(d) *the set* $\mathcal{O}(m,0)$ *is the union* $\mathcal{O}(m,0)^+ \cup \mathcal{O}(m,0)^-$ *of two orbits under* $GL(m,\mathbb{C}) \times SO(2l-2m,\mathbb{C})$ *if* $l = 2m$, *and in this case the topological closure of each of* $\mathcal{O}(m,0)^+$ *and* $\mathcal{O}(m,0)^-$ *is an algebraic variety*.

Example. Let us write members of $M_{m,2l-2m}$ as block-type row vectors, with m columns grouped first, then $l-m$ more, m more, and the last $l-m$. Let X be the matrix

$$X = (\text{diag}(1,\dots,1,1,\dots,1,0,\dots,0) \quad 0 \quad \text{diag}(i,\dots,i,0,\dots,0,0,\dots,0) \quad 0) \tag{3.1}$$

with u, q, and v members in the respective segments of diagonal entries of the indicated diagonal matrices. This X has rank $u+q$, and

$$XX^{\text{tr}} = \text{diag}(0,\dots,0,1,\dots,1,0,\dots,0)$$

has rank q. Therefore X is in $\mathcal{O}(u+q,q)$. We shall make use of this X in the proof below and in Sections 8 and 9.

Proposition 3.1 is a kind of canonical-form result, though not a basic one of linear algebra. We take it as a result that is in principle well known but in practice maybe less well known. Accordingly we give a sketch of the proof, providing detail about only some of the points of the argument. We make repeated use of a variant of the Gram–Schmidt orthogonalization process, beginning with a lemma.

Lemma 3.2. *If* $w_1,\dots,w_c$ *is a linearly independent set in* $\mathbb{C}^n$ *with* $w_a \cdot w_b = \delta_{ab}$ *and if* $c<n$, *then there exists* w_{c+1} *with* $w_1,\dots,w_{c+1}$ *linearly independent and* $w_{c+1} \cdot w_a$ *equal to* 0 *for* $a<c$ *and equal to* 1 *for* $a = c+1$.

Remark. Here δ_{ab} indicates the Kronecker delta, which is 1 if $a = b$ and 0 if $a \neq b$.

Proof of Lemma 3.2. Let $v \in \mathbb{C}^n$ be independent of $w_1,\dots,w_c$, and define $w = v - \sum_{a=1}^{c} (v \cdot w_a) w_a$. Computation shows that $w \cdot w_a = 0$ for $a \leqslant c$ and that $w \cdot w = v \cdot v - \sum_{a=1}^{c} (v \cdot w_a)^2$. If $w \cdot w \neq 0$, then a suitable multiple of w will serve as w_{c+1}.

Arguing by contradiction, we may thus assume that

$$v \cdot v = \sum_{a=1}^{c} (v \cdot w_a)^2 \tag{3.2}$$

for every v such that $v, w_1, \dots, w_c$ is linearly independent. If $v, w_1, \dots, w_c$ is linearly dependent, we must have $v = \sum_{a=1}^{c} (v \cdot w_a) w_a$. Computation shows that we again have (3.2). Thus we may assume that (3.2) holds for every $v \in \mathbb{C}^n$.

Polarization of (3.2) gives $u \cdot v = \sum_{a=1}^{c} (u \cdot w_a)(v \cdot w_a)$ for all u and v in $\mathbb{C}^n$. Define $M_{rs} = e_r \cdot w_s$ for $r \leqslant n$ and $s \leqslant c$, where $\{e_r\}$ is the standard basis of $\mathbb{C}^n$. Then

$$\delta_{ab} = e_a \cdot e_b = \sum_{s=1}^{c} (e_a \cdot w_s)(e_b \cdot w_s) = \sum_{s=1}^{c} M_{as} M_{bs}$$

for a and $b \leqslant n$. Hence MM^{tr} equals the identity, and M has rank $\geqslant n$. However, M has rank equal at most to its number of columns, which is c and is $< n$. Thus we have a contradiction, and the lemma follows.

Proof of Proposition 3.1 (Sketch). For (a), it is a simple matter to check that each $\mathcal{O}(p, q)$ is stable under $GL(m, \mathbb{C}) \times O(2l - 2m, \mathbb{C})$. We are to check that $\mathcal{O}(p, q)$ is a single orbit. Start with a given matrix in $M_{m,2l-2m}$. Any elementary row operation on that matrix amounts to an operation by $GL(m, \mathbb{C})$ and hence transforms the matrix within the same orbit. Doing row reduction, we may therefore assume that the first p rows of the matrix are linearly independent and the last $m - p$ rows are 0. Let V be the linear span of the rows, and regard V as a subspace of $\mathbb{C}^{2l-2m}$. Further row operations show that we may replace the nonzero rows of our matrix by any basis of V without leaving the orbit.

Let U be the subspace of elements w of V such that $w \cdot w' = 0$ for all $w' \in V$, and put $u = \dim_{\mathbb{C}} U$. Write $u + q$ for the dimension of V. We shall construct $2u + q$ linearly independent members of $\mathbb{C}^{2l-2m}$ denoted x_a and y_a for $1 \leqslant a \leqslant u$, denoted z_a for $u + 1 \leqslant a \leqslant u + q$, and having the following properties:

(i) distinct members of the set have dot product 0, and each member of the set has dot product 1 with itself,
(ii) the elements $z_1, \dots, z_u$ form a basis of U when z_a is defined as $x_a + iy_a$,
(iii) the elements $z_1, \dots, z_u, z_{u+1}, \dots, z_{u+q}$, with $z_1, \dots, z_u$ as in (ii), form a basis of V.

Then we define an ordered basis $\{v_a \mid 1 \leqslant a \leqslant l - m\} \cup \{v_{a'} \mid 1 \leqslant a \leqslant l - m\}$ of $\mathbb{C}^{2l-2m}$ as follows: We let $v_a = x_a$ and $v_{a'} = y_a$ for $1 \leqslant a \leqslant u$, and we let $v_a = z_a$ for $u + 1 \leqslant a \leqslant u + q$. Taking (i) into account, we use Lemma 3.2 to construct the remaining v_a's and $v_{a'}$'s so that distinct members of the basis have dot product 0 and each member of the basis has dot product 1 with itself. Let g be the $(2l - 2m)$-by-$(2l - 2m)$ matrix whose rows are the basis vectors v_a and $v_{a'}$ for $1 \leqslant a \leqslant l - m$. Then g is in $O(2l - 2m, \mathbb{C})$, and (ii) shows that if X is as in (3.1), then Xg is the matrix whose rows are $z_1, \dots, z_u, z_{u+1}, \dots, z_{u+q}, 0, \dots, 0$. By (iii), Xg is in the orbit of

$O(2l-2m,\mathbb{C})$ that we are studying. Hence every element of $\mathcal{O}(u+q,q)$ is in the same orbit as X.

Thus the proof of (a) in the proposition will be complete once a construction is made so that (i)–(iii) are valid. We begin with an inductive construction that starts with any basis $w_1,\dots,w_u$ of U and transforms it into the desired basis $z_1,\dots,z_u$. Decompose $w_1=x_1+iy_1$ into its real and imaginary parts. From the real and imaginary parts of $w_1\cdot w_1=0$, we obtain $x_1\cdot x_1=y_1\cdot y_1>0$ and $x_1\cdot y_1=0$. Renormalizing w_1 suitably, we may assume that $x_1\cdot x_1=y_1\cdot y_1=1$. We take z_1 to be this renormalized version of w_1. Proceeding inductively, suppose that $z_1,\dots,z_t$ have been constructed so that $\operatorname{span}\{z_1,\dots,z_t\}=\operatorname{span}\{w_1,\dots,w_t\}$ and each z_a decomposes into real and imaginary parts as $z_a=x_a+iy_a$ with $x_a\cdot x_b=y_a\cdot y_b=\delta_{ab}$ and $x_a\cdot y_b=0$ whenever the indices are all $\leqslant t$. Define

$$z=w_{t+1}-\sum_{a=1}^{t}(w_{t+1}\cdot x_a)w_a.$$

This is a nonzero element of U, and we readily check that $x_a\cdot z=y_a\cdot z=0$ for $a\leqslant t$. Since z is in U, $z\cdot z=0$. We decompose z into its real and imaginary parts as $z=x+iy$, and we obtain $x\cdot x=y\cdot y>0$ and $x\cdot y=0$. It is not asserted that x and y are in U. Normalizing z to z_{t+1}, we may assume that $z_{t+1}=x_{t+1}+iy_{t+1}$ with $x_{t+1}\cdot x_{t+1}=y_{t+1}\cdot y_{t+1}=1$ and $x_{t+1}\cdot y_{t+1}=0$. From $x_a\cdot z=y_a\cdot z=0$ for $a\leqslant t$, we have $x_a\cdot z_{t+1}=y_a\cdot z_{t+1}=0$. Extraction of real and imaginary parts shows that x_a and y_a have dot product 0 with x_{t+1} and y_{t+1}. This completes the induction and shows that the basis $z_1,\dots,z_u$ of U has the required properties.

Extend $z_1,\dots,z_u$ to an ordered basis $z_1,\dots,z_u,w_{u+1},\dots,w_{u+q}$ of V. For $u+1\leqslant a\leqslant u+q$, define inductively

$$z=w_a-\sum_{b=1}^{u}(x_b\cdot w_a)z_b-\sum_{b=u+1}^{a-1}(z_b\cdot w_a)z_b.$$

We readily check that z is orthogonal to $x_1,y_1,\dots,x_u,y_u,z_{u+1},\dots z_{a-1}$, and then we normalize z to become z_a. The inductive construction of a basis $z_1,\dots,z_{u+q}$ of V is complete, and we see that it has all the required properties.

For (b) in the proposition, we compute the Lie algebra of the isotropy subgroup at each of the points X in (3.1), using the method of Sato–Kimura [SaK, pp. 109–110], and the dimension formula follows. We shall not need the actual formula, and we omit the details.

For (c) we observe that except in the case of $\mathcal{O}(m,0)$ when $l=2m$, the special X in (3.1) has a column of 0's. Thus X is fixed by a certain diagonal matrix in $O(2l-2m,\mathbb{C})$ that has one diagonal entry -1 and all other diagonal entries $+1$, and (c) follows.

For (d) we suppose that $l=2m$. Let $v_1,\dots,v_m$ be the rows of a matrix X in $M_{m,2m}$, and consider the map $X\mapsto v_1\wedge\cdots\wedge v_m$ of $M_{m,2m}$ into $\bigwedge^m\mathbb{C}^{2m}$. This map is given by a vector-valued polynomial p equivariant under $O(2m,\mathbb{C})$ and invariant up to a

determinant factor under $GL(m,\mathbb{C})$. The representation of $SO(2m,\mathbb{C})$ on $\bigwedge^m \mathbb{C}^{2m}$ is reducible, with two irreducible invariant subspaces. Write $\bigwedge^m \mathbb{C}^{2m} = V_1 \oplus V_2$ accordingly, and let E_1 and E_2 be the projections. The set where $E_1 \circ p$ is 0 is stable under $GL(m,\mathbb{C}) \times SO(n,\mathbb{C})$, and the same is true of $E_2 \circ p$. Using formulas of Section 1 of [Kn4] for the action of root vectors of $\mathfrak{so}(2m,\mathbb{C})$, we can see that the X of (3.1) for $\mathcal{O}(m,0)$ is a highest weight vector of one of the two irreducible subspaces V_1 and V_2, say V_1, and that X with its last column negated (call it X') is a highest weight vector of the other irreducible subspace V_2. Then we have $E_1(p(X)) \neq 0$, $E_2(p(X)) = 0$, $E_1(p(X')) = 0$, and $E_2(p(X')) \neq 0$. Thus the zero loci of $E_1 \circ p$ and $E_2 \circ p$ within $\mathcal{O}(m,0)$ are nontrivial, and (d) follows. This completes the proof of the proposition.

We shall be interested in the dual coordinate ring $R(\mathcal{O}) = S(\mathfrak{u} \cap \mathfrak{p})/J_{\mathcal{O}^{\mathrm{cl}}}$ of various orbits. Recall from Section 1 that a $(\mathfrak{g}, K)$ module with a unique minimal K type Λ is *associated* to the orbit $\mathcal{O}$ if, for every L type σ, the multiplicity of the K type $\Lambda + \sigma$ in π' equals the multiplicity of the L type σ in $S(\mathfrak{u} \cap \mathfrak{p})/J_{\mathcal{O}^{\mathrm{cl}}}$ and if all K types in π' and L types in $S(\mathfrak{u} \cap \mathfrak{p})/J_{\mathcal{O}^{\mathrm{cl}}}$ are accounted for by this correspondence.

The dual coordinate ring was defined in Section 1 by passing between $S(\mathfrak{u} \cap \mathfrak{p})$ and $P(\mathfrak{u} \cap \mathfrak{p})$ by means of the canonical conjugate-linear algebra automorphism given by *bar* in (1.2). In order to work with ideals in $S(\mathfrak{u} \cap \mathfrak{p})$, we need a way of handling the mapping *bar*. This does not seem to be so easy to do directly, and our approach will be indirect. We shall take advantage of the identification of $\mathfrak{u} \cap \mathfrak{p}$ with $M_{m,2l-2m}$ in order to define another isomorphism $x(\cdot)$ of $S(\mathfrak{u} \cap \mathfrak{p})$ with $P(\mathfrak{u} \cap \mathfrak{p})$. This isomorphism will be complex linear, but it will be noncanonical because the identification of $\mathfrak{u} \cap \mathfrak{p}$ with $M_{m,2l-2m}$ is noncanonical. In Proposition 3.3 we address the problem of relating *bar* and $x(\cdot)$.

For X in $M_{m,2l-2n}$, we let

$$x_{ab}(X) \text{ and } x_{ab'}(X) \quad \text{for } 1 \leqslant a \leqslant m \text{ and } 1 \leqslant b \leqslant l-m \tag{3.3}$$

be the entry functions on the matrices in $\mathfrak{u} \cap \mathfrak{p}$. Each of these is a linear function on $\mathfrak{u} \cap \mathfrak{p}$, and together they form a basis of the dual space $(\mathfrak{u} \cap \mathfrak{p})^*$, which is the same as $P^1(\mathfrak{u} \cap \mathfrak{p})$.

We now introduce the basis of $\mathfrak{u} \cap \mathfrak{p}$ of which (3.3) is the dual basis. Define

$$X_{ab} \text{ and } X_{ab'} \quad \text{for } 1 \leqslant a \leqslant m \text{ and } 1 \leqslant b \leqslant l-m \tag{3.4}$$

to be matrices that are 1 in the indicated entries and are 0 elsewhere. We define $x(\cdot)$ on $\mathfrak{u} \cap \mathfrak{p}$ by sending basis to dual basis: $x(X_{ab}) = x_{ab}$ and $x(X_{ab'}) = x_{ab'}$. Then we extend $x(\cdot)$ to a complex-vector-space isomorphism of $\mathfrak{u} \cap \mathfrak{p}$ onto $P^1(\mathfrak{u} \cap \mathfrak{p})$ and from there to a complex-linear algebra isomorphism of $S(\mathfrak{u} \cap \mathfrak{p})$ onto $P(\mathfrak{u} \cap \mathfrak{p})$.

The paper [Kn4] makes extensive use of a certain formal "dot product" that will come up again below in Section 9. For $1 \leqslant a \leqslant m$, let X_a denote the row vector

$$X_a = (X_{a1} \quad \cdots \quad X_{a,l-m} \quad X_{a1'} \quad \cdots \quad X_{a,(l-m)'})$$

whose entries are m-by-$(2l-2m)$ matrices. For a and b between 1 and m, define

$$X_a \cdot X_b = \sum_{c=1}^{l-m} (X_{ac} X_{bc} + X_{ac'} X_{bc'})$$

as a member of $S^2(\mathfrak{u} \cap \mathfrak{p})$. The function $x(\cdot)$ relates this formal dot product to ordinary dot products. Namely $X \mapsto x(X_a \cdot X_b)(X)$, for $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant m$, picks out the ordinary dot product of the ath and bth rows of the matrix X in $M_{m,2l-2m}$. This, as we have already noted, is the same as the (a,b)th entry of the matrix product XX^{tr}.

The identification of $\mathfrak{u} \cap \mathfrak{p}$ with $M_{m,2l-2m}$ in effect picks out a real form of $\mathfrak{u} \cap \mathfrak{p}$, namely the real subspace of members of $\mathfrak{u} \cap \mathfrak{p}$ that correspond to real matrices. Accordingly we can use this identification to define a conjugate-linear involution $(\cdot)^{\mathrm{conj}}$ of $\mathfrak{u} \cap \mathfrak{p}$ by complex conjugation of the corresponding matrix. We extend this to a conjugate-linear involution of $S(\mathfrak{u} \cap \mathfrak{p})$. In addition, the map $(\cdot)^{\mathrm{conj}}$ on $\mathfrak{u} \cap \mathfrak{p}$ yields a conjugate-linear involution $(\cdot)^{\mathrm{conj}}$ of $P(\mathfrak{u} \cap \mathfrak{p})$ by $p^{\mathrm{conj}}(X) = (p(X^{\mathrm{conj}}))^{\mathrm{conj}}$ for p in $P(\mathfrak{u} \cap \mathfrak{p})$, the outer conj on the right-hand side indicating complex conjugation of the numerical values of p.

Proposition 3.3. *Up to a global nonzero constant, bar can be computed as the composition of $x(\cdot)$ followed by $(\cdot)^{\mathrm{conj}}$.*

Remark. In using *bar* to pass from $S(\mathfrak{u} \cap \mathfrak{p})$ to $P(\mathfrak{u} \cap \mathfrak{p})$, we shall really be interested in the effect of *bar* on complex vector subspaces of $S(\mathfrak{u} \cap \mathfrak{p})$. For this purpose the global constant in the proposition plays no role. We shall therefore work with *bar*, $x(\cdot)$, and $(\cdot)^{\mathrm{conj}}$ as if this constant were 1. In cases where we have closure under complex conjugation, such as with all polynomials vanishing on a set defined by polynomials with real coefficients, we can consequently treat *bar* and $x(\cdot)$ as if they produce the same results. The sets $\mathcal{O}(p,q)$ are examples of sets defined by polynomials with real coefficients; however, $\mathcal{O}(m,0)^+$ and $\mathcal{O}(m,0)^-$ are not asserted to have this property.

Proof. It will be more convenient to work with the version of $(\cdot)^{\mathrm{conj}}$ on $\mathfrak{u} \cap \mathfrak{p}$. Let us check that

$$(\cdot)^{\mathrm{conj}} \circ x(\cdot) = x(\cdot) \circ (\cdot)^{\mathrm{conj}}, \tag{3.5}$$

the $(\cdot)^{\mathrm{conj}}$ on the left being the one on $P(\mathfrak{u} \cap \mathfrak{p})$ and the one on the right being the one on $S(\mathfrak{u} \cap \mathfrak{p})$. Both sides of (3.5) are conjugate-linear, multiplicative, and linear, and thus it is enough to check their equality on members X of $\mathfrak{u} \cap \mathfrak{p}$. The question is whether

$$(x(X))^{\mathrm{conj}} \stackrel{?}{=} x(X^{\mathrm{conj}}).$$

On an element $Y \in \mathfrak{u} \cap \mathfrak{p}$, the respective sides are

$$(x(X)(Y^{\text{conj}}))^{\text{conj}} \quad \text{and} \quad x(X^{\text{conj}})(Y).$$

Both expressions are conjugate-linear in X and linear in Y, and it is enough to check their equality for X and Y equal to basis vectors. This is routine, and (3.5) follows.

Thus we are to compare *bar* with $x(\cdot)\circ(\cdot)^{\text{conj}}$. Let l be a member of L, which we regard as $U(m) \times SO(2l-2m)$, the action on matrices being as in (1.8). If X is in $\mathfrak{u} \cap \mathfrak{p}$, then (1.6) shows that $bar(\text{Ad}(l)X) = l(bar(X))$, i.e.,

$$bar \text{ commutes with the action of } L. \tag{3.6}$$

Since we are realizing L concretely as $U(m) \times SO(2l-2m)$, it is meaningful to speak of complex conjugation and transpose of members of L. Writing $l(X)$ as a matrix product uXs^{-1} with $u \in U(m)$ and $s \in SO(2l-2m)$ when X is in $M_{m,2l-2m}$, we compute the complex conjugates of the entries $x_{ab}(l(X))$ and $x_{ab'}(l(X))$ of $l(X)$, and we compare the results with the entries of $l(X^{\text{conj}})$. The result is that

$$(l(X))^{\text{conj}} = l^{\text{conj}}(X^{\text{conj}}). \tag{3.7}$$

Similarly, we check by using bases that

$$l(x(X)) = x((l^{-1})^{\text{tr}}(X)) \quad \text{for } X \in \mathfrak{u} \cap \mathfrak{p}. \tag{3.8}$$

Combining (3.7) and (3.8), we see that $x(\cdot)\circ(\cdot)^{\text{conj}}$ commutes with the action of L. Then we bring (3.6) to bear and conclude that

$$bar^{-1} \circ x(\cdot)\circ(\cdot)^{\text{conj}} \tag{3.9}$$

is a complex-linear self map of $\mathfrak{u} \cap \mathfrak{p}$ that commutes with L. Since L acts irreducibly on $\mathfrak{u} \cap \mathfrak{p}$, Schur's Lemma says that (3.9) is a scalar. This completes the proof.

