Deformations of nilpotent cones and Springer correspondences*

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Abstract

Let $G = Sp(2n)$ be the symplectic group over $\mathbb{Z}$. We present a certain kind of deformation of the nilpotent cone of G with G-action. This enables us to make direct links between the Springer correspondence of $\mathfrak{sp}_{2n}$ over $\mathbb{C}$, that over characteristic two, and our exotic Springer correspondence. As a by-product, we obtain a complete description of our exotic Springer correspondence.

Introduction

Let $G = Sp(2n)$ be the symplectic group over $\mathbb{Z}$. Let $\Bbbk$ be an algebraically closed field. Let $\mathfrak{g}$ be the Lie algebra of G defined over $\mathbb{Z}$. Let $\mathcal{N}$ denote the subscheme of nilpotent elements of $\mathfrak{g}$. Let $G_\Bbbk$, $\mathfrak{g}_\Bbbk$, and $\mathcal{N}_\Bbbk$ denote the specializations of G, $\mathfrak{g}$, and $\mathcal{N}$ to $\Bbbk$.

Springer [Spr76] defines a correspondence between the set of $G_\Bbbk$-orbits in $\mathcal{N}_\Bbbk$ and a certain set of Weyl group representations (with a basis) when $\mathsf{char}\Bbbk$ is good (ie. not equal to 2). This correspondence, together with the so-called "A-group data", lifts to a one-to-one correspondence.

This story is later deepen in two ways. One is Lusztig's generalized Springer correspondence [Lus84], which serves as a basis of his theories on Chevalley groups. The other is Joseph's realization [Jos83], which serves a model of the structure of the primitive spectrum of the enveloping algebra of $\mathfrak{g}_\mathbb{C}$.

In our previous papers [K06a, K06b], we found that a certain Hilbert nilcone $\mathfrak{N}$ gives a variant of one aspect of the above mentioned Lusztig's theory (cf. [KL87] and [Lus88]). Quite unexpectedly, our correspondence gives a one-to-one correspondence without the "A-group data", which is needed in the original Springer correspondence for Weyl groups of type C. Therefore, it seems natural to seek some meaning of $\mathfrak{N}$.

The main theme of this paper is to give one explanation of $\mathfrak{N}$. Roughly speaking, our conclusion is that $\mathfrak{N}$ is a model of $\mathcal{N}_{\mathbb{F}_2}$ over $\mathbb{Z}$, which is "better" than $\mathcal{N}$ in a certain sense.

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To see what we mean by this, we need a more precise formulation: Let T be a maximal torus of G. We define the Weyl group of (G,T) as $W := N_G(T)/T$. We denote the set of irreducible representations of W by $W^\vee$. Let V_1 be the vector representation. Put $V_2 := \wedge^2 V_1$. We denote $V_1 \oplus V_2$ by $\mathbb{V}$. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard choice of T-weight basis of G (see eg. Bourbaki [Bou02]). We denote the "positive part" of $\mathfrak{g}$ and $\mathbb{V}$ by $\mathfrak{n}$ and $\mathbb{V}^+$, respectively. (Cf. §1.2) Let $\mathfrak{N}$ be the Hilbert nilcone of $(G,\mathbb{V})$ over $\mathbb{Z}$. We have a natural map

$$\nu : G \times^B \mathbb{V}^+ \longrightarrow \mathfrak{N},$$

which we regard as a counter-part of the Springer resolution.

Theorem A. *The variety $\mathfrak{N}$ is normal and flat over $\mathbb{Z}$. Moreover, the number of $G_\Bbbk$-orbits of $\mathfrak{N}_\Bbbk$ is independent of the characteristic of $\Bbbk$.*

Theorem B. *Let $\Bbbk = \overline{\mathbb{F}}_2$. There exists a $G_\Bbbk$-equivariant flat family $\pi : \mathcal{N}_S \longrightarrow \mathbb{A}^1_\Bbbk$ with the following properties:*

1. *We have $\pi^{-1}(t) \cong \mathcal{N}_\Bbbk$ for $t \neq 0$;*
2. *There exists an isogeny $\mathsf{F}_1 : \mathfrak{N}_\Bbbk \longrightarrow \pi^{-1}(0)$, which is an endomorphism as varieties.*

Moreover, for a $G_\Bbbk$-orbit $\mathcal{O}_\Bbbk \subset \mathfrak{N}_\Bbbk$, there exists a flat subfamily of single $G_\Bbbk$-orbits $\mathcal{O}_S \subset \mathcal{N}_S$ such that $\mathcal{O}_S \cap \pi^{-1}(0) = \mathsf{F}_1(\mathcal{O}_\Bbbk)$.

Theorem A claims that our variety $\mathfrak{N}$ behaves well with respect to the specializations. Theorem B claims that we can regard $\mathfrak{N}$ as a model of $\mathcal{N}_\Bbbk$ in a certain sense.

To illustrate these, let us describe the orbit correspondence of Theorem B, together with the corresponding Springer correspondences:

Example C (The orbit correspondence for $n = 2$). We put $\alpha_1 := \epsilon_1 - \epsilon_2$. Let $\mathbf{x}[\lambda] \in \mathfrak{g}$ and $\mathbf{v}[\lambda] \in \mathbb{V}$ be T-eigenvectors with T-weight λ. We refer the Springer correspondence of $\mathcal{N}$ by ordinary and that of $\mathfrak{N}$ by exotic. Then, we have:

$W^\vee$	dim.	ordinary ($\mathsf{char}\Bbbk \neq 2$)	ordinary ($\mathsf{char}\Bbbk = 2$)	exotic
sign	1	0	0	0
Ssign	1	$\mathbf{x}[2\epsilon_1]$	$\mathbf{x}[2\epsilon_1]$	$\mathbf{v}[\epsilon_1]$
Lsign	1	$\mathbf{x}[\alpha_1]$	$\mathbf{x}[\alpha_1]$	$\mathbf{v}[\alpha_1]$
regular	2	$\mathbf{x}[\alpha_1]$	$\mathbf{x}[\alpha_1] + \mathbf{x}[2\epsilon_1]$	$\mathbf{v}[\alpha_1] + \mathbf{v}[\epsilon_1]$
triv	1	$\mathbf{x}[\alpha_1] + \mathbf{x}[2\epsilon_2]$	$\mathbf{x}[\alpha_1] + \mathbf{x}[2\epsilon_2]$	$\mathbf{v}[\alpha_1] + \mathbf{v}[\epsilon_2]$

Theorem B gives an isogeny between the Springer fibers of $\mathcal{N}_\Bbbk$ and $\mathfrak{N}_\Bbbk$ when $\mathsf{char}\Bbbk = 2$. This implies that the Springer correspondences associated to $\mathcal{N}_\Bbbk$ and $\mathfrak{N}_\Bbbk$ must coincide up to scalar multiplication of their basis.

To see this phenomena more closely, we employ the Joseph model of the Springer representations. Following Joseph [Jos83], we define the orbital variety attached to a $G_\mathbb{C}$-orbit $\mathbb{O}_\mathbb{C} \subset \mathcal{N}_\mathbb{C}$ as an irreducible component of the intersection $\mathbb{O}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C}$. Let us denote the set of orbital varieties attached to $\mathbb{O}_\mathbb{C}$ by $\mathrm{Comp}(\mathbb{O}_\mathbb{C})$. Similarly, let $\mathcal{O}_\mathbb{C} \subset \mathfrak{N}_\mathbb{C}$ be a $G_\mathbb{C}$-orbit and let $\mathrm{Comp}(\mathcal{O}_\mathbb{C})$ be the set of irreducible components of $\mathcal{O}_\mathbb{C} \cap \mathbb{V}^+_\mathbb{C}$. We also call a member of $\mathrm{Comp}(\mathcal{O}_\mathbb{C})$ an orbital variety (attached to $\mathcal{O}_\mathbb{C}$).

Joseph found that the T-equivariant Hilbert polynomials of $\mathrm{Comp}(\mathbb{O}_{\mathbb{C}})$ yield an irreducible W-module which is contained in the Springer representation attached to $\mathbb{O}_{\mathbb{C}}$. These polynomials are usually called the Joseph polynomials.

In view of Joseph [Jos89] (cf. Chriss-Ginzburg [CG97]), it is straightforward to see that Joseph's construction extends to the case of our exotic Springer correspondence. In particular, we have the notion of Joseph polynomials attached to each orbit of $\mathfrak{N}$.

