An exotic Deligne-Langlands correspondence for symplectic groups

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Abstract

Let $G = Sp(2n, \mathbb{C})$ be a complex symplectic group. We introduce a $G \times (\mathbb{C}^\times)^{\ell+1}$-variety $\mathfrak{N}_\ell$, which we call the ℓ-exotic nilpotent cone. Then, we realize the Hecke algebra $\mathbb{H}$ of type $C_n^{(1)}$ with three parameters via equivariant algebraic K-theory in terms of the geometry of $\mathfrak{N}_2$. This enables us to establish a Deligne-Langlands type classification of simple $\mathbb{H}$-modules under a mild assumption on parameters. As applications, we present a character formula and multiplicity formulas of $\mathbb{H}$-modules.

Table of Contents

Introduction

In their celebrated paper [KL87], Kazhdan and Lusztig gave a classification of simple modules of an affine Hecke algebra $\mathbb{H}$ with one-parameter in terms of the geometry of nilpotent cones. (It is also done by Ginzburg, cf. [CG97].) Since some of the affine Hecke algebras admit two or three parameters, it is natural to extend their result to multi-parameter cases. (It is called the unequal parameter case.) Lusztig realized the "graded version" of $\mathbb{H}$ (with unequal parameters) via several geometric means [Lu88, Lu89, Lu95b] (cf. [Lu03]) and classified their representations in certain cases. Unfortunately, his geometries admit essentially only one parameter. As a result, his classification is restricted to the case where all of the parameters are certain integral power of a single parameter. It is

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enough for his main interest, the study of representations of p-adic groups (cf. [Lu95a]). However, there are many areas of mathematics which wait for the full-representation theory of Hecke algebras with unequal parameters (see e.g. [Mc03] and its featured review in MathSciNet).

In this paper, we give a realization of all simple modules of the Hecke algebra of type $C_n^{(1)}$ with three parameters by introducing a variety which we call *the ℓ-exotic nilpotent cone* (cf. §1.1). Our framework works for all parameters and realizes the whole Hecke algebra (Theorem A) and its specialization to each central character. Unfortunately, the study of our geometry becomes harder for some parameters and the result becomes less explicit in such cases (cf. Theorem D). Even so, our result, coupled with that of Lusztig [*loc. cit.*], gives a definitive classification of simple modules of the extended Hecke algebra of type $B_n^{(1)}$ with two-parameters for almost all parameters (cf. Theorem E and the argument after that).

Let G be the complex symplectic group $Sp(2n,\mathbb{C})$. We fix its Borel subgroup B and a maximal torus $T \subset B$. Let R be the root system of (G,T). We embed R into a n-dimensional Euclid space $\oplus_i \mathbb{C}\epsilon_i$ as $R = \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\}$. We define $V_1 := \mathbb{C}^{2n}$ and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. We put $\mathbb{V}_\ell := V_1^{\oplus \ell} \oplus V_2$ and call it *the ℓ-exotic representation.* Let $\mathbb{V}_\ell^+$ be the positive part of $\mathbb{V}_\ell$ (for precise definition, see §1). We define

$$F_\ell := G \times^B \mathbb{V}_\ell^+ \subset G \times^B \mathbb{V}_\ell \cong G/B \times \mathbb{V}_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow \mathbb{V}_\ell.$$

We denote the image of μ_ℓ by $\mathfrak{N}_\ell$. This is the G-variety which we refer as *the ℓ-exotic nilpotent cone.* We put $Z_\ell := F_\ell \times_{\mathfrak{N}_\ell} F_\ell$. Let $G_\ell := G \times (\mathbb{C}^\times)^{\ell+1}$. We have a natural G_ℓ-action on F_ℓ (and Z_ℓ). (In fact, the variety F_ℓ admits an action of $G \times GL(\ell,\mathbb{C}) \times \mathbb{C}^\times$. We use only a restricted action in this paper.)

Assume that $\mathbb{H}$ is the Hecke algebra with unequal parameters of type $C_n^{(1)}$ (cf. Definition 2.1). This algebra has three parameters q_0, q_1, q_2. All affine Hecke algebras of classical type with two parameters are obtained from $\mathbb{H}$ by suitable specializations of parameters (cf. Remark 2.2).

Theorem A (= Theorem 2.8)**.** *We have an isomorphism*

$$\mathbb{H} \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} K^{G_2}(Z_2)$$

as algebras.

The Ginzburg theory suggests the classification of simple $\mathbb{H}$-modules by the G-conjugacy classes of the following Langlands parameters:

Definition B (Langlands parameters)**.**

1) A pair $(\mathbf{a}, X) = (s, q_0, q_1, q_2, X_0 \oplus X_1 \oplus X_2) \in G_2 \times \mathfrak{N}_2$ is called an admissible parameter iff s is semisimple, $q_0 \neq q_1$, q_2 is not a root of unity of order $\leq 2n$, and $sX_i = q_i X_i$ for $i = 0, 1, 2$;

For an admissible parameter $(\mathbf{a}, X)$, we denote by $\mu_2^{\mathbf{a}}$ be the restriction of μ_2 to the set of $\mathbf{a}$-fixed points $F_2^{\mathbf{a}}$ of F_2. Let $G_2(\mathbf{a}) := Z_{G_2}(\mathbf{a})$.

2) An admissible parameter $(\mathbf{a}, X)$ is called regular iff there exists a direct summand $A[d] \subset (\mu_2^{\mathbf{a}})_* \mathbb{C}_{F_2^{\mathbf{a}}}$, where A is a simple $G_2(\mathbf{a})$-equivariant perverse sheaf on $\mathfrak{N}_2^{\mathbf{a}}$ such that $\mathrm{supp} A = \overline{G_2(\mathbf{a})X}$ and d is an integer.

Notice that our Langlands parameters do not have additional data as in the usual Deligne-Langlands-Lusztig correspondence. This is because the (equivariant) fundamental groups of orbits are always trivial (cf. Theorem 7.14). Instead, the regularity condition poses subtlety:

Example C (Non-regular parameters). Let $G = Sp(4, \mathbb{C})$ and let $\mathbf{a} = (\exp(r\epsilon_1 + (r + \pi\sqrt{-1})\epsilon_2), e^r, -e^r, -e^{2r}) \in T \times (\mathbb{C}^\times)^3$ $(r \in \mathbb{C} \backslash \pi\sqrt{-1}\mathbb{Q})$. Then, the number of $G_2(\mathbf{a})$-orbits in $\mathfrak{N}_2^{\mathbf{a}}$ is eight, while the number of corresponding representations of $\mathbb{H}$ is six. (cf. Enomoto [En06]) In fact, there are two non-regular admissible parameters in this case. These parameters correspond to weight vectors of $\epsilon_1 + \epsilon_2$ or "ϵ_1 & ϵ_2".

Theorem D (= Corollary 6.3)**.** *Let (s, q_0, q_1, q_2, X) be an admissible parameter. If $q_0^2 \neq q_2^{\pm l} \neq q_1^2$ $(0 \le \forall l < 2n)$ and $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ $(0 \le \forall m < n)$ hold, then it is regular.*

Now we state the main theorem of this paper:

Theorem E (= Part of Theorem 8.1 + Lemma 8.2)**.** *The set of G-conjugacy classes of regular admissible parameters is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}$-modules if q_2 is not a root of unity of order $\le 2n$, and $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ holds for every $0 \le m < n$.*

We treat a slightly more general case in Theorem 8.1 including Example C. Since the general condition is rather technical, we state only a part of it here.

By imposing an additional relation $q_0 + q_1 = 0$, the algebra $\mathbb{H}$ specializes to an extended Hecke algebra $\mathbb{H}_B$ of type $B_n^{(1)}$ with two-parameters. (cf. Remark 2.2) In this case, almost all of the exception of Theorem D is covered by the description of Lusztig [*loc. cit.*]. (I learned this from Prof. Lusztig. The author wants to express his gratitude to him for this kind information.) Therefore, Theorems D-E complete a definitive classification of simple $\mathbb{H}_B$-modules except for $-q_0^2 \neq q_2^m$ $(|m| < n)$ or $q_2^{(2n)!} \neq 1$.

Let us illustrate an example which (partly) explains the title "exotic":

Example F (Equal parameter case). Let $G = Sp(4, \mathbb{C})$. Let $s = \exp(r\epsilon_1 + r\epsilon_2) \in T$ $(r \in \mathbb{C} \backslash \pi\sqrt{-1}\mathbb{Q})$. Fix $\mathbf{a}_0 = (s, e^r) \in G \times \mathbb{C}^\times$ and $\mathbf{a} = (s, e^r, -e^r, e^{2r}) \in G_2$. Let $\mathcal{N}$ be the nilpotent cone of G. Then, the sets of $G(s)$-orbits of $\mathcal{N}^{\mathbf{a}_0}$ and $\mathfrak{N}_2^{\mathbf{a}}$ are responsible for the usual and our exotic Deligne-Langlands correspondences. The number of $G(s)$-orbits in $\mathcal{N}^{\mathbf{a}_0}$ is three. (Corresponding to root vectors of $\emptyset$, $2\epsilon_1$, and "$2\epsilon_1$ & $2\epsilon_2$") The number of $G(s)$-orbits in $\mathfrak{N}_2^{\mathbf{a}}$ is four. (Corresponding to weight vectors of $\emptyset$, ϵ_1, $\epsilon_1 + \epsilon_2$, and "ϵ_1 & $\epsilon_1 + \epsilon_2$") On the other hand, the actual number of simple modules arising in this way is four (cf. Ram [Ra01] and [En06]).

The organization of this paper is as follows: In §1, we fix notation and introduce our main geometric objects including the ℓ-exotic nilpotent cone. Then, we give a rough classification of orbits of $\mathfrak{N}_2^{\mathbf{a}}$. In §1.3, we review Ginzburg's convolution realization of simple $K^{G_2}(Z_2)$-modules. Theorem A is proved in §2. The proof itself is similar to that of Chriss-Ginzburg [CG97] §7. The main point is to identify $K^{G_2}(F_2)$ with a basic representation of $\mathbb{H}$. Here we also introduce a way to regard $\mathfrak{N}_2^{\mathbf{a}}$ as a subvariety of $\mathfrak{N}_1$, provided if $q_0 \neq q_1$. In §3, we prove a reduction theorem (Corollary 3.10) which reduces the proofs of Theorems D and E to their spacial cases. To analyze the primitive parameters arising from this reduction procedure, we reformulate the description of parameters in §4. In particular, we introduce the notion of height functions adapted to a parameter, which is a rough analogue of the Jacobson-Morozov theorem used in Kazhdan-Lusztig's work ([KL87]). In §5, we formulate abstract regularity criteria of parameters (Proposition 5.2 and Lemma 5.4) involving height functions. At the same time, we introduce several notation and lemmas which help us to check the assumptions of the criteria. Using this, we give a sufficient condition for the regularity of parameters (Corollary 6.3) by checking our criteria with a case-by-case analysis (Propositions 6.1 and 6.2). In §7, we present a proof that the equivariant fundamental groups of orbits are always trivial. Its main ingredients are: the identification of $\mathfrak{N}_0^{\mathbf{a}}$ with the representation space of some quivers, the representation theory of quivers, and the fact that a linearly defined subgroup of $GL(n, \mathbb{C})$ is connected. We formulate and prove (the precise form of) Theorem E in §8. After proving Theorem E, we present a Deligne-Langlands type classification in type $B_n^{(1)}$-case. In §9, we study geometric standard modules as consequences of our results and the Ginzburg theory (cf. [CG97] §8). We present its multiplicity formula of simple modules, its character formula, and a special case of the multiplicity formula of simple modules in projective modules. With an aid of [CG97] §8, the last result follows from the odd homology vanishing of relevant varieties, for which we provide a proof under the same assumption as in Theorem D. Also, we briefly mention a connection with the canonical basis of quantum groups of type ADE. We finish this paper by supplying an appendix which is devoted to proofs of some geometric facts about exotic nilpotent cones.

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1 Preparatory materials

Let $G := Sp(2n, \mathbb{C})$. Let B be a Borel subgroup of G. Let T be a maximal torus of B. Let $X^*(T)$ be the character group of T. Let R be the root system of (G, T) and let R^+ be its positive part defined by B. We embed R and R^+ into a n-dimensional Euclid space $\mathbb{E} = \oplus_i \mathbb{C}\epsilon_i$ with standard inner product as:

$$R^+ = \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\} \subset \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\} = R \subset \mathbb{E}.$$

By the inner product, we identify ϵ_i with its dual basis. We put $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(i = 1, \ldots, n-1)$ or $2\epsilon_n$ $(i = n)$. Let W be the Weyl group of (G, T). For each α_i, we denote the reflection of $\mathbb{E}$ corresponding to α_i by s_i. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function with respect to (B, T). We denote by $\dot{w} \in N_G(T)$ a lift of $w \in W$. For a subgroup $H \subset G$ containing T, we put ${}^wH := \dot{w}H\dot{w}^{-1}$. For a group H and its element h, we put $H(h) := Z_H(h)$. For each $\alpha \in R$, we denote the corresponding one-parameter unipotent subgroup of G (with respect to T) by U_α. We define $\mathfrak{g}, \mathfrak{t}, \mathfrak{u}_\alpha, \mathfrak{g}(s)$, etc... to be the Lie algebras of $G, T, U_\alpha, G(s)$, etc..., respectively.

For a T-module V, we define its weight λ-part (with respect to T) as $V[\lambda]$. We define the positive part V^+ of V as

$$V^+ := \bigoplus_{\lambda \in \mathbb{Q}_{\geq 0} R^+ - \{0\}} V[\lambda].$$

We denote the set of T-weights of V by $\Psi(V)$.

In this paper, a segment is a set of integers I written as $I = [i_1, i_2] \cap \mathbb{Z}$ for some integers $i_1 \leq i_2$. By abuse of notation, we may denote I by $[i_1, i_2]$. For a variety $\mathcal{X}$, we denote by $H_\bullet(\mathcal{X})$ and $H_\bullet(\mathcal{X}, \mathbb{Z})$ the Borel-Moore homology groups with coefficients $\mathbb{C}$ and $\mathbb{Z}$, respectively.

1.1 Exotic nilpotent cones

Let $\ell \geq 0$ be an integer. We define $V_1 := \mathbb{C}^{2n}$ (vector representation) and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. These representations have B-highest weights ϵ_1 and $\epsilon_1 + \epsilon_2$, respectively. We put $\mathbb{V}_\ell := V_1^{\oplus \ell} \oplus V_2$ and call it *the ℓ-exotic representation of $Sp(2n)$*. For $\ell \geq 1$, the set of non-zero weights of $\mathbb{V}_\ell$ is in one-to-one correspondence with R as

$$R \ni \begin{cases} \pm 2\epsilon_i \leftrightarrow \pm \epsilon_i & \in \Psi(V_1) \\ \pm \epsilon_i \pm \epsilon_j \leftrightarrow \pm \epsilon_i \pm \epsilon_j & \in \Psi(V_2) \end{cases}. \tag{1.1}$$

We define

$$F_\ell := G \times^B \mathbb{V}_\ell^+ \subset G \times^B \mathbb{V}_\ell \cong G/B \times \mathbb{V}_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow \mathbb{V}_\ell.$$

We denote the image of μ_ℓ by $\mathfrak{N}_\ell$. We call this variety *the ℓ-exotic nilpotent cone*. By abuse of notation, we may denote the map $F_\ell \to \mathfrak{N}_\ell$ also by μ_ℓ.

Convention 1.1. For the sake of simplicity, we define objects F, $\mathfrak{N}$, $\mathbb{V}$, μ, etc... to be the objects F_ℓ, $\mathfrak{N}_\ell$, $\mathbb{V}_\ell$, μ_ℓ etc... with $\ell = 1$.

We summarize some basic geometric properties of $\mathfrak{N} = \mathfrak{N}_1$:

Theorem 1.2 (Geometric properties of $\mathfrak{N}$)**.** *We have the following:*

1. *The defining ideal of $\mathfrak{N}$ is $(\mathbb{C}[\mathbb{V}]^G_+)\mathbb{C}[\mathbb{V}] = (\mathbb{C}[V_2]^G_+)\mathbb{C}[\mathbb{V}]$;*
2. *The variety $\mathfrak{N}$ is normal;*
3. *The set of G-orbits in $\mathfrak{N}$ is finite;*
4. *For each $\ell \geq 1$, the map μ_ℓ is a birational projective morphism onto $\mathfrak{N}_\ell$;*
5. *Every fiber of the map μ is connected;*

6. *The map μ is stratified semi-small with respect to the stratification of $\mathfrak{N}$ given by G-orbits.*

Proof. Postponed to Appendix A. $\square$

Lemma 1.3. *We have a natural identification*

$$F_\ell \cong \{(gB, X) \in G/B \times \mathbb{V}_\ell; X \in g\mathbb{V}_\ell^+\}.$$

Proof. Straightforward. $\square$

Let $G_\ell := G \times (\mathbb{C}^\times)^{\ell+1}$. We define a G_ℓ-action on $\mathfrak{N}_\ell$ as

$$G_\ell \times \mathfrak{N}_\ell \ni (g, q_{2-\ell}, \ldots, q_2) \times (X_{2-\ell} \oplus \cdots \oplus X_2) \mapsto (q_{2-\ell}^{-1} g X_{2-\ell} \oplus \cdots \oplus q_2^{-1} g X_2) \in \mathfrak{N}_\ell.$$

(Here we always regard $X_{2-\ell}, \ldots, X_1 \in V_1$ and $X_2 \in V_2$.) Similarly, we have a natural G_ℓ-action on F_ℓ which makes μ_ℓ a G_ℓ-equivariant map. We define $Z_\ell := F_\ell \times_{\mathfrak{N}_\ell} F_\ell$. By Lemma 1.3, we have

$$Z_\ell := \{(g_1B, g_2B, X) \in (G/B)^2 \times \mathbb{V}_\ell; X \in g_1\mathbb{V}_\ell^+ \cap g_2\mathbb{V}_\ell^+\}.$$

We put

$$Z_\ell^{123} := \{(g_1B, g_2B, g_3B, X) \in (G/B)^3 \times \mathbb{V}_\ell; X \in g_1\mathbb{V}_\ell^+ \cap g_2\mathbb{V}_\ell^+ \cap g_3\mathbb{V}_\ell^+\}.$$

We define $p_i : Z_\ell \ni (g_1B, g_2B, X) \mapsto (g_iB, X) \in F_\ell$ and $p_{ij} : Z_\ell^{123} \ni (g_1B, g_2B, g_3B, X) \mapsto (g_iB, g_jB, X) \in Z_\ell$ ($i, j \in \{1, 2, 3\}$). We also put $\tilde{p}_i : F_\ell \times F_\ell \to F_\ell$ as the first and second projections ($i = 1, 2$). (Notice that the meaning of $p_i, \tilde{p}_i, p_{ij}$ depends on ℓ. The author hopes that there occurs no confusion on it.)

Lemma 1.4. *The maps p_i and p_{ij} $(1 \le i < j \le 3)$ are projective.*

Proof. The fibers of the above maps are given as the subsets of G/B defined by incidence relations. It is automatically closed and we obtain the result. $\square$

We have a projection

$$\pi_\ell : Z_\ell \ni (g_1B, g_2B, X) \mapsto (g_1B, g_2B) \in G/B \times G/B.$$

For each $w \in W$, we define a point $\mathsf{p}_w := B \times \dot{w}B \in G/B \times G/B$. This point is independent of the choice of $\dot{w}$. We put $\mathsf{O}_w := G\mathsf{p}_w \subset G/B \times G/B$. By the Bruhat decomposition, we have

$$G/B \times G/B = \bigsqcup_{w \in W} \mathsf{O}_w. \tag{1.2}$$

Lemma 1.5. *For each $\ell \ge 1$, the variety Z_ℓ consists of $|W|$-irreducible components. Moreover, all of the irreducible components of Z have the same dimension.*

Proof. We first prove the assertion for Z. By (1.2), the structure of Z is determined by the fibers over p_w. We have

$$\pi^{-1}(\mathsf{p}_w) = \mathbb{V}^+ \cap \dot{w}\mathbb{V}^+.$$

By the dimension counting using (1.1), we deduce

$$\begin{aligned}\dim \mathbb{V}^+ \cap \dot{w}\mathbb{V}^+ =& \dim V_1^+ \cap \dot{w}V_1^+ + \dim V_2^+ \cap \dot{w}V_2^+ \\ =& \#(R_l^+ \cap wR_l^+) + \#(R_s^+ \cap wR_s^+) = N - \ell(w),\end{aligned}$$

where $N := \dim \mathbb{V}^+ = \dim G/B$ and R_l^+, R_s^+ are the sets of long and short positive roots, respectively. As a consequence, we deduce

$$\dim \pi^{-1}(\mathsf{O}_w) = N + \ell(w) + N - \ell(w) = 2N.$$

Thus, each $\overline{\pi^{-1}(\mathsf{O}_w)}$ is an irreducible component of Z.
Next, we prove the assertion for Z_ℓ ($\ell \geq 2$). By forgetting the first ($\ell - 1$) V_1-factors, we have a surjective map $\eta : Z_\ell \to Z$. We have a surjective map $\eta' : Z \to Z_0$ given by forgetting the V_1-factor. The fiber of η at $x \in Z$ is isomorphic to the $(\ell-1)$-fold product of the fiber of η' at $\eta'(x)$. The latter fiber is isomorphic to the vector space $V_1^+ \cap g V_1^+$ when $\pi(x) = (1, g)\mathsf{p}_1$. Therefore, the preimage of each irreducible component of Z gives an irreducible component of Z_ℓ. These irreducible components are distinct since their images under η must be distinct. Hence, the number of irreducible components of Z_ℓ is equal to the number of irreducible components of Z as desired. $\Box$

By a general result of [Gi97] p135 (cf. [CG97] 2.7), the G_ℓ-equivariant K-group of Z_ℓ becomes an associative algebra via the map

$$\star : K^{G_\ell}(Z_\ell) \times K^{G_\ell}(Z_\ell) \ni ([\mathcal{E}], [\mathcal{F}]) \mapsto \sum_{i \geq 0} (-1)^i [\mathbb{R}^i (p_{13})_* (p_{12}^* \mathcal{E} \otimes^{\mathbb{L}} p_{23}^* \mathcal{F})] \in K^{G_\ell}(Z_\ell).$$

Moreover, the G_ℓ-equivariant K-group of F_ℓ becomes a representation of $K^{G_\ell}(Z_\ell)$ as

$$\circ : K^{G_\ell}(Z_\ell) \times K^{G_\ell}(F_\ell) \ni ([\mathcal{E}], [\mathcal{K}]) \mapsto \sum_{i \geq 0} (-1)^i [\mathbb{R}^i (p_1)_* (\mathcal{E} \otimes^{\mathbb{L}} \tilde{p}_2^* \mathcal{K})] \in K^{G_\ell}(F_\ell).$$

Here we regard $\mathcal{E}$ as a sheaf over $F_\ell \times F_\ell$ via the natural embedding $Z_\ell \subset F_\ell \times F_\ell$.

1.2 Definition of parameters

In this subsection, we present a (rough) classification of orbits of $\mathfrak{N}_2^{\mathfrak{a}}$, which is needed in the sequel. A complete classification of G-orbits of $\mathfrak{N}$ is given by the set of bi-partitions of n as is proved in [Ka06b].

For each $\lambda \in X^*(T) \backslash \{0\}$, we fix a basis element $\mathbf{v}[\lambda] \in \mathbb{V}[\lambda]$. For each $X \in \mathbb{V}$, we write

$$X := \mathbf{v}[0] + \sum_{\lambda \in X^*(T)} X(\lambda) \mathbf{v}[\lambda],$$

where $X(\lambda) \in \mathbb{C}$ and $\mathbf{v}[0] \in \mathbb{V}[0]$. We define the total support of X as

$$\|X\| := \{i \in [1, n]; X(\pm\epsilon_i) \neq 0 \text{ or } X(\pm\epsilon_i \pm \epsilon_j) \neq 0 \text{ for some sign and } j \in [1, n]\}.$$

The following is a slight enhancement of the good basis of Ohta [Oh86] (1.3).

Definition 1.6 (ℓ-normal form)**.** A ℓ-block of length m is a $N_G(T)$-translate of one of the following vectors in $\mathbb{V}$:

$$\mathbf{v}^{(\vec{\jmath},\sigma)}(m)_i := \sum_{k=1}^{\ell}(1-\delta_{j_k,0})\mathbf{v}[\sigma_k\epsilon_{i+j_k}] + \sum_{k=1}^{\lambda-1}\mathbf{v}[\alpha_{i+k}],$$

where $\vec{\jmath} = (j_1 \le j_2 \le \cdots) \in [0,m]^{\ell}$, $\sigma = \{\sigma_k\}_k \in \{\pm 1\}^{\ell}$, $i \in [0,n]$, and $\delta_{j,0}$ is Kronecker's delta. Here we interpret $\mathbf{v}[\pm\epsilon_j] = 0 = \mathbf{v}[\alpha_{j-1}]$ if $j > n$. It is clear that

$$\|\mathbf{v}^{(\vec{\jmath},\sigma)}(m)_i\| = [i+1, m+i] \text{ or } \emptyset.$$

A ℓ-normal form of $\mathbb{V}$ is a sum $\mathbf{v} = \sum_i \mathbf{v}_i$ of ℓ-blocks $\mathbf{v}_i$ such that

$$\|\mathbf{v}_i\| \cap \|\mathbf{v}_{i'}\| = \emptyset \text{ if } i \neq i'.$$

We define the support of a ℓ-normal form $\mathbf{v}$ as:

$$|\mathbf{v}| := \{\|\mathbf{v}_i\|; i \in [1,n] \text{ s.t. } \|\mathbf{v}_i\| \neq \emptyset\} \cup \bigcup_{k \in ([1,n]-\|\mathbf{v}\|)} \{k\}.$$

By abuse of notation, we may call 1-blocks or 1-normal forms merely by blocks or normal forms. For the sake of simplicity, we may denote $\mathbf{v}^{(0,+)}(m)_i$ by $\mathbf{v}(m)_i$.