4. Enright–Parthasarathy–Wallach–Wolf theory

In this section we return to the general setting described in the first part of Section 1, with $\mathfrak{g}_0$ not necessarily equal to $\mathfrak{so}(2m, 2l-2m)$. The essential assumption about the setting is that the positive system Δ^+ is of Borel–de Siebenthal type, so that $[\mathfrak{u}, \mathfrak{u} \cap \mathfrak{k}] = 0$ as in (1.1).

The heart of [EPWW] for our purposes is Sections 3 and 6. The proof of the first result in Section 3 of that paper seems less than convincing, and we supply our own argument. Our argument ends at the end of the remarks following Lemma 4.3 below. We begin with quite a general result; its content is well known, but it does not seem to have been completely proved in the literature.

Lemma 4.1. *Let $Z(\mathfrak{k})$ be the center of the universal enveloping algebra of $\mathfrak{k}$. Suppose that V is a $U(\mathfrak{k})$ module and U is a $U(\mathfrak{k})$ submodule. If U and V/U are $Z(\mathfrak{k})$ finite, then so is V. Suppose in addition that $\chi_0, \chi_1, \dots, \chi_n$ are distinct nonzero homomorphisms of $Z(\mathfrak{k})$ into $\mathbb{C}$, that $z - \chi_0(z)$ acts as 0 on V/U, and that $\prod_{j=1}^n (z - \chi_i(z))$ acts as 0 on U. Then $\prod_{j=0}^n (z - \chi_i(z))$ acts as 0 on V, and there exists a $U(\mathfrak{k})$ stable subspace Y of V such that*

(a) $V = U \oplus Y$,
(b) *U is the sum of the primary subspaces for $\chi_1, \dots, \chi_n$,*
(c) *every element of Y is annihilated by $z - \chi_0(z)$ for all $z \in Z(\mathfrak{k})$.*

Remark. The primary subspace for χ_i is defined to be the subspace of $v \in V$ for which $(z - \chi_i(z))^n v = 0$ for all $z \in Z(\mathfrak{k})$ and for some $n = n(v, z)$. Under the assumption of $Z(\mathfrak{k})$ finiteness, the n can be taken independent of v and z.

Proof. Let $I_1 = \mathrm{Ann}_{Z(\mathfrak{k})}(U)$ and $I_2 = \mathrm{Ann}_{Z(\mathfrak{k})}(V/U)$ be the annihilators of U and V/U in $Z(\mathfrak{k})$. These have finite codimension in $Z(\mathfrak{k})$ by the assumed $Z(\mathfrak{k})$ finiteness. Define $I_1 I_2$ to be the set of sums of products from I_1 and I_2. If z_1 is in I_1 and z_2 is in I_2, then any v in V has $z_2(v + U) = U$ and thus $z_2(v) \in U$. Since $z_1(u) = 0$ for all $u \in U$, $z_1 z_2$ is in $\mathrm{Ann}_{Z(\mathfrak{k})}(V)$. In other words, $I_1 I_2$ is contained in $\mathrm{Ann}_{Z(\mathfrak{k})}(V)$.

By a theorem of Chevalley [KnV, Theorem 7.30], $Z(\mathfrak{k})$ is isomorphic as an algebra to a full polynomial algebra and is therefore Noetherian. We shall use a standard argument for Noetherian algebras to prove that $I_1 I_2$ has finite codimension in $Z(\mathfrak{k})$, and then it will follow that V is $Z(\mathfrak{k})$ finite.

The vector space $Z(\mathfrak{k})/I_1$ is finite dimensional since the $Z(\mathfrak{k})$ module U is $Z(\mathfrak{k})$ finite, and we let $x_1 + I_1, \dots, x_r + I_1$ be a vector-space basis. Since $Z(\mathfrak{k})$ is Noetherian, the ideal I_2 is finitely generated, say with $y_1, \dots, y_s$ as generators. Let us show that $\{x_i y_j + I_1 I_2\}$ is a spanning set for the vector space $I_2/I_1 I_2$. In fact, any x in I_2 is of the form $x = \sum_{j=1}^s z_j y_j$ with z_j in $Z(\mathfrak{k})$. For each j, write $z_j + I_1 = \sum_{i=1}^r c_{ij} x_i + I_1$ with $c_{ij} \in \mathbb{C}$. Then $z_j y_j = \sum_{i=1}^r c_{ij} x_i y_j + I_1 I_2$, and $x = \sum_{j=1}^s \sum_{i=1}^r c_{ij} x_i y_j + I_1 I_2$. Thus $I_2/I_1 I_2$ is finite dimensional. Since $\dim Z(\mathfrak{k})/I_1 I_2 = \dim Z(\mathfrak{k})/I_2 + \dim I_2/I_1 I_2$, we conclude that $Z(\mathfrak{k})/I_1 I_2$ is finite dimensional. Consequently $\mathrm{Ann}_{Z(\mathfrak{k})}(V)$ has finite codimension in $Z(\mathfrak{k})$, and the $Z(\mathfrak{k})$ module V is $Z(\mathfrak{k})$ finite. This proves the first conclusion of the lemma.

Now suppose in addition that $\chi_0, \chi_1, \dots, \chi_n$ are distinct nonzero homomorphisms of $Z(\mathfrak{k})$ into $\mathbb{C}$, that $z - \chi_0(z)$ acts as 0 on V/U, and that $\prod_{j=1}^n (z - \chi_i(z))$ acts as 0 on U. If v is in V, then $(z - \chi_0(z))(v + U) = U$ and hence $u = (z - \chi_0(z))(v)$ is in U. Applying $\prod_{j=1}^n (z - \chi_i(z))$ to u, we see that $\prod_{j=0}^n (z - \chi_i(z))(v) = 0$. This proves the second conclusion of the lemma.

If the primary subspace V' of V for some homomorphism $\chi' : Z(\mathfrak{k}) \to \mathbb{C}$ is nonzero, then Corollary 7.27 of [KnV] shows that there exists $v' \neq 0$ in V' with $zv' = \chi'(z)v'$ for all $z \in Z(\mathfrak{k})$. Substituting into $\prod_{j=0}^n (z - \chi_i(z))(v') = 0$, we see that $\prod_{j=0}^n (\chi'(z) - \chi_i(z))(v') = 0$ for all z and therefore that $\prod_{j=0}^n (\chi'(z) - \chi_i(z)) = 0$ for all z. From Lemma 7.14 of [KnV], we can then conclude that $\chi' = \chi_i$ for some i.

Consequently the primary decomposition of V, given by Proposition 7.20 of [KnV], is $V = \sum \oplus_{i=0}^{n} V_i$, where V_i is the primary subspace of V for χ_i, $0 \leqslant i \leqslant n$. Define $Y = V_0$, and then $V = (\sum \oplus_{i=1}^{n} V_i) \oplus Y$.

To prove (a) and (b), let us show that $U = \sum \oplus_{i=1}^{n} V_i$. Since $\prod_{j=1}^{n} (z - \chi_j(z))$ is 0 on U, the primary decomposition for U shows that $U \subseteq \sum \oplus_{i=1}^{n} V_i$. For the reverse inclusion let us observe that if $v \in V$ satisfies $c(z)(z - \chi_i(z))P(z)(v) \in U$ for a scalar-valued function $c(z)$ and a polynomial $P(z)$, then the fact that

$$c(z)(z - \chi_0(z))P(z)(v) = c(z)P(z)(z - \chi_0(z))(v) \in P(z)(U) \subseteq U$$

implies, upon subtraction, that $c(z)(\chi_0(z) - \chi_i(z))P(z)(v)$ is in U. Let $i \neq 0$. It follows by induction from this observation that if $v \in V$ has $(z - \chi_i(z))^k(v)$ in U, then $(\chi_0(z) - \chi_i(z))^k(v)$ is in U. In particular, if $(z - \chi_i(z))^k(v) = 0$ for all z, then $(\chi_0(z) - \chi_i(z))^k(v)$ is in U. By Lemma 7.14 of [KnV], there exists $z \in Z(\mathfrak{k})$ with $\chi_0(z) \neq \chi_i(z)$, and we see that v is in U. Hence V_i is contained in U. This proves (a) and (b).

Since $V = U \oplus Y$, Y is isomorphic to V/U. Therefore $z - \chi_0(z)$ acts as 0 on Y. This proves (c).

We turn to the setting of the first part of Section 1, with a Borel–de Siebenthal positive system Δ^+ in place. We work with a one-dimensional $(\mathfrak{l}, L)$ module $\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$. For the parameter λ under consideration, the essential assumption is (1.4), namely that $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant.

Define

$$d_0 = \langle \lambda + 2\delta(\mathfrak{u}), \delta(\mathfrak{u}) \rangle, \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on it_0^* defined in Section 1. We know that there is a positive constant c such that

$$\langle \beta, \delta(\mathfrak{u}) \rangle = \begin{cases} 0 & \text{for all } \beta \in \Delta(\mathfrak{l}), \\ c & \text{for all } \beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}), \\ 2c & \text{for all } \beta \in \Delta(\mathfrak{u} \cap \mathfrak{k}). \end{cases} \tag{4.2}$$

We have a $(\mathfrak{q}, L)$ isomorphism

$$N(\lambda + 2\delta(\mathfrak{u})) = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} \mathbb{C}_{\lambda+2\delta(\mathfrak{u})} \cong U(\mathfrak{u}) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda+2\delta(\mathfrak{u})},$$

and it follows that every weight of $N(\lambda + 2\delta(\mathfrak{u}))$ is of the form

$$\omega = \lambda + 2\delta(\mathfrak{u}) + \sum_{\beta \in \Delta(\mathfrak{u})} n_\beta \beta, \tag{4.3}$$

where the n_β are integers $\geqslant 0$. Referring to (4.1) and (4.2), we see that every weight ω of $N(\lambda + 2\delta(\mathfrak{u}))$ has $\langle \omega, \delta(\mathfrak{u}) \rangle = d_0 + kc$ for some integer $k \geqslant 0$. Let N_k be the direct

sum of all weight spaces of $N(\lambda+2\delta(\mathfrak{u}))$ for which $\langle\omega,\delta(\mathfrak{u})\rangle = d_0+kc$. The subspace N_k is stable under L.

Let $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ be the $(\mathfrak{l},L)$ module consisting of all $\bar{\mathfrak{u}}\cap\mathfrak{k}$ invariant vectors of $N(\lambda+2\delta(\mathfrak{u}))$. If $F_v = F_v^L$ is an irreducible L stable subspace of $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ with highest weight v, then it follows from the above considerations that F_v lies in a single N_k and that any other irreducible L stable subspace F_v' with the same highest weight lies in the same N_k.

Lemma 4.2 (Cf. Enright et al. [EPWW, Lemma 3.1]). *With λ as above, if F_v is an irreducible L stable subspace of $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ with highest weight v, then $U(\mathfrak{k})(F_v)$ is an irreducible $(\mathfrak{k},L)$ submodule of $N(\lambda+2\delta(\mathfrak{u}))$ isomorphic to $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_v$ and having $\mathfrak{k}$ infinitesimal character $v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$. The expression $v-2\delta(\mathfrak{u}\cap\mathfrak{k})$ is $\Delta^+(\mathfrak{k})$ dominant, and hence so is $v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k}) = (v-2\delta(\mathfrak{u}\cap\mathfrak{k}))+\delta(\mathfrak{k})$. If v' is another irreducible L stable subspace of $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ and if it has highest weight v' with $v'\neq v$, then the infinitesimal-character parameters of $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_v$ and $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_{v'}$ are not conjugate by the Weyl group of $\mathfrak{k}$.*

Proof. With v as in the statement of the lemma, $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_v$ has $\mathfrak{k}$ infinitesimal character $v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$, according to Theorem 5.24 of [KnV]. By assumption, v is $\Delta^+(\mathfrak{l})$ dominant. Since $\delta(\mathfrak{u}\cap\mathfrak{k})$ is orthogonal to $\Delta(\mathfrak{l})$, $v-2\delta(\mathfrak{u}\cap\mathfrak{k})$ is $\Delta^+(\mathfrak{l})$ dominant. By (4.3) we can write $v=\lambda+2\delta(\mathfrak{u})+\sum_{\beta\in\Delta(\mathfrak{u})} n_\beta\beta$ with all $n_\beta\geqslant 0$. Then

$$v-2\delta(\mathfrak{u}\cap\mathfrak{k}) = \lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})+\sum_{\beta\in\Delta(\mathfrak{u})} n_\beta\beta = \Lambda+\sum_{\beta\in\Delta(\mathfrak{u})} n_\beta\beta.$$

If α is in $\Delta(\mathfrak{u}\cap\mathfrak{k})$, then $\langle\Lambda,\alpha\rangle\geqslant 0$ by (1.4); also $\langle\beta,\alpha\rangle\geqslant 0$ for $\beta\in\Delta(\mathfrak{u})$ by (1.1). Hence $\langle v-2\delta(\mathfrak{u}\cap\mathfrak{k}),\alpha\rangle\geqslant 0$. Thus $v-2\delta(\mathfrak{u}\cap\mathfrak{k})$ is $\Delta^+(\mathfrak{k})$ dominant, and it follows that $v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$ is $\Delta^+(\mathfrak{k})$ dominant.

In the case of v' with $v'\neq v$, we obtain $v'+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})\neq v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$ with both expressions $\Delta^+(\mathfrak{k})$ dominant. Then it follows that $v'+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$ and $v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$ cannot be conjugate by the Weyl group of $\mathfrak{k}$.

To see that $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_v$ is irreducible, we can quote the general Corollary 5.105 of [KnV] or we can argue more simply by contradiction as follows: If M is a proper nonzero $U(\mathfrak{k})$ submodule, then M contains an irreducible L stable subspace $F_{v''}$ for which $\langle v'',\delta(\mathfrak{u})\rangle$ is a minimum, by (4.2) and (4.3). This v'' cannot equal v. By Frobenius reciprocity (Propositions 2.34 and 2.57e of [KnV]), the nonzero $(\bar{\mathfrak{q}}\cap\mathfrak{k},L)$ inclusion of $F_{v''}$ into $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_v$ yields a nonzero $(\mathfrak{k},L)$ map of $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_{v''}$ into $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_v$. Since $v''\neq v$, the $\mathfrak{k}$ infinitesimal characters $v''+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$ and $v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$ are not conjugate by the Weyl group of $\mathfrak{k}$. From this contradiction we conclude that $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_v$ is irreducible.

We can now return to the $(\mathfrak{k},L)$ submodule $U(\mathfrak{k})(F_v)$ of $N(\lambda+2\delta(\mathfrak{u}))$. Since F_v is assumed to consist of $\bar{\mathfrak{u}}\cap\mathfrak{k}$ invariant vectors, the same kind of argument with Frobenius reciprocity uses the inclusion of F_v into $U(\mathfrak{k})(F_v)$ to obtain a nonzero

$(\mathfrak{k}, L)$ map of $U(\mathfrak{k}) \otimes_{\bar{\mathfrak{q}} \cap \mathfrak{k}} F_v$ into $U(\mathfrak{k})(F_v)$. This map is clearly onto $U(\mathfrak{k})(F_v)$, and it has to be one–one since $U(\mathfrak{k}) \otimes_{\bar{\mathfrak{q}} \cap \mathfrak{k}} F_v$ is irreducible. Therefore $U(\mathfrak{k})(F_v)$ is isomorphic to $U(\mathfrak{k}) \otimes_{\bar{\mathfrak{q}} \cap \mathfrak{k}} F_v$.

Lemma 4.3 (Cf. Enright et al. [EPWW, Lemma 3.1]). *With λ as above, $N(\lambda + 2\delta(\mathfrak{u}))$ is semisimple as a $(\mathfrak{k}, L)$ module. Every irreducible $(\mathfrak{k}, L)$ submodule contains, for a unique v, an irreducible L stable subspace F_v of $N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ with highest weight v, is isomorphic to $U(\mathfrak{k}) \otimes_{\bar{\mathfrak{q}} \cap \mathfrak{k}} F_v$ for that v, and has $\mathfrak{k}$ infinitesimal character the $\Delta^+(\mathfrak{k})$ dominant expression $v + \delta(\mathfrak{l}) - \delta(\mathfrak{u} \cap \mathfrak{k})$.*

Proof. Recall that N_k is the sum of the weight spaces of $N(\lambda + 2\delta(\mathfrak{u}))$ for which the weight ω has $\langle \omega, \delta(\mathfrak{u}) \rangle = d_0 + kc$. Define $V_k = U(\mathfrak{k})(\sum_{j=0}^{k} N_j)$. We shall prove inductively on k that V_k is a sum of irreducible $\mathfrak{k}$ submodules. Then it follows that every member of $N(\lambda + 2\delta(\mathfrak{u}))$ lies in a finite sum of irreducible $\mathfrak{k}$ modules and hence that $N(\lambda + 2\delta(\mathfrak{u}))$ is semisimple as a $(\mathfrak{k}, L)$ module. The rest will be easy.

In the case of V_0, every member of N_0 is invariant under $\bar{\mathfrak{q}} \cap \mathfrak{k}$. By Lemma 4.2 any irreducible L stable subspace of N_0 therefore generates an irreducible $U(\mathfrak{k})$ submodule. Thus V_0 is a sum of irreducible $\mathfrak{k}$ submodules. Fix an L type v occurring in N_0, and let $\tilde{F}_v$ be the isotypic subspace of N_0 of type v. Then V_0 is the direct sum of the $U(\mathfrak{k})$ stable subspaces $U(\mathfrak{k})(\tilde{F}_v)$, and the vth such space has infinitesimal character $v + \delta(\mathfrak{l}) - \delta(\mathfrak{u} \cap \mathfrak{k})$. As v varies, these infinitesimal-character parameters are nonconjugate under the Weyl group of $\mathfrak{k}$, according to Lemma 4.2, and therefore iteration of Lemma 4.1 shows that V_0 is $Z(\mathfrak{k})$ finite and $\prod_v (z - \chi_{v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})}(z))$ acts as 0 on it.

Inductively assume that V_{k-1} is $Z(\mathfrak{k})$ finite and that the product of all expressions $z - \chi_{v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})}(z)$, $z \in Z(\mathfrak{k})$, acts as 0 on it, where v runs through the L types that occur in $N_0^{\bar{\mathfrak{u}} \cap \mathfrak{k}}, \dots, N_{k-1}^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$. We shall prove the corresponding statement for k.

The idea is to apply Lemma 4.1 with U as V_{k-1} and V as V_k, but the possibility of having more than one L type in N_k that is not fully accounted for by $V_{k-1} \cap N_k$ complicates matters. Thus let L_k be an L stable complement to $V_{k-1} \cap N_k$ in N_k. We shall pass from V_{k-1} to V_k in a number of steps equal to the number of L types appearing in L_k. For each such L type v, let $\tilde{F}_v$ be the isotypic subspace of L_k of type v. Let $v_1, \dots, v_r$ be these L types. Suppose inductively on r that $U = V_{k-1} + U(\mathfrak{k})(\tilde{F}_{v_1} + \cdots + \tilde{F}_{v_{r-1}})$ is a sum of irreducible $\mathfrak{k}$ submodules, that U is $Z(\mathfrak{k})$ finite, and that the product of all $z - \chi_{v+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})}(z)$, $z \in Z(\mathfrak{k})$, acts as 0 on it, where v runs through the L types that occur in $N_0^{\bar{\mathfrak{u}} \cap \mathfrak{k}}, \dots, N_{k-1}^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ and the L types $v_1, \dots, v_{r-1}$.

Put $V = V_{k-1} + U(\mathfrak{k})(\tilde{F}_{v_1} + \cdots + \tilde{F}_{v_r})$. In V/U, all weights that minimize $\langle \omega, \delta(\mathfrak{u}) \rangle$ have to be $\bar{\mathfrak{u}} \cap \mathfrak{k}$ invariant vectors. Since U and V both contain all weight spaces of $N(\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{k}))$ for weights ω with $\langle \omega, \delta(\mathfrak{u}) \rangle < d_0 + kc$ and since V contains $\tilde{F}_{v_r}$ but U does not, the expression $\langle \omega, \delta(\mathfrak{u}) \rangle$ is minimized at value $d_0 + kc$ by $\omega = v_r$ among all weights of V/U. The space $\tilde{F}_{v_r}$ maps one-one into V/U, and its image therefore consists entirely of $\bar{\mathfrak{u}} \cap \mathfrak{k}$ invariant vectors. Write this image as

the direct sum of irreducible L stable subspaces. Each of these subspaces, by the same argument as in the proof of Lemma 4.2, generates an irreducible $\mathfrak{k}$ submodule of V/U, necessarily of infinitesimal character $\nu_r + \delta(\mathfrak{l}) - \delta(\mathfrak{u}\cap\mathfrak{k})$. Form the pre-image in V of the sum of these subspaces. This pre-image contains U and $\tilde{F}_{\nu_r}$, and it therefore must be all of V. Consequently V/U has infinitesimal character $\nu_r + \delta(\mathfrak{l}) - \delta(\mathfrak{u}\cap\mathfrak{k})$ and is the sum of irreducible $\mathfrak{k}$ submodules. Applying Lemma 4.1, we find a $\mathfrak{k}$ submodule Y of V such that $V = U \oplus Y$. Necessarily Y is isomorphic to V/U, and hence V is still the sum of irreducible $\mathfrak{k}$ submodules. Although Y is not asserted to contain $\tilde{F}_{\nu_r}$, it does contain a subspace of N_k isomorphic to $\tilde{F}_{\nu_r}$ and disjoint from U. Every vector of this subspace has to be $\bar{\mathfrak{u}}\cap\mathfrak{k}$ invariant, and thus ν_r is an L type occurring in $N_k^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$. Lemma 4.1 shows that the product of all $z - \chi_{\nu+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})}(z)$, $z\in Z(\mathfrak{k})$, acts as 0 on V, where ν runs through the L types that occur in $N_0^{\bar{\mathfrak{u}}\cap\mathfrak{k}}, \dots, N_{k-1}^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ and the L types $\nu_1, \dots, \nu_r$.

