An exotic Deligne-Langlands correspondence for symplectic groups

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Abstract

Let $G = Sp(2n, \mathbb{C})$ be a complex symplectic group. We introduce a $G \times (\mathbb{C}^\times)^{\ell+1}$-variety $\mathfrak{N}_\ell$, which we call the ℓ-exotic nilpotent cone. Then, we realize the Hecke algebra $\mathbb{H}$ of type $C_n^{(1)}$ with three parameters via equivariant algebraic K-theory in terms of the geometry of $\mathfrak{N}_2$. This enables us to establish a Deligne-Langlands type classification of simple $\mathbb{H}$-modules under a mild assumption on parameters. As applications, we present a character formula and multiplicity formulas of $\mathbb{H}$-modules.

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Introduction

In their celebrated paper [KL87], Kazhdan and Lusztig gave a classification of simple modules of an affine Hecke algebra $\mathbb{H}$ with one-parameter in terms of the geometry of nilpotent cones. (It is also done by Ginzburg, c.f. [CG97].) Since some of the affine Hecke algebras admit two or three parameters, it is natural to extend their result to multi-parameter cases. (It is called the unequal parameter case.) Lusztig realized the "graded version" of $\mathbb{H}$ (with unequal parameters) via several geometric means [Lu88, Lu89, Lu95b] (c.f. [Lu03]) and classified their representations in certain cases. Unfortunately, his geometries admit essentially

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only one parameter. As a result, his classification is restricted to the case where all of the parameters are certain integral power of a single parameter. It is enough for his main interest, the study of representations of p-adic groups (c.f. [Lu95a]). However, there are many areas of mathematics which wait for the full-representation theory of Hecke algebras with unequal parameters (see e.g. Macdonald's book [Mc03] and its featured review in MathSciNet).

In this paper, we give a realization of all simple modules of the Hecke algebra of type $C_n^{(1)}$ with three parameters by introducing a variety which we call *the ℓ-exotic nilpotent cone* (c.f. §1.1). Our framework works for all parameters and realizes the whole Hecke algebra (Theorem A) and its specialization to each central character. Unfortunately, the study of our geometry becomes harder for some parameters and the result becomes less explicit in such cases. Even so, our result gives a definitive classification of simple modules of affine Hecke algebras of type $B_n^{(1)}$ and $C_n^{(1)}$ for almost all parameters including so-called real central character case. (See the argument after Theorem D.)

Let G be the complex symplectic group $Sp(2n, \mathbb{C})$. We fix its Borel subgroup B and a maximal torus $T \subset B$. Let R be the root system of (G, T). We embed R into a n-dimensional Euclid space $\oplus_i \mathbb{C}\epsilon_i$ as $R = \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\}$. We define $V_1 := \mathbb{C}^{2n}$ and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. We put $\mathbb{V}_\ell := V_1^{\oplus \ell} \oplus V_2$ and call it *the ℓ-exotic representation*. Let $\mathbb{V}_\ell^+$ be the positive part of $\mathbb{V}_\ell$ (for precise definition, see §1). We define

$$F_\ell := G \times^B \mathbb{V}_\ell^+ \subset G \times^B \mathbb{V}_\ell \cong G/B \times \mathbb{V}_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow \mathbb{V}_\ell.$$

We denote the image of μ_ℓ by $\mathfrak{N}_\ell$. This is the G-variety which we refer as *the ℓ-exotic nilpotent cone*. We put $Z_\ell := F_\ell \times_{\mathfrak{N}_\ell} F_\ell$. Let $G_\ell := G \times (\mathbb{C}^\times)^{\ell+1}$. We have a natural G_ℓ-action on F_ℓ (and Z_ℓ). (In fact, the variety F_ℓ admits an action of $G \times GL(\ell, \mathbb{C}) \times \mathbb{C}^\times$. We use only a restricted action in this paper.)

Assume that $\mathbb{H}$ is the Hecke algebra with unequal parameters of type $C_n^{(1)}$ (c.f. Definition 2.1). This algebra has three parameters q_0, q_1, q_2. All affine Hecke algebras of classical type with two parameters are obtained from $\mathbb{H}$ by suitable specializations of parameters (c.f. Remark 2.2).

Theorem A (= Theorem 2.8)**.** *We have an isomorphism*

$$\mathbb{H} \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} K^{G_2}(Z_2)$$

as algebras.

The Ginzburg theory suggests the classification of simple $\mathbb{H}$-modules by the G-conjugacy classes of the following Langlands parameters:

Definition B (Langlands parameters)**.**

1) A triple $\vec{q} := (q_0, q_1, q_2) \in (\mathbb{C}^\times)^3$ is said to be admissible if $q_0 \neq q_1$, q_2 is not a root of unity of order $\leq 2n$, $q_0 q_1^{\pm 1} \neq q_2^m$ for $|m| < n$;

2) A pair $(a, X) = (s, \vec{q}, X_0 \oplus X_1 \oplus X_2) \in G_2 \times \mathfrak{N}_2$ is called an admissible parameter iff s is semisimple, $\vec{q}$ is admissible, and $sX_i = q_i X_i$ for $i = 0, 1, 2$.

For an admissible parameter (a, X), we put $G_2(a) := Z_{G_2}(a)$.

Notice that our Langlands parameters do not have additional data as in the usual Deligne-Langlands-Lusztig correspondence. This is because the (equivariant) fundamental groups of orbits are always trivial (c.f. Theorem 4.7). Instead, we have the following kind of difficulty:

Example C (Non-regular parameters). Let $G = Sp(4, \mathbb{C})$ and let $a = (\exp(r\epsilon_1 + (r + \pi\sqrt{-1})\epsilon_2), e^r, -e^r, -e^{2r}) \in T \times (\mathbb{C}^\times)^3$ $(r \in \mathbb{C} \backslash \pi\sqrt{-1}\mathbb{Q})$. Then, the number of $G_2(a)$-orbits in $\mathfrak{N}_2^a$ is eight, while the number of corresponding representations of $\mathbb{H}$ is six. (c.f. Enomoto [En06]) In fact, there are two non-regular admissible parameters in this case. These parameters correspond to weight vectors of $\epsilon_1 + \epsilon_2$ or "ϵ_1 & ϵ_2".

Now we state the main theorem of this paper:

Theorem D (= Theorem 10.2)**.** *The set of G-conjugacy classes of admissible parameters is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}$-modules if q_2 is not a root of unity of order $\leq 2n$, and $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ holds for every $0 \leq m < n$.*

We treat a slightly more general case in Theorem 10.1 including Example C. Since the general condition is rather technical, we state only a part of it here.

By imposing an additional relation $q_0 + q_1 = 0$, the algebra $\mathbb{H}$ specializes to an extended Hecke algebra $\mathbb{H}_B$ of type $B_n^{(1)}$ with two-parameters. (c.f. Remark 2.2.) Therefore, Theorems D also gives a definitive classification of simple $\mathbb{H}_B$-modules except for $-q_0^2 = q_2^m$ $(|m| < n)$ or q_2 is a root of order $\leq n$.

Let us illustrate an example which (partly) explains the title "exotic":

Example E (Equal parameter case). Let $G = Sp(4, \mathbb{C})$. Let $s = \exp(r\epsilon_1 + r\epsilon_2) \in T$ $(r \in \mathbb{C} \backslash \pi\sqrt{-1}\mathbb{Q})$. Fix $a_0 = (s, e^r) \in G \times \mathbb{C}^\times$ and $a = (s, e^r, -e^r, e^{2r}) \in G_2$. Let $\mathcal{N}$ be the nilpotent cone of G. Then, the sets of $G(s)$-orbits of $\mathcal{N}^{a_0}$ and $\mathfrak{N}_2^a$ are responsible for the usual and our exotic Deligne-Langlands correspondences. The number of $G(s)$-orbits in $\mathcal{N}^{a_0}$ is three. (Corresponding to root vectors of $\emptyset$, $2\epsilon_1$, and "$2\epsilon_1$ & $2\epsilon_2$") The number of $G(s)$-orbits in $\mathfrak{N}_2^{\mathbf{a}}$ is four. (Corresponding to weight vectors of $\emptyset$, ϵ_1, $\epsilon_1 + \epsilon_2$, and "ϵ_1 & $\epsilon_1 + \epsilon_2$") On the other hand, the actual number of simple modules arising in this way is four (c.f. Ram [Ra01] and [En06]).

The organization of this paper is as follows:

In §1, we fix notation and introduce exotic nilcones and related varieties. In particular, we present the geometric structure involved in our varieties as much as we need in the later section. In §2, we prove Theorem A, which connects our varieties with an affine Hecke algebra $\mathbb{H}$. In order to simplify the study of representation theory of $\mathbb{H}$, we divide our varieties into a product of primitive ones in §3. In §4, we prove that the stabilizers of exotic nilpotent orbits are connected, which implies that the "Lusztig" part of the Deligne-Langlands-Lusztig parameter should be always trivial in our situation. Unfortunately, we have no nice parabolic subgroup as Kazhdan-Lusztig employed in [KL87]. We construct some explicit semisimple element out of each orbit in §5 for the sake of compensation. We introduce the notion of exotic Springer fibers and prove its odd-term vanishing result in §6, under the assumption that the parameters are sufficiently nice (admissible). Its proof essentially relies on the argument of §5. We define our standard modules as the total homology group of an exotic

Springer fibers in §7. At the same time, we present an induction theorem, which claims that they behave well under the induction. In §8, we present an analogue of Springer correspondence for our exotic nilcones. In order to prove Theorem D, we still need two additional structural results. One is that our geometric structure is preserved by replacing the infinitesimal character by a suitable real positive one. The other is that we can embed the corresponding finite Weyl group into the graded version of $\mathbb{H}$. Both are false in non-admissible parameter range. These results occupy §9. With the knowledge of all of the previous sections except for §7, we prove Theorem D in §10. The last section §11 concerns with applications, which are straight-forward consequences of Ginzburg theory assuming the results presented in earlier sections.

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1 Preparatory materials

Let $G := Sp(2n, \mathbb{C})$. Let B be a Borel subgroup of G. Let T be a maximal torus of B. Let $X^*(T)$ be the character group of T. Let R be the root system of (G, T) and let R^+ be its positive part defined by B. We embed R and R^+ into a n-dimensional Euclid space $\mathbb{E} = \oplus_i \mathbb{C}\epsilon_i$ with standard inner product as:

$$R^+ = \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\} \subset \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\} = R \subset \mathbb{E}.$$

By the inner product, we identify ϵ_i with its dual basis. We put $\epsilon_i := -\epsilon_{-i}$ when $i < 0$. We put $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(i = 1, \ldots, n-1)$ or $2\epsilon_n$ $(i = n)$. Let W be the Weyl group of (G, T). For each α_i, we denote the reflection of $\mathbb{E}$ corresponding to α_i by s_i. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function with respect to (B, T). We denote by $\dot{w} \in N_G(T)$ a lift of $w \in W$. For a subgroup $H \subset G$ containing T, we put ${}^wH := \dot{w}H\dot{w}^{-1}$. For a group H and its element h, we put $H(h) := Z_H(h)$. We denote the identity component of H by H°. If H acts on a space $\mathcal{X}$ and $x \in \mathcal{X}$, then we denote the stabilizer of the H-action on x by $\mathsf{Stab}_H x$. For each $\alpha \in R$, we denote the corresponding one-parameter unipotent subgroup of G (with respect to T) by U_α. We define $\mathfrak{g}, \mathfrak{t}, \mathfrak{u}_\alpha, \mathfrak{g}(s)$, etc... to be the Lie algebras of $G, T, U_\alpha, G(s)$, etc..., respectively.

[1]**Note:** After the original version of this paper is circulated (in 2006, with different argument and weaker conclusion in Theorem D), there appeared two kind of related works. One is the study of geometry which is connected to our nilcone by Achar-Henderson [AH07], Finkelberg-Ginzburg-Travkin [FGT08], Springer [Sp07], Travkin [Tr08], and the other is the classification of tempered dual by Opdam and Solleveld [OS07, So07]. For the former, I have included explanation about the situation as much as I could in order to avoid potential problems. For the latter, I think it is a very interesting to connect our result with theirs. I hope to make some contribution to this direction in my future work.

For a T-module V, we define its weight λ-part (with respect to T) as $V[\lambda]$. We define the positive part V^+ and negative part V^- of V as

$$V^+ := \bigoplus_{\lambda \in \mathbb{Q}_{\geq 0} R^+ - \{0\}} V[\lambda], \text{ and } V^- := \bigoplus_{\lambda \in \mathbb{Q}_{\leq 0} R^+ - \{0\}} V[\lambda],$$

respectively. We denote the set of T-weights of V by $\Psi(V)$.

In this paper, a segment is a set of integers I written as $I = [i_1, i_2] \cap \mathbb{Z}$ for some integers $i_1 \leq i_2$. By abuse of notation, we may denote I by $[i_1, i_2]$. For a segment I, we set $I^* := I$ (if $0 \notin I$) or $I - \{0\}$ (if $0 \in I$).

We denote the absolute value function by $| \bullet | : \mathbb{C} \to \mathbb{R}_{\geq 0}$.

We set $\Gamma_0 := 2\pi\sqrt{-1}\mathbb{Z} \subset \mathbb{C}$ and set $\exp : \mathbb{E} \to T$ to be the exponential map. We normalize the map exp so that $\ker \exp \cong \sum_{i=1}^n \Gamma_0 \epsilon_i$.

For a variety $\mathcal{X}$, we denote by $H_\bullet(\mathcal{X})$ the Borel-Moore homology groups with coefficients $\mathbb{C}$.

1.1 Exotic nilpotent cones

Let $\ell = 0, 1$, or 2 be an integer. We define $V_1 := \mathbb{C}^{2n}$ (vector representation) and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. These representations have B-highest weights ϵ_1 and $\epsilon_1 + \epsilon_2$, respectively. We put $\mathbb{V}_\ell := V_1^{\oplus \ell} \oplus V_2$ and call it *the ℓ-exotic representation of $Sp(2n)$*. For $\ell \geq 1$, the set of non-zero weights of $\mathbb{V}_\ell$ is in one-to-one correspondence with R as

$$R \ni \begin{cases} \pm 2\epsilon_i \leftrightarrow \pm \epsilon_i & \in \Psi(V_1) \\ \pm\epsilon_i \pm \epsilon_j \leftrightarrow \pm\epsilon_i \pm \epsilon_j & \in \Psi(V_2) \end{cases}. \tag{1.1}$$

We define

$$F_\ell := G \times^B \mathbb{V}_\ell^+ \subset G \times^B \mathbb{V}_\ell \cong G/B \times \mathbb{V}_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow \mathbb{V}_\ell.$$

We denote the image of μ_ℓ by $\mathfrak{N}_\ell$. We call this variety *the ℓ-exotic nilpotent cone*. By abuse of notation, we may denote the map $F_\ell \to \mathfrak{N}_\ell$ also by μ_ℓ.

Convention 1.1. For the sake of simplicity, we define objects F, $\mathfrak{N}$, $\mathbb{V}$, μ, etc... to be the objects F_ℓ, $\mathfrak{N}_\ell$, $\mathbb{V}_\ell$, μ_ℓ etc... with $\ell = 1$.

We summarize some basic geometric properties of $\mathfrak{N}_\ell$:

Theorem 1.2 (Geometric properties of $\mathfrak{N}_\ell$)**.** *We have the following:*

1. *The defining ideal of $\mathfrak{N}_\ell$ is generated by G-invariant polynomials of $\mathbb{C}[\mathbb{V}_\ell]$ without constant terms;*
2. *The variety $\mathfrak{N}_\ell$ is normal;*
3. *For $\ell = 1, 2$, the map μ_ℓ is a birational projective morphism onto $\mathfrak{N}_\ell$;*
4. *Every fiber of the map μ_ℓ is connected;*

In the below, we present properties which is valid only for the $\ell = 1$ case.

5. *The set of G-orbits in $\mathfrak{N}_1$ is finite;*
6. *The map μ_1 is stratified semi-small with respect to the stratification of $\mathfrak{N}_1$ given by G-orbits.*

Proof. The weight distribution of $\mathbb{V}^+$ and the Hesselink theory (c.f. [Po04] Theorem 1) claims that μ_ℓ gives a birational projective morphism onto an irreducible component of the Hilbert nilcone of $\mathbb{V}_\ell$. Here the Hilbert nilcone of $\mathbb{V}_\ell$ is irreducible normal variety by Schwarz [Sc78], which implies that $\mathfrak{N}_\ell \subset \mathbb{V}_\ell$ is the Hilbert nilcone. Therefore, we obtain 1–3). 4) is an immediate consequence of 2), 3), and the Zariski main theorem (c.f. [CG97] 3.3.26). 5) is proved as a part of Proposition 1.15. We show 6). Let $\widehat{\mathbb{O}}$ be the inverse image of a G-orbit $G.X = \mathbb{O} \subset \mathfrak{N}$ under the map $\mu \circ p_2$. Then, we have

$$\dim \mathbb{O} + 2 \dim \mu^{-1}(X) \le \dim \widehat{\mathbb{O}}.$$

The dimension of the RHS is less than or equal to $\dim F$, which is the (constant) dimension of irreducible components of Z. In particular, we have

$$\dim \mathbb{O} + 2 \dim \mu^{-1}(X) \le \dim \mathfrak{N} = \dim F,$$

which implies that μ is semi-small. □

Lemma 1.3. *We have a natural identification*

$$F_\ell \cong \{(gB, X) \in G/B \times \mathbb{V}_\ell ; X \in g\mathbb{V}_\ell^+\}.$$

Proof. Straightforward. □

Let $G_\ell := G \times (\mathbb{C}^\times)^{\ell+1}$. We define a G_ℓ-action on $\mathfrak{N}_\ell$ as

$$G_\ell \times \mathfrak{N}_\ell \ni (g, q_{2-\ell}, \ldots, q_2) \times (X_{2-\ell} \oplus \cdots \oplus X_2) \mapsto (q_{2-\ell}^{-1} g X_{2-\ell} \oplus \cdots \oplus q_2^{-1} g X_2) \in \mathfrak{N}_\ell.$$

(Here we always regard $X_{2-\ell}, \ldots, X_1 \in V_1$ and $X_2 \in V_2$.) Similarly, we have a natural G_ℓ-action on F_ℓ which makes μ_ℓ a G_ℓ-equivariant map. We define $Z_\ell := F_\ell \times_{\mathfrak{N}_\ell} F_\ell$. By Lemma 1.3, we have

$$Z_\ell := \{(g_1 B, g_2 B, X) \in (G/B)^2 \times \mathbb{V}_\ell ; X \in g_1 \mathbb{V}_\ell^+ \cap g_2 \mathbb{V}_\ell^+\}.$$

We put

$$Z_\ell^{123} := \{(g_1 B, g_2 B, g_3 B, X) \in (G/B)^3 \times \mathbb{V}_\ell ; X \in g_1 \mathbb{V}_\ell^+ \cap g_2 \mathbb{V}_\ell^+ \cap g_3 \mathbb{V}_\ell^+\}.$$

We define $p_i : Z_\ell \ni (g_1 B, g_2 B, X) \mapsto (g_i B, X) \in F_\ell$ and $p_{ij} : Z_\ell^{123} \ni (g_1 B, g_2 B, g_3 B, X) \mapsto (g_i B, g_j B, X) \in Z_\ell$ $(i, j \in \{1, 2, 3\})$. We also put $\tilde{p}_i : F_\ell \times F_\ell \to F_\ell$ as the first and second projections $(i = 1, 2)$. (Notice that the meaning of $p_i, \tilde{p}_i, p_{ij}$ depends on ℓ. The author hopes that there occurs no confusion on it.)

Lemma 1.4. *The maps p_i and p_{ij} $(1 \le i < j \le 3)$ are projective.*

Proof. The fibers of the above maps are given as the subsets of G/B defined by incidence relations. It is automatically closed and we obtain the result. □

We have a projection

$$\pi_\ell : Z_\ell \ni (g_1 B, g_2 B, X) \mapsto (g_1 B, g_2 B) \in G/B \times G/B.$$

For each $w \in W$, we define a point $\mathsf{p}_w := B \times \dot{w} B \in G/B \times G/B$. This point is independent of the choice of $\dot{w}$. We put $\mathsf{O}_w := G\mathsf{p}_w \subset G/B \times G/B$. By the Bruhat decomposition, we have

$$G/B \times G/B = \bigsqcup_{w \in W} \mathsf{O}_w. \tag{1.2}$$

Lemma 1.5. *The variety Z_ℓ consists of $|W|$-irreducible components. Moreover, all of the irreducible components of Z have the same dimension.*

Proof. We first prove the assertion for $Z = Z_1$. By (1.2), the structure of Z is determined by the fibers over p_w. We have

$$\pi^{-1}(\mathsf{p}_w) = \mathbb{V}^+ \cap \dot{w}\mathbb{V}^+.$$

By the dimension counting using (1.1), we deduce

$$\begin{aligned}\dim \mathbb{V}^+ \cap \dot{w}\mathbb{V}^+ =& \dim V_1^+ \cap \dot{w} V_1^+ + \dim V_2^+ \cap \dot{w} V_2^+ \\ =& \#(R_l^+ \cap w R_l^+) + \#(R_s^+ \cap w R_s^+) = N - \ell(w),\end{aligned}$$

where $N := \dim \mathbb{V}^+ = \dim G/B$ and R_l^+, R_s^+ are the sets of long and short positive roots, respectively. As a consequence, we deduce

$$\dim \pi^{-1}(\mathsf{O}_w) = N + \ell(w) + N - \ell(w) = 2N.$$

Thus, each $\overline{\pi^{-1}(\mathsf{O}_w)}$ is an irreducible component of Z.
Next, we prove the assertion for Z_2. By forgetting the first V_1-factor, we have a surjective map $\eta : Z_2 \to Z$. We have a surjective map $\eta' : Z \to Z_0$ given by forgetting the V_1-factor. The fiber of η at $x \in Z$ is isomorphic to the two-fold product of the fiber of η' at $\eta'(x)$. The latter fiber is isomorphic to the vector space $V_1^+ \cap g V_1^+$ when $\pi(x) = (1, g)\mathsf{p}_1$. Therefore, the preimage of each irreducible component of Z gives an irreducible component of Z_2. These irreducible components are distinct since their images under η must be distinct. Hence, the number of irreducible components of Z_2 is equal to the number of irreducible components of Z as desired. $\square$

By a general result of [Gi97] p135 (c.f. [CG97] 2.7), the G_ℓ-equivariant K-group of Z_ℓ becomes an associative algebra via the map

$$\star : K^{G_\ell}(Z_\ell) \times K^{G_\ell}(Z_\ell) \ni ([\mathcal{E}], [\mathcal{F}]) \mapsto \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (p_{13})_* (p_{12}^* \mathcal{E} \otimes^{\mathbb{L}} p_{23}^* \mathcal{F})] \in K^{G_\ell}(Z_\ell).$$

Moreover, the G_ℓ-equivariant K-group of F_ℓ becomes a representation of $K^{G_\ell}(Z_\ell)$ as

$$\circ : K^{G_\ell}(Z_\ell) \times K^{G_\ell}(F_\ell) \ni ([\mathcal{E}], [\mathcal{K}]) \mapsto \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (p_1)_* (\mathcal{E} \otimes^{\mathbb{L}} \tilde{p}_2^* \mathcal{K})] \in K^{G_\ell}(F_\ell).$$

Here we regard $\mathcal{E}$ as a sheaf over $F_\ell \times F_\ell$ via the natural embedding $Z_\ell \subset F_\ell \times F_\ell$.

1.2 Definition of parameters

In this subsection, we present the definitions of parameters which we need in the sequel. First, we put $a_0 := (1, 1, -1, 1) \in G_2$.