Theorem D. *Let $\Bbbk = \overline{\mathbb{F}}_2$. Let $\mathbb{O}_{\mathbb{C}}$ be a $G_{\mathbb{C}}$-orbit of $\mathcal{N}_{\mathbb{C}}$. Then, there exists G-stable locally closed subsets $\mathbb{O} \subset \mathcal{N}$ and $\mathcal{O} \subset \mathfrak{N}$ such that*

1. *We have $\mathbb{O}_{\mathbb{C}} = \mathbb{O} \otimes \mathbb{C}$;*
2. *The set $\mathcal{O} \otimes \Bbbk$ is a single $G_{\Bbbk}$-orbit which corresponds to a unique dense open $G_{\Bbbk}$-orbit of $\mathbb{O} \otimes \Bbbk$;*
3. *The Joseph polynomials of $\mathcal{O}_{\mathbb{C}}$ and that of $\mathbb{O}_{\mathbb{C}}$ are equal up to scalar.*

It may worth to mention that there exists some orbit $\mathcal{O}$ of $\mathfrak{N}$ which does not correspond to an orbit of $\mathcal{N}_{\mathbb{C}}$. In this case, our version of Joseph polynomials realize Weyl group representations which cannot be realized by the usual Joseph polynomials. To illustrate these, we compare Joseph polynomials for $Sp(4)$:

Example E (Joseph polynomials for $n = 2$). Keep the setting of Example C. We have:

$W^\vee$	dim.	ordinary ($\mathsf{char}\Bbbk \neq 2$)	ordinary ($\mathsf{char}\Bbbk = 2$)	exotic
sign	1	$4\epsilon_1\epsilon_2(\epsilon_1^2 - \epsilon_2^2)$	$4\epsilon_1\epsilon_2(\epsilon_1^2 - \epsilon_2^2)$	$\epsilon_1\epsilon_2(\epsilon_1^2 - \epsilon_2^2)$
Ssign	1	$2(\epsilon_1^2 - \epsilon_2^2)$	$2(\epsilon_1^2 - \epsilon_2^2)$	$(\epsilon_1^2 - \epsilon_2^2)$
Lsign	1	N/A	$4\epsilon_1\epsilon_2$	$\epsilon_1\epsilon_2$
regular	2	$\alpha_1, 2\epsilon_2$	$\alpha_1, 2\epsilon_2$	α_1, ϵ_2
triv	1	1	1	1

Since our exotic Springer correspondence shares a similar flavor with the usual Springer correspondence of type A, it is natural to expect a combinatorial description. To state this, we need:

Definition F. Let (μ, ν) be a pair of partitions such that $|\mu| + |\nu| = n$. For a partition λ, we put $\lambda_i^< := \sum_{j<i} \lambda_j$ and $\lambda_i^\leq := \sum_{j \leq i} \lambda_j$. We define

$$\mathsf{D}_i^0(\mu) := \prod_{\mu_i^< < k < l \leq \mu_i^\leq} (\epsilon_k^2 - \epsilon_l^2),\ \mathsf{D}_i^+(\mu,\nu) := \prod_{\nu_i^< < k < l \leq \nu_i^\leq} (\epsilon_{k+|\mu|}^2 - \epsilon_{l+|\mu|}^2), \text{ and}$$

$$\mathsf{D}(\mu,\nu) := \prod_{i > |\mu|} \epsilon_i \times \prod_{i=1}^{\infty} \mathsf{D}_i^0(\mu)\mathsf{D}_i^+(\mu,\nu).$$

Theorem G. *For each G-orbit $\mathcal{O}$ of $\mathfrak{N}$, there exists a pair of partitions (μ, ν) such that there exists $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$ whose Joseph polynomial is a scalar multiplication of $\mathsf{D}(\mu, \nu)$.*

Since (μ, ν) in Theorem G is easily computable, this completes a determination of our exotic Springer correspondence. Taking account into Theorem D, we have determined some special Joseph polynomials which we cannot compute easily from their naive definitions.

The organization of this paper is as follows:

In §1, we fix convention and introduce our variety $\mathfrak{N}$. Then, we describe its set of defining equations in §2. Our system of defining equations is explicit and behaves nice with respect to the restriction to certain linear subvariety. These facts enable us to prove that $\mathfrak{N}$ is normal in §3. This proves the first part of Theorem A. Also, we introduce a parameterization of orbits of $\mathfrak{N}$ over $\mathbb{Z}$ or $\Bbbk$. The §4 contains the main observations of this paper. Namely, we observe:

- the adjoint representation $\mathfrak{g}$ of a symplectic group over characteristic two is not irreducible;
- this reducibility enables one to define a natural deformation of $\mathfrak{g}$, and its subvariety $\mathcal{N}$ in characteristic two;
- the special fiber such that (the deformation of) $\mathfrak{g}$ becomes decomposable is isogenous to $\mathbb{V}$;
- the above three observations are sufficient to construct a "deformation" from $\mathcal{N}$ (general fiber) to $\mathfrak{N}$ (special fiber) in characteristic two.

These observations enable us to prove Theorem B. In §5, we see that every orbit of $\mathfrak{N}_{\Bbbk}$ extends to an orbit of $\mathfrak{N}$ in order to prove the second part of Theorem A. The §§6–7 are devoted to the equi-dimensionality of the orbital varieties attached to $\mathfrak{N}$. Its proof is nothing but a minor modification of the Steinberg-Spaltenstein-Joseph theorem, which we present here for the reference purpose. (So I claim no originality here.) These are preparatory steps to the later sections. In §8, we use the results in the previous sections to prove Theorem D. With the help of previous sections and Joseph's theory, the only missing piece boils down to the rigidity of the torus character. In §9, we construct a special orbital variety from an orbit of $\mathfrak{N}$ in order to prove Theorem G. The main difficulty in the couse of its proof is that we cannot expect some orbital variety to be a linear subspace contrary to the type A case. We make a trick coming from the symmetry of Joseph polynomials to avoid this difficulty.

With the technique developed in this paper, a similar construction applied to $G^{\vee} = SO(2n+1)$ yields an analogue of Theorem D for special representations of the Weyl groups of $Sp(2n)$ and $SO(2n+1)$. However, the orbit correspondence is rather unclear since the number of orbits are different[1] (cf. [Hes79]). We hope to settle this in our future work.

Finally, one word of caution is in order. We work not over Spec$\mathbb{Z}$ but a neighborhood of Spec$\overline{\mathbb{F}}_2$ in the main body of this paper. The reason is that two is the only bad prime for symplectic groups and the corresponding statements are more or less trivial (or inexistent) with respect to the reduction to the other primes.

[1]This point is bit clarified by a recent result of Tian Xue [Xue08], who established the Springer correspondence in this setting.

1 Preliminaries

1.1 Convention

Consider a ring

$$A := \mathbb{Z}[p^{-1}, \zeta_N; p, N \in \mathbb{Z}_{>0}, (p,2) = 1, \zeta_N^{2^N-1} = 1] \subset \overline{\mathbb{Q}}.$$

This is a local ring with a unique maximal ideal (2). Let $\mathbb{K}$ be the quotient field of A and let $\Bbbk$ be the residual field of A. We have $\Bbbk = \overline{\mathbb{F}}_2$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we define $\lambda_i^< := \sum_{j<i} \lambda_j$ and $\lambda_i^> := \sum_{j>i} \lambda_j$ for each i. We also use the notation $\lambda_i^\geq := \lambda_{i-1}^>$ and $\lambda_i^\leq := \lambda_{i+1}^<$. We put $|\lambda| := (\lambda)_1^\geq$. We denote the dual partition of λ by ${}^t\lambda$.

For a scheme $\mathcal{X}$ over A, we denote its specializations to $\Bbbk$ and $\mathbb{K}$ by $\mathcal{X}_\Bbbk$ and $\mathcal{X}_\mathbb{K}$, respectively. In addition, assume that $\mathcal{X}$ admits an action of the group scheme G over A. By a G-orbit on $\mathcal{X}$, we refer a flat subfamily $\mathcal{O}$ of $\mathcal{X}$ over A such that $\mathcal{O}_\mathbb{K}$ is a single $G_\mathbb{K}$-orbit. For a map of commutative rings $A \to D$, we define $\mathcal{X}(D)$ the set of D-valued points of $\mathcal{X}$. We denote by $H_\bullet(\mathcal{X}, \mathbb{C})$ the Borel-Moore homology of $\mathcal{X}_\mathbb{C}$.

We understand that the intersection $\cap$ of two (sub-)schemes are set-theoretic. (I.e. we consider the reduced part of the scheme-theoretic intersection.) The scheme-theoretic intersection is denoted by $\dot{\cap}$.

For a scheme $\mathcal{Y}$ over $\Bbbk$, we denote its (geometric) Frobenius endomorphism by Fr. Here geometric means that the induced map $\mathsf{Fr}^* : \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{Y}$ is $\Bbbk$-linear and (suitable) local coordinates are changed to its 2nd power.