This completes the inner induction and allows us to conclude that V_k is $Z(\mathfrak{k})$ finite and that the product of all $z - \chi_{\nu+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})}(z)$, $z\in Z(\mathfrak{k})$, acts as 0 on V_k, where ν runs through the L types that occur in $N_0^{\bar{\mathfrak{u}}\cap\mathfrak{k}}, \dots, N_k^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$. Thus the induction on k is complete, and $N(\lambda + 2\delta(\mathfrak{u}))$ is semisimple.

If an irreducible $(\mathfrak{k}, L)$ submodule V is given, let F_ν be an irreducible L invariant subspace of V with $\langle \nu, \delta(\mathfrak{u})\rangle$ as small as possible. The members of F_ν are necessarily $\bar{\mathfrak{u}}\cap\mathfrak{k}$ invariant, and Lemma 4.2 shows that V is isomorphic to $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_\nu$ and has infinitesimal character the $\Delta^+(\mathfrak{k})$ dominant expression $\nu + \delta(\mathfrak{l}) - \delta(\mathfrak{u}\cap\mathfrak{k})$. If V contains another irreducible L stable subspace $F_{\nu'}$ of $\bar{\mathfrak{u}}\cap\mathfrak{k}$ invariant vectors, then Lemma 4.2 shows $U(\mathfrak{k})(F_{\nu'})$ has infinitesimal character $\nu' + \delta(\mathfrak{l}) - \delta(\mathfrak{u}\cap\mathfrak{k})$. For this to be conjugate to $\nu + \delta(\mathfrak{l}) - \delta(\mathfrak{u}\cap\mathfrak{k})$, Lemma 4.2 says that we must have $\nu' = \nu$. But the L type ν occurs just once in $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_\nu$, and hence $F_{\nu'} = F_\nu$. This completes the proof of Lemma 4.3.

Remark. Lemma 4.3 proves everything that is asserted in the statement of Lemma 3.1 of [EPWW]. However, the proof of Lemma 3.1 in [EPWW] contains an additional statement that will be useful to us. The assumptions on λ are unchanged. From Lemma 4.3 we know that $N(\lambda + 2\delta(\mathfrak{u}))$ is isomorphic as a $(\mathfrak{k}, L)$ module to a direct sum of $(\mathfrak{k}, L)$ modules $U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} F_\nu$, the F_ν's and their multiplicities being given by the decomposition of the $(\mathfrak{l}, L)$ module $N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$. The additional statement in the proof of Lemma 3.1 of [EPWW] identifies this $(\mathfrak{l}, L)$ module concretely as

$$N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}} \cong S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}, \tag{4.4}$$

and hence we obtain a $(\mathfrak{k}, L)$ isomorphism

$$N(\lambda + 2\delta(\mathfrak{u})) \cong U(\mathfrak{k})\otimes_{\bar{\mathfrak{q}}\cap\mathfrak{k}} (S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}). \tag{4.5}$$

The action of $\bar{\mathfrak{u}}\cap\mathfrak{k}$ on $S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$ is understood to be trivial. The isomorphism in (4.5) is extremely complicated and in particular appears to depend

on λ; it was asserted without proof in [EPWW], and Lemmas 4.1–4.3 in the present paper provide a way to derive it that we give in a moment. Possibly the authors of [EPWW] intended that (4.5) is initially to be viewed with a certain understood nontrivial action of $\bar{\mathfrak{u}} \cap \mathfrak{k}$ on $S(\mathfrak{u} \cap \mathfrak{p})$ and then that (4.5) holds also with the trivial action. The usual action of $\bar{\mathfrak{u}} \cap \mathfrak{k}$ on $S(\mathfrak{u} \cap \mathfrak{p})$ is obtained by identifying $\mathfrak{u} \cap \mathfrak{p}$ with $\mathfrak{p}/(\bar{\mathfrak{q}} \cap \mathfrak{p})$ and having $\bar{\mathfrak{u}} \cap \mathfrak{k}$ act on $\mathfrak{p}$ by the adjoint representation, but this quotient action is 0 for our setting because $[\bar{\mathfrak{u}} \cap \mathfrak{k}, \mathfrak{u} \cap \mathfrak{p}] \subseteq [\bar{\mathfrak{u}} \cap \mathfrak{p}]$.

One can derive (4.5), with the trivial action of $\bar{\mathfrak{u}} \cap \mathfrak{k}$, fairly easily from Lemmas 4.2 and 4.3. Corresponding to the Bernstein functor $\Pi_{\mathfrak{g},L}^{\mathfrak{g},K}$ on the $\mathfrak{g}$ level is a Bernstein functor $\Pi_{\mathfrak{k},L}^{\mathfrak{k},K}$ on the $\mathfrak{k}$ level, and Proposition 2.69 of [KnV] shows that these are related by forgetful functors according to the formula $\Pi_{\mathfrak{k},L}^{\mathfrak{k},K} \circ \mathscr{F}_{\mathfrak{g},L}^{\mathfrak{k},L} \cong \mathscr{F}_{\mathfrak{g},K}^{\mathfrak{k},L} \circ \Pi_{\mathfrak{g},L}^{\mathfrak{g},K}$. More importantly Proposition 2.115 of [KnV] shows that this relationship extends to the Sth derived functor:

$$(\Pi_{\mathfrak{k},L}^{\mathfrak{k},K})_S \circ \mathscr{F}_{\mathfrak{g},L}^{\mathfrak{k},L} \cong \mathscr{F}_{\mathfrak{g},K}^{\mathfrak{k},L} \circ (\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S. \tag{4.6}$$

Formula (4.6) says that the K decomposition of the effect of $(\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S$ on a $(\mathfrak{g}, L)$ module can be computed by using the derived Bernstein functor on the $\mathfrak{k}$ level. The effect of a derived Bernstein functor on the $\mathfrak{k}$ level may be computed by the algebraic version of the Borel–Weil–Bott theorem given as (4.170) of [KnV]. In our case the functor is to be applied in the top dimension, which is S, and the result is

$$(\Pi_{\mathfrak{k},L}^{\mathfrak{k},K})_S (U(\mathfrak{k}) \otimes_{\bar{\mathfrak{q}} \cap \mathfrak{k}} F_\nu)^L \cong \begin{cases} F^K_{\nu - 2\delta(\mathfrak{u} \cap \mathfrak{k})} & \text{if } \nu - 2\delta(\mathfrak{u} \cap \mathfrak{k}) \text{ is } \Delta^+(\mathfrak{k}) \text{ dominant,} \\ 0 & \text{if not.} \end{cases} \tag{4.7}$$

Lemma 4.2 shows that $\nu - 2\delta(\mathfrak{u} \cap \mathfrak{k})$ is necessarily $\Delta^+(\mathfrak{k})$ dominant, and hence we are always in the first case. Combining this fact with Lemma 4.3, we obtain the following equality of multiplicities:

$$[N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}} : F_\nu^L]_L = [\pi(\lambda) : F^K_{\nu - 2\delta(\mathfrak{u} \cap \mathfrak{k})}]_K. \tag{4.8}$$

On the other hand, Corollary 7 of [Kn1] shows that

$$[\pi(\lambda) : F^K_{\nu - 2\delta(\mathfrak{u} \cap \mathfrak{k})}]_K = [S(\mathfrak{u} \cap \mathfrak{p}) \otimes \mathbb{C}_{\lambda + 2\delta(\mathfrak{u})} : F_\nu^L]_L, \tag{4.9}$$

and (4.4) follows by combining (4.8) and (4.9).

Let us restate (4.9) as a proposition.

Proposition 4.4. *Under the assumption that $\mathbb{C}_\lambda$ is a one-dimensional $(\mathfrak{l}, L)$ module such that $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant, the K types that occur in $\pi(\lambda)$ are of the form $F^K_{\Lambda + \sigma}$, where F^L_σ is an L type in $S(\mathfrak{u} \cap \mathfrak{p})$. Moreover the multiplicities of $F^K_{\Lambda+\sigma}$ in $\pi(\lambda)$ and of F^L_σ in $S(\mathfrak{u} \cap \mathfrak{p})$ match.*

Proof. The only thing left unsaid is that the form of $S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$, with $\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$ one dimensional, allows us to read off the L types as the sum of $\lambda+2\delta(\mathfrak{u})$ and any highest weight of $S(\mathfrak{u}\cap\mathfrak{p})$.

The remainder of Section 3 and much of Section 6 of [EPWW] examine what happens to the Shapovalov form on $N(\lambda+2\delta(\mathfrak{u}))$. The paper shows that the restriction of this form to the $\bar{\mathfrak{u}}\cap\mathfrak{k}$ invariant vectors is what is important. Each L-type multiplicity in $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ matches a corresponding K-type multiplicity after application of the derived Bernstein functor on the K level, the Shapovalov form is transported functorially, and the signatures match. It is then shown that the forms for the various K types have been transported consistently, so that the form is meaningful on $\pi(\lambda)$. The details of this construction will not be needed here.

A feature of the Shapovalov form is that it descends to a nonsingular form on the unique irreducible quotient $N'(\lambda+2\delta(\mathfrak{u}))$. A consequence of this fact and the controlling behavior of the $\bar{\mathfrak{u}}\cap\mathfrak{k}$ invariant vectors is that if the Shapovalov form can be shown to be positive definite on $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$, then $N(\lambda+2\delta(\mathfrak{u}))$ is irreducible and equals $N'(\lambda+2\delta(\mathfrak{u}))$. The paper [EPWW] makes use of this observation to prove irreducibility of $N(\lambda+2\delta(\mathfrak{u}))$ in certain cases, and the Gross–Wallach papers [GrW1,GrW2] use the observation to identify $N'(\lambda+2\delta(\mathfrak{u}))$ when $N(\lambda+2\delta(\mathfrak{u}))$ is reducible.

5. Gross–Wallach theory

We continue with the setting of Section 4: Δ^+ is a Borel–de Siebenthal positive system, and $\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$ is a one-dimensional $(\mathfrak{l},L)$ module such that $\Lambda=\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant.

The Gross–Wallach theory in [GrW1,GrW2] nominally applies to a Borel–de Siebenthal positive system only if the system is compatible with a quaternionic structure on G/K, but in fact some of the theory's tools apply to all Borel–de Siebenthal positive systems.

According to (4.4), $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ can be identified with $S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$. When the Shapovalov form is transferred to $S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$, the effect of applying the theory of [EPWW] is to allow the unitarity of $\pi'(\lambda)$ to be investigated by making an appropriate study of $S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$. In the Gross–Wallach theory this is done by induction on the degree of the $S(\mathfrak{u}\cap\mathfrak{p})$ part of members of $S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$.

Let σ be an L type. The subspace of $S(\mathfrak{u}\cap\mathfrak{p})$ transforming according to σ will be denoted V^σ. If $V^\sigma\neq 0$, then V^σ lies completely within one homogeneous component $S^d(\mathfrak{u}\cap\mathfrak{p})$, namely the one with

$$\langle\sigma,\delta(\mathfrak{u})\rangle=dc, \tag{5.1}$$

where c is the positive constant in (4.2). Formula (5.1) allows us to associate the *homogeneous degree* d to the L type σ appearing in $S(\mathfrak{u}\cap\mathfrak{p})$. In the notation of

Section 4, if σ has homogeneous degree d, then σ is one of the L types appearing in $N_d \cap N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{p}}$.

Proposition 10.3 of [GrW2] is the critical result that is valid for our general setting but is proved in that paper only in the context of a positive system compatible with a quaternionic structure on G/K. The statement in our general setting is as follows.

Proposition 5.1 (Gross–Wallach). *Normalize root vectors so that* $\theta \bar{X}_\alpha = -X_{-\alpha}$ *and* $[X_\alpha, X_{-\alpha}] = H_\alpha$ *for every root* α. *Let* σ *be an* L *type with* $V^\sigma \subseteq S^d(\mathfrak{u} \cap \mathfrak{p})$, *and define*

$$c_\lambda(\sigma) = |\sigma + \lambda + \delta|^2 - |\lambda + \delta|^2, \tag{5.2}$$

where $|\cdot|^2$ *is the quadratic form induced on* $i\mathfrak{t}_0^*$ *by the symmetric invariant bilinear form* C. *Let* v *be a member of* $N_d \cap N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{p}}$ *corresponding to a member of* V^σ. *Then the Shapovalov form* $\{\cdot, \cdot\}$ *on* $N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ *has*

$$c_\lambda(\sigma)\{v, v\} = 2 \sum_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{p})} \{X_{-\alpha} v, X_{-\alpha} v\}. \tag{5.3}$$

Proof (Sketch). We give the part of the proof where [GrW2] uses notation that is more special than necessary. Let Ω and $\Omega(\mathfrak{l})$ be the Casimir elements for $\mathfrak{g}$ and $\mathfrak{l}$. The relevant formula that replaces (10.4) of [GrW2] is

$$\Omega = \Omega(\mathfrak{l}) - 2H_{\delta(\mathfrak{u})} + 2 \sum_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{p})} X_\alpha X_{-\alpha} + 2 \sum_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k})} X_\beta X_{-\beta}. \tag{5.4}$$

The various properties of v make

$$X_{-\beta} v = 0 \quad \text{for all } \beta \in \Delta(\mathfrak{u} \cap \mathfrak{k}),$$

$$\Omega v = (\,|\lambda + \delta|^2 - |\delta|^2) v,$$

$$\Omega(\mathfrak{l}) v = (\,|\lambda + 2\delta(\mathfrak{u}) + \sigma + \delta(\mathfrak{l})|^2 - |\delta(\mathfrak{l})|^2) v,$$

$$2H_{\delta(\mathfrak{u})}(v) = 2\langle \lambda + 2\delta(\mathfrak{u}) + \sigma, \delta(\mathfrak{u}) \rangle v.$$

Substitution into (5.4) gives

$$c_\lambda(\sigma) v + 2 \sum_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{p})} X_\alpha X_{-\alpha} v = 0,$$

and then one arrives at (5.3) by the same kind of argument as in the last paragraph of the proof in [GrW2, p. 104].

The reason that Proposition 5.1 lends itself to an induction is that $X_{-\alpha} v$ on the right-hand side of (5.3) is in $N_{d-1} \cap N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ by (1.1). As long as $c_\lambda(\sigma)$ is not 0, (5.3) is a formula for the Shapovalov form on $N_d \cap N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ in terms of

the form on $N_{d-1} \cap N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$. The following two theorems are restatements of the consequences that [GrW2] derives in its Propositions 11.1 and 11.5 from Proposition 5.1 above. Our standard assumptions about λ are in force.

Theorem 5.2 (Gross–Wallach). *Suppose that every L highest weight $\sigma \neq 0$ in $S(\mathfrak{u} \cap \mathfrak{p})$ has $c_\lambda(\sigma) > 0$. Then $N'(\lambda + 2\delta(\mathfrak{u})) = N(\lambda + 2\delta(\mathfrak{u}))$, and the Shapovalov form on $\pi(\lambda)$ exhibits $\pi(\lambda)$ as infinitesimally unitary.*

Theorem 5.3 (Gross–Wallach). *Suppose that every L highest weight $\sigma \neq 0$ in $S(\mathfrak{u} \cap \mathfrak{p})$ of homogeneous degree $< d$ has $c_\lambda(\sigma) > 0$ and that $\tilde{\sigma}$ is an L highest weight of homogeneous degree d with $c_\lambda(\tilde{\sigma}) = 0$. Let $Q(\lambda + 2\delta(\mathfrak{u}))$ be the quotient of $N(\lambda + 2\delta(\mathfrak{u}))$ by the smallest $(\mathfrak{g}, L)$ submodule containing all $\bar{\mathfrak{u}} \cap \mathfrak{k}$ invariant vectors of L type $\tilde{\sigma} + \lambda + 2\delta(\mathfrak{u})$. Then the L types of $Q(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$, with their multiplicities, are the sum of $\lambda + 2\delta(\mathfrak{u})$ and the L types of $S(\mathfrak{u} \cap \mathfrak{p})/V^{\tilde{\sigma}} S(\mathfrak{u} \cap \mathfrak{p})$. If the L types $\sigma + \lambda + 2\delta(\mathfrak{u})$ of $Q(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ all have $c_\lambda(\sigma) > 0$ for $\sigma \neq 0$, then $Q(\lambda + 2\delta(\mathfrak{u})) = N'(\lambda + 2\delta(\mathfrak{u}))$, and the Shapovalov form on $\pi'(\lambda)$ exhibits $\pi'(\lambda)$ as infinitesimally unitary.*

Among the universal covering groups of indefinite orthogonal groups, the ones that satisfy the assumptions of [GrW1,GrW2] concerning a quaternionic structure are the covers of $SO(4,n)_0$ with $n \geqslant 3$. For n even, the representations studied by Gross and Wallach are π_2', π_1', and π_0'. The representations π_2' are studied by means of Theorem 5.2; π_1' and π_0', except for π_0' in $SO(4,4)_0$, are studied by means of Theorem 5.3. The one exceptional representation π_0' in $SO(4,4)_0$ requires a slight refinement of Theorem 5.3 that we shall not bother to formulate. The results below for the representations π_s' of the universal covering group of $SO(2m, 2l-2m)_0$ with $2 \leqslant m \leqslant l/2$ will need the slight refinement only when $s = 0$ and $l = 2m$.

6. Inequalities, Part I

In this section we specialize our considerations to $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m)$, using the notation of Section 1. Then $\pi_s = \pi(\lambda_s)$ and $\pi_s' = \pi'(\lambda_s)$ with λ_s as in (1.6). The paper [EPWW] proved that $\pi_s' = \pi_s$ for $s \geqslant m$ and that, for these s's, the Shapovalov form exhibits π_s' as infinitesimally unitary. The proof in [EPWW] was based on techniques that we did not discuss in detail in Section 4.

A different proof of this theorem can be based on Theorem 5.2, and two lemmas in this different proof are needed for our work with π_s' when $s < m$. Put

$$c_s(\sigma) = c_{\lambda_s}(\sigma) \tag{6.1}$$

with $c_{\lambda_s}(\sigma)$ as in (5.2). According to Theorem 5.2, the results about π_s for $s \geqslant m$ will follow if it is shown that every L type $\sigma \neq 0$ in $S(\mathfrak{u} \cap \mathfrak{p})$ has $c_s(\sigma) > 0$. The two lemmas in this section establish that inequality and something more.

Following the notation in (1.9), we write L types σ as $\sigma = (\xi, \nu)$, where $\xi = (a_1, \dots, a_m)$ is a $U(m)$ type and $\nu = (b_1, \dots, b_m, 0, \dots, 0)$ is an $SO(2l-2m)$ type. If $l = 2m$, then b_m will be the last entry of ν, and it can in principle be negative when ν is dominant; the cases with b_m negative can be handled by a simple trick, and we return to them in Section 8. For now, we shall assume that $b_m \geqslant 0$.

With b_m understood to be $\geqslant 0$, define

$$A = \sum_{j=1}^{m} a_j \quad \text{and} \quad B = \sum_{j=1}^{m} b_j. \tag{6.2}$$

The quadratic form $|\cdot|^2$ for these groups is given by the sum of squares of the entries, as in (1.5).

Lemma 6.1. $\sum_{j=1}^{m} [(b_j + l - m - j)^2 - (l - m - j)^2] \geqslant mB + 2(l-2m)B$.

Proof. The left-hand side is

$$= \sum b_j^2 + 2\sum b_j(l - m - j)$$

$$= \sum (b_j^2 - b_j) - B + 2\sum b_j(m + 1 - j) + 2\sum b_j(l - 2m). \tag{6.3}$$

In the third term of the right-hand side of (6.3), when $j < m + 1 - j$,

$$\begin{aligned} b_j(m+1-j) + b_{m+1-j}j &= \tfrac{1}{2}(b_j + b_{m+1-j})(m+1) + \tfrac{1}{2}(b_j - b_{m+1-j})(m+1-2j) \\ &\geqslant \tfrac{1}{2}(b_j + b_{m+1-j})(m+1). \end{aligned} \tag{6.4}$$

If m is odd, so that there is a middle term with $j = m + 1 - j$, then that j has

$$b_j(m + 1 - j) = \tfrac{1}{2} b_j(m+1). \tag{6.5}$$

Summing over j in (6.4) and (6.5), we obtain

$$2\sum b_j(m + 1 - j) \geqslant (m+1)B.$$

Thus (6.3) is

$$\geqslant \sum (b_j^2 - b_j) + mB + 2(l-2m)B \geqslant mB + 2(l-2m)B,$$

the final inequality holding since $b_j^2 \geqslant b_j$ for all j.

Lemma 6.2. *Assume* $(\xi, v) \neq (0,0)$. *Define* r *by the condition that* $a_r > 0$ *and* $a_{r+1} = \cdots = a_m = 0$. *Then*

$$|\sigma + \lambda_s + \delta|^2 - |\lambda_s + \delta|^2$$
$$\geqslant \sum_{j=1}^{r} (a_j^2 - a_j) + (s-r)A + \sum_{1 \leqslant u < v \leqslant r} (a_u - a_v) + mB + 2(l-2m)B,$$

and the right-hand side is > 0 *if* $s \geqslant r$.

Remark. In particular the right-hand side is > 0 if $s \geqslant m$. In this case Theorem 5.2 is applicable, showing for $s \geqslant m$ that $\pi_s = \pi_s'$ and that the Shapovalov form on π_s' exhibits π_s' as infinitesimally unitary. In the terminology we are using concerning orbits, Proposition 4.4 therefore says that π_s' is associated to $\mathcal{O}(m,m)$ if $s \geqslant m$.