Definition 1.6 (Configuration of semisimple elements)**.**

1) An element $a = (s, q_0, q_1, q_2) \in G_2$ is called pre-admissible iff s is semisimple, $q_0 \neq q_1$, q_2 is not a root of unity of order $\leq 2n$.

2) An element $a \in G_2$ is called finite if $\mathfrak{N}_2^a$ has only finitely many $G_2(a)$-orbit.

3) A pre-admissible element $a = (s, q_0, q_1, q_2)$ is called admissible if $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ holds for every $0 \leq m < n$.

For a pre-admissible element $a = (s, q_0, q_1, q_2)$, we put

$$\mathbb{V}_2^a = V_1^{(s,q_0)} \oplus V_1^{(s,q_1)} \oplus V_2^{(s,q_2)} \subset V_1 \oplus V_1 \oplus V_2 = \mathbb{V}_2.$$

In the below, we may denote $(q_0, q_1, q_2) \in (\mathbb{C}^\times)^3$ by $\vec{q}$ for the sake of simplicity.

Let $a = (s, \vec{q}) \in G_2$ be a pre-admissible element such that $s \in T$. We sometimes denote it as

$$s = \exp\left(\sum_{i=1}^{n} \log_i(s)\epsilon_i\right) \in \exp(\mathbb{E}) \cong T,$$

where $\log_i(s) \in \mathbb{C}$.

Remark 1.7. The values of $\log_i(s)$ are determined modulo Γ_0. Here we understand that $\log_i(s)$ is a fixed choice of a representative in $\log_i(s) + \Gamma_0$.

Definition 1.8 (Admissible parameters)**.**

1) A pre-admissible parameter is a pair

$$\nu = (a, X) = (s, \vec{q}, X_1 \oplus X_2) \in G_2 \times \mathfrak{N}_1$$

such that a is pre-admissible, $(s - q_0)(s - q_1)X_1 = 0$, and $sX_2 = q_2 X_2$;

For a pre-admissible $a \in G_2$, we denote by Λ_a the set of $G(s)$-conjugacy classes of pre-admissible parameters of the form (a, Y), where $Y \in \mathbb{V}$.

2) A pre-admissible parameter $\nu = (a, X)$ is called admissible if a is admissible.

1.3 Orbit structures arising from $\mathfrak{N}_\ell$

In the following, we fix vectors in V_1 and V_2 as follows:

- For each $i \in [-n, n]^*$, we define $0 \neq x_i, \bar{x}_i \in V_1$ to be non-zero vectors of weights ϵ_i;
- For each distinct $i, j \in [-n, n]^*$, we define $y_{ij} \in V_2$ to be a non-zero vector of weight $\epsilon_i - \epsilon_j$.

The following is a slight enhancement of the good basis of Ohta [Oh86] (1.3).

Definition 1.9 (Signed partitions)**.** Let $J = (J^1, J^2, J^3, \ldots)$ be a sequence of elements of $[-n, n]^*$. Let $\mathbf{J} := \{J_1, J_2, \ldots\}$ be a collection of sequence of elements of $[-n, n]^*$. (I.e. each J_k is of the form $(J_k^1, J_k^2, \ldots)$.) It is called a signed partition of n if and only if the following condition hold:

- We have $[1, n] = \bigsqcup_{k \ge 1} \{|j|; j \in J_k\}$.

For each member J of a signed partition $\mathbf{J}$, we define

$$T_J := \exp \sum_{i \in J} \mathbb{C}\epsilon_i \subset T.$$

Let $\lambda := (\lambda_1 \ge \lambda_2 \ge \cdots)$ be a partition of n. Then, we regard it as a signed partition by setting

$$J_j^i := i + \sum_{k=1}^{j-1} \lambda_k \text{ if } \lambda_j \neq 0 \text{ and } 1 \le i \le \lambda_j.$$

Definition 1.10 (Foot functions)**.** Let $\ell = 0, 1$, or 2. A collection of ℓ-tuple of functions $\delta_k : [-n, n]^* \to \{0, 1\}$ for $1 \le k \le \ell$ is called a ℓ-foot function of n. We denote a ℓ-foot function $\{\delta_k\}_{k=1}^{\ell}$ as $\vec{\delta}$.

Notice that Definition 1.10 claims that $\vec{\delta} \equiv \emptyset$ when $\ell = 0$.

Definition 1.11 (Marked partitions, blocks and normal forms)**.** Let ℓ be as in Definition 1.10. We refer a pair $\sigma = (\mathbf{J}, \vec{\delta})$ consisting of a signed partition and a ℓ-foot function of n as a ℓ-marked partition if the following condition hold:

- For each $J \in \mathbf{J}$ and $m = 1, \ldots, \ell$, we have

$$\#\{j \in J; \delta_m(j) = 1\} + \#\{j \in J; \delta_m(-j) = 1\} \le 1.$$

For each $J \in \mathbf{J}$, we define the ℓ-block $\mathbf{v}_\sigma^J = \mathbf{v}_{\sigma,1}^J + \mathbf{v}_{\sigma,2}^J \in \mathbb{V}$ associated to (σ, J) as:

$$\mathbf{v}_{\sigma,1}^J := \sum_{j \in J} \sum_{k=1}^{\ell} (\delta_k(j) x_j + \delta_k(-j) x_{-j}) \in V_1$$
$$\mathbf{v}_{\sigma,2}^J := \sum_{j \ge 1} y_{J_j, J_{j+1}} \in V_2,$$

where we regard $y_{J_j^k, J_{j+1}^k} \equiv 0$ whenever J^k nor J_{j+1}^k is non-existent.
A ℓ-normal form $\mathbf{v}_\sigma = \mathbf{v}_{\sigma,1} + \mathbf{v}_{\sigma,2} \in \mathbb{V}$ associated to σ is defined as:

$$\mathbf{v}_{\sigma,1} := \sum_{J \in \mathbf{J}} \mathbf{v}_{\sigma,1}^J \in V_1, \text{ and } \mathbf{v}_{\sigma,2} := \sum_{J \in \mathbf{J}} \mathbf{v}_{\sigma,2}^J \in V_2.$$

Definition 1.12 (Strict normal forms)**.** A ℓ-marked partition $\sigma = (\mathbf{J}, \vec{\delta})$ is called strict if and only if the following conditions hold:

1. $\mathbf{J}$ is obtained from a partition λ of n;
2. We have $\delta_2 \equiv 0$ and $\delta_1(j) = 0$ for every $j \in [-n, -1]$;

Assume the above two conditions. If we have $\delta_1(j) = 1$ for $j \in J$, then we set $\underline{\#}J := \#\{j' \in J; j' \le j\}$ and $\overline{\#}J := \#\{j' \in J; j' > j\}$.

3. Let $k < m$ be two integers and let $\mathbf{J} = \{J_1, J_2, \ldots\}$. Then, we have $\delta_1|_{J_m} \equiv 0$ if $\#J_k = \#J_m$;

4. Let $J, J' \in \mathbf{J}$ be a pair such that $\delta_1(j) = 1 = \delta_1(j')$ for some $j \in J$ and $j' \in J'$. If $\#J > \#J'$, then we have

$$\underline{\#}J > \underline{\#}J' \text{ and } \overline{\#}J > \overline{\#}J'.$$

Conditions are not applicable when δ_2 or δ_1 are non-existent. Notice that only the first condition survives when $\ell = 0$. A normal form attached to a strict marked partition is called a strict normal form.

In the below, we refer foot funtions, blocks, normal forms..., to be the 1-foot funtions, 1-blocks, 1-normal forms..., respectively. Moreover, we naturally identify strict 1-normal forms and strict 2-normal forms since $\delta_2 \equiv 0$ for 2-strict marked partitions.

Theorem 1.13 (Orbit description of $\mathfrak{N}_1$)**.** *We have the following:*

1. *The set of strict* 1*-normal forms is in one-to-one correspondence with the the set of G-orbits of $\mathfrak{N}_1$;*

2. *We have $\#(G\backslash\mathfrak{N}_1) = \#\mathsf{Irrep}W$, where $\mathsf{Irrep}W$ is the set of isomorphism classes of irreducible W-modules;*

3. *For each $X \in \mathfrak{N}_1$, the group $\mathsf{Stab}_G X$ is connected.*

Remark 1.14. The original form of the proof of Theorem 1.13 (in [Ka06b]) employs explicit calculation using basis. In the meantime, Springer [Sp07] gives a base-free proof (with stronger consequences). The proof given here is somewhat the mixture of the both, which the author give it basically for the sake of completeness. Note that the closure relation of the orbit structure of $\mathfrak{N}_1$ is calculated by Achar-Henderson [AH07].

The proof of Theorem 1.13 is obtained as a combinations of Proposition 4.4 and Theorem 8.3 by using the knowledge of the following:

Proposition 1.15 (Weak version of Theorem 1.13)**.** *We have the following:*

1. *Each G-orbit of $\mathfrak{N}_1$ contains a strict normal form;*

2. *The number of elements of the set of strict marked partitions is less than or equal to $\#\mathsf{Irrep}W$, where $\mathsf{Irrep}W$ is the set of isomorphism classes of irreducible W-modules.*

Proof. By a result of Ohta-Sekiguchi [Se84, Oh86], the set of strict 0-marked partitions are in one-to-one correspondence with the set of G-orbits of $\mathfrak{N}_0$ via the assignment $\sigma \mapsto G\mathbf{v}_\sigma$. We have

$$\mathbb{C}[\mathbb{V}]^G \cap \mathbb{C}[\mathbb{V}_0] = \mathbb{C}[\mathbb{V}_0]^G,$$

which gives the natural projection map

$$\mathfrak{N}_1 \longrightarrow \mathfrak{N}_0$$

obtained from the natural projection $\mathbb{V}_1 \to \mathbb{V}_0$. (In fact we have $\mathbb{C}[\mathbb{V}]^G = \mathbb{C}[\mathbb{V}_0]^G$. But this fact is not used here.) It follows that each orbit of $\mathfrak{N}_1$ contains a vector of type $v = v_1 \oplus \mathbf{v}_{2,\lambda}$, where λ is a partition of n regarded as a strict 0-marked partition in a natural way.
Consider the action of

$$G' := Sp(2\lambda_1) \times Sp(2\lambda_2) \times \cdots \subset Sp(2n),$$

which are embedded so that $T \subset G'$ and V_1 restricted to G' has the form

$$\mathsf{Res}^G_{G'} V_1 = \bigoplus_{k \geq 1} V_1^{(k)}$$

such that $V_1^{(k)}$ is a vector representation of $Sp(2\lambda_k)$ with T-weights $\pm\epsilon_i$ for $i = 1 + \sum_{j=1}^{k-1} \lambda_j, \ldots, \sum_{j=1}^{k} \lambda_j$.
We decompose $v = \sum_{k \geq 1} v_k$, where $v_k = v_{1,k} \oplus \mathbf{v}_{2,\lambda}^{J_k} \in V_1^k$. We regard $\mathbf{v}_{2,\lambda}^{J_k} \in \wedge^2 V_1^k \cong \mathrm{Alt}(V_1^k)$ as a linear endomorphism on V_1^k which preserves the symplectic form on V_1^k compatible with $Sp(2\lambda_k)$. Ohta-Sekigushi result asserts that such identification gives an identification of $Sp(2\lambda_k)\mathbf{v}_{2,\lambda}^{J_k}$ and the set of linear endomorphism on V_1^k which preserves the symplectic form. Since $v_{1,k}$ can be complemented to a suitable choice of standard basis of symplectic space, we obtain that a suitable change of symplectic basis makes $v_{1,k}$ into one of x_i $(i > 0)$. This implies that v can be transformed into a 1-normal form which satisfies 1.12 1) and 2).
Now we examine the orbit structure of $v_k + v_{k'} \in V_1^k \oplus V_1^{k'}$ under the action of $Sp(2\lambda_k + 2\lambda_{k'})$ for $k < k'$. We have $\lambda_k \geq \lambda_{k'}$ by 1.12 1). We put $\xi := \mathbf{v}_{2,\sigma}^{J_k} + \mathbf{v}_{2,\sigma}^{J_{k'}}$. The $Sp(2\lambda_k + 2\lambda_{k'})$-conjugacy class of ξ is the set of nilpotent endomorphism of $V_1^k \oplus V_1^{k'}$ preserving the natural symplectic form with its Jordan form $(\lambda_k, \lambda_k, \lambda_{k'}, \lambda_{k'})$. If $v_{1,k} = 0$ or $v_{1,k'} = 0$ hold, then 1.12 3) and 4) are satisfied for the pair $(J_k, J_{k'})$. Hence, we assume $v_{1,k} \neq 0 \neq v_{1,k'}$ in the below.
We have

$$\xi^{\underline{\#} J_k} v_{1,k} = 0, \xi^{\underline{\#} J_k - 1} v_{1,k} \neq 0, \text{ and } \xi^{\underline{\#} J_{k'}} v_{1,k'} = 0, \xi^{\underline{\#} J_{k'} - 1} v_{1,k'} \neq 0$$
$$v_{1,k} \in \mathrm{Im}\xi^{\overline{\#} J_k}, v_{1,k} \notin \mathrm{Im}\xi^{\overline{\#} J_k + 1}, \text{ and } v_{1,k'} \in \mathrm{Im}\xi^{\overline{\#} J_{k'}}, v_{1,k'} \notin \mathrm{Im}\xi^{\overline{\#} J_{k'} + 1}.$$

If $\underline{\#} J_k \leq \underline{\#} J_{k'}$ or $\overline{\#} J_k \leq \overline{\#} J_{k'}$ holds, then we can regard $v_{1,k} + v_{1,k'}$ as a part of the standard basis of $V_1^{k'}$ or V_1^k by suitable coordinate changes, respectively. When $\lambda_k = \lambda_{k'}$, we use this to acheive 1.12 3). When $\lambda_k > \lambda_{k'}$, we use this to transform our normal form into another normal form which satisfies 1.12 4) for the pair $(J_k, J_{k'})$. Repeating these procedure completes the proof of the first assertion.
For the second assertion, recall that $\mathsf{Irrep} W$ is parametrised by the set of ordered pair of partitions (λ^1, λ^2) which sum up to n. We define two-partitions out of a strict marked partition σ as

$$\lambda_k^1 + \lambda_k^2 = \lambda_k, \text{ and } \lambda_k^2 := \begin{cases} \underline{\#} J_k & (\mathbf{v}_{1,\sigma}^{J_k} \neq 0) \\ \max\{\underline{\#} J_{k'}, \lambda_k - \overline{\#} J_{k''}; k' > k > k''\} & (otherwise) \end{cases}$$

for each k, where the set we choose its maximal is formed only from these $J_{k'}$ and $J_{k''}$ for which $\overline{\#}$ and $\underline{\#}$ are defined. It is clear that two sequences λ^1, λ^2

sum up to n. By 1.12 4), we deduce that

$$\lambda_k - \overline{\#} J_{k''} < \underline{\#} J_k \text{ (this is equivalent to } \overline{\#} J_{k''} > \overline{\#} J_k)$$

holds for $k'' < k$ (such that $\overline{\#}$ and $\underline{\#}$ are defined for both J_k and $J_{k''}$). It follows that λ^2 is a partition. (I.e. $\{\lambda^2_i\}_i$ is a decreasing sequence.) By the symmetry of $\overline{\#}$ and $\underline{\#}$ in 1.12, we conclude that λ^1 is also a partition.
Therefore, it suffices to prove that the pairs of partitions formed by strict marked partitions are equal only if the marked partitions are equal. (Since this gives the injectivity of the above assignment.) For this, we assume that two strict marked partitions $\sigma = (\mathbf{J}, \delta_1)$ and $\sigma' = (\mathbf{J}', \delta'_1)$ gives the same pair (λ^1, λ^2) to deduce contradiction. We can assume that $\mathbf{J} = \mathbf{J}'$ since $\lambda = \lambda^1 + \lambda^2$. Hence, their difference is concentrated in their foot function. By 1.12 3) and 4), we deduce that the foot functions are non-trivial on $J_k = J_{k'}$ if and only if

$$\lambda^2_k \neq \max\{\lambda^2_j, \lambda^2_k - \lambda^1_i; \lambda_j \neq \lambda_k \neq \lambda_i, i < k < j\}$$

and $\lambda_k \neq \lambda_{k-1}$. Moreover, the foot functions is determined by the value of λ^2_k if it is non-trivial. Since this system has a unique solution, we deduce $\sigma = \sigma'$, which is contradiction. Thus, the pair of partitions recovers a strict marked partition uniquely, which completes the proof of the second assertion. □

Theorem 1.16 (Orbit structure of $\mathfrak{N}^a_2$). *Let $\nu = (a, X) = (s, \vec{q}, X)$ be an admissible parameter. Then, there exists $g \in G$ such that:*

$$gsg^{-1} \in T \textit{ and } gX \textit{ is a normal form.}$$

Proof. Postponed to §4. □

We have a natural W-action $\cdot$ on $[-n, n]^*$ by setting

$$s_i \cdot j := \begin{cases} -(i+1) & (j = \pm i) \\ -(i-1) & (j = \pm(i+1)) \\ j & (\textit{otherwise}) \end{cases} \text{ for } i = 1, \ldots, n-1, \text{ and } s_n \cdot j = \begin{cases} -j & (j = \pm n) \\ j & (\textit{otherwise}) \end{cases}.$$

Using this, we define the W-action $\cdot$ on the set of ℓ-marked partitions as:
For $w \in W$ and $\sigma = (\mathbf{J}, \vec{\delta}) = (\{J_1, J_2, \ldots, \}, \{\delta_1, \ldots, \delta_\ell\})$, we set

$$w \cdot \sigma := (\{w \cdot J_1, w \cdot J_2, \ldots, \}, \{w \cdot \delta_1, \ldots, w \cdot \delta_\ell\}),$$

where we set

$$w \cdot (J^1_1, J^2_1, \ldots) = ((w \cdot J)^1_1, (w \cdot J)^2_1, \ldots) := (w \cdot J^1_1, w \cdot J^1_1, \ldots)$$

and $w \cdot \delta_k(j) := \delta_k(w \cdot j)$. Notice that we have $w \cdot J_k = (w \cdot J)_k$ and $w \cdot J^j_k = (w \cdot J)^j_k$ for every k, j in this action.

Lemma 1.17. *Let $\sigma = (\mathbf{J}, \vec{\delta})$ is a marked partition which is a W-translation of a strict marked partition. Then, we have*

$$\mathbb{C}^\times \mathbf{v}_{1,\sigma} \oplus \mathbb{C}^\times \mathbf{v}_{2,\sigma} \subset T\mathbf{v}_\sigma, \textit{ and } \mathbb{C}^\times \mathbf{v}^J_{1,\sigma} \oplus \mathbb{C}^\times \mathbf{v}^J_{2,\sigma} \subset T_J \mathbf{v}^J_\sigma \textit{ for each } J \in \mathbf{J}.$$

Proof. Since $T_J \cap T_{J'} = \{1\}$, it suffices to prove the second assertion. Let $\mathbf{v}_\sigma^J = \sum_{\xi \in \Xi} v_\xi$ be the T-eigen-decomposition of $\mathbf{v}_\sigma^J$. Then, we have $\# \Xi = \# J$ and $\dim T_J = \# J$. Moreover, the weights appearing in Ξ are linearly independent. Hence, we have the scalar multiplications of each v_ξ, which implies the result. □

Corollary 1.18 (of the proof of Lemma 1.17)**.** *Let σ be a strict marked partition. Let $w \in W$. Then, we have $\mathbf{v}_{w \cdot \sigma} \in G\mathbf{v}_\sigma$.* □

1.4 Structure of simple modules

We put $T_\ell := T \times (\mathbb{C}^\times)^{\ell+1}$. Let $a \in T_\ell$. Let Z_ℓ^a, F_ℓ^a, and $\mathfrak{N}_\ell^a$ be the set of a-fixed points of Z_ℓ, F_ℓ, and $\mathfrak{N}_\ell$, respectively. Let $\mu^a : F_\ell^a \to \mathfrak{N}_\ell^a$ denote the restriction of μ_ℓ to a-fixed points.

We review the convolution realization of simple modules in our situation. The detailed constructions are found in [CG97] 5.11, 8.4 or [Gi97] §5. For its variant, see [Jo98].

The properties we used to apply the Ginzburg theory are: 1) $Z_\ell = F_\ell \times_{\mathfrak{N}_\ell} F_\ell$; 2) F_ℓ is smooth; 3) μ_ℓ is projective; 4) $R(G_\ell) \subset K^{G_\ell}(Z_\ell)$ is central; and 5) $H_\bullet(Z_\ell)$ is spanned by algebraic cycles.

Let $\mathbb{C}_a$ be the quotient of $\mathbb{C} \otimes_{\mathbb{Z}} R(G_\ell)$ or $\mathbb{C} \otimes_{\mathbb{Z}} R(T_\ell)$ by the ideal defined by the evaluation at a. The Thomason localization theorem yields ring isomorphisms

$$\mathbb{C}_a \otimes_{R(G_\ell)} K^{G_\ell}(Z_\ell) \xrightarrow{\cong} \mathbb{C}_a \otimes_{R(G_\ell(a))} K^{G_\ell(a)}(Z_\ell^a) \xrightarrow{\cong} \mathbb{C}_a \otimes_{R(T_\ell)} K^{T_\ell}(Z_\ell^a).$$

Moreover, we have the Riemann-Roch isomorphism

$$\mathbb{C}_a \otimes_{R(T_\ell)} K^{T_\ell}(Z_\ell^a) \cong K(Z_\ell^a) \xrightarrow{RR} H_\bullet(Z_\ell^a) \cong \mathrm{Ext}^\bullet(\mu_*^a \mathbb{C}_{F_\ell^a}, \mu_*^a \mathbb{C}_{F_\ell^a}).$$

By the equivariant Beilinson-Bernstein-Deligne (-Gabber) decomposition theorem (c.f. Saito [Sa88] 5.4.8.2), we have

$$\mu_*^a \mathbb{C}_{F_\ell^a} \cong \bigoplus_{\mathbb{O} \subset \mathfrak{N}_\ell^a, \chi, d} L_{\mathbb{O},\chi,d} \boxtimes IC(\mathbb{O}, \chi)[d],$$

where $\mathbb{O} \subset \mathfrak{N}_\ell^a$ is a $G(s)$-stable subset such that μ^a is locally trivial along $\mathbb{O}$, χ is an irreducible local system on $\mathbb{O}$, d is an integer, $L_{\mathbb{O},\chi,d}$ is a finite dimensional vector space, and $IC(\mathbb{O}, \chi)$ is the minimal extension of χ. Moreover, the set of $\mathbb{O}$'s such that $L_{\mathbb{O},\chi,d} \neq 0$ (for some χ and d) forms a subset of an algebraic stratification in the sense of [CG97] 3.2.23. It follows that:

Theorem 1.19 (Ginzburg [Gi97] Theorem 5.2)**.** *The set of simple modules of $K^{G_\ell}(Z_\ell)$ for which $R(G_\ell)$ acts as the evaluation at a is in one-to-one correspondence with the set of isomorphism classes of irreducible $G_\ell(a)$-equivariant perverse sheaves appearing in $\mu_*^a \mathbb{C}_{F_\ell^a}$ (up to degree shift).* □

2 Hecke algebras and exotic nilpotent cones

We retain the setting of the previous section. We put $\boldsymbol{G} = G_2$, $\boldsymbol{G} := T_2$, $\boldsymbol{G} := F_2$, $\boldsymbol{\mu} := \mu_2$, $\boldsymbol{Z} := Z_2$, and $\boldsymbol{\pi} := \pi_2$. Most of the arguments in this section are exactly the same as [CG97] 7.6 if we replace $\boldsymbol{G}$ by $G \times \mathbb{C}^\times$, $\mathfrak{N}_2$ by the usual nilpotent cone, $\boldsymbol{\mu}$ by the moment map, $\boldsymbol{F}$ by the cotangent bundle of the flag variety, and $\boldsymbol{Z}$ by the Steinberg variety. Therefore, we frequently omit the detail and make pointers to [CG97] 7.6 in which the reader can obtain a correct proof merely replacing the meaning of symbols as mentioned above.