Definition 1.7 (Configuration of semisimple elements)**.**

1) An element $\mathbf{a} = (s, q_0, q_1, q_2) \in G_2$ is called pre-admissible iff s is semisimple, $q_0 \neq q_1$, q_2 is not a root of unity of order $\le 2n$.

2) An element $\mathbf{a} \in G_2$ is called finite if $\mathfrak{N}_2^{\mathbf{a}}$ has only finitely many $G_2(\mathbf{a})$-orbit.

3) A pre-admissible element $\mathbf{a} = (s, q_0, q_1, q_2)$ is called general if $q_i^2 \neq q_2^{\pm l}$ $(i = 1, 2, 0 \le \forall l < 2n)$, and $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ $(0 \le \forall m < n)$.

For a pre-admissible element $\mathbf{a} = (s, q_0, q_1, q_2)$, we put

$$\mathbb{V}_2^{\mathbf{a}} = V_1^{(s,q_0)} \oplus V_1^{(s,q_1)} \oplus V_2^{(s,q_2)} \subset V_1 \oplus V_1 \oplus V_2 = \mathbb{V}_2.$$

In the below, we may denote $(q_0, q_1, q_2) \in (\mathbb{C}^{\times})^3$ by $\vec{q}$ for the sake of simplicity.

Definition 1.8 (Admissible parameters)**.**

1) An admissible parameter is a pair

$$\nu = (\mathbf{a}, X) = (s, \vec{q}, X_1 \oplus X_2) \in G_2 \times \mathfrak{N}$$

such that $\mathbf{a}$ is pre-admissible, $(s - q_0)(s - q_1)X_1 = 0$, and $sX_2 = q_2 X_2$;

For a pre-admissible $\mathbf{a} \in G_2$, we denote by $\mathfrak{P}_{\mathbf{a}}$ the set of admissible parameters of the form $(\mathbf{a}, Y)$ $(Y \in \mathbb{V})$;

2) An admissible parameter ν is called standard if $s \in T$, and X is a $2n$-normal form.

The following theorems are exotic and equivariant analogues of a result of Sekiguchi [Se84] (cf. Theorem A.1). For the sake of completeness, we provide a full-proof in Appendix A.

Theorem 1.9 (Normal forms). *Let $X \in \mathfrak{N}$. Then, there exists $g \in G$ such that gX is a normal form.*

Proof. Postponed to Appendix A. $\square$

Theorem 1.10 (Standard parameters). *Let ν be an admissible parameter. Then, there exists $g \in G$ such that $g\nu$ is a standard parameter.*

Proof. Postponed to Appendix A. $\square$

1.3 Structure of simple modules

We put $T_\ell := T \times (\mathbb{C}^\times)^{\ell+1}$. Let $\mathbf{a} \in T_\ell$. Let $Z_\ell^{\mathbf{a}}$, $F_\ell^{\mathbf{a}}$, and $\mathfrak{N}_\ell^{\mathbf{a}}$ be the set of $\mathbf{a}$-fixed points of Z_ℓ, F_ℓ, and $\mathfrak{N}_\ell$, respectively. Let $\mu^{\mathbf{a}} : F_\ell^{\mathbf{a}} \to \mathfrak{N}_\ell^{\mathbf{a}}$ denote the restriction of μ_ℓ to $\mathbf{a}$-fixed points.

We review the convolution realization of simple modules in our situation. The detailed constructions are found in [CG97] 5.11, 8.4 or [Gi97] §5. For its variant, see [Jo98].

The properties we used to apply the Ginzburg theory are: 1) $Z_\ell = F_\ell \times_{\mathfrak{N}_\ell} F_\ell$; 2) F_ℓ is smooth; 3) μ_ℓ is projective; 4) $R(G_\ell) \subset K^{G_\ell}(Z_\ell)$ is central; and 5) $H_\bullet(Z_\ell)$ is spanned by algebraic cycles.

Let $\mathbb{C}_{\mathbf{a}}$ be the quotient of $\mathbb{C} \otimes_{\mathbb{Z}} R(G_\ell)$ or $\mathbb{C} \otimes_{\mathbb{Z}} R(T_\ell)$ by the ideal defined by the evaluation at $\mathbf{a}$. The Thomason localization theorem yields ring isomorphisms

$$\mathbb{C}_{\mathbf{a}} \otimes_{R(G_\ell)} K^{G_\ell}(Z_\ell) \xrightarrow{\cong} \mathbb{C}_{\mathbf{a}} \otimes_{R(G_\ell(\mathbf{a}))} K^{G_\ell(\mathbf{a})}(Z_\ell^{\mathbf{a}}) \xrightarrow{\cong} \mathbb{C}_{\mathbf{a}} \otimes_{R(T_\ell)} K^{T_\ell}(Z_\ell^{\mathbf{a}}).$$

Moreover, we have the Riemann-Roch isomorphism

$$\mathbb{C}_{\mathbf{a}} \otimes_{R(T_\ell)} K^{T_\ell}(Z_\ell^{\mathbf{a}}) \cong K(Z_\ell^{\mathbf{a}}) \xrightarrow{RR} H_\bullet(Z_\ell^{\mathbf{a}}) \cong \mathrm{Ext}^\bullet(\mu_*^{\mathbf{a}}\mathbb{C}_{F_\ell^{\mathbf{a}}}, \mu_*^{\mathbf{a}}\mathbb{C}_{F_\ell^{\mathbf{a}}}).$$

By the equivariant Beilinson-Bernstein-Deligne (-Gabber) decomposition theorem (cf. Saito [Sa88] 5.4.8.2), we have

$$\mu_*^{\mathbf{a}}\mathbb{C}_{F_\ell^{\mathbf{a}}} \cong \bigoplus_{\mathbb{O} \subset \mathfrak{N}_\ell^{\mathbf{a}}, \chi, d} L_{\mathbb{O},\chi,d} \boxtimes IC(\mathbb{O}, \chi)[d],$$

where $\mathbb{O} \subset \mathfrak{N}_\ell^{\mathbf{a}}$ is a $G(s)$-stable subset such that $\mu^{\mathbf{a}}$ is locally trivial along $\mathbb{O}$, χ is an irreducible local system on $\mathbb{O}$, d is an integer, $L_{\mathbb{O},\chi,d}$ is a finite dimensional vector space, and $IC(\mathbb{O}, \chi)$ is the minimal extension of χ. Moreover, the set of $\mathbb{O}$'s such that $L_{\mathbb{O},\chi,d} \neq 0$ (for some χ and d) forms a subset of an algebraic stratification in the sense of [CG97] 3.2.23. It follows that:

Theorem 1.11 (Ginzburg [Gi97] Theorem 5.2). *The set of simple modules of $K^{G_\ell}(Z_\ell)$ for which $R(G_\ell)$ acts as the evaluation at $\mathbf{a}$ is in one-to-one correspondence with the set of isomorphism classes of irreducible $G_\ell(\mathbf{a})$-equivariant perverse sheaves appearing in $\mu_*^{\mathbf{a}}\mathbb{C}_{F_\ell^{\mathbf{a}}}$ (up to degree shift).* $\square$

2 Hecke algebras and exotic nilpotent cones

We retain the setting of the previous section. We put $\mathbf{G} = G_2$ and $\mathbf{T} := T_2$. Most of the arguments in this section are exactly the same as [CG97] 7.6 if we replace $\mathbf{G}$ by $G \times \mathbb{C}^\times$, $\mathfrak{N}_2$ by the usual nilpotent cone, μ_2 by the moment map, F_2 by the cotangent bundle of the flag variety, and Z_2 by the Steinberg variety. Therefore, we frequently omit the detail and make pointers to [CG97] 7.6 in which the reader can obtain a correct proof merely replacing the meaning of symbols as mentioned above.

We put $\mathcal{A}_\mathbb{Z} := \mathbb{Z}[q_0^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}]$ and $\mathcal{A} := \mathbb{C} \otimes_\mathbb{Z} \mathcal{A}_\mathbb{Z} = \mathbb{C}[q_0^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}]$.

Definition 2.1 (Hecke algebras of type $C_n^{(1)}$)**.** A Hecke algebra of type $C_n^{(1)}$ with three parameters is an associative algebra $\mathbb{H}$ over $\mathcal{A}$ generated by $\{T_i\}_{i=1}^n$ and $\{e^\lambda\}_{\lambda \in X^*(T)}$ subject to the following relations:

(Toric relations) For each $\lambda, \mu \in X^*(T)$, we have $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ (and $e^0 = 1$);

(The Hecke relations) We have

$$(T_i + 1)(T_i - q_2) = 0 \ (1 \le i < n) \text{ and } (T_n + 1)(T_n + q_0 q_1) = 0;$$

(The braid relations) We have

$$\begin{gathered} T_i T_j = T_j T_i \ (\text{if } |i-j| > 1), \ (T_n T_{n-1})^2 = (T_{n-1} T_n)^2, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \ (\text{if } 1 \le i < n-1); \end{gathered}$$

(The Bernstein-Lusztig relations) For each $\lambda \in X^*(T)$, we have

$$T_i e^\lambda - e^{s_i \lambda} T_i = \begin{cases} (1 - q_2) \frac{e^\lambda - e^{s_i \lambda}}{e^{\alpha_i} - 1} & (i \neq n) \\ \frac{(1 + q_0 q_1) - (q_0 + q_1) e^{\epsilon_n}}{e^{\alpha_n} - 1} (e^\lambda - e^{s_n \lambda}) & (i = n) \end{cases}.$$

Remark 2.2. **1)** The standard choice of parameters (t_0, t_1, t_n) is: $t_1^2 = q_2$, $t_n^2 = -q_0 q_1$, and $t_n(t_0 - t_0^{-1}) = (q_0 + q_1)$. This yields

$$T_n e^\lambda - e^{s_n \lambda} T_n = \frac{1 - t_n^2 - t_n(t_0 - t_0^{-1}) e^{\epsilon_n}}{e^{2\epsilon_n} - 1} (e^\lambda - e^{s_n \lambda});$$

2) If $n = 1$, then we have $T_1 = T_n$ in Definition 2.1. In this case, we have $\mathbb{H} \cong \mathbb{C}[q_2^{\pm 1}] \otimes_\mathbb{C} \mathbb{H}_0$, where $\mathbb{H}_0$ is the Hecke algebra of type $A_1^{(1)}$ with two-parameters (q_0, q_1);

3) An extended Hecke algebra of type $B_n^{(1)}$ with two-parameters considered in [En06] is obtained by requiring $q_0 + q_1 = 0$. An equal parameter extended Hecke algebra of type $B_n^{(1)}$ is obtained by requiring $q_0 + q_1 = 0$ and $q_1^2 = q_2$. An equal parameter Hecke algebra of type $C_n^{(1)}$ is obtained by requiring $q_2 = -q_0 q_1$ and $(1 + q_0)(1 + q_1) = 0$.

For each $w \in W$, we define two closed subvarieties of Z_2 as

$$Z_{\le w} := \pi_2^{-1}(\overline{\mathsf{O}_w}) \text{ and } Z_{< w} := Z_{\le w} \backslash \pi_2^{-1}(\mathsf{O}_w).$$

Let $\lambda \in X^*(T)$. Let $\mathcal{L}_\lambda$ be the pullback of the line bundle $G \times^B \lambda^{-1}$ over G/B to F_2. Clearly $\mathcal{L}_\lambda$ admits a $\mathbf{G}$-action by letting $(\mathbb{C}^\times)^3$ act on $\mathcal{L}_\lambda$ trivially. We denote the operator $[\tilde{p}_1^* \mathcal{L}_\lambda \otimes^\mathbb{L} \bullet]$ by $\mathbf{e}^\lambda$. By abuse of notation, we may denote $\mathbf{e}^\lambda(1)$ by $\mathbf{e}^\lambda$ (in $K^\mathbf{G}(Z)$). Let $q_0 \in R(\{1\} \times \mathbb{C}^\times \times \{1\} \times \{1\}) \subset R(\mathbf{G})$, $q_1 \in R(\{1\} \times \{1\} \times \mathbb{C}^\times \times \{1\}) \subset R(\mathbf{G})$, and $q_2 \in R(\{1\} \times \{1\} \times \{1\} \times \mathbb{C}^\times) \subset R(\mathbf{G})$ be the inverse of degree-one characters. (I.e. q_2 corresponds to the inverse of the scalar multiplication on V_2.) By the operation $\mathbf{e}^\lambda$ and the multiplication by q_i, each of $K^\mathbf{G}(Z_{\le w})$ admits a structure of $R(\mathbf{T})$-modules.

Each $Z_{\leq w} \backslash Z_{<w}$ is a $\mathbf{G}$-equivariant vector bundle over an affine fibration over G/B via the composition of π_2 and the second projection. Therefore, the cellular fibration Lemma (or the successive application of localization sequence) yields:

Theorem 2.3 (cf. [CG97] 7.6.11)**.** *We have*

$$K^{\mathbf{G}}(Z_{\leq w}) = \bigoplus_{v \in W; \mathsf{O}_v \subset \overline{\mathsf{O}}_w} R(\mathbf{T})[\mathcal{O}_{Z_{\leq v}}].$$

For each $i = 1, 2, \ldots, n$, we put $\mathbb{O}_i := \overline{\pi_2^{-1}(\mathsf{O}_{s_i})}$. We define $\tilde{T}_i := [\mathcal{O}_{\mathbb{O}_i}]$ for each $i = 1, \ldots, n$.

Theorem 2.4 (cf. Proof of [CG97] 7.6.12)**.** *The set* $\{[\mathcal{O}_{Z_{\leq 1}}], \tilde{T}_i, \mathbf{e}^\lambda; 1 \leq i \leq n, \lambda \in X^*(T)\}$ *is a generator set of* $K^{\mathbf{G}}(Z_2)$ *as* $\mathcal{A}_{\mathbb{Z}}$*-algebras.*

Proof. The tensor product of structure sheaves corresponding to vector subspaces of a vector space is the structure sheaf of their intersection. Taking account into that, the proof of the assertion is exactly the same as [CG97] 7.6.12. □

By the Thom isomorphism, we have an identification

$$K^{\mathbf{G}}(F_2) \cong K^{\mathbf{G}}(G/B) \cong R(\mathbf{T}) = \mathcal{A}_{\mathbb{Z}}[T]. \tag{2.1}$$

We normalize the image of $[\mathcal{L}_\lambda]$ under (2.1) as e^λ.

Theorem 2.5 (cf. [CG97] Claim 7.6.7)**.** *The homomorphism*

$$\circ : K^{\mathbf{G}}(Z_2) \longrightarrow \mathrm{End}_{R(\mathbf{G})} K^{\mathbf{G}}(F_2)$$

is injective. □

Proposition 2.6. *We have*

1. $[\mathcal{O}_{Z_{\leq 1}}] = 1 \in \mathrm{End}_{R(\mathbf{G})} K^{\mathbf{G}}(F_2)$*;*
2. $\tilde{T}_i \circ \mathbf{e}^\lambda = (1 - q_2 \mathbf{e}^{\alpha_i}) \frac{\mathbf{e}^\lambda - \mathbf{e}^{s_i \lambda - \alpha_i}}{1 - \mathbf{e}^{-\alpha_i}}$ *for every* $\lambda \in X^*(T)$ *and every* $1 \leq i < n$*;*
3. $\tilde{T}_n \circ \mathbf{e}^\lambda = (1 - q_0 \mathbf{e}^{\frac{1}{2}\alpha_n})(1 - q_1 \mathbf{e}^{\frac{1}{2}\alpha_n}) \frac{\mathbf{e}^\lambda - \mathbf{e}^{s_n \lambda - \alpha_n}}{1 - \mathbf{e}^{-\alpha_n}}$ *for every* $\lambda \in X^*(T)$*.*

Proof. The component $Z_{\leq 1}$ is equal to the diagonal embedding of F_2. In particular, both of the first and the second projections give isomorphisms between $Z_{\leq 1}$ and F_2. It follows that

$$\begin{aligned}
[\mathcal{O}_{Z_{\leq 1}}] \circ [\mathcal{L}_\lambda] &= \sum_{i \geq 0} (-1)^i [\mathbb{R}^i (p_1)_* \left(\mathcal{O}_{Z_{\leq 1}} \otimes^{\mathbb{L}} \tilde{p}_2^* \mathcal{L}_\lambda \right)] \\
&= [\mathbb{R}^0 (p_1)_* \left(\mathcal{O}_{Z_{\leq 1}} \otimes \tilde{p}_2^* \mathcal{L}_\lambda \right)] = [\mathcal{L}_\lambda],
\end{aligned}$$

which proves 1). For each $i = 1, \ldots, n$, we define $\mathbb{V}^+(i) := \mathbb{V}_2^+ \cap \dot{s}_i \mathbb{V}_2^+$. Let $P_i := B \dot{s}_i B \sqcup B$ be a parabolic subgroup of G corresponding to s_i. Each $\mathbb{V}^+(i)$ is B-stable. Hence, it is P_i-stable. We have

$$\pi_2(\mathbb{O}_i) = \overline{\mathsf{O}}_{s_i} = (1 \times P_i) \mathsf{O}_1 \subset G/B \times G/B.$$

The product $(1 \times P_i)\mathsf{p}_1 \times \mathbb{V}^+(i)$ is a B-equivariant vector bundle. Here we have $G \cap (B \times P_i) = B$. Hence, we can induce it up to a G-equivariant vector bundle $\tilde{\mathbb{V}}(i)$ on $\pi_2(\mathbb{O}_i)$. By means of the natural embedding of G-equivariant vector bundles

$$F_2 = G \times^B \mathbb{V}_2^+ \hookrightarrow G \times^B \mathbb{V}_2 \cong G \times \mathbb{V}_2,$$

we can naturally identify $\pi_2^{-1}(\mathsf{p}_{s_i})$ with $\mathbb{V}^+(i)$. Since $\mathbb{V}^+(i)$ is P_i-stable, we conclude $\pi_2^{-1}(\mathsf{p}_{s_i}) \cong \mathbb{V}^+(i)$ as P_i-modules. As a consequence, we conclude $\tilde{\mathbb{V}}(i) \cong \mathbb{O}_i$. Let $\breve{F}(i) := G \times^B (\mathbb{V}_2^+/\mathbb{V}^+(i))$. It is a $\mathbf{G}$-equivariant quotient bundle of F_2. The rank of $\breve{F}(i)$ is one $(1 \le i < n)$ or two $(i = n)$. Let $\breve{Z}_{\le s_i}$ be the image of $Z_{\le s_i}$ under the quotient map $F_2 \times F_2 \to \breve{F}(i) \times \breve{F}(i)$. We obtain the following commutative diagram:

$$\begin{tikzcd}
F_2 \arrow[d] & Z_{\le s_i} \arrow[l] \arrow[r] \arrow[d] & F_2 \arrow[d] \\
\breve{F}(i) & \breve{Z}_{\le s_i} \arrow[l] \arrow[r] & \breve{F}(i)
\end{tikzcd}$$

Here the above objects are smooth $\mathbb{V}^+(i)$-fibrations over the bottom objects. Therefore, it suffices to compute the convolution operation of the bottom line. We have $\breve{Z}_{\le s_i} = \overline{\mathsf{O}}_{s_i} \cup \triangle(\breve{F}(i))$, where $\triangle : \breve{F}(i) \hookrightarrow \breve{F}(i)^2$ is the diagonal embedding. Let $\breve{p}_j : \overline{\mathsf{O}}_{s_i} \to G/B$ $(j = 1, 2)$ be projections induced by the natural projections of $G/B \times G/B$. By construction, each $\breve{p}_j$ is a $\mathbf{G}$-equivariant $\mathbb{P}^1$-fibration. Let $\breve{\mathcal{L}}_\lambda$ be the pullback of $G \times^B \lambda^{-1}$ to $\breve{F}(i)$. We deduce

$$\tilde{T}_i \circ [\breve{\mathcal{L}}_\lambda] = \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (\breve{p}_1)_* (\mathcal{O}_{\overline{\mathsf{O}}_{s_i}} \otimes^{\mathbb{L}} (\mathcal{O}_{\breve{F}(i)} \boxtimes \breve{\mathcal{L}}_\lambda)]$$
$$= \sum_{i \ge 0} (-1)^i [\mathbb{R}^i (\breve{p}_1)_* \breve{p}_2^* (G \times^B \lambda^{-1})] = \left[G \times^B [\frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}] \right],$$

where $[\frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}] \in R(T) \cong R(B)$ is a virtual B-module. Here the ideal sheaf associated to $G/B \subset \breve{F}(i)$ represents $q_2[\breve{\mathcal{L}}_{\alpha_i}]$ in $K^{\mathbf{G}}(\breve{F}(i))$ $(1 \le i < n)$ or corresponds to $q_0 \breve{\mathcal{L}}_{\epsilon_n} + q_1 \breve{\mathcal{L}}_{\epsilon_n} \subset \mathcal{O}_{\breve{F}(i)}$ $(i = n)$. In the latter case, divisors corresponding to $q_0 \breve{\mathcal{L}}_{\epsilon_n}$ and $q_1 \breve{\mathcal{L}}_{\epsilon_n}$ are normal crossing. Thus, we have $[q_0 \breve{\mathcal{L}}_{\epsilon_n} \cap q_1 \breve{\mathcal{L}}_{\epsilon_n}] = q_0 q_1 [\breve{\mathcal{L}}_{2\epsilon_n}]$. In particular, we deduce

$$[q_0 \breve{\mathcal{L}}_{\epsilon_n} + q_1 \breve{\mathcal{L}}_{\epsilon_n}] = q_0 [\breve{\mathcal{L}}_{\epsilon_n}] + q_1 [\breve{\mathcal{L}}_{\epsilon_n}] - q_0 q_1 [\breve{\mathcal{L}}_{2\epsilon_n}] \in K^{\mathbf{G}}(\breve{F}(n)).$$

Therefore, we conclude

$$\tilde{T}_i \circ \mathbf{e}^\lambda = \begin{cases} (1 - q_2 \mathbf{e}^{\alpha_i}) \frac{\mathbf{e}^\lambda - \mathbf{e}^{s_i \lambda - \alpha_i}}{1 - \mathbf{e}^{-\alpha_i}} & (\text{if } 1 \le i < n) \\ (1 - q_0 \mathbf{e}^{\frac{\alpha_n}{2}})(1 - q_1 \mathbf{e}^{\frac{\alpha_n}{2}}) \frac{e^\lambda - \mathbf{e}^{s_n \lambda - \alpha_n}}{1 - \mathbf{e}^{-\alpha_n}} & (\text{if } i = n) \end{cases}$$

as desired. □

The following representation of $\mathbb{H}$ is usually called the basic representation or the anti-spherical representation:

Theorem 2.7 (Basic representation cf. [Mc03] 4.3.10)**.** *There is an injective $\mathcal{A}$-algebra homomorphism*

$$\varepsilon : \mathbb{H} \to \mathrm{End}_{\mathcal{A}}\mathcal{A}[T],$$

defined as $\varepsilon(e^{\lambda}) := e^{\lambda}\cdot$ $(\lambda \in X^*(T))$ *and*

$$\varepsilon(T_i)e^{\lambda} := \begin{cases} \frac{e^{\lambda}-e^{s_i\lambda}}{e^{\alpha_i}-1} - q_2\frac{e^{\lambda}-e^{s_i\lambda+\alpha_i}}{e^{\alpha_i}-1} & (\textit{if } 1 \le i < n) \\ \frac{e^{\lambda}-e^{s_n\lambda}}{e^{\alpha_n}-1} + q_0q_1\frac{e^{\lambda}-e^{s_n\lambda+\alpha_n}}{e^{\alpha_n}-1} - (q_0+q_1)e^{\epsilon_n}\frac{e^{\lambda}-e^{s_n\lambda}}{e^{\alpha_n}-1} & (\textit{if } i = n) \end{cases}.$$

Theorem 2.8 (Exotic geometric realization of Hecke algebras)**.** *We have an isomorphism*

$$\mathbb{H} \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} K^{\mathbf{G}}(Z_2),$$

as algebras.