Proof. Using Lemma 6.1, we have

$$\begin{aligned}
&|\sigma + \lambda_s + \delta|^2 - |\lambda_s + \delta|^2 \\
&= \sum_{j=1}^{m} \left[\left(a_j - j + \tfrac{1}{2}s\right)^2 - \left(-j + \tfrac{1}{2}s\right)^2 \right] + \sum_{j=1}^{m} \left[(b_j + l - m - j)^2 - (l - m - j)^2 \right] \\
&\geqslant \sum_{j=1}^{m} \left[\left(a_j - j + \tfrac{1}{2}s\right)^2 - \left(-j + \tfrac{1}{2}s\right)^2 \right] + mB + 2(l-2m)B \\
&= \sum_{j=1}^{r} \left[\left(a_j - j + \tfrac{1}{2}s\right)^2 - \left(-j + \tfrac{1}{2}s\right)^2 \right] + mB + 2(l-2m)B \\
&= \sum_{j=1}^{r} a_j^2 + \sum_{j=1}^{r} a_j(s-2j) + mB + 2(l-2m)B \\
&= \sum_{j=1}^{r} (a_j^2 - a_j) + \sum_{j=1}^{r} a_j s - r \sum_{j=1}^{r} a_j + \sum_{j=1}^{r} (r+1-2j)a_j + mB + 2(l-2m)B \\
&= \sum_{j=1}^{r} (a_j^2 - a_j) + (s-r) \sum_{j=1}^{r} a_j + \sum_{1 \leqslant u < v \leqslant r} (a_u - a_v) + mB + 2(l-2m)B.
\end{aligned}$$

When $s \geqslant r$, each term on the right-hand side is $\geqslant 0$. For the right-hand side to be 0, we must have $B = 0$ and $a_j^2 = a_j$ for all j. From $B = 0$, we obtain $v = 0$. Then Theorem 1.2 implies that ξ has to be even. Since $a_j^2 = a_j$ for all j, we obtain $\xi = 0$. Since the hypotheses exclude $(\xi, v) = (0,0)$, the argument is complete.

7. Inequalities, Part II

In this section we continue with the notation of Section 6, using the notation of Section 1 specialized to $\mathfrak{g}_0 = \mathfrak{so}(2m, 2(l-m))$. Then $\pi_s = \pi(\lambda_s)$ and $\pi'_s = \pi'(\lambda_s)$ with λ_s as in (1.6).

Our goal in this section and the next is to use the Gross–Wallach theory to prove that the Shapovalov form exhibits π'_s as infinitesimally unitary for $0 \leqslant s < m$. The main new step in the proof of unitarity is Theorem 7.1 below. We shall prove Theorem 7.1a in this section and Theorem 7.1b in Section 8. Afterward, still in Section 8, we shall apply the Gross–Wallach theory, particularly Theorem 5.3, and deduce the first consequences.

As in Section 6 we write L types in $S(\mathfrak{u} \cap \mathfrak{p})$ as $\sigma = (\xi, \nu)$ with ξ and ν as in (1.9). We continue with $c_s(\sigma)$ as defined in (6.1), with the mth entry b_m of ν understood to be $\geqslant 0$, and with the sums A and B as defined in (6.2). Recall from Section 5 that V^σ is defined to be the isotypic subspace of $S(\mathfrak{u} \cap \mathfrak{p})$ of type σ. Put

$$\sigma_s = \left(\sum_{i=1}^{s+1} 2e_i,\ 0 \right) \quad \text{for } 0 \leqslant s < m. \tag{7.1}$$

Theorem 1.2 and fact (1.10) show that σ_s occurs exactly once in $S(\mathfrak{u} \cap \mathfrak{p})$, and direct calculation shows that $c_s(\sigma_s) = 0$.

Theorem 7.1. *For $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m)$ with $2 \leqslant m \leqslant l/2$, suppose that $0 \leqslant s < m$. Let $\sigma = (\xi, \nu) \neq (0;0)$ be an L type occurring in $S(\mathfrak{u} \cap \mathfrak{p})$, and define t by $2t = A - B$.*

(a) *If $[a_1/2] + \cdots + [a_s/2] \geqslant t$, then $c_s(\sigma) \geqslant 0$ with equality only if $l = 2m$, $s = 0$, $a_1 = \cdots = a_m = 1$, and $b_1 = \cdots = b_m = 1$.*
(b) *If $[a_1/2] + \cdots + [a_s/2] < t$, then V^σ is contained in $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$.*

Remark. The number t in the theorem is an integer $\geqslant 0$ by (1.10) and Theorem 1.2. The function $[\cdot]$ is the greatest-integer function.

Lemma 7.2. *Assume $\sigma \neq 0$. Define r by the condition that $a_r > 0$ and $a_{r+1} = \cdots = a_m = 0$. Then*

$$c_s(\sigma) = |\sigma + \lambda_s + \delta|^2 - |\lambda_s + \delta|^2 \geqslant \sum_{j=1}^{r} (a_j s - 2t) + \sum_{1 \leqslant u < v \leqslant r} (a_u - a_v)$$

with equality only if $l = 2m$, $r = m$, $a_1 = \cdots = a_m = 1$, and $b_1 = \cdots = b_m = 1$.

Proof. Using $A = B + 2t$, we have

$$(s-r)A + mB = sA - rB - 2tr + mB = \sum_{j=1}^{r} (a_j s - 2t) + (m-r)B.$$

Substitution into Lemma 6.2 gives

$$\begin{aligned}
&|\sigma+\lambda_s+\delta|^2-|\lambda_s+\delta|^2\\
&\quad\geqslant \sum_{j=1}^{r}(a_j^2-a_j)+\sum_{j=1}^{r}(a_j s-2t)+\sum_{1\leqslant u<v\leqslant r}(a_u-a_v)\\
&\qquad +(m-r)B+2(l-2m)B\\
&\quad\geqslant \sum_{j=1}^{r}(a_j s-2t)+\sum_{1\leqslant u<v\leqslant r}(a_u-a_v).
\end{aligned}$$

For equality to hold at the last step, we must have $a_j^2=a_j$ for all j and also $(m-r)B+2(l-2m)B=0$. Thus $a_j=1$ or 0 for each j. In this case, (1.10) and Theorem 1.2 show that $a_j=b_j$ for each j; hence $B\neq 0$ unless $\sigma=0$. Since we are assuming that $\sigma\neq 0$, we conclude from $(m-r)B+2(l-2m)B=0$ that $m-r=0$ and $l-2m=0$. The lemma follows.

Lemma 7.3. *Define r as in Lemma 7.2. If $s<r$ and $[a_1/2]+\cdots+[a_s/2]\geqslant t$, then*

$$\sum_{j=1}^{r}(a_j s-2t)+\sum_{1\leqslant u<v\leqslant r}(a_u-a_v)\geqslant 0.$$

Equality holds only if no a_j equals 1 or else all a_u are even for $1\leqslant u\leqslant s$.

Proof. Define r' by the conditions that $a_j\geqslant 2$ for $1\leqslant j\leqslant r'$ and $a_j=1$ for $r'+1\leqslant j\leqslant r$. We distinguish two cases, $r'\leqslant s$ and $s<r'$.

First suppose $r'\leqslant s$. Then we have

$$\sum_{j=1}^{r}(a_j s-2t)=\sum_{j=1}^{r'}(a_j s-2t)+\sum_{j=r'+1}^{r}(s-2t)=\sum_{j=1}^{r'}(a_j s-2t)+(r-r')(s-2t)$$

and

$$\begin{aligned}
\sum_{1\leqslant u<v\leqslant r}(a_u-a_v)&\geqslant \sum_{u=1}^{r'}\sum_{v=r'+1}^{r}(a_u-a_v)=(r-r')\sum_{u=1}^{r'}(a_u-1)\\
&=(r-r')\sum_{u=1}^{s}(a_u-1)\\
&\geqslant (r-r')\left(\sum_{u=1}^{s}2[a_u/2]-s\right)\\
&\geqslant (r-r')(2t-s),
\end{aligned}$$

with equality at the next-to-last step only if $r'=r$ or else all a_u are even for $1\leqslant u\leqslant s$.

Adding we obtain

$$\begin{aligned}\sum_{j=1}^{r}(a_j s-2t)+\sum_{1\leqslant u<v\leqslant r}(a_u-a_v) &\geqslant \sum_{j=1}^{r'}(a_j s-2t)\\ &= s\sum_{j=1}^{r'}a_j-2r't\\ &\geqslant s\sum_{j=1}^{r'}a_j-r'\sum_{j=1}^{s}2[a_j/2]\\ &= s\sum_{j=1}^{r'}a_j-r'\sum_{j=1}^{r'}2[a_j/2]\\ &\geqslant s\sum_{j=1}^{r'}a_j-r'\sum_{j=1}^{r'}a_j\\ &= (s-r')\sum_{j=1}^{r'}a_j\\ &\geqslant 0\end{aligned}$$

since $r'\leqslant s$. Again equality holds only if $r'=r$ or else all a_u are even for $1\leqslant u\leqslant s$.

Second suppose $s<r'$. Then we still have

$$\sum_{j=1}^{r}(a_j s-2t)=\sum_{j=1}^{r'}(a_j s-2t)+\sum_{j=r'+1}^{r}(s-2t)=\sum_{j=1}^{r'}(a_j s-2t)+(r-r')(s-2t),$$

and this time we have

$$\begin{aligned}\sum_{1\leqslant u<v\leqslant r}(a_u-a_v) &\geqslant \sum_{u=1}^{r'}\sum_{v=r'+1}^{r}(a_u-a_v)+\sum_{u=1}^{s}\sum_{v=s+1}^{r'}(a_u-a_v)\\ &= (r-r')\sum_{u=1}^{r'}(a_u-1)+(r'-s)\sum_{u=1}^{s}a_u-s\sum_{u=s+1}^{r'}a_u\\ &\geqslant (r-r')\sum_{u=1}^{s}(a_u-1)+(r'-s)\sum_{u=1}^{s}a_u-s\sum_{u=s+1}^{r'}a_u\\ &\geqslant (r-r')\left(\sum_{u=1}^{s}2[a_u/2]-s\right)+(r'-s)\sum_{u=1}^{s}a_u-s\sum_{u=s+1}^{r'}a_u\\ &\geqslant (r-r')(2t-s)+(r'-s)\sum_{u=1}^{s}a_u-s\sum_{u=s+1}^{r'}a_u,\end{aligned}$$

with equality at the next-to-last step only if $r'=r$ or else all a_u are even for $1\leqslant u\leqslant s$.

Adding, we obtain

$$\begin{aligned}
&\sum_{j=1}^{r} (a_j s - 2t) + \sum_{1 \leqslant u < v \leqslant r} (a_u - a_v) \\
&\quad \geqslant \sum_{j=1}^{r'} (a_j s - 2t) + (r' - s) \sum_{j=1}^{s} a_j - s \sum_{j=s+1}^{r'} a_j \\
&\quad \geqslant s \sum_{j=1}^{r'} a_j - r' \sum_{j=1}^{s} 2[a_j/2] + (r' - s) \sum_{j=1}^{s} a_j - s \sum_{j=s+1}^{r'} a_j \\
&\quad \geqslant s \sum_{j=1}^{s} a_j + s \sum_{j=s+1}^{r'} a_j - r' \sum_{j=1}^{s} a_j + (r' - s) \sum_{j=1}^{s} a_j - s \sum_{j=s+1}^{r'} a_j \\
&\quad = 0,
\end{aligned}$$

with equality at the first $\geqslant$ sign only if $r' = r$ or else all a_u are even for $1 \leqslant u \leqslant s$. This completes the proof.

Proof of Theorem 7.1(a). If $s \geqslant r$, then strict inequality holds in Theorem 7.1a as a result of Lemma 6.2. If $s < r$, then Lemmas 7.2 and 7.3 combine to give the inequality with $\geqslant$ in place. If equality holds with $s < r$, then Lemma 7.2 says that $l = 2m$, $a_1 = \cdots = a_m = 1$, and $b_1 = \cdots = b_m$. From equality in Lemma 7.3, we conclude that no a_j equals 1 (which is false) or else that all a_u are even for $1 \leqslant u \leqslant s$. Since all a_u are in fact odd, s must equal 0.

8. Unitarity

In this section we prove Theorem 7.1b concerning $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l - 2m)_0$, and we use the Gross–Wallach theory to derive two consequences of the theorem. One consequence is the unitarity of π'_s for $0 \leqslant s < m$, and the other is a first description of the K spectrum of π'_s for these same values of s.

Proof of Theorem 7.1(b). We begin with some preliminary remarks. By (1.10) and Theorem 1.2 every even nonnegative dominant integral $\mu = (\mu_1, \ldots, \mu_m)$ has the property that $(\mu, 0)$ occurs with multiplicity one in $S(\mathfrak{u} \cap \mathfrak{p})$. That is, $V^{(\mu,0)}$ is irreducible. If $\mu = \mu' + \mu''$ with μ' and μ'' nonnegative dominant integral and even, then the product in $S(\mathfrak{u} \cap \mathfrak{p})$ of a nonzero highest weight vector in $V^{(\mu',0)}$ and a nonzero highest weight vector in $V^{(\mu'',0)}$ is a nonzero highest weight vector in $V^{(\mu,0)}$. Since $V^{(\mu,0)}$ is irreducible, it follows that

$$V^{(\mu,0)} \subseteq V^{(\mu',0)} V^{(\mu'',0)}. \tag{8.1}$$

Inclusion (8.1) leads one to consider the spaces $V^{\sigma_0}, \dots, V^{\sigma_{m-1}}$, which are special instances of $V^{(\mu,0)}$; here $\sigma_0, \dots, \sigma_{m-1}$ are as in (7.1). The form of the highest weight vectors of these spaces is given in Proposition 3.2 of [Kn4] as certain determinants whose entries are in $S^2(\mathfrak{u}\cap\mathfrak{p})$, and, in combination with the expansion-by-cofactors formula, it implies inclusions

$$V^{\sigma_1} \subseteq V^{\sigma_0} S^2(\mathfrak{u}\cap\mathfrak{p}), \quad V^{\sigma_2} \subseteq V^{\sigma_1} S^2(\mathfrak{u}\cap\mathfrak{p}), \dots, \quad V^{\sigma_{m-1}} \subseteq V^{\sigma_{m-2}} S^2(\mathfrak{u}\cap\mathfrak{p}). \tag{8.2}$$

Suppose that μ has depth r, so that $\mu = (\mu_1, \dots, \mu_r, 0, \dots, 0)$. Then we can write

$$(\mu, 0) = \tfrac{1}{2}(\mu_1 - \mu_2)\sigma_0 + \cdots + \tfrac{1}{2}(\mu_{r-1} - \mu_r)\sigma_{r-2} + \tfrac{1}{2}\mu_r\sigma_{r-1}.$$

Since $\mu_r > 0$, iteration of (8.1) and use of (8.2) yields

$$\begin{aligned} V^{(\mu,0)} &\subseteq \underbrace{V^{\sigma_0} \dots V^{\sigma_0}}_{\frac{1}{2}(\mu_1-\mu_2)\ \text{times}} \cdots \underbrace{V^{\sigma_{r-2}} \dots V^{\sigma_{r-2}}}_{\frac{1}{2}(\mu_{r-1}-\mu_r)\ \text{times}} V^{\sigma_{r-1}} \underbrace{V^{\sigma_{r-1}} \dots V^{\sigma_{r-1}}}_{\frac{1}{2}\mu_r - 1\ \text{times}} \\ &\subseteq V^{\sigma_{r-1}} S(\mathfrak{u}\cap\mathfrak{p}), \quad \text{where } r = \text{depth } \mu. \end{aligned} \tag{8.3}$$

Returning to the proof of the theorem, suppose that $\sigma = (\xi, \nu) \neq (0,0)$ occurs in $S(\mathfrak{u}\cap\mathfrak{p})$ and has the property that $[a_1/2] + \cdots + [a_s/2] < t$. By (1.9), (1.10), and Theorem 1.2, the multiplicity of σ in $S(\mathfrak{u}\cap\mathfrak{p})$ equals the sum of the Littlewood–Richardson coefficients $c_{\mu\nu}^{\xi}$ over all even nonnegative dominant integral μ such $||\xi|| = ||\mu|| + ||\nu||$, $\xi - \mu$ is nonnegative, and μ has depth $\leqslant m$. For such a μ, write $\mu = (\mu_1, \dots, \mu_m)$. Then $2t = A - B = ||\xi|| - ||\nu|| = ||\mu||$. Since $\xi - \mu$ is nonnegative, we have $\mu_i \leqslant a_i$ for $1 \leqslant i \leqslant m$, and since μ is even, we have $\mu_i \leqslant 2[a_i/2]$ for each i. The given condition $[a_1/2] + \cdots + [a_s/2] < t$ therefore implies that any μ with $c_{\mu\nu}^{\xi} > 0$ has

$$\mu_1 + \cdots + \mu_s \leqslant 2[a_1/2] + \cdots + 2[a_s/2] < 2t = ||\mu||.$$

This condition implies that

$$\mu_{s+1} > 0 \quad \text{if } c_{\mu\nu}^{\xi} > 0.$$

That is, the depth r of μ satisfies $s + 1 \leqslant r$. Combining this conclusion with (8.3) and (8.2), we obtain

$$V^{(\mu,0)} \subseteq V^{\sigma_{r-1}} S(\mathfrak{u}\cap\mathfrak{p}) \subseteq V^{\sigma_s} S(\mathfrak{u}\cap\mathfrak{p}) \quad \text{if } c_{\mu\nu}^{\xi} > 0. \tag{8.4}$$

We now appeal to Theorem 0.1 of [Kn4], which implies that

$$V^{(\xi,\nu)} \subseteq V^{(\nu,\nu)} \sum_{\substack{\mu \text{ with } c_{\mu,\nu}^{\xi} > 0, \\ \mu \text{ even}}} V^{(\mu,0)}. \tag{8.5}$$

Combining (8.4) and (8.5), we obtain

$$V^{(\xi,v)} \subseteq V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p}),$$

and the proof is complete.

Before applying Theorem 7.1 to the unitarity problem, let us return to the fact that when $l = 2m$, an L type $\sigma = (\xi, v)$ with $v = (b_1, \dots, b_m)$ that occurs in $S(\mathfrak{u} \cap \mathfrak{p})$ need not have $b_m \geqslant 0$. We introduce the notation

$$v^{\#} = (b_1, \dots, b_{m-1}, -b_m) \quad \text{and} \quad |v| = (b_1, \dots, b_{m-1}, |b_m|). \tag{8.6}$$

This $|v|$ cannot be squared, and thus there should be no confusion with the definition of $|v|^2$ as a sum of squares. Restriction from $U(2m)$ to $SO(2m)$ of the representation ξ always yields the $SO(2m)$ types v and $v^{\#}$ with the same multiplicity because one can restrict first to $O(2m)$ and then to $SO(2m)$. Theorem 1.2 always applies to $(\xi, |v|)$, and although v itself need not be nonnegative, we can always conclude that the multiplicity with which the irreducible representation of $SO(2m)$ with highest weight v appears in the restriction to $SO(2m)$ of the irreducible representation of $U(2m)$ with highest weight ξ is equal to the sum over all nonnegative even dominant integral forms μ of the Littlewood–Richardson coefficients $c^{\xi}_{\mu,|v|}$.

The multiplicity formula (1.10) for $S(\mathfrak{u} \cap \mathfrak{p})$ remains valid whether or not v is nonnegative, and thus $V^{(\xi,v)}$ has the same dimension as $V^{(\xi,v^{\#})}$. In fact, we can pass from $V^{(\xi,v)}$ to $V^{(\xi,v^{\#})}$ by

$$\begin{aligned} &\text{extending the vector-space isomorphism } X \mapsto X \operatorname{diag}(1, \dots, 1, -1) \\ &\text{of } \mathfrak{u} \cap \mathfrak{p} \text{ with itself to an algebra automorphism of } S(\mathfrak{u} \cap \mathfrak{p}). \end{aligned} \tag{8.7}$$

Finally it is immediate from (6.1) and (5.2) that

$$c_s(v) = c_s(v^{\#}) = c_s(|v|). \tag{8.8}$$

If we redefine B as $\sum_{j=1}^{m} |b_j|$, then Theorem 7.1 remains true for $l = 2m$ whether or not v is nonnegative. In fact, if v is nonnegative, we know the theorem to be true. To transfer the conclusion for v into one for $v^{\#}$, we have only to use (8.8) to obtain (a) in Theorem 7.1 and then (8.7) to obtain (b).

Theorem 8.1. *For* $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l - 2m)$ *with* $2 \leqslant m \leqslant l/2$, *if* $0 \leqslant s < m$ *but* $(l, s) \neq (2m, 0)$, *then* π'_s *is infinitesimally unitary and the* K *spectrum of* π'_s *is given by the sum of* $\Lambda_s = \lambda_s + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ *and the* L *spectrum of* $S(\mathfrak{u} \cap \mathfrak{p})/V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$.

Proof. First assume that $l > 2m$. We have $c_s(\sigma_s) = 0$, and V^{σ_s} lies in $S^d(\mathfrak{u} \cap \mathfrak{p})$ for $d = 2(s + 1)$. Suppose that σ occurs in $S^d(\mathfrak{u} \cap \mathfrak{p})$ for some d and that $c_s(\sigma) \leqslant 0$. Then Theorem 7.1 shows that $V^{\sigma} \subseteq V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$, and it follows that $d \geqslant 2(s + 1)$. Thus the hypotheses of Theorem 5.3 are satisfied for $\tilde{\sigma} = \sigma_s$.