1.2 Notation and Terminology

Let $G = Sp(2n, A)$ be the symplectic group of rank n over A. Let $B \supset T$ be its Borel subgroup and a maximal torus defined over A. Put $N := [B, B]$. Denote the opposite unipotent radical of N (with respect to T) by N^-. Let $W := N_G(T)/T$ be the Weyl group of G. We denote by $X^*(T)$ the weight lattice of T. Let R be the set of roots of (G, T) with its positive part R^+ determined by B. Consider an A-module $V_1 := A^{2n}$, for which G acts by the multiplication of matrices. Let $V_2 := \wedge^2 V_1$ ($\subset \wedge^2 (V_1)_\mathbb{C}$) be the A-module with the natural G-module structure. Let $\mathfrak{g}$ be the Lie algebra of G over $\mathbb{Z}$, whose integral structure is $\mathsf{Sym}^2 V_1 = (V_1 \otimes_A V_1)^{\mathfrak{S}_2}$. Let $\mathfrak{b}, \mathfrak{t}, \mathfrak{n}$ be the intersections of Lie algebras corresponding to $B_\mathbb{C}, T_\mathbb{C}, N_\mathbb{C}$ with $\mathfrak{g}$ inside of $\mathfrak{g}_\mathbb{C}$, respectively.

Fix a $\mathbb{Z}$-basis $\epsilon_1, \ldots, \epsilon_n$ of $X^*(T)$ such that $R^+ = \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\}_i \subset X^*(T)$. For each i, we put $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(1 \le i < n)$, $2\epsilon_n$ $(i = n)$. Let s_i be the reflection of W corresponding to α_i. Let $\ell : W \to \mathbb{Z}$ denote the length function on W with respect to $s_1, \ldots, s_n$.

We put $\mathbb{V} := V_1 \oplus V_2$. Consider the sum $\mathbb{V}^+$ of T-weight spaces of $\mathbb{V}$ with its weights in $\mathbb{Q}_{\geq 0} R^+ - \{0\}$. For a T-weight $\lambda \neq 0$, we denote a non-zero T-eigenvector of $\mathbb{V}$ with T-weight λ by $\mathbf{v}[\lambda]$. (It is unique up to scalar.)

For each $w \in W$, we denote (one of) its lift by $\dot{w} \in N_G(T)$. For a T-stable subset $\mathcal{S}$ in $\mathbb{V}$ or $\mathfrak{g}$, we define ${}^w\mathcal{S} := \dot{w}\mathcal{S}$.

We denote the flag variety G/B by $\mathcal{B}$.

Let $\mathfrak{N}$ be the G-subscheme of $\mathbb{V}$ defined by the positive degree part $A[\mathbb{V}]^G_+$ of $A[\mathbb{V}]^G$.

Let $\mathcal{N}$ be the space of ad-nilpotent elements of $\mathfrak{g}$.

Theorem 1.1 (Hesselink [Hes79]). *We have $\mathcal{N}(\Bbbk) = \bigcup_{g \in G(\Bbbk)} \mathrm{Ad}(g)\mathfrak{n}(\Bbbk)$.*

Theorem 1.1 implies that the natural map (the Springer resolution over A)

$$\mu : G \times^B \mathfrak{n} \longrightarrow \mathcal{N}$$

is surjective at the level of points.

For a $G_\Bbbk$-module V over $\Bbbk$, we have its Frobenius twist $V^{[1]}$ as the composition map

$$G_\Bbbk \longrightarrow GL(V) \xrightarrow{\mathsf{Fr}} GL(V) \subset \mathrm{End}_\Bbbk(V).$$

2 Defining equations of $\mathfrak{N}$

Let $e \in T$ be an element such that $\epsilon_i(e) = c$ (for every $1 \le i \le n$), where $c \in A$ is an element with sufficiently high order after taking modulo two. In particular, we assume $Z_G(e) \cong GL(n, A)$, $V_1^e = \{0\}$, and $V_2^e \cong \mathrm{Mat}(n, A)$. Put $G_0 := Z_G(e)$. Consider a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \text{ and } \mathbb{V} = \mathbb{V}_{-2} \oplus \mathbb{V}_{-1} \oplus \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \tag{2.1}$$

determined by the eigenvalues of the action of e (indicated as subscript). Here $\mathfrak{g}_{\pm 2}$ and $\mathbb{V}_i$ $(-2 \le i \le 2)$ are G_0-modules. We have $\mathrm{Lie}(G_0)_\mathbb{C} \cap \mathfrak{g} = \mathfrak{g}_0 \cong \mathbb{V}_0$ as G_0-modules. Let $\mathfrak{n}_0 := \mathrm{Lie} G_0 \cap \mathfrak{n}$, which we may regard as a subspace of $\mathbb{V}_0$. We define G_{-2}, G_2, N_0 to be the unipotent subgroups of G corresponding to $\mathfrak{g}_{-2}, \mathfrak{g}_2$, and $\mathfrak{n}_0$, respectively. We fix an identification $\mathfrak{S}_n = N_{G_0}(T)/T \cong \langle s_i ; i < n \rangle \subset W$. We define

$$\mathbf{J} := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

We have $V_2 \cong \mathrm{Alt}(2n, A)$ as $GL(2n, A)$-modules. Hence, it restricts to a G-module isomorphism. Let Pf be the Pfaffian associated to $X = \{x_{ij}\}_{ij} = \{-x_{ji}\}_{ij} \in \mathrm{Alt}(2n)$. It is defined as

$$\mathsf{Pf}(X) = \frac{1}{n!} \sum_{\sigma} \mathrm{sgn}(\sigma) x_{\sigma(1)\sigma(2)} \cdots x_{\sigma(2n-1)\sigma(2n)}, \tag{2.2}$$

where σ runs over all permutations of $\mathfrak{S}_{2n}$ such that $\sigma(2m-1) < \sigma(2m)$ for every $1 \le m \le n$. By using Pf, we define polynomials $1 = P_0, P_1, \ldots P_n$ on V_2 as

$$\sum_{i=0}^{n} t^{n-i} P_i(X) = \mathsf{Pf}(t\mathbf{J} - X) \qquad (X \in V_2 \cong \mathrm{Alt}(2n)).$$

We have $P_i \in A[V_2]^G$ by

$$\mathsf{Pf}(t\mathbf{J} - X) = \det(g)\mathsf{Pf}(t\mathbf{J} - X) = \mathsf{Pf}(tg\mathbf{J}^t g - gX^t g) = \mathsf{Pf}(t\mathbf{J} - gX^t g).$$

Proposition 2.1. *By means of the G-module isomorphism $V_2 \cong \mathrm{Alt}(2n)$, the variety $(\mathfrak{N} \cap V_2)$ is identified with the common zeros of $P_1, \ldots, P_n$.*

Proof. Under the isomorphism $V_2 \cong \mathrm{Alt}(2n)$, the subspace $\mathbb{V}_0 \subset V_2$ corresponds to

$$\mathrm{Alt}(2n)_0 = \{\{x_{ij}\} \in \mathrm{Alt}(2n); x_{ij} = 0 \text{ for } 1 \le i, j \le n \text{ or } n < i, j \le 2n\}.$$

Substituting them into the definition of Pfaffians, we deduce

$$\mathsf{Pf}(t\mathbf{J} - X) = (-1)^n \det(t1_n - Y) \quad \text{where } X = \begin{pmatrix} 0 & -Y \\ {}^tY & 0 \end{pmatrix}.$$

This implies

$$A[V_2]^G \supset A[P_i; i \ge 1] \cong A[V_2[0]]^{\mathfrak{S}_n}$$

via the restriction map. By the Dadok-Kac classification [DK85] table 2, we know that

$$\mathbb{C}[V_2]^G \cong \mathbb{C}[\mathbb{V}_0]^{G_0} \cong \mathbb{C}[V_2[0]]^{\mathfrak{S}_n}.$$

This implies that $\{P_i\}_{i=1}^n$ generates the ideal $\langle A[V_2]_+^G \rangle$ as desired. $\square$

Corollary 2.2. *We have $\mathfrak{N} \cong A[\mathbb{V}]/(P_i; 1 \le i \le n)$.*

Proof. It is clear from the isomorphism $\mathbb{C}[\mathbb{V}]^G \cong \mathbb{C}[V_2]^G$, which can be read off from the Dadok-Kac classification ([DK85] table 2). $\square$

Lemma 2.3. *Let $Y \in \mathrm{Alt}(n)$ and let $Z \in \mathrm{Mat}(n)$. Then, we have*

$$P_i \begin{pmatrix} Y & Z \\ -{}^tZ & 0 \end{pmatrix} = P_i \begin{pmatrix} 0 & Z \\ -{}^tZ & 0 \end{pmatrix}$$

for each i.