Proof. Consider an assignment ϑ

$$\underline{e}^{\lambda} \mapsto \mathbf{e}^{\lambda}, \underline{T_i} \mapsto \begin{cases} \tilde{T}_i - (1 - q_2(\mathbf{e}^{\alpha_i}+1)) & (1 \le i < n) \\ \tilde{T}_i + (q_0+q_1)\mathbf{e}^{\epsilon_n} - (1 + q_0q_1(\mathbf{e}^{\alpha_n}+1)) & (i = n) \end{cases}.$$

By means of the Thom isomorphism, the above assignment gives an action of an element of the set $\{\underline{e}^{\lambda}\} \cup \{\underline{T_i}\}_{i=1}^{n}$ on $\mathcal{A}[T]$. We have

$$\begin{aligned}
\vartheta(\underline{e}^{\lambda})e^{\mu} =& e^{\lambda+\mu} \\
\vartheta(\underline{T_i})e^{\lambda} =& \left(\tilde{T}_i - (1 - q_2(e^{\alpha_i}+1))\right)e^{\lambda} = (1 - q_2e^{\alpha_i})\frac{e^{\lambda} - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - e^{\lambda} + q_2(e^{\alpha_i}+1)e^{\lambda} \\
=& \left(\frac{e^{\lambda} - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - \frac{e^{\lambda} - e^{\lambda-\alpha_i}}{1-e^{-\alpha_i}}\right) - q_2e^{\alpha_i}\left(\frac{e^{\lambda} - e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}} - \frac{e^{\lambda} - e^{\lambda-2\alpha_i}}{1-e^{-\alpha_i}}\right) = \varepsilon(T_i)e^{\lambda} \\
\vartheta(\underline{T_n})e^{\lambda} =& \left(\tilde{T}_n + (q_0+q_1)e^{\epsilon_n} - (1 + q_0q_1(e^{\alpha_n}+1))\right)e^{\lambda} \\
=& (1 - q_0e^{\epsilon_n})(1 - q_1e^{\epsilon_n})\frac{e^{\lambda} - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - e^{\lambda} + (q_0+q_1)e^{\lambda+\epsilon_n} - q_0q_1(e^{\alpha_n}+1)e^{\lambda} \\
=& \left(\frac{e^{\lambda} - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - \frac{e^{\lambda} - e^{\lambda-\alpha_n}}{1-e^{-\alpha_n}}\right) + q_0q_1e^{\alpha_n}\left(\frac{e^{\lambda} - e^{s_n\lambda-\alpha_n}}{1-e^{-\alpha_n}} - \frac{e^{\lambda} - e^{\lambda-2\alpha_n}}{1-e^{-\alpha_n}}\right) \\
&- (q_0+q_1)\left(\frac{e^{\lambda+\epsilon_n} - e^{s_n\lambda-\epsilon_n}}{1-e^{-\alpha_n}} - \frac{e^{\lambda+\epsilon_n} - e^{\lambda-\epsilon_n}}{1-e^{-\alpha_n}}\right) = \varepsilon(T_n)e^{\lambda}.
\end{aligned}$$

This identifies $\mathbb{C} \otimes_{\mathbb{Z}} K^{\mathbf{G}}(F_2)$ with the basic representation of $\mathbb{H}$ via the correspondence $e^{\lambda} \mapsto \underline{e}^{\lambda}$ and $T_i \mapsto \underline{T_i}$. In particular, it gives an inclusion $\mathbb{H} \subset \mathbb{C} \otimes_{\mathbb{Z}} K^{\mathbf{G}}(Z_2)$. Here we have $T_i \in \tilde{T}_i + \mathcal{A}[T]$ for $1 \le i \le n$. It follows that $\mathbb{C} \otimes_{\mathbb{Z}} K^{\mathbf{G}}(Z_2) \subset \mathbb{H}$, which yields the result. □

Theorem 2.9 (Bernstein cf. [CG97] 7.1.14 and [Mc03] 4.2.10)**.** *The center $Z(\mathbb{H})$ of $\mathbb{H}$ is naturally isomorphic to $\mathbb{C} \otimes_{\mathbb{Z}} R(\mathbf{G})$.* □

Corollary 2.10. *The center of $K^{\mathbf{G}}(Z_2)$ is $R(\mathbf{G})$.* □

For a semisimple element $\mathbf{a} \in \mathbf{G}$, we define

$$\mathbb{H}_{\mathbf{a}} := \mathbb{C}_{\mathbf{a}} \otimes_{Z(\mathbb{H})} \mathbb{H} \quad (\text{cf. } \S 1.3)$$

and call it the specialized Hecke algebra.

Theorem 2.11. *Let $\mathbf{a} \in \mathbf{G}$ be a semisimple element. We have an isomorphism*

$$\mathbb{H}_{\mathbf{a}} \cong \mathbb{C} \otimes_{\mathbb{Z}} K(Z_2^{\mathbf{a}})$$

as algebras.

Proof. This is a combination of [CG97] 6.2.3 and 5.10.11. (See also [CG97] 8.1.6.) □

Convention 2.12. Let $\mathbf{a} = (s, \vec{q}) \in \mathbf{G}$ be a pre-admissible element. We define $Z_+^{\mathbf{a}}$ to be the image of $Z_2^{\mathbf{a}}$ under the natural projection defined by

$$Z_2 \ni (g_1 B, g_2 B, X_0, X_1, X_2) \mapsto (g_1 B, g_2 B, X_0 + X_1, X_2) \in Z.$$

Let $F_+^{\mathbf{a}}$ be the image of $Z_+^{\mathbf{a}}$ via the first (or the second) projection. Let $\mu_+^{\mathbf{a}}$ be the restriction of μ to $F_+^{\mathbf{a}}$. We denote its image by $\mathfrak{N}_+^{\mathbf{a}}$. By the assumption $q_0 \neq q_1$, we have $F_+^{\mathbf{a}} \cong F_2^{\mathbf{a}}$, $Z_+^{\mathbf{a}} \cong Z_2^{\mathbf{a}}$, and $\mathfrak{N}_+^{\mathbf{a}} \cong \mathfrak{N}_2^{\mathbf{a}}$.

Corollary 2.13. *Keep the setting of Convention 2.12. We have an isomorphism*

$$\mathbb{H}_{\mathbf{a}} \cong \mathbb{C} \otimes_{\mathbb{Z}} K(Z_+^{\mathbf{a}})$$

as algebras. □

3 Clan decomposition

We work under the same setting as in §2.

Definition 3.1 (Clans)**.** Let $\mathbf{a} = (s, \vec{q}) \in \mathbf{T}$ be a pre-admissible element. We denote $s = \exp(\lambda)$, where

$$\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \mathbb{E} \cong \mathfrak{t}.$$

Let $q_2 = \exp r_2$. We put $\Gamma_0 := 2\pi\sqrt{-1}\mathbb{Z}$ and $\Gamma := r_2\mathbb{Z} + \Gamma_0$. A clan associated to $\mathbf{a}$ is a maximal subset $\mathbf{c} \subset [1, n]$ with the following property: For each two elements $i, j \in \mathbf{c}$, there exists a sequence $i = i_0, i_1, \ldots, i_m = j$ (in $\mathbf{c}$) such that

$$\{\lambda_{i_k} \pm \lambda_{i_{k+1}}\} \cap \{\pm r_2 + \Gamma_0, \Gamma_0\} \neq \emptyset \quad \text{for each } 0 \leq k < m.$$

We have a disjoint decomposition

$$[1, n] = \bigsqcup_{\mathbf{c} \in I(\mathbf{a})} \mathbf{c},$$

where each $\mathbf{c}$ is a clan associated to $\mathbf{a}$ and $I(\mathbf{a})$ is the set of clans associated to $\mathbf{a}$. For a clan $\mathbf{c}$, we put $n^{\mathbf{c}} := \#\mathbf{c}$.

We assume the setting of Definition 3.1 in the rest of this section unless stated otherwise. At the level of Lie algebras, we have a decomposition

$$\mathfrak{g}(s) := \mathfrak{t} \oplus \bigoplus_{\substack{i < j, \sigma_1, \sigma_2 \in \{\pm\}, \\ \sigma_1 \lambda_i + \sigma_2 \lambda_j \equiv 0}} \mathfrak{g}(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{\substack{i \in [1, n], \sigma \in \{\pm\}, \\ 2\lambda_i \equiv 0}} \mathfrak{g}(s)[\sigma 2\epsilon_i],$$

where $\equiv$ means modulo Γ_0. For each $\mathbf{c} \in I(\mathbf{a})$, we define a Lie algebra $\mathfrak{g}(s)_{\mathbf{c}}$ as the Lie subalgebra of $\mathfrak{g}(s)$ defined as

$$\bigoplus_{i \in \mathbf{c}} \mathbb{C}\epsilon_i \oplus \bigoplus_{\substack{i<j \in \mathbf{c}, \sigma_1, \sigma_2 \in \{\pm\}, \\ \sigma_1 \lambda_i + \sigma_2 \lambda_j \equiv 0}} \mathfrak{g}(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{\substack{i \in \mathbf{c}, \sigma \in \{\pm\}, \\ 2\lambda_i \equiv 0}} \mathfrak{g}(s)[\sigma 2 \epsilon_i],$$

where $\equiv$ means modulo Γ_0. Moreover, we have

$$\mathfrak{g}(s) = \bigoplus_{\mathbf{c} \in I(\mathbf{a})} \mathfrak{g}(s)_{\mathbf{c}}. \tag{3.1}$$

In particular, we have $[\mathfrak{g}(s)_{\mathbf{c}}, \mathfrak{g}(s)_{\mathbf{c}'}] = 0$ unless $\mathbf{c} = \mathbf{c}'$. Let $G(s)_{\mathbf{c}}$ be the connected subgroup of $G(s)$ which has $\mathfrak{g}(s)_{\mathbf{c}}$ as its Lie algebra.

Lemma 3.2. *We have* $G(s) = \prod_{\mathbf{c} \in I(\mathbf{a})} G(s)_{\mathbf{c}}$.

Proof. By (3.1), it is clear that $\prod_{\mathbf{c} \in I(\mathbf{a})} G(s)_{\mathbf{c}}$ is equal to the identity component of $G(s)$. Since G is a simply connected semi-simple group, it follows that $G(s)$ is connected by Steinberg's centralizer theorem (cf. [Ca85] 3.5.6). In particular, we have $G(s) \subset \prod_{\mathbf{c} \in I(\mathbf{a})} G(s)_{\mathbf{c}}$ as desired. □

We denote $B \cap G(s)_{\mathbf{c}}$ and ${}^{w}B \cap G(s)_{\mathbf{c}}$ by $B(s)_{\mathbf{c}}$ and ${}^{w}B(s)_{\mathbf{c}}$, respectively.

Convention 3.3. We denote by $\mathbb{V}^{\mathbf{a}}$ the image of $\mathbb{V}_2^{\mathbf{a}}$ to $\mathbb{V}$ via the map

$$\mathbb{V}_2 \ni (X_0 \oplus X_1 \oplus X_2) \mapsto ((X_0 + X_1) \oplus X_2) \in \mathbb{V}.$$

Since $q_0 \neq q_1$, we have $\mathbb{V}^{\mathbf{a}} \cong \mathbb{V}_2^{\mathbf{a}}$.

For each $\mathbf{c} \in I(\mathbf{a})$, we define

$$\mathbb{V}_{\mathbf{c}}^{\mathbf{a}} := \sum_{i,j \in \mathbf{c}, \sigma_1, \sigma_2, \sigma_3 \in \{\pm\}} \mathbb{V}^{\mathbf{a}}[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \mathbb{V}^{\mathbf{a}}[\sigma_3 \epsilon_i].$$

It is clear that $\mathbb{V}^{\mathbf{a}} = \bigoplus_{\mathbf{c} \in I(\mathbf{a})} \mathbb{V}_{\mathbf{c}}^{\mathbf{a}}$. By the comparison of weights, the $\mathfrak{g}(s)_{\mathbf{c}}$-action on $\mathbb{V}_{\mathbf{c}'}^{\mathbf{a}}$ is trivial unless $\mathbf{c} = \mathbf{c}'$.

Remark 3.4. Since $\mathbf{c}$ is not an integer and we do not use $\mathbb{V}_\ell$ in the rest of this paper, we use the notation $\mathbb{V}_{\mathbf{c}}^{\mathbf{a}}$. The author hopes the reader not to confuse $\mathbb{V}_{\mathbf{c}}^{\mathbf{a}}$ with $(\mathbb{V}_\ell)^{\mathbf{a}}$.

Lemma 3.5. *Let* $\mathbb{O} \subset \mathfrak{N}_+^{\mathbf{a}}$ *be a* $\mathbf{G}(\mathbf{a})$*-orbit. Let* $\mathbb{O}_{\mathbf{c}}$ *denote the image of* $\mathbb{O}$ *under the natural projection* $\mathbb{V}^{\mathbf{a}} \to \mathbb{V}_{\mathbf{c}}^{\mathbf{a}}$*. Then, we have a product decomposition* $\mathbb{O} = \oplus_{\mathbf{c} \in I(\mathbf{a})} \mathbb{O}_{\mathbf{c}}$.

Proof. Let $X \in \mathbb{V}^{\mathbf{a}}$. There exists a family $\{X_{\mathbf{c}}\}_{\mathbf{c} \in I(\mathbf{a})}$ ($X_{\mathbf{c}} \in \mathbb{V}_{\mathbf{c}}^{\mathbf{a}}$) such that $X = \sum_{\mathbf{c} \in I(\mathbf{a})} X_{\mathbf{c}}$. We have $G(s)X = \bigoplus_{\mathbf{c} \in I(\mathbf{a})} G(s)_{\mathbf{c}} X_{\mathbf{c}}$. Let $\mathbf{c}^i$ ($i = 1, 2$) be the unique clan such that $V_1^{(s,q_i)}[\sigma \epsilon_j] \neq 0$ for some $j \in \mathbf{c}^i$ and $\sigma \in \{\pm\}$. Let $\mathbb{G}_{\mathbf{c}}$ be the product of scalar multiplications of $V_1^{(s,q_i)}$ such that $V_1^{(s,q_i)} \cap \mathbb{V}_{\mathbf{c}}^{\mathbf{a}} \neq \{0\}$. Since the set of $\mathbf{a}$-fixed points of a conic variety in $\mathbb{V}$ is conic, we have $(G(s)_{\mathbf{c}} \times (\mathbb{C}^{\times})^3) X_{\mathbf{c}} = (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}}) X_{\mathbf{c}}$. We have $\prod_{\mathbf{c} \in I(\mathbf{a})} (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}}) \subset \mathbf{G}(\mathbf{a})$. It follows that

$$\mathbf{G}(\mathbf{a}) X = \bigoplus_{\mathbf{c} \in I(\mathbf{a})} \mathbf{G}(\mathbf{a}) X_{\mathbf{c}} = \bigoplus_{\mathbf{c} \in I(\mathbf{a})} (G(s)_{\mathbf{c}} \times \mathbb{G}_{\mathbf{c}}) X_{\mathbf{c}} = \bigoplus_{\mathbf{c} \in I(\mathbf{a})} \mathbb{O}_{\mathbf{c}}$$

as desired. □

For each $w \in W$, we define

$$F_+^{\mathbf{a}}(w) := G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}}).$$

Similarly, we define

$$F_+^{\mathbf{a}}(w, \mathbf{c}) := G(s)_{\mathbf{c}} \times^{{}^w B(s)_{\mathbf{c}}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^{\mathbf{a}})$$

for each $\mathbf{c} \in I(\mathbf{a})$.

Lemma 3.6. *We have* $F_+^{\mathbf{a}} = \cup_{w \in W} F_+^{\mathbf{a}}(w)$.

Proof. The set of $\mathbf{a}$-fixed points of G/B is a disjoint union of flag varieties of $G(s)$. It follows that each point of $F_+^{\mathbf{a}}$ is $G(s)$-conjugate to a point in the fiber over a T-fixed point of G/B. □

The local structures of these connected components are as follows.

Lemma 3.7. *For each* $w \in W$*, we have*

$$F_+^{\mathbf{a}}(w) \cong \prod_{\mathbf{c} \in I(\mathbf{a})} F_+^{\mathbf{a}}(w, \mathbf{c}).$$

Proof. The set $\mathbb{V}_{\mathbf{c}}^{\mathbf{a}}$ is T-stable for each $\mathbf{c} \in I(\mathbf{a})$. Hence, we have

$$F_+^{\mathbf{a}}(w) = G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}}) \cong G(s) \times^{{}^w B(s)} \Big(\bigoplus_{\mathbf{c} \in I(\mathbf{a})} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^{\mathbf{a}})\Big).$$

Since we have $G(s)/B(s) \cong \prod_{\mathbf{c} \in I(\mathbf{a})} G(s)_{\mathbf{c}}/B(s)_{\mathbf{c}}$, we deduce

$$G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^{\mathbf{a}}) \cong \prod_{\mathbf{c}' \in I(\mathbf{a})} G(s)_{\mathbf{c}'} \times^{{}^w B(s)_{\mathbf{c}'}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^{\mathbf{a}} \cap \mathbb{V}_{\mathbf{c}'}^{\mathbf{a}}).$$

Here the RHS is isomorphic to

$$F_+^{\mathbf{a}}(w, \mathbf{c}) \times \prod_{\mathbf{c} \neq \mathbf{c}'} G(s)_{\mathbf{c}'} / {}^w B(s)_{\mathbf{c}'}.$$

Gathering these information yields the result. □

We define a map ${}^w\mu_{\mathbf{c}}^{\mathbf{a}}$ by

$${}^w\mu_{\mathbf{c}}^{\mathbf{a}} : F_+^{\mathbf{a}}(w, \mathbf{c}) = G(s)_{\mathbf{c}} \times^{{}^w B(s)_{\mathbf{c}}} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}_{\mathbf{c}}^{\mathbf{a}}) \longrightarrow \mathbb{V}_{\mathbf{c}}^{\mathbf{a}}.$$

Definition 3.8 (Regular parameters)**.** An admissible parameter $(\mathbf{a}, X)$ is called regular iff there exists a direct factor $A[d] \subset (\mu_+^{\mathbf{a}})_* \mathbb{C}_{F_+^{\mathbf{a}}}$, where A is a simple $\mathbf{G}(\mathbf{a})$-equivariant perverse sheaf on $\mathfrak{N}_+^{\mathbf{a}}$ such that $\mathrm{supp} A = \overline{\mathbf{G}(\mathbf{a})X}$ and d is an integer. For a pre-admissible element $\mathbf{a} \in \mathbf{G}$, we define

$$\Lambda_{\mathbf{a}} := \{X \in \mathfrak{N}_+^{\mathbf{a}}; (\mathbf{a}, X) \text{ is regular admissible}\}/\mathbf{G}(\mathbf{a}).$$

Proposition 3.9 (Clan decomposition)**.** *For each* $w \in W$*, we have*

$$\mu_+^{\mathbf{a}}|_{F_+^{\mathbf{a}}(w)} \cong \prod_{\mathbf{c} \in I(\mathbf{a})} {}^w\mu_{\mathbf{c}}^{\mathbf{a}}.$$

In particular, every irreducible direct summand A *of* $(\mu_+^{\mathbf{a}})_* \mathbb{C}_{F_+^{\mathbf{a}}}$ *is written as an external product of* $G(s)$*-equivariant sheaves appearing in* $({}^w\mu_{\mathbf{c}}^{\mathbf{a}})_* \mathbb{C}_{F_+^{\mathbf{a}}(w,\mathbf{c})}$ *(up to degree shift).*

Proof. The first assertion follows from the combination of Lemma 3.5, Lemma 3.7, and the definition of ${}^{w}\mu^{\mathbf{a}}_{\mathbf{c}}$. We have $\mathbb{C}_{F^{\mathbf{a}}_{+}} = \bigoplus_{F^{\mathbf{a}}_{+}(w) \subset F^{\mathbf{a}}_{+}} \mathbb{C}_{F^{\mathbf{a}}_{+}(w)}$. A direct summand of $(\mu^{\mathbf{a}}_{+})_{*}\mathbb{C}_{F^{\mathbf{a}}_{+}}$ is a direct summand of $(\mu^{\mathbf{a}}_{+})_{*}\mathbb{C}_{F^{\mathbf{a}}_{+}(w)}$ for some $w \in W$. Since

$$(\mu^{\mathbf{a}}_{+})_{*}\mathbb{C}_{F^{\mathbf{a}}_{+}(w)} \cong \boxtimes_{\mathbf{c}} ({}^{w}\mu^{\mathbf{a}}_{\mathbf{c}})_{*}\mathbb{C}_{F^{\mathbf{a}}_{+}(w,\mathbf{c})},$$

the second assertion follows. □

We put $G_{\mathbf{c}} := Sp(2n^{\mathbf{c}})$ and $s_{\mathbf{c}} := \exp(\sum_{i \in \mathbf{c}} \lambda_i \epsilon_i) \in T$. We have embeddings

$$s = \prod_{\mathbf{c} \in I(\mathbf{a})} s_{\mathbf{c}} \in \prod_{\mathbf{c} \in I(\mathbf{a})} Sp(2n^{\mathbf{c}}) \subset Sp(2n),$$

induced by the following identifications:

$$\mathfrak{g}(s)_{\mathbf{c}} = \mathfrak{g}_{\mathbf{c}}(s_{\mathbf{c}}) \subset \left(\bigoplus_{i \in \mathbf{c}} \mathbb{C}\epsilon_i \right) \oplus \bigoplus_{\substack{\alpha = \sigma_1 \epsilon_i + \sigma_2 \epsilon_j \neq 0 \\ \sigma_1, \sigma_2 \in \{\pm\}, i, j \in \mathbf{c}}} \mathfrak{g}[\alpha] = \mathfrak{g}_{\mathbf{c}}. \tag{3.2}$$

It follows that $G(s)_{\mathbf{c}} = G_{\mathbf{c}}(s_{\mathbf{c}}) \subsetneq G_{\mathbf{c}}$ in general.

Let $\mathbb{V}(\mathbf{c})$ be the 1-exotic representation of $G_{\mathbf{c}}$. We have a natural embedding $\mathbb{V}^{\mathbf{a}}_{\mathbf{c}} \subset \mathbb{V}(\mathbf{c})$ which is compatible with (3.2).

Let $\nu = (\mathbf{a}, X)$ be a standard parameter. We have a family of admissible parameters $\nu_{\mathbf{c}} := (s_{\mathbf{c}}, \vec{q}, X_{\mathbf{c}})$ of $G_{\mathbf{c}}$'s such that $s = \prod_{\mathbf{c}} s_{\mathbf{c}}$, $X = \oplus_{\mathbf{c}} X_{\mathbf{c}}$. Let $W_{\mathbf{a}} := \prod_{\mathbf{c} \in I(\mathbf{a})} N_{G_{\mathbf{c}}}(T)/T$. By Lemma 3.7, we conclude that

$$\bigcup_{w \in W_{\mathbf{a}}} F^{\mathbf{a}}_{+}(w) \subset F^{\mathbf{a}}_{+} \tag{3.3}$$

is the product of the $F^{\mathbf{a}}_{+}$'s obtained by replacing the pair (G, ν) by $(G_{\mathbf{c}}, \nu_{\mathbf{c}})$ for all $\mathbf{c} \in I(\mathbf{a})$.

Corollary 3.10. *Let $\nu = (\mathbf{a}, X)$ be a standard parameter. Then, it is regular if and only if $\nu_{\mathbf{c}}$ is a regular admissible parameter of $G_{\mathbf{c}}$ for every $\mathbf{c} \in I(\mathbf{a})$.*

Proof. Let $W_0 := N_{G(s)}(T)/T \subset W$. We have a natural inclusion $W_0 \subset W_{\mathbf{a}}$. Here we have

$$\mu^{\mathbf{a}}_{+} = \bigsqcup_{w \in W/W_0} \mu^{\mathbf{a}}_{+}|_{F^{\mathbf{a}}_{+}(w)},$$

where we regard $W/W_0 \subset W$ by taking some representative. For each $w \in W$, there exists $v \in W_{\mathbf{a}}$ such that ${}^{w}\mathbb{V}^{+} \cap \mathbb{V}^{\mathbf{a}} = {}^{v}\mathbb{V}^{+} \cap \mathbb{V}^{\mathbf{a}} \subset \mathbb{V}^{\mathbf{a}}$. Moreover, we can choose v so that ${}^{w}B(s)_{\mathbf{c}} = {}^{v}B(s)_{\mathbf{c}}$ holds for each $\mathbf{c} \in I(\mathbf{a})$. As a consequence, all $F^{\mathbf{a}}_{+}(w)$ are isomorphic to one of $F^{\mathbf{a}}_{+}(w)$ ($w \in W_{\mathbf{a}}$) as $\mathbf{G}(\mathbf{a})$-varieties, together with maps $\mu^{\mathbf{a}}_{+}|_{F^{\mathbf{a}}_{+}(w)}$ to $\mathbb{V}^{\mathbf{a}}$. Therefore, ν is regular if and only if an intersection cohomology complex with its support $\overline{\mathbf{G}(\mathbf{a})\nu}$ (with degree shift) appears in $(\mu^{\mathbf{a}}_{+})_{*}\mathbb{C}_{F^{\mathbf{a}}_{+}(w)}$ for some $w \in W_{\mathbf{a}}$. Hence, Proposition 3.9 implies the result. □

Corollary 3.10 reduces the analysis of the decomposition pattern of $(\mu^{\mathbf{a}}_{+})_{*}\mathbb{C}_{F^{\mathbf{a}}_{+}}$ into the case that ν has a unique clan.

4 Rearrangement of parameters

We work under the same setting as in the previous section. In particular, we fix a pre-admissible $\mathbf{a} = (s, \vec{q}) \in \mathbf{T}$ such that $s = \exp(\sum_i \lambda_i \epsilon_i)$. In the below, we assume that $\mathbf{c} = [1, n]$ is the unique clan associated to $\mathbf{a}$ unless stated otherwise.

For each $\xi \in \mathbb{C}$, we put $\mathbf{c}(\xi) := \{i \in \mathbf{c}; \lambda_i \in (\xi + \Gamma_0) \cup (-\xi + \Gamma_0)\}$ and $\widetilde{\mathbf{c}}(\xi) := \mathbf{c}(\xi) \cup \mathbf{c}(\xi + \pi\sqrt{-1})$.

Lemma 4.1. *Let $\mathbf{c}$ and λ be as the above. If $\mathbf{c}(\lambda) \neq \emptyset$, then there exists $\lambda_0 \in \mathbb{C}$ such that*

$$\mathbf{c}(\lambda_0) \neq \emptyset, \mathbf{c}(\lambda_0 + nr_2) = \emptyset, \text{ and } \mathbf{c} = \bigsqcup_{0 \le m < n} \mathbf{c}(\lambda_0 + mr_2).$$

Moreover, the choice of λ_0 is unique up to sign and translation by Γ_0.