That theorem says that if $Q(\lambda_s + 2\delta(\mathfrak{u}))$ denotes the quotient of $N(\lambda_s + 2\delta(\mathfrak{u}))$ by the smallest $(\mathfrak{g}, L)$ submodule containing all $\bar{\mathfrak{u}} \cap \mathfrak{k}$ invariant vectors of L type $\sigma_s + \lambda_s + 2\delta(\mathfrak{u})$, then the L types of $Q(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$, with their multiplicities, are the sum of $\lambda_s + 2\delta(\mathfrak{u})$ and the L types of $S(\mathfrak{u} \cap \mathfrak{p})/V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$.

Theorem 7.1 shows that the L types of this kind are limited to at most those σ's occurring in $S(\mathfrak{u} \cap \mathfrak{p})$ such that $[a_1/2] + \cdots + [a_s/2] \geqslant t$, and it shows further that, apart from $\sigma = (0, 0)$, these L types all have $c_s(\sigma) > 0$. From the remaining part of Theorem 5.3, it follows that $Q(\lambda_s + 2\delta(\mathfrak{u})) = N'(\lambda_s + 2\delta(\mathfrak{u}))$ and that the Shapovalov form on π'_s exhibits π'_s as infinitesimally unitary.

Since $N(\lambda + 2\delta(\mathfrak{u}))$ is semisimple (Lemma 4.3), we know that the K types of π'_s are a subset of the K types of π_s and in particular are the sum of $-2\delta(\mathfrak{u} \cap \mathfrak{k})$ and the L types of $N'(\lambda + 2\delta(\mathfrak{u}))$. Consequently, they are the sum of Λ_s and the L types of $S(\mathfrak{u} \cap \mathfrak{p})/V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$.

Now assume that $l = 2m$ but that $s \neq 0$. The new ingredient is that $\nu = (b_1, \ldots, b_m)$ can have $b_m < 0$. If we take into account the remarks concerning ν and $\nu^{\#}$ that precede the present theorem, then we see that Theorem 7.1 applies here and that the above part of the present proof goes through.

Theorem 8.2. *For $\mathfrak{g}_0 = \mathfrak{so}(2m, 2m)$ with $2 \leqslant m$, π'_0 is infinitesimally unitary and the K spectrum of π'_0 is given by the sum of $\Lambda_0 = \lambda_0 + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ and the L spectrum of*

$$S(\mathfrak{u} \cap \mathfrak{p})/(V^{\sigma_0} + V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^{\#})}) S(\mathfrak{u} \cap \mathfrak{p}),$$

where $\mathbf{1} = (1, \ldots, 1)$.

Proof. As in the proof of Theorem 8.1, but with $l = 2m$ and with s taken to be 0, we see that the hypotheses of Theorem 5.3 are satisfied for $\tilde{\sigma} = \sigma_0$.

That theorem says that if $Q(\lambda_0 + 2\delta(\mathfrak{u}))$ denotes the quotient of $N(\lambda_0 + 2\delta(\mathfrak{u}))$ by the smallest $(\mathfrak{g}, L)$ submodule containing all $\bar{\mathfrak{u}} \cap \mathfrak{k}$ invariant vectors of L type $\sigma_0 + \lambda_0 + 2\delta(\mathfrak{u})$, then the L types of $Q(\lambda_0 + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$, with their multiplicities, are the sum of $\lambda_0 + 2\delta(\mathfrak{u})$ and the L types of $S(\mathfrak{u} \cap \mathfrak{p})/V^{\sigma_0} S(\mathfrak{u} \cap \mathfrak{p})$.

Theorem 7.1, as amended before the statement of Theorem 8.1, shows that the L types of this kind are limited to at most those σ occurring in $S(\mathfrak{u} \cap \mathfrak{p})$ such that $t = 0$, and it shows further that, apart from

$$\sigma = (0, 0), \quad \sigma = (\mathbf{1}, \mathbf{1}), \quad \text{and} \quad \sigma = (\mathbf{1}, \mathbf{1}^{\#}),$$

these L types all have $c_0(\sigma) > 0$. For the two exceptional σ's, we have $c_0((\mathbf{1}, \mathbf{1})) = 0$ and $c_0((\mathbf{1}, \mathbf{1}^{\#})) = 0$.

Since there are exceptional σ's, the remaining part of Theorem 5.3 does not apply. But a slight refinement of the theorem, which allows for a second iteration of the quotient process and which we shall not state, applies and leads us immediately to the conclusion of the theorem.

9. K spectrum

In this section we continue with the notation of Sections 6–8 concerning the Lie algebra $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m)$ for $2 \leqslant m \leqslant l/2$. Theorems 8.1 and 8.2 gave us a first formula for the K spectrum of π'_s when $0 \leqslant s < m$, showing that the K spectrum, with multiplicities, is the sum of the minimal K type parameter Λ_s and the L spectrum of either

$$S(\mathfrak{u} \cap \mathfrak{p}) / V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p}) \tag{9.1a}$$

or

$$S(\mathfrak{u} \cap \mathfrak{p}) / (V^{\sigma_0} + V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^\#)}) S(\mathfrak{u} \cap \mathfrak{p}). \tag{9.1b}$$

Our goal in this section is to improve on these results by identifying the L spectrum of each quotient (9.1) with the L spectrum of the dual coordinate ring of a suitable orbit closure $\mathcal{O}(p,q)^{\mathrm{cl}}$. To make this identification, we shall prove an ostensibly more difficult result, which gives an explicit formula for the multiplicity of each L type of (9.1) in terms of Littlewood–Richardson coefficients. The main results are Theorems 9.4 and 9.5.

This section will make extensive use of material from [Kn4]. What is needed from that paper is not only isolated results but also the overall approach. We shall quote specific results from [Kn4] as we need them.

Lemma 1.1 above identified $S(\mathfrak{u} \cap \mathfrak{p})$ with the space $M_{m,2l-2m}$ of rectangular complex matrices, and we shall take this identification for granted in this section. Typically we shall continue to write the space of symmetric tensors as $S(\mathfrak{u} \cap \mathfrak{p})$ even though it might sometimes better be viewed as $S(M_{m,2l-2m})$.

The mapping *bar* of (1.2) gave us a canonical conjugate-linear algebra isomorphism of $S(\mathfrak{u} \cap \mathfrak{p})$ onto $P(\mathfrak{u} \cap \mathfrak{p})$. An effect of our isomorphism of $\mathfrak{u} \cap \mathfrak{p}$ with $M_{m,2l-2m}$ is to pick out a real form of $\mathfrak{u} \cap \mathfrak{p}$, namely the real vector space of members of $\mathfrak{u} \cap \mathfrak{p}$ that correspond to real matrices. From this isomorphism we constructed in Section 3 a noncanonical complex-linear algebra isomorphism $x(\cdot)$ of $S(\mathfrak{u} \cap \mathfrak{p})$ onto $P(\mathfrak{u} \cap \mathfrak{p})$. For getting at dual coordinate rings, we shall need to be able to work with the function *bar*. This function seems rather difficult to compute directly, and we shall instead compute it indirectly with the aid of $x(\cdot)$. We therefore review the construction in Section 3 of $x(\cdot)$ briefly.

The rows of $M_{m,2l-2m}$ are numbered $1, \dots, m$, and the columns are numbered $1, \dots, l-m, 1', \dots (l-m)'$. For X in $M_{m,2l-2n}$, we let $x_{ab}(X)$ and $x_{ab'}(X)$, with $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant l-m$, denote the entry functions on the matrices in $\mathfrak{u} \cap \mathfrak{p}$. Each of these is a member of the space $P^1(\mathfrak{u} \cap \mathfrak{p})$ of homogeneous first-order polynomials on $\mathfrak{u} \cap \mathfrak{p}$. Define X_{ab} and $X_{ab'}$, for $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant l-m$, to be matrices that are 1 in the indicated entries and are 0 elsewhere. For future reference, we set

$$Z_{ab} = X_{ab} - iX_{ab'} \quad \text{for } 1 \leqslant a \leqslant m \text{ and } 1 \leqslant b \leqslant l-m. \tag{9.2}$$

We define $x(\cdot)$ on a basis of matrices by $x(X_{ab}) = x_{ab}$ and $x(X_{ab'}) = x_{ab'}$, and we extend $x(\cdot)$ to a complex-vector-space isomorphism of $\mathfrak{u} \cap \mathfrak{p}$ onto $P^1(\mathfrak{u} \cap \mathfrak{p})$. The further extension of $x(\cdot)$ to $S(\mathfrak{u} \cap \mathfrak{p})$ is our complex-linear algebra isomorphism of $S(\mathfrak{u} \cap \mathfrak{p})$ onto $P(\mathfrak{u} \cap \mathfrak{p})$.

Of particular interest is the effect of $x(\cdot)$ on the "dot products" in Section 3, namely that $X \mapsto x(X_a \cdot X_b)(X)$, for $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant m$, picks out the ordinary dot product of the ath and bth rows of the matrix X. This is the same as the (a, b)th entry of the matrix product XX^{tr}.

The presence of a real form of $\mathfrak{u} \cap \mathfrak{p}$ allowed us also to define in Section 3 a complex conjugation mapping $(\cdot)^{\mathrm{conj}}$ on $\mathfrak{u} \cap \mathfrak{p}$, regarded as a space of complex matrices, and to extend it to a conjugate-linear algebra automorphism of $S(\mathfrak{u} \cap \mathfrak{p})$ and $P(\mathfrak{u} \cap \mathfrak{p})$. It was shown in Proposition 3.3 that *bar* can be computed, up to a nonzero global constant, as $x(\cdot)$ followed by $(\cdot)^{\mathrm{conj}}$. As far as *bar* is concerned, we shall really be interested only in its effect on vector spaces, and this global constant will play no role. Thus we shall work with the global constant as if it is 1. We can then compute *bar* by means of the easier function $x(\cdot)$ whenever we have closure under $(\cdot)^{\mathrm{conj}}$.

One technique of [Kn4] was to concentrate especially on matrices X with a certain triangular property. The following definition quantifies a version of this triangular property.

Call a member X of $M_{m,2l-2m}$ *quasi-upper triangular* if it has the following properties:

(a) $x_{bb}(X) = 1$ for $1 \leqslant b \leqslant m$,
(b) $x_{bb'}(X) \neq -i$ for $1 \leqslant b \leqslant m$,
(c) $x_{ab}(X) = 0$ for $a > b$,
(d) $x_{ab'}(X) = 0$ for $a \neq b$,
(e) $x_{ab}(X) = 0$ if $m < b \leqslant l - m$.

Condition (b) is arranged so that if Z_{pp} is as in (9.2), then $x(Z_{pp})(X)$ cannot be 0 when X is quasi-upper triangular. Every quasi-upper triangular matrix X has rank m. An example of a quasi-upper triangular matrix is given in (3.1) when $v = 0$.

Lemma 9.1. *Let $T = \{t_{ab}\}$ be any m-by-m symmetric matrix over $\mathbb{C}$. Then there exists a quasi-upper triangular matrix X in $M_{m,2l-2m}$ for which the matrix product XX^{tr} equals T.*

Proof. We define $x_{ab}(X)$ and $x_{ab'}(X)$ inductively downward on a and, for each fixed a, inductively downward on b for $a \leqslant b \leqslant m$. The base case of the induction is $a = m$, and we take, partly as required by (a) in the definition above,

$$x_{mm}(X) = 1 \quad \text{and} \quad x_{mm'}^2(X) = t_{mm} - 1.$$

For $x_{mm'}(X)$, we use either square root except that we disallow $x_{mm'}(X) = -i$ because of (b) in the definition of quasi-upper triangular. Suppose inductively on a

that we have defined row indices $>a$. Define inductively on b

$$\begin{aligned} x_{am}(X) &= t_{am}, \\ x_{a,m-1}(X) &= t_{a,m-1} - x_{am}(X)x_{m-1,m}(X), \\ &\vdots \\ x_{ab}(X) &= t_{ab} - x_{am}(X)x_{bm}(X) - \cdots - x_{a,b+1}(X)x_{b,b+1}(X) \quad \text{for } a<b<m, \end{aligned}$$

and then put

$$x_{aa}(X) = 1 \quad \text{and} \quad x_{aa'}^2(X) = t_{aa} - 1 - x_{a,a+1}^2(X) - \cdots - x_{am}^2(X),$$

avoiding $-i$ as a choice for $x_{aa'}(X)$ in order that (b) will be satisfied. This completes the inductive definition of the entries of X that might be nonzero, and the entries of X specified as 0 in (c)–(e) of the definition of quasi-upper triangular are taken to be 0. The resulting X has the required properties.

Let $D = \{D_{ab}\}_{1\leqslant a\leqslant b\leqslant m}$ be an $\binom{m}{2}$ tuple of nonnegative integers, and put $||D|| = \sum_{a\leqslant b} D_{ab}$. The paper [Kn4] made extensive use of the members $P(D)$ of $S(\mathfrak{u}\cap\mathfrak{p})$ defined by

$$P(D) = \prod_{a\leqslant b} (X_a \cdot X_b)^{D_{ab}}. \tag{9.3}$$

The element $P(D)$ lies in the homogeneous component $S^d(\mathfrak{u}\cap\mathfrak{p})$ of $S(\mathfrak{u}\cap\mathfrak{p})$ with $d = 2||D||$. Proposition 3.4 of [Kn4] shows that the elements $P(D)$ with $2||D||$ equal to a fixed d span the space of $SO(2l-2m)$ invariant members of $S^d(\mathfrak{u}\cap\mathfrak{p})$ and that this span coincides with the direct sum of all $V^{(\mu,0)}$ for even nonnegative dominant integral μ with $||\mu|| = d$, the spaces $V^{(\mu,0)}$ each being of multiplicity one.

Lemma 9.2. *If the polynomial $X\mapsto x(\sum a_D P(D))(X)$ vanishes on all quasi-upper triangular $X\in M_{m,2l-2m}$ with* rank $XX^{\mathrm{tr}} = s$, *then it vanishes everywhere on $\mathcal{O}(m,s)$.*

Proof. The value at X of the polynomial function $X\mapsto x(X_a\cdot X_b)(X)$ depends only on the entries of the matrix product XX^{tr}, and hence the same thing is true of the value at X of $X\mapsto x(\sum a_D P(D))(X)$. Let $X'\in\mathcal{O}(m,s)$ be given, and use Lemma 9.1 to choose a quasi-upper triangular matrix $X''\in M_{m,2l-2m}$ such that the matrix products $X'X'^{\mathrm{tr}}$ and $X''X''^{\mathrm{tr}}$ are equal. Since any quasi-upper triangular matrix has rank m, X'' lies in $\mathcal{O}(m,s)$. Also $x(\sum a_D P(D))(X') = x(\sum a_D P(D))(X'')$. The lemma follows.

The key tool of [Kn4] was a certain endomorphism φ of $S(\mathfrak{u}\cap\mathfrak{p})$ that extracted a totally upper-triangular part. Define a linear mapping φ of $M_{m,2l-2m}$ to itself by its

values on a basis:

$$\begin{aligned}\varphi(X_{ab}) &= \begin{cases} X_{ab} & \text{if } a \leqslant b, \\ 0 & \text{if } a > b, \end{cases} \\ \varphi(X_{ab'}) &= \begin{cases} X_{ab'} & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}\end{aligned} \tag{9.4}$$

Then extend φ, without changing its name, to an algebra endomorphism of $S(\mathfrak{u} \cap \mathfrak{p})$ sending 1 to 1. An important property of φ for our current purposes is that

$$\varphi(X) = X \quad \text{if } X \text{ is quasi-upper triangular in } \mathfrak{u} \cap \mathfrak{p}. \tag{9.5}$$

Lemma 9.3. *If M is in $S(\mathfrak{u} \cap \mathfrak{p})$ and X is in $\mathfrak{u} \cap \mathfrak{p}$, then*

$$x(M)(\varphi(X)) = x(\varphi(M))(X).$$

Proof. If $M = M_1 M_2$, then

$$x(M)(\varphi(X)) = x(M_1 M_2)(\varphi(X)) = x(M_1)(\varphi(X))\ x(M_2)(\varphi(X))$$

and

$$\begin{aligned} x(\varphi(M))(X) &= x(\varphi(M_1 M_2))(X) = x(\varphi(M_1)\varphi(M_2))(X) \\ &= x(\varphi(M_1))(X)\ x(\varphi(M_2))(X). \end{aligned}$$

It follows that there is no loss in generality in assuming that M is homogeneous of degree 1. Since both sides of the desired formula are bilinear in M and X, we may assume that M is X_{ab} or $X_{ab'}$ and that X is X_{cd} or $X_{cd'}$. In the two cases (i) $M = X_{ab}$ and $X = X_{cd'}$ and (ii) $M = X_{ab'}$ and $X = X_{cd}$, $x(M)(\varphi(X))$ and $x(\varphi(M))(X)$ are both 0.

Case (iii) is that $M = X_{ab}$ and $X = X_{cd}$. In this case we find that

$$x(X_{ab})(\varphi(X_{cd})) = \left\{ \begin{array}{cc} x_{ab}(X_{cd}) & \text{if } c \leqslant d \\ 0 & \text{if } c > d \end{array} \right\} = \begin{cases} \delta_{ac}\delta_{bd} & \text{if } c \leqslant d, \\ 0 & \text{if not,} \end{cases}$$

whereas

$$x(\varphi(X_{ab}))(X_{cd}) = \left\{ \begin{array}{cc} x(X_{ab})(X_{cd}) & \text{if } a \leqslant b, \\ 0 & \text{if } a > b \end{array} \right\} = \begin{cases} \delta_{ac}\delta_{bd} & \text{if } a \leqslant b, \\ 0 & \text{if not.} \end{cases}$$

If $\delta_{ac}\delta_{bd} = 1$, then $a = c$ and $b = d$, so that $a \leqslant b$ if and only if $c \leqslant d$. Our two expressions are thus equal in this case. If $\delta_{ac}\delta_{bd} = 0$, then both expressions are 0 and we again have equality. This completes case (iii).

Case (iv) is that $M = X_{ab'}$ and $X = X_{cd'}$. In this case we find that

$$x(X_{ab'})(\varphi(X_{cd'})) = \left\{ \begin{array}{cc} x_{ab'}(X_{cd'}) & \text{if } c = d \\ 0 & \text{if } c \neq d \end{array} \right\} = \left\{ \begin{array}{ll} \delta_{ac}\delta_{bd} & \text{if } c = d, \\ 0 & \text{if not,} \end{array} \right.$$

whereas

$$x(\varphi(X_{ab'}))(X_{cd'}) = \left\{ \begin{array}{cc} x(X_{ab'})(X_{cd'}) & \text{if } a = b \\ 0 & \text{if } a \neq b \end{array} \right\} = \left\{ \begin{array}{ll} \delta_{ac}\delta_{bd} & \text{if } a = b, \\ 0 & \text{if not.} \end{array} \right.$$

If $\delta_{ac}\delta_{bd} = 1$, then $a = c$ and $b = d$, so that $a = b$ if and only if $c = d$. Our two expressions are thus equal in this case. If $\delta_{ac}\delta_{bd} = 0$, then both expressions are 0 and we again have equality. This completes case (iv).

Theorem 9.4. *For* $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l - 2m)$ *with* $2 \leqslant m \leqslant l/2$, *if* $0 \leqslant s < m$ *but* $(l, s) \neq (2m, 0)$, *then*

(a) $\mathcal{O}(m, s)$ *is the locus of common zeros of the ideal in* $P(\mathfrak{u} \cap \mathfrak{p})$ *that corresponds to the ideal* $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$ *in* $S(\mathfrak{u} \cap \mathfrak{p})$ *under bar*,

(b) *the ideal* $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$ *is prime in* $S(\mathfrak{u} \cap \mathfrak{p})$, *and therefore* $R(\mathcal{O}(m, s)^{\mathrm{cl}})$ *equals* $S(\mathfrak{u} \cap \mathfrak{p})/V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$,

(c) *the multiplicity of the L type* (ξ, v) *in* $R(\mathcal{O}(m, s)^{\mathrm{cl}})$ *is* $\sum_\mu c^{\xi}_{\mu,|v|}$, *the sum being over all even nonnegative dominant integral* μ *for* $U(m)$ *such that* $\mu_{s+1} = \cdots = \mu_m = 0$. *Here* $c^{\xi}_{\mu,|v|}$ *is the Littlewood–Richardson coefficient for multiplicities in a tensor product for* $U(m)$.

Remark. In view of Theorem 8.1, conclusion (b) says for $0 \leqslant s < m$ and $(l, s) \neq (2m, 0)$ that π'_s is associated to the orbit $\mathcal{O}(m, s)$. The conclusion that $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$ is prime in this case is not new; see Theorem 5.7 of [DeP].