Proof. By the pigeon hole principle, if a Pfaffian term

$$\mathrm{sgn}(\sigma) x_{\sigma(1)\sigma(2)} \cdots x_{\sigma(2n-1)\sigma(2n)}$$

in (2.2) satisfies $\sigma(2m-1), \sigma(2m) \le n$ for some m, then there exists m' such that $\sigma(2m'-1), \sigma(2m') > n$. Then, this term cannot contribute to $\mathsf{Pf}\begin{pmatrix} Y & Z \\ -{}^tZ & 0 \end{pmatrix}$, which implies the result. $\square$

Corollary 2.4. *We have $\mathfrak{N}(A) = (G\mathbb{V}^+)(A) \subset \mathbb{V}(A)$.*

Proof. By the proof of Proposition 2.1, the variety $(\mathfrak{N} \dot{\cap} \mathbb{V}_0)$ is isomorphic to the nilpotent cone of $\mathfrak{g}_0$. By Lemma 2.3, it follows that

$$(\mathfrak{N} \cap \mathbb{V}_0) \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 = \mathfrak{N} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2) \subset \mathfrak{N}.$$

We know

$$\mathbb{C}[\mathbb{V}]^G \cong \mathbb{C}[\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2]^{G_0B} \cong \mathbb{C}[\mathbb{V}_0]^{G_0}.$$

Here the inclusion $G(\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2) \subset \mathbb{V}$ is a closed subset. As a consequence, $G(\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2)(A) = \mathbb{V}(A)$ follows. Therefore, we deduce

$$(G((\mathfrak{N} \cap \mathbb{V}_0) \oplus \mathbb{V}_1 \oplus \mathbb{V}_2))(A) = \mathfrak{N}(A).$$

Here we have $\mathfrak{N} \cap \mathbb{V}_0 = G_0(\mathbb{V}_0 \cap \mathbb{V}^+)$, which implies the result. $\square$

Corollary 2.5. *We have $\mathfrak{N}(\Bbbk) = G(\Bbbk)\mathbb{V}^+(\Bbbk) \subset \mathbb{V}(\Bbbk)$.* $\square$

3 Geometric construction of $\mathfrak{N}$

We retain the setting of the previous section. Let $F := G \times^B \mathbb{V}^+$. Consider the map

$$\nu : F = G \times^B \mathbb{V}^+ \longrightarrow \mathbb{V}.$$

We denote the specialization of ν to $\mathbb{K}$ and $\Bbbk$ by $\nu_{\mathbb{K}}$ and $\nu_{\Bbbk}$, respectively. Since the fiber of ν is naturally isomorphic to a closed subscheme of a flag variety, ν is projective.

Lemma 3.1. *The map ν is semi-small with respect to the stratification given by G-orbits.*

Proof. This is a straight-forward generalization of the results in K [K06a] §1.1. □

Remark 3.2. By a result of Borho-MacPherson (cf. [CG97] §8.9), our exotic Springer correspondence (a bijection between $G_{\mathbb{C}}$-orbits of $\mathfrak{N}_{\mathbb{C}}$ and irreducible representations of W) implies that $\nu_{\mathbb{C}}$ must be strictly semi-small. (Otherwise there must be some $G_{\mathbb{C}}$-orbit which does not correspond to an irreducible representation of W.)

Proposition 3.3. *The differentials $dP_1, \ldots, dP_n$ of the polynomials $P_1, \ldots, P_n$ are linearly independent up to codimension two subscheme of $\mathfrak{N}$.*

Proof. By [K06a] 1.2 6), a non-dense orbit of $G_{\mathbb{K}}$ in $\mathfrak{N}_{\mathbb{K}}$ has codimension two. Hence it suffices to check the assertion for the open orbit of $\mathfrak{N}_{\Bbbk}$. (The existence of an open $G_{\Bbbk}$-orbit in $\mathfrak{N}_{\Bbbk}$ follows from Corollary 2.4 and a modification of [K06a] 1.8 or Theorem 4.1 in this paper.) The linear independence is an open condition. By the proof of Proposition 2.1, it suffices to prove that the assertion on the dense open $(G_0)_{\Bbbk}$-orbit of $(\mathfrak{N}_{\Bbbk} \cap (\mathbb{V}_0)_{\Bbbk})$, which is the regular nilpotent orbits for $GL(n)_{\Bbbk}$. This is well-known (or is easily checked). □

Corollary 3.4. *The scheme $\mathfrak{N}$ is regular in codimension one.*

Proof. The reduced induced scheme of $\mathfrak{N}$ is a complete intersection up to codimension two locus. □

Proposition 3.5. *The scheme $\mathfrak{N}$ is Cohen-Macaulay.*

Proof. We have

$$A[\mathbb{V}[0]]^{\mathfrak{S}_n} \cong A[\mathbb{V}]^G = A[P_1, \ldots P_n].$$

As a consequence, we deduce that $A[\mathbb{V}[0]]$ is a free $A[\mathbb{V}[0]]^{\mathfrak{S}_n}$-module by the Pittie-Steinberg theorem. It follows that the multiplication map $A[\mathbb{V}/\mathbb{V}[0]] \otimes_A A[\mathbb{V}]^G \to A[\mathbb{V}]$ equip $A[\mathbb{V}]$ the structure of a free $A[\mathbb{V}/\mathbb{V}[0]] \otimes_A A[\mathbb{V}]^G$-module. We have $A[\mathfrak{N}] \cong A[\mathbb{V}] \otimes_{A[\mathbb{V}]^G} A$. This is a free $A[\mathbb{V}/\mathbb{V}[0]]$-module, which implies that $\mathfrak{N}$ is Cohen-Macaulay. □

Corollary 3.6. *The scheme $\mathfrak{N}$ is flat over A.*

Proof. A free $A[\mathbb{V}/\mathbb{V}[0]]$-module is automatically flat over $A[\mathbb{V}/\mathbb{V}[0]]$. □

Theorem 3.7. *The scheme $\mathfrak{N}$ is normal.*

Proof. By the Serre criterion and Propositions 3.3 and 3.5, it suffices to show that $\mathfrak{N}$ is integral. The intersection

$$\mathfrak{N}\dot{\cap}(V_1 \oplus \mathbb{V}_0 \oplus \mathbb{V}_2)$$

is integral since $\mathfrak{N}\dot{\cap}\mathbb{V}_0$ is so. Let $I_1 = (P_i)$ and $I_2 := (x_{ij} = 0; i, j > n)$ be ideals of $A[\mathbb{V}]$, where $\{x_{ij}\} \in \mathrm{Alt}(2n) \cong V_2$. Then, the ideal $I := I_0 + I_1$ is prime. The set of A-valued points $T(A) \cong (A^\times)^n$ is Zariski dense in T. By the Bruhat decomposition, it follows that $G(A)$ is Zariski dense in G. Since $G(\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2) = \mathbb{V}$, this implies that

$$\begin{aligned} I_1 = I_1 + \bigcap_{g \in G(A)} (g^* I_2) &= \bigcap_{g \in G(A)} (I_1 + g^* I_2) = \bigcap_{g \in G(A)} g^* I \\ &= \bigcap_{g \in G(A)} (g^* I \otimes_A \mathbb{K}) \cap A[\mathbb{V}] = \langle \mathbb{K}[\mathbb{V}]_+^G \rangle \cap A[\mathbb{V}]. \end{aligned}$$

Since $\mathbb{K}[\mathfrak{N}]$ is integral, the RHS is an ideal whose quotient does not contain a zero divisor. □

Theorem 3.8. *The image of ν is equal to $\mathfrak{N}$.*

Proof. Since F is smooth over A, the A-algebra

$$B := \Gamma(\mathbb{V}, \nu_* \mathcal{O}_F) = \Gamma(F, \mathcal{O}_F)$$

is torsion-free over A. Hence, B is flat and integral over A.
We have $\mathbb{C} \otimes_A B \cong (\nu_{\mathbb{C}})_* \mathcal{O}_{F_{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{N}]$ by [K06a] 1.2 and the Zariski main theorem. By the proof of Theorem 3.7, we have

$$\left\langle A[\mathbb{V}]_+^G \right\rangle = \left\langle \mathbb{C}[\mathbb{V}]_+^{G_{\mathbb{C}}} \right\rangle \cap A[\mathbb{V}].$$

In particular, the natural map $A[\mathbb{V}] \ni f \mapsto f.1 \in B$ factors through $A[\mathfrak{N}]$. Here we have $(\mathrm{Im}\nu)(\Bbbk) = \mathfrak{N}(\Bbbk)$ as sets. This implies that B must be normal since $A[\mathfrak{N}]$ is so. □

Definition 3.9 (Marked partitions)**.** A marked partition $\boldsymbol{\lambda} = (\lambda, a)$ is a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of n, together with a sequence $a = (a_1, a_2, \ldots)$ of integers such that:

1. $0 \leq a_k \leq \lambda_k$ for each k;
2. $a_k = 0$ if $\lambda_{k+1} = \lambda_k$;
3. $\lambda_p - \lambda_q > a_p - a_q > 0$ if $p < q$ and $a_p \neq 0 \neq a_q$.