Proof. By the definition of clans, we have $\mathbf{c} = \bigcup_{0 \le m} \mathbf{c}(\lambda_0' + mr_2)$ for some $\lambda_0' \in \mathbb{C}$. We can rearrange λ_0' if necessary to assume

$$(\lambda_0' + \mathbb{Z}_{\ge 0} r_2 + \Gamma_0) \cap (-\lambda_0' - \mathbb{Z}_{\ge 0} r_2 + \Gamma_0) \subset \Gamma_0.$$

(Notice that the LHS is non-empty only if $2\lambda_i \in \Gamma$ for every i.) Since we have $\mathbf{c}(\lambda_0' + mr_2) = \mathbf{c}(-\lambda_0' - mr_2)$ and q_2 is not a root of unity of order $\le 2n$, this implies

$$\bigsqcup_{0 \le m \le n} \mathbf{c}(\lambda_0' + (m + k)r_2) \subset \mathbf{c}$$

for each $k \in \mathbb{Z}_{\ge 0}$. We have $\mathbf{c}(\lambda_0' + mr_2) \neq \emptyset$ only if $\mathbf{c}(\lambda_0' + (m \pm 1)r_2) \neq \emptyset$ or $\mathbf{c}(\lambda_0' + mr_2) = \mathbf{c}$. Hence, there exists unique m_0 such that $\lambda_0 = \lambda_0' + m_0 r_2$ satisfies $\mathbf{c} = \bigsqcup_{0 \le m \le n} \mathbf{c}(\lambda_0 + mr_2)$ and $\mathbf{c}(\lambda_0) \neq \emptyset$. This implies $\mathbf{c}(\lambda_0 + nr_2) = \emptyset$ as desired. $\square$

Definition 4.2. Keep the setting of Lemma 4.1. We define $\mathbf{c}_m := \mathbf{c}(\lambda_0 + mr_2)$ for each $0 \le m < n$.

Definition 4.3. Let $\mathbf{a}$ and $\mathbf{c}$ be as the above. The clan $\mathbf{c}$ is called type II iff $\mathbf{c}_0 = \widetilde{\mathbf{c}}(r_2/2)$, and called type III iff $\mathbf{c}_0 = \widetilde{\mathbf{c}}(0)$. Otherwise, we call $\mathbf{c}$ type I.

Lemma 4.4. *We have $\mathbf{c}_m = \widetilde{\mathbf{c}}(0)$ if and only if $\mathbf{c}$ is type III and $m = 0$.*

Proof. The "if" part is definition. We prove "only if" part. We assume $\widetilde{\mathbf{c}}(0) \neq \emptyset$. It follows that $\widetilde{\mathbf{c}}(-nr_2) = \widetilde{\mathbf{c}}(nr_2) = \emptyset$ and $\widetilde{\mathbf{c}}(-r_2) = \widetilde{\mathbf{c}}(r_2)$. Thus, the description of Lemma 4.1 forces $2\lambda_0 \in \Gamma_0$, which implies $\mathbf{c}_0 = \widetilde{\mathbf{c}}(0)$ as desired. $\square$

Lemma 4.5. *We have $\mathbf{c}_m = \widetilde{\mathbf{c}}(r_2/2)$ if and only if $\mathbf{c}$ is type II and $m = 0$.*

Proof. The "if" part is definition. We prove "only if" part. We assume $\widetilde{\mathbf{c}}(r_2/2) \neq \emptyset$. It follows that $\widetilde{\mathbf{c}}(-\frac{n+1}{2} r_2) = \widetilde{\mathbf{c}}(\frac{n+1}{2} r_2) = \emptyset$ and $\widetilde{\mathbf{c}}(-r_2/2) = \widetilde{\mathbf{c}}(r_2/2)$. Thus, the description of Lemma 4.1 forces $2\lambda_0 \in r_2 + \Gamma_0$, which implies $\mathbf{c}_0 = \widetilde{\mathbf{c}}(r_2/2)$ as desired. $\square$

For each $J \subset [1, n]$ and $0 \le m < n$, we define $J_m := J \cap \mathbf{c}_m$. By abuse of notation, we may denote by J_m the unique member of J_m if J satisfies $\# J_m \le 1$ for each m. We put

$$\underline{J} := \{m \in [0, n); J_m \neq \emptyset\} \subset [0, n).$$

Lemma 4.6. *Let $\nu = (\mathbf{a}, X)$ be a standard parameter. Assume that $J \in |X|$. There exists a unique partition $J = J_+ \cup J_-$ such that $\underline{J_\pm}$ are segments, $\underline{J_-} \subset \underline{J_+}$, $(J_\pm)_m = \emptyset$ or $\{j_m^\pm\}$ (signs are the same), and $j_m^+ \geq j_m^-$ holds for each $0 \leq m < n$.*

Proof. By the definition of standard parameters, we have $i, j \in J_m$ only if $\lambda_i \in \sigma_j \lambda_j + l_j r_2 + \Gamma_0$ for some $\sigma_j \in \{\pm\}$ and $0 < |l_j| < n$. Since $\mathbf{c}(\lambda_i) = \mathbf{c}(\lambda_j)$ and r_2 is not a root of unity of order $\leq 2n$, we have $2\lambda_i \equiv l_j r_2 \mod \Gamma_0$. Let $i, j, k \in J_m$ be distinct members to deduce contradiction. Then, we have

$$\sigma_k \lambda_k + l_k r_2 \equiv \lambda_i \equiv \sigma_j \lambda_j + l_j r_2 \mod \Gamma_0,$$

for some $\sigma_k \in \{\pm\}$ and $0 < |l_k| < n$. We have $2\lambda_i \equiv l_k r_2 \mod \Gamma_0$. This forces $l_j = l_k$ since r_2 is not a root of unity of order $\leq 2n$. This is contradiction. It follows that we have $\# J_m \leq 2$. Hence, we have a unique partition $J = J_+ \cup J_-$ with $\underline{J_-} \subset \underline{J_+}$, $\#(J_\pm)_m = \emptyset$ or $\{j_m^\pm\}$, and $j_m^+ \geq j_m^-$. Since J is the support of a block, it follows that each of $\underline{J_\pm}$ must be a segment. $\square$

For a standard parameter $\nu = (\mathbf{a}, X)$, we define

$$\mathcal{J}(\nu) := \{J_\pm ; J \in |X|\}.$$

Here $J_\pm$ are defined as in Lemma 4.6. We define $m(J) = \min \underline{J}$. For each $J, J' \in \mathcal{J}(\nu)$, we put $\alpha_{J,J'} := \epsilon_{J_{m(J)}} + \epsilon_{J'_{m(J')}}$.

Definition 4.7 (Supports and normal forms)**.** We assume the same setting as in Lemma 4.6. For $J \in \mathcal{J}(\nu)$, we define $\mathbf{v}_J := \sum_{m, m-1 \in \underline{I}} \mathbf{v}[\epsilon_{J_m} - \epsilon_{J_{m-1}}]$. For $J \in |X|$, we define

$$\mathbf{v}_J := \mathbf{v}[\alpha_{J_+, J_-}] + \mathbf{v}_{J_+} + \mathbf{v}_{J_-},$$

where we understand that $\mathbf{v}[\alpha_{J_+, J_-}] = 0$ unless $J_- \neq \emptyset$.

Let $\nu = (\mathbf{a}, X)$ be an admissible parameter. The action of $N_G(T)$ exchanges λ_i with $\pm\lambda_j$. Adding an element of Γ_0 to λ_i does not change $s = e^\lambda$. Hence, we can rearrange ν and $\{\lambda_i\}_i$ to satisfy the following condition $(\star)$:

$(\star)_1$ We have $\lambda_i - m r_2 = \lambda_j$ if $i \in \mathbf{c}_{k+m}$ and $j \in \mathbf{c}_k$ for some k;

$(\star)_2$ We have $i < j$ if $i \in \mathbf{c}_{k+m}$ and $j \in \mathbf{c}_k$ for some k and $m > 0$;

$(\star)_3$ We have $X_2 = \sum_{J \in |X|} \mathbf{v}_J$.

By construction, we can assume $(\star)$ for a representative of every $G(s)$-conjugacy class in $\mathfrak{P}_\mathbf{a}$ without the loss of generality.

Lemma 4.8. *Assume the condition $(\star)$. Then, we have*

$$\Psi(\mathbb{V}^\mathbf{a} \cap V_2) = \Psi(\nu) \cup \bigcup_{m \geq 0} \{\epsilon_i - \epsilon_j ; i \in \mathbf{c}_{m+1}, j \in \mathbf{c}_m\},$$

where we have

$$\Psi(\nu) = \begin{cases} \emptyset & (\mathbf{c} \textit{ is type I}) \\ \{\epsilon_i + \epsilon_j ; i \in \mathbf{c}_1, j \in \mathbf{c}_0\} & (\mathbf{c} \textit{ is type II}) \\ \{\epsilon_i + \epsilon_j ; i, j \in \mathbf{c}_0, i \neq j\} & (\mathbf{c} \textit{ is type III}) \end{cases}.$$

In particular, we have $\mathbb{V}^\mathbf{a} \cap V_2 \subset V_2^+$.

Proof. We have $(\mathbb{V}^{\mathbf{a}} \cap V_2)[\sigma_1\epsilon_i + \sigma_2\epsilon_j] \neq 0$ $(i < j,\ \sigma_1, \sigma_2 \in \{\pm\})$ if and only if $\sigma_1\lambda_i + \sigma_2\lambda_j \in r_2 + \Gamma_0$. By $(\star)_1$, this happens only if a) $\sigma_1 = \sigma_2 = +$ and $i, j \in \widetilde{\mathbf{c}}(r_2/2)$, b) $\sigma_1 = \sigma_2 = +$ and $i \in \widetilde{\mathbf{c}}(r_0), j \in \widetilde{\mathbf{c}}(0)$, or c) $\sigma_1 = +, \sigma_2 = -$ and $\lambda_i = \lambda_j + r_2$. These roots are positive by $(\star)_2$. □

Corollary 4.9. *Keep the setting of Lemma 4.8. Then, we have $X_2 \in V_2^+$ for every $(\mathbf{a}, X) \in \mathfrak{P}_{\mathbf{a}}$.* □

For each $0 \le m < n$, we put

$$\mathfrak{g}(m) := \begin{cases} \bigoplus_{i \in \mathbf{c}_m} \mathbb{C}\epsilon_i \oplus \bigoplus_{i \in \mathbf{c}_m, \sigma \in \{\pm\}} \mathfrak{g}[\sigma 2\epsilon_i] \oplus \bigoplus_{\substack{i,j \in \mathbf{c}_m, i<j \\ \sigma_1, \sigma_2 \in \{\pm\}}} \mathfrak{g}[\sigma_1\epsilon_i + \sigma_2\epsilon_j] & (\mathbf{c}_m = \widetilde{\mathbf{c}}(0)) \\ \bigoplus_{i \in \mathbf{c}_m} \mathbb{C}\epsilon_i \oplus \bigoplus_{\substack{i,j \in \mathbf{c}_m, i<j \\ \sigma \in \{\pm\}}} \mathfrak{g}[\sigma(\epsilon_i - \epsilon_j)] & (\text{otherwise}) \end{cases}.$$

These are Lie subalgebras of $\mathfrak{g}(s)_{\mathbf{c}} = \mathfrak{g}(s)$. By a weight comparison, we conclude $[\mathfrak{g}(m), \mathfrak{g}(m')] = 0$ unless $m = m'$. We define $G(m)$ to be the connected subgroup of G with its Lie algebra $\mathfrak{g}(m)$. We have

$$G(m) \cong \begin{cases} GL(d_m^{\mathbf{a}}) & (\mathbf{c}_m \neq \widetilde{\mathbf{c}}(0)) \\ Sp(d_m^{\mathbf{a}}) & (\mathbf{c}_m = \widetilde{\mathbf{c}}(0)) \end{cases}, \tag{4.1}$$

where $d_m^{\mathbf{a}} = \#\mathbf{c}_m$ $(\mathbf{c}_m \neq \widetilde{\mathbf{c}}(0))$, or $2(\#\mathbf{c}_m)$ $(\mathbf{c}_m = \widetilde{\mathbf{c}}(0))$.

Lemma 4.10. *We have $G(s) = G(s)_{\mathbf{c}} = \prod_{m \ge 0} G(m)$.*

Proof. The first identity follows from the assumption $[1, n] = \mathbf{c}$. Let $i, j \in \mathbf{c}$ and let $\sigma_1, \sigma_2 \in \{\pm\}$. We have $\mathfrak{g}(s)[\sigma_1\epsilon_i + \sigma_2\epsilon_j] \neq 0$ if and only if $\sigma_1\lambda_i + \sigma_2\lambda_j \in \Gamma_0$ holds. By $(\star)_1$, this implies $i, j \in \mathbf{c}_m$ for some m. Moreover, $\sigma_1\sigma_2 = +$ occurs in the above condition if and only if $i, j \in \widetilde{\mathbf{c}}(0)$. Therefore, we deduce $G(s) \supset \prod_{m \ge 0} G(m)$. The reverse inclusion exists since $G(s)$ is connected. □

Definition 4.11 (Height function)**.** Let $\nu = (s, \vec{q}, X)$ be a standard parameter. Let $h : \mathcal{J}(\nu) \to \mathbb{R}$ be a function. We put

$$s_h := \exp\Big(\sum_{i \in J, J \in \mathcal{J}(X)} h(J)\epsilon_i\Big).$$

We call h a height function adapted to ν if $s_h X = X$ holds. Let $L^h := Z_{G(s)}(s_h)$. We put

$$U^h := \exp\left\langle Z \in \mathfrak{g}(s); \lim_{n \to -\infty} s_h^n Z = 0 \right\rangle \text{ and } \mathbb{V}_h := \{v \in \mathbb{V}; \lim_{n \to -\infty} s_h^n v \text{ converges.}\}.$$

The group $P^h := L^h U^h$ is a subgroup of $G(s)$. We put $\mathbb{V}^h := \mathbb{V}_h \cap \mathbb{V}^{\mathbf{a}}$. It is clear that P^h acts on $\mathbb{V}^h$. Let $w_h \in W$ be the shortest element which sends $\mathbb{V}^+$ into $\mathbb{V}_h$.

Lemma 4.12. *Keep the setting of Definition 4.11. The subgroup $P^h \subset G(s)$ is parabolic.*

Proof. By the choice of s_h, we have $\alpha(s_h) \in \mathbb{R}_{>0}$ for every $\alpha \in R$. Let $R_h := \{\alpha \in R; \alpha(s_h) \geq 1\}$. The set R_h contains wR^+ for some $w \in W$. The set $R_h^s := \{\alpha \in R_h; \alpha(s) = 1\}$ defines the set of T-roots of P^h. This contains $wR^+ \cap R_h^s$, which is the set of T-roots of a Borel subgroup of $G(s)$ as desired. $\square$

Corollary 4.13. *Keep the setting of Definition 4.11. A function $h : \mathcal{J}(\nu) \to \mathbb{R}$ is a height function adapted to ν only if we have $h(J_+) + h(J_-) = 0$ for each $J \in |X|$. In addition, we have a reverse implication when $X = X_2$.*

Proof. Since h is a function on $\mathcal{J}(\nu)$, we have

$$s_h(\sum_{I \in \mathcal{J}(\nu)} v_I) = s_h(\sum_{J \in |X|} (\mathbf{v}_{J_-} + \mathbf{v}_{J_+})) = (\sum_{J \in |X|} (\mathbf{v}_{J_-} + \mathbf{v}_{J_+})) = \sum_{I \in \mathcal{J}(\nu)} v_I.$$

In order to have $s_h X_2 = X_2$, it suffices to check

$$s_h(\sum_{J \in |X|} v_{J_+, J_-}) = \sum_{J \in |X|} e^{h(J_+) + h(J_-)} v_{J_+, J_-} = \sum_{J \in |X|} v_{J_+, J_-}$$

by $(\star)_3$. This is equivalent to $h(J_+) + h(J_-) = 0$, which implies the results. $\square$

5 Abstract criteria of regularity

We work under the same settings as in the previous section. In the below, we fix a standard parameter $\nu = (\mathbf{a}, X)$ which satisfies $(\star)$ and a height function h adapted to ν.

The goal of this section is to present Proposition 5.2, together with its local counter-parts.

Lemma 5.1. *Assume that there exists $w \in W$ such that*

$$\mathbf{G}(\mathbf{a})X \cap \dot{w}\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}} \subset \dot{w}\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}}$$

is dense. Then, ν is regular.

Proof. The subspace $\widehat{\mathbb{O}} := G(s) \times^{{}^w B(s)} (\dot{w}\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}}) \subset F_+^{\mathbf{a}}$ is a connected component of $F_+^{\mathbf{a}}$. By assumption, we have

$$\overline{\dot{w}\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}}} = \overline{\mathbf{G}(\mathbf{a})X \cap \dot{w}\mathbb{V}^+ \cap \mathbb{V}^{\mathbf{a}}} \subset \overline{\mathbf{G}(\mathbf{a})X}.$$

Hence, we have $\mu_+^{\mathbf{a}}(\widehat{\mathbb{O}}) = \overline{\mathbf{G}(\mathbf{a})X}$. Therefore, the Beilinson-Bernstein-Deligne (-Gabber) decomposition theorem yields the result. $\square$

Proposition 5.2. *Let $\nu = (s, \vec{q}, X)$ be a standard parameter such that $X \in \mathbb{V}^+$. Let h be a height function adapted to ν. If we have $\mathfrak{p}^h X = \mathbb{V}^h$, then we have $\overline{P^h X} = \mathbb{V}^h$. In particular, we have*

$$X \in (\mathbb{V}^{\mathbf{a}} \cap \dot{w}_h \mathbb{V}^+) \subset \overline{P^h X} = \mathbb{V}^h.$$

Proof. By construction, we have $X \in \mathbb{V}^h$. Since $\mathfrak{p}^h X$ is the tangent space of $P^h X$ at X, the equality $\mathfrak{p}^h X = \mathbb{V}^h$ implies $\dim P^h X = \dim \mathbb{V}^h$. Since $\mathbb{V}^h$ is an irreducible variety, we conclude $\overline{P^h X} = \mathbb{V}^h$. This proves the first assertion. We prove the second assertion. The inclusion $(\mathbb{V}^{\mathbf{a}} \cap \dot{w}_h \mathbb{V}^+) \subset \mathbb{V}^h$ is clear from the definition of $\dot{w}_h$. Since $s_h X = X$, the vector X belongs to the zero weight space with respect to the s_h-action. Since $\dot{w}_h$ is the shortest element which sends $\mathbb{V}^+$ to $\mathbb{V}_h$, the $\dot{w}_h$-action gives an automorphism on the s_h-weight zero part of $\mathbb{V}^+$. This means $X \in \dot{w}_h \mathbb{V}^+$, which implies the result. $\square$

Corollary 5.3. *Keep the setting of Proposition 5.2. Then, ν is regular.* $\square$

Proof. Since $\mathbf{G}(\mathbf{a})X \cap (\mathbb{V}^{\mathbf{a}} \cap \dot{w}_h \mathbb{V}^+) \subset (\mathbb{V}^{\mathbf{a}} \cap \dot{w}_h \mathbb{V}^+)$ is clearly open, Lemma 5.1 implies the result. $\square$

It is not easy to check the assumption $\mathfrak{p}^h X = \mathbb{V}^h$ in Proposition 5.2 directly. To remedy this, we introduce some Lie subalgebras of $\mathfrak{p}^h$ and linear subspaces of $\mathbb{V}^h$, which enable us to check the assumption of Proposition 5.2 in more "localized" form.

Let J, J' be subsets of $[1, n]$. We define

$$\mathfrak{p}^h_{J,J'} := \sum_{i,j \in J \cup J'} \mathfrak{p}^h[\pm\epsilon_i \pm \epsilon_j] \subset \mathfrak{p}, \quad \mathfrak{g}(s)_{J,J'} := \sum_{i,j \in J \cup J'} \mathfrak{g}(s)[\pm\epsilon_i \pm \epsilon_j] \subset \mathfrak{g}(s)$$

$$\mathbb{V}^h_{J,J'} := \sum_{i,j \in J \cup J'} (\mathbb{V}^h[\pm\epsilon_i] \oplus \mathbb{V}^h[\pm\epsilon_i \pm \epsilon_j]) \subset \mathbb{V}^h, \text{ and}$$

$$V_{J,J'} := \bigoplus_{m \geq 0} \bigoplus_{i \in J_m, j \in J'_{m-1}} \mathbb{V}[\epsilon_i - \epsilon_j] \subset V_2.$$

It is clear that $\mathfrak{p}^h_{J,J'}$ and $\mathfrak{g}(s)_{J,J'}$ are Lie subalgebras of $\mathfrak{g}(s)$. We denote the corresponding connected algebraic subgroups of G by $P^h_{J,J'}$ and $G(s)_{J,J'}$, respectively. For the sake of simplicity, we may write the subscript as J when $J' = J$.

Lemma 5.4. *Let $\nu = (\mathbf{a}, X)$ be a standard parameter which satisfies $(\star)$. Let h be the height function adapted to ν. Then, we have $\mathfrak{p}^h X = \mathbb{V}^h$ if*

$$\mathbf{1)}\ (\mathbb{V}^h \cap V_1) \subset \mathfrak{p}^h X \text{ and } \mathbf{2)}\ (\mathbb{V}^h_{J,J'} \cap V_2) \subset \mathfrak{p}^h_{J,J'} X_2$$

holds for every $J, J' \in |X|$.

Proof. We have $\mathbb{V}^h = (\mathbb{V}^h \cap V_1) \oplus (\mathbb{V}^h \cap V_2)$. We have $\mathfrak{p}^h X \subset \mathbb{V}^h$ by construction. Hence, the assertion $(\mathbb{V}^h \cap V_1) \subset \mathfrak{p}^h X$ implies that $(\mathbb{V}^h \cap V_2) \subset \mathfrak{p}^h X_2$ is equivalent to $(\mathbb{V}^h \cap V_2) \subset \mathfrak{p}^h X$. Therefore, we obtain

$$\mathbb{V}^h = (\mathbb{V}^h \cap V_1) + \sum_{J,J'} (\mathbb{V}^h_{J,J'} \cap V_2) \subset \mathfrak{p}^h X + \sum_{J,J'} \mathfrak{p}^h_{J,J'} X_2 \subset \mathfrak{p}^h X + \mathfrak{p}^h X \subset \mathfrak{p}^h X,$$

which implies the result. $\square$

Corollary 5.5. *Keep the setting of Lemma 5.4. If $X \in \mathbb{V}^+$, then ν is regular.* $\square$

Proof. See Proposition 5.2 and Corollary 5.3. $\square$

In the below, we present several notation and lemmas which help us to check Lemma 5.4 **1–2)** in the next section.

Let $I = [i_1, i_2]$ and $I' = [i_3, i_4]$ be two segments in $[0, n)$. We denote $I \leq I'$ iff $i_1 \leq i_3 \leq i_2 \leq i_4$ holds. We denote $I \prec I'$ iff $i_1 < i_3 \leq i_2 + 1 \leq i_4$ holds.

Definition and Lemma 5.6. Let $J, J' \in \mathcal{J}(\nu)$. For each $l \in \underline{J} \cap \underline{J'}$, we have $\xi_l^{J,J'} \in \mathfrak{g}$ such that $0 \neq \xi_l^{J,J'} = \xi_l \in \mathfrak{g}[\epsilon_{J_l} - \epsilon_{J'_l}] \subset \mathfrak{g}(s)$. Moreover, we have $\xi_l^{J,J'} \in \mathfrak{p}^h$ if and only if $h(J) \geq h(J')$.

Proof. The first inclusion follows by $(\star)_1$. The second assertion follows by the construction of s_h and $\mathfrak{p}^h$. □

Definition and Lemma 5.7. Let $J, J' \in \mathcal{J}(\nu)$. Let

$$v_k^{J,J'} = v_k = \mathbf{v}[\epsilon_{J_k} - \epsilon_{J'_{k-1}}] \in \mathbb{V}[\epsilon_{J_k} - \epsilon_{J'_{k-1}}] \subset V_{J,J'}$$

be an element which we understand it to be zero when $J_k = \emptyset$ or $J'_{k-1} = \emptyset$ hold. For each $l \in \underline{J} \cap \underline{J'}$, we have

$$\xi_l^{J,J'} \mathbf{v}_J = -v_{l+1}, \text{ and } \xi_l^{J,J'} \mathbf{v}_{J'} = v_l$$

up to normalizations of ξ_l and $\mathbf{v}[\beta]$ for $\beta \in \Psi(\mathbb{V}^{\mathbf{a}})$. Moreover, $\{v_k^{J,J'}; v_k^{J,J'} \neq 0\}_k$ is a basis of $V_{J,J'}$.

Proof. Notice that v_k is uniquely defined since $\# J_m, \# J'_m \leq 1$. We have $\mathbf{v}_J = \sum_{m \geq 0} \mathbf{v}[\epsilon_{J_m} - \epsilon_{J_{m-1}}]$ and $\mathbf{v}_{J'} = \sum_{m \geq 0} \mathbf{v}[\epsilon_{J'_m} - \epsilon_{J'_{m-1}}]$. It follows that

$$\begin{aligned} \xi_l \mathbf{v}_J &= \xi_l \mathbf{v}[\epsilon_{J_{l+1}} - \epsilon_{J_l}] \in \mathbb{C} v_{l+1} = \mathbb{V}[\epsilon_{J_{l+1}} - \epsilon_{J'_l}] \\ \xi_l \mathbf{v}_{J'} &= \xi_l \mathbf{v}[\epsilon_{J'_l} - \epsilon_{J'_{l-1}}] \in \mathbb{C} v_l = \mathbb{V}[\epsilon_{J_l} - \epsilon_{J'_{l-1}}]. \end{aligned}$$

This proves the first assertion by taking account into weights. The second assertion is an immediate consequence of the definition. □

Lemma 5.8. *Let $J, J' \in \mathcal{J}(\nu)$. If $\underline{J} \leq \underline{J}'$ and $J'_m \neq \emptyset$ holds, then we have*

$$V_1[\epsilon_{J_m}] \oplus V_{J,J'} \subset \mathfrak{g}(s)_{J,J'}(\mathbf{v}[\epsilon_{J'_m}] + \mathbf{v}_J + \mathbf{v}_{J'}).$$

In addition, we have

$$V_1[\epsilon_{J_m}] \oplus V_{J,J'} \subset \mathfrak{p}^h_{J,J'}(\mathbf{v}[\epsilon_{J'_m}] + \mathbf{v}_J + \mathbf{v}_{J'}).$$

when $h(J) \geq h(J')$. Moreover, these inclusions still hold if we add $\mathbf{v}[\epsilon_{J_m}]$ to $(\mathbf{v}[\epsilon_{J'_m}] + \mathbf{v}_J + \mathbf{v}_{J'})$.