We put $\mathcal{A}_{\mathbb{Z}} := \mathbb{Z}[\mathbf{q}_0^{\pm 1}, \mathbf{q}_1^{\pm 1}, \mathbf{q}_2^{\pm 1}]$ and $\mathcal{A} := \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}_{\mathbb{Z}} = \mathbb{C}[\mathbf{q}_0^{\pm 1}, \mathbf{q}_1^{\pm 1}, \mathbf{q}_2^{\pm 1}]$.

Definition 2.1 (Hecke algebras of type $C_n^{(1)}$)**.** A Hecke algebra of type $C_n^{(1)}$ with three parameters is an associative algebra $\mathbb{H}$ over $\mathcal{A}$ generated by $\{T_i\}_{i=1}^n$ and $\{e^\lambda\}_{\lambda \in X^*(T)}$ subject to the following relations:

(Toric relations) For each $\lambda, \mu \in X^*(T)$, we have $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ (and $e^0 = 1$);

(The Hecke relations) We have

$$(T_i + 1)(T_i - \mathbf{q}_2) = 0 \ (1 \le i < n) \text{ and } (T_n + 1)(T_n + \mathbf{q}_0\mathbf{q}_1) = 0;$$

(The braid relations) We have

$$\begin{gathered} T_iT_j = T_jT_i \ (\text{if } |i-j| > 1),\ (T_nT_{n-1})^2 = (T_{n-1}T_n)^2, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \ (\text{if } 1 \le i < n-1); \end{gathered}$$

(The Bernstein-Lusztig relations) For each $\lambda \in X^*(T)$, we have

$$T_ie^\lambda - e^{s_i\lambda}T_i = \begin{cases} (1-\mathbf{q}_2)\frac{e^\lambda - e^{s_i\lambda}}{e^{\alpha_i}-1} & (i \neq n) \\ \frac{(1+\mathbf{q}_0\mathbf{q}_1)-(\mathbf{q}_0+\mathbf{q}_1)e^{\epsilon_n}}{e^{\alpha_n}-1}(e^\lambda - e^{s_n\lambda}) & (i = n) \end{cases}.$$

Remark 2.2. **1)** The standard choice of parameters (t_0, t_1, t_n) is: $t_1^2 = \mathbf{q}_2$, $t_n^2 = -\mathbf{q}_0\mathbf{q}_1$, and $t_n(t_0 - t_0^{-1}) = (\mathbf{q}_0 + \mathbf{q}_1)$. This yields

$$T_ne^\lambda - e^{s_n\lambda}T_n = \frac{1 - t_n^2 - t_n(t_0 - t_0^{-1})e^{\epsilon_n}}{e^{2\epsilon_n} - 1}(e^\lambda - e^{s_n\lambda});$$

2) If $n = 1$, then we have $T_1 = T_n$ in Definition 2.1. In this case, we have $\mathbb{H} \cong \mathbb{C}[\mathbf{q}_2^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{H}_0$, where $\mathbb{H}_0$ is the Hecke algebra of type $A_1^{(1)}$ with two-parameters $(\mathbf{q}_0, \mathbf{q}_1)$;

3) An extended Hecke algebra of type $B_n^{(1)}$ with two-parameters considered in [En06] is obtained by requiring $\mathbf{q}_0+\mathbf{q}_1 = 0$. An equal parameter extended Hecke algebra of type $B_n^{(1)}$ is obtained by requiring $\mathbf{q}_0+\mathbf{q}_1 = 0$ and $\mathbf{q}_1^2 = \mathbf{q}_2$. An equal parameter Hecke algebra of type $C_n^{(1)}$ is obtained by requiring $\mathbf{q}_2 = -\mathbf{q}_0\mathbf{q}_1$ and $(1 + \mathbf{q}_0)(1 + \mathbf{q}_1) = 0$.

For each $w \in W$, we define two closed subvarieties of $\boldsymbol{Z}$ as

$$Z_{\le w} := \boldsymbol{\pi}^{-1}(\overline{\mathsf{O}_w}) \text{ and } Z_{<w} := Z_{\le w} \backslash \boldsymbol{\pi}^{-1}(\mathsf{O}_w).$$

Let $\lambda \in X^*(T)$. Let $\mathcal{L}_\lambda$ be the pullback of the line bundle $G \times^B \lambda^{-1}$ over G/B to F_2. Clearly $\mathcal{L}_\lambda$ admits a $\mathbf{G}$-action by letting $(\mathbb{C}^\times)^3$ act on $\mathcal{L}_\lambda$ trivially. We denote the operator $[\tilde{p}_1^*\mathcal{L}_\lambda \otimes^{\mathbb{L}} \bullet]$ by e^λ. By abuse of notation, we may denote $\mathsf{e}^\lambda(1)$ by e^λ (in $K^{\mathbf{G}}(\boldsymbol{Z})$). Let $\mathsf{q}_0 \in R(\{1\} \times \mathbb{C}^\times \times \{1\} \times \{1\}) \subset R(\boldsymbol{G})$, $\mathsf{q}_1 \in R(\{1\}\times\{1\}\times\mathbb{C}^\times\times\{1\}) \subset R(\boldsymbol{G})$, and $\mathsf{q}_2 \in R(\{1\}\times\{1\}\times\{1\}\times\mathbb{C}^\times) \subset R(\boldsymbol{G})$ be the inverse of degree-one characters. (I.e. q_2 corresponds to the inverse of the scalar multiplication on V_2.) By the operation e^λ and the multiplication by q_i, each of $K^{\boldsymbol{G}}(Z_{\le w})$ admits a structure of $R(\boldsymbol{T})$-modules.

Each $Z_{\le w} \backslash Z_{<w}$ is a $\boldsymbol{G}$-equivariant vector bundle over an affine fibration over G/B via the composition of $\boldsymbol{\pi}$ and the second projection. Therefore, the cellular fibration Lemma (or the successive application of localization sequence) yields:

Theorem 2.3 (c.f. [CG97] 7.6.11)**.** *We have*

$$K^{\boldsymbol{G}}(Z_{\leq w}) = \bigoplus_{v \in W; \mathsf{O}_v \subset \overline{\mathsf{O}}_w} R(\boldsymbol{T})[\mathcal{O}_{Z_{\leq v}}].$$

For each $i = 1, 2, \ldots, n$, we put $\mathbb{O}_i := \overline{\boldsymbol{\pi}^{-1}(\mathsf{O}_{s_i})}$. We define $\tilde{T}_i := [\mathcal{O}_{\mathbb{O}_i}]$ for each $i = 1, \ldots, n$.

Theorem 2.4 (c.f. Proof of [CG97] 7.6.12)**.** *The set* $\{[\mathcal{O}_{Z_{\leq 1}}], \tilde{T}_i, \mathsf{e}^\lambda; 1 \leq i \leq n, \lambda \in X^*(T)\}$ *is a generator set of* $K^{\boldsymbol{G}}(\boldsymbol{Z})$ *as* $\mathcal{A}_{\mathbb{Z}}$*-algebras.*

Proof. The tensor product of structure sheaves corresponding to vector subspaces of a vector space is the structure sheaf of their intersection. Taking account into that, the proof of the assertion is exactly the same as [CG97] 7.6.12. □

By the Thom isomorphism, we have an identification

$$K^{\boldsymbol{G}}(\boldsymbol{F}) \cong K^{\boldsymbol{G}}(G/B) \cong R(\boldsymbol{T}) = \mathcal{A}_{\mathbb{Z}}[T]. \tag{2.1}$$

We normalize the images of $[\mathcal{L}_\lambda]$ and q_i $(i = 0, 1, 2)$ under (2.1) as e^λ and $\mathbf{q}_i$, respectively.

Theorem 2.5 (c.f. [CG97] Claim 7.6.7)**.** *The homomorphism*

$$\circ : K^{\boldsymbol{G}}(\boldsymbol{Z}) \longrightarrow \mathrm{End}_{R(\boldsymbol{G})} K^{\boldsymbol{G}}(\boldsymbol{G})$$

is injective. □

Proposition 2.6. *We have*

1. $[\mathcal{O}_{Z_{\leq 1}}] = 1 \in \mathrm{End}_{R(\boldsymbol{G})} K^{\boldsymbol{G}}(\boldsymbol{F})$*;*

2. $\tilde{T}_i \circ e^\lambda = (1 - \mathbf{q}_2 e^{\alpha_i}) \frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}$ *for every* $\lambda \in X^*(T)$ *and every* $1 \leq i < n$*;*

3. $\tilde{T}_n \circ e^\lambda = (1 - \mathbf{q}_0 e^{\frac{1}{2}\alpha_n})(1 - \mathbf{q}_1 e^{\frac{1}{2}\alpha_n}) \frac{e^\lambda - e^{s_n \lambda - \alpha_n}}{1 - e^{-\alpha_n}}$ *for every* $\lambda \in X^*(T)$*.*

Proof. The component $Z_{\leq 1}$ is equal to the diagonal embedding of $\boldsymbol{F}$. In particular, both of the first and the second projections give isomorphisms between $Z_{\leq 1}$ and $\boldsymbol{F}$. It follows that

$$\begin{aligned}
[\mathcal{O}_{Z_{\leq 1}}] \circ [\mathcal{L}_\lambda] &= \sum_{i \geq 0} (-1)^i [\mathbb{R}^i (p_1)_* \left(\mathcal{O}_{Z_{\leq 1}} \otimes^{\mathbb{L}} \tilde{p}_2^* \mathcal{L}_\lambda \right)] \\
&= [\mathbb{R}^0 (p_1)_* \left(\mathcal{O}_{Z_{\leq 1}} \otimes \tilde{p}_2^* \mathcal{L}_\lambda \right)] = [\mathcal{L}_\lambda],
\end{aligned}$$

which proves 1). For each $i = 1, \ldots, n$, we define $\mathbb{V}^+(i) := \mathbb{V}_2^+ \cap \dot{s}_i \mathbb{V}_2^+$. Let $P_i := B\dot{s}_i B \sqcup B$ be a parabolic subgroup of G corresponding to s_i. Each $\mathbb{V}^+(i)$ is B-stable. Hence, it is P_i-stable. We have

$$\pi_2(\mathbb{O}_i) = \overline{\mathsf{O}}_{s_i} = (1 \times P_i)\mathsf{O}_1 \subset G/B \times G/B.$$

The product $(1 \times P_i)\mathsf{p}_1 \times \mathbb{V}^+(i)$ is a B-equivariant vector bundle. Here we have $G \cap (B \times P_i) = B$. Hence, we can induce it up to a G-equivariant vector bundle

$\tilde{\mathbb{V}}(i)$ on $\boldsymbol{\pi}(\mathbb{O}_i)$. By means of the natural embedding of G-equivariant vector bundles

$$\boldsymbol{F} = G \times^B \mathbb{V}_2^+ \hookrightarrow G \times^B \mathbb{V}_2 \cong G \times \mathbb{V}_2,$$

we can naturally identify $\boldsymbol{\pi}^{-1}(\mathsf{p}_{s_i})$ with $\mathbb{V}^+(i)$. Since $\mathbb{V}^+(i)$ is P_i-stable, we conclude $\boldsymbol{\pi}^{-1}(\mathsf{p}_{s_i}) \cong \mathbb{V}^+(i)$ as P_i-modules. As a consequence, we conclude $\tilde{\mathbb{V}}(i) \cong \mathbb{O}_i$. Let $\breve{F}(i) := G \times^B (\mathbb{V}_2^+/\mathbb{V}^+(i))$. It is a $\boldsymbol{G}$-equivariant quotient bundle of $\boldsymbol{F}$. The rank of $\breve{F}(i)$ is one $(1 \le i < n)$ or two $(i = n)$. Let $\breve{Z}_{\le s_i}$ be the image of $Z_{\le s_i}$ under the quotient map $\boldsymbol{F} \times \boldsymbol{F} \to \breve{F}(i) \times \breve{F}(i)$. We obtain the following commutative diagram:

$$\begin{tikzcd}
\boldsymbol{F} \arrow[d] & Z_{\le s_i} \arrow[l] \arrow[r] \arrow[d] & \boldsymbol{F} \arrow[d] \\
\breve{F}(i) & \breve{Z}_{\le s_i} \arrow[l] \arrow[r] & \breve{F}(i)
\end{tikzcd}$$

Here the above objects are smooth $\mathbb{V}^+(i)$-fibrations over the bottom objects. Therefore, it suffices to compute the convolution operation of the bottom line. We have $\breve{Z}_{\le s_i} = \overline{\mathsf{O}}_{s_i} \cup \triangle(\breve{F}(i))$, where $\triangle : \breve{F}(i) \hookrightarrow \breve{F}(i)^2$ is the diagonal embedding. Let $\breve{p}_j : \overline{\mathsf{O}}_{s_i} \to G/B$ $(j = 1, 2)$ be projections induced by the natural projections of $G/B \times G/B$. By construction, each $\breve{p}_j$ is a $\boldsymbol{G}$-equivariant $\mathbb{P}^1$-fibration. Let $\breve{\mathcal{L}}_\lambda$ be the pullback of $G \times^B \lambda^{-1}$ to $\breve{F}(i)$. We deduce

$$\tilde{T}_i \circ [\breve{\mathcal{L}}_\lambda] = \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (\breve{p}_1)_* (\mathcal{O}_{\overline{\mathsf{O}}_{s_i}} \otimes^{\mathbb{L}} (\mathcal{O}_{\breve{F}(i)} \boxtimes \breve{\mathcal{L}}_\lambda)]$$
$$= \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (\breve{p}_1)_* \breve{p}_2^* (G \times^B \lambda^{-1})] = \left[G \times^B [\frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}] \right],$$

where $[\frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}] \in R(T) \cong R(B)$ is a virtual B-module. Here the ideal sheaf associated to $G/B \subset \breve{F}(i)$ represents $\mathbf{q}_2[\breve{\mathcal{L}}_{\alpha_i}]$ in $K^{\boldsymbol{G}}(\breve{F}(i))$ $(1 \le i < n)$ or corresponds to $\mathbf{q}_0 \breve{\mathcal{L}}_{\epsilon_n} + \mathbf{q}_1 \breve{\mathcal{L}}_{\epsilon_n} \subset \mathcal{O}_{\breve{F}(i)}$ $(i = n)$. In the latter case, divisors corresponding to $\mathbf{q}_0 \breve{\mathcal{L}}_{\epsilon_n}$ and $\mathbf{q}_1 \breve{\mathcal{L}}_{\epsilon_n}$ are normal crossing. Thus, we have $[\mathbf{q}_0 \breve{\mathcal{L}}_{\epsilon_n} \cap \mathbf{q}_1 \breve{\mathcal{L}}_{\epsilon_n}] = \mathbf{q}_0 \mathbf{q}_1 [\breve{\mathcal{L}}_{2\epsilon_n}]$. In particular, we deduce

$$[\mathbf{q}_0 \breve{\mathcal{L}}_{\epsilon_n} + \mathbf{q}_1 \breve{\mathcal{L}}_{\epsilon_n}] = \mathbf{q}_0 [\breve{\mathcal{L}}_{\epsilon_n}] + \mathbf{q}_1 [\breve{\mathcal{L}}_{\epsilon_n}] - \mathbf{q}_0 \mathbf{q}_1 [\breve{\mathcal{L}}_{2\epsilon_n}] \in K^{\boldsymbol{G}}(\breve{F}(n)).$$

Therefore, we conclude

$$\tilde{T}_i \circ e^\lambda = \begin{cases} (1 - \mathbf{q}_2 e^{\alpha_i}) \frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}} & (1 \le i < n) \\ (1 - \mathbf{q}_0 \mathbf{e}^{\frac{\alpha_n}{2}})(1 - \mathbf{q}_1 \mathbf{e}^{\frac{\alpha_n}{2}}) \frac{e^\lambda - e^{s_n \lambda - \alpha_n}}{1 - e^{-\alpha_n}} & (i = n) \end{cases}$$

as desired. $\square$

The following representation of $\mathbb{H}$ is usually called the basic representation or the anti-spherical representation:

Theorem 2.7 (Basic representation c.f. [Mc03] 4.3.10)**.** *There is an injective $\mathcal{A}$-algebra homomorphism*

$$\varepsilon : \mathbb{H} \to \mathrm{End}_{\mathcal{A}} \mathcal{A}[T],$$

defined as $\varepsilon(e^\lambda) := e^\lambda \cdot$ $(\lambda \in X^*(T))$ *and*

$$\varepsilon(T_i)e^\lambda := \begin{cases} \frac{e^\lambda - e^{s_i\lambda}}{e^{\alpha_i}-1} - \mathbf{q}_2 \frac{e^\lambda - e^{s_i\lambda+\alpha_i}}{e^{\alpha_i}-1} & (1 \le i < n) \\ \frac{e^\lambda - e^{s_n\lambda}}{e^{\alpha_n}-1} + \mathbf{q}_0\mathbf{q}_1 \frac{e^\lambda - e^{s_n\lambda+\alpha_n}}{e^{\alpha_n}-1} - (\mathbf{q}_0+\mathbf{q}_1)e^{\epsilon_n}\frac{e^\lambda - e^{s_n\lambda}}{e^{\alpha_n}-1} & (i = n) \end{cases}.$$

Theorem 2.8 (Exotic geometric realization of Hecke algebras)**.** *We have an isomorphism*

$$\mathbb{H} \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{Z}),$$

as algebras.

Proof. Consider an assignment ϑ

$$\underline{e}^\lambda \mapsto \mathbf{e}^\lambda, \underline{T_i} \mapsto \begin{cases} \tilde{T}_i - (1 - \mathsf{q}_2(\mathbf{e}^{\alpha_i}+1)) & (1 \le i < n) \\ \tilde{T}_i + (\mathsf{q}_0+\mathsf{q}_1)\mathbf{e}^{\epsilon_n} - (1 + \mathsf{q}_0\mathsf{q}_1(\mathbf{e}^{\alpha_n}+1)) & (i = n) \end{cases}.$$

By means of the Thom isomorphism, the above assignment gives an action of an element of the set $\{\underline{e}^\lambda\} \cup \{\underline{T_i}\}_{i=1}^n$ on $\mathcal{A}[T]$. We have

$$\begin{aligned}
\vartheta(\underline{e}^\lambda)e^\mu =& e^{\lambda+\mu} \\
\vartheta(\underline{T_i})e^\lambda =& \Big(\tilde{T}_i - (1 - \mathbf{q}_2(e^{\alpha_i}+1))\Big) e^\lambda = (1 - \mathbf{q}_2 e^{\alpha_i})\frac{e^\lambda - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - e^\lambda + \mathbf{q}_2(e^{\alpha_i}+1)e^\lambda \\
=& (\frac{e^\lambda - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - \frac{e^\lambda - e^{\lambda-\alpha_i}}{1-e^{-\alpha_i}}) - \mathbf{q}_2 e^{\alpha_i}(\frac{e^\lambda - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - \frac{e^\lambda - e^{\lambda-2\alpha_i}}{1-e^{-\alpha_i}}) = \varepsilon(T_i)e^\lambda \\
\vartheta(\underline{T_n})e^\lambda =& \Big(\tilde{T}_n + (\mathbf{q}_0+\mathbf{q}_1)e^{\epsilon_n} - (1 + \mathbf{q}_0\mathbf{q}_1(e^{\alpha_n}+1))\Big) e^\lambda \\
=& (1-\mathbf{q}_0 e^{\epsilon_n})(1-\mathbf{q}_1 e^{\epsilon_n})\frac{e^\lambda - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - e^\lambda + (\mathbf{q}_0+\mathbf{q}_1)e^{\lambda+\epsilon_n} - \mathbf{q}_0\mathbf{q}_1(e^{\alpha_n}+1)e^\lambda \\
=& (\frac{e^\lambda - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - \frac{e^\lambda - e^{\lambda-\alpha_n}}{1-e^{-\alpha_n}}) + \mathbf{q}_0\mathbf{q}_1 e^{\alpha_n}(\frac{e^\lambda - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - \frac{e^\lambda - e^{\lambda-2\alpha_n}}{1-e^{-\alpha_n}}) \\
&- (\mathbf{q}_0+\mathbf{q}_1)(\frac{e^{\lambda+\epsilon_n} - e^{s_n\lambda-\epsilon_n}}{1-e^{-\alpha_n}} - \frac{e^{\lambda+\epsilon_n} - e^{\lambda-\epsilon_n}}{1-e^{-\alpha_n}}) = \varepsilon(T_n)e^\lambda.
\end{aligned}$$

This identifies $\mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{F})$ with the basic representation of $\mathbb{H}$ via the correspondence $e^\lambda \mapsto \underline{e}^\lambda$ and $T_i \mapsto \underline{T_i}$. In particular, it gives an inclusion $\mathbb{H} \subset \mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{Z})$. Here we have $T_i \in \tilde{T}_i + \mathcal{A}[T]$ for $1 \le i \le n$. It follows that $\mathbb{C} \otimes_{\mathbb{Z}} K^{\boldsymbol{G}}(\boldsymbol{Z}) \subset \mathbb{H}$, which yields the result. □

Theorem 2.9 (Bernstein c.f. [CG97] 7.1.14 and [Mc03] 4.2.10)**.** *The center* $Z(\mathbb{H})$ *of* $\mathbb{H}$ *is naturally isomorphic to* $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbf{G})$. □

Corollary 2.10. *The center of* $K^{\boldsymbol{G}}(\boldsymbol{Z})$ *is* $R(\boldsymbol{G})$. □

For a semisimple element $a \in \boldsymbol{G}$, we define

$$\mathbb{H}_a := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H} \quad \text{(c.f. §1.4)}$$

and call it the specialized Hecke algebra.

Theorem 2.11. *Let* $a \in \boldsymbol{G}$ *be a semisimple element. We have an isomorphism*

$$\mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^a)$$

as algebras.

Proof. This is a combination of [CG97] 6.2.3 and 5.10.11. (See also [CG97] 8.1.6.) $\square$

Convention 2.12. Let $a = (s, \vec{q}) \in \boldsymbol{G}$ be a pre-admissible element. We define Z_+^a to be the image of $\boldsymbol{Z}^a$ under the natural projection defined by

$$\boldsymbol{Z} \ni (g_1 B, g_2 B, X_0, X_1, X_2) \mapsto (g_1 B, g_2 B, X_0 + X_1, X_2) \in Z.$$

Let F_+^a be the image of Z_+^a via the first (or the second) projection. Let μ_+^a be the restriction of μ to F_+^a. We denote its image by $\mathfrak{N}_+^a$. By the assumption $q_0 \neq q_1$, we have $F_+^a \cong \boldsymbol{F}^a$, $Z_+^a \cong \boldsymbol{Z}^a$, and $\mathfrak{N}_+^a \cong \mathfrak{N}_2^a$.

Corollary 2.13. *Keep the setting of Convention 2.12. We have an isomorphism*

$$\mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(Z_+^a)$$

as algebras. $\square$

3 Clan decomposition

We work under the same setting as in §2.