Proof. First we prove (a). By Corollary 3.6 of [Kn4], the space V^{σ_s} equals the linear span of all $(s + 1)$-by-$(s + 1)$ minors of $\det(\{X_a \cdot X_b\}_{\substack{1 \leqslant a \leqslant m \\ 1 \leqslant b \leqslant m}})$. Let us use the function $x(\cdot)$ to set up the correspondence of ideals between $S(\mathfrak{u} \cap \mathfrak{p})$ and $P(\mathfrak{u} \cap \mathfrak{p})$, checking afterward that *bar* and $x(\cdot)$ yield the same result. Generators of the ideal in question in $P(\mathfrak{u} \cap \mathfrak{p})$ are then all polynomials $X \mapsto \det(\{x(X_{i_a} \cdot X_{j_b})(X)\}_{\substack{1 \leqslant a \leqslant s+1 \\ 1 \leqslant b \leqslant s+1}})$. These are the $(s + 1)$-by-$(s + 1)$ minors of the matrix XX^{tr}, and their common locus of zeros is the set of all $X \in M_{m,2l-2m}$ for which $\mathrm{rank}(XX^{\mathrm{tr}}) \leqslant s$, i.e., the union of all $\mathcal{O}(p, q)$ with $0 \leqslant q \leqslant \min(s, p)$. On the other hand, the zero locus contains $\mathcal{O}(m, s)$ and is a union of orbits. Adjusting matrix (3.1) for $q = s$ and $v = 0$ by making some of the nonzero entries tend to 0 in a suitable fashion, we see that the closure of $\mathcal{O}(m, s)$ contains all orbits $\mathcal{O}(p, q)$ with $0 \leqslant q \leqslant \min(s, p)$. Thus $x(V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p}))$ has $\mathcal{O}(m, s)^{\mathrm{cl}}$ as its common locus of zeros. The use of $x(\cdot)$ in place of *bar* was legitimate because $x(V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p}))$ is

closed under complex conjugation; specifically the alternating tensors that generate V^{σ_s} take real values on real matrices. This proves (a).

The part of the proof just completed identifies the zero locus of the ideal $x(V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p}))$ in $P(\mathfrak{u}\cap\mathfrak{p})$ as $\mathcal{O}(m,s)^{\mathrm{cl}}$. Let I be the ideal of all polynomials vanishing on $\mathcal{O}(m,s)^{\mathrm{cl}}$, and let $J=\mathrm{bar}^{-1}(I)$ be the corresponding ideal of $S(\mathfrak{u}\cap\mathfrak{p})$. From the proof just completed, we know that $I=x(J)$.

We turn to (b) and (c). Fix an L type (ξ,v). First we show that

$$\begin{aligned}&\text{the multiplicity of } (\xi,v) \text{ in } V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p}) \text{ is at least } \sum_{\mu} c^{\xi}_{\mu,|v|},\\ &\text{the sum being over all even nonnegative dominant integral}\\ &\mu \text{ for } U(m) \text{ such that } \mu_{s+1}>0.\end{aligned}\tag{9.6}$$

Then we show that

$$\begin{aligned}&\text{the multiplicity of } (\xi,v) \text{ in } J \text{ is at most } \sum_{\mu} c^{\xi}_{\mu,|v|},\\ &\text{the sum being over the same } \mu\text{'s as in (9.6).}\end{aligned}\tag{9.7}$$

Since the multiplicities in question are finite and since the multiplicity in (9.6) has to be $\leqslant$ the multiplicity in (9.7), it follows that $J=V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p})$ and that

$$\begin{aligned}&(\xi,v) \text{ has multiplicity in } J \text{ equal to } \sum_{\mu} c^{\xi}_{\mu,|v|}, \text{ the sum being over all}\\ &\text{even nonnegative dominant integral } \mu \text{ for } U(m) \text{ such that } \mu_{s+1}>0.\end{aligned}\tag{9.8}$$

The zero locus in question, namely $\mathcal{O}(m,s)^{\mathrm{cl}}$, is the closure of an orbit of a connected complex Lie group acting holomorphically; it follows that $\mathcal{O}(m,s)^{\mathrm{cl}}$ is an irreducible variety. Therefore J is prime. Hence $V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p})$ is prime. Modulo proofs of (9.6) and (9.7), this proves (b). By (1.10) and Theorem 1.2,

$$\begin{aligned}&\text{the total multiplicity of the } L \text{ type } (\xi,v) \text{ in } S(\mathfrak{u}\cap\mathfrak{p}) \text{ is equal to}\\ &\sum_{\mu} c^{\xi}_{\mu,|v|}, \text{ the sum being over all even nonnegative dominant}\\ &\text{integral } \mu \text{ for } U(m).\end{aligned}\tag{9.9}$$

Subtracting the multiplicity formula in (9.8) from the formula in (9.9), we obtain conclusion (c).

Thus we are to prove (9.6) and (9.7). For (9.6) it is enough to handle v nonnegative. We observe from (8.2) and (8.3) that $V^{(\mu,0)}$ is contained in the ideal $V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p})$ whenever μ is an even nonnegative dominant integral form for $U(m)$ such that $\mu_{s+1}>0$. By Theorem 0.1 of [Kn4], we have

$$V^{(\xi,v)}\subseteq\sum\bigoplus_{\mu} V^{(v,v)}V^{(\mu,0)},$$

the sum being over all even nonnegative dominant integral μ for $U(m)$. The sum on the right is direct, and therefore the multiplicity of the L type (ξ, ν) in $S(\mathfrak{u}\cap\mathfrak{p})$ equals the sum of the multiplicities for that L type in each product $V^{(\nu,\nu)}V^{(\mu,0)}$. The multiplicity for a particular product is just $c_{\mu\nu}^{\xi}$ since Theorem 0.1 of [Kn4] shows multiplication from $V^{(\nu,\nu)}\otimes V^{(\mu,0)}$ to $V^{(\nu,\nu)}V^{(\mu,0)}$ to be one-one. Since we have just seen that a product $V^{(\nu,\nu)}V^{(\mu,0)}$ for which μ_{s+1} is >0 is necessarily contained in $V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p})$, the lower bound (9.6) on the multiplicity of (ξ,ν) in $V^{\sigma_s}S(\mathfrak{u}\cap\mathfrak{p})$ follows.

Now consider (9.7). Again it is enough to handle ν nonnegative. The multiplicity of the L type (ξ,ν) in J equals the dimension of the space of highest weight vectors in $V^{(\xi,\nu)}$ such that the corresponding polynomial vanishes on $\mathcal{O}(m,s)$. Theorem 4.1 of [Kn4] gives a fairly explicit formula for each highest weight vector $\phi_{\xi,\nu}$ of type (ξ,ν) in the tensor product $V^{\nu,\nu}\otimes\sum_\mu V^{(\mu,0)}$, and Theorem 0.1 of [Kn4] says that, under the multiplication mapping $\mathscr{M}$ on $S(\mathfrak{u}\cap\mathfrak{p})$, this space of highest weight vectors in the tensor product maps one-one onto the space of highest weight vectors in $V^{(\xi,\nu)}$.

Suppose that $\mathscr{M}(\phi_{\xi,\nu})$ is in J, i.e., that $x(\mathscr{M}(\phi_{\xi,\nu}))$ vanishes on $\mathcal{O}(m,s)$, and suppose that $X\in\mathcal{O}(m,s)$ is quasi-upper triangular. The formula in Theorem 4.1 of [Kn4] says that we can write

$$\mathscr{M}(\phi_{\xi,\nu}) = Z(\nu)\sum a_D P(D) + \text{other terms}, \tag{9.10}$$

where

$$\begin{aligned} &\varphi(Z(\nu)) = \prod_p Z_{pp}^{\nu_p}, \quad \varphi(\text{other terms}) = 0,\\ &\text{and} \quad \textstyle\sum a_D P(D)\neq 0 \text{ if } \phi_{\xi,\nu}\neq 0. \end{aligned} \tag{9.11}$$

Here φ is the homomorphism of $S(\mathfrak{u}\cap\mathfrak{p})$ into itself defined in (9.4). Since X is quasi-upper triangular, (9.5) shows that $\varphi(X)=X$. The fact that X is in $\mathcal{O}(m,s)$ then implies that

$$\begin{aligned} 0 &= x(\mathscr{M}(\phi_{\xi,\nu}))(X) \\ &= x(\mathscr{M}(\phi_{\xi,\nu}))(\varphi(X)) && \text{since } \varphi(X)=X \\ &= x(\varphi(\mathscr{M}(\phi_{\xi,\nu})))(X) && \text{by Lemma 9.3} \\ &= x\left(\prod_p Z_{pp}^{\nu_p}\right)(X)\ x(\varphi(\textstyle\sum a_D P(D)))(X) && \text{by (9.10) and (9.11)} \\ &= \prod_p (x_{pp}(X) - i x_{pp'}(X))^{\nu_p}\, x(\textstyle\sum a_D P(D))(\varphi(X)) && \text{by (9.2) and Lemma 9.3} \\ &= \prod_p (x_{pp}(X) - i x_{pp'}(X))^{\nu_p}\, x(\textstyle\sum a_D P(D))(X) && \text{since } \varphi(X)=X. \end{aligned}$$

The factor $\prod_p (x_{pp}(X) - x_{pp'}(X))^{\nu_p}$ is not zero since X is quasi-upper triangular, and thus the function $x(\sum a_D P(D))$ vanishes at every quasi-upper triangular X in $\mathcal{O}(m,s)$. By Lemma 9.2, the function $x(\sum a_D P(D))$ vanishes identically on $\mathcal{O}(m,s)$.

The element $\sum a_D P(D)$ is a member of $\sum \oplus V^{(\mu,0)}$, the sum being over all even nonnegative dominant integral μ such that $c_{\mu\nu}^{\xi} \neq 0$. Any average of translates by L of the function $x(\sum a_D P(D))$ has to vanish identically on $\mathcal{O}(m,s)$, and in particular the projection operator corresponding to a single $V^{(\mu,0)}$, when applied to $x(\sum a_D P(D))$, has to give 0. We are interested in seeing what conditions are imposed on μ by this requirement.

Thus suppose that $\sum a_D P(D)$ is a nonzero vector of a single $V^{(\mu,0)}$ and that $x(\sum a_D P(D))$ vanishes identically on $\mathcal{O}(m,s)$. The subspace of members M of $V^{(\mu,0)}$ such that $x(M)$ vanishes identically on $\mathcal{O}(m,s)$ is stable under L, and we know from (1.10) and Theorem 1.2 that $V^{(\mu,0)}$ is irreducible. Consequently $x(M)$ vanishes identically on $\mathcal{O}(m,s)$ for every M in $V^{(\mu,0)}$.

We apply this conclusion with M equal to a nonzero highest weight vector of $V^{(\mu,0)}$. According to Proposition 3.2 of [Kn4], M has to be a nonzero multiple of

$$\prod_{p=1}^{m} \left(\det \left(\{X_a \cdot X_b\}_{\substack{1 \leqslant a \leqslant p \\ 1 \leqslant b \leqslant p}} \right)^{\frac{1}{2}(\mu_p - \mu_{p+1})} \right).$$

Thus

$$\prod_{p=1}^{m} \det \left(\{x(X_a \cdot X_b)(X)\}_{\substack{1 \leqslant a \leqslant p \\ 1 \leqslant b \leqslant p}} \right)^{\frac{1}{2}(\mu_p - \mu_{p+1})} \tag{9.12}$$

has to vanish for all X in $\mathcal{O}(m,s)$. For X of the form

$$X = (\operatorname{diag}(1, \ldots, 1, 1, \ldots, 1) \quad 0 \quad \operatorname{diag}(0, \ldots, 0, i, \ldots, i) \ 0),$$

with $m - s$ entries of i, the matrix whose upper left determinants figure into (9.12) is $\operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$, with s diagonal entries of 1. The determinant is 1 for $p = 1, 2, \ldots, s$ and is 0 for $p > s$. For that X, we see that some μ_p has to > 0 with $p > s$ for (9.12) to be 0, i.e., that μ_{s+1} has to be > 0. The upper bound (9.7) on the multiplicity of the L type (ξ, ν) in J follows, and the proof of the theorem is complete.

Theorem 9.5. *For* $\mathfrak{g}_0 = \mathfrak{so}(2m, 2m)$ *with* $2 \leqslant m$ *and* $\mathbf{1} = (1, \ldots, 1)$,

(a) $\mathcal{O}(m-1,0)$ *is the locus of common zeros of the ideal in* $P(\mathfrak{u} \cap \mathfrak{p})$ *that corresponds to the ideal* $(V^{\sigma_0} + V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^{\#})})S(\mathfrak{u} \cap \mathfrak{p})$ *in* $S(\mathfrak{u} \cap \mathfrak{p})$ *under bar*,

(b) *the ideal* $(V^{\sigma_0} + V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^{\#})})S(\mathfrak{u} \cap \mathfrak{p})$ *is prime in* $S(\mathfrak{u} \cap \mathfrak{p})$, *and therefore* $R(\mathcal{O}(m-1,0)^{\mathrm{cl}})$ equals $S(\mathfrak{u} \cap \mathfrak{p})/(V^{\sigma_0} + V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^{\#})})S(\mathfrak{u} \cap \mathfrak{p})$,

(c) *the multiplicity of the* L *type* (ξ, ν) *in* $R(\mathcal{O}(m-1,0)^{\mathrm{cl}})$ *is* 1 *if* $\xi = \nu$ *with* $\nu_m = 0$, *and it is* 0 *otherwise*.

Remark. In view of Theorem 8.2, conclusion (b) says for $l = 2m$ and $s = 0$ that π_0' is associated to the orbit $\mathcal{O}(m-1,0)$.

Proof. If (ξ, ν) is an L type in the quotient

$$S(\mathfrak{u}\cap\mathfrak{p})/(V^{\sigma_0} + V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^\#)})S(\mathfrak{u}\cap\mathfrak{p}), \tag{9.13}$$

it is already an L type in the quotient

$$S(\mathfrak{u}\cap\mathfrak{p})/V^{\sigma_0}S(\mathfrak{u}\cap\mathfrak{p}). \tag{9.14}$$

The same argument as for Theorem 9.4 shows that all the conclusions of that theorem hold for $(l,s) = (2m,0)$ except for the primeness of the ideal $V^{\sigma_0}S(\mathfrak{u}\cap\mathfrak{p})$. Thus the multiplicity of the L type (ξ,ν) in (9.14) is $\sum_\mu c^\xi_{\mu,|\nu|}$, the sum being over all even nonnegative dominant integral μ for $U(m)$ such that $\mu_1 = 0$. The only possibility is that $\mu = 0$, and thus (1.10) and Theorem 1.2 show that $\nu = \xi$ or $\nu = \xi^\#$. So the decomposition of (9.14) under L is into all L types (ξ,ξ) and $(\xi,\xi^\#)$, each with multiplicity one.

Put $\xi = (\xi_1,\dots,\xi_m)$. If $\xi_m > 0$, then we can write $\xi = \mathbf{1} + \xi'$, where $\xi' = (\xi_1 - 1,\dots,\xi_m - 1)$ is nonnegative dominant integral for $U(m)$. The product of highest weight vectors of $V^{(\mathbf{1},\mathbf{1})}$ and $V^{(\xi',\xi')}$ is a highest weight vector for $V^{(\xi,\xi)}$. Since (1.10) and Theorem 1.2 show (ξ,ξ) to have multiplicity one in $V^{(\xi,\xi)}$, it follows that $V^{(\xi,\xi)} \subseteq V^{(\mathbf{1},\mathbf{1})}V^{(\xi',\xi')}$. Consequently the L type (ξ,ξ) does not occur in (9.13). Similarly $(\xi,\xi^\#)$ does not occur. Thus the only L types that can occur in (9.13) are the ones (ξ,ξ) with $\xi_m = 0$, and they have multiplicity at most one.

Let us now prove (a). The locus of common zeros of the ideal in $P(\mathfrak{u}\cap\mathfrak{p})$ corresponding to $V^{\sigma_0}S(\mathfrak{u}\cap\mathfrak{p})$ is $\mathcal{O}(m,0)^{\mathrm{cl}}$. Proposition 1.3 of [Kn4] shows that the nonzero highest weight vectors of $V^{(\mathbf{1},\mathbf{1})}$ are the nonzero multiples of

$$\det\left(\{Z_{ab}\}_{\substack{1\le a\le m\\ 1\le b\le m}}\right),$$

with Z_{ab} as in (9.2), and a similar argument shows that the nonzero highest weight vectors of $V^{(\mathbf{1},\mathbf{1}^\#)}$ are the nonzero multiples of the same determinant except that for $b = m$ we use

$$Z^{\mathrm{conj}}_{ab} = X_{ab} + iX_{ab'}$$

in place of Z_{ab}. Application of root vectors of $\mathfrak{so}(2l-2m)$, using the formulas of Section 1 of [Kn4], shows that $V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^\#)}$ is spanned by all determinants whose b^{th} column consists of either Z_{ab} or $Z^{\mathrm{conj}}_{ab'}$, the same one for all a. The corresponding polynomials are the functions $X \mapsto \det\Big(x(Z_{ab})(X) \text{ or } x(Z^{\mathrm{conj}}_{ab})(X)\Big)$, and a few column operations readily show that these vanish simultaneously

exactly when X has rank $\leqslant m-1$. Therefore the locus of common zeros of the ideal corresponding to

$$(V^{\sigma_0} + V^{(\mathbf{1},\mathbf{1})} + V^{(\mathbf{1},\mathbf{1}^{\#})})S(\mathfrak{u}\cap\mathfrak{p})$$

equals the closure of $\mathcal{O}(m,0)\cap\bigcup_{p=0}^{m-1}\mathcal{O}(m-1,p)$, which equals the closure of $\mathcal{O}(m-1,0)$.

Now we can prove (b) and (c). We have seen that the only possible L types in (9.13), hence in $R(\mathcal{O}(m-1,0)^{\mathrm{cl}})$, are the various (ξ,ξ) with $\xi_m = 0$. For any such L type, Proposition 1.4 of [Kn4] shows that the nonzero highest weight vectors of $V^{(\xi,\xi)}$ are the nonzero multiples of

$$\prod_{p=1}^{m-1}\det\left(\{Z_{ab}\}_{\substack{1\leqslant a\leqslant p\\ 1\leqslant b\leqslant p}}\right)^{\xi_p-\xi_{p+1}},$$

with Z_{ab} as in (9.2). The corresponding polynomial is

$$X\mapsto\prod_{p=1}^{m-1}\det\left(\{x_{ab}(X)-ix_{ab'}(X)\}_{\substack{1\leqslant a\leqslant p\\ 1\leqslant b\leqslant p}}\right)^{\xi_p-\xi_{p+1}}. \tag{9.15}$$

The m-by-$2m$ matrix given in block form by

$$(\mathrm{diag}(1,\ldots,1,0)\quad \mathrm{diag}(i,\ldots,i,0))$$

is in $\mathcal{O}(m-1,0)$. If we take X in (9.15) to be this matrix, we obtain 2^{m-1} as the value of (9.15). Thus the polynomial (9.15) does not vanish on $\mathcal{O}(m-1,0)$, and we see that every L type (ξ,ξ) with $\xi_m = 0$ occurs in $R(\mathcal{O}(m-1,0)^{\mathrm{cl}})$. Then (b) and (c) follow, and the proof of the theorem is complete.

10. Converse results

In this section we continue with the notation of Sections 6–8 concerning the Lie algebra $\mathfrak{g}_0 = \mathfrak{so}(2m,2l-2m)$ for $2\leqslant m\leqslant l/2$. In the previous sections we have established the unitarity of π'_s for $s\geqslant 0$ and identified the K spectrum completely. In the present section we deal with $s<0$. By (1.7) we then have $2(2m-l)\leqslant s<0$.

Proposition 10.1. *For $\mathfrak{g}_0 = \mathfrak{so}(2m,2l-2m)$ with $2\leqslant m\leqslant l/2$, suppose that s satisfies $2(2m-l)\leqslant s<0$. Then $c_s(\sigma)>0$ for all σ of homogeneous degree 1, and $c_s(\sigma)<0$ for some σ of homogeneous degree 2. Therefore the Shapovalov form on π'_s is indefinite and does not exhibit π'_s as unitary.*

Proof. By (1.10) and Theorem 1.2 the only σ of homogeneous degree 1 is $\sigma = (1,0,\dots,0;\,1,0,\dots,0)$. This has

$$\begin{aligned} c_s(\sigma) &= |\sigma+\lambda_s+\delta|^2 - |\lambda_s+\delta|^2 \\ &= \left(\left(1-1+\tfrac{1}{2}s\right)^2 - \left(-1+\tfrac{1}{2}s\right)^2\right) + \left((1+l-m-1)^2 - (l-m-1)^2\right) \\ &= (-1+s) + (2l-2m-1) \\ &\geqslant (-1+2(2m-l)) + (2l-2m-1) \\ &= 2(m-1) \\ &> 0. \end{aligned}$$

Now put $\sigma = \sigma_0 = (2,0,\dots,0;\,0,\dots,0)$. This σ has homogeneous degree 2, and

$$\begin{aligned} c_s(\sigma_0) &= |\sigma_0+\lambda_s+\delta|^2 - |\lambda_s+\delta|^2 \\ &= \left(2-1+\tfrac{1}{2}s\right)^2 - \left(-1+\tfrac{1}{2}s\right)^2 \\ &= 2s \\ &< 0. \end{aligned}$$

The argument in [GrW2] that proves Theorems 5.2 and 5.3 above then allows us to conclude that the Shapovalov form on π'_s is indefinite.[10]

Proposition 10.2. *For $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m)$ with $2 \leqslant m \leqslant l/2$, suppose that s satisfies $2(2m-l) \leqslant s < 0$.*

(a) *If s is even, let $r = \frac{1}{2}(-s)$, so that r is an integer with $1 \leqslant r \leqslant l-2m$. Put*

$$\sigma' = (r+1)\sigma_0 = (2(r+1),0,\dots,0;\,0,\dots,0).$$

Then $\sigma_0 + \lambda_s + 2\delta(\mathfrak{u})$ is an L type of $N'(\lambda_s+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ but $\sigma'+\lambda_s+2\delta(\mathfrak{u})$ is not, and it follows that π'_s is associated to no orbit (or union of orbits).