Proof. By explicit calculation, we have $\xi_l \mathbf{v}[\epsilon_{J_m}] = 0$ and $\xi_l \mathbf{v}[\epsilon_{J'_m}] = \delta_{l,m} \mathbf{v}[\epsilon_{J_m}]$ up to non-zero scalars. This implies the last assertion provided if we prove the other assertions only by using ξ_l's. By the comparison of dimensions, it follows that the multiplication gives an isomorphism

$$\sum_{l \in \underline{J} \cap \underline{J'}} \mathbb{C} \xi_l \ni \xi \mapsto \xi X \in V_1[\epsilon_{J_m}] \oplus V_{J,J'},$$

which implies the first result. By Definition and Lemma 5.6, we have $\sum_{l \in \underline{J} \cap \underline{J'}} \mathbb{C} \xi_l \subset \mathfrak{p}^h$ when $h(J) \geq h(J')$. This proves the second assertion. □

Corollary 5.9. *Keep the setting of Lemma 5.8. We have*

$$(\mathbf{v}[\epsilon_{J'_m}] + \mathbf{v}_J + \mathbf{v}_{J'}) + V_1[\epsilon_{J_m}] \subset G(s)_{J,J'}(\mathbf{v}[\epsilon_{J'_m}] + \mathbf{v}_J + \mathbf{v}_{J'}).$$

Proof. Admitting the proof of Lemma 5.8, the result follows by merely exponentiating $\mathbb{C}\xi \subset \sum_{l \in \underline{J} \cap \underline{J'}} \mathbb{C}\xi_l$ which annihilates $\mathbf{v}_J + \mathbf{v}_{J'}$. $\square$

Lemma 5.10. *Let $J, J' \in \mathcal{J}(\nu)$. Assume $h(J) \geq h(J')$ and $\underline{J} \not\succ \underline{J'}$. Then, we have an inclusion*

$$V_{J,J'} \subset \mathfrak{p}^h_{J,J'}(\mathbf{v}_J + \mathbf{v}_{J'}).$$

Proof. We put $[k_-, k_+] = \underline{J} \cap \underline{J'}$. If $[k_-, k_+] = \emptyset$ and $V_{J,J'} \neq \{0\}$, then we have $\underline{J} \not\succ \underline{J'}$, which is excluded by the assumption. If $[k_-, k_+] \neq \emptyset$, then the assumption $\underline{J} \not\succ \underline{J'}$ is equivalent to $J'_{k_- - 1} = \emptyset$ or $J_{k_+ + 1} = \emptyset$. It follows that $v_k = 0$ $(k > k_+)$ or $v_k = 0$ $(k \leq k_-)$ holds. Therefore, we deduce

$$\dim V_{J,J'} = \#\{v_k; v_k \neq 0\}_k \leq \#(\underline{J} \cap \underline{J'}).$$

By Definition and Lemma 5.7, the multiplication

$$\sum_{l \in \underline{J} \cap \underline{J'}} \mathbb{C}\xi_l \ni \xi \mapsto \xi X \in V_{J,J'}$$

is surjective as desired. $\square$

Lemma 5.11. *Let $J, J' \in |X|$. Assume that $h(J_+) \geq h(J'_+)$, $J'_+ \neq J'$, and $\underline{J} \leq \underline{J'}$. Then, we have $\mathbf{v}[\alpha_{J_+, J'_-}] \in \mathfrak{p}^h_{J,J'} X_2$ if and only if $J_0 \neq \emptyset$.*

Proof. Let $y := \mathbf{v}_J + \mathbf{v}_{J'} + \mathbf{v}[\alpha_{J_+,J_-}] + \mathbf{v}[\alpha_{J'_+,J'_-}]$. By $J'_+ \neq J'$, we deduce $J'_0 \neq \emptyset$. By the comparison of weights, we have $\mathfrak{p}^h_{J,J'} X_2 = \mathfrak{p}^h_{J,J'} y$. By $(\star)_1$, we deduce $0 \neq \mathbf{v}[\alpha_{J_+,J'_-}] \in \mathbb{V}^{\mathbf{a}}$ only if $J_0 \neq \emptyset$. By explicit computation, we have

$$\xi_l^{J_+,J'_+} \mathbf{v}[\alpha_{J_+,J_-}] = 0, \text{ and } \xi_l^{J_+,J'_+} \mathbf{v}[\alpha_{J'_+,J'_-}] = \begin{cases} \delta_{l,0} \mathbf{v}[\alpha_{J_+,J'_-}] & (J_0 \neq \emptyset) \\ 0 & (otherwise) \end{cases}.$$

Therefore, the assertion falls into Lemma 5.8 applied to the pair (J_+, J'_+). $\square$

Lemma 5.12. *Let $J \in \mathcal{J}(\nu)$. we have an equality*

$$\mathfrak{p}^h_J \mathbf{v}_J = (\mathbb{V}^h_J \cap V_2).$$

Proof. A weight consideration implies the inclusion $\subset$. Hence, we prove $\supset$. By the definition of $\mathbf{v}_J$, $\mathfrak{t}\mathbf{v}_J$ contains T-eigenspaces of weight $\epsilon_{J_m} - \epsilon_{J_{m-1}}$. By Lemma 4.8, a T-weight of $\mathbb{V}^h_J \cap V_2$ is given as $\epsilon_{J_l} - \epsilon_{J_{l-1}}$ for some l, or $\epsilon_{J_1} + \epsilon_{J_2}$. The latter occurs only if ν is type III. Since we have $\mathfrak{t}\mathbf{v}_J = (\mathbb{V}^h_J \cap V_2)$ if the latter does not occur, we assume $\mathbb{V}^h_J[\epsilon_{J_0} + \epsilon_{J_1}] \neq 0$. In this case, we have $\mathfrak{g}[2\epsilon_{J_0}] \subset \mathfrak{p}^h$. This gives $\mathbb{V}^h_J[\epsilon_{J_0} + \epsilon_{J_1}] \subset \mathfrak{p}^h_J \mathbf{v}_J$, which implies

$$(\mathbb{V}^h_J \cap V_2) = \mathbb{V}^h_J[\epsilon_{J_0} + \epsilon_{J_1}] + \mathfrak{t}\mathbf{v}_J \subset \mathfrak{p}^h_J \mathbf{v}_J$$

as required. $\square$

6 Proof of regularity of parameters

The goal of this section is to prove Corollary 6.3, which guarantees the regularity of parameters for general parameters. The core of the argument lies on Propositions 6.1 and 6.2, which is proved by the case-by-case argument.

We retain the setting of §3.

Proposition 6.1. *Let $\nu = (s, \vec{q}, X)$ be a standard parameter which satisfies $(\star)$. Assume that either $V_1^{(s,q_0)} = \{0\}$ or $V_1^{(s,q_1)} = \{0\}$. If $\mathbf{c}$ is type I, then there exists a height function h adapted to ν such that $\overline{P^h X} = \mathbb{V}^h$. In particular, ν is regular.*

Proposition 6.2. *Let $\nu = (s, \vec{q}, X)$ be a standard parameter which satisfies $(\star)$. Assume that $V_1^{(s,q_0)} = \{0\} = V_1^{(s,q_1)}$. Then, there exists a height function h adapted to ν such that $\overline{P^h X} = \mathbb{V}^h$. In particular, ν is regular.*

Before giving the proofs of Propositions 6.1 and 6.2, we prove the following consequence by admitting them.

Corollary 6.3. *An admissible parameter $\nu = (s, \vec{q}, X)$ is regular if $\vec{q}$ is general.*

Proof of Corollary 6.3. The assertion is unchanged by the conjugation by G. Hence, we rearrange ν by the G-action to assume that ν is a standard parameter by Theorem 1.10. By Corollary 3.10, we have only to show the case that $(s, \vec{q})$ has a unique clan. By taking $N_G(T)$-conjugation if necessary, we can assume that ν satisfies $(\star)$ without the loss of generality. The set of s-eigenvalues of V_1 is contained in $\{(e^{\lambda_0} q_2^k)^{\pm 1}; 0 \le k < n\}$. By the relations $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ $(0 \le m < n)$, it follows that either $V_1^{(s,q_0)} = \{0\}$ or $V_1^{(s,q_1)} = \{0\}$ holds. Therefore, we deduce the result by Proposition 6.1 if $\mathbf{c}$ is type I. Otherwise, we have $\mathbf{c}_0 = \widetilde{\mathbf{c}}(0)$ or $\widetilde{\mathbf{c}}(r_2/2)$. Then, the relation $q_0^2 \neq q_2^{\pm l} \neq q_1^2$ $(0 \le l < 2n)$ claims $\mathbb{V}^{\mathbf{a}} \cap V_1 = \{0\}$. Thus, Proposition 6.2 yields the result in this case. □

Proof of Proposition 6.1. By Lemma 4.8, we have $\epsilon_i + \epsilon_j \notin \Psi(\mathbb{V}^{\mathbf{a}})$ for each $i, j \in \mathbf{c}$. In particular, we have $J_- = \emptyset$ for every $J \in |X|$. (Which in turn implies $|X| = \mathcal{J}(\nu)$.)

By $(\star)_1$, the set of T-weights of $\mathbb{V}^{\mathbf{a}} \cap V_1$ are given as the form $\{\epsilon_i\}_{i \in \mathbf{c}_m}$ or $\{-\epsilon_i\}_{i \in \mathbf{c}_m}$ for some m. By $V_1^{(s,q_0)} = \{0\}$ or $V_1^{(s,q_1)} = \{0\}$, the choice of m is at most one. By $\mathbf{c}_0 \neq \widetilde{\mathbf{c}}(0)$, the sign σ of weight for which $V_1^{(s,q_*)}[\sigma \epsilon_i] \neq 0$ $(i \in \mathbf{c}_m)$ is at most one. Hence, we apply $N_G(T)$-action (as $\lambda_i \mapsto -\lambda_{n-i}$ for all i) if necessary to assume $(\mathbb{V}^{\mathbf{a}} \cap V_1) \subset V_1^+$ holds. Then, we have $\mathbb{V}^{\mathbf{a}} \subset \mathbb{V}^+$ by Lemma 4.8. We put

$$m_0 := \max\{m; \forall i \in \mathbf{c}_m, V_1^{(s,q_0)}[\epsilon_i] \neq 0 \text{ or } V_1^{(s,q_1)}[\epsilon_i] \neq 0\} \cup \{-1\}.$$

We have $X(\epsilon_i) = 0$ unless $i \in \mathbf{c}_{m_0}$. We define

$$\begin{aligned} \mathcal{D}^0 :=& \{J \in \mathcal{J}(\nu); X(\epsilon_{J_{m_0}}) \neq 0, \text{ and } \nexists J' \in \mathcal{J}(\nu) \text{ s.t. } \underline{J} \le \underline{J'}, X(\epsilon_{J'_{m_0}}) \neq 0\}, \\ \mathcal{D}^- :=& \{J \in \mathcal{J}(\nu); J_{m_0} \neq \emptyset, X(\epsilon_{J_{m_0}}) = 0, \text{ and } \nexists J' \in \mathcal{D}^0 \text{ s.t. } \underline{J} \le \underline{J'}\}, \end{aligned}$$

and $\mathcal{D}^+ := \mathcal{J}(\nu) \backslash (\mathcal{D}^- \cup \mathcal{D}^0)$. For each $J \in \mathcal{D}^+$ such that $J_{m_0} \neq \emptyset$, there exists $J' \in \mathcal{D}^0$ such that $\underline{J} \le \underline{J'}$ (otherwise we have $J \in \mathcal{D}^-$ by the definition of $\mathcal{D}^-$).

Applying Corollary 5.9 to $J \in \mathcal{D}^+$ such that $X(\epsilon_{J_{m_0}}) \neq 0$ (with some $J' \in \mathcal{D}^0$ such that $\underline{J} \leq \underline{J'}$), we rearrange X by the $\mathrm{Stab}_{G(s)} X_2$-action if necessary to assume

$$X_1 = \sum_{J \in \mathcal{D}^0} \mathbf{v}[\epsilon_{J_{m_0}}] \tag{6.1}$$

without the loss of generality.

We choose a sequence $\gamma_0, \gamma_1, \ldots \in \mathbb{R}$ such that: 1) $\gamma_i > \gamma_{i+1}$ for every i; and 2) $\gamma_{m_0+1} < \gamma_{m_0} < 0 < \gamma_{m_0-1}$. Fix $\frac{1}{n}\gamma_{m_0} < \delta < 0$. We define a height function h as

$$h(J) := \begin{cases} (m(J)+1)\delta & (\text{if } J \in \mathcal{D}^-) \\ 0 & (\text{if } J \in \mathcal{D}^0) \\ \gamma_{m(J)} & (\text{if } J \in \mathcal{D}^+). \end{cases}$$

By (6.1), we conclude that h is adapted to ν. By Corollary 5.5, it suffices to check Lemma 5.4 **1–2)** for proofs of the assertions. We have

$$\mathbb{V}^h \cap V_1 = \bigoplus_{J \in \mathcal{D}^+; h(J) > 0} V_1[\epsilon_{J_{m_0}}] \oplus \bigoplus_{J \in \mathcal{D}^0} V_1[\epsilon_{J_{m_0}}]. \tag{6.2}$$

The second term of the RHS of (6.2) is contained in $\mathfrak{t}X$. For each $J \in \mathcal{D}^+$, we apply Lemma 5.8 to the pair (J, J') such that $J' \in \mathcal{D}^0$ and $\underline{J} \leq \underline{J'}$. It follows that

$$\mathbb{C}\mathbf{v}[\epsilon_{J_{m_0}}] \subset \mathfrak{p}^h_{J,J'}(X_1 \oplus (\mathbf{v}_J + \mathbf{v}_{J'})) \subset \mathfrak{p}^h X.$$

This implies that Lemma 5.4 **1)** holds. Therefore, we have only to show Lemma 5.4 **2)** for each pair (J, J') in $\mathcal{J}(\nu)$. Since we have $\epsilon_i + \epsilon_j \notin \Psi(\mathbb{V}^h)$ for every $i, j \in \mathbf{c}$, we have

$$\mathbb{V}^h_{J,J'} \cap V_2 = \sum_{K,K' \in \{J,J'\}; h(K) \geq h(K')} V^h_{K,K'}.$$

We have $V_J \subset \mathfrak{t}X_2$ by the definition of standard parameters. In particular, Lemma 5.4 **2)** holds if we have

$$V_{J,J'} \subset \mathfrak{p}^h_{J,J'}(\mathbf{v}_J + \mathbf{v}_{J'})$$

for every distinct $J, J' \in \mathcal{J}(\nu)$ such that $h(J) \geq h(J')$. By Lemma 5.10, it suffices to check that no pair (J, J') in $\mathcal{J}(\nu)$ satisfies $h(J) \geq h(J')$ and $\underline{J} \succ \underline{J'}$ simultaneously. The condition $\underline{J} \succ \underline{J'}$ implies $m(J) > m(J')$. We assume the existence of such pair (J, J') to deduce contradiction. There are three cases:

($J \in \mathcal{D}^-$) We have $J' \notin \mathcal{D}^-$ since $h(J) \geq h(J')$ implies $m(J) \leq m(J')$ in this case. The case $J' \in \mathcal{D}^0$ is impossible by $h(J) < 0 = h(J')$ in this case. We have $J' \notin \mathcal{D}^+$ since $m_0 \geq m(J) > m(J')$ implies $h(J) < 0 < h(J')$ in this case.

($J \in \mathcal{D}^0$) We have $J' \notin \mathcal{D}^- \cup \mathcal{D}^0$ since $\underline{J} \succ \underline{J'}$ is prohibited by the definitions of the sets $\mathcal{D}^-$ and $\mathcal{D}^0$. We have $J' \notin \mathcal{D}^+$ since $m_0 \geq m(J) > m(J')$ implies $h(J) = 0 < h(J')$ in this case.

($J \in \mathcal{D}^+$) Assume $J' \in \mathcal{D}^- \cup \mathcal{D}^0$ to deduce contradiction. We have $h(J) \geq h(J') > \gamma_{m_0}$, which implies $m(J) < m_0$ in this case. By $\underline{J} \succ \underline{J'}$, we have $0 \in \underline{J}$. In particular, there exists $J'' \in \mathcal{D}^0$ such that $\underline{J'} \leq \underline{J} \leq \underline{J''}$. Since $0 \in \underline{J'} \cap \underline{J''}$, we deduce $\underline{J'} \leq \underline{J''}$, which contradicts with the definition of $\mathcal{D}^-$ or $\mathcal{D}^0$. Hence, $J' \notin \mathcal{D}^- \cup \mathcal{D}^0$. We have $J' \notin \mathcal{D}^+$ since $m(J) > m(J')$ implies $h(J) < h(J')$ in this case.

By the above case-by-case analysis, we cannot have a desired pair (J, J'), which completes the proof. $\square$

Proof of Proposition 6.2. We assume that $\mathbf{c}$ is not type I. We choose a sequence $\gamma_0, \gamma_1, \ldots \in \mathbb{R}$ such that:

$$0 > \gamma_0 > \gamma_1 > \cdots > \gamma_i > \gamma_{i+1} > \cdots .$$

Fix a negative real number δ such that $\frac{1}{n}\gamma_0 < \delta$. For each $K \in \mathcal{J}(\nu)$, we define $l(K) := \max \underline{K}$. We define a height function as

$$h(K) := \begin{cases} (l(J_+) + 1)\delta & (\text{if } J_+ = K \neq J \text{ for some } J \in |X|) \\ -(l(J_+) + 1)\delta & (\text{if } K = J_- \text{ for some } J \in |X|) \\ \gamma_{m(K)} & (\text{otherwise}). \end{cases}$$

By Corollary 4.13, we deduce that h is adapted to ν. We apply Lemma 5.4 and Corollary 5.5 to deduce the assertions. Lemma 5.4 **1)** is a void condition in this case. We check Lemma 5.4 **2)** for every pair $(J, J') \in |X|$ by the case-by-case analysis:

(The case $J = J'$) Since $h(J_+) < 0$, we have $\epsilon_{(J_+)_0} + \epsilon_{(J_+)_1} \not\in \Psi(\mathbb{V}^h)$. We have $h(J_+) < h(J_-)$. Compared with Lemma 4.8, it follows that

$$\mathbb{V}^h_J = V_{J_+,J_+} \oplus V_{J_-,J_+} \oplus V_{J_-,J_-} \oplus \mathbb{V}^h[\alpha_{J_+,J_-}].$$

By $(\star)_3$, we have

$$V_{J_+,J_+} \oplus V_{J_-,J_-} \oplus \mathbb{V}^h[\alpha_{J_+,J_-}] \subset \mathfrak{t}X.$$

Hence, it suffices to show $V_{J_-,J_+} \subset \mathfrak{p}^h X$. For each $l \in \underline{J_+} \cap \underline{J_-}$, we have

$$\xi_l \mathbf{v}[\alpha_{J_+,J_-}] = \xi_l \mathbf{v}[\epsilon_{m(J_+)} + \epsilon_{m(J_-)}] = 0$$

since the RHS must have a weight outside of $\Psi(V_2)$. Therefore, we deduce $V_{J_-,J_+} \subset \mathfrak{p}^h_J X$ by Lemma 5.10 applied to the pair (J_-, J_+) in $\mathcal{J}(\nu)$. This implies $\mathfrak{p}^h_J X = \mathbb{V}^h_J$ as desired.

(The case $J = J_+$ or $J' = J_+$) By swapping the roles of J and J' if necessary, we assume $J_+ = J$. We divide this proof into two subcases:

i) If we have $J'_+ = J'$, then we have $h(J) + h(J') < 0$. Hence, we have $\epsilon_i + \epsilon_j \not\in \Psi(\mathbb{V}^h)$ if $i \in J$ and $j \in J'$. Compared with Lemma 4.8, it follows that

$$\mathbb{V}^h_{J,J'} = \sum_{K,K' \in \{J,J'\}; h(K) \geq h(K')} V_{K,K'}.$$

By the definition of h, we have $m(K) \leq m(K')$ if $h(K) \geq h(K')$. In particular, we have $K \not\succ K'$ when $h(K) \geq h(K')$. Therefore, we deduce $\mathfrak{p}^h_{J,J'} X = \mathbb{V}^h_{J,J'}$ by applying Lemma 5.10 to all the pairs (K, K') such that $h(K) \geq h(K')$ and $K \neq K'$.

ii) If we have $J'_+ \neq J'$, then we have $h(J) + h(J'_+) < h(J) + h(J'_-) < 0$. Hence, we have $\epsilon_i + \epsilon_j \not\in \Psi(\mathbb{V}^h)$ if $i \in J$ and $j \in J'$. We have $h(J'_+), h(J'_-) > h(J)$. It follows that

$$\mathbb{V}^h_{J,J'} = \mathbb{V}^h_J \oplus \mathbb{V}^h_{J'} \oplus V_{J'_+,J} \oplus V_{J'_-,J}.$$

Since $m(J) \geq m(J'_+) = 0$, we deduce $J'_+ \not> J$. Applying Lemma 5.10 to (J'_+, J), we conclude $V_{J'_+,J} \subset \mathfrak{p}^h_{J,J'}X$. The same argument applies to the pair (J'_-, J) if $m(J) \geq m(J'_-)$. Otherwise, we have necessarily have $0 = m(J) < m(J'_-) = 1$. This implies that $\mathbf{c}$ is type Ⅲ. We have $0 \neq \xi \in \mathfrak{p}^h[-\epsilon_{(J'_+)_0} - \epsilon_{J_0}]$ by $(-h(J'_+) - h(J)) > 0$. We have $\xi v_{J'_+} = 0$ and

$$\xi \mathbf{v}[\alpha_{J'_+,J'_-}] = \mathbf{v}[\epsilon_{(J'_-)_1} - \epsilon_{J_0}]$$

by explicit calculation. We have $\dim V_{J'_-,J} \leq \dim(\underline{J'_-} \cap \underline{J}) + 1$. Taking account into the proof of Lemma 5.10, we conclude $V_{J'_-,J} \subset \mathfrak{p}^h_{J'_-,J}X$. By Lemma 5.12, we conclude

$$\mathbb{V}^h_{J,J'} = \mathbb{V}^h_J \oplus \mathbb{V}^h_{J'} \oplus V_{J'_+,J} \oplus V_{J'_-,J} \subset \mathfrak{p}^h_{J,J'}X$$

in this case.