Let $X = X_1 \oplus X_2 \in \mathfrak{N}$. Since $V_1^* \cong V_1$, we can regard $X_2 \in \mathrm{End}(V_1)$ via the embedding

$$X_2 \in V_2 \xrightarrow{\cong} \mathrm{Alt}(V_1) \subset (V_1) \boxtimes (V_1) \cong \mathrm{End}_A(V_1). \tag{3.1}$$

Definition 3.10. Let $\boldsymbol{\lambda} = (\lambda, a)$ be a marked partition of n. An element $X \in \mathfrak{N}$ is said to have $\mathbb{K}$-invariant $\boldsymbol{\lambda}$ if the following conditions hold:

- X_2 is a nilpotent element with its Jordan type $(\lambda_1^2, \lambda_2^2, \ldots)$;

- There exists a family of vectors $\{\xi(i)\}_{i\geq 0} \in (V_1)_{\mathbb{K}}$ such that:
  1. $X_1 = \sum_j X_2^{\lambda_j - a_j}\xi(j)$;
  2. $X_2^{\lambda_i - 1}\xi(i) \neq 0$ and $X_2^{\lambda_i}\xi(i) = 0$ for each i such that $\xi(i) \neq 0$.

We say that $X \in \mathfrak{N}$ have $\Bbbk$-invariant $\boldsymbol{\lambda}$ if the same conditions hold by replacing every $\mathbb{K}$ by $\Bbbk$. The $\mathbb{K}$-invariant (resp. the $\Bbbk$-invariant) of X is denoted by $\boldsymbol{\lambda}(X_{\mathbb{K}}) = (\lambda(X_{\mathbb{K}}), a(X_{\mathbb{K}}))$ (resp. $\boldsymbol{\lambda}(X_{\Bbbk})$). We define $\mathcal{O}_{\boldsymbol{\lambda}}$ to be the locally closed subscheme whose $\mathbb{K}$-valued points shares the $\mathbb{K}$-invariant $\boldsymbol{\lambda}$.

It is standard that the $\mathbb{K}$-invariants and $\Bbbk$-invariants are invariants under the G-action.

Theorem 3.11 ([K06b] Theorem B)**.** *Two points $X, Y \in \mathfrak{N}_{\mathbb{K}}$ are $G_{\mathbb{K}}$-conjugate iff X and Y share the same $\mathbb{K}$-invariant. In particular, $(\mathcal{O}_{\boldsymbol{\lambda}})_{\mathbb{K}}$ is a single $G_{\mathbb{K}}$-orbit.* □

4 A geometric family of nilcones

We assume the same setting as in the previous section. The $G_{\Bbbk}$-module $\mathfrak{g}_{\Bbbk}$ has a non-trivial B-eigenvector with its highest weight $\epsilon_1 + \epsilon_2$. This yields the following short exact sequence of $G_{\Bbbk}$-modules:

$$0 \longrightarrow (V_2)_{\Bbbk} \longrightarrow \mathfrak{g}_{\Bbbk} \longrightarrow (V_1)_{\Bbbk}^{[1]} \longrightarrow 0. \tag{4.1}$$

Thus, we have a $G_{\Bbbk}$-equivariant flat deformation

$$\pi_V : \mathcal{V} \longrightarrow \mathbb{A}^1_{\Bbbk}$$

of $\mathfrak{g}_{\Bbbk}$ such that $\pi_V^{-1}(0) \cong (V_1)_{\Bbbk}^{[1]} \oplus (V_2)_{\Bbbk}$ and $\pi_V^{-1}(x) \cong \mathfrak{g}_{\Bbbk}$ ($x \neq 0$) as $G_{\Bbbk}$-modules. Let $\mathcal{V}^+ \subset \mathcal{V}$ be its $B_{\Bbbk}$-equivariant smooth subfamily such that $\mathcal{V}^+ \cap \pi_V^{-1}(0) \cong \left(V_1^+\right)_{\Bbbk}^{[1]} \oplus (V_2^+)_{\Bbbk}$ and $\mathcal{V}^+ \cap \pi_V^{-1}(x) \cong \mathfrak{n}_{\Bbbk}$ ($x \neq 0$).

By (4.1) and the isomorphism $\mathsf{Sym}^2 V_1 \cong \mathfrak{g}$ over A, we have the following $G_{\Bbbk}$-equivariant map

$$ml : \mathbb{V}_{\Bbbk} = (V_1)_{\Bbbk} \oplus (V_2)_{\Bbbk} \ni (X_1 \oplus X_2) \mapsto \mathsf{Sym}^2 X_1 + X_2 \in \mathfrak{g}_{\Bbbk}.$$

This map is $G_{\Bbbk}$-equivariant and finite as a map between affine algebraic varieties. By restriction, we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{A}^1_{\Bbbk} \times \mathbb{V}^+_{\Bbbk} \arrow[r, "\underline{\mathsf{ml}}"] & \mathcal{V}^+ \\ \{0\} \times \mathbb{V}^+_{\Bbbk} \arrow[u] \arrow[r, "\mathsf{F}_1"] & \{0\} \times (V_1^+)^{[1]}_{\Bbbk} \oplus (V_2^+)_{\Bbbk} \arrow[u] \end{tikzcd}, \tag{4.2}$$

where the map $\underline{\mathsf{ml}}$ is the natural prolongization of the map ml, and the map F_1 is the product of the Frobenius map of the first component of $\mathbb{V}^+_{\Bbbk}$ and the identity map of the second component of $\mathbb{V}^+_{\Bbbk}$.

Let $F := G_{\Bbbk} \times^{B_{\Bbbk}} \mathbb{V}_{\Bbbk}^{+}$ and $\mathcal{F} := G_{\Bbbk} \times^{B_{\Bbbk}} \mathcal{V}^{+}$. The latter is a flat family over $\mathbb{A}^1_{\Bbbk}$. We define $\mathcal{N}_S := G_{\Bbbk}\mathcal{V}^{+} \subset \mathcal{V}$. We define $\pi := \pi_V|_{\mathcal{N}_S}$. The diagram (4.2) gives a $G_{\Bbbk}$-equivariant commutative diagram

$$\begin{tikzcd} \mathbb{A}^1_{\Bbbk} \times F \arrow[r, "\widetilde{\mathsf{ml}}"] \arrow[d] & \mathcal{F} \arrow[d] & \{0\} \times F \arrow[l] \arrow[d, "\nu_{\Bbbk}"] \\ \mathbb{A}^1_{\Bbbk} \times \mathfrak{N}_{\Bbbk} \arrow[r, "\mathsf{ml}"] & \mathcal{N}_S & \{0\} \times \mathfrak{N}_{\Bbbk} \arrow[l] \end{tikzcd} . \tag{4.3}$$

Here the vertical arrows are defined as $G_{\Bbbk}$-translation of (4.2) inside $\mathbb{A}^1_{\Bbbk} \times \mathbb{V}_{\Bbbk}$, $\mathcal{V}$, or $\mathbb{V}_{\Bbbk}$, respectively. Thanks to the surjectivity of $\underline{\mathsf{ml}}$ at (4.2) and Theorem 1.1 and 3.8, the map ml at (4.3) is surjective at the level of points. Since $\mathcal{F}$ is flat over A, it follows that $\mathcal{N}_S$ is flat over $\mathbb{A}^1_{\Bbbk}$. Let $\mathsf{m} : \mathfrak{N}_{\Bbbk} \longrightarrow \mathcal{N}_{\Bbbk}$ be the map obtained by the specialization of ml to the fiber at the point $\{1\} \in \mathbb{A}^1_{\Bbbk}$.

Theorem 4.1. *Each $G_{\Bbbk}$-orbit $\mathcal{O}$ of $\mathfrak{N}_{\Bbbk}$ extends to a flat family of $G_{\Bbbk}$-orbits in $\mathcal{N}_S$ with its general fiber isomorphic to $\mathsf{m}(\mathcal{O})$. Moreover, this yields a one-to-one correspondence between $G_{\Bbbk}$-orbits of $\mathfrak{N}_{\Bbbk}$ and $\mathcal{N}_{\Bbbk}$ which preserves the closure relations.*

Proof. The map m is an isomorphism at the level of $\Bbbk$-valued points. Hence, we have an equi-dimensional one-to-one correspondence between the $G_{\Bbbk}$-orbits of $\mathfrak{N}_{\Bbbk}$ and $\mathcal{N}_{\Bbbk}$.

We have an equi-dimensional family $\mathbb{O}_S := \mathsf{ml}(\mathbb{A}^1_{\Bbbk} \times \mathcal{O})$ for each $G_{\Bbbk}$-orbit $\mathcal{O}$. Here the map F_1 is finite. As a result, $\overline{\mathbb{O}}_S$ is an equi-dimensional family such that each fiber contains a unique dense open $G_{\Bbbk}$-orbit. Therefore, each $G_{\Bbbk}$-orbit $\mathbb{O} \subset \mathcal{N}_{\Bbbk}$ determines a $G_{\Bbbk}$-orbit $\mathcal{O}' \subset \mathsf{F}_1(\mathfrak{N})$ via

$$\mathcal{O}' \overset{\text{open}}{\hookrightarrow} \mathsf{F}_1(\overline{\mathcal{O}}) = \pi^{-1}(0) \cap \overline{\mathcal{O}} \subset \overline{\mathbb{O}}_S = \overline{\pi^{-1}((\mathbb{A}^1_{\Bbbk} - \{0\}) \times \mathbb{O})}. \tag{4.4}$$

This establishes a one-to-finite correspondence between $G_{\Bbbk}$-orbits in $\mathfrak{N}_{\Bbbk}$ and that of $\mathcal{N}_{\Bbbk}$.

We have $\mathsf{F}_1(\mathcal{O})(\Bbbk) \cong \mathbb{O}(\Bbbk)$ as sets with $G(\Bbbk)$-actions. Therefore, the correspondences given by m and (4.4) are identical (and hence one-to-one). As a consequence, $\mathcal{O}_S$ must coincide with the desired family at the level of points.

Since $\mathbb{A}^1_{\Bbbk}$ is one-dimensional, taking quotient by the nilpotent ideal of $\overline{\mathcal{O}}_S$ yields the required flat family $\widetilde{\mathcal{O}}_S$. ☐

In the below, we denote the one-to-one correspondence from the set of $G_{\Bbbk}$-orbits of $\mathfrak{N}_{\Bbbk}$ to that of $\mathcal{N}_{\Bbbk}$ described in Theorem 4.1 by df.

Corollary 4.2. *The numbers of $G_{\Bbbk}$-orbits in $\mathcal{N}_{\Bbbk}$ and $\mathfrak{N}_{\Bbbk}$ are equal to the number of $G_{\mathbb{K}}$-orbits in $\mathfrak{N}_{\mathbb{K}}$.* ☐

Corollary 4.3. *Two $G_{\Bbbk}$-orbits in $\mathfrak{N}_{\Bbbk}$ are equal if and only if their $\Bbbk$-invariants are equal.* ☐

5 Normality of nilcones in characteristic two

We retain the setting of the previous section.

Theorem 5.1. *Let $X = X_1 \oplus X_2 \in \mathfrak{N}$ be a point with $\Bbbk$-invariant (λ, a). Let $\mathcal{O}_{\mathbb{K}} \subset \mathfrak{N}_{\mathbb{K}}$ be a $G_{\mathbb{K}}$-orbit with $\mathbb{K}$-invariant (λ, a). Then, there exists a $G_{\mathbb{K}}$-orbit $\mathcal{O}'_{\mathbb{K}}$ such that $X_{\mathbb{K}} \in \mathcal{O}'_{\mathbb{K}}$ and $\mathcal{O}_{\mathbb{K}} \subset \overline{\mathcal{O}'}_{\mathbb{K}}$ holds.*

Proof. Let $Y \in \mathcal{O}$ be a point with $\mathbb{K}$-invariant (λ, a) such that $Y_{\Bbbk} = (X_1 \oplus X_2)_{\Bbbk}$. First, we assume $X_1 = 0$. We regard $X_2 \in \mathrm{End}(V_1)_{\mathbb{K}}$ as in Definition 3.10. We have $({}^t\lambda(X))^{\le}_i := \frac{1}{2} \dim \ker X_2^i$. By the upper-semicontinuity of the dimensions of closed subsets, we have $({}^t\lambda(X))^{<}_i \le ({}^t\lambda)^{<}_i$ for each i. By the closure relation of G-orbits of V_2 (cf. Ohta [Oht86] (1.4)), we conclude the result when $X_1 = 0$. Now we consider the case $(X_1)_{\Bbbk} \neq 0$. We prove this case by the induction on the rank n of G. Hence, we assume (♣): the assertion holds for $Sp(2m)$ $(m < n)$ or $(X_1)_{\Bbbk} = 0$. By rearranging X and Y if necessary by the G-action, we can assume that $X_1 = Y_1 = \mathbf{v}[\epsilon_1]$. It suffices to prove the following assertion inductively on m (by also using the above induction hypothesis):

Claim 5.2. *Let $m \ge 0$ be an integer. Let*

$$Y_m := \mathbf{v}[\epsilon_1] + \sum_{p=1}^{m-1} \mathbf{v}[(-1)^p(\epsilon_p + \epsilon_{p+1})].$$

Here we understand that $Y_1 = \mathbf{v}[\epsilon_1]$. If we have

$$X, Y \in Y_m + \sum_{p>m} (V_2[\epsilon_1 \pm \epsilon_p] + V_2[(-1)^m \epsilon_m \pm \epsilon_p]) + \sum_{p,q>m} V_2[\pm\epsilon_p \pm \epsilon_q], \quad (5.1)$$

then we have $(GY)_{\mathbb{C}} \subset \overline{(GX)_{\mathbb{C}}}$.

Notice that $m = 1$ case of Claim 5.2 implies Theorem 5.1. We prove Claim 5.2. Define

$$V_+^m := \sum_{p>m} V_2[\epsilon_1 \pm \epsilon_p], V_-^m := \sum_{p>m} V_2[(-1)^m \epsilon_m \pm \epsilon_p], \text{ and } V^m := \sum_{p,q>m} V_2[\pm\epsilon_p \pm \epsilon_q].$$

We define $\mathbb{V}^m := V_+^m \oplus V_-^m \oplus V^m$. Let $P_m := \mathrm{Stab}_G(Y_m)$. We define

$$G_m := Sp(2(n-m)) \xrightarrow{1 \times \mathrm{id}} Sp(2m) \times Sp(2(n-m)) \hookrightarrow G,$$

where the middle factor contains T and $\epsilon_i(T \cap G_m) = 1$ for $1 \le i \le m$. Let U_m be the unipotent group generated by unipotent one-parameter subgroups of G with T-weights

$$(-1)^{m-1}\epsilon_m \pm \epsilon_p \quad (p > m), \text{ and } (-1)^{m-1} 2\epsilon_m.$$

It is clear that $G_m U_m \subset P_m$ and G_m normalizes U_m. In addition, there exists $g_m \in Z_{P_m}(G_m)$ such that

$$g_m \mathbf{v}[\dot{\epsilon}_i] = \begin{cases} \mathbf{v}[\epsilon_1] + \mathbf{v}[(-1)^m \epsilon_m] & (i = m, \dot{\epsilon}_m = (-1)^m \epsilon_m) \\ \mathbf{v}[\dot{\epsilon}_i] & (\mathit{otherwise}) \end{cases},$$

where $\dot{\epsilon}_i = \epsilon_i$ or $-\epsilon_i$. The element g_m is given as an appropriate product of elements of unipotent one-parameter subgroups with T-weights

$$\epsilon_1 + (-1)^{m-1}\epsilon_m, -\epsilon_2 + (-1)^{m-2}\epsilon_{m-1}, \epsilon_3 + (-1)^{m-3}\epsilon_{m-2}, \ldots.$$

Let $X = Y_m + X_+ + X_- + X_0$ and $Y = Y_m + Y_+ + Y_- + Y_0$ be the decomposition of X and Y corresponding to (5.1). It is unique by the T-weight consideration. Assume that $Y_+ = 0$ or $Y_- = 0$ holds. If $X_+ = 0$ ($Y_+ = 0$ case) or $X_- = 0$ ($Y_- = 0$ case) holds, then the assertion follows from the induction hypothesis ($\clubsuit$) applied to G_m and $Y - Y_m \in V_1^m \oplus V^m$. Here $V_1^m = V_+^m$ (if $X_- = 0 = Y_-$) or V_-^m (if $X_+ = 0 = Y_+$). We define $X' = X - X_*$ ($* = +$ if $Y_+ = 0$ and $-$ if $Y_- = 0$). If $X_+ \neq 0 \neq X_-$, then the assertion reduces to the induction hypothesis if the following subclaim holds:

Subclaim 5.3. *We have $X'_{\mathbb{C}} \in \overline{(GX)_{\mathbb{C}}}$.*

Proof of Subclaim 5.3. Let $\{\gamma_i\}_{i=1}^n$ be the basis of $\mathfrak{t}$ such that $\langle \epsilon_i, \gamma_j \rangle = \delta_{ij}$. Let $c \in \mathbb{Q} - \{0\}$. We put

$$s = \exp(\sum_{i=1}^{m} (-1)^i c\gamma_i) \in T(\mathbb{C}), \text{ and } q_1 = \exp(c), q_2 = 1 \in \mathbb{C}^\times.$$

Consider an action $a : \mathfrak{N}(\mathbb{C}) \ni Z_1 \oplus Z_2 \mapsto q_1 s Z_1 \oplus q_2 s Z_2 \in \mathfrak{N}(\mathbb{C})$. It is straightforward to see that a fixes $Y_m + X_0$ and $Y_m + Y_0$. Here the a-action dilates V_+^m by q_1^{-1} and dilates V_-^m by q_1. A nilpotent G_m-orbit of $V_1^m \oplus V_2^m$ is stable under the action of $\mathbb{G}_m^2$ defined as the scalar multiplications of V_1^m and V_2^m. It follows that there exists $a' \in G_m$ such that a' fixes $Y_m + X_0$, $aX_- = q_0^{\pm 1} X_-$, and moves $X_+ = X'_+ + X''_+$ (a sum of vectors in V_+^m) as $aX_+ = q_0 X'_+ + q_0^{-1} X''_+$. Letting q_0 and q_1 sufficiently generic, we conclude $Y_m + X_+ + X_0 \in \overline{\langle a, a' \rangle X}$. By swapping the roles of $X_\pm$, we conclude that $Y_m + X_+ + X_0, Y_m + X_- + X_0 \in \overline{(GX)_{\mathbb{C}}}$ as required. $\square$

We return to the proof of Claim 5.2. In the below, we always assume $Y_+ \neq 0 \neq Y_-$. This implies $X_+ \neq 0 \neq X_-$ since X is a lift of $Y_{\mathbb{k}}$. By rearranging X and Y by the G_m-action if necessary, we can assume $X_- = Y_- = \mathbf{v}[(-1)^m(\epsilon_m + \epsilon_{m+1})]$ without the loss of generality. By rearranging X and Y by means of the unipotent one-parameter subgroups of P_{m+1} with its weights $(-1)^{m+2}\epsilon_{m+1} \pm \epsilon_p$ ($p > m+1$), we assume that

$$X, Y \in Y_{m+1} + \mathbb{V}^{m+1} + V_2[\epsilon_1 \pm \epsilon_{m+1}].$$

By rearranging X and Y by some powers of $g_m \in P_m$, we assume that

$$X, Y \in Y_{m+1} + \mathbb{V}^{m+1} + V_2[\epsilon_1 + (-1)^{m+1}\epsilon_{m+1}].$$

Let ${}^+E_m := \sum_{i=1}^m V_1[(-1)^i \epsilon_i]$ and ${}^-E_m := \sum_{i=1}^m V_1[(-1)^{i+1}\epsilon_i]$. We define ${}^+E_m^\perp := {}^-E_m + \sum_{i>m} V_i[\pm\epsilon_i]$ and ${}^-E_m^\perp := {}^+E_m + \sum_{i>m} V_i[\pm\epsilon_i]$. We have composition maps

$${}^\sigma E_m \hookrightarrow V_1 \to V_1 \twoheadrightarrow V_1/{}^\sigma E_m^\perp \cong {}^\sigma E_m \text{ for } \sigma = \pm,$$

which yields a surjection $\theta_{m+1}^- : \mathrm{End}(V_1) \to \mathrm{End}({}^-E_{m+1})$. Since $X, Y \in \mathrm{End}(V_1)$ are nilpotent elements which preserve ${}^-E_{m+1}^\perp$, it follows that $\theta_{m+1}^-(X_2)$ and $\theta_{m+1}^-(Y_2)$ must be nilpotent. Let $Y'_{m+1} := \sum_{p=1}^m \mathbf{v}[(-1)^p(\epsilon_p + \epsilon_{p+1})]$. (This is the V_2-part of Y_{m+1}.) An element $Z \in \theta_{m+1}^-(Y'_{m+1} + V_2[\epsilon_1 + (-1)^{m+1}\epsilon_{m+1}])$ is nilpotent if and only if $Z = \theta_{m+1}^-(Y'_{m+1})$. Therefore, we deduce

$$X, Y \in Y_{m+1} + \mathbb{V}^{m+1},$$

which reduces the assertion from m to $m+1$. By repeating these reductions, the assertion follows from the case $m = n-1$. This case is easy since the resulting elements Y_{n-1} defines a dense open orbit in $\mathfrak{N}$. This completes the proof of Claim 5.2. Letting $m = 1$, we deduce Theorem 5.1 as desired. $\square$

Corollary 5.4. *For each $G_{\Bbbk}$-orbit $\mathcal{O}_{\Bbbk}$ of $\mathfrak{N}_{\Bbbk}$, there exists a G-orbit $\mathcal{O}$ of $\mathfrak{N}$ such that $\mathcal{O} \otimes \Bbbk = \mathcal{O}_{\Bbbk}$.*

Proof. Assume that $\mathcal{O}_{\Bbbk}$ has $\Bbbk$-invariant (λ, a). Let $\mathcal{O}$ be the G-orbit of $\mathfrak{N}$ with $\mathbb{K}$-invariant (λ, a). By the semi-continuity of the dimension, we have $\dim \overline{\mathcal{O} \otimes \Bbbk} \geq \dim \mathcal{O}_{\mathbb{K}}$. Here we have a section $s : \mathrm{Spec} A \to \overline{\mathcal{O}_{\mathbb{K}}}$, where the RHS is the closure in a scheme $\mathfrak{N}$ over A. Applying Lemma 3.1, we conclude that

$$\dim G_{\Bbbk} s(\Bbbk) \leq N - 2 \dim \nu_{\Bbbk}^{-1}(s(\Bbbk)) \leq N - 2 \dim \nu_{\mathbb{K}}^{-1}(s(\mathbb{K})) = \dim \mathcal{O}_{\mathbb{K}}.$$

Therefore, $\overline{\mathcal{O}_{\mathbb{K}}}$ is equi-dimensional over A. By Corollary 4.3 and Theorem 5.1, we have $\overline{\mathcal{O}_{\mathbb{K}}} \otimes \Bbbk = \overline{\mathcal{O}}_{\Bbbk}$. Since A is one-dimensional, taking the reduced quotient of

$$\mathcal{O} := \overline{\mathcal{O}_{\mathbb{K}}} - \bigcup_{\mathcal{O}' : G_{\mathbb{K}}\text{-orbit}; \mathcal{O}' \subset \overline{\mathcal{O}}} \overline{\mathcal{O}'_{\mathbb{K}}}$$

yields the result. $\square$

Corollary 5.5. *The variety $\mathfrak{N}_{\Bbbk}$ is normal.*

Proof. By the proof of Corollary 3.8, the space of defining equations of $\mathfrak{N}$ in $\mathbb{V}$ is flat. Hence, $\mathfrak{N}_{\Bbbk}$ is regular in codimension one. By Corollary 5.4, it follows that every non-dense $G_{\Bbbk}$-orbit in $\mathfrak{N}_{\Bbbk}$ is codimension two. Therefore, we deduce the result. $\square$

Theorem 5.6. *The variety $\mathcal{N}_{\Bbbk}$ is normal.*

Proof. The isogeny $ml : \mathbb{V} \mapsto \mathfrak{g}$ induces an injective map $\Bbbk[\mathfrak{g}_{\Bbbk}] \hookrightarrow \Bbbk[\mathbb{V}_{\Bbbk}]$. Since this map is $G(\Bbbk)$-equivariant, we have an inclusion $\Bbbk[\mathfrak{g}_{\Bbbk}]^G \hookrightarrow \Bbbk[\mathbb{V}_{\Bbbk}]^G$. The variety $\mathfrak{N}_{\Bbbk}$ is defined from $\mathfrak{N}$ by reduction modulo 2. Hence, its defining equations are coming from $\mathbb{Q}[\mathfrak{g}_{\mathbb{Q}}]^G$, which are given by polynomials of degree $2, 4, \ldots, 2n$ (cf. [Bou02]). Here we have $P_k(a_{ij}^2 - v_i^2 v_j^2) \in \Bbbk[\mathfrak{g}_{\Bbbk}] \cap \Bbbk[\mathbb{V}_{\Bbbk}]^G$, where $\{a_{ij}\}_{ij} \in \mathrm{Alt}(V_1)_{\Bbbk} \cong (V_2)_{\Bbbk}$ and $\{v_i, \overline{v}_i\}_i \in (V_1)_{\Bbbk}$ are the coordinates with respect to the T-eigenbasis. We assume that v_i is of weight ϵ_i and $v_{n+i} := \overline{v}_i$ is of weight $-\epsilon_i$ for each $1 \leq i \leq n$. By explicit calculation, we have

$$P_k(a_{ij}^2 - v_i^2 v_j^2) = \sum_{I \subset [1,n]; \# I = k} \prod_{i \in I} (v_i \overline{v}_i)^2 + \text{ lower terms with respect to } \{v_i\}.$$

It follows that the differentials of $P_k(a_{ij}^2 - v_i^2 v_j^2)$ with respect to $(V_1)_{\Bbbk}^{[1]} = \mathrm{Spec} \Bbbk[v_i^2]$ defines a collection of linear independent differentials along the generic point, regardless the values of a_{ij}.

By degree counting using the inclusions $\Bbbk[\mathbb{V}_{\Bbbk}^{[1]}] \subset \Bbbk[\mathfrak{g}_{\Bbbk}] \subset \Bbbk[\mathbb{V}_{\Bbbk}]$, these are precisely the defining equations of $\mathcal{N}_{\Bbbk}$ embedded into $\Bbbk[\mathbb{V}_{\Bbbk}]$. The union of non-dense orbit of $\mathcal{N}_{\Bbbk}$ has codimension two (cf. [Hes79] or Theorem 4.1). Therefore, the defining equations of $\mathcal{N}_{\Bbbk}$ defines a set of linearly independent differentials up to codimension two. In conclusion, the same proof as the normality of $\mathfrak{N}_{\Bbbk}$ implies the result. $\square$

Remark 5.7. The normality of the nilpotent cone of $\mathfrak{sp}(2n)$ over a field of characteristic $\neq 2$ is well-known. (See eg. Brion-Kumar [BK04] §5.)

6 Exotic orbital varieties: statement

In this section, the term "flat" means that the object is a flat scheme over Spec$A$.

Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$. We denote the set of irreducible components of $\mathcal{O} \cap \mathbb{V}^+$ by Comp$(\mathcal{O})$. We define Comp$(\mathbb{O})$ for a G-orbit of $\mathbb{O} \subset \mathcal{N}$ by replacing $\mathbb{V}^+$ with $\mathfrak{n}$.

Similarly, we denote the set of irreducible components of $\mathcal{O}_* \cap \mathbb{V}_*^+$ (or $\mathbb{O}_* \cap \mathfrak{n}_*$) by Comp$(\mathcal{O}_*)$ or Comp$(\mathbb{O}_*)$ for $* = A, \mathbb{K}, \Bbbk$.

An element of Comp$(\mathcal{O})$ or Comp$(\mathbb{O})$ is called an orbital variety.

Theorem 6.1. *Let $\mathcal{O}_{\mathbb{K}}$ be a $G_{\mathbb{K}}$-orbit. Let $\mathfrak{X}_{\mathbb{K}} \in$ Comp$(\mathcal{O}_{\mathbb{K}})$. Then, we have*

1. $\dim \mathfrak{X}_{\mathbb{K}} = \frac{1}{2} \dim \mathcal{O}_{\mathbb{K}}$*;*

2. *There exists $w \in W$ such that $\overline{\mathfrak{X}}_{\mathbb{K}} = \overline{B_{\mathbb{K}}(\mathbb{V}_{\mathbb{K}}^+ \cap {}^w\mathbb{V}_{\mathbb{K}}^+)}$.*

Proof. Postponed to §7. □

Remark 6.2. Theorem 6.1 is an "exotic" analogue of Joseph's version of the Steinberg-Spaltenstein theorem.

Corollary 6.3. *Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$. Then, every element of* Comp$(\mathcal{O})$ *has relative dimension $\frac{1}{2} \dim \mathcal{O}_{\mathbb{K}}$ over A.*

Proof. By Theorem 5.4, there exists a G-orbit $\mathcal{O}$ such that $\mathcal{O} \otimes \Bbbk \cong \mathbb{O}$. The scheme-theoretic intersection $\mathcal{O} \dot{\cap} \mathbb{V}^+$ is flat since each variety is flat. In particular, each member of Comp$(\mathcal{O})$ gives rise to (possibly several) irreducible components of $\mathbb{O}$ of the same dimension. Therefore, Theorem 6.1 implies the result. □

Corollary 6.4. *Let $\mathcal{O}_{\Bbbk}$ is a $G_{\Bbbk}$-orbit of $\mathcal{N}_{\Bbbk}$. Then, every element of* Comp$(\mathcal{O}_{\Bbbk})$ *has dimension $\frac{1}{2} \dim \mathcal{O}_{\Bbbk}$.* □

Remark 6.5. If the variety $G \times^B \mathbb{V}_{\mathbb{C}}^+$ nor $\mathcal{O}_{\mathbb{C}}$ admits a symplectic structure, then Theorem 6.1 follows from Kaledin [Kal06] nor [CG97]. However, there exists no G-invariant holomorphic symplectic form on both of them. We do not know whether it exists when we drop the invariance.

Lemma 6.6. *Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$ and let $\mathbb{O}$ be a G-orbit of $\mathfrak{N}$.*

1) *Let $\mathfrak{X} \in$ Comp$(\mathcal{O})$. The variety $\mathfrak{X}_{\Bbbk}$ is irreducible.*

2) *Let $\mathfrak{Y} \in$ Comp$(\mathbb{O})$. The variety $\mathfrak{Y}_{\Bbbk}$ is irreducible.*

Proof. Since the proof for $\mathbb{O}$ is obtained by the proof for $\mathcal{O}$ merely replacing the meaning of symbols as $\mathcal{O}$ by $\mathbb{O}$, $\mathfrak{X}$ by $\mathfrak{Y}$ and $\mathbb{V}^+$ by $\mathfrak{n}$, we provide the proof only for $\mathcal{O}$.

By theorem 6.1, we have a dense inclusion $\mathfrak{X}_{\mathbb{K}} \subset \overline{B(\mathbb{V}^+ \cap {}^w\mathbb{V}^+)}$ for some $w \in W$. We put

$$X_w := B \times^{(B \cap {}^w B)} (\mathbb{V}^+ \cap {}^w\mathbb{V}^+) \hookrightarrow G \times^{(B \cap {}^w B)} (\mathbb{V}^+ \cap {}^w\mathbb{V}^+).$$

Let $Z := \{(g_1B, g_2B, v) \in \mathcal{B}^2 \times \mathbb{V}; v \in g_1\mathbb{V}^+ \cap g_2\mathbb{V}^+\}$. We have an embedding

$$G \times^{(B \cap {}^w B)} (\mathbb{V}^+ \cap {}^w\mathbb{V}^+) \ni (g, v) \mapsto (gB, g\dot{w}B, v) \in Z.$$

Let $\overline{X_w}$ be the closure of X_w in Z. This is clearly an irreducible scheme. It is straight-forward to see

$$\widetilde{\nu} : Z \ni (g_1B, g_2B, v) \mapsto v \in \mathfrak{N}$$

gives a projective morphism. Since the image of a closed subset is closed under the projective map, we deduce that

$$\overline{\mathfrak{X}}_{\mathbb{K}} = \widetilde{\nu}(\overline{X_w})_{\mathbb{K}}.$$

By construction, $(X_w)_{\Bbbk}$ is irreducible. It follows that

$$\overline{\mathfrak{X}}_{\Bbbk} = \widetilde{\nu}((\overline{X_w})_{\Bbbk}).$$

This implies that $\overline{\mathfrak{X}}_{\Bbbk}$ is irreducible as desired. □

Definition 6.7. Let $X \in \mathfrak{N}$. Then, we define a set

$$\mathcal{G}_X := \{g \in G; X \in g\mathbb{V}^+\}.$$

It is clear that $\mathcal{G}_X$ admits a free left $\mathrm{Stab}_G(X)$-action and a free right B-action.

Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$. For each $X \in \mathcal{O}$, we define

$$\mathcal{E}_X := \mathcal{G}_X/B \subset \mathcal{B}$$

and call it the (exotic) Springer fiber along X. By taking conjugation, we know that $\mathcal{E}_X \otimes \mathbb{K} \cong \mathcal{E}_Y \otimes \mathbb{K}$ and $\mathcal{E}_X \otimes \Bbbk \cong \mathcal{E}_Y \otimes \Bbbk$ holds if $X, Y \in \mathcal{O}$.

Lemma 6.8. *Keep the setting of Definition 6.7. Let $\mathcal{O}$ be a G-orbit such that $X \in \mathcal{