(b) *If s is odd, let $r = \frac{1}{2}(1-s)$, so that r is an integer with $1 \leqslant r \leqslant l-2m$. Put*

$$\sigma' = (r+1)\sigma_1 = (2(r+1),2(r+1),0,\dots,0;0,\dots,0).$$

Then $\sigma_1 + \lambda_s + 2\delta(\mathfrak{u})$ is an L type of $N'(\lambda_s+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ but $\sigma'+\lambda_s+2\delta(\mathfrak{u})$ is not, and it follows that π'_s is associated to no orbit (or union of orbits).

[10] Cf. [GrW2, top of p. 109].

The proof will occupy the remainder of this section. We shall prove all of (a) but shall limit part of the proof of (b) to a sketch. By way of preliminaries, let $J(\lambda_s + 2\delta(u))$ be the largest proper $(\mathfrak{g}, L)$ submodule of $N(\lambda_s + 2\delta(\mathfrak{u}))$. Lemma 4.3 shows that $N(\lambda_s + 2\delta(\mathfrak{u}))$ is semisimple as a $(\mathfrak{k}, L)$ module, and therefore $N(\lambda_s + 2\delta(\mathfrak{u})) \cong N'(\lambda_s + 2\delta(\mathfrak{u})) \oplus J(\lambda_s + 2\delta(\mathfrak{u}))$ as $(\mathfrak{k}, L)$ modules. Consequently

$$N(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}} \cong N'(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}} \oplus J(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$$

as L modules. By (4.4) the left-hand side is $\cong S(\mathfrak{u} \cap \mathfrak{p}) \otimes \mathbb{C}_{\lambda_s + 2\delta(\mathfrak{u})}$ as an L module, and it follows that

$$N'(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}} \cong S' \otimes \mathbb{C}_{\lambda_s + 2\delta(\mathfrak{u})} \quad \text{and} \quad J(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}} \cong J \otimes \mathbb{C}_{\lambda_s + 2\delta(\mathfrak{u})}$$

for some L submodules S' and J of $S(\mathfrak{u} \cap \mathfrak{p})$. The statement that π'_s is associated to some orbit or union of orbits Y would mean that J can be taken to be an ideal in $S(\mathfrak{u} \cap \mathfrak{p})$ in such a way that $S' \cong S(\mathfrak{u} \cap \mathfrak{p})/J$ is the dual coordinate ring of Y. In particular, it would not be possible for S' to contain nonzero nilpotent elements. If $M \in S(\mathfrak{u} \cap \mathfrak{p})$ is a nonzero highest weight vector of V^{σ_0} in the case of (a), or of V^{σ_1} in the case of (b), then M descends to a nonzero element of S' according to the conclusion of the proposition while $M^{2(r+1)}$ descends to 0. Thus the coset $M + S(\mathfrak{u} \cap \mathfrak{p})$ is a nonzero nilpotent element in S', and S' cannot be the dual coordinate ring of any subset of $\mathfrak{u} \cap \mathfrak{p}$.

Thus we have only to prove the statements in the proposition about L types in $N'(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$. First we prove the assertions about σ_0 and σ_1. Let σ be an L type of lowest possible homogenous degree in the L submodule J defined in the previous paragraph. Then every member of L type $\sigma + \lambda_s + 2\delta(\mathfrak{u})$ in $J(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ is $\bar{\mathfrak{u}}$ invariant, and the same argument as in the third paragraph of the proof of Lemma 4.2 shows that $\sigma + \lambda_s + 2\delta(\mathfrak{u})$ is conjugate to $\lambda_s + 2\delta(\mathfrak{u})$ under the Weyl group of $\mathfrak{k}$. Hence $|\sigma + \lambda_s + 2\delta(\mathfrak{u})|^2 = |\lambda_s + 2\delta(\mathfrak{u})|^2$, i.e., $c_s(\sigma) = 0$.

In the case of (a), we first check that this lowest possible degree is $\geqslant 2$ by computing $c_s(\sigma)$ for all σ of lower nonzero homogeneous degree. The only such σ is $(1, 0, \ldots, 0; 1, 0, \ldots, 0)$, and we saw in the proof of Proposition 10.1 that this σ has $c_s(\sigma) > 0$. We next check that the L type σ_0, which has homogeneous degree 2, does not occur in J by checking that $c_s(\sigma_0) \neq 0$. Indeed, we saw that $c_s(\sigma_0) = 2s < 0$ in the proof of Proposition 10.1, and thus σ_0 does not occur in J.

In the case of (b), we first check that the lowest possible degree is $\geqslant 4$. The σ's of lower degree, for which we need to check that $c_s(\sigma) \neq 0$, are $(1, 0, \ldots, 0; 1, 0, \ldots, 0)$, $(1, 1, 0, \ldots, 0; 1, 1, 0, \ldots, 0)$, $(2, 0, \ldots, 0; 0, \ldots, 0)$, $(2, 0, \ldots, 0; 2, 0, \ldots, 0)$, and the ones

$$(a_1, \ldots, a_m; b_1, \ldots, b_m, 0, \ldots, 0)$$

of homogeneous degree 3. We have already checked the first and third of these, and direct computation shows that the second and fourth have $c_s(\sigma) > 0$. If σ has homogeneous degree 3, put $A = \sum_{j=1}^m a_j$ and $B = \sum_{j=1}^m b_j$ as in (6.2). We compute

modulo 2 and find that

$$\begin{aligned}
c_s(\sigma) &= \sum_{j=1}^{m} \left(\left(a_j - j + \frac{1}{2}s \right)^2 - \left(-j + \frac{1}{2}s \right)^2 \right) \\
&\quad + \sum_{j=1}^{m} \left((b_j - j + l - m)^2 - (-j + l - m)^2 \right) \\
&= \sum_{j=1}^{m} a_j(a_j - 2j + s) + \sum_{j=1}^{m} b_j(b_j - 2j + 2l - 2m) \\
&\equiv \sum_{j=1}^{m} a_j^2 + As + \sum_{j=1}^{m} b_j^2 \\
&\equiv \sum_{j=1}^{m} a_j + As + \sum_{j=1}^{m} b_j \\
&= (A + B) + As \\
&\equiv (A - B) + As \\
&\equiv As,
\end{aligned}$$

the last congruence following from the next-to-last paragraph of Section 1. Here $A = 3$ and s is odd, and we see consequently that $c_s(\sigma)$ cannot be 0. Finally we check that the L type σ_1, which has homogeneous degree 4, does not occur in J by checking that $c_s(\sigma_1) \neq 0$. Indeed, we compute that $c_s(\sigma_1) = 2s + 2(-2 + s) < 0$, and thus σ_1 does not occur in J.

The main step in the proof of Proposition 10.2 is to show that the L type $\sigma' + \lambda_s + 2\delta(\mathfrak{u})$, which has multiplicity 1 in $N(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$, lies in $J(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$. We do so by checking that a nonzero highest weight vector of L type $\sigma' + \lambda_s + 2\delta(\mathfrak{u})$ in $N(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$ is $\bar{\mathfrak{u}}$ invariant. Being $\bar{\mathfrak{u}}$ invariant, it has the property that $U(\mathfrak{l} + \mathfrak{u})$ of it is a proper $(\mathfrak{g}, L)$ submodule of $N(\lambda_s + 2\delta(\mathfrak{u}))$ and therefore lies in $J(\lambda_s + 2\delta(\mathfrak{u}))$. Again by the $\bar{\mathfrak{u}}$ invariance, it lies in $J(\lambda_s + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}} \cap \mathfrak{k}}$. Thus the proof will be complete once we have identified such a vector and proved that it is invariant under $\bar{\mathfrak{u}}$.

For this step it will be convenient to have a multiplication table for $\mathfrak{so}(2l, \mathbb{C})$. First of all, with the invariant form $C(\cdot, \cdot)$ chosen so that $|\alpha|^2 = 1$ for all roots α, we appeal to Theorem 6.6 of [Kn2] and to [Gre] in order to choose root vectors E_α for the roots α in such a way that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for all α and that the constants $N_{\alpha,\beta}$ such that $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ whenever α, β, and $\alpha + \beta$ are roots satisfy

(i) $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$ for all roots α and β,
(ii) $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$ for all roots α, β, and γ of sum 0, and
(iii) the formulas for $N_{\alpha,\beta}$ in Table 3 hold whenever α and β are roots and $\alpha + \beta$ is a positive root.

Table 3
Multiplication table in $\mathfrak{so}(2l,\mathbb{C})$ for roots α and β involving indices $i<j<k$ when $\alpha+\beta$ is a positive root

Formula	α	β	$N_{\alpha,\beta}$
(1)	e_i+e_k	e_j-e_k	-1
(2)	e_i-e_k	e_j+e_k	-1
(3)	e_i+e_k	$-e_j-e_k$	$+1$
(4)	e_i-e_k	$-e_j+e_k$	$+1$
(5)	e_i+e_j	$-e_j+e_k$	-1
(6)	e_i-e_j	e_j+e_k	$+1$
(7)	e_i+e_j	$-e_j-e_k$	-1
(8)	e_i-e_j	e_j-e_k	$+1$
(9)	e_i+e_j	$-e_i+e_k$	$+1$
(10)	$-e_i+e_j$	e_i+e_k	$+1$
(11)	e_i+e_j	$-e_i-e_k$	$+1$
(12)	$-e_i+e_j$	e_i-e_k	$+1$

Consider (a) in the proposition. In $N(\lambda_s+2\delta(\mathfrak{u}))$, let $1_{\lambda_s+2\delta(\mathfrak{u})}$ denote the canonical generator $1\otimes 1$, the latter factor of 1 denoting the unit element of $\mathbb{C}_{\lambda_s+2\delta(\mathfrak{u})}$. Put

$$E=\sum_{k=m+1}^{l} E_{e_1+e_k}E_{e_1-e_k}$$

as a member of $U(\mathfrak{g})$. This has weight σ_0. To complete the proof of (a), we show that the nonzero element $E^{r+1}1_{\lambda_s+2\delta(\mathfrak{u})}$ of $N(\lambda_s+2\delta(\mathfrak{u}))$ is a highest weight vector under $\mathfrak{l}$, evidently of weight $(r+1)\sigma_0+\lambda_s+2\delta(\mathfrak{u})$, and is annihilated by $\bar{\mathfrak{u}}$.

Concerning the interaction of E and root vectors in $\mathfrak{l}$, let us check that

$$[E_{e_1-e_p},E]=0 \quad \text{for } 1<p\leqslant m, \tag{10.1}$$

$$[E_{\pm(e_p-e_q)},E]=0 \quad \text{for } 1<p<q\leqslant m, \tag{10.2}$$

$$[E_{e_p\pm e_l},E]=0 \quad \text{for } m<p<l, \tag{10.3}$$

$$[E_{-e_p\pm e_l},E]=0 \quad \text{for } m<p<l. \tag{10.4}$$

In fact, (10.1) and (10.2) are trivial. For (10.3) with the plus sign, we have

$$\begin{aligned}[E_{e_p+e_l},E]&=\sum_{k=m+1}^{l}[E_{e_p+e_l},E_{e_1+e_k}]E_{e_1-e_k}+\sum_{k=m+1}^{l}E_{e_1+e_k}[E_{e_p+e_l},E_{e_1-e_k}]\\&=0+E_{e_1+e_p}[E_{e_p+e_l},E_{e_1-e_p}]+E_{e_1+e_l}[E_{e_p+e_l},E_{e_1-e_l}]\\&=-E_{e_1+e_p}E_{e_1+e_l}+E_{e_1+e_l}E_{e_1+e_p}=0\end{aligned}$$

by (6) and (2) in Table 3. For (10.3) with the minus sign, we have

$$\begin{aligned}[E_{e_p-e_l}, E] &= \sum_{k=m+1}^{l} [E_{e_p-e_l}, E_{e_1+e_k}]E_{e_1-e_k} + \sum_{k=m+1}^{l} E_{e_1+e_k}[E_{e_p-e_l}, E_{e_1-e_k}] \\ &= [E_{e_p-e_l}, E_{e_1+e_l}]E_{e_1-e_l} + E_{e_1+e_p}[E_{e_p-e_l}, E_{e_1-e_p}] \\ &= E_{e_1+e_p}E_{e_1-e_l} - E_{e_1+e_p}E_{e_1-e_l} = 0\end{aligned}$$

by (1) and (8) in Table 3. This proves (10.3). For (10.4) with the plus sign, we have

$$\begin{aligned}[E_{-e_p+e_l}, E] &= \sum_{k=m+1}^{l} [E_{-e_p+e_l}, E_{e_1+e_k}]E_{e_1-e_k} + \sum_{k=m+1}^{l} E_{e_1+e_k}[E_{-e_p+e_l}, E_{e_1-e_k}] \\ &= [E_{-e_p+e_l}, E_{e_1+e_p}]E_{e_1-e_p} + E_{e_1+e_l}[E_{-e_p+e_l}, E_{e_1-e_l}] \\ &= E_{e_1+e_l}E_{e_1-e_p} - E_{e_1+e_l}E_{e_1-e_p} = 0\end{aligned}$$

by (5) and (4) in Table 3. For (10.4) with the minus sign, we have

$$\begin{aligned}[E_{-e_p-e_l}, E] &= \sum_{k=m+1}^{l} [E_{-e_p-e_l}, E_{e_1+e_k}]E_{e_1-e_k} + \sum_{k=m+1}^{l} E_{e_1+e_k}[E_{-e_p-e_l}, E_{e_1-e_k}] \\ &= [E_{-e_p-e_l}, E_{e_1+e_p}]E_{e_1-e_p} + [E_{-e_p-e_l}, E_{e_1+e_l}]E_{e_1-e_l} + 0 \\ &= E_{e_1-e_l}E_{e_1-e_p} - E_{e_1-e_p}E_{e_1-e_l} = 0\end{aligned}$$

by (7) and (3) in Table 3. This proves (10.4). If α and β are roots, then

$$[[E_\alpha, E_\beta], E] = [E_\alpha, [E_\beta, E]] - [E_\beta, [E_\alpha, E]],$$

and it follows from (10.3) and (10.4) that

$$[E_{\pm e_p \pm e_q}, E] = 0 \quad \text{for } m<p<q\leqslant l \tag{10.5}$$

for all choices of signs. If α is any member of $\Delta(\mathfrak{l})$ other than a root $-e_1+e_p$ with $1<p\leqslant m$, then it follows from (10.1), (10.2), (10.5), and the identity $E_\alpha\, 1_{\lambda_s+2\delta(\mathfrak{u})} = 0$ that

$$E_\alpha(E^{t+1}\, 1_{\lambda_s+2\delta(\mathfrak{u})}) = \sum_{k=0}^{t} E^k[E_\alpha, E]E^{t-k}\, 1_{\lambda_s+2\delta(\mathfrak{u})} + E^{t+1}E_\alpha\, 1_{\lambda_s+2\delta(\mathfrak{u})} = 0 \tag{10.6}$$

for all $t\geqslant -1$. In particular this formula is valid for any α in $\Delta^+(\mathfrak{l})$ if $t=r$. Consequently $E^{r+1}\, 1_{\lambda_s+2\delta(\mathfrak{u})}$ is a highest weight vector under $\mathfrak{l}$.

Now let us show that $E^{r+1}\, 1_{\lambda_s+2\delta(\mathfrak{u})}$ is annihilated by $\bar{\mathfrak{u}}$. First we consider the effect of $E_{-e_1+e_l}$. We use lines (9) and (12) of Table 3, the latter with α and β replaced by

their negatives, and then we use line (6) of Table 3 to obtain

$$\begin{aligned}
[E_{-e_1+e_l}, E] &= \sum_{k=m+1}^{l} \left([E_{-e_1+e_l}, E_{e_1+e_k}]E_{e_1-e_k} + E_{e_1+e_k}[E_{-e_1+e_l}, E_{e_1-e_k}]\right) \\
&= \sum_{k=m+1}^{l-1} \left([E_{-e_1+e_l}, E_{e_1+e_k}]E_{e_1-e_k} + E_{e_1+e_k}[E_{-e_1+e_l}, E_{e_1-e_k}]\right) \\
&\quad - E_{e_1+e_l}H_{e_1-e_l} \\
&= -\sum_{k=m+1}^{l-1} E_{e_k+e_l}E_{e_1-e_k} + \sum_{k=m+1}^{l-1} E_{e_1+e_k}E_{-e_k+e_l} - E_{e_1+e_l}H_{e_1-e_l} \\
&= -\sum_{k=m+1}^{l-1} E_{e_1-e_k}E_{e_k+e_l} + \sum_{k=m+1}^{l-1} E_{e_1+e_k}E_{-e_k+e_l} \\
&\quad - \sum_{k=m+1}^{l-1} [E_{e_k+e_l}, E_{e_1-e_k}] - E_{e_1+e_l}H_{e_1-e_l} \\
&= -\sum_{k=m+1}^{l-1} E_{e_1-e_k}E_{e_k+e_l} + \sum_{k=m+1}^{l-1} E_{e_1+e_k}E_{-e_k+e_l} \\
&\quad + (l-m-1)E_{e_1+e_l} - E_{e_1+e_l}H_{e_1-e_l}. \qquad (10.7)
\end{aligned}$$

Since $1_{\lambda_s+2\delta(\mathfrak{u})}$ is $\bar{\mathfrak{u}}$ invariant, we have

$$E_{-e_1+e_l}E^{r+1}\,1_{\lambda_s+2\delta(\mathfrak{u})} = \sum_{j=0}^{r} E^j[E_{-e_1+e_l}, E]E^{r-j}\,1_{\lambda_s+2\delta(\mathfrak{u})}. \qquad (10.8)$$

When we substitute from (10.7) for $[E_{-e_1+e_l}, E]$ in (10.8), the two sums on the right-hand side of (10.7) contribute 0 because of (10.6) with α equal to e_k+e_l or $-e_k+e_l$ and with $t = r-j-1$. Thus

$$\begin{aligned}
E_{-e_1+e_l}E^{r+1}\,1_{\lambda_s+2\delta(\mathfrak{u})} &= (l-m-1)\sum_{j=0}^{r} E_{e_1+e_l}E^r\,1_{\lambda_s+2\delta(\mathfrak{u})} \\
&\quad - \sum_{j=0}^{r} E_{e_1+e_l}E^jH_{e_1-e_l}E^{r-j}\,1_{\lambda_s+2\delta(\mathfrak{u})} \\
&= (l-m-1)(r+1)E_{e_1+e_l}E^r\,1_{\lambda_s+2\delta(\mathfrak{u})} \\
&\quad - \sum_{j=0}^{r} \langle e_1-e_l,\ (r-j)2e_1\rangle E_{e_1+e_l}E^r\,1_{\lambda_s+2\delta(\mathfrak{u})} \\
&\quad - \sum_{j=0}^{r} E_{e_1+e_l}E^rH_{e_1-e_l}\,1_{\lambda_s+2\delta(\mathfrak{u})}
\end{aligned}$$

$$= \Bigg((l-m-1)(r+1) - \sum_{j=0}^{r} \langle e_1 - e_l,\ (r-j)2e_1 \rangle - (r+1)\langle e_1 - e_l, \lambda_s + 2\delta(\mathfrak{u}) \rangle \Bigg) E_{e_1+e_l} E^r\, 1_{\lambda_s+2\delta(\mathfrak{u})}$$
$$= 0. \tag{10.9}$$

The set of members X of $\mathfrak{g}$ with $X(E^{r+1}\, 1_{\lambda_s+2\delta(\mathfrak{u})}) = 0$ is closed under brackets. It contains root vectors for $-e_1 + e_l$ and all $\pm e_q - e_l$ with $m<q<l$ and therefore contains root vectors for all $-e_1 \pm e_q$ with $m<q<l$. Since it contains also root vectors for $\mp e_q \pm e_l$, it contains root vectors for all $-e_1 \pm e_q$ with $m<q\leqslant l$. By (10.1) it contains $E_{e_1-e_p}$ for $1<p\leqslant m$, and therefore it contains root vectors for all $-e_p \pm e_q$ with $1\leqslant p\leqslant m<q\leqslant l$. In other words it contains $\bar{\mathfrak{u}} \cap \mathfrak{p}$. Since any member of $\bar{\mathfrak{u}} \cap \mathfrak{k}$ is the sum of two members of $\bar{\mathfrak{u}} \cap \mathfrak{p}$, it contains all of $\bar{\mathfrak{u}}$. This completes the proof of (a) in the proposition.

Consider (b) in the proposition. We continue to denote the canonical generator of $N(\lambda_s + 2\delta(\mathfrak{u}))$ by $1_{\lambda_s+2\delta(\mathfrak{u})}$. The idea is to produce a member Ω of $U(\mathfrak{u})$ of weight $(r+1)\sigma_1$ such that $\Omega(1_{\lambda_s+2\delta(\mathfrak{u})})$ is a highest weight vector in $N(\lambda_s + 2\delta(\mathfrak{u}))$ under $\mathfrak{l}$, evidently of weight $(r+1)\sigma_1 + \lambda_s + 2\delta(\mathfrak{u})$, and is annihilated by $\bar{\mathfrak{u}}$. In outline the argument is similar to what was done for (a), but the details of the argument are considerably more complicated than for (a) because the individual root vectors that enter the definition of Ω do not commute. Accordingly we shall give only a sketch of the argument.