(The case $J \neq J_+$ and $J' \neq J_+$) By swapping the roles of J and J' if necessary, we assume $h(J_+) \leq h(J'_+)$. By the choice of h, we have

$$h(J_+) \leq h(J'_+) < 0 < h(J'_-) \leq h(J_-), \text{ and } \underline{J'_-} \subset \underline{J'_+} = \underline{J'} \subset \underline{J} = \underline{J_+}. \tag{6.3}$$

We divide the proof into three subcases **i)** $\mathbf{c}$ is type Ⅱ, **ii)** $\mathbf{c}$ is type Ⅲ and $h(J) \neq h(J')$, and **iii)** $\mathbf{c}$ is type Ⅲ and $h(J) = h(J')$:

i) Assume that $\mathbf{c}$ is type Ⅱ. Compared with Lemma 4.8, we have

$$\mathbb{V}^h_{J,J'} = \mathbb{V}^h_J \oplus \mathbb{V}^h_{J'} \oplus \left(\sum_{\substack{K \in \{J_\pm\}, K' \in \{J'_\pm\} \\ h(K) \geq h(K')}} V_{K,K'} \right) \oplus \left(\sum_{\substack{K \in \{J_\pm\}, K' \in \{J'_\pm\} \\ h(K) + h(K') \geq 0}} \mathbb{V}^h[\alpha_{K,K'}] \right).$$

We have $K_0 \neq \emptyset$ for every $K \in \{J_\pm, J'_\pm\}$. It follows that we have $K \not> K'$ for all the pairs (K, K') in $\{J_\pm, J'_\pm\}$. Therefore, we deduce $\mathbb{V}^h_{J,J'} \subset \mathfrak{p}^h_{J,J'}X$ by a successive application of Lemma 5.10 and $\mathbb{V}^h_J \subset \mathfrak{p}^h_J X$ (the first case treated in the above) provided if we have

$$\sum_{\substack{K \in \{J_\pm\}, K' \in \{J'_\pm\} \\ h(K) + h(K') \geq 0}} \mathbb{V}^h[\alpha_{K,K'}] \subset \mathfrak{p}^h X. \tag{6.4}$$

Claim A. *The inclusion* (6.4) *holds when the pair* $(J, J') \in |X|$ *satisfies* $J \neq J_+$, $J' \neq J'_+$, *and* $\mathbf{c}$ *is type Ⅱ.*

Proof of Claim A. For $(K, K') = (J_+, J'_+)$, we have $\mathbb{V}^h[\alpha_{K,K'}] = 0$ by $h(J_+) + h(J'_+) < 0$. For $(K, K') = (J_+, J'_-)$, we have $\mathbb{V}^h[\alpha_{K,K'}] \neq 0$ only if $h(J_+) \geq h(J'_+)$. We have $\underline{J'_-} \leq \underline{J_+}$ and $J_0 \neq \emptyset$. Hence, we have $\mathbb{V}^h[\alpha_{K,K'}] \subset \mathfrak{p}^h_{J,J'}X$ by Lemma 5.11. The case $(K, K') = (J_-, J'_+)$ is the same as the previous case by swapping the roles of K and K'. For the case $(K, K') = (J_-, J'_-)$, we always have $\mathbb{V}^h[\alpha_{J_-,J'_-}] \neq 0$ by $h(J_-) + h(J'_-) > 0$. We have $\underline{J'_-} \subset \underline{J_+}$. The vectors $\mathbf{v}[\alpha_{J_+,J_-}] = \mathbf{v}[\epsilon_{(J_+)_0} + \epsilon_{(J_-)_0}]$ and $\mathbf{v}[\alpha_{J'_+,J'_-}] = \mathbf{v}[\epsilon_{(J'_+)_0} + \epsilon_{(J'_-)_0}]$ behave as elements of vector representation of $G(s)_{J_+,J'_-}$ with its weight $\epsilon_{(J_+)_0}$ and $\epsilon_{(J'_-)_0}$, respectively. Hence, we apply Lemma 5.8 to the pair (J'_-, J_+) to deduce

$$\mathbb{V}^h[\alpha_{J_-,J'_-}] \oplus V_{J'_-,J_+} \subset \mathfrak{p}^h_{J'_-,J_+}X.$$

In particular, we obtain $\mathbb{V}^h[\alpha_{K,K'}] \subset \mathfrak{p}^h_{K',J_+}X$ when $(K, K') = (J_-, J'_-)$. These case-by-case checking yields Claim A as desired. $\square$

We return to the proof of Proposition 6.2. Now (6.4) implies $\mathbb{V}^h_{J,J'} \subset \mathfrak{p}^h_{J,J'}X$ when $\mathbf{c}$ is type Ⅱ as desired.

ii) Assume that $\mathbf{c}$ is type Ⅲ and $h(J_+) < h(J'_+)$. We define

$$V'_{J,J'} := \bigoplus_{i \in \mathbf{c}, j \in \{(J_+)_0, (J'_+)_0\}} \mathbb{V}^h_{J,J'}[\epsilon_i \pm \epsilon_j].$$

We put $\dot{J} := J_+ - \{(J_+)_0\}$ and $\dot{J}' := J'_+ - \{(J'_+)_0\}$. It is clear that

$$m(\dot{J}) = m(J_-) = m(\dot{J}') = m(J'_-) = 1. \tag{6.5}$$

It follows that $K \not\succ K'$ for every $K, K' \in \{\dot{J}, J_-, \dot{J}', J'_-\}$. Compared with Lemma 4.8, we have

$$\mathbb{V}^h_{J,J'} = V'_{J,J'} \oplus \sum_{K,K' \in \{\dot{J}, J_-, \dot{J}', J'_-\}; h(K) \geq h(K')} V_{K,K'},$$

where we extend the domain of h so that $h(\dot{J}) = h(J_+)$ and $h(\dot{J}') = h(J'_+)$ hold. By forgetting J_0 and J'_0, we can apply Lemma 5.10 for all pairs (K, K') in $\{\dot{J}, J_-, \dot{J}', J'_-\}$ such that $h(K) \geq h(K')$. Then, we conclude $\mathbb{V}^h_{J,J'} \subset \mathfrak{p}^h_{J,J'}X$ provided if

$$V'_{J,J'} \subset \mathfrak{p}^h_{J,J'}X \tag{6.6}$$

holds.

Claim B. *The inclusion* (6.6) *holds when the pair* $(J, J') \in |X|$ *satisfies* $J \neq J_+$, $J' \neq J'_+$, $\mathbf{c}$ *is type Ⅲ, and* $h(J_+) < h(J'_+)$.

Proof of Claim B. By (6.3) and (6.5), we deduce $\underline{J'_-} \leq \underline{\dot{J}}$ and $\underline{\dot{J}'} \leq \underline{\dot{J}}$. Here the vectors

$$\mathbf{v}[\epsilon_{(J_+)_1} - \epsilon_{(J_+)_0}], \mathbf{v}[\alpha_{J_+.J_-}], \mathbf{v}[\epsilon_{(J'_+)_1} - \epsilon_{(J'_+)_0}], \mathbf{v}[\alpha_{J'_+,J'_-}],$$

behaves as vectors in the vector representation of $G(s)_{\dot{J},\dot{J}'}$ or $G(s)_{\dot{J},J'_-}$ of weights $\epsilon_{(J_+)_1}$, 0, $\epsilon_{(J'_+)_1}$, or $\epsilon_{(J'_-)_1}$, respectively. Applying Lemma 5.8 to the pairs $(\dot{J}', \dot{J})$ and $(J'_-, \dot{J})$ by forgetting J_0 and J'_0, we deduce that

$$\mathbf{v}[\epsilon_{(J'_+)_1} - \epsilon_{(J_+)_0}], \mathbf{v}[\epsilon_{(J'_-)_1} - \epsilon_{(J_+)_0}] \in \mathfrak{p}^h_{J,J'}X. \tag{6.7}$$

We have

$$\Psi(\mathfrak{g}(0) \cap \mathfrak{p}^h_{J,J'}) = \{0, -2\epsilon_{(J_+)_0}, -2\epsilon_{(J'_+)_0}, -\epsilon_{(J_+)_0} \pm \epsilon_{(J'_+)_0}\}.$$

Moreover, we have

$$X = \mathbf{v}[\epsilon_{(J_+)_1} - \epsilon_{(J_+)_0}] + \mathbf{v}[\alpha_{J_+,J_-}] + \mathbf{v}[\epsilon_{(J'_+)_1} - \epsilon_{(J'_+)_0}] + \mathbf{v}[\alpha_{J'_+,J'_-}] + X_{ot},$$

where X_{ot} is a sum of T-eigenvectors of weight $\epsilon_k \pm \epsilon_l$ ($k \not\in \{(J_+)_0, (J'_+)_0\} \not\ni l$). It is clear that $(\mathfrak{g}(0) \cap \mathfrak{p}^h_{J,J'}) X_{ot} = 0$. Now an explicit computation gives

$$\mathbf{v}[\epsilon_{(J_-)_1} - \epsilon_{(J_+)_0}], \mathbf{v}[\epsilon_{(J'_-)_1} - \epsilon_{(J'_+)_0}], \mathbf{v}[\epsilon_{(J_-)_1} - \epsilon_{(J'_+)_0}] + \mathbf{v}[\epsilon_{(J'_-)_1} - \epsilon_{(J_+)_0}],$$
$$\mathbf{v}[\epsilon_{(J_-)_1} + \epsilon_{(J'_+)_0}] - \mathbf{v}[\epsilon_{(J'_+)_1} - \epsilon_{(J_+)_0}] \in (\mathfrak{g}(0) \cap \mathfrak{p}^h_{J,J'}) X$$

Taking account into (6.7) and $\dim(\mathfrak{t}X \cap V'_{J,J'}) = 4$, we deduce

$$\dim(V'_{J,J'} \cap \mathfrak{p}^h X) \geq 10.$$

Hence, it suffices to prove $\dim V'_{J,J'} = 10$. Consider the following numbers:

$$h(J_\pm) \pm h(J_+), h(J_\pm) \pm h(J'_+), h(J'_\pm) \pm h(J'_+), h(J'_\pm) \pm h(J_+). \tag{6.8}$$

Here the above numbers are responsible for the non-negativity of $\log(s_h)$-eigenvalues of the weights $\epsilon_i \pm \epsilon_j$, where $(i,j) \in \mathbf{c}_1 \times \mathbf{c}_0$ is taken from (J,J), (J,J'), (J',J'), and (J',J), respectively. Hence, it suffices to show that exactly 10 out of the 16 numbers in (6.8) are non-negative when $-h(J_-) = h(J_+) < h(J'_+) = -h(J'_-) < 0$. We have $3,2,3,2$ non-negative numbers out of each blocks consisting of four. Hence, Claim B follows. □

We return to the proof of Proposition 6.2. Now (6.6) implies $\mathbb{V}^h_{J,J'} \subset \mathfrak{p}^h_{J,J'} X$ when $\mathbf{c}$ is type Ⅲ and $h(J_+) > h(J'_+)$ as desired.

ⅲ) Assume that $\mathbf{c}$ is type Ⅲ and $h(J_+) = h(J'_+)$. Consider small perturbations h^+ and h^- of h such that $h^+(J_+) > h^+(J'_+)$ and $h^-(J_+) < h^-(J'_+)$, respectively. Then, we have

$$\mathbb{V}^{h^?}_{J,J'} \subset \mathbb{V}^h_{J,J'} \text{ and } \mathfrak{p}^{h^?}_{J,J'} \subset \mathfrak{p}^h_{J,J'}$$

for $? = \pm$. Moreover, we have

$$\mathbb{V}^{h^+}_{J,J'} + \mathbb{V}^{h^-}_{J,J'} = \mathbb{V}^h_{J,J'}.$$

Since the arguments of the case $\mathbf{c}$ is type Ⅲ and $h(J_+) > h(J'_+)$ carries over to $h^?$ ($? = \pm$), we conclude

$$\mathbb{V}^h_{J,J'} = \mathbb{V}^{h^+}_{J,J'} + \mathbb{V}^{h^-}_{J,J'} \subset \mathfrak{p}^{h^+}_{J,J'} X + \mathfrak{p}^{h^-}_{J,J'} X \subset \mathfrak{p}^h_{J,J'} X.$$

These case-by-case analysis complete the verification of the condition of Lemma 5.4 2) as desired. □

7 Stabilizers of exotic nilpotent orbits

We work under the setting of §4. In particular, we fix a standard parameter $\nu = (\mathbf{a}, X) = (s, \vec{q}, X_1 \oplus X_2)$ which satisfies ($\star$). Let $\mathbf{c} = [1,n]$ be the unique clan associated to $\mathbf{a}$. We assign the following quiver $Q^{\mathbf{a}}$ to $\mathbf{a}$:

- The vertexes of $Q^{\mathbf{a}}$ is $\{m \in \mathbb{Z}_{\geq 0}; \mathbf{c}_m \neq 0\}$;
- The edges of $Q^{\mathbf{a}}$ connect from $(m-1)$ (source or start) to m (target or terminal) for each m.

We put $V(m) := \mathbb{C}^{d^{\mathbf{a}}_m}$ for each $m \geq 0$. We fix a basis of $V(m)$ as $\{v_i, \overline{v}_i\}_{i \in \mathbf{c}_m}$ (if $\mathbf{c}_m = \widetilde{\mathbf{c}}(0)$), or $\{v_i\}_{i \in \mathbf{c}_m}$ (otherwise). We form a dimension vector $\mathbf{d}^{\mathbf{a}}$ of $Q^{\mathbf{a}}$ as $\{d^{\mathbf{a}}_m\}_m$. Let $\mathrm{Rep}(Q^{\mathbf{a}}, \mathbf{d}^{\mathbf{a}})$ be the representation space of $Q^{\mathbf{a}}$ with dimension vector $\mathbf{d}^{\mathbf{a}}$.

Construction 7.1 (Quiver presentation of $\mathbb{V}^{\mathbf{a}}$)**.** We assign $\mathbf{v}[\epsilon_i - \epsilon_j]$ ($i \in \mathbf{c}_m$, $j \in \mathbf{c}_{m-1}$) a linear map

$$\upsilon[\epsilon_i - \epsilon_j] : V(m-1) \longrightarrow V(m)$$

such that

$$\upsilon[\epsilon_i - \epsilon_j]\overline{v}_k = 0, \quad \upsilon[\epsilon_i - \epsilon_j]v_k = \begin{cases} v_i & (j = k) \\ 0 & (\text{otherwise}) \end{cases}.$$

We assign $\mathbf{v}[\epsilon_i + \epsilon_j]$ ($i \in \mathbf{c}_1$, $j \in \mathbf{c}_0$; type III) a linear map

$$\upsilon[\epsilon_i + \epsilon_j] : V(0) \longrightarrow V(1)$$

defined as

$$\upsilon[\epsilon_i + \epsilon_j]v_k = 0, \quad \upsilon[\epsilon_i + \epsilon_j]\overline{v}_k = \begin{cases} v_i & (j = k) \\ 0 & (\text{otherwise}) \end{cases}.$$

These assignments give rise to a map

$$\begin{aligned} \Xi : V_2^{(s,q_2)} \ni X_2 = & \sum_{i,j \in [1,n], \sigma \in \{\pm\}} c^{\sigma}_{i,j} \mathbf{v}[\epsilon_i + \sigma\epsilon_j] \\ & \mapsto \Xi(X_2) := \sum_{i,j \in [1,n], \sigma \in \{\pm\}} c^{\sigma}_{i,j} \upsilon[\epsilon_i + \sigma\epsilon_j] \in \mathrm{Rep}(Q^{\mathbf{a}}, \mathbf{d}^{\mathbf{a}}). \end{aligned}$$

Here we understand that $c^{+}_{i,j} = 0$ unless $i < j$. We assign $\mathbf{v}[\epsilon_i + \epsilon_j]$ ($i, j \in \mathbf{c}_0, i < j$; type II) a two-form $\theta[\epsilon_i + \epsilon_j] \in \wedge^2 V(0)$ defined as

$$\theta[\epsilon_i + \epsilon_j] = v_i \wedge v_j.$$

We define a two-form $\theta = \theta(X_2) \in \wedge^2 V(0)$ associated to X_2 as

$$\theta := \sum_{i,j \in \mathbf{c}_0} c^{+}_{i,j} \theta[\epsilon_i + \epsilon_j].$$

Remark 7.2. Since ν is a standard parameter, we have

$$\mathrm{dim}\theta^{\perp} = \dim V(0)^* - 2\#\{(i,j) \in \mathbf{c}_0 \times \mathbf{c}_0; c^{+}_{i,j} \neq 0\}$$

if θ is defined (i.e. $\mathbf{c}$ is type II).

By Construction 7.1, we have a $Q^{\mathbf{a}}$-representation structure on the space $\mathsf{M} := \oplus_{k \geq 0} V(k)$ by letting $\Xi(X_2)$ act. For each $J \in |X|$ and $\sigma \in \{\pm\}$, we define

$$\mathsf{M}^{\sigma}_J := \begin{cases} (\sum_{i \in J_\sigma} \mathbb{C}v_i) \oplus \mathbb{C}\overline{v}_{J_0} & (\sigma = -, (J_-) \neq \emptyset, \text{ and } (J_-)_0 = \emptyset) \\ \sum_{i \in J_\sigma} \mathbb{C}v_i & (\textit{otherwise}) \end{cases}.$$

Lemma 7.3. *For each $J \in |X|$, the subspaces $\mathsf{M}_J^{\pm}$ of M are $\Xi(X_2)$-stable. In particular, $\mathsf{M}_J^{\pm}$ are submodules of M as $Q^{\mathbf{a}}$-representations.*

Proof. Straight-forward. □

Lemma 7.4. *For each $J \in |X|$, the $Q^{\mathbf{a}}$-representations $\mathsf{M}_J^{\pm}$ are indecomposable. In particular, the decomposition*

$$\mathsf{M} = \bigoplus_{J \in |X|} (\mathsf{M}_J^{+} \oplus \mathsf{M}_J^{-})$$

is an indecomposable direct sum decomposition of $Q^{\mathbf{a}}$-representations.

Proof. The first assertion is clear since the both $J_{\pm}$ and $J_{-} \cup \{0\}$ are segments. The second assertion follows by $[1,n] = \#\|X\|$ and $\dim \mathsf{M} = n + \#\widetilde{\mathbf{c}}(0)$. □

We put $G_{\mathbf{a}} := \prod_{i \geq 0} GL(d_i^{\mathbf{a}})$. The space $\mathrm{Rep}(Q^{\mathbf{a}}, \mathbf{d}^{\mathbf{a}})$ admits a natural $G_{\mathbf{a}}$-action.

Lemma 7.5. *We have a natural embedding $G(s) \hookrightarrow G_{\mathbf{a}}$, which makes Ξ a $G(s)$-equivariant map. Moreover, it acts on θ as the base change of two forms.*

Proof. Consider the action of $G(m)$ on $V_2^{(s,q_2)}$. By a weight consideration, we have

$$V_2^{(s,q_2)} \cong \begin{cases} \mathrm{Mat}(d_{m+1}^{\mathbf{a}}, d_m^{\mathbf{a}}) \oplus \wedge^2 V(0) \oplus triv & (\text{if } \mathbf{c} \text{ is type II and } m = 0) \\ \mathrm{Mat}(d_{m+1}^{\mathbf{a}}, d_m^{\mathbf{a}}) \oplus \mathrm{Mat}(d_m^{\mathbf{a}}, d_{m-1}^{\mathbf{a}}) \oplus triv & (\text{otherwise}) \end{cases}$$

as $GL(d_m^{\mathbf{a}})$-modules, where we understand $d_m^{\mathbf{a}} = 0$ if $m \notin Q^{\mathbf{a}}$ and $triv$ is the trivial representation of some dimension. The natural action of $G(m) = GL(d_m^{\mathbf{a}})$ or $Sp(d_0^{\mathbf{a}})$ is

$$\begin{aligned} G(m) \times \mathrm{Mat}(d_{m+1}^{\mathbf{a}}, d_m^{\mathbf{a}}) \ni &(g, A) \mapsto Ag^{-1} \in \mathrm{Mat}(d_{m+1}^{\mathbf{a}}, d_m^{\mathbf{a}}), \\ G(m) \times \mathrm{Mat}(d_m^{\mathbf{a}}, d_{m-1}^{\mathbf{a}}) \ni &(g, A) \mapsto gA \in \mathrm{Mat}(d_m^{\mathbf{a}}, d_{m-1}^{\mathbf{a}}), \text{ and} \\ G(0) \times \mathrm{Alt}(d_0^{\mathbf{a}}) \ni &(g, A) \mapsto gA^t g \in \mathrm{Alt}(d_0^{\mathbf{a}}) \cong \wedge^2 V(0). \end{aligned}$$

The first two actions commute with the composition of matrices. Hence, we have an injective map $G(m) \to GL(d_m^{\mathbf{a}})$ which commutes with the embedding Ξ. The last action claims that $G(0)$ acts on the $\wedge^2 V(0)$ by the coordinate transformation, which forms a subgroup of $GL(d_0^{\mathbf{a}})$. □

Corollary 7.6 (of proof of Lemma 7.5)**.** *Keep the setting of Lemma 7.5. Then, we have a natural embedding $\mathsf{M} \subset V_1$ which is compatible with the $G(s)$-actions on $\Xi(X_2)$ and V_1.*

Proof. The assignments $v_i \mapsto \mathbf{v}[\epsilon_i]$ and $\overline{v}_i \mapsto \mathbf{v}[-\epsilon_i]$ gives an embedding $\mathsf{M} \subset V_1$ as vector spaces. This respects the action of $G(s)$ by the proof of Lemma 7.5. □

Corollary 7.7. *We have a natural $G_{\mathbf{a}}$-action on $V_1^{(s,q_0)} \oplus V_1^{(s,q_1)}$, which factors through $G(m) \times G(m')$ for some pair (m, m').*

Proof. Choose m so that $V_1^{(s,q_0)}[\epsilon_i] \neq 0$ or $V_1^{(s,q_0)}[-\epsilon_i] \neq 0$ for some $i \in \mathbf{c}_m$. Then, we have $V_1^{(s,q_0)} \cong V(m)$ or $V(m)^*$ as $G(s)$-modules by the weight comparison. Hence, the natural action of $G(s)$ on $V_1^{(s,q_0)}$ must factor through $G(m)$. Applying the same argument to the (s,q_1)-fixed points of V_1, we conclude the result. $\square$

Lemma 7.8. *Let $Sp(2l,\mathbb{C})$ be a symplectic group embedded into $GL(2l,\mathbb{C})$. Let $v_1, v_2 \in \mathbb{C}^{2l}$ and let P be the simultaneous stabilizer of (v_1, v_2) in $GL(2l,\mathbb{C})$. Then, the group $Sp(2l,\mathbb{C}) \cap P$ is connected.*

Proof. We put $P_0 := Sp(2l,\mathbb{C}) \cap P$. Let ω be a symplectic form on $\mathbb{C}^{2m}$ associated to $Sp(2l,\mathbb{C})$. Let $v^{\perp}$ be the orthogonal compliment of v with respect to ω. The Levi subgroup of P_0 is the stabilizer of

$$(v_1^{\perp} \cap v_2^{\perp})/((\mathbb{C}v_1 + \mathbb{C}v_2) \cap (v_1^{\perp} \cap v_2^{\perp})).$$

This is always a symplectic vector space. Therefore, the Levi subgroup of P_0 is isomorphic to a symplectic group, which completes the proof. $\square$

For each pair $\gamma = (I, I')$ of segments (maybe emptysets) and a $Q^{\mathbf{a}}$-submodule N of M, we define

$$n_{\gamma} := \#\{J \in |X| \,; I = \underline{J_+}, I' = \underline{J_-}\}, \;\; \mathsf{M}_{\gamma}^{\pm} := \bigoplus_{J \in |X|, I = \underline{J_+}, I' = \underline{J_-}} \mathsf{M}_J^{\pm}$$

$$\mathsf{M}_{\gamma} := \mathsf{M}_{\gamma}^{+} \oplus \mathsf{M}_{\gamma}^{-}, \;\; \mathsf{N}(m) := \mathsf{N} \cap V(m) \quad (m \geq 0),$$

and $\theta_{\gamma} := \theta(X_2)|_{\mathsf{M}_{\gamma}(0)}$ (the last definition applies only if $\mathbf{c}$ is type II).

Lemma 7.9. *We have*

$$\theta(X_2) = \sum_{\gamma} \theta_{\gamma},$$

where $\gamma = (I, I')$ runs over all pairs of segments.

Proof. Immediate from definition. $\square$

Definition 7.10. Assume that $\mathbf{c}$ is type III. For each pair $\gamma = (I, I')$ of segments (maybe emptysets), we define

$$\theta_{\gamma} := \omega|_{\mathsf{M}_{\gamma}(0)},$$

where ω is a symplectic form of V_1 which defines G.

Lemma 7.11. *Keep the setting of Definition 7.10. Then, we have*

$$\omega|_{V(0)} = \sum_{\gamma} \theta_{\gamma},$$

where $\gamma = (I, I')$ runs over all pairs of segments. In particular, θ_{γ} is non-degenerate.

Proof. We put $\mathsf{M}_J := \mathsf{M}_J^+ \oplus \mathsf{M}_J^-$ for each $J \in |X|$. The decomposition $\mathsf{M} = \oplus_{J \in |X|} \mathsf{M}_J$ is T-stable. Since $\mathbf{c}$ is type III, we have $\epsilon_i \in \Psi(\mathsf{M}_J(0))$ if and only if $-\epsilon_i \in \Psi(\mathsf{M}_J(0))$. Hence, we have

$$\omega(\mathsf{M}_J(0), \mathsf{M}_{J'}(0)) \equiv 0$$

unless $J = J'$. This yields the first equality. By construction, we can identify $V(0)$ with a subspace of V_1 with its T-weights $\{\pm\epsilon_i\}_{i \in \widetilde{\mathbf{c}}(0)}$. It follows that $\omega|_{V(0)}$ is non-degenerate as desired. □

Corollary 7.12. *The group $G(s)$ is the subgroup of $G_{\mathbf{a}}$ which respects $\omega|_{V(0)}$.*

Proof. The inclusion $\subset$ is trivial. The inclusion $\supset$ is (4.1). □

Proposition 7.13. *Assume that $X_1 = 0$. The subgroup L of $G_{\mathbf{a}}$ which preserves each M_γ (as $Q^{\mathbf{a}}$-representations) and θ_γ contains a Levi subgroup of the stabilizer of X_2 in $G(s)$.*

Proof. The group $G_{\mathbf{a}}(\Xi(X_2))$ is equal to the group of automorphism of M as $Q^{\mathbf{a}}$-representations. Hence, L fixes $\Xi(X_2)$. By a standard quiver theory (cf. Crawley-Boevey [CB92], p12), it is a semi-direct product of general linear groups and an unipotent subgroup of $G_{\mathbf{a}}$. Moreover, the size of general linear groups are equal to the multiplicity of an indecomposable module appearing in M. If $\mathbf{c}$ is type I, then $\mathsf{M}_\gamma \neq 0$ only if $\gamma = (I, \emptyset)$ for some segment I. Since $G_{\mathbf{a}} \cong G(s)$ if $\mathbf{c}$ is type I, we conclude the result in this case. Assume that $\mathbf{c}$ is type II or II I. By Corollary 7.12, we have $L \subset G(s)$. By Lemma 7.9 and 7.11, the group L fixes X_2. By construction, θ_γ is zero or non-degenerate for each γ.