Definition 3.1 (Clans)**.** Let $a = (s, \vec{q})$ be a pre-admissible element such that $s \in T$. Let $q_2 = e^{r_2}$. We put $\Gamma := r_2\mathbb{Z} + \Gamma_0$. A clan associated to a is a maximal subset $\mathbf{c} \subset [1, n]$ with the following property: For each two elements $i, j \in \mathbf{c}$, there exists a sequence $i = i_0, i_1, \dots, i_m = j$ (in $\mathbf{c}$) such that

$$\{\log_{i_k}(s) \pm \log_{i_{k+1}}(s)\} \cap \{\pm r_2 + \Gamma_0, \Gamma_0\} \neq \emptyset \quad \text{for each } 0 \le k < m.$$

We have a disjoint decomposition

$$[1, n] = \bigsqcup_{\mathbf{c} \in \mathcal{C}_a} \mathbf{c},$$

where each $\mathbf{c}$ is a clan associated to a and $\mathcal{C}_a$ is the set of clans associated to a. For a clan $\mathbf{c}$, we put $n^{\mathbf{c}} := \#\mathbf{c}$.

We assume the setting of Definition 3.1 in the rest of this section unless stated otherwise. At the level of Lie algebras, we have a decomposition

$$\mathfrak{g}(s) := \mathfrak{t} \oplus \bigoplus_{\substack{i<j,\, \sigma_1, \sigma_2 \in \{\pm 1\},\\ \sigma_1 \log_i(s) + \sigma_2 \log_j(s) \equiv 0}} \mathfrak{g}(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{\substack{i \in [1,n],\, \sigma \in \{\pm 1\},\\ 2\log_i(s) \equiv 0}} \mathfrak{g}(s)[\sigma 2 \epsilon_i],$$

where $\equiv$ means modulo Γ_0. For each $\mathbf{c} \in \mathcal{C}_a$, we define a Lie algebra $\mathfrak{g}(s)_{\mathbf{c}}$ as the Lie subalgebra of $\mathfrak{g}(s)$ defined as

$$\bigoplus_{i \in \mathbf{c}} \mathbb{C}\epsilon_i \oplus \bigoplus_{\substack{i<j \in \mathbf{c},\, \sigma_1, \sigma_2 \in \{\pm 1\},\\ \sigma_1 \log_i(s) + \sigma_2 \log_j(s) \equiv 0}} \mathfrak{g}(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{\substack{i \in \mathbf{c},\, \sigma \in \{\pm 1\},\\ 2\log_i(s) \equiv 0}} \mathfrak{g}(s)[\sigma 2 \epsilon_i],$$

where $\equiv$ means modulo Γ_0. Moreover, we have

$$\mathfrak{g}(s) = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \mathfrak{g}(s)_{\mathbf{c}}. \tag{3.1}$$

In particular, we have $[\mathfrak{g}(s)_{\mathbf{c}}, \mathfrak{g}(s)_{\mathbf{c}'}] = 0$ unless $\mathbf{c} = \mathbf{c}'$. Let $G(s)_{\mathbf{c}}$ be the connected subgroup of $G(s)$ which has $\mathfrak{g}(s)_{\mathbf{c}}$ as its Lie algebra.

Lemma 3.2. *We have* $G(s) = \prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}$.

Proof. By (3.1), it is clear that $\prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}$ is equal to the identity component of $G(s)$. Since G is a simply connected semi-simple group, it follows that $G(s)$ is connected by Steinberg's centralizer theorem (c.f. [Ca85] 3.5.6). In particular, we have $G(s) \subset \prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}$ as desired. □

We denote $B \cap G(s)_{\mathbf{c}}$ and ${}^{w}B \cap G(s)_{\mathbf{c}}$ by $B(s)_{\mathbf{c}}$ and ${}^{w}B(s)_{\mathbf{c}}$, respectively.

Convention 3.3. We denote by $\mathbb{V}^a$ the image of $\mathbb{V}_2^a$ to $\mathbb{V}$ via the map

$$\mathbb{V}_2 \ni (X_0 \oplus X_1 \oplus X_2) \mapsto ((X_0 + X_1) \oplus X_2) \in \mathbb{V}.$$

Since $q_0 \neq q_1$, we have $\mathbb{V}^a \cong \mathbb{V}_2^a$.

For each $\mathbf{c} \in \mathcal{C}_a$, we define

$$\mathbb{V}_{\mathbf{c}}^a := \sum_{i,j \in \mathbf{c}, \sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}} \mathbb{V}^a[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \mathbb{V}^a[\sigma_3 \epsilon_i].$$

It is clear that $\mathbb{V}^a = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \mathbb{V}_{\mathbf{c}}^a$. By the comparison of weights, the $\mathfrak{g}(s)_{\mathbf{c}}$-action on $\mathbb{V}_{\mathbf{c}'}^a$ is trivial unless $\mathbf{c} = \mathbf{c}'$.

Remark 3.4. Since $\mathbf{c}$ is not an integer and we do not use $\mathbb{V}_\ell$ in the rest of this paper, we use the notation $\mathbb{V}_{\mathbf{c}}^a$. The author hopes the reader not to confuse $\mathbb{V}_{\mathbf{c}}^a$ with $(\mathbb{V}_\ell)^a$.

Lemma 3.5. *Let* $\mathbb{O} \subset \mathfrak{N}_+^a$ *be a* $\boldsymbol{G}(a)$*-orbit. Let* $\mathbb{O}_{\mathbf{c}}$ *denote the image of* $\mathbb{O}$ *under the natural projection* $\mathbb{V}^a \to \mathbb{V}_{\mathbf{c}}^a$*. Then, we have a product decomposition* $\mathbb{O} = \oplus_{\mathbf{c} \in \mathcal{C}_a} \mathbb{O}_{\mathbf{c}}$.

Proof. Let $X \in \mathbb{V}^a$. There exists a family $\{X_{\mathbf{c}}\}_{\mathbf{c} \in \mathcal{C}_a}$ ($X_{\mathbf{c}} \in \mathbb{V}_{\mathbf{c}}^a$) such that $X = \sum_{\mathbf{c} \in \mathcal{C}_a} X_{\mathbf{c}}$. We have $G(s)X = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}} X_{\mathbf{c}}$. For each of $i = 0, 1$, the clan $\mathbf{c} \in \mathcal{C}_a$ such that $(V_1^{(s,q_i)} \cap \mathbb{V}_{\mathbf{c}}) \neq \{0\}$ is at most one since clans are determined by the s-eigenvalues of V_1. Let $\mathbf{c}^i$ ($i = 1, 2$) be the unique clan such that $(V_1^{(s,q_i)} \cap \mathbb{V}_{\mathbf{c}^i}) \neq \{0\}$. Let $\mathbb{G}_{\mathbf{c}}$ be the product of scalar multiplications of $V_1^{(s,q_i)}$ such that $V_1^{(s,q_i)} \cap \mathbb{V}_{\mathbf{c}}^{\mathbf{a}} \neq \{0\}$. Since the set of a-fixed points of a conic variety in $\mathbb{V}$ is conic, we have $(G(s)_{\mathbf{c}} \times (\mathbb{C}^\times)^3) X_{\mathbf{c}} = (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}}) X_{\mathbf{c}}$. We have $\prod_{\mathbf{c} \in \mathcal{C}_a} (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}}) \subset \boldsymbol{G}(a)$. It follows that

$$\boldsymbol{G}(a) X = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \boldsymbol{G}(a) X_{\mathbf{c}} = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}}) X_{\mathbf{c}} = \bigoplus_{\mathbf{c} \in \mathcal{C}_a} \mathbb{O}_{\mathbf{c}}$$

as desired. □

For each $w \in W$, we define

$$F_+^a(w) := G(s) \times^{{}^{w}B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^a).$$

Similarly, we define

$$F_+^a(w, \mathbf{c}) := G(s)_{\mathbf{c}} \times^{{}^{w}B(s)_{\mathbf{c}}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a)$$

for each $\mathbf{c} \in \mathcal{C}_a$.

Lemma 3.6. *We have $F_+^a = \cup_{w \in W} F_+^a(w)$.*

Proof. The set of a-fixed points of G/B is a disjoint union of flag varieties of $G(s)$. It follows that each point of F_+^a is $G(s)$-conjugate to a point in the fiber over a T-fixed point of G/B. □

The local structures of these connected components are as follows.

Lemma 3.7. *For each $w \in W$, we have*

$$F_+^a(w) \cong \prod_{\mathbf{c} \in \mathcal{C}_a} F_+^a(w, \mathbf{c}).$$

Proof. The set $\mathbb{V}_{\mathbf{c}}^a$ is T-stable for each $\mathbf{c} \in \mathcal{C}_a$. Hence, we have

$$F_+^a(w) = G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^a) \cong G(s) \times^{{}^w B(s)} (\bigoplus_{\mathbf{c} \in \mathcal{C}_a} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a)).$$

Since we have $G(s)/B(s) \cong \prod_{\mathbf{c} \in \mathcal{C}_a} G(s)_{\mathbf{c}}/B(s)_{\mathbf{c}}$, we deduce

$$G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a) \cong \prod_{\mathbf{c}' \in \mathcal{C}_a} G(s)_{\mathbf{c}'} \times^{{}^w B(s)_{\mathbf{c}'}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a \cap \mathbb{V}_{\mathbf{c}'}^a).$$

Here the RHS is isomorphic to

$$F_+^a(w, \mathbf{c}) \times \prod_{\mathbf{c} \neq \mathbf{c}'} G(s)_{\mathbf{c}'}/{}^w B(s)_{\mathbf{c}'}.$$

Gathering these information yields the result. □

We define a map ${}^w\mu_{\mathbf{c}}^a$ by

$${}^w\mu_{\mathbf{c}}^a : F_+^a(w, \mathbf{c}) = G(s)_{\mathbf{c}} \times^{{}^w B(s)_{\mathbf{c}}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^a) \longrightarrow \mathbb{V}_{\mathbf{c}}^a.$$

Definition 3.8 (Regular parameters)**.** A pre-admissible parameter (a, X) is called regular iff there exists a direct factor $A[d] \subset (\mu_+^a)_* \mathbb{C}_{F_+^a}$, where A is a simple $\boldsymbol{G}(a)$-equivariant perverse sheaf on $\mathfrak{N}_+^a$ such that $\mathrm{supp} A = \overline{\boldsymbol{G}(a)X}$ and d is an integer.
We denote by $\mathfrak{R}_a$ the set of $\boldsymbol{G}(a)$-conjugacy classes of regular parameters of the form (a, X) $(X \in \mathfrak{N}_+^a)$.

Proposition 3.9 (Clan decomposition)**.** *For each $w \in W$, we have*

$$\mu_+^a|_{F_+^a(w)} \cong \prod_{\mathbf{c} \in \mathcal{C}_a} {}^w\mu_{\mathbf{c}}^a.$$

In particular, every irreducible direct summand A of $(\mu_+^a)_* \mathbb{C}_{F_+^a}$ is written as an external product of $G(s)$-equivariant sheaves appearing in $({}^w\mu_{\mathbf{c}}^a)_* \mathbb{C}_{F_+^a(w, \mathbf{c})}$ (up to degree shift).

Proof. The first assertion follows from the combination of Lemma 3.5, Lemma 3.7, and the definition of ${}^w\mu_{\mathbf{c}}^a$. We have $\mathbb{C}_{F_+^a} = \bigoplus_{F_+^a(w) \subset F_+^a} \mathbb{C}_{F_+^a(w)}$. A direct summand of $(\mu_+^a)_* \mathbb{C}_{F_+^a}$ is a direct summand of $(\mu_+^a)_* \mathbb{C}_{F_+^a(w)}$ for some $w \in W$. Since

$$(\mu_+^a)_* \mathbb{C}_{F_+^a(w)} \cong \boxtimes_{\mathbf{c}} ({}^w\mu_{\mathbf{c}}^a)_* \mathbb{C}_{F_+^a(w, \mathbf{c})},$$

the second assertion follows. □

We put $G_{\mathbf{c}} := Sp(2n^{\mathbf{c}})$ and $s_{\mathbf{c}} := \exp(\sum_{i\in\mathbf{c}}(\log_i(s))\epsilon_i) \in T$. We have embeddings

$$s = \prod_{\mathbf{c}\in\mathcal{C}_a} s_{\mathbf{c}} \in \prod_{\mathbf{c}\in\mathcal{C}_a} Sp(2n^{\mathbf{c}}) \subset Sp(2n),$$

induced by the following identifications:

$$\mathfrak{g}(s)_{\mathbf{c}} = \mathfrak{g}_{\mathbf{c}}(s_{\mathbf{c}}) \subset \left(\bigoplus_{i\in\mathbf{c}} \mathbb{C}\epsilon_i\right) \oplus \bigoplus_{\substack{\alpha = \sigma_1\epsilon_i + \sigma_2\epsilon_j \neq 0 \\ \sigma_1, \sigma_2 \in \{\pm 1\}, i, j \in \mathbf{c}}} \mathfrak{g}[\alpha] = \mathfrak{g}_{\mathbf{c}}. \tag{3.2}$$

It follows that $G(s)_{\mathbf{c}} = G_{\mathbf{c}}(s_{\mathbf{c}}) \subsetneq G_{\mathbf{c}}$ in general.

Let $\mathbb{V}(\mathbf{c})$ be the 1-exotic representation of $G_{\mathbf{c}}$. We have a natural embedding $\mathbb{V}^a_{\mathbf{c}} \subset \mathbb{V}(\mathbf{c})$ of $G(s)_{\mathbf{c}}$-modules. (The $G(s)_{\mathbf{c}}$-module structure on the RHS is given by restriction of $G_{\mathbf{c}}$-action.)

Let $\nu = (a, X)$ be a pre-admissible parameter. We have a family of pre-admissible parameters $\nu_{\mathbf{c}} := (s_{\mathbf{c}}, \vec{q}, X_{\mathbf{c}})$ of $G_{\mathbf{c}}$'s such that $s = \prod_{\mathbf{c}} s_{\mathbf{c}}$, $X = \oplus_{\mathbf{c}} X_{\mathbf{c}}$. We denote

$$\nu = \prod_{\mathbf{c}\in\mathcal{C}_a} \nu_{\mathbf{c}}$$

and call it the clan decomposition of ν. Let $W_a := \prod_{\mathbf{c}\in\mathcal{C}_a} N_{G_{\mathbf{c}}}(T)/T$. By Lemma 3.7, we conclude that

$$\bigcup_{w\in W_a} F^a_+(w) \subset F^a_+ \tag{3.3}$$

is the product of the F^a_+'s obtained by replacing the pair (G, ν) by $(G_{\mathbf{c}}, \nu_{\mathbf{c}})$ for all $\mathbf{c} \in \mathcal{C}_a$.

Corollary 3.10. *Let $\nu = (a, X)$ be a pre-admissible parameter. Then, it is regular if and only if $\nu_{\mathbf{c}}$ is a regular pre-admissible parameter of $G_{\mathbf{c}}$ for every $\mathbf{c} \in \mathcal{C}_a$.*

Proof. Let $W_0 := N_{G(s)}(T)/T \subset W$. We have a natural inclusion $W_0 \subset W_a$. Here we have

$$\mu^a_+ = \bigsqcup_{w\in W/W_0} \mu^a_+|_{F^a_+(w)},$$

where we regard $W/W_0 \subset W$ by taking some representative. For each $w \in W$, there exists $v \in W_a$ such that ${}^w\mathbb{V}^+ \cap \mathbb{V}^a = {}^v\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}} \subset \mathbb{V}^a$. Moreover, we can choose v so that ${}^wB(s)_{\mathbf{c}} = {}^vB(s)_{\mathbf{c}}$ holds for each $\mathbf{c} \in \mathcal{C}_a$. As a consequence, all $F^a_+(w)$ are isomorphic to one of $F^a_+(w)$ ($w \in W_a$) as $\boldsymbol{G}(a)$-varieties, together with maps $\mu^a_+|_{F^a_+(w)}$ to $\mathbb{V}^a$. Therefore, ν is regular if and only if an intersection cohomology complex with its support $\overline{\boldsymbol{G}(a)\nu}$ (with degree shift) appears in $(\mu^a_+)_*\mathbb{C}_{F^a_+(w)}$ for some $w \in W_a$. Hence, Proposition 3.9 implies the result. $\square$

Corollary 3.10 reduces the analysis of the decomposition pattern of $(\mu^a_+)_*\mathbb{C}_{F^a_+}$ into the case that ν has a unique clan.

4 On stabilizers of exotic nilpotent orbits

We retain the setting of §2.

Lemma 4.1. *Let H be an algebraic group and let $\mathcal{X}$ be an algebraic variety with H-action. Let $H = H_r H_u$ be a Levi decomposition of H with H_r its reductive part. If $\mathsf{Stab}_{H_r} x$ is connected for $x \in \mathcal{X}$, then so is $\mathsf{Stab}_H x$.*

Proof. Assume to the contrary to deduce contradiction. Let $h \in \mathsf{Stab}_H x$ be an element which is not in the identity component. Let $h = h_r h_u \in H_r H_u$ be its Levi decomposition. For some $k > 1$, we have $h^k \in (\mathsf{Stab}_H x)^\circ$. This implies the existence of $g \in (\mathsf{Stab}_H x)^\circ$ which satisfies $h^k = g^k$. Let $g = g_r g_u$ be the Levi decomposition. We have $H_u \triangleleft H$, which claims $h_r^k = g_r^k$. Replacing h by $g^{-1}h$, we further assume $h_r^k = 1$. Then, the assumption implies $h_u \neq 1$. Moreover, we have $h^k \in (\mathsf{Stab}_{H_u} x)^\circ = \mathsf{Stab}_{H_u} x$. Consider the map

$$\mathsf{Stab}_{H_u} x \ni u \mapsto (h_r^{1-k} u h_r^{k-1}) \cdots (h_r^{-1} u h_r) u \in \mathsf{Stab}_{H_u} x.$$

By a Lie algebra calculation using the eigenvalues of h_r, the image of this map is $(\mathsf{Stab}_{H_u} x)^{h_r}$, the h_r-fixed part of $\mathsf{Stab}_{H_u} x$. Here simple calculation shows $h^k \in (\mathsf{Stab}_{H_u} x)^{h_r}$. Thus, we can adjust h by some element of $(\mathsf{Stab}_{H_u} x)^{h_r}$ to assume $h^k = 1$. Since an element of finite order is necessarily semisimple, we conclude that $h = m h_r m^{-1}$, where $m \in H_u$. This gives an element of $\mathsf{Stab}_{H_r} x - (\mathsf{Stab}_{H_r} x)^\circ$, which contradicts the initial assumption of this proof. As a consequence, we deduce the result. $\square$

The following result is not exactly the same as the original, but we can easily deduce it from the proof:

Theorem 4.2 (Igusa [Ig73] Lemma 8)**.** *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition of n. We regard it as a 0-marked partition. Then, the reductive part of $\mathsf{Stab}_G \mathbf{v}_\lambda$ is*

$$L_\lambda := Sp(2n_1, \mathbb{C}) \times Sp(2n_2, \mathbb{C}) \times \cdots,$$

where the sequence $(n_1, n_2, \ldots)$ are the number of λ_i's which share the same value. Moreover, we have

$$\mathsf{Res}^G_{L_\lambda} V_1 = \bigoplus_{i \geq 1} V(i)^{\oplus \lambda_i},$$

where $V(i)$ is the vector representation of $Sp(2n_i)$ with trivial actions of $Sp(2n_j)$ $(j \neq i)$. $\square$

Corollary 4.3. *Keep the setting of Theorem 4.2. Then, we can choose maximal torus of L_λ inside T.*

Proof. Let $\mathbf{J}$ be the signed partition corresponding to λ. By Lemma 1.17, we have $\mathbb{C}^\times \subset \mathsf{Stab}_{T_J} \mathbf{v}^J$ for each $J \in \mathbf{J}$. It follows that $L_\lambda \cap T$ contains a torus of dimension $(\sum_{i \geq 0} n_i)$, which implies the result. $\square$

Proposition 4.4. *Let $X \in \mathfrak{N}_2$. Then, $\mathsf{Stab}_G X$ is connected.*

Proof. Let $X = (X_0 \oplus X_1) \oplus \mathbf{v}_{2,\lambda}$, where λ is a partition of n regarded as a 0-marked partition. By Theorem 4.2, it suffices to show that the action of $\mathsf{Stab}_G \mathbf{v}_{2,\lambda}$ on $(X_0 \oplus X_1)$ has connected stabilizer. Let $L_\lambda U_\lambda$ be the Levi decomposition of $\mathsf{Stab}_G \mathbf{v}_{2,\lambda}$. By Lemma 4.1, it is sufficient to show that the stabilizer of L_λ on $(X_0 \oplus X_1)$ is connected. By Theorem 4.2, it suffices to prove that the G-stabilizer of finite set of elements in V_1 is connected. By a repeated use of Lemma 4.1, it suffices to prove that the G-stabilizer of one element in V_1 has $Sp(2n-2)$ as its (reductive) Levi factor. We denote the element $v \in V_1$ and fix a symplectic form on V_1 which is preserved by G. Then, it is easy to see that $\mathsf{Stab}_G v$ preserves $\mathbb{C}v$, the compliment space $v^\perp$ of V_1 with respect to the symplectic form. Thus, its Levi component is given as a subgroup of

$$\mathbb{C}^\times \times Sp(2n-2) = \left(\mathbb{C}^\times \times GL(2n-2,\mathbb{C}) \times \mathbb{C}^\times\right) \cap Sp(2n) \subset GL(V_1),$$

which fixes v. (Here the middle group is the Levi component of $GL(V_1)$ which preserves a partial flag $\{0\} \subset \mathbb{C}v \subset v^\perp \subset V_1$.) Therefore, it is $Sp(2n-2)$ as desired. $\square$

Remark 4.5. Springer [Sp07] contains an explicit description of the G-stabilizer of each strict normal form. As is seen easily from the proof of Proposition 4.4, it is not hard to write down the G-stabilizer of a point of $\mathfrak{N}_2$ assuming [Sp07].

Corollary 4.6 (of the proof of Proposition 4.4)**.** *For each $X \in \mathfrak{N}_2$, the reductive part of $\mathsf{Stab}_G X$ is a product of symplectic groups.* $\square$

Corollary 4.7. *Let $(a, X) = (s, \vec{q}, X)$ be a pre-admissible parameter. Then, $(\mathsf{Stab}_G X)(s)$ is connected.*

Proof. A Levi decomposition $\boldsymbol{L}_X \boldsymbol{U}_X = \mathsf{Stab}_{\boldsymbol{G}} X$ is preserved by the a-action by requiring $a \in \boldsymbol{L}_X$. Since a fixed part of a unipotent group is again unipotent, we can forget about the unipotent part. Here we have $L_X = G \cap \mathsf{Stab}_{\boldsymbol{G}} X$. Since L_X is a product of symplectic groups, the Steinberg centralizer theorem asserts that $L_X(s)$ is connected as desired. $\square$

A by-product of the above argument is the following:

Theorem 4.8 (Refined form of Theorem 1.16)**.** *Let $\nu = (a, X) = (s, \vec{q}, X_1 \oplus X_2)$ be an admissible parameter or $a = a_0$. Then, we have a clan decomposition*

$$\nu = \prod_{\mathbf{c} \in \mathcal{C}_a} \nu^{\mathbf{c}} = \prod_{\mathbf{c} \in \mathcal{C}_a} (s_{\mathbf{c}}, \vec{q}, X_{\mathbf{c}})$$

with the following properties:

- *Each $\nu_{\mathbf{c}}$ is an admissible parameter;*
- *There exists $g \in G$ such that:*

$$gsg^{-1} \in T \text{ and each } gX_{\mathbf{c}} \text{ is a strict normal form.}$$

Proof. If $a = a_0$, then we have $\mathfrak{N}_2^a \cong \mathfrak{N}$. Hence, the result reduces to Proposition 1.15 1).