Let $S(\mathfrak{g})$ be the symmetric algebra, and let $\mathfrak{S} : S(\mathfrak{g}) \to U(\mathfrak{g})$ be symmetrization. Define

$$X_{11} = \sum_{k=m+1}^{l} E_{e_1+e_k} E_{e_1-e_k},$$
$$X_{12} = \sum_{k=m+1}^{l} (E_{e_1+e_k} E_{e_2-e_k} + E_{e_1-e_k} E_{e_2+e_k}),$$
$$X_{22} = \sum_{k=m+1}^{l} E_{e_2+e_k} E_{e_2-e_k},$$
$$X = X_{11} X_{22} - \tfrac{1}{4} X_{12}^2,$$
$$Y = E_{e_1+e_l} X_{22} - \tfrac{1}{2} E_{e_2+e_l} X_{12}$$

as members of $S(\mathfrak{g})$. Here $\mathfrak{S}(X)$ has weight σ_1 in $U(\mathfrak{u})$, and so does $E_{e_1+e_2}^2$. The elements $\mathfrak{S}(X)$ and $E_{e_1+e_2}^2$ commute. The intention is to take as Ω a suitable linear combination of terms $(E_{e_1+e_2}^2)^j \mathfrak{S}(X)^{r+1-j}$.

The first computations are done in $S(\mathfrak{g})$. One shows that the root vectors $E_{\pm e_p \pm e_q}$ for $m<p<q\leqslant l$, $E_{\pm(e_p-e_q)}$ for $2<p<q\leqslant m$, $E_{\pm(e_1-e_2)}$, and $E_{e_1-e_p}$ and $E_{e_2-e_p}$ for

$2<p\leqslant m$ have 0 bracket with X. Since $\mathfrak{S}$ commutes with ad, $\mathfrak{S}(X)$ is in particular a highest weight vector under $\mathfrak{l}$.

Next one shows with some effort that

(i) $[E_{e_2+e_l}, \mathfrak{S}(X)] = E_{e_1+e_2}\mathfrak{S}(Y)$ and
(ii) $[\mathfrak{S}(X), \mathfrak{S}(Y)] = -\frac{1}{2}E_{e_1+e_2}(R_1 + aE_{e_1+e_2}^2 R_0)$ for a certain constant a,

where R_p for $p\geqslant 0$ is the member of $U(\mathfrak{g})$ defined by

$$R_p = E_{e_2+e_l}\mathfrak{S}(X)^p + \mathfrak{S}(X)^p E_{e_2+e_l}. \tag{10.10}$$

Meanwhile one computes in $S(\mathfrak{g})$ that

$$\begin{aligned}[E_{-e_1+e_l}, X] = &- E_{e_1+e_l}H_{e_1-e_l}X_{22} - E_{e_2+e_l}E_{-e_1+e_2}X_{11}\\ &+ \tfrac{1}{2}H_{e_1-e_l}E_{e_2+e_l}X_{12} + \tfrac{1}{2}E_{e_1+e_l}E_{-e_1+e_2}X_{12}\\ &+ \sum_{k=m+1}^{l-1}(-E_{e_k+e_l}E_{e_1-e_k} + E_{e_1+e_k}E_{-e_k+e_l})X_{22}\\ &+ \sum_{k=m+1}^{l-1}\tfrac{1}{2}(E_{e_k+e_l}E_{e_2-e_k} - E_{-e_k+e_l}E_{e_2+e_k})X_{12}.\end{aligned}$$

It will be convenient to abbreviate these eight sums of terms as Ia, IIa, Ib, IIb, IIIa, IVa, IIIb, and IVb. When we apply symmetrization to these eight sums of terms, each monomial $ABCD$ in $\mathfrak{S}(\mathfrak{g})$ has a particular factor A that we want to have at the right end, namely

$$H_{e_1-e_l} \quad \text{in the case of Ia and Ib,}$$

$$E_{-e_1+e_2} \quad \text{in the case of IIa and IIb,}$$

$$E_{e_k+e_l} \quad \text{in the case of IIIa and IIIb,}$$

$$E_{-e_k+e_l} \quad \text{in the case of IVa and IVb.}$$

We then use the formula

$$\begin{aligned}24\mathfrak{S}(ABCD) = &\,12\mathfrak{S}(BCD)A + 2B\mathfrak{S}(CD)A + 2C\mathfrak{S}(BD)A + 2D\mathfrak{S}(BC)A\\ &+ 2\mathfrak{S}(CD)BA + 2\mathfrak{S}(BD)CA + 2\mathfrak{S}(BC)DA\\ &+ 6[A, \mathfrak{S}(BCD)] + 2B[A, \mathfrak{S}(CD)] + 2C[A, \mathfrak{S}(BD)] + 2D[A, \mathfrak{S}(BC)]\\ &+ 2\mathfrak{S}(CD)[A, B] + 2\mathfrak{S}(BD)[A, C] + 2\mathfrak{S}(BC)[A, D].\end{aligned}$$

After a tedious and hazardous calculation, we are able to conclude that

(iii) $[E_{-e_1+e_l}, \mathfrak{S}(X)]v = (j-r)\mathfrak{S}(Y)v + bE_{e_1+e_2}R_0 v$ for any $v\in N(\lambda_s + 2\delta(\mathfrak{u}))$ of weight $\lambda_s + 2\delta(\mathfrak{u}) + j(e_1+e_2)$ with the property that v is invariant under

$E_{-e_1+e_2}$ and under all $E_{\pm e_k+e_l}$ with $m<k<l$; the constant b is independent of j and v.

Next an induction on $q\geqslant 0$ produces simultaneous proofs of the following four formulas:

- For all p with $0\leqslant p\leqslant q$,

$$R_p\mathfrak{S}(X)+\mathfrak{S}(X)R_p=2R_{p+1}+\sum_{j=1}^{p+1}c_jE_{e_1+e_2}^{2j}R_{p+1-j}$$

  for suitable constants c_j depending on p.
- For all p with $1\leqslant p\leqslant q$,

$$\mathfrak{S}(X)R_{p-1}\mathfrak{S}(X)=R_{p+1}+\sum_{j=1}^{p+1}c_jE_{e_1+e_2}^{2j}R_{p+1-j}$$

  for suitable constants c_j depending on p.
- If t is 0 or 1, then

$$\sum_{j=0}^{q-1}\mathfrak{S}(X)^jR_t\mathfrak{S}(X)^{q-j-1}=\sum_{i=0}^{q+t-1}c_iE_{e_1+e_2}^{2i}R_{q+t-1-i}$$

  for suitable constants c_i depending on t.
- For suitable constants c_i,

$$[\mathfrak{S}(Y),\mathfrak{S}(X)^q]=E_{e_1+e_2}\sum_{i=0}^{q}c_iE_{e_1+e_2}^{2i}R_{q-i}.$$

Putting all these facts together, one obtains, for $0\leqslant p\leqslant r+1$, the formula

$$\begin{aligned}&[E_{-e_1+e_l},E_{e_1+e_2}^{2r-2p+2}\mathfrak{S}(X)^p]\,1_{\lambda_s+2\delta(\mathfrak{u})}\\&\quad=\left(-(r-p+1)E_{e_1+e_2}^{2r-2p+1}R_p+\sum_{k\geqslant 0}b_kE_{e_1+e_2}^{2r-2p+3+2k}R_{p-1-k}\right)1_{\lambda_s+2\delta(\mathfrak{u})}\end{aligned}$$

for suitable constants b_k depending on p. Finally an inductive construction from this formula allows one to exhibit constants ω_p for $0\leqslant p\leqslant r+1$ with $\omega_{r+1}=1$ such that the operator $\Omega=\sum_{p=0}^{r+1}\omega_pE_{e_1+e_2}^{2r-2p+2}\mathfrak{S}(X)^p$ has $E_{-e_1+e_l}(\Omega 1_{\lambda_s+2\delta(\mathfrak{u})})=0$. It is then a simple matter to see that the root vectors for the other roots in $\Delta(\bar{\mathfrak{u}}\cap\mathfrak{p})$ annihilate $\Omega 1_{\lambda_s+2\delta(\mathfrak{u})}$ and to conclude that $\bar{\mathfrak{u}}$ annihilates $\Omega 1_{\lambda_s+2\delta(\mathfrak{u})}$. This concludes the sketch of the argument for the main step of (b) in the proposition.

11. Theory for $SO(2m, 2l-2m+1)_0$

This section establishes most of the comparable results for the simply connected covering group of $SO(2m, 2l-2m+1)_0$ with $2 \leqslant m \leqslant l/2$, noting any significant differences in the proofs between what happens in these groups and what happened in the previous sections. For the most part the theory is easier for these new groups. Lemma 1.1 gets replaced by a slightly tidier result, namely that $\mathrm{Ad}_{\mathfrak{g}}(L)$ is isomorphic to $U(m) \times SO(2l-2m+1)$ and $\mathfrak{u} \cap \mathfrak{p}$ is isomorphic to $M_{m,2l-2m+1}$ in such a way that the action of L on $\mathfrak{u} \cap \mathfrak{p}$ corresponds to the action of $U(m) \times SO(2l-2m+1)$ on matrices with $U(m)$ acting on the left and $SO(2l-2m+1)$ acting on the right.

Fact (1.10) about this action, quoted from [Kn4], remains true with only notational changes:

$$\begin{aligned} &\text{the } L \text{ type } \sigma = (\xi, \nu) \text{ occurs in } S(\mathfrak{u} \cap \mathfrak{p}) \text{ as many times as the} \\ &SO(2l-2m+1) \text{ type } \nu \text{ occurs in the restriction of the } U(2l-2m+1) \\ &\text{type } \xi \text{ from } U(2l-2m+1) \text{ to } SO(2l-2m+1). \end{aligned} \tag{11.1}$$

Littlewood's theorem (Theorem 1.2) is valid independently of the parity of n, and in combination with (11.1) it gives us an explicit multiplicity formula for each L type in $S(\mathfrak{u} \cap \mathfrak{p})$. Highest weights for the action of L on $S(\mathfrak{u} \cap \mathfrak{p})$ are in particular of the form $\sigma = (\xi, \nu)$ with ξ nonnegative dominant integral for $U(m)$ and ν dominant integral for $SO(2l-2m+1)$.

One simplification for $\mathfrak{so}(2m, 2l-2m+1)$, as opposed to $\mathfrak{so}(2m, 2l-2m)$, is that the representations of L, written again as $\sigma = (\xi, \nu)$, have ν nonnegative since ν is the highest weight of a rotation group $SO(2l-2m+1)$. Another simplification is that there is nothing special about π_s and π'_s in the case $(l, s) = (2m, 0)$; this case can be handled at the same time as the other unitary cases.

Now let us be specific about notation. Let

$$\lambda = \lambda_s = \left(-l + \tfrac{1}{2}(s-1), \dots, -l + \tfrac{1}{2}(s-1);\ 0, \dots, 0\right) \quad \text{with } s \in \mathbb{Z}.$$

Then

$$\lambda_s + \delta = \left(-1 + \tfrac{1}{2}s, -2 + \tfrac{1}{2}s, \dots, -m + \tfrac{1}{2}s; l - m - \tfrac{1}{2}, \dots, \tfrac{1}{2}\right),$$

$$\lambda_s + \delta(\mathfrak{u}) = \left(\tfrac{1}{2}(s-1-m), \dots, \tfrac{1}{2}(s-1-m); 0, \dots, 0\right),$$

$$\Lambda_s = \lambda_s + 2\delta(\mathfrak{u} \cap \mathfrak{p}) = \left(l - 2m + 1 + \tfrac{1}{2}(s-1), \dots, l - 2m + 1 + \tfrac{1}{2}(s-1); 0, \dots, 0\right).$$

In the notation of Section 1, we take $\pi_s = \pi(\lambda_s)$ and $\pi'_s = \pi'(\lambda_s)$. Assumption (1.4) concerning $\Lambda = \Lambda_s$ is valid for

$$s \geqslant 2(2m - l) - 1. \tag{11.2}$$

Under this condition, Λ_s is $\Delta^+(\mathfrak{k})$ dominant, and the K type Λ_s occurs with multiplicity 1 in π_s and π_s'. It is the unique minimal K type of these $(\mathfrak{g}, K)$ modules.

The unitarity/nonunitarity of the representations π_s' is summarized in Table 4, with the same reservations about notation as in Section 1. Quantitative statements of the results appear later in this section. The new results in Table 4 concern $s<m$: for $0\leqslant s\leqslant m-1$, there is unitarity, and for $s<0$, there is not. The paper [EPWW] had settled $s\geqslant m$, and the paper [GrW2] had settled $s\geqslant 0$ for $m=2$.

The analog of Proposition 3.1 is simpler for $SO(2m, 2l-2m+1)_0$, and the proof requires only small notational changes. The orbits of $GL(m,\mathbb{C})\times O(2l-2m+1,\mathbb{C})$ on $M_{m,2l-2m+1}$ are given by

$$\mathcal{O}(p,q)=\{X\in M_{m,2l-2m+1}\mid \text{rank } X=p \text{ and rank } XX^{\text{tr}}=q\}$$

for $0\leqslant q\leqslant p\leqslant m$. All of the $\mathcal{O}(p,q)$'s remain single orbits when the action is restricted to $GL(m,\mathbb{C})\times SO(2l-2m+1,\mathbb{C})$. The association of representations π_s' to orbits is summarized in Table 5. The first line of Table 5 follows from [EPWW] for all $m\geqslant 2$, and the second line of the table is proved in [GrW2] for the case $m=2$. There has been no attempt to examine $s<0$ for this table; the presence of roots of two different lengths would complicate the already-complicated analysis in the proof of Proposition 10.2.

Now let us examine the statements and proofs in Sections 6–10 to see what changes are needed in handling $\mathfrak{so}(2m, 2l-2m+1)$. For the analog of Section 6, we put $c_s(\sigma)=c_{\lambda_s}(\sigma)$ as in (6.1), and we write the L types in $S(\mathfrak{u}\cap\mathfrak{p})$ as $\sigma=(\xi,\nu)$ with $\xi=(a_1,\dots,a_m)$, $\nu=(b_1,\dots,b_m,0,\dots,0)$, and b_m always $\geqslant 0$. Let $A=\sum_{j=1}^m a_j$ and

Table 4
Unitary line-bundle cases π_s' for universal cover of $SO(2m, 2l-2m+1)_0$, $2\leqslant m\leqslant l/2$

Discrete series:	$s>2l-1$
Limit of discrete series:	$s=2l-1$
Last [Vo2] unitary point:	$s=m+1$
Last [EPWW] unitary point:	$s=m$
Last unitary point:	$s=0$
First nonunitary point:	$s=-1$
Last point under study:	$s=-2(l-2m)-1$

Table 5
Association of π_s' to orbits for universal cover of $SO(2m, 2l-2m+1)_0$, $2\leqslant m\leqslant l/2$

π_s' for $s\geqslant m$ is associated to	$\mathcal{O}(m,m)$
π_s' for $0\leqslant s<m$ is associated to	$\mathcal{O}(m,s)$

$B = \sum_{j=1}^m b_j$ as in (6.2). In place of Lemma 6.1, we have

$$\sum_{j=1}^{m} \left[\left(b_j + l - m - j + \tfrac{1}{2}\right)^2 - \left(l - m - j + \tfrac{1}{2}\right)^2 \right] \geqslant (m+1)B + 2(l-2m)B, \qquad (11.3)$$

the adjustment resulting from the contribution of $\frac{1}{2}$ from various places in δ. To adjust Lemma 6.2, we leave the hypotheses alone, and the conclusion is that

$$|\sigma + \lambda_s + \delta|^2 - |\lambda_s + \delta|^2$$

$$\geqslant \sum_{j=1}^{r} (a_j^2 - a_j) + (s-r)A + \sum_{1 \leqslant u < v \leqslant r} (a_u - a_v) + (m+1)B + 2(l-2m)B, \quad (11.4)$$

the right side being >0 if $s \geqslant r$. The only change in the proof is that we use (11.3) instead of Lemma 6.1. Then the previously known results follow: that $\pi_s = \pi_s'$ for $s \geqslant m$, that π_s' is unitary in these cases, and that π_s' is associated to $\mathcal{O}(m,m)$ in these cases.

For the analogs of results in Section 7, we continue to define σ_s as in (7.1), and we still have $c_s(\sigma_s) = 0$ for $0 \leqslant s < m$. To adjust Theorem 7.1, we leave the hypotheses alone, except for changing $\mathfrak{so}(2m, 2l-2m)$ to $\mathfrak{so}(2m, 2l-2m+1)$, and the new conclusions are that

(a) If $[a_1/2] + \cdots + [a_s/2] \geqslant t$, then $c_s(\sigma) > 0$.
(b) If $[a_1/2] + \cdots + [a_s/2] < t$, then V^σ is contained in $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$.

The analog of Lemma 7.2 is unchanged except that equality never holds. The reason is that equality implies $(m+1-r)B = 0$ and $a_j^2 = a_j$ for all j. Since $r \leqslant m$, we see that $B = 0$. When all a_j are $\leqslant 1$, each a_j has to equal b_j by (11.1) and Theorem 1.2; thus $\xi = 0$ and $\nu = 0$ in contradiction to the hypothesis that $\sigma \neq (0,0)$. Lemma 7.3 needs no change, except that we no longer need the statement about what happens when equality holds. Then the proof of Theorem 7.1a goes over in the new setting with only minor changes.

This brings us to Section 8. The proof of Theorem 7.1b in that section requires no change, and we arrive at the analog of Theorem 8.1, which is as follows.

Theorem 11.1. *For $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m+1)$ with $2 \leqslant m \leqslant l/2$, if $0 \leqslant s < m$, then π_s' is infinitesimally unitary and the K spectrum of π_s' is given by the sum of $\Lambda_s = \lambda_s + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ and the L spectrum of $S(\mathfrak{u} \cap \mathfrak{p})/V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$.*

The first four paragraphs of the proof of Theorem 8.1 apply here, and the fact that there are no exceptional L types in the analog of Theorem 7.1a means that the case $(l,s) = (2m,0)$ needs no special treatment in Theorem 11.1. As a consequence, there is no need for an analog of Theorem 8.2.

Adjusting Section 9 involves some slight notational complications. A compensating simplification is that the L spectrum in question is the one in (9.1a), and (9.1b) plays no role.

The columns of members of $M_{m,2l-2m+1}$ need to be indexed in a more complicated way than was used in Section 9. Namely we number them as

$$1, \dots, l-m, 1', \dots, (l-m)', \infty .$$

We define $X_{a,\infty}$ for $1 \leqslant a \leqslant m$ to be the matrix that is 1 in the indicated entry and 0 elsewhere. The definition of Z_{ab} is unchanged from (9.2), and column ∞ plays no role. In the definition of a quasi-upper triangular matrix X, we add the further property

(f) $x_{a,\infty}(X) = 0$ for $1 \leqslant a \leqslant m$.

Lemma 9.1 needs no change in its statement to translate into the current setting, but we need to add in the proof the definition $x_{a,\infty}(X) = 0$ for $1 \leqslant a \leqslant m$. Lemma 9.2 requires no change at all, and the definition of the homomorphism φ in (9.4) has to be expanded to include the additional line $\varphi(X_{a,\infty}) = 0$. Then, as in (9.5), it is still true that $\varphi(X) = X$ if X is quasi-upper triangular in $\mathfrak{u} \cap \mathfrak{p}$.

The statement of Lemma 9.3 needs no change, but extra cases must be considered in the proof. We have to allow for the possibility that M is $X_{a,\infty}$ or X is $X_{c,\infty}$. In these cases the conclusion of the lemma, namely that $x(M)(\varphi(X)) = x(\varphi(M))(X)$, is true because both sides can be checked to be 0. That brings us to the analog of Theorem 9.4, which is as follows.

Theorem 11.2. *If $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m+1)$ with $2 \leqslant m \leqslant l/2$ and if $0 \leqslant s < m$, then*

(a) *$\mathcal{O}(m,s)$ is the locus of common zeros of the ideal in $P(\mathfrak{u} \cap \mathfrak{p})$ that corresponds to the ideal $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$ in $S(\mathfrak{u} \cap \mathfrak{p})$ under bar,*

(b) *the ideal $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$ is prime in $S(\mathfrak{u} \cap \mathfrak{p})$, and therefore $R(\mathcal{O}(m,s)^{\mathrm{cl}})$ equals $S(\mathfrak{u} \cap \mathfrak{p})/V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$,*

(c) *the multiplicity of the L type (ξ, v) in $R(\mathcal{O}(m,s)^{\mathrm{cl}})$ is $\sum_\mu c^{\xi}_{\mu,v}$, the sum being over all even nonnegative dominant integral μ for $U(m)$ such that $\mu_{s+1} = \cdots = \mu_m = 0$. Here $c^{\xi}_{\mu,v}$ is the Littlewood–Richardson coefficient for multiplicities in a tensor product for $U(m)$.*

The proof of Theorem 11.2 requires only minor notational changes in the proof of Theorem 9.4. Conclusion (b) of Theorem 11.2 says for $0 \leqslant s < m$ that π'_s is associated to the orbit $\mathcal{O}(m,s)$, as is asserted in Table 5. As with $\mathfrak{so}(2m, 2l-2m)$, the conclusion that $V^{\sigma_s} S(\mathfrak{u} \cap \mathfrak{p})$ is prime for this situation is not new; see Theorem 5.7 of [DeP].

Finally we come to Section 10. The analog of Proposition 10.1 is the following; the new proof involves only minor changes to the proof of Proposition 10.1.

Proposition 11.3. *In $\mathfrak{g}_0 = \mathfrak{so}(2m, 2l-2m+1)$ with $2 \leqslant m \leqslant l/2$, suppose that s satisfies $2(2m-l) - 1 \leqslant s < 0$. Then $c_s(\sigma) > 0$ for all σ of homogeneous degree 1, and $c_s(\sigma) < 0$ for some σ of homogeneous degree 2. Therefore the Shapovalov form on π'_s is indefinite and does not exhibit π'_s as unitary.*

Proposition 11.3 verifies the last line of Table 4.

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