We put L' be a Levi subgroup of L. Let L'' be a Levi subgroup of the stabilizer of X_2 in $G(s)$ which contains L'. Write $g \in \mathrm{Aut}_{Q^{\mathbf{a}}}(\mathsf{M})$ as

$$g = \{g_{\gamma,\gamma'}\}_{\gamma,\gamma'} = \{g_{\gamma,\gamma'}^{\sigma,\sigma'}\}_{\gamma,\gamma'}^{\sigma,\sigma'} \text{ , where } \sigma, \sigma' \in \{\pm\}, \text{ and}$$
$$g_{\gamma,\gamma'} = \bigoplus_{\sigma,\sigma' \in \{\pm\}} g_{\gamma,\gamma'}^{\sigma,\sigma'} \in \bigoplus_{\sigma,\sigma' \in \{\pm\}} \mathrm{Hom}_{Q^{\mathbf{a}}}(\mathsf{M}_\gamma^\sigma, \mathsf{M}_{\gamma'}^{\sigma'}) = \mathrm{Hom}_{Q^{\mathbf{a}}}(\mathsf{M}_\gamma, \mathsf{M}_{\gamma'})$$

for every pair of segments γ, γ'. It is easy to see that θ_γ gives a non-degenerate pairing between $\mathsf{M}_\gamma^+(0)$ and $\mathsf{M}_\gamma^-(0)$ if $\mathsf{M}_\gamma^- \neq \{0\}$. It follows that respecting two-forms $\theta(X_2)$ ($\mathbf{c}$ is type II) or $\omega|_{V(0)}$ ($\mathbf{c}$ is type III) gives rise to the constraint

$$g_{\gamma,\gamma'}^{\sigma,\sigma'} \neq 0 \Leftrightarrow (g^{-1})_{\gamma',\gamma}^{-\sigma',-\sigma} \neq 0 \quad \text{if} \quad (\diamondsuit) : g \in L'', \, \mathsf{M}_\gamma^- \neq \{0\} \text{ and } \mathsf{M}_{\gamma'}^- \neq \{0\}.$$

Claim C. *Assume $(\diamondsuit)$. We put $\gamma = (I, I')$ and $\gamma' = (K, K')$. Then, we have $g_{\gamma,\gamma'} \neq 0$ only if*

$$I \geq K, I' \leq K', \text{ or } I \leq K, I' \geq K'.$$

Proof of Claim C. Since g respects two-forms, we have $g_{\gamma,\gamma'}^{\sigma,\sigma'} \neq 0$ only if

$$\mathrm{Hom}_{Q^{\mathbf{a}}}(\mathsf{M}_\gamma^\sigma, \mathsf{M}_{\gamma'}^{\sigma'}) \neq \{0\} \text{ and } \mathrm{Hom}_{Q^{\mathbf{a}}}(\mathsf{M}_{\gamma'}^{-\sigma'}, \mathsf{M}_\gamma^{-\sigma}) \neq \{0\}.$$

By a quiver-theoretic consideration, we have

$$\mathrm{Hom}_{Q^{\mathbf{a}}}(\mathsf{M}_J^\sigma, \mathsf{M}_{J'}^{\sigma'}) \neq \{0\} \quad (J, J' \in |X|, \sigma, \sigma' \in \{\pm\})$$

only if $\underline{J_\sigma} \leq \underline{J'_{\sigma'}}$. Rewriting this by M_γ yields the result. □

We return to the proof of Proposition 7.13. Consider the map

$$f_{ss} : \mathrm{Aut}_{Q^{\mathbf{a}}}(\mathsf{M}) \longrightarrow \mathrm{Aut}_{Q^{\mathbf{a}}}(\mathsf{M})/U(\mathsf{M}),$$

where $U(\mathsf{M})$ be the unipotent radical of $\mathrm{Aut}_{Q^{\mathbf{a}}}(\mathsf{M})$. Since L'' injects to the RHS, we can rearrange L'' if necessary to sit in a splitting of f_{ss}. In particular, we can assume $g^{\sigma,\sigma'}_{\gamma,\gamma'} = 0$ holds for every $g \in L''$ unless at least one of

$$\mathsf{M}^{\sigma}_{\gamma} \cong \mathsf{M}^{\sigma'}_{\gamma'}, \quad \text{or} \quad \mathsf{M}^{-\sigma}_{\gamma} \cong \mathsf{M}^{-\sigma'}_{\gamma'} \qquad (\text{as } Q^{\mathbf{a}}\text{-representations})$$

holds. It follows that $\gamma \neq \gamma'$ implies that each $g \in L''$ satisfies $g^{\sigma,\sigma'}_{\gamma,\gamma'} = 0$ or $g^{\sigma',\sigma}_{\gamma',\gamma} = 0$ for all possible choices of σ, σ'. Since L'' is reductive, we can further rearrange L'' if necessary to assume $g_{\gamma,\gamma'} = 0$ ($g \in L''$) unless $\gamma \neq \gamma'$. Therefore, L'' preserves each M_{γ}. This forces $L' = L''$ as desired. ☐

For an admissible parameter $\nu_0 = (s_0, \vec{q}, X_0)$, we define $G(\nu_0)$ to be the simultaneous stabilizer of (s_0, X_0) in G. Let $C(\nu_0)$ be the component group of $G(\nu_0)$.

Theorem 7.14. *We have $C(\nu_0) = \{1\}$ for every admissible parameter ν_0.*

Proof. The assertion is unchanged by the conjugation action of G. Take G-conjugate to assume that ν_0 is a standard parameter. By Lemma 3.5, we have only to verify the case that ν_0 has a unique clan. Hence, we further assume that $\nu_0 = \nu$. Let $G_{\mathbf{a}}(X_2)$ be the stabilizer of $\Xi(X_2)$ in $G_{\mathbf{a}}$. Since $G_{\mathbf{a}}(X_2)$ is the group of $Q^{\mathbf{a}}$-automorphisms of M, we deduce its connectivity. Let $G_{\mathbf{a}}(X_1)$ be the stabilizer of X_1 in $G_{\mathbf{a}}$. The group $G_{\mathbf{a}}(X_1)$ is realized as the pullback of a linearly defined subgroup of the RHS of the natural map

$$f : G_{\mathbf{a}} = \prod_{m \geq 0} GL(d^{\mathbf{a}}_m) \longrightarrow GL(V(m_1)) \times GL(V(m_2)), \tag{7.1}$$

where $m_1 \neq m_2$ are some integers given by Corollary 7.7. (If they do not exist, then we set $GL(V(m_k)) = \{1\}$.) Denote the RHS of (7.1) by H. The $G_{\mathbf{a}}$-stabilizer $G_{\mathbf{a}}(X)$ of $X_1 \oplus \Xi(X_2)$ is given as $G_{\mathbf{a}}(X_1) \cap G_{\mathbf{a}}(X_2)$. By a standard quiver theory, $G_{\mathbf{a}}(X_2)$ is a linearly defined subgroup of $G_{\mathbf{a}}$. In particular, $f(G_{\mathbf{a}}(X_2))$ is the common zeros of linear functions on H. Hence, the image of $G_{\mathbf{a}}(X_1) \cap G_{\mathbf{a}}(X_2)$ in H is connected. Therefore, we conclude that $G_{\mathbf{a}}(X)$ is connected. Since $G_{\mathbf{a}} \cong G(s)$ if $\mathbf{c}$ is type I, we conclude the result in this case. Assume that $\mathbf{c}$ is type II or III. Let $G^{\theta}_{\mathbf{a}}(X_2)$ be the $G_{\mathbf{a}}$-stabilizer of $\Xi(X_2)$ which respects $\sum_{\gamma} \theta_{\gamma}$. By Corollary 7.12, we have $G^{\theta}_{\mathbf{a}}(X_2) \subset G(s)$. By Proposition 7.13, a Levi subgroup L of $G^{\theta}_{\mathbf{a}}(X_2)$ is contained in the simultaneous stabilizer of M_{γ} and θ_{γ} for every possible pairs of segments γ. It suffices to prove that the stabilizer of X_1 in L is connected. We have an inclusion

$$L \hookrightarrow \prod_{\gamma} \mathrm{Aut}_{Q^{\mathbf{a}}} \mathsf{M}_{\gamma} \hookrightarrow \prod_{m \geq 0} \prod_{\gamma} GL(\mathsf{M}_{\gamma}(m)) \subset G_{\mathbf{a}}.$$

If $\gamma = (I, \emptyset)$ for some segment I, then $\mathsf{M}_{\gamma}(m)$ is contained in the direct sum of $Q^{\mathbf{a}}$-modules M^{+}_{J} such that $\gamma = (\underline{J}, \emptyset)$. It follows that $\theta_{\gamma} = 0$. Hence, the action of L on M_{γ} gives a direct factor of L isomorphic to $GL(n_{\gamma})$. If $\gamma = (I, I')$

for some two non-empty segments I, I', then the condition $\mathsf{M}_\gamma \neq \{0\}$ forces $0 \in I \cap I'$ by definition. The action of $\mathrm{Aut}_{Q^{\mathbf{a}}} \mathsf{M}_\gamma$ on M_γ factors through

$$GL(\mathsf{M}_\gamma(0)) \cong GL(2n_\gamma) \subset GL(d_0^{\mathbf{a}}).$$

We have two cases:

($\mathsf{M}_J^+ \cong \mathsf{M}_J^-$ as $Q^{\mathbf{a}}$-representation) The action of $GL(\mathsf{M}_\gamma(0))$ preserves θ_γ if and only if it respects a non-degenerate alternating form. It follows that the L-action on M_γ gives its direct factor isomorphic to $Sp(2n_\gamma)$.

(otherwise) Since $\mathsf{M}_J^+ \not\cong \mathsf{M}_J^-$, the group $\mathrm{Aut}_{Q^{\mathbf{a}}} \mathsf{M}_\gamma$ is a parabolic subgroup of $GL(\mathsf{M}_\gamma(0))$ with its Levi component $GL(n_\gamma) \times GL(n_\gamma)$. Since respecting θ_γ reduces $\mathrm{Aut}_{Q^{\mathbf{a}}} \mathsf{M}_\gamma$ to a subgroup of $Sp(2n_\gamma)$, we deduce that the L-action on M_γ gives its direct factor isomorphic to $GL(n_\gamma)$.

It follows that L is a product of general linear groups and symplectic groups. Applying Lemma 7.8, we conclude the result. $\square$

8 Main Theorems

We retain the setting of §2.

Theorem 8.1 (A Deligne-Langlands type classification)**.** *Let $\mathbf{a} \in \mathbf{G}$ be a finite element. Then, the set $\Lambda_{\mathbf{a}}$ is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{H}_{\mathbf{a}}$-modules.*

Proof. The definition of regular parameters asserts that we have at least one simple module corresponding to each element of $\Lambda_{\mathbf{a}}$. Each irreducible direct summand of $(\mu_+^{\mathbf{a}})_* \mathbb{C}_{F_+^{\mathbf{a}}}$ is the minimal extension of a local system (up to degree shift) from some smooth $\mathbf{G}(\mathbf{a})$-stable locally closed subvariety $\tilde{\mathbb{O}}$ of $\mathfrak{N}_+^{\mathbf{a}}$ which contains a dense open $\mathbf{G}(\mathbf{a})$-orbit $\mathbb{O}$. (cf. §1.3) By Theorem 7.14, a $\mathbf{G}(\mathbf{a})$-equivariant local system on $\mathbb{O}$ is a constant sheaf. Since $\tilde{\mathbb{O}} \backslash \mathbb{O}$ is real codimension two in $\tilde{\mathbb{O}}$, we deduce that the natural map $\pi_1(\mathbb{O}, *) \to \pi_1(\tilde{\mathbb{O}}, *)$ is surjective. In particular, every $\mathbf{G}(\mathbf{a})$-equivariant local system on $\tilde{\mathbb{O}}$ is constant. As a result, every G-conjugacy class of regular admissible parameter of the form $(\mathbf{a}, X)$ corresponds to at most one irreducible module as desired. $\square$

In order to give more effective versions of Theorem 8.1, we need

Lemma 8.2. *Let $\mathbf{a} = (s, q_0, q_1, q_2) \in \mathbf{G}$ be a pre-admissible element. If $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ holds for every $0 \le m < n$, then $\mathbf{a}$ is finite.*

Proof. By taking G-conjugate, we can assume $\mathbf{a} \in \mathbf{T}$ without the loss of generality. The number of standard parameters of type $(\mathbf{a}, X)$ (X is a 1-normal form) are finite (up to T-action). By Theorem 1.10, there are infinitely many $\mathbf{G}(\mathbf{a})$-orbits in $\mathfrak{N}_+^{\mathbf{a}}$ only if there exists a standard parameter $\nu' = (\mathbf{a}, X')$ which satisfies ($\star$) such that X' is not $G(s)$-conjugate to a 1-normal form. Here we have

$$X_1 \in V_1^{(s,q_0)} + V_1^{(s,q_1)} \subset V_1.$$

By the definition of 0-normal forms and Corollary 5.9, it suffices to prove that

$$V_1^{(s,q_i)}[\sigma \epsilon_j] \neq \{0\} \quad (i \in \{0,1\}, \sigma \in \{\pm\}, \text{ and } j \in \mathbf{c}_m)$$

holds for a unique choice of i, m. This is equivalent to the case that the s-eigenvalues of V_1 does not contain both of q_0 and q_1. This follows from $q_0 q_1^{\pm 1} \neq q_2^{\pm m}$ $(0 \leq m < n)$. Hence, all ℓ-normal forms $X' \in \mathfrak{N}_+^{\mathbf{a}}$ are $G(s)$-conjugate to 1-normal forms as desired. $\square$

Corollary 8.3. *A general element of* $\mathbf{G}$ *is finite.* $\square$

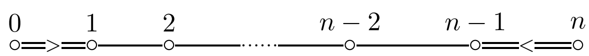
As in Remark 2.2, the quotient $\mathbb{H}/(q_0 + q_1)$ is isomorphic to an extended Hecke algebra $\mathbb{H}_B$ of type $B_n^{(1)}$ with two parameters. Combined with Lemma 8.2, we have

Corollary 8.4 (Type B case)**.** *Let* $\mathbf{a} = (s, q_0, -q_0, q_2) \in \mathbf{G}$ *be a pre-admissible element such that* $-q_0^2 \neq q_2^{\pm m}$ *holds for every* $0 \leq m < n$*. Then, the set* $\Lambda_{\mathbf{a}}$ *is in one-to-one correspondence with the set of isomorphism classes of simple* $\mathbb{H}_{\mathbf{a}}$*-modules.* $\square$

Corollary 8.5 (General case)**.** *Let* $\mathbf{a} = (s, \vec{q}) \in \mathbf{G}$ *be a general element. Then, the set of* $G(s)$*-orbits in* $\mathfrak{N}_+^{\mathbf{a}}$ *is in one-to-one correspondence with the set of isomorphism classes of simple* $\mathbb{H}_{\mathbf{a}}$*-modules.*

Proof. Assuming Theorem 8.1, the assertion is an immediate consequence of Corollary 8.3 and Corollary 6.3. $\square$

Remark 8.6. The Dynkin diagram of type $C_n^{(1)}$ is written as:



This Dynkin diagram has a unique non-trivial involution φ. We define t_0, t_1, t_n to be

$$t_1^2 = q_2, t_n^2 = -q_0 q_1, t_n(t_0 - t_0^{-1}) = q_0 + q_1 \quad \text{(cf. Remark 2.2 1)).}$$

Let $T_0, \ldots, T_n$ be the Iwahori-Matsumoto generators of $\mathbb{H}$ (cf. [Mc03, Lu03]). Their Hecke relations read

$$(T_0 + 1)(T_0 - t_0^2) = (T_i + 1)(T_i - t_1^2) = (T_n + 1)(T_n - t_n^2) = 0,$$

where $1 \leq i < n$. The natural map $\varphi(T_i) = T_{n-i}$ $(0 \leq i \leq n)$ extends to an algebra map $\varphi : \mathbb{H} \to \mathbb{H}'$, where $\mathbb{H}'$ is the Hecke algebra of type $C_n^{(1)}$ with parameters t_n, t_1, t_0. We have $t_n = \pm\sqrt{-q_0 q_1}$ and $t_0 = \pm\sqrt{-q_0/q_1}$ or $\pm\sqrt{-q_1/q_0}$. In particular, φ changes the parameters as $(q_0, q_1, q_2) \mapsto (q_0, q_1^{-1}, q_2)$ or (q_0^{-1}, q_1, q_2). Therefore, the representation theory of $\mathbb{H}_{\mathbf{a}}$ is unchanged if we replace q_0 with q_0^{-1}, or q_1 with q_1^{-1}.

9 Consequences

In this section, we present some of the consequences of our results. We retain the setting of the previous section. For an admissible parameter $\nu = (\mathbf{a}, X)$, we define

$$F(\nu) := (\mu_+^{\mathbf{a}})^{-1}(X) \subset F.$$

By means of the isomorphism $F_+^{\mathbf{a}} \cong F_2^{\mathbf{a}}$ (cf. 2.12), we may regard $F(\nu) \subset F_2$.

Definition 9.1 (Standard modules)**.** Let $\nu = (\mathbf{a}, X)$ be an admissible parameter. We define

$$M_\nu := H_\bullet(F(\nu), \mathbb{C}) \text{ and } M^\nu := H^\bullet(F(\nu), \mathbb{C}).$$

By the Ginzburg theory [CG97] 8.6, each of M_ν or M^ν is $\mathbb{H}$-module. Let $\nu' = (\mathbf{a}, X')$ be a regular admissible parameter. Let $L_{\nu'}$ be the corresponding simple module of $\mathbb{H}$. Let $IC(\nu')$ be the corresponding $\mathbf{G}(\mathbf{a})$-equivariant simple perverse sheaf on $\mathfrak{N}_+^{\mathbf{a}}$. (cf. §1.3) We denote by $P_{\nu'}$ the projective cover of $L_{\nu'}$ as $\mathbb{H}_{\mathbf{a}}$-modules. (It exists since $\mathbb{H}_{\mathbf{a}}$ is finite dimensional.)

Applying [CG97] 8.6.23 to our situation, we obtain:

Theorem 9.2 (The multiplicity formula of standard modules)**.** *Let $\nu = (\mathbf{a}, X)$, $\nu' = (\mathbf{a}, X')$ be regular admissible parameters. We have:*

$$[M_\nu : L_{\nu'}] = \sum_k \dim H^k(i_X^! IC(\nu')) \text{ and } [M^\nu : L_{\nu'}] = \sum_k \dim H^k(i_X^* IC(\nu')),$$

where $i_X : \{X\} \hookrightarrow \mathfrak{N}_+^{\mathbf{a}}$ is an inclusion. □

The following result is a variant of the Lusztig-Ginzburg character formula of standard modules in our setting.

Theorem 9.3 (The character formula of standard modules)**.** *Let $\nu = (s, \vec{q}, X)$ be an admissible parameter. Let $\mathfrak{B}_\nu$ be the set of connected components of $F(\nu)$. For each $\mathcal{B} \in \mathfrak{B}_\nu$, we define a linear form $\langle \bullet, s\rangle_{\mathcal{B}}$ as a composition map*

$$\begin{array}{ccccc} \langle \bullet, s\rangle_{\mathcal{B}} : R(T) & \xrightarrow{\cong} & R(gBg^{-1}) & \xrightarrow{\mathrm{ev}_s} & \mathbb{C} \\ \uparrow & & \uparrow & & \\ R^+ & \longrightarrow & \{\textit{weights of } gBg^{-1}\} & & \end{array}$$

by some $g \in G$ such that $gB \in \mathcal{B}$. Then, $\langle \bullet, s\rangle_{\mathcal{B}}$ is independent of the choice of g and the restriction of M_ν to $R(T)$ is given as

$$\mathrm{Tr}(e^\lambda; M_\nu) := \sum_{\mathcal{B} \in \mathfrak{B}_\nu} \langle \lambda, s\rangle_{\mathcal{B}} \sum_{j \geq 0} \dim H_j(\mathcal{B}, \mathbb{C}).$$

Proof. Taking account into Theorem 7.14, the proof is exactly the same as in [CG97] §8.2. □

In order to provide a standard form of the above character formula and to apply a general theory on projective modules, we need the following analogue of the De Concini-Lusztig-Procesi version of the Spaltenstein-Shoji's vanishing theorem [Sp82, Sh83]:

Theorem 9.4. *Let $\nu = (s, \vec{q}, X_1 \oplus X_2)$ be an admissible parameter such that $\vec{q}$ is general. Then, we have $H_{odd}(F(\nu), \mathbb{Z}) = 0$.*

Proof. Postponed to §10. □

Let A be a $\mathbb{H}$-module and let L be a simple $\mathbb{H}$-module. We denote by $[A : L]$ the multiplicity of L in A.

Definition 9.5. Let $\mathbf{a} = (s, \vec{q}) \in \mathbf{T}$ be a general element. We form three $|\Lambda_{\mathbf{a}}| \times |\Lambda_{\mathbf{a}}|$-matrices

$$[P : L]^{\mathbf{a}}_{\nu,\nu'} := [P_\nu, L_{\nu'}], D^{\mathbf{a}}_{\nu,\nu'} := \delta_{\nu,\nu'}\chi_c(\nu), \text{ and } IC^{\mathbf{a}}_{\nu,\nu'} := [M^\nu, L_{\nu'}],$$

where $\chi_c(\nu) := \sum_{i\geq 0}(-1)^i \dim H^i(\mathbf{G}(\mathbf{a})X, \mathbb{C})$ $(\nu = (\mathbf{a}, X))$.

The following result is a special case of the Ginzburg theory [CG97] Theorem 8.7.5 applied to our particular setting:

Theorem 9.6 (The multiplicity formula of projective modules)**.** *Keep the setting of Definition 9.5. We have*

$$[P : L]^{\mathbf{a}} = IC^{\mathbf{a}} \cdot D \cdot {}^t IC^{\mathbf{a}},$$

where ^{t} *denotes the transposition of matrices.* □

Remark 9.7 (Relation with canonical bases of quantum groups). Assume the setting of §7. Let $\nu = (\mathbf{a}, X) = (s, \vec{q}, X)$ be a standard parameter of type I. Let $Q^{\mathbf{a}}$ be the quiver attached to $\mathbf{a}$. We define a quiver $Q^{\mathbf{a}}_+$ by adding new vertexes and arrows to $Q^{\mathbf{a}}$ as follows:

- A vertex $\underline{i}$ and an arrow $\underline{i} \to i$ if $\mathbb{V}^{\mathbf{a}}[\epsilon_k] \neq 0$ for some $k \in \mathbf{c}_i$;
- A vertex $\overline{i}$ and an arrow $i \to \overline{i}$ if $\mathbb{V}^{\mathbf{a}}[-\epsilon_k] \neq 0$ for some $k \in \mathbf{c}_i$.

We define $\mathbf{d}^{\mathbf{a}}_+$ to be the dimension vector of $Q^{\mathbf{a}}_+$ obtained from $\mathbf{d}^{\mathbf{a}}$ by adding 1's to all vertexes of $Q^{\mathbf{a}}_+ \backslash Q^{\mathbf{a}}$. Then, we have a natural isomorphism

$$\mathfrak{N}^{\mathbf{a}}_+ \cong \mathrm{Rep}(Q^{\mathbf{a}}_+, \mathbf{d}^{\mathbf{a}}_+)$$

extending that of Construction 7.1. We further assume that $Q^{\mathbf{a}}_+$ is of finite or affine type. This assumption includes all affine quivers of type ADE except for $A^{(1)}_\ell$. In this case, the variety $\mathrm{Rep}(Q^{\mathbf{a}}_+, \mathbf{d}^{\mathbf{a}}_+)$ has only finitely many orbit. By Theorem 8.1, each simple module of $\mathbb{H}_{\mathbf{a}}$ determines a $\mathbf{G}(\mathbf{a})$-orbit of $\mathfrak{N}^{\mathbf{a}}_+$.
Let U^+_v be the plus part of the corresponding quantized enveloping algebra. Then, the coefficients of Theorem 9.2 are identified with the coefficients of the PBW bases of U^+_v in terms of the canonical bases specialized to 1. (For more details, consult Lusztig [Lu90] and Ariki [Ar96].)

10 Proof of Theorem 9.4

In this section, we recall several key definitions and results of [DLP88] and apply them to prove Theorem 9.4.

Definition 10.1 (α-partitions)**.** A partition of a variety $\mathcal{X}$ over $\mathbb{C}$ is said to be an α-partition if it is indexed as $\mathcal{X}_1, \mathcal{X}_2, \ldots \mathcal{X}_k$ in such a way that $\mathcal{X}_1 \cup \ldots \cup \mathcal{X}_i$ is closed for every $i = 1, \ldots, k$.

The following is a slight modification of the condition (S) in [DLP88] 1.7:

Definition 10.2 (Condition (S'))**.** A variety $\mathcal{X}$ is said to satisfy condition (S') if there exists an α-partition $\{\mathcal{X}_i\}_i$ of $\mathcal{X}$ such that each $\mathcal{X}_i$ satisfy the condition (S'). We declare that the following varieties satisfy the condition (S'):

1. A smooth projective variety with $\mathbb{G}_m$-action with finite fixed points;
2. A vector bundle over a variety with (S');
3. A smooth deformation of a smooth projective variety with (S').

Theorem 10.3 (Bialynichi-Birula cf. [DLP88] 1.2)**.** *A smooth projective variety $\mathcal{X}$ with $\mathbb{G}_m$-action satisfies (S') if $\mathcal{X}^{\mathbb{G}_m}$ satisfies (S').*

Using the arguments of [DLP88] 1.7–1.10 and Theorem 10.3, it is straightforward to see that a variety with (S') satisfies the homological property (the first condition) of the condition (S). In particular, we have:

Theorem 10.4 ([DLP88] 1.7–1.10)**.** *For a veriety $\mathcal{X}$ with (S'), we have $H_{odd}(\mathcal{X}, \mathbb{Z}) = 0$.*

Theorem 10.5 ([DLP88] 1.5)**.** *Let $\pi : \mathcal{E} \to \mathcal{X}$ be a vector bundle over a smooth variety $\mathcal{X}$, with a fiber preserving linear $\mathbb{C}^\times$-action on $\mathcal{E}$ with strictly positive weights. Let $\mathcal{Z} \subset \mathcal{E}$ be a $\mathbb{C}^\times$-stable smooth closed subvariety. Then, $\pi(\mathcal{Z})$ is smooth and $\mathcal{Z}$ is a subbundle of $\mathcal{E}$ restricted to $\pi(\mathcal{Z})$.*

The rest of this section is devoted to the proof of Theorem 9.4. In the below, we first assume the setting of §9 and assume more notation from previous sections as indicated.

Let $Y \in \mathfrak{N}^{\mathbf{a}}_+$. Then, $\nu' := (\mathbf{a}, Y)$ is an admissible parameter. We put $\mathcal{B} := (G/B)^s$. It is isomorphic to a disjoint union of copies of $G(s)/B(s)$ as $G(s)$-varieties. Let $\mathcal{B}_Y$ denote the projection of $F(\nu')$ to $\mathcal{B}$. (It is clear that $F(\nu') \cong \mathcal{B}_Y$.)