Thus, we assume that a is admissible. Since the admissibility condition depends only on the configuration of $\vec{q}$, the clan decomposition preserves admissibility. Hence, it suffices to prove the case that $\mathbf{c} = [1, n]$ is the unique clan of $\mathcal{C}_a$. Then, two distinct eigenvalues t_1, t_2 of s on V_1 satisfies

$$t_1 t_2 \text{ or } t_1/t_2 = q_2^m, \text{ where } |m| < 2n.$$

It follows that at least one of q_0 or q_1 does not appear as a s-eigenvalue of V_1 by the admissibility condition. Therefore, we can assume $(s - q_1)X_1 = 0$ by swapping the roles of q_0 and q_1 if necessary.

Let us take G-conjugate to assume that $X = \mathbf{v}_\sigma$ for a strict marked partition $\sigma = (\mathbf{J}, \vec{\delta})$. By the description of the G-stabilizer of $\mathbf{v}_\sigma$, we deduce that we can choose a maximal torus of $\mathsf{Stab}_G \mathbf{v}_\sigma$ inside of T. By Lemma 1.17 and the fact $\mathbf{v}_\sigma$ is a strict normal form, we deduce that a (possibly disconnected) maximal torus of $\mathsf{Stab}_{\boldsymbol{G}} \mathbf{v}_\sigma$ is taken inside $\boldsymbol{T}$. Therefore, we conclude that $(a, \mathbf{v}_\sigma)$ is a strict normal form after taking conjugate of a by the $\mathsf{Stab}_{\boldsymbol{G}} \mathbf{v}_\sigma$-action (or the $\mathsf{Stab}_G \mathbf{v}_\sigma$-action). $\square$

Corollary 4.9. *Let $a = (s, q_0, q_1, q_2)$ be an admissible element. If $\mathcal{C}_a$ consists of a unique clan $[1, n]$, then we have either $V_1^{(s,q_0)} = \{0\}$ or $V_1^{(s,q_1)} = \{0\}$.*

Proof. See the second paragraph of the proof of Proposition 4.8. $\square$

5 Semisimple elements attached to $G \backslash \mathfrak{N}_1$

We keep the setting of the previous section.

Let $\sigma := (\mathbf{J}, \vec{\delta})$ be a strict marked partition. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be the partition of n corresponding to $\mathbf{J} = \{J_1, J_2, \ldots\}$.

We fix a sequence of positive real numbers $\gamma_0, \gamma_1, \ldots, \gamma_n > (n + 1)\gamma$ such that

$$\{\gamma_i + \gamma_j, \gamma_i - \gamma_j\} \cap (\Gamma + \mathbb{Z}\gamma) = \emptyset \tag{5.1}$$

holds for every pair of (not necessarily distinct) numbers in $[0, n]$.

Remark 5.1. Our choice of $\{\gamma_k\}_k$ and γ are possible since $\mathbb{C}$ is an extension of the field $\mathbb{Q}(q_2, \sqrt{-1}, \pi)$ with infinite transcendental degree.

We define a semi-simple element $s_\sigma \in T$ as follows:

- If $\delta_1|_{J_k} \equiv 0$, then we set $\log_j s_\sigma = \gamma_k - j\gamma$ for each $j \in J_k$;
- If $\delta_1(j_0) = 1$ for $j_0 \in J_k$, then we set $\log_j s_\sigma = \gamma_0 - (j - j_0)\gamma$ for each $j \in J_k$.

By the definition of strict marked partitions, the choice of j_0 is unique for each $J \in \mathbf{J}$. Hence, s_σ is uniquely determined. We put $a_\sigma := (s_\sigma, e^{\gamma_0}, 1, e^{\gamma}) \in \boldsymbol{T}$.

Lemma 5.2. *In the above setting, we have $a_\sigma \mathbf{v}_\sigma = \mathbf{v}_\sigma$.*

Proof. It suffices to prove $(s_\sigma, e^{\gamma_0}, 1, e^{\gamma})\mathbf{v}_\sigma^J = \mathbf{v}_\sigma^J$ for each $J \in \mathbf{J}$. Let $J = J_k$. Then, $\mathbf{v}_{2,\sigma}^{J_k}$ is a sum of $y_{i,i+1}$ for $i, i+1 \in J_k$, which has s_σ-eigenvalue

$$e^{(\gamma_k - i\gamma - (\gamma_k - (i+1)\gamma))} = e^{\gamma}.$$

Hence, we have $s_\sigma \mathbf{v}_{2,\sigma}^{J_k} = e^{\gamma} \mathbf{v}_{2,\sigma}^{J_k}$. Moreover, we have $s_\sigma x_i = e^{\gamma_0} x_i$ for every $i \in J_k$. In particular, we have $s_\sigma \mathbf{v}_{1,\sigma}^{J_k} = e^{\gamma_0} \mathbf{v}_{1,\sigma}^{J_k}$. These calculations imply the desired result. $\square$

Fix a real number $\mathsf{r} > 0$. We define $D_\sigma \in T$ to be

$$\log_i D_\sigma = \begin{cases} 0 & (\log_i s_\sigma \neq \gamma_0) \\ -\mathsf{r}(\underline{\#} J_k) & (i \in J_k \ni \exists j_0, \delta_1(j_0) = 1) \end{cases}$$

Consider a parabolic subgroup P_σ of $G(s_\sigma)$:

$$P_\sigma := \{g \in G(s_\sigma); \lim_{N \to \infty} \mathrm{Ad}(D_\sigma^N) g \in G(s_\sigma)\}.$$

It is well-known that P_σ is a parabolic subgroup of $G(s_\sigma)$. Let w_σ be the shortest element of W such that

$$\left\langle w_\sigma R^+, D_\sigma \right\rangle \leq 1.$$

It is straight-forward to see

$$^{w_\sigma} B \cap G(s_\sigma) \subset P_\sigma.$$

Lemma 5.3. *For a strict marked partition σ, we have $\mathbf{v}_\sigma \in \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+$.*

Proof. By Lemma 5.2, it suffices to prove $\mathbf{v}_\sigma \in {}^{w_\sigma}\mathbb{V}^+$. The definition of w_σ implies that

1. $x_i \in {}^{w_\sigma}\mathbb{V}^+$ if and only if **a)** $D_\sigma(\epsilon_i) < 1$ or **b)** $D_\sigma(\epsilon_i) = 1$ and $i > 0$;
2. $y_{ij} \in {}^{w_\sigma}\mathbb{V}^+$ if and only if **a)** $D_\sigma(\epsilon_i - \epsilon_j) < 1$ or **b)** $D_\sigma(\epsilon_i - \epsilon_j) = 1$ and $\epsilon_i - \epsilon_j \in R^+$.

Since $\mathbf{v}_{1,\sigma}$ is sum of x_i with D_σ-eigenvalue < 1, we have $\mathbf{v}_{1,\sigma} \in {}^{w_\sigma}\mathbb{V}^+$. The vector $\mathbf{v}_{2,\sigma}$ have D_σ-eigenvalue 1. By construction, a strict normal form is contained in $\mathbb{V}^+$. There, we conclude $\mathbf{v}_{2,\sigma} \in {}^{w_\sigma}\mathbb{V}^+$, which completes the proof. $\square$

Proposition 5.4. *Let $\sigma = (\mathbf{J}, \vec{\delta})$ be a strict marked partition. Then, we have an inclusion*

$$P_\sigma \mathbf{v}_\sigma \subset \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+,$$

which is dense open.

Before giving the proof of Proposition 5.4, we count the set of weights we concern in its proof:

Lemma 5.5. *Keep the setting of Proposition 5.4. Then, the set $\Psi(\mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+)$ is given by the following list:*

1. *$\epsilon_i - \epsilon_{i+1}$ for each $i, i+1 \in J \in \mathbf{J}$;*
2. *$\epsilon_{i+j_0} - \epsilon_{i+j_1+1}$ if the following conditions hold:*
    - *$i + j_0, j_0 \in J_k$, and $i + j_1 + 1, j_1 \in J_{k'}$ for some k, k';*
    - *$\delta_1(j_0) = 1 = \delta_1(j_1)$, and $\underline{\#} J_k > \underline{\#} J_{k'}$;*

3. ϵ_{j_0} *for each* $j_0 \in J_k$ *such that* $\delta(j_0) = 1$.

Proof. In this proof, we assume all integer index (which are *a priori* not necessarily positive) to be positive. By the choice of the sequence $\{\gamma_k\}_k$, we have

$$|\langle s_\sigma, \epsilon_j + \epsilon_{j'} \rangle| \geq e^{-2n\gamma} \min\{e^{\gamma_k + \gamma_{k'}}; k, k' \in [1, n]\} > e^{\gamma}$$

for each j, j'. It follows that weights of the form $\epsilon_j + \epsilon_{j'}$ does not belong to $\Psi(\mathbb{V}^{a_\sigma})$. We examine the assertion $\pm\epsilon_j, \epsilon_j - \epsilon_{j'} \in \Psi(\mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+)$ by the case-by-case analysis. We have three cases:

(Case $j, j' \in J_k$) We have $\epsilon_j - \epsilon_j' \in \Psi(\mathbb{V}^{a_\sigma})$ if and only if

$$\langle \epsilon_j - \epsilon_{j'}, s_\sigma \rangle = e^{(j'-j)\gamma} = e^{\gamma}.$$

Hence, we have $j' - j = 1$. Hence, we put $i := j, j' = i + 1$. We have $\langle \epsilon_i - \epsilon_{i+1}, D_\sigma \rangle = 1 \leq 1$, which verifies the first part of the assertion.

(Case $i \in J_k \neq J_m \ni j$) By (5.1), we deduce that

$$\langle \epsilon_j - \epsilon_{j'}, s_\sigma \rangle \in e^{\Gamma}$$

if and only if $j, j' \in J_k$ or $\delta_1(J_k) = \{0, 1\} = \delta_1(J_m)$ holds. We choose $j_0 \in J_k$ and $j_1 \in J_m$ such that $\delta_1(j_0) = 1 = \delta_1(j_1)$. We write $j = i + j_0$ and $j' := i' + j_1$. Then, we need

$$\langle \epsilon_{i+j_0} - \epsilon_{i'+j_1}, s_\sigma \rangle = e^{(i'-i)\gamma} = e^{\gamma}.$$

This happens if and only if $i' = i + 1$. By the definition of D_σ, we have

$$\langle \epsilon_{i+j_0} - \epsilon_{i+j_1+1}, D_\sigma \rangle \leq 1$$

if and only if $\underline{\#} J_k \geq \underline{\#} J_{k'}$. Since we assume $(\mathbf{J}, \vec{\delta})$ to be a strict marked partition, it follows that $\underline{\#} J_k \neq \underline{\#} J_{k'}$ by 1.12 4). This verifies the second part of the assertion.

(Case $j \in J_k$) If $\epsilon_j \in \Psi(\mathbb{V}^{a_\sigma})$ or $-\epsilon_j \in \Psi(\mathbb{V}^{a_\sigma})$, then we have $\langle s, \epsilon_j \rangle = e^{\gamma_0}$ or $e^{-\gamma_0}$, respectively. By (5.1), this forces $\epsilon_j \in \Psi(\mathbb{V}^{a_\sigma})$ and $\delta_1(J_k) = \{0, 1\}$. Let $j_0 \in J_k$ be such that $\delta_1(j_0) = 1$. Put $j = i + j_0$ for some $i \in \mathbb{Z}$. Then, we have $\langle \epsilon_{i+j_0}, D_\sigma \rangle = e^{i\gamma + \gamma_0} = e^{\gamma_0}$ if and only if $i = 0$. Moreover, we have $\langle \epsilon_{j_0}, D_\sigma \rangle = 1 \leq 1$ and $j_0 > 0$, which verifies the final part of the assertion. □

Lemma 5.6. *The group* P_σ *satisfies the following conditions:*

1. $P_\sigma = TU_\sigma \subset B$*, where* U_σ *is a unipotent subgroup of* G*;*
2. *The Lie algebra* $\mathfrak{u}_\sigma$ *of* U_σ *contains a root subspace* $\mathfrak{g}_\alpha \subset \mathfrak{g}$ *if and only if* $\alpha = \epsilon_j - \epsilon_{j'}$*, where* j, j' *are as follows:*
   - $j \in J_k \neq J_{k'} \ni j'$ *for some* k, k'*;*
   - *There exists* $j_0 \in J_k$ *and* $j_1 \in J_{k'}$ *such that* $\delta_1(j_0) = 1 = \delta_1(j_1)$*;*
   - $j - j_0 = j' - j_1$ *and* $j < j'$*.*

Proof. Let L_σ be the semisimple Levi component of L_σ which contains T. We have $L_\sigma \subset \bigcap_{m_1,m_2\in\mathbb{Z}} G(D_\sigma^{m_1} s_\sigma^{m_2})$. Each $G(D_\sigma^{m_1} s_\sigma^{m_2})$ is connected by Steinberg's centralizer theorem. Thus, we have $L_\sigma = T$ if we have $\alpha(D_\sigma^{m_1} s_\sigma^{m_2}) \neq 1$ for each $\alpha \in R$ and generic choices of m_1, m_2. This is equivalent to $\alpha(D_\sigma) \neq 1$ or $\alpha(s_\sigma) \neq 1$ holds for each $\alpha \in R$ since R is a finite set. In the below, we assume all integer index to be positive. By (5.1), we have $\langle \epsilon_i - \epsilon_j, s_\sigma \rangle = 1$ if and only if

$$\exists \sigma \in \{\pm 1\} \text{ s.t. } \sigma i \in J_k \neq J_{k'} \ni \sigma j \text{ and } \delta_1(J_k) = \{0,1\} = \delta_1(J_{k'}), \qquad (5.2)$$

where $J_k, J_{k'} \in \mathbf{J}$. By 1.12 4), we deduce that

$$\langle \epsilon_i, D_\sigma \rangle \neq \langle \epsilon_j, D_\sigma \rangle \text{ for each } i \in J_k, j \in J_{k'}.$$

Therefore, we deduce $L_\sigma = T$.

Since T normalizes the unipotent part of P_σ, we describe all one-parameter unipotent subgroup of G belongs to P_σ in order to prove the assertion. This is equivalent to count the set of one-parameter unipotent subalgebra $\mathfrak{g}_\alpha \subset \mathfrak{g}$ which commutes with s_σ and has eigenvalue ≤ 1 with respect to D_σ. We examine the case $\alpha = \epsilon_i - \epsilon_j$ with the assumption (5.2) for $\sigma = +1$. (This last part of the assumption is achieved by swapping the roles of i and j if necessary.) Fix $j_0 \in J_k$ and $j_1 \in J_{k'}$ such that $\delta_1(j_0) = 1 = \delta_1(j_1)$. Then, the definition of s_σ further asserts $i - j_0 = j - j_1$. In order that D_σ has eigenvalue ≤ 1, we need to have

$$\langle \epsilon_i, D_\sigma \rangle \leq \langle \epsilon_j, D_\sigma \rangle,$$

which is equivalent to $\underline{\#} J_k \geq \underline{\#} J_{k'}$. This implies $\# J_k > \# J_{k'}$ by 1.12 4). It follows that $i < j$, which verifies the second condition. Since $\alpha = \epsilon_i - \epsilon_j \in R^+$ in this case, we also deduce the first condition. $\square$

Proof of Propositon 5.4. Since $P_\sigma \subset G(s_\sigma)$, we have $P_\sigma \mathbf{v}_\sigma \subset \mathbb{V}^{a_\sigma}$. Since the reductive part of P_σ is equal to T, we deduce $P_\sigma \mathbf{v}_\sigma \subset {}^{w_\sigma}\mathbb{V}^+$. Therefore, it suffices to prove the following equality at the level of tangent space

$$T_{\mathbf{v}_\sigma}(P_\sigma \mathbf{v}_\sigma) \cong \mathfrak{p}_\sigma \mathbf{v}_\sigma = \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+ \qquad (5.3)$$

in order to deduce the assertion. Consider a T-weight decomposition $\mathbf{v}_\sigma^J = \sum_{\beta \in \Xi_J} v_\beta$, where $J \in \mathbf{J}$ and $0 \neq v_\beta \in \mathbb{V}[\beta]$. Since each Ξ_J consists of linearly independent weights of $X^*(T_J)$ or $X^*(T)$, we deduce

$$\mathfrak{t}\mathbf{v}_\sigma^J = \sum_{k\geq 1} \sum_{\beta \in \Xi_{J_k}} \mathbb{C} v_\beta.$$

It is easy to see that $\bigcup_{k\geq 1} \Xi_{J_k}$ is precisely the set of T-weights described in Lemma 5.5 1) and 3).

In the below, we apply the action of $\mathfrak{u}_\sigma$ (c.f. Lemma 5.6) to fill out each $\mathbb{V}[\beta]$ for each T-weight β described in Lemma 5.5 2). Such a β is written as $\epsilon_i - \epsilon_j$, where $i \in J_k, j \in J_{k'}$ are as in Lemma 5.5 2). By explicit calculation, we have a non-zero element of $\mathfrak{g}$ of weight $\epsilon_{m+j_0} - \epsilon_{m+j_1}$ which satisfies

$$\xi_m \mathbf{v}_\sigma = \begin{cases} y_{m-1+j_0, m+j_1} - y_{m+j_0, m+j_1+1} & (m + j_0 + 1 \in J_{k'}) \\ y_{m-1+j_0, m+j_1} & (m + j_0 + 1 \notin J_{k'}) \end{cases}.$$

for each $m + j_0 \in J_{k'}$. (Here we implicitly used $\overline{\#} J_k > \overline{\#} J_{k'}$, which is deduced from $\underline{\#} J_k > \underline{\#} J_{k'}$ by 1.12 4).) It is clear that $\xi_m \in \mathfrak{p}_\sigma$. We have

$$\left(\sum_{m \in \mathbb{Z}; m + j_1 \in J_{k'}} \mathbb{C} \xi_m \right) \mathbf{v}_\sigma = \sum_{m \in \mathbb{Z}; m + j_1 \in J_{k'}} \mathbb{V}[\epsilon_{m-1+j_0} - \epsilon_{m+j_1}].$$

By summing up for all possible pairs $(J_k, J_{k'}) \in \mathbf{J}$, the set of T-weights appearing in the RHS exhausts the T-weights described in Lemma 5.5 2). $\square$

Corollary 5.7. *Keep the setting of Proposition 5.4. Let $\nu = (a, \mathbf{v}_\sigma) = (s, \vec{q}, \mathbf{v}_\sigma)$ be an admissible parameter. Then, the natural embedding*

$$P_\sigma(s) \mathbf{v}_\sigma \subset \mathbb{V}^a \cap \mathbb{V}^{a_\sigma} \cap {}^{w_\sigma}\mathbb{V}^+$$

is dense open.

Proof. The assertion follows merely by taking a-fixed part of (5.3) in the proof of Proposition 5.4. $\square$

6 A vanishing theorem

We retain the setting of the previous section.

Let $\nu = (a, X)$ be a pre-admissible parameter. Associated to ν, we have a subvariety

$$\boldsymbol{\mu}^{-1}(X)^a \subset F = G \times^B \mathbb{V}^+.$$

By construction, it is immediate to see that the image of $\boldsymbol{\mu}^{-1}(X)^a$ under the projection to the base space G/B is an isomorphism. We denote this image by $\mathcal{E}_X^a$. We may also write $\mathcal{E}_X$ instead of $\mathcal{E}_X^a$ when $a = (1, 1, 1, 1)$.

Theorem 6.1 (Cohomology vanishing theorem)**.** *Let $\nu = (a, X)$ be an admissible parameter or $a = a_0$. Then, we have*

$$H_{2i+1}(\mathcal{E}_X^a) = 0 \textit{ for every } i = 0, 1, \ldots.$$

Moreover, we have an isomorphism

$$\mathsf{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{E}_X^a) \xrightarrow{\cong} H_\bullet(\mathcal{E}_X^a).$$

Remark 6.2. **1)** The map ch in Theorem 6.1 is the homology Chern character map. (See e.g. [CG97] §5.8.) It sends the class of the (embedded) structure sheaf $\mathcal{O}_C$ for a closed subvariety $C \subset \mathcal{E}_X^a$ to

$$\mathsf{ch}[\mathcal{O}_C] = [C] + \text{lower degree terms} \in H_{2 \dim C}(\mathcal{E}_X^a) \oplus \cdots \oplus H_0(\mathcal{E}_X^a).$$

2) The first part of Theorem 6.1 is valid even for integral coefficient case when $G(s) \subset GL(n, \mathbb{C})$ (c.f. [AH07])[2]. Here we present a proof along the line of earlier versions of this paper, with a slight enhancement informed to the author by Eric Vasserot.

[2] Previous versions of this paper also contain such a result. The author decided to drop it since [AH07] contains slightly stronger statement and a better proof.

6.1 Review of general theory on cohomology vanishing

In this subsection, we recall several definitions and results of [KL87] and [DLP88] which we need in the course of our proof of Theorem 6.1.

Definition 6.3 (α-partitions)**.** A partition of a variety $\mathcal{X}$ over $\mathbb{C}$ is said to be an α-partition if it is indexed as $\mathcal{X}_1, \mathcal{X}_2, \ldots \mathcal{X}_k$ in such a way that $\mathcal{X}_1 \cup \ldots \cup \mathcal{X}_i$ is closed for every $i = 1, \ldots, k$.

Theorem 6.4 ([DLP88] 1.7–1.10)**.** *Let $\mathcal{X}$ be a variety with α-partition $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k$. If we have*

$$H_{2i+1}(\mathcal{X}_m) = 0 \text{ for every } i = 0, 1, \ldots$$

for each $m = 1, \ldots, k$, then we have

$$H_{2i+1}(\mathcal{X}) = 0 \text{ for every } i = 0, 1, \ldots.$$

Moreover, we have

$$\sum_{i \geq 0} \dim H_{2i}(\mathcal{X}) = \sum_{m \geq 1} \sum_{i \geq 0} \dim H_{2i}(\mathcal{X}_m).$$

Theorem 6.5 ([DLP88] 1.5)**.** *Let $\pi : \mathcal{E} \to \mathcal{X}$ be a vector bundle over a smooth variety $\mathcal{X}$, with a fiber preserving linear $\mathbb{C}^\times$-action on $\mathcal{E}$ with strictly positive weights. Let $\mathcal{Z} \subset \mathcal{E}$ be a $\mathbb{C}^\times$-stable smooth closed subvariety. Then, $\pi(\mathcal{Z})$ is smooth and $\mathcal{Z}$ is a subbundle of $\mathcal{E}$ restricted to $\pi(\mathcal{Z})$.*

Theorem 6.6 ([KL87] Lemma 4.4)**.** *Let $\mathcal{Z}$ be a smooth variety with $\mathbb{G}_m$-action. Assume that some $t \in \mathbb{G}_m$ satisfies*

$$\mathcal{Z}^{\mathbb{G}_m} = \mathcal{Z}^t = \{\lim_{n \to \infty} t^n z \in \mathcal{Z} \cup \{\emptyset\}; \forall z \in \mathcal{Z}\}.$$

(I.e. every point of $\mathcal{Z}$ converges to a point of $\mathcal{Z}^t$.) Let $\mathcal{X} \subset \mathcal{Z}$ be a (possibly singular) subvariety such that $\mathcal{X}^{\mathbb{G}_m} = \mathcal{Z}^{\mathbb{G}_m}$. Then, the two assertions

$$H_{2i+1}(\mathcal{Z}) = 0 \text{ for every } i = 0, 1, \ldots, \text{ and } H_{2i+1}(\mathcal{X}) \text{ for every } i = 0, 1, \ldots$$

are equivalent. Moreover, we have

$$\sum_{i \geq 0} \dim H_{2i}(\mathcal{Z}) = \sum_{i \geq 0} \dim H_{2i}(\mathcal{X}).$$

Remark 6.7. Notice that in [KL87], the statement is given by topological K-groups (with complex coefficient). Here we identify them with our odd-part of the Borel-Moore homology by the isomorphism

$$K_1^{top}(\mathcal{Z}) \cong \bigoplus_{i \geq 0} H_{2i+1}(\mathcal{Z}),$$

which is valid for smooth varieties.

6.2 Proof of vanishing theorem

Proposition 6.8 (Weak version of Theorem 6.1)**.** *Let* $(a, X) = (s, \vec{q}, X)$ *be an admissible parameter or* $a = a_0$*. Then, we have some semi-simple element* $s_X \in G(s)$ *with the following properties:*

- s_X *has only positive real eigenvalues on* V_1*;*
- *We have*
$$H_{2i+1}((\mathcal{E}_X^a)^{s_X}) = 0 \text{ for every } i = 0, 1, \dots;$$
- *Each connected component of* $(\mathcal{E}_X^a)^{s_X}$ *is smooth projective;*
- *We have an isomorphism*
$$\mathsf{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K((\mathcal{E}_X^a)^{s_X}) \xrightarrow{\cong} H_\bullet((\mathcal{E}_X^a)^{s_X}).$$

Before giving the proof of Proposition 6.8, we present a proof of Theorem 6.1 for (a, X) assuming Proposition 6.8 for an element (a, X).

Proof of Theorem 6.1 for (a, X). Let $\mathcal{E}_1, \mathcal{E}_2, \dots$ be a sequence of all connected components of $(\mathcal{E}_X^a)^{s_X}$. For each $\mathcal{E}_k$, we set

$$\mathcal{B}_k := \{gB \in G/B; \lim_{N \to \infty} s_X^N gB \in \mathcal{E}_k\}.$$

Let
$$P := \{g \in G; \lim_{N \to \infty} \mathrm{Ad}(s_X^N) g \in G\}$$
be a parabolic subgroup of G. It is well-known that

$$\lim_{N \to \infty} \mathrm{Ad}(s_X^N) g \in G(s_X)$$

for $g \in P$. It follows that each $\mathcal{B}_k$ intersects with a unique P-orbit in G/B. In particular, we can assume that the sequence $\mathcal{B}_1, \mathcal{B}_2, \dots$ forms an α-partition of $\bigcup_{k \ge 1} \mathcal{B}_k \subset G/B$ by rearranging the sequence if necessary. By regarding each P-orbit of G/B as a vector bundle over P/B, we can regard $\mathcal{B}_k$ as a vector bundle over $\mathcal{E}_k$ by restriction. By Theorem 6.5, we deduce that $(\mathcal{B}_k \cap \mathcal{E}_X^a)$ is a vector bundle over $\mathcal{E}_k$, hence we have $K(\mathcal{E}_k) \cong K(\mathcal{B}_k \cap \mathcal{E}_X^a)$.

We have obvious decomposition

$$\mathcal{E}_X^a = \bigsqcup_{k \ge 1} (\mathcal{E}_X^a \cap \mathcal{B}_k).$$

This decomposition is an α-partition.

For the first assertion, it suffices to prove that

$$H_{2i+1}(\mathcal{E}_k) = 0 \text{ for every } i = 0, 1, \dots$$

for each k. For the second assertion, it suffices to prove isomorphisms

$$\mathsf{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{E}_k) \xrightarrow{\cong} H_\bullet(\mathcal{E}_k)$$

for each k, since the Chern character map commutes with the localization sequence and pullback along the fibers of vector bundles. (Thom isomorphism for K-theory.) Both assertions follow from Proposition 6.8. □

The rest of this section is devoted to the proof of Proposition 6.8.

The natural projection induces an isomorphism $(\boldsymbol{\mu}^{-1}(X)^a)^{(s_X,\vec{q}_X)} \cong (\mathcal{E}_X^a)^{s_X}$ for every $\vec{q}_X \in \mathbb{G}_m^3$ such that $s_X X = \vec{q}_X X$. We have

$$(a, X) = (s, \vec{q}, X) = \prod_{\mathbf{c} \in \mathcal{C}_a} (s_{\mathbf{c}}, \vec{q}, X_{\mathbf{c}}).$$

By the same argument as in the proof of Corollary 3.10, each connected component of $\boldsymbol{\mu}^{-1}(X)^a$ is a product of connected components of

$$\mathcal{E}_{X_{\mathbf{c}}}^{(s_{\mathbf{c}},\vec{q})} \subset Sp(2n^{\mathbf{c}})/(B \cap Sp(2n^{\mathbf{c}})).$$

Therefore, by the Künneth formula, it suffices to prove the assertion when $\mathcal{C}_a$ consists of a unique clan $[1, n]$. By Proposition 4.8, we further assume that $s \in T$, and $X = \mathbf{v}_\sigma$ for a strict marked partition $\sigma = (\mathbf{J}, \vec{\delta})$ by taking G-conjugate if necessary.

Now we take $(s_X, \vec{q}_X) := a_\sigma$. Let A be the Zarkiski closure of the subgroup of $\boldsymbol{T}$ generated by a and a_σ. We put $F^A(w) := G(A) \times^{B(A)} (\mathbb{V}^A \cap {}^w\mathbb{V}^+)$ for each $w \in W$. We have $\bigcup_{w \in W} F^A(w) = (G \times^B \mathbb{V}^+)^A$. Consider the map

$${}^w\mu^A : F^A(w) = G(A) \times^{B(A)} (\mathbb{V}^A \cap {}^{vw}\mathbb{V}^+) \longrightarrow \mathbb{V}^A \cap {}^w\mathbb{V}^+,$$

where $v \in N_{G(A)}(T)/T \subset W$ is a unique element such that $B(A) \subset {}^{vw}B$.

Lemma 6.9 (Part of Proposition 6.8). *Each connected component of $(\mathcal{E}_X^a)^{a_\sigma}$ is smooth projective.*

Proof. Projectivity follows from that of $\mathcal{E}_X$, which itself follows by Theorem 1.2 3). By Lemma 5.6 1) and Corollary 5.7, we deduce that

$$\overline{B(A)\mathbf{v}_\sigma} \subset \mathbb{V}^A$$

is a linear subspace. It follows that $({}^w\mu^A)^{-1}(\overline{B(A)\mathbf{v}_\sigma})$ is a smooth subvariety of $G(A) \times^{B(A)} (\mathbb{V}^A \cap {}^w\mathbb{V}^+)$. Hence, $({}^w\mu^A)^{-1}(B(A)\mathbf{v}_\sigma)$ is a smooth subvariety of F_A^w. Since changing $\mathbf{v}_\sigma$ by $B(A)$-action gives an isomorphic fibers, we deduce that $({}^w\mu^A)^{-1}(\mathbf{v}_\sigma)$ is a smooth subvariety of F_A^w as required. $\square$

Corollary 6.10 (of the Proof of Lamma 6.9). *The variety $({}^w\mu^A)^{-1}(\overline{B(A)\mathbf{v}_\sigma})$ is smooth.* $\square$

We return to the proof of Proposition 6.8.

We prove the rest of assertions by the induction on the cardinality $n(\sigma)$ of the set

$$\mathtt{N}(\sigma) := \{J \in \mathbf{J}; \delta_1(J) = \{0, 1\}\}.$$

Notice that this implies that we can assume Theorem 6.1 for all admissible parameter of the form $(a, \mathbf{v}_\sigma)$ such that $n(\sigma) < n(\sigma')$. If $n(\sigma) = 0$, then Lemma 5.6 2) asserts that $G(s_\sigma) = T$. This implies that $(\mathcal{E}_X^a)^{s_\sigma}$ is a union of points. Thus, we obtained the assertion for $n(\sigma) = 0$.

We prove the assertion for $n(\sigma) = k$ by assuming that the assertion holds for all $n(\sigma) < k$. Let $J \in \mathtt{N}(\sigma)$ be the member such that $\#J \geq \#J'$ for every $J' \in \mathtt{N}(\sigma)$. Let σ' be a strict marked partition obtained from σ by replacing δ_1 by δ_1' defined as:

$$\delta_1'(J) = \{0\}, \text{ and } \delta_1'(j) = \delta_1(j) \text{ for all } j \in [1, n] \backslash J.$$

Let $j_0 \in J$ be its unique element such that $\delta_1(j_0) = 1$. By Lemma 1.17, there exists $t \in T_J$ such that

$$\lim_{N \to \infty} t^N \mathbf{v}_\sigma = \mathbf{v}_\sigma - \mathbf{v}^J_{1,\sigma} = \mathbf{v}_{\sigma'}.$$

By Lemma 5.6 2), every T-weights of P_σ containing $i \in J$ is of the form $\epsilon_i - \epsilon_j$ for some $j \in J'$. Moreover, we have $P_{\sigma'} \subset P_\sigma$. It follows that the action of $t \in T$ contracts P_σ to $P_{\sigma'}$. By Corollary 5.7, the t-action also contracts $\mathbb{V}^A \cap {}^{w_\sigma}\mathbb{V}^+$ to

$$\mathbb{V}^A \cap \mathbb{V}^{a_{\sigma'}} \cap {}^{w_{\sigma'}}\mathbb{V}^+ = \mathbb{V}^{A'} \cap {}^{w_{\sigma'}}\mathbb{V}^+,$$

where A' is the Zariski closure of $\langle a, a_{\sigma'} \rangle \subset \boldsymbol{T}$.

Therefore, the t-action contracts $({}^w\mu^A)^{-1}(\overline{B(A)\mathbf{v}_\sigma})$ to $({}^w\mu^{A'})^{-1}(\overline{B(A')\mathbf{v}_{\sigma'}})$. By taking the quotient of

$$\mathcal{S} := B(A)\mathbf{v}_\sigma \cup B(A)\mathbf{v}_{\sigma'}$$

by $\mathsf{Stab}_{B(A)}\mathbf{v}^J_{1,\sigma}$, we obtain an affine plane $\mathbb{A}^1$ with contracting t-action to the origin. Therefore, we obtain a smooth family of smooth projective varieties over $\mathbb{A}^1$ whose fiber over $0 \in \mathbb{A}^1$ is $\mathcal{E}^A_{\mathbf{v}_{\sigma'}}$ and whose general fiber $\mathcal{E}^A_{\mathbf{v}_\sigma}$ contracting to $\mathcal{E}^{A'}_{\mathbf{v}_{\sigma'}}$. Thus, applying Theorem 6.6 proceeds the induction for homologies. For K-theory, moving smooth projective varieties is the same as moving all cycles by rational equivalence. Therefore, it suffices to prove

$$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{E}^A_{\mathbf{v}'_\sigma}) \xrightarrow{\cong} H_\bullet(\mathcal{E}^A_{\mathbf{v}'_\sigma}).$$

This is guaranteed by Theorem 6.1 for $(a, \mathbf{v}'_\sigma)$, which is proved by the induction hypothesis. Therefore, we have Proposition 6.8 for the pair $(a, \mathbf{v}_\sigma)$ for all strict marked partition with fixed $n(\sigma)$. Hence, the induction proceeds and we have proved Proposition 6.8 (and hence Theorem 6.1).

7 Standard modules and an induction theorem

We retain the setting of the previous section.

Definition 7.1 (Standard modules)**.** Let $\nu = (a, X)$ be a pre-admissible parameter. We define

$$M_\nu := H_\bullet(\mathcal{E}^a_X) \text{ and } M^\nu := H^\bullet(\mathcal{E}^a_X).$$

By the Ginzburg theory [CG97] 8.6, each of M_ν or M^ν is a $\mathbb{H}$-module.

By the symmetry of the construction of varieties involved in $M_{(a,X)}$ and $\mathbb{H}_a$, we deduce $M_{(a,X)} \cong M_{(\mathrm{Ad}(g)a, gX)}$ as $\mathbb{H}_a = \mathbb{H}_{\mathrm{Ad}(g)a}$-modules for each $g \in \boldsymbol{G}(a)$.

Let $s_Q \in T(\mathbb{R})$ be an element such that

$$0 < \langle \alpha, s_Q \rangle \leq 1 \text{ for all } \alpha \in R^+. \tag{7.1}$$

Let $Q := G(s_Q)$ and $\boldsymbol{Q} := Q \times (\mathbb{C}^\times)^3$. These are subgroups of G and $\boldsymbol{G}$, respectively. We put $\mathbb{V}_Q := \mathbb{V}^{s_Q}$ and $\mathfrak{N}_Q = \mathfrak{N}_2^{s_Q} \subset \mathbb{V}_Q$. We have a map

$$\mu_Q : F_Q := Q \times^{(Q \cap B)} (\mathbb{V}_Q \cap \mathbb{V}^+) \longrightarrow \mathfrak{N}_Q.$$

We define $Z_Q := F_Q \times_{\mathfrak{N}_Q} F_Q$.

The natural inclusion map

$$Q \times^{(Q\cap B)} (\mathbb{V}_Q \cap \mathbb{V}^+) \hookrightarrow \bigcup_{w\in W} Q \times^{{}^w B(s_Q)} (\mathbb{V}_Q \cap {}^w\mathbb{V}^+) = \boldsymbol{F}^{s_Q}$$

gives an identification of F_Q with a connected component of $\boldsymbol{F}^{s_Q}$. This equips an action of $\boldsymbol{Q}$ on $\mathfrak{N}_Q$, F_Q, and Z_Q by restricting the $\boldsymbol{G}$-actions on their ambient spaces.

We put

$$\mathbb{H}_Q := \mathbb{C} \otimes_{\mathbb{Z}} K^{Q\times(\mathbb{C}^\times)^3}(Z_Q),$$

where the convolution algebra structure on $K^{\boldsymbol{Q}}(Z_Q)$ are equipped by the restriction of the maps p_1 and p_2 from $\boldsymbol{Z} \to \boldsymbol{F}$ to $Z_Q \to F_Q$.

Lemma 7.2. *Keep the above setting. Form an increasing sequence of integers*

$$1 \le n_1 \le n_2 \le \cdots$$

by requiring that

$$\alpha_i(s_Q) < 1 \text{ if and only if } i = n_k \text{ for some } k.$$

Then, we have

1. *$\mathbb{H}_Q$ is a subalgebra of $\mathbb{H}$ generated by $\mathcal{A}[T]$ and the set*

$$\{T_i; i \neq n_k \text{ for some } k\};$$

2. *For a pre-admissible parameter $\nu = (s, \vec{q}, X)$ such that $s \in T$ and $X \in \mathfrak{N}_Q$, the vector space*

$$M_\nu^Q := H_\bullet(\mu_Q^{-1}(X)^{(s,\vec{q})})$$

is a $\mathbb{H}_Q$-module.

Proof. By the condition (7.1), we have $\langle \alpha + \beta, s_Q\rangle = 1$ for $\alpha, \beta \in R^+$ if and only if $\langle \alpha, s_Q \rangle = 1$ and $\langle \beta, s_Q\rangle = 1$. This implies that the Q is generated by T and the one-parameter unipotent subgroups corresponding to simple roots α_i (and $-\alpha_i$) such that $\alpha_i(s_Q) = 1$.

The variety F_Q decomposes into a product of vector bundles over the flag varieties of simple components of Q. By explicit computation, we deduce that the vector bundles we concern are either **a)** the cotangent bundle of the flag variety when the simple component is type A, or **b)** the variety $\boldsymbol{F}$ for a (possibly smaller) symplectic group which arose as a simple component of Q. Moreover, the map μ_Q is the product of the moment maps of cotangent bundles of flag varieties of type A and our map $\boldsymbol{\mu}$ (for some symplectic group).

Hence, taking account into the argument in §2, both statements are straightforward modifications of [CG97] §7.6 and §8.6. Thus, we leave the details to the reader. $\square$

Let $\mathbb{V}_U$ be the unique T-equivariant splitting of the map $\mathbb{V}^+ \longrightarrow \mathbb{V}^+/\mathbb{V}_Q^+$. If $X \in \mathbb{V}$ satisfies $s_Q X = X$, then s_Q has eigenvalue < 1 on $\mathfrak{u}X$. Hence, we have necessarily $\mathfrak{u}X \subset \mathbb{V}_U$.

Theorem 7.3 (Kazhdan-Lusztig type induction theorem)**.** *Let $\nu = (s, \vec{q}, X)$ be an admissible parameter. Let P be a parabolic subgroup of G with its Levi decomposition $P = QU$ such that $s \in Q$ and $X \in \mathfrak{N}_Q$. If we have*

$$\mathbb{V}_U^a \subset \mathfrak{u}X, \tag{7.2}$$

then we have an isomorphism

$$\mathsf{Ind}_{\mathbb{H}_Q}^{\mathbb{H}} M_\nu^Q \cong M_\nu$$

as $\mathbb{H}$-modules, where M_ν^Q is in Lemma 7.2.

Remark 7.4. As we observe in the proof, one may obtain the strong induction theorem in the sense of Lusztig [Lu02] (i.e. Theorem 7.3 without (7.2)) if we have sufficiently nice slice instead of our $\mathcal{E}_X^{\sim}$.

The rest of this section is devoted to the proof of Theorem 7.3.

We choose $s_Q \in T(\mathbb{R})$ so that $G(s_Q) = Q$ and (7.1) holds (so that we can freely use the notation and terminology in the earlier part of this section). By taking Q-conjugation if necessary, we assume $X \in \mathbb{V}^+$.

Let $W_Q := N_Q(T)/T \subset W$. We define

$$W^Q := \{w \in W; \ell(w) \leq \ell(vw) \text{ for all } v \in W_Q\}.$$

Let $w \in W$. Let $\mathcal{O}_w$ be the P-orbit of G/B which contains $\dot{w}B$. By counting the weights, we have $(\mathbb{V}^+ \cap \mathbb{V}_Q) \subset (\mathbb{V}^+ \cap {}^w\mathbb{V}^+)$. It follows that $X \in (\mathbb{V}^+ \cap {}^w\mathbb{V}^+)$. Hence, the map

$$(\mathcal{E}_X \cap \mathcal{O}_1) = \mu_Q^{-1}(X) \ni gB \mapsto g\dot{w}B \in \mathcal{E}_X \cap \mathcal{O}_w$$

gives rise to an isomorphism $(\mathcal{E}_X \cap \mathcal{O}_1) \cong \mathcal{E}_X \cap \mathcal{O}_w^{s_Q}$. Let B^- be the opposite Borel subgroup of B with respect to T. We put $U_w := U \cap {}^wB^-$. Since s_Q attracts points of $\mathcal{O}_w$, we obtain a map

$$\psi_w : \mathcal{E}_X \cap \mathcal{O}_w \to (\mathcal{E}_X \cap \mathcal{O}_1)$$

by sending each point p to $\lim_{N \to \infty} s_Q^N \mathsf{p}$. We have an expression of a point $gu\dot{w}B \in \mathcal{E}_X \cap \mathcal{O}_w$ as $g \in Q$, $gB \in \mu_Q^{-1}(X)$, and $u \in U_w$. Let w_Q be the longest element of W^Q.

Lemma 7.5. *The fiber of the map ψ_w at $gB \in QB$ is given as*

$$\psi_w^{-1}(gB) = \{u \in gU_wg^{-1}; uX - X \in g^w\mathbb{V}^+ \cap \mathbb{V}_U\}.$$

In particular, $\psi_{w_Q}^{-1}(gB)$ is isomorphic to $\mathsf{Stab}_U(X)$.

Proof. The variety $\mathcal{O}_w$ is a U_w-fibration over $\mathcal{O}_1$. The condition $X \in gu\dot{w}\mathbb{V}^+$ is equivalent to $(gu^{-1}g^{-1})X - X \in g\dot{w}\mathbb{V}^+$. Moreover, U is Q-stable and $(gu^{-1}g^{-1})X - X \in \mathbb{V}_U$, which implies the first result. Since $U_{w_Q} = U$ and $g^{w_Q}\mathbb{V}^+ \cap \mathbb{V}_U = \{0\}$, we conclude the second assertion. □

Corollary 7.6. *We have*

$$\dim H_\bullet(\mathcal{E}_X \cap \mathcal{O}_w) = \dim H_\bullet(\mathcal{E}_X \cap \mathcal{O}_1).$$

Proof. By the proof of Theorem 6.1 and [KL87] Theorem 4.4, we have an α-partition of $\mathcal{E}_X \cap \mathcal{O}_1$ into an α-partition $\mathcal{X}_1, \mathcal{X}_2, \dots$ such that each term is a smooth variety without odd-term homology. Applying Theorem 6.6 by setting $\mathcal{Z}$ to be the pullback of each piece along the contraction map $\mathcal{O}_w \to \mathcal{O}_w^{s_Q}$, we deduce that

$$\sum_{i \geq 0} \dim H_{2i}(\mathcal{X}_m) = \sum_{i \geq 0} \dim H_{2i}(\psi_w^{-1}(\mathcal{X}_m))$$

and

$$H_{2i+1}(\mathcal{X}_m) = H_{2i+1}(\psi_w^{-1}(\mathcal{X}_m)) = 0 \text{ for } i = 1, 2, \dots$$

for each m. Since $\psi_w^{-1}(\mathcal{X}_1), \psi_w^{-1}(\mathcal{X}_2), \dots$ forms an α-partition of $\mathcal{E}_X \cap \mathcal{O}_w$, we obtain the result. $\square$

We return to the proof of Theorem 7.3.

It is easy to see that

$$\mathcal{E}_X = \bigsqcup_{w \in W^Q} (\mathcal{E}_X \cap \mathcal{O}_w)$$

forms an α-partition. Together with Theorem 6.1 and Corollary 7.6, this implies

$$\dim M_\nu = (\# W^Q) \dim M_\nu^Q = (\# W / \# W_Q) \dim M_\nu^Q. \tag{7.3}$$

Moreover, the natural map

$$\imath : M_\nu^Q = H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1) \hookrightarrow H_\bullet(\mathcal{E}_X^a) = M_\nu$$

is injective. Since we have

$$p_1(Z_{\leq s_i} \cap p_2^{-1}(\mathcal{O}_1)) \subset \mathcal{O}_1 \text{ if } i \neq n_k \text{ for some } k = 1, 2, \dots,$$

the map $\imath$ is an embedding of $\mathbb{H}_Q$-modules. (The sequence $\{n_k\}_k$ is borrowed from Lemma 7.2.) Hence, we have an induced map

$$\phi : \mathsf{Ind}_{\mathbb{H}_Q}^{\mathbb{H}} M_\nu^Q \longrightarrow M_\nu.$$

Thanks to (7.3), we have:

Lemma 7.7. *Theorem 7.3 follows if ϕ is surjective.* $\square$

We return to the proof of Theorem 7.3.

For each $w \in W^Q$, we define

$$R_w := [\mathcal{O}_{Z_{\leq w^{-1}}}] \in K^{\boldsymbol{G}}(\boldsymbol{Z}).$$

By the construction of §2, we have

$$R_w H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1) \subset H_\bullet(\mathcal{E}_X^a \cap \overline{\mathcal{O}_w}) \subset H_\bullet(\mathcal{E}_X^a).$$

Since W^Q has a partial order $\leq_Q$ induced by the Bruhat order, we put

$$H_\bullet(\mathcal{E}_X^a)_{\leq w} := \sum_{v \leq_Q w; v \in W^Q} R_v H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1).$$

Consider the composition map

$$\tau_w : H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_1) \xrightarrow{R_w \circ} H_\bullet(\mathcal{E}_X^a \cap \overline{\mathcal{O}_w}) \xrightarrow{\mathsf{res}} H_\bullet(\mathcal{E}_X^a \cap \mathcal{O}_w).$$

Lemma 7.8. *Theorem 7.3 follows if each τ_w is surjective.*

Proof. We have

$$\dim \mathrm{Im}\phi \geq \sum_{w \in W^Q} \dim \mathsf{gr} H_\bullet(\mathcal{E}_X^a)_{\leq w} = \sum_{w \in W^Q} \dim M_\nu^Q = (\# W^Q) \dim M_\nu^Q .$$

Here gr stands for the graded quotient with respect to some completed order on W^Q which extends $\leq_Q$. By (7.3), we conclude that ϕ must be surjective under this assumption. $\square$

We return to the proof of Theorem 7.3.

We have only to prove that each τ_w is surjective provided if (7.2) holds. We have an open enbedding

$$\left(p_2^{-1}(\mathcal{E}_X \cap \mathcal{O}_1) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})\right) \subset \left(p_2^{-1}(\mathcal{E}_X \cap \mathcal{O}_1) \cap Z_{\leq w^{-1}}\right).$$

Lemma 7.9. *For each subset $\mathcal{E} \subset (\mathcal{E}_X \cap \mathcal{O}_1)$, we have $\left(p_2^{-1}(\mathcal{E}) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})\right) \cong \psi_w^{-1}(\mathcal{E})$.*

Proof. By definition, the LHS is written as:

$$\{(g_1 B, g_2 B) \in \boldsymbol{\mu}^{-1}(X) \times (\mathcal{E}_X \cap \mathcal{O}_1); g_1^{-1} g_2 \in B \dot{w}^{-1} B\}.$$

Since $B \cap Q = {}^w B \cap Q$, we have $B\dot{w}^{-1}B = B\dot{w}^{-1}U$. By taking the right B-translation if necessary, we can assume $g_1 \in g_2 U \dot{w}$. This forces $g_1 B$ to live in the fiber of the map ψ_w. This implies that $g_2 B$ is completely determined by the data of $\psi_w^{-1}(\mathcal{E})$ and vice versa. $\square$

Let A be the Zariski closure of $\langle a, s_Q \rangle \subset \boldsymbol{T}$. The set $\mathfrak{u}X \subset \mathbb{V}_U$ is an A-stable linear subspace. It follows that

$$S := \mathbb{V}/\mathfrak{u}X$$

has a A-stable splitting in $\mathbb{V}_U$. Using this splitting, we define

$$\mathcal{E}_X^{\sim} := \{(gB, X + y) \in \boldsymbol{F}; gB \in (\mathcal{E}_X \cap \mathcal{O}_1), y \in S\}.$$

Each element of $\mathbb{V}_U$ is contracted to 0 by the s_Q-action. Hence, S has a contraction to $0 \in S$. This gives a contraction

$$\theta : \mathcal{E}_X^{\sim} \longrightarrow (\mathcal{E}_X \cap \mathcal{O}_1)$$

given by collecting s_Q-attracting points.

Proposition 7.10. *For each $w \in W^Q$, the intersection of $\boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})$ and $(\boldsymbol{F} \times \mathcal{E}_X^{\sim})$ is transversal inside $\boldsymbol{F}^2$.*

Proof. We prove the assertion by induction. The case $w = 1$ is clear. Assume that

- $w = w's$ by $w' \in W^Q$ and $s = s_i$ such that $\ell(w) = \ell(w') + \ell(s)$;
- The assertion holds for w';

and prove the assertion for w. For $v = w, w'$, we set

$$\mathcal{E}^{\sim}(v) := \{(g\dot{v}B, X + y) \in p_1((\boldsymbol{F} \times \mathcal{E}_X^{\sim}) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{v^{-1}}))\}.$$

We denote the fibers of the maps $\mathcal{E}^{\sim}(v) \to (\mathcal{E}_X \cap \mathcal{O}_v^{s_Q})$ over $g\dot{v}B$ as:

$$F_v(gB) := \{(u, X + y) \in gU_v g^{-1} \times S; u^{-1}X - X \in g^v\mathbb{V}^+, y \in S \cap gu^v\mathbb{V}^+\}.$$

Assume to the contrary to deduce contradiction. The failure of the dimension condition of transversal intersection implies

$$\begin{aligned} &\dim F_w(gB) > \dim F_{w'}(gB) \text{ if } s \neq s_n \text{ or} \\ &\dim F_w(gB) \geq \dim F_{w'}(gB) \text{ if } s = s_n \end{aligned} \tag{7.4}$$

for some $gB \in (\mathcal{E}_X \cap \mathcal{O}_1)$. Being transversal intersection is an open condition. Since the intersection has a contraction to its s_Q-fixed point, it suffices to consider the situation near s_Q-fixed points. Hence, we replace $F_v(gB)$ in (7.4) by the following tangent space version

$$\mathfrak{f}_v(gB) := \{(\xi, y) \in \mathrm{Ad}(g)\mathfrak{u}_v \times S; \xi X \in g^v\mathbb{V}^+, y \in S \cap g^v\mathbb{V}^+\}.$$

Let $\mathfrak{u}^v := \mathrm{Lie}(U \cap {}^vB)$. It is clear that $\mathfrak{u}^v X \subset {}^v\mathbb{V}^+$ since $X \in {}^v\mathbb{V}^+$. We have $\mathfrak{u} = \mathfrak{u}^v \oplus \mathfrak{u}_v$. It follows that

$$\dim \mathfrak{f}_v(gB) = \dim \mathsf{Stab}_{\mathfrak{u}_v} g^{-1}X + \dim(\mathbb{V}_U \cap {}^v\mathbb{V}^+)/\mathfrak{u}^v X.$$

It is impossible to achieve the infinitesimal version of (7.4) since the dimensions compensates each other. Hence, we have contradiction. It follows that the intersection of $\boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})$ and $(\boldsymbol{F} \times \mathcal{E}_X^{\sim})$ must have proper dimension inside $\boldsymbol{F}^2$ under the induction hypothesis.

Now the linear independence of the normal vectors follows as an immediate consequence of the fact that they are concentrated on the first factor and S on the second factor (of $\boldsymbol{F} \times \mathcal{E}_X^{\sim}$), or the diagonal part (of $\boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}})$), respectively.

Therefore, the induction proceeds and we obtain the result. $\square$

Lemma 7.11. *The map τ_w is an isomorphism.*

Proof. By [CG97] 2.7.26 and Proposition 7.10, we deduce that the map τ_w induces an isomorphism

$$H_\bullet((\mathcal{E}_X^{\sim})^a) \xrightarrow{\cong} H_\bullet(p_1((\boldsymbol{F} \times \mathcal{E}_X^{\sim}) \cap \boldsymbol{\pi}^{-1}(\mathsf{O}_{w^{-1}}))^a). \tag{7.5}$$

The spaces appearing in the homologies are given as fibrations over $(\mathcal{E}_X \cap \mathcal{O}_1)$ and $(\mathcal{E}_X \cap \mathcal{O}_w)$ with its fiber linear subspaces of S. Here $\mathcal{E}_X^{\sim}$ has larger fiber. Hence, the map τ_w itself is surjective if $[Y], [\theta^{-1}(Y)] \in K^A(\mathcal{E}_X^{\sim})$ define the same cycle up to an invertible factor for each A-stable closed subvariety $Y \subset (\mathcal{E}_X \cap \mathcal{O}_1)$. (Here we switched to algebraic K-theory thanks to Theorem 6.1 and Lemma 7.6.) This is true if the alternating sum of the Koszul complex of S is invertible in $R(A)_a$. This is equivalent to $S^a = 0$, which is further re-pherased as

$$\mathbb{V}_U^a \subset \mathfrak{u}X.$$

This is (7.2). $\square$

We return to the proof of Theorem 7.3.

Thanks to Lemma 7.11, we have finished the proof of Theorem 7.3 by Lemma 7.8.

8 Exotic Springer correspondence

We keep the setting of the previous section.

As we see in §1.4, we know that the action of $\mathbb{H}$ on M_ν factors through the isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^a) \xrightarrow{\mathrm{RR}} H_\bullet(\boldsymbol{Z}^a) \cong \mathbb{H}_a.$$

Let $\langle \mathbb{C}[\mathfrak{t}]^W_+ \rangle$ be the ideal of $\mathbb{C}[\mathfrak{t}]$ generated by the set of W-invariant polynoimals without constant terms. Let $\mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\langle \mathbb{C}[\mathfrak{t}]^W_+ \rangle\right)$ be the smash-product, which means that its product is given as

$$(w_1, f_1)(w_2, f_2) := (w_1 w_2, f_1 w_1(f_2)) \text{ for } w_1, w_2 \in W, f_1, f_2 \in \mathbb{C}[\mathfrak{t}]/\langle \mathbb{C}[\mathfrak{t}]^W_+ \rangle.$$

It is clear that $\boldsymbol{F}^{a_0} \cong F$, $\boldsymbol{Z}^{a_0} \cong Z$, and the restriction of the natural projections $\boldsymbol{Z} \to \boldsymbol{F}$ restricts to natural projections $Z \to F$.

Proposition 8.1. *We have an isomorphism*

$$\mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\langle \mathbb{C}[\mathfrak{t}]^W_+ \rangle\right) \cong H_\bullet(\boldsymbol{Z}^{a_0})$$

as algebras.

Proof. We have

$$H_\bullet(\boldsymbol{Z}^{a_0}) \cong \mathbb{C}_{a_0} \otimes_{R(\boldsymbol{G})} K^{\boldsymbol{G}}(Z).$$

Here the RHS is written as

$$\mathbb{C} \otimes_{R(G)} \mathbb{H}/(\mathbf{q}_0 = -\mathbf{q}_1 = \mathbf{q}_2 = 1).$$

Thus, we have

$$H_\bullet(\boldsymbol{Z}^{a_0}) \cong \mathbb{C} \otimes_{R(G)} \mathbb{C}[\widetilde{W}],$$

where $\widetilde{W} := W \ltimes X^*(T)$ is the affine Weyl group of type $C_n^{(1)}$. (Here $\mathbb{C}$ is the $R(G)$-module given by the evaluation at $1 \in G$. The algebra $R(G)$ acts on $\mathbb{C}[\widetilde{W}]$ by $R(G) \cong \mathbb{Z}[X^*(T)]^W$.) Thus, it suffices to show

$$\mathbb{C}[X^*(T)]/\mathbb{C}[X^*(T)]\mathfrak{m}_1^{\sim} \cong \mathbb{C}[\mathfrak{t}]/\langle \mathbb{C}[\mathfrak{t}]^W_+ \rangle,$$

where $\mathfrak{m}_1^{\sim} \subset \mathbb{C}[X^*(T)]^W = \mathbb{C}[T]^W$ is the defining ideal of the image of $1 \in T$ in $\mathrm{Spec}\mathbb{C}[T]^W$. This follows from the fact that the neighborhoods of $1 \in T$ and $0 \in \mathfrak{t}$ are W-equivariantly diffeomorphic through the exponential map. $\square$

Corollary 8.2. *Keep the setting of 8.1. We have a surjection*

$$H_\bullet(\boldsymbol{Z}^{a_0}) \twoheadrightarrow \mathbb{C}[W].$$

Proof. Keep the notation of the proof of Theorem 8.1. We have

$$\langle \mathbb{C}[\mathfrak{t}]^W_+ \rangle \subset \mathfrak{m}_1 \subset \mathbb{C}[\mathfrak{t}],$$

where $\mathfrak{m}_1$ is the defining ideal of $0 \in \mathfrak{t}$. Since 0 is a W-fixed point of $\mathfrak{t}$, we deduce that $\mathfrak{m}_1$ is a W-invariant maximal ideal. It follows that

$$H_\bullet(\boldsymbol{Z}^{a_0}) \cong \mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\langle \mathbb{C}[\mathfrak{t}]^W_+ \rangle\right) \twoheadrightarrow \mathbb{C}[W]\#\left(\mathbb{C}[\mathfrak{t}]/\mathfrak{m}_1\right) \cong \mathbb{C}[W]$$

as desired. $\square$

Theorem 8.3 (Exotic Springer correspondence)**.** *There exists one-to-one correspondences between the sets of the following three kinds of objects:*

- *a strict marked partition σ;*
- *the G-orbit of $\mathfrak{N}$ given as $G\mathbf{v}_\sigma$;*
- *an irreducible W-module.*

Remark 8.4. Our proof of Theorem 8.3 does not tell which representation is obtained from a given orbit. Such information can be found in [Ka08], which employs totally different argument.

Proof of Theorem 8.3. Let $\mathcal{P}$ be the set of isomorphism classes G-equivariant irreducible perverse sheaves on $\mathfrak{N}$. Each $I \in \mathcal{P}$ is isomorphic to the minimal extension from a smooth G-orbit of $\mathfrak{N}$. By Proposition 4.4, the (perverse) sheaf I must be the extension of a constant sheaf on a G-orbit. This implies a $\#\mathcal{P} \le \#(G\backslash\mathfrak{N})$. Let S be the set of strict normal forms. By Proposition 1.15 1), we have $\#(G\backslash\mathfrak{N}) \le \#S$. Hence, we have

$$\#\mathsf{Irrep}W \le \#\mathcal{P} \le \#(G\backslash\mathfrak{N}) \le \#S \le \#\mathsf{Irrep}W \tag{8.1}$$

where the first inequality comes from Theorem 1.19 and the last inequality is Proposition 1.15 2). This forces all the inequalities in (8.1) to be equalities, which implies the result. □

The following is a summary of the consequences of §1.4:

Theorem 8.5 (Ginzburg, [CG97] §8.5)**.** *Let a be a finite pre-admissible element. Let L be an irreducible $\mathbb{H}_a$-module. Then, there exists unique $\boldsymbol{G}(a)$-orbit $\mathcal{O} \subset \mathfrak{N}^a$ with the following properties:*

1. *There exists a surjective $\mathbb{H}_a$-module homomorphism $M_{(a,X)} \to L$ for every $X \in \mathcal{O}$;*
2. *If we have a non-trivial map $M_{(a,Y)} \to L$ of $\mathbb{H}_a$-modules for some $Y \in \mathfrak{N}^a$, then we have $Y \in \overline{\mathcal{O}}$.* □

Theorems 8.3 and 8.5 claim that each strict marked partition σ gives a unique simple quotient of $M_{(a_0,\mathbf{v}_\sigma)}$. We denote this W-module by L_σ or L_X for $X \in G\mathbf{v}_\sigma$, depending on the situation.

Corollary 8.6. *Keep the setting of Theorem 8.3. A $\mathbb{C}[W]$-module $M_{(a_0,X)}$ contains L_σ only if $X \in \overline{G\mathbf{v}_\sigma}$ holds.* □

9 A deformation argument on parameters

We retain the setting of the previous section.

Theorem 9.1. *Let $a = (s, \vec{q})$ be an admissible element such that $\mathcal{C}_a = \{[1, n]\}$. Then, there exists an admissible element $a' := (s', \vec{q}')$ such that*

- *The s'-action on V_1 has only positive real eigenvalue;*

- *We have $q_0', q_1', q_2' \in \mathbb{R}^{\times}_{\geq 0}$;*
- *We have equalities $\mathfrak{N}^a_+ = \mathfrak{N}^{a'}_+$ and $G(s) = G(s')$.*

Moreover, we have an isomorphism $\mathbb{H}_a \cong \mathbb{H}_{a'}$ as algebras.

Proof. For each $i = 1, \ldots, n$, we set $\chi_i := \epsilon_i(s)$. We set $\mathtt{E} := \{\chi_i, \chi_i^{-1}; 1 \leq i \leq n\}$. We choose a representative $j_0 \in [1, n]$ which satisfies the following condition:

- If $\pm 1 \in \mathtt{E}$, then we require $\chi_{j_0} = \pm 1$;
- If $\pm q_2^{1/2} \in \mathtt{E}$, then we require $\chi_{j_0} = \pm q_2^{1/2}$;
- If $\pm q_2^{-1/2} \in \mathtt{E}$, then we require $\chi_{j_0} = \pm q_2^{-1/2}$.

For each pair $i, j \in [1, n]$, we have

$$\chi_i^{\kappa_{i,j}} = \chi_j^{\kappa'_{i,j}} q_2^{m_{i,j}} \text{ for some } \sigma_{i,j} \in \{\pm 1\}, m_{i,j} \in [1, n]. \tag{9.1}$$

Since q_2 is not a root of unity of order $\leq 2n$, it follows that $m_{i,j}$ are uniquely determined if $(\kappa_{i,j}, \kappa'_{i,j})$ is fixed. Let N be the largest positive integer such that $1, q_2, \ldots, q_2^N$ are distinct. (If q_2 is not a root of unity, then we regard $N = \infty$.) For each pair (i, j) in $[1, n]$, we set

$$\mathtt{I}_{(i,j)} := \begin{cases} \{(\kappa_{i,j}, \kappa'_{i,j}, m_{i,j}); (9.1) \text{ and } \chi_i^{\kappa_{i,j}}, \chi_j^{\kappa'_{i,j}} \in q_2^{\frac{1}{2}[1,N]}\} & (\pm 1, \pm q_2^{\pm 1/2} \not\in \mathtt{E}) \\ \{(\kappa_{i,j}, \kappa'_{i,j}, m_{i,j}); (9.1)\} & (\textit{otherwise}) \end{cases}.$$

Notice that the all relations of the type (9.1) with $m_{i,j} = 0, \pm 1$ are contained in $\mathtt{I}_{(i,j)}$. Choose two real numbers $q \gg q_2' \gg 1$ such that q and q_2' have no algebraic relation. Then, we set

$$(\chi_i')^{\kappa_{i,j_0}} := \begin{cases} (q_2')^{m_{i,j_0}} & (\chi_{j_0} = \pm 1) \\ (q_2')^{m_{i,j_0} + \kappa'_{i,j_0}/2} & (\chi_{j_0} = \pm q_2^{1/2}) \\ (q_2')^{m_{i,j_0} - \kappa'_{i,j_0}/2} & (\chi_{j_0} = \pm q_2^{-1/2}) \\ q(q_2')^{m_{i,j_0}} & (\chi_{j_0} \neq \pm 1, \pm q_2^{\pm 1/2}) \end{cases}.$$

Since the relation (9.1) for (i, j) is determined by that of (i, j_0) and (j, j_0) for each pair i, j in $[1, n]$, it follows that

$$(\chi_i')^{\kappa_{i,j}} = (\chi_j')^{\kappa'_{i,j}} q_2^{m_{i,j}} \text{ for some } \kappa_{i,j}, \kappa'_{i,j} \in \{\pm 1\}, m_{i,j} \in [1, n]$$

for all $(\kappa_{i,j}, \kappa'_{i,j}, m_{i,j}) \in \mathtt{I}_{(i,j)}$. It is clear that $\chi_i^2 = 1$ if and only if $(\chi_i')^2 = 1$. We put $s' \in T$ so that $\epsilon_i(s') = \chi_i'$ for each $i = 1, 2, \ldots, n$. By the above consideration, it follows that $\mathfrak{g}(s') = \mathfrak{g}(s)$. Since both $G(s')$ and $G(s)$ are connected by Steinberg's centralizer theorem, we deduce $G(s) = G(s')$.

Since the relation of (9.1) is preserved, we have $V_2^{(s,q_2)} = V_2^{(s',q_2')}$. If we have $\chi_i^{\sigma_i} = q_k$ for some $i \in [1, n]$, $\sigma_i \in \{\pm 1\}$, and $k = 0, 1$, then we set $q_k' := (\chi_i')^{\sigma_i}$. Otherwise, we set q_k' $(i = 0, 1)$ to be an arbitrary real number which is not an eigenvalue of s' on V_1. (I.e. not equal to any of $(\chi_i')^{\pm 1}$.) Since we have infinitely many possibilities, we can assume $q_0' \neq q_1'$ and $q_k' \gg 1$ in this case. This give $\mathbb{V}^a = \mathbb{V}^{a'}$ by setting $a' := (s', \vec{q}')$. We have $q_0' \neq q_1'$ in all cases since $q_0 \neq q_1$. Hence, the isomorphism $\mathbb{V}_2^a \cong \mathbb{V}^a$ implies $\mathbb{V}_2^{a'} \cong \mathbb{V}^{a'}$.

Therefore, as subvarieties of $\boldsymbol{F}$ and $\mathfrak{N}_2$, we have equalities

$$\boldsymbol{F}^a = \bigcup_{w \in W} G(s) \times^{{}^w B(s)} ({}^w\mathbb{V}^+ \cap \mathbb{V}^a) = \bigcup_{w \in W} G(s') \times^{{}^w B(s')} ({}^w\mathbb{V}^+ \cap \mathbb{V}^{a'}) = \boldsymbol{F}^{a'}$$

and $\mathfrak{N}_+^a = \mathfrak{N}_+^{a'}$.

The projection map $\boldsymbol{F}^a \to \mathfrak{N}_+^a$ are induced by the projection $\boldsymbol{\mu}$. Hence, so is $\boldsymbol{F}^{a'} \to \mathfrak{N}_+^{a'}$. Therefore, we have an equality of convolution algebras

$$\mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^a) = \mathbb{C} \otimes_{\mathbb{Z}} K(\boldsymbol{Z}^{a'}) \cong \mathbb{H}_{a'},$$

which proves the last assertion.

Since $q, q_2' \gg 1$, each of q_i' $(i = 0, 1)$ is positive real. This verifies the requirement about $\vec{q}'$ as desired. $\square$

Proposition 9.2. *Let $a = (s, q_0, q_1, q_2)$ be an admissible element such that:*

- *We have $\mathcal{C}_a = [1, n]$;*
- *The s-action on V_1 has only positive real eigenvalue;*
- *We have $V_1^{(s,q_1)} = \{0\}$;*
- *Each q_i $(i = 0, 1, 2)$ is a positive real number;*

Let $\underline{a} := (s, q_0, q_2)$ and let

$$\log \underline{a} := (\log s, r_0, r_2), \text{ where } q_0 = e^{r_0}, q_2 = e^{r_2}.$$

Let A be the Zariski closure of $\langle \underline{a} \rangle \subset \boldsymbol{T}$. Then $H_\bullet^A(Z)$ is a $\mathbb{C}[\mathfrak{a}]$-algebra such that

1. *The quotient of $H_\bullet^A(Z)$ by the ideal generated by functions of $\mathbb{C}[\mathfrak{a}]$ which is zero along $\log \underline{a}$ is isomorphic to $H_\bullet(Z^{\underline{a}})$;*
2. *The images of the natural inclusions $\mathbb{C}[W] \subset H_\bullet(Z) \subset H_\bullet^A(Z)$ induces an injection*
$$\mathbb{C}[W] \hookrightarrow H_\bullet(Z^{\underline{a}}) = H_\bullet(\boldsymbol{Z}^a).$$

Moreover, we have

$$\mathbb{C}[\mathfrak{a}] \otimes H_\bullet(\mathcal{E}_X) \cong H_\bullet^A(\mathcal{E}_X)$$

as a compatible $(\mathbb{C}[W], \mathbb{C}[\mathfrak{a}])$-module, where W acts on $\mathfrak{a}$ trivially.

Corollary 9.3. *Keep the setting of Proposition 9.2. We have*

$$M_{(a_0,X)} = H_\bullet(\mathcal{E}_X) \cong H_\bullet(\mathcal{E}_X^a) = M_{(a,X)}$$

as $\mathbb{C}[W]$-modules. $\square$

The rest of this section is devoted to the proof of Proposition 9.2.

Lemma 9.4. *Keep the setting of Proposition 9.2. Then, A is connected.*

Proof. The group A is defined to be the quotient of $\mathbb{C}[T_1]$ by the monomials m such that $m(s, q_0, q_2) = 1$. Since all the values of $\epsilon_i(s)$, q_0, q_1 are positive real number, the condition $m(s, q_0, q_2) = 0$ and $m(s^r, q_0^r, q_2^r) = 0$ is the same for all $r \in R_{>0}$, where the branch of powers are taken so that all of $\epsilon_i(s^r), q_0^r, q_2^r$ $(i = 1, \ldots, n)$ are positive real numbers. It follows that a monomial $m \in \mathbb{C}[T_1]$ satisfies $m(s, q_0, q_2)^k = 1$ for some positive integer k if and only if $m(s, q_0, q_2) = 1$. Therefore, such monomials form a saturated $\mathbb{Z}$-sublattice of $X^*(T_1)$. In particular, its quotient lattice is a free $\mathbb{Z}$-lattice, which implies that A is connected. $\square$

We return to the proof of Proposition 9.2.

For each $m \geq 0$, let $ET_m := (\mathbb{C}^m \backslash \{0\})^{\dim \boldsymbol{T}}$ be a variety such that i-th $\mathbb{C}^\times$-factor of $\boldsymbol{T} = (\mathbb{C}^\times)^{\dim \boldsymbol{T}}$ acts as dilation of the i-th factor for each $1 \leq i \leq n+3$. By the standard embedding $\mathbb{C}^m \hookrightarrow \mathbb{C}^{m+1}$ sending (x) to $(x, 0)$, we form a sequence of A-varieties

$$\emptyset = ET_0 \hookrightarrow ET_1 \hookrightarrow ET_2 \hookrightarrow \cdots .$$

We define $ET := \varinjlim_m ET_m$, which is an ind-quasiaffine scheme with free A-action. Since EA is contractible manifold with respect to the classical topology, we regard ET as the universal vector bundle of each subgroup of $\boldsymbol{T}$. (Hence we regard $BA := A \backslash ET$ in the below.)

Corollary 9.5 (of Lemma 9.4)**.** *Keep the above setting. We have $H^{odd}(BA) = 0$.*

Proof. It is well-known that $B\mathbb{C}^\times$ is homotopic to $\mathbb{P}^\infty$, which has no odd-cohomology. $\square$

We return to the proof of Proposition 9.2.

For a A-variety $\mathcal{X}$, we set

$$\mathcal{X}_A := \triangle A \backslash (ET \times \mathcal{X}).$$

We have a forgetful map

$$f_{\mathcal{X}}^A : \mathcal{X}_A \to BA = A \backslash ET.$$

Let $\mathbb{D}_X^A$ be the relative dualizing sheaf with respect to $f_{\mathcal{X}}$. We define

$$H_i^A(\mathcal{X}) \cong H^{-i}(\mathcal{X}_A, \mathbb{D}_X^A).$$

We have the Leray spectral sequence

$$H^i(BA) \otimes H_j(\mathcal{X}) \Rightarrow H_{-i+j}^A(\mathcal{X}).$$

In the below, we understand that $H_\bullet^A(\mathcal{X}) := \bigoplus_m H_m^A(\mathcal{X})$. The projection maps $p_i : Z_A \to F_A$ $(i = 1, 2)$ equip $H_\bullet^A(Z)$ a structure of convolution algebra. It is straight-forward to see that the diagonal subsets $\triangle F \subset Z$ and $(\triangle F)_A \subset Z_A$ represents $1 \in H_\bullet(Z)$ and $1 \in H_\bullet^A(Z)$, respectively.

Lemma 9.6. *The algebra $H_\bullet^A(Z)$ contains $H_\bullet(Z)$ as its subalgebra. In particular, we have $\mathbb{C}[W] \subset H_\bullet^A(Z)$ as subalgebras. Moreover, the center of $H_\bullet^A(Z)$ contains $H^\bullet(BA)[(\triangle F)_A] \subset H_\bullet^A(Z)$.*

Proof. In the Leray spectral sequence

$$H^i(BA) \otimes H_j(Z) \Rightarrow H^A_{-i+j}(Z),$$

we have $H^{odd}(BA) = 0$ and $H_{odd}(Z) = 0$ (since Z is paved by affine spaces). It follows that this spectral sequence degenerates at the level of E_2-terms. Moreover, the image of the natural map $\imath : H_j(Z) \hookrightarrow H^A_j(Z)$ represents cycles which are locally constant fibration over the base BA. It follows that the map $\imath$ is an embedding of convolution algebras.
Multiplying $H^\bullet(BA)$ is an operation along the base BA, which commutes with the convolution operation (along the fibers of f^A_Z). It follows that $H^\bullet(BA) \to H^\bullet(BA)[(\triangle F)_A] \subset H^A_\bullet(Z)$ is central subalgebra as desired. □

We return to the proof of Proposition 9.2.

Let $R(A)_a$ be the localization of $R(A)$ at the point $\underline{a}$. By the Thomason localization theorem (see e.g. [CG97] §8.2), we have an isomorphism

$$R(A)_a \otimes_{R(A)} K^A(Z^a) \cong R(A)_a \otimes_{R(A)} K^A(Z)$$

as algebras. For each of $\mathcal{X} = Z$, or Z^a, we have a dense open embedding

$$K^A(\mathcal{X}) \hookrightarrow \varprojlim_m K^A(EA_m \times \mathcal{X}) \cong \varprojlim_m K(A \backslash (EA_m \times \mathcal{X})).$$

We regard the RHS as a substitute of $K(\mathcal{X}_A)$. It follows that the Chern character map relative to BA gives an isomorphism

$$\mathbb{C}[[\mathfrak{a}]]_a \otimes_{\mathbb{C}[\mathfrak{a}]_a} H^A_\bullet(Z^a)_a \cong \mathbb{C}[[\mathfrak{a}]]_a \otimes_{\mathbb{C}[\mathfrak{a}]_a} H^A_\bullet(Z)_a,$$

where $\mathbb{C}[[\mathfrak{a}]]_a$ is the formal power series ring of $\mathbb{C}[\mathfrak{a}]$ along $\log a$. By restricting this to the sum of vectors of finitely many degrees, we obtain

$$H^A_\bullet(Z^a)_a \cong H^A_\bullet(Z)_a. \tag{9.2}$$

Since localization along $\mathbb{C}[\mathfrak{a}]_a$ commutes with the quotient by its unique maximal ideal, we deduce the first assertion.

The isomophism (9.2) is an algebra isomorphism, it follows that $1 \in \mathbb{C}[W]$ goes to $1 \in H_\bullet(Z^a)$. It follows that each of s_i goes to a non-zero element of $H_\bullet(Z^a)$ with its square equal to 1. By construction, there exists $f_i \in \mathbb{C}(\mathfrak{a})$ $(i = 1, \ldots n)$ such that $1, f_1 s_1, \ldots f_n s_n \in H^A_\bullet(Z^a)_a$ define linearly independent vectors in $H_\bullet(Z^a)$. It follows that $f_i^2 \in \mathbb{C}[\mathfrak{a}]$. This forces $f_i \in \mathbb{C}[\mathfrak{a}]$, which implies that the images of $1, s_1, \ldots, s_n \in H_\bullet(Z^a)$ are linearly independent. This verifies the second assertion.

The vector space $H^A_\bullet(\mu^{-1}(X))$ admits an action of $H^A_\bullet(Z)$. By the Leray spectral sequence, we have

$$H^\bullet(BA) \otimes H_\bullet(\mu^{-1}(X)) \Rightarrow H^A_\bullet(\mu^{-1}(X)). \tag{9.3}$$

By Theorem 6.1 and Lemma 9.5, we know that $H_{odd}(\mu^{-1}(X)) = 0 = H^{odd}(BA)$. It follows that (9.3) is E_2-degenerate, which proves the last part of Proposition 9.2.

10 Main Theorems

We retain the setting of §2.

Theorem 10.1 (Deligne-Langlands type classification)**.** *Let $a \in \boldsymbol{G}$ be a finite pre-admissible element. Then, $\mathfrak{R}_a$ is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}_a$-modules.*

Proof. By definition, each element of $\mathfrak{R}_a$ corresponds to at least one isomorphism classes of $\mathbb{H}_a$-modules. Since a is finite, each irreducible direct summand of $(\mu_+^a)_* \mathbb{C}_{F_+^a}$ is the minimal extension of a local system (up to degree shift) from $\boldsymbol{G}(a)$-orbit $\mathbb{O}$. By Corollary 4.7, a $\boldsymbol{G}(a)$-equivariant local system on $\mathbb{O}$ is a constant sheaf. As a result, every element of $\mathfrak{R}_a$ corresponds to at most one irreducible module as desired. $\square$

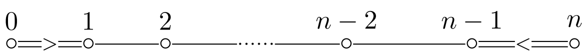
Theorem 10.2 (Effective Deligne-Langlands type classification)**.** *Let $a \in \boldsymbol{G}$ be an admissible element. Then, the set Λ_a is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}_a$-modules.*

The proof of Theorem 10.3 is given at the end of this section.

As in Remark 2.2, the quotient $\mathbb{H}/(q_0 + q_1)$ is isomorphic to an extended Hecke algebra $\mathbb{H}_B$ of type $B_n^{(1)}$ with two parameters. Hence, we have

Corollary 10.3 (Effective Deligne-Langlands type classification for type B)**.** *Let $a = (s, q_0, -q_0, q_2) \in \boldsymbol{G}$ be a pre-admissible element such that $-q_0^2 \neq q_2^{\pm m}$ holds for every $0 \le m < n$. Then, the set Λ_a is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}_a$-modules.* $\square$

Remark 10.4. The Dynkin diagram of type $C_n^{(1)}$ is written as:



This Dynkin diagram has a unique non-trivial involution φ. We define t_0, t_1, t_n to be

$$t_1^2 = \mathbf{q}_2, t_n^2 = -\mathbf{q}_0\mathbf{q}_1, t_n(t_0 - t_0^{-1}) = \mathbf{q}_0 + \mathbf{q}_1 \quad \text{(c.f. Remark 2.2 1)).}$$

Let $T_0, \ldots, T_n$ be the Iwahori-Matsumoto generators of $\mathbb{H}$ (c.f. [Mc03, Lu03]). Their Hecke relations read

$$(T_0 + 1)(T_0 - t_0^2) = (T_i + 1)(T_i - t_1^2) = (T_n + 1)(T_n - t_n^2) = 0,$$

where $1 \le i < n$. The natural map $\varphi(T_i) = T_{n-i}$ $(0 \le i \le n)$ extends to an algebra map $\varphi : \mathbb{H} \to \mathbb{H}'$, where $\mathbb{H}'$ is the Hecke algebra of type $C_n^{(1)}$ with parameters t_n, t_1, t_0. We have $t_n = \pm\sqrt{-\mathbf{q}_0\mathbf{q}_1}$ and $t_0 = \pm\sqrt{-\mathbf{q}_0/\mathbf{q}_1}$ or $\pm\sqrt{-q_1/q_0}$. In particular, φ changes the parameters as $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2) \mapsto (\mathbf{q}_0, \mathbf{q}_1^{-1}, \mathbf{q}_2)$ or $(\mathbf{q}_0^{-1}, \mathbf{q}_1, \mathbf{q}_2)$. Therefore, the representation theory of $\mathbb{H}_a$ ($a = (s, \vec{q})$) is unchanged if we replace q_0 with q_0^{-1}, or q_1 with q_1^{-1}.

The rest of this section is devoted to the proof of Theorem 10.3. In the course of the proof, we use:

Proposition 10.5. *Let a be an admissible element. Let $\mathcal{O} \subset \mathfrak{N}$ be a G-orbit. For any two distinct $G(s)$-orbits $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O} \cap \mathfrak{N}_+^a$, we have*

$$\overline{\mathcal{O}_1} \cap \mathcal{O}_2 = \emptyset.$$

Proof. By Proposition 4.8 and Lemma 1.17, we deduce that the scalar multiplication of a normal form of $\mathfrak{N}$ is achieved by the action of T. It follows that each $G(s)$-orbit of $\mathfrak{N}^a$ is a $Z_{G\times(\mathbb{C}^\times)^2}(a)$-orbit. Let $X \in \mathcal{O}_1$. Let G_X be the stabilizer of X in $G\times(\mathbb{C}^\times)^2$. Assume that $\mathcal{O}_2 \cap \overline{\mathcal{O}_1} \neq \emptyset$ to deduce contradiction. Since $\mathcal{O}_2$ is a $Z_{G\times(\mathbb{C}^\times)^2}(a)$-orbit, we have $\mathcal{O}_2 \subset \overline{\mathcal{O}_1}$. Fix $X' \in \mathcal{O}_2$. Consider an open neighborhood $\mathcal{U}$ of 1 in G (as complex analytic manifolds). Then, $\mathcal{U}X' \in \mathcal{O}$ is an open neighborhood of X'. It follows that $\mathcal{U}X' \cap \mathcal{O}_1 \neq \emptyset$. We put $\mathfrak{g}_{a,X'} := \mathrm{Lie} G_{X'} + \mathrm{Lie} Z_{G\times(\mathbb{C}^\times)^2}(a)$. We have

$$N_{\mathcal{O}_2/\mathcal{O},X'} = \mathfrak{g}/\mathfrak{g}_{a,X'}.$$

Every non-zero vectors of $N_{\mathcal{O}_2/\mathcal{O},X'}$ is expressed as a linear combination of eigenvectors with respect to the a-action. These a-eigenvectors can be taken to have non-zero weights and does not contained in $G_{X'}$. It follows that

$$\mathcal{U}X' \cap \mathcal{O}_1 \not\subset \mathbb{V}^a,$$

which is contradiction (for an arbitrary sufficiently small $\mathcal{U}$). Hence, we have necessarily $\mathcal{O}_2 \cap \overline{\mathcal{O}_1} = \emptyset$ as desired. □

By Corollary 3.10, it suffices to prove Theorem 10.3 when $\mathcal{C}_a$ consists of a unique clan $[1, n]$. By Corollary 4.9, we can further assume $V_1^{(s,q_1)} = \{0\}$ by swapping the roles of q_0 and q_1 if necessary. By Theorem 8.5 (c.f. Theorem 1.19), an admissible parameter (a, X) is regular if there exists a simple $\mathbb{H}_a$-constituent of $M_{(a,X)}$ which does not appear in any $M_{(a,X')}$ such that $G(s)X \subsetneq \overline{G(s)X'}$.

We apply Propotisium 9.1 (if necessary) to modify a so that the assumption of Proposition 9.2 is fillfulled. By Proposition 9.2, each $M_{(a,X)}$ has a W-module structure given by the restriction of the $\mathbb{H}_a$-module structure. Moreover, the simple W-module L_X corresponding to the G-orbit $GX \subset \mathfrak{N}$ (by the exotic Springer correspondence) appears in $M_{(a,X)}$. By Proposition 10.5, we have $GX \neq GX'$ for every $X' \in \mathfrak{N}^a$ such that $GX \subsetneq \overline{GX'}$. By Corollary 9.3 and Corollary 8.6, $M_{(a,X')}$ does not contain L_X as W-modules. Hence, the simple $\mathbb{H}_a$-constituent of $M_{(a,X)}$ which contains L_X as W-type does not occur in any $M_{(a,X')}$ such that $G(s)X \subsetneq \overline{G(s)X'}$ as required.

11 Consequences

In this section, we present some of the consequences of our results. We retain the setting of the previous section.

Definition 11.1. Let $\nu = (a, X)$ be an admissible parameter. Let L_ν be the simple module of $\mathbb{H}$ corresponding to ν. Let $IC(\nu)$ be the corresponding $\boldsymbol{G}(a)$-equivariant simple perverse sheaf on $\mathfrak{N}^a_+$. (c.f. §1.4) We denote by P_ν the projective cover of L_ν as $\mathbb{H}_a$-modules. (It exists since $\mathbb{H}_a$ is finite dimensional.)

Let K be a $\mathbb{H}$-module and let L be a simple $\mathbb{H}$-module. We denote by $[K : L]$ the multiplicity of L in A.

Definition 11.2. Let $a = (s, \vec{q}) \in \mathbf{T}$ be an admissible element. We form three $|\Lambda_a| \times |\Lambda_a|$-matrices

$$[P : L]^a_{\nu,\nu'} := [P_\nu, L_{\nu'}], D^a_{\nu,\nu'} := \delta_{\nu,\nu'}\chi_c(\nu), \text{ and } IC^a_{\nu,\nu'} := [M^\nu, L_{\nu'}],$$

where $\chi_c(\nu) := \sum_{i\geq 0}(-1)^i \dim H^i(\boldsymbol{G}(a)X, \mathbb{C})$ $(\nu = (a, X))$.

Applying [CG97] 8.6.23 to our situation, we obtain:

Theorem 11.3 (The multiplicity formula of standard modules)**.** *Let $\nu = (a, X)$, $\nu' = (a, X')$ be regular admissible parameters. We have:*

$$[M_\nu : L_{\nu'}] = \sum_k \dim H^k(i_X^! IC(\nu')) \text{ and } [M^\nu : L_{\nu'}] = \sum_k \dim H^k(i_X^* IC(\nu')),$$

where $i_X : \{X\} \hookrightarrow \mathfrak{N}_+^a$ is an inclusion. □

The following result is a variant of the Lusztig-Ginzburg character formula of standard modules in our setting.

Theorem 11.4 (The character formula of standard modules)**.** *Let $\nu = (a, X) = (s, \vec{q}, X)$ be an admissible parameter. Let $\mathfrak{B}_\nu$ be the set of connected components of $\mathcal{E}_X^a$. For each $\mathcal{B} \in \mathfrak{B}_\nu$, we define a linear form $\langle \bullet, s \rangle_{\mathcal{B}}$ as a composition map*

$$\begin{tikzcd} \langle \bullet, s \rangle_{\mathcal{B}} : R(T) \arrow[r, "\cong"] & R(gBg^{-1}) \arrow[r, "\mathrm{ev}_s"] & \mathbb{C} \\ R^+ \arrow[u, hook] \arrow[r] & \{\textit{weights of } gBg^{-1}\} \arrow[u, hook] & \end{tikzcd}$$

by some $g \in G$ such that $gB \in \mathcal{B}$. Then, $\langle \bullet, s \rangle_{\mathcal{B}}$ is independent of the choice of g and the restriction of M_ν to $R(T)$ is given as

$$\mathrm{Tr}(e^\lambda; M_\nu) := \sum_{\mathcal{B} \in \mathfrak{B}_\nu} \langle \lambda, s \rangle_{\mathcal{B}} \sum_{j \geq 0} \dim H_{2j}(\mathcal{B}, \mathbb{C}).$$

Proof. Taking account into Corollary 4.7, the proof is exactly the same as in [CG97] §8.2. □

The following result is a special case of the Ginzburg theory [CG97] Theorem 8.7.5 applied to our particular setting:

Theorem 11.5 (The multiplicity formula of projective modules)**.** *Keep the setting of Definition 11.2. We have*

$$[P : L]^a = IC^a \cdot D^a \cdot {}^t IC^a,$$

where ^{t} denotes the transposition of matrices. □

Index of notation

(Sorted by the order of appearance)

References

[Ar96] Susumu Ariki, On the decomposition numbers of the Hecke algebras of $G(m,1,n)$, J. Math. Kyoto Univ. **36** (1996) no.4, 789–808.

[AH07] Pramod Achar, and Anthony Henderson, Orbit closures in the enhanced nilpotent cone, to appear.

[Ca85] Roger W. Cartier, Finite Groups of Lie type: Conjugacy classes and complex characters, Wiley, (1985). ISBN 0-471-50683-4

[CG97] Neil Chriss, and Victor Ginzburg, Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997. x+495 pp. ISBN 0-8176-3792-3

[DLP88] Corrado De Concini, George Lusztig, and Claudio Procesi, Homology of the zero-set of a nilpotent vector field on a flag manifold, J. Amer. Math. Soc. **1**, (1988), no.1, 15–34.

[DK85] Jiri Dadok, and Victor Kac, Polar representations. J. Algebra 92 (1985), no. 2, 504–524.

[En06] Naoya Enomoto, Classification of the Irreducible Representations of Affine Hecke Algebras of Type B_2 with unequal parameters, J. Math. Kyoto Univ. **46** no.2. (2006) 259–273.

[FGT08] Michael Finkelberg, Victor Ginzburg, and Roman Travkin, Mirabolic affine Grassmannian and character sheaves, arXiv:0802.1652.

[Gi97] Victor Ginzburg, Geometric methods in representation theory of Hecke algebras and quantum groups. Notes by Vladimir Baranovsky. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 127–183, Kluwer Acad. Publ., Dordrecht, 1998.

[Ha77] Robin Hartshorne, Algebraic Geometry, GTM 52. Springer-Verlag, 1977. xvi+496 pp. ISBN 0-387-90244-9.

[Ig73] Jun-ichi, Igusa, Geometry of absolutely admissible representations, in: Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, 373–452, Kinokuniya, Tokyo, (1973).

[Jo98] Roy Joshua, Modules over convolution algebras from derived categories, I, J. Algebra, **203**, 385–446, (1998).

[Ka06b] Syu Kato, An exotic Springer correspondence for symplectic groups, math.RT/0607478, preprint (never to appear).

[Ka08] Syu Kato, Deformations of nilpotent cones and Springer correspondences, arXiv:0801.3707, preprint.

[KL87] David Kazhdan, and George Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987), no. 1, 153–215.

[KR71] Bertram Kostant, and Stephen Rallis, Orbits and representations associated with symmetric spaces, Amer. J. Math. 93 (1971), 753–809.

[Lu88] George Lusztig, Cuspidal local systems and graded Hecke algebras. I, Inst. Hautes Études Sci. Publ. Math., **67**, (1988), 145–202.

[Lu89] George Lusztig, Affine Hecke algebras and their graded version, J. Amer. Math. Soc. **2**, (1989), no.3, 599–635.

[Lu90] George Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. **3**, (1990), no.2, 447–498.

[Lu95a] George Lusztig, Classification of unipotent representations of simple p-adic groups. Internat. Math. Res. Notices 1995, no. 11, 517–589.

[Lu95b] George Lusztig, Cuspidal local systems and graded Hecke algebras. II, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., **16**, 217–275. (1995)

[Lu02] George Lusztig, Cuspidal local systems and graded Hecke algebras. III, Representations Theory **6**, 202–242. (2002)

[Lu03] George Lusztig, Hecke algebras with unequal parameters, CRM Monograph Series, **18**, American Mathematical Society, Providence, RI, (2003), vi+136, ISBN: 0-8218-3356-1

[Mc03] Ian G. Macdonald, Affine Hecke algebras and orhogonal polynomials, Cambridge Tracts in Mathematics 157 (2003)

[Oh86] Takuya Ohta, The singularities of the closures of nilpotent orbits in certain symmetric pairs, Tôhoku Math. J., 38 (1986), 441-468.

[OS07] Eric Opdam, and Maarten Solleveld, Homological algebra for affine Hecke algebras, arXiv:0708.1372.

[Po04] Vladimir L. Popov, The cone of Hilbert nullforms, Proc. Steklov Math. Inst. 241 (2003), 177-194.

[Ra01] Arun Ram, Representations of rank two affine Hecke algebras, in "Advances in Algebra and Geometry, University of Hyderabad conference 2001", C. Musili ed., Hindustan Book Agency, 2003, 57-91, available online at `http://www.math.wisc.edu/~ram/bib.html`

[Sa88] Morihiko Saito, Modules de Hodge polarisables, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989).

[Sc78] Gerald W. Schwarz, Representations of simple groups with a free module of covariants, Invent. Math. 49, 1–12 (1978).

[Se84] Jiro Sekiguchi, The nilpotent subvariety of the vector space associated to a symmetric pair, Publ. Res. Inst. Math. Sci. **20** (1984), 155–212.

[Sh83] Toshiaki Shoji, On the Green polynomials of classical groups. Invent. Math. **74** (1983), no. 2, 239–267

[So07] Maarten Solleveld, Periodic cyclic homology of reductive p-adic groups, arXiv:0710.5815.

[Sp07] Tonny A. Springer, The exotic nilcone of a symplectic group, preprint.

[Sp82] Nicolas Spaltenstein, Classes unipotents et sous groupes de Borel, Lect. Notes. Math. 946 Springer-Verlag, 1982.

[Th86] Robert W. Thomason, Lefschetz-Riemann-Roch theorem and coherent trace formula, Invent. Math. 85 515–543 (1986).

[Tr08] Roman Travkin, Mirabolic Robinson-Schensted-Knuth correspondence, preprint.