By the argument in §3 (cf. (3.3) and Corollary 3.10), $F(\nu)$ is isomorphic to a disjoint union of $\prod_{\mathbf{c} \in I(\mathbf{a})} ({}^w\mu^{\mathbf{a}}_{\mathbf{c}})^{-1}(X_{\mathbf{c}})$ for $w \in W_{\mathbf{a}}$. Therefore, it is enough to show $H_{odd}(({}^w\mu^{\mathbf{a}}_{\mathbf{c}})^{-1}(X_{\mathbf{c}})), \mathbb{Z}) = 0$ for each $w \in W_{\mathbf{a}}$ and $\mathbf{c} \in I(\mathbf{a})$. In particular, we can assume that $\mathbf{c}$ is the unique clan associated to $\mathbf{a}$. Since $\vec{q}$ is general, we deduce that the assumption of Proposition 6.1 or Proposition 6.2 holds by the proof of Corollary 6.3. In particular, we can assume that

- ν is a standard parameter which satisfies $(\star)$;
- We have a height function h adapted to ν.

Proposition 10.6 (cf. Proposition 5.2)**.** *The space $\overline{P^h X} \subset \mathbb{V}^{\mathbf{a}}$ is a linear subspace.*

We return to the proof of Theorem 9.4.

We assume $Y \in \overline{P^h X}$ and $s_h Y = Y$ in the below. Let $\mathcal{O}$ be a P^h-orbit of $\mathcal{B}$. Since $\mathcal{B}$ is a union of finitely many P^h-orbits, $\mathcal{B}_Y = \bigsqcup_{\mathcal{O}} (\mathcal{O} \cap \mathcal{B}_Y)$ is an α-partition. Thus, $\mathcal{B}_Y$ satisfies (S') if $(\mathcal{O} \cap \mathcal{B}_Y)$ satisfies (S') for each $\mathcal{O}$.

We define

$$E(Y) := \{(gB, Z) \in \mathcal{B} \times \overline{P^h Y}, Z \in \mathbb{V}^{\mathbf{a}} \cap g\mathbb{V}^+\}.$$

The second projection $\breve{p}_2 : E(Y) \ni (gB, Z) \mapsto Z \in \overline{P^h X}$ is P^h-equivariant. It is clear that $\breve{p}_2^{-1}(Y) \cong \mathcal{B}_Y$.

Proposition 10.7. *Assume that the locally closed subset*

$$(\overline{P^h Y} \cap g\mathbb{V}^+) \cap G(s)Y \subset \mathbb{V}$$

is smooth for every $gB \in \mathcal{B}$. Then, $(\mathcal{O} \cap \mathcal{B}_Y)$ is smooth for each P^h-orbit $\mathcal{O}$.

Proof. We define a P^h-stable subvariety of $E(Y)$ as:

$$E_{\mathcal{O}}(Y) := \{(gB, Z) \in E(Y), gB \in \mathcal{O}, Z \in G(s)Y\}.$$

Since $\breve{p}_2$ is a P^h-equivariant fibration over $P^h Y$, the fiber $(\breve{p}_2^{-1}(Y) \cap E_{\mathcal{O}}(Y))$ is smooth if $E_{\mathcal{O}}(Y)$ is smooth. Here $E_{\mathcal{O}}(Y)$ is an open subset of a smooth fibration over $\mathcal{O}$. Thus, $E_{\mathcal{O}}(Y)$ must be smooth. As a result, we deduce that $(\mathcal{O} \cap \mathcal{B}_Y)$ is smooth. □

Corollary 10.8. *Keep the setting of Proposition 10.7. The variety $\mathcal{B}_Y^{s_h}$ is smooth projective.* □

Proof. The variety $\mathcal{O}^{s_h}$ is isomorphic to the flag variety of L^h. Since $\mathcal{B}_Y$ is closed, the intersection $(\mathcal{O}^{s_h} \cap \mathcal{B}_Y) = (\mathcal{O} \cap \mathcal{B}_Y^{s_h})$ is projective. The smoothness follows by Proposition 10.7. □

Corollary 10.9. *The varieties $\mathcal{B}_X^{s_h}$ and $(\mathcal{O} \cap \mathcal{B}_X)$ are smooth for each P^h-orbit $\mathcal{O}$ of $\mathcal{B}$.* □

Proof. Since $\overline{P^h X}$ is a linear subspace of $\mathbb{V}$, we deduce that $\overline{P^h X} \cap g\mathbb{V}^+$ is a linear subspace of $\mathbb{V}$. Hence, its open subset is smooth. □

We return to the proof of Theorem 9.4.

For a P^h-orbit $\mathcal{O}$ in $\mathcal{B}$, we have a vector bundle $\mathcal{O} \to \mathcal{O}^{s_h}$ whose fiber is given as the U^h-translate. This vector bundle satisfies the first condition of Theorem 10.5 by using some one-parameter subgroup which contains s_h and fixes X. It follows that we have a vector bundle structure

$$\mathcal{O} \cap \mathcal{B}_X \to (\mathcal{O} \cap \mathcal{B}_X)^{s_h}.$$

Therefore, $\mathcal{B}_X$ satisfies (S') if each connected component of $\mathcal{B}_X^{s_h}$ satisfies (S').

Let T_0 be the maximal torus of the stabilizer of X in $\mathbf{G}(\mathbf{a})$ which contains s_h. We have $T_0 \subset P^h \times (\mathbb{C}^\times)^3$. Each $\mathcal{B}_X \cap \mathcal{O}$ is T_0-stable. By Theorem 10.3, it suffices to show that each connected component of $\mathcal{B}_X^{T_0} \cap \mathcal{O}$ satisfies (S').

For a 1-block v, we have

$$\dim \mathrm{Stab}_T v = \begin{cases} \dim T - \#\|v\| + 1 & (v \text{ is a 0-block}) \\ \dim T - \#\|v\| & (\text{otherwise}) \end{cases}.$$

It follows that we can choose T_0 so that the $Z_{G(s)}(T_0)$-action separates the supports of 0-blocks appearing in X. Hence, we can concentrate into the cases where 1) X has a unique block, or 2) X is a sum of blocks and $\overline{TX}_1 = \oplus_{i \in \|X\|} \mathbb{V}_1^{\mathbf{a}}[\epsilon_i]$. Since $\vec{q}$ is general, genuine 1-blocks can appear only if $\mathbf{c}$ is type I (cf. Proof of Corollary 6.3). Thus, the case 2) occurs only if $\mathbf{c}$ is type I.

In these cases, the group $Z_{G(s)}(T_0)$ is always a product of tori and groups of type A_m. (The latter occurs only if 2) holds.) In the first case, every connected component of $\mathcal{B}_X^{T_0}$ is a point, which satisfies (S').

Proposition 10.10. *Assume that $Z_{G(s)}(T_0)$ contains type A_m-factor. Then, the variety $\mathcal{B}_X^{T_0}$ satisfies (S').*

Proof. We prove the assertion by the induction on $\# |X|$. The case $\# |X| \le 1$ is covered by the case 1). We assume the assertion for $|X| < l$. We prove the case $|X| = l$.

Let $G_0 := Z_{G(s)}(T_0) \subset P^h$. Let $B_0 := Z_{G(s)}(T_0) \cap B$. We put $V := (\mathbb{V}^{\mathbf{a}})^{T_0}$. We write $X_2 = \sum_{J \in |X|} \mathbf{v}_J$. By Corollary 5.9, we rearrange X if necessary to assume $\underline{J} \subsetneq \underline{J}'$ or $\underline{J}' \subsetneq \underline{J}$ for each $J \neq J' \in |X|$. It follows that $\mathrm{Stab}_{G_0} X_2$ is a torus T_1 of rank l. Let J^0 be the maximal element of $|X|$ with respect to the inclusion relations of $\underline{J}$. We put

$$Y = \sum_{J^0 \neq J \in |X|} \mathbf{v}[\epsilon_{J_m}] \oplus X_2 \in \overline{P^h X},$$

where $J_m \in J$ is the unique member such that $\mathbf{v}[\epsilon_{J_m}] \in \overline{TX}_1$. We have $s_h Y = Y$. We put $\mathbb{G} := \exp \mathbb{C}\epsilon_{J_m^0}$. We identify V with a representation space of a type A-quiver with one extra vertex as in Remark 9.7. It follows that $G_0 X$ is dense in V and the $G_0 Y$ is codimension one in V.

For each $w \in W$, we define a map

$$\phi_w : G_0 \times^{{}^w B_0} (V \cap \dot{w}\mathbb{V}^+) \to V.$$

It suffices to prove that $\phi_w^{-1}(X)$ satisfies (S') for each $w \in W$. Each fiber of ϕ_w is projective. By Corollary 10.9, we know that $\phi_w^{-1}(X)$ is smooth.

By the semi-continuity of the dimensions of fibers, we have $\dim \phi_w^{-1}(X) = \dim \phi_w^{-1}(Y)$. The variety $\phi_w^{-1}(\overline{G_0 X})$ is reduced. It follows that $\phi_w^{-1}(\overline{\mathbb{G}X})$ is reduced. By Hartshorne [Ha77] II 9.7, we deduce that $\phi_w^{-1}(\overline{\mathbb{G}X})$ gives a flat family over $\mathbb{A}^1$ with its general fiber isomorphic to $\phi_w^{-1}(X)$ and its special fiber isomorphic to $\phi_w^{-1}(Y)$.

We have $Y \in \overline{\mathbb{G}X} \subset V$. Let $V \to V \cap V_2$ be the projection, which we regard as a vector bundle $\mathcal{V}$. By construction, $\overline{G_0 Y}$ is a vector subbundle of $\mathcal{V}$ of corank one. Therefore, we deduce that $G_0 Y \cap g\dot{w}\mathbb{V}^+$ is smooth for each $g \in G_0$. By Proposition 10.7, we conclude that $\phi_w^{-1}(Y)$ is smooth. Since each fiber is smooth, $\phi_w^{-1}(\overline{\mathbb{G}X})$ is a smooth family over $\mathbb{A}^1$.

The set of $\mathbb{G}$-fixed points of $\phi_w^{-1}(Y)$ satisfies (S') by the induction assumption. It follows that $\phi_w^{-1}(Y)$ satisfies (S'). This implies that $\phi_w^{-1}(X)$ itself satisfies (S'). Thus, the induction proceeds and we conclude the result. $\square$

We return to the proof of Theorem 9.4.

By Theorem 10.3, we conclude that $\mathcal{B}_X$ itself satisfies (S'). Therefore, Theorem 10.4 implies the result.

A Proofs of geometric results on $\mathfrak{N}$

In this appendix, we prove three geometric assertions whose proofs are not given in the main body of this paper. We retain the setting of §1.1 and §1.2.

The following proof of Theorem 1.2 is suggested by referees of a previous version of this paper. The author wishes to thank the referees for these kind information.

Proof of Theorem 1.2. 1) is a direct consequence of Dadok-Kac [DK85] Table 1. 1) implies that $\mathfrak{N}$ is a product of V_1 and the Hilbert null-cone (cf. Popov [Po04]) of V_2. Since the latter is normal by a result of Kostant-Rallis [KR71], 2) follows. 3) is a corollary of Theorem 1.9, which we postpone to §1.2. (Its proof is given in the below.) 4) is a direct consequence of the weight distribution of $\mathbb{V}^+$ and the Hesselink theory (cf. [Po04] Theorem 1). 5) is an immediate consequence of 2), 4), and the Zariski main theorem. (cf. [CG97] 3.3.26) We show 6). Let $\widehat{\mathbb{O}}$ be the inverse image of a G-orbit $G.X = \mathbb{O} \subset \mathfrak{N}$ under the map $\mu \circ p_2$. Then, we have

$$\dim \mathbb{O} + 2 \dim \mu^{-1}(X) \le \dim \widehat{\mathbb{O}}.$$

The dimension of the RHS is less than or equal to $\dim F$, which is the (constant) dimension of irreducible components of Z. In particular, we have

$$\dim \mathbb{O} + 2 \dim \mu^{-1}(X) \le \dim \mathfrak{N} = \dim F,$$

which implies that μ is semi-small. □

For each $p \in [1, n]$, we define elements $(-)_p, \dot{w}_p \in N_G(T)$ as lifts of elements of W given by

$$(-)_p \epsilon_i = \begin{cases} \epsilon_i & (p \neq i), \\ -\epsilon_p & (p = i) \end{cases} \text{ and } \dot{w}_p \epsilon_i = \begin{cases} \epsilon_i & (i \le p) \\ -\epsilon_{n+p+1-i} & (i > p) \end{cases}.$$

Let $G_0 \subset G$ be the subgroup generated by T and the unipotent one-parameter subgroups of G corresponding to the roots $\pm\alpha_1, \ldots, \pm\alpha_{n-1}$. ($G_0 \cong GL(n, \mathbb{C})$) We have a natural decomposition

$$V_1 = V_1^+ \oplus V_1^-, V_2 = V_2^\flat \oplus V_2^0 \oplus V_2^\sharp$$

as G_0-modules, where $V_1^\pm = \bigoplus_i V_1[\pm\epsilon_i]$, $V_2^\sharp = \bigoplus_{i<j} V_2[\epsilon_i + \epsilon_j]$, and $V_2^\flat = \bigoplus_{i<j} V_2[-(\epsilon_i + \epsilon_j)]$. Let $u_{i,j}$ be a non-zero element of the unipotent one-parameter subgroup of G with its weight $\epsilon_i + \epsilon_j$.

Proof of Theorem 1.10. We start by assuming $s \in T$ and $X \in V_1 \oplus V_2^+$ by the G-action (cf. Lemma 3.5 and 4.8). Let $X_2 = X_2^0 + X_2^\sharp \in V_2^0 \oplus V_2^\sharp$ be the isotypical decomposition. We may regard X_2^0 as an element of $\mathfrak{gl}_n$ via the embedding $V_2^0 \cong \mathfrak{sl}_n \subset \mathfrak{gl}_n$. We put $d_X := \#(\|X_2\| \backslash \|X_2^0\|)$. We define n_X as an integer such that $(X_2^0)^{n_X} \neq 0$ and $(X_2^0)^{n_X+1} = 0$. We define r_X as the rank of X_2^0. (Since X_2^0 does not naturally act on some space, the definition of r_X and n_X might look strange. But we only use them as invariants under the G_0-action, which make sense in this setting.) We divide the proof into five steps:

(Step 1) By means of $G_0(s)$-action and the $N_{G_0}(T)$-action, we can assume that $s \in T$ and X_2^0 is a Jordan normal form of $\mathfrak{gl}_n$. I.e.

$$X_2(\epsilon_i - \epsilon_j) \neq 0 \text{ only if } j = i + 1.$$

(Step 2) Assume that $d_X > 0$. We choose $j \in \|X_2\| \backslash \|X_2^0\|$. By multiplying $\dot{w} \in N_{G_0}(T)$ appropriately, we assume that $\dot{w}X_2 \in V_2^+$, $\dot{w}X_2^0$ is still a Jordan normal form, and $\dot{w}\epsilon_j = \epsilon_n$. We have $\dot{w}X(\epsilon_i + \epsilon_n) \neq 0$ for some i. If $i \in \|X_2\| \backslash \|X_2^0\|$, then $(-)_n \dot{w} X_2 \in V_2^+$ and $r_{(-)_n \dot{w} X} > r_X$. Assume that $i \in \|X_2^0\|$. We have $u_{i+1,j}^c X(\epsilon_i + \epsilon_j) = 0$ for some $c \in \mathbb{C}$ if $X(\epsilon_i - \epsilon_{i+1}) \neq 0$. Thus, we can assume

$X(\epsilon_i - \epsilon_{i+1}) = 0$. Here $(-)_n \dot{w} X_2 \in V_2^+$ and we have $r_{(-)_n \dot{w} X} > r_X$. We replace X with $(-)_n \dot{w} X$ and again apply the procedure of (Step 1). We can do this operation only finitely many times since $r_X \le n$. As a consequence, we can assume that $\| X_2 \| = \| X_2^0 \|$.

(Step 3) The action of $u_{i,j}$ is trivial on $X_2^{\sharp}$. Moreover, we have $[u_{i,j}, u_{i',j'}] = 1$. We have

$$u_{i,j} X_2^0 = X_2^0 + c_{i,j}^1 \mathbf{v}[\epsilon_{i-1} + \epsilon_j] + c_{i,j}^2 \mathbf{v}[\epsilon_i + \epsilon_{j-1}]$$

for some constants $c_{i,j}^k$ $(k = 1, 2)$. Moreover, we have $c_{i.j}^1 = c_{i,j}^2 = 0$ only if

$$X(\epsilon_{i-1} - \epsilon_i) = X(\epsilon_{j-1} - \epsilon_j) = 0.$$

It follows that we can rearrange X by an appropriate product

$$u = \prod_{b_{i,j} \in \mathbb{C}} u_{i,j}^{b_{i,j}} \in G(s) \quad (b_{i,j} \in \mathbb{C}),$$

so that $u X_2 = X_2^0 + \dot{X}_2^{\sharp}$ satisfies

$$\dot{X}_2^{\sharp}(\epsilon_i + \epsilon_j) = 0$$

if $X(\epsilon_p - \epsilon_{p+1}) \neq 0$ $(i \le p \le j-1)$ holds. We replace X with uX. Notice that this operation is consistent with the operations in (Step 1) and (Step 2). Therefore, the repeated use of procedures in (Step 1)–(Step 3) terminates.

(Step 4) We can assume

$$X_2^0 = \sum_{i=0}^{n-1} \mathbf{v}(\mu_i)_i, \ \| X_2^{\sharp} \| \subset \| X_2^0 \|, \text{ and } X^{\sharp}(\epsilon_p + \epsilon_q) = 0 \text{ (if } p, q \in \| \mathbf{v}(\mu_i)_i \|)$$

for an appropriate choice of $\mu_i \in \mathbb{Z}_{\ge 0}$. Rearranging X_2 by the action of $N_{G_0}(T)$, we can assume $\mu_0 \ge \mu_i$ for every i. If $X(\epsilon_i + \epsilon_j) = 0$ for every $i \in [1, \mu_0]$ and $j \notin [1, \mu_0]$, then we forget $\mathbf{v}(\mu_0)_0$ and restrict our attention to $X_2 - \mathbf{v}(\mu_0)_0$ under the action of $Sp(2(n-\mu_0))$. Repeating this procedure, we deduce either $X_2^0 = X_2$ or $X(\epsilon_i + \epsilon_j) \neq 0$ for some $i \in [1, \mu_0]$ and $j \notin [1, \mu_0]$. By rearranging X by the action of $N_{G_0}(T)$, we assume that $j \in [p+1, n]$, where $\mu_p = n - p$. By means of $u_{i,j}$-action ($i \in [1, \mu_0]$ and $j \in [p+1, n]$), we can rearrange X if necessary to assume $X(\epsilon_i + \epsilon_j) = 0$ for $i \neq \mu_0$. Then, we have $(-)_{p+1} \cdots (-)_n \dot{w}_p X \in V_1 \oplus V_2^+$ and $n_{(-)_{p+1} \cdots (-)_n \dot{w}_p X} > n_X$. Since we have $n_X \le n$, a repeated use of this operation terminates. Repeating this procedure and that of (Step 1), we conclude that (s, X) is expressed as the form

$$s \in T, X_1 \in V_1 \text{ and } X_2^0 = X_2 = \sum_{i=0}^{n-1} \mathbf{v}(\mu_i')_i \tag{A.1}$$

up to G-conjugation.

(Step 5) By the description of (Step 4), it suffices to consider the case $X_2 = \mathbf{v}(n)_0$ or $\emptyset$. (Since here we consider all n.) In this setting, X is at worst $2n$-normal form, which implies the result. $\square$

By forgetting the semisimple element s in the proof of Theorem 1.10 at steps 1–4, we re-prove a result of Sekiguchi [Se84]:

Theorem A.1 (Sekiguchi cf. [Oh86] Proposition 1)**.** *The following assignment gives a one-to-one correspondence between the set of partitions of n and the set of G-orbits in $\mathfrak{N}_0$:*

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \mapsto [G(\sum_{i\geq 1} \mathbf{v}(\lambda_i)_{\lambda_i^{\leq}})] \in G\backslash\mathfrak{N}_0,$$

where $\lambda_i^{\leq} = \sum_{j\leq i}\lambda_i$.

Proof of Theorem 1.9. By Theorem A.1, it suffices to prove that $X_1 \oplus \mathbf{v}(n)_0$ is G-conjugate to a 1-normal form for every n and $X_1 \in V_1$. By a weight consideration, there exists elements $c_{i,j} \in \mathbb{C}$ such that

$$g_1 := u_{1,1}, g_2 := u_{1,2}, g_3 := u_{1,3}u_{2,2}^{c_{2,2}}, g_4 := u_{1,4}u_{2,3}^{c_{2,3}}, g_5 := u_{1,5}u_{2,4}^{c_{2,4}}u_{3,3}^{c_{3,3}}, \ldots, g_n$$

fixes $\mathbf{v}(n)_0$. Let $X_1 = X_1^+ \oplus X_1^- \in V_1^+ \oplus V_1^-$ be the direct decomposition. We have $g_p X_1^+ = X_1^+$ for each $p \in [1,n]$. The stabilizer U of $\mathbf{v}(n)_0$ in $SL(n)$ is a $(n-1)$-dimensional unipotent subgroup of $SL(n)$. By means of U-action, we assume that $X_1(-\epsilon_p) \neq 0$ holds for a unique $p \in [1,n]$. By rearranging X by $(-)_1 \cdots (-)_n \dot{w}_0^0$ if necessary before the U-conjugation, we further assume $X_1(\epsilon_{n-q}) = 0$ for $q \leq p$. Thus, we can choose appropriate $c_n, c_{n-1}, \ldots, c_p$ to obtain

$$g_n^{c_n} g_{n-1}^{c_{n-1}} \cdots g_p^{c_p} X_1(\epsilon_{n-q}) = 0$$

for each $1 \leq q \leq n$. Therefore, we obtain

$$(-)_1 \cdots (-)_n \dot{w}_0^0 g_n^{c_n} g_{n-1}^{c_{n-1}} \cdots g_p^{c_p} X = \mathbf{v}[\epsilon_{n-p}] + \mathbf{v}(n)_0$$

as desired. $\square$

Index of notation

(Sorted by the order of appearance)

$G, B, T, G(s), U_\alpha$	§1	$\star, \circ$	§1.1	$T_i, e^\lambda \in \mathbb{H}$	§2
$R, R^+, \mathbb{E}, \epsilon_i, \alpha_i$	§1	$X(\lambda), \mathbf{v}[\lambda]$ $(\lambda \neq 0)$	§1.2	$Z_{\leq w}, \mathbb{O}_i, \widetilde{T}_i, \ldots$	§2
$W, \dot{w} \in N_G(T), s_i, \ell$	§1	$\|X\|$: total support	§1.2	$\mathbb{H}_{\mathbf{a}}, F_+^{\mathbf{a}}, \mu_+^{\mathbf{a}}, \mathfrak{N}_+^{\mathbf{a}}, \ldots$	§2
${}^wH := \dot{w}H\dot{w}^{-1}$	§1	$\mathbf{v}^{\vec{\jmath},\sigma}(m)_i$: ℓ-block	§1.2	$\lambda = \sum \lambda_i \epsilon_i$	§3
$\mathfrak{g}, \mathfrak{t}, \mathfrak{g}(s), \mathfrak{u}_\alpha$	§1	$\mathbf{v}$: ℓ-normal form	§1.2	$\Gamma_0 \subset \Gamma$	§3
$V[\lambda], V^+, \Psi(V)$	§1	$\lvert\mathbf{v}\rvert$: support	§1.2	$\mathbf{c}$: a clan of $\mathbf{a}$	§3
$H_\bullet(\mathcal{X}), H_\bullet(\mathcal{X}, \mathbb{Z})$	§1	$\mathbf{a}$: pre-admissible	§1.2	$n^{\mathbf{c}}$: size of $\mathbf{c}$	§3
$V_1 = \mathbb{C}^n, V_2 = \wedge^2 V_1$	§1.1	$\mathbf{a}$: finite, general	§1.2	$\mathfrak{g}(s)_{\mathbf{c}}, G(s)_{\mathbf{c}}$	§3
$\mathbb{V}_\ell$: ℓ-exotic rep.	§1.1	ν: admissible param.	§1.2	$\mathbb{V}^{\mathbf{a}}, \mathbb{V}_{\mathbf{c}}^{\mathbf{a}}, F_+^{\mathbf{a}}, F_+^{\mathbf{a}}(w)$	§3
$F_\ell, \mu_\ell, \mathfrak{N}_\ell$	§1.1	$\nu \in \mathfrak{P}_{\mathbf{a}}$	§1.2	ν: regular param.	§3
$F, \mu, \mathfrak{N}, \ldots$	§1.1	ν: standard param.	§1.2	${}^w\mu_{\mathbf{c}}^{\mathbf{a}}, \Lambda_{\mathbf{a}}$	§3
$G_\ell, Z_\ell, p_i, \pi_\ell$	§1.1	$T_\ell, F_\ell^{\mathbf{a}}, \nu_\ell^{\mathbf{a}}, \mathfrak{N}_\ell^{\mathbf{a}}, \ldots$	§1.3	$G_{\mathbf{c}}, \mathbb{V}(\mathbf{c}), X_{\mathbf{c}}, \ldots$	§3
$\mathbb{C}_{\mathbf{a}}$	§1.1	$\mathbf{G} = G_2, \mathbf{T} = T_2, \mathcal{A}, \ldots$	§2	$\mathbf{c}_m, \mathbf{c}(\xi), \widetilde{\mathbf{c}}(\xi) \subset \mathbf{c}$	§4
$\mathsf{p}_w \in \mathsf{O}_w$	§1.1	$\mathbb{H}$: Hecke algebra	§2	$\mathbf{c}$: type I, II, III	§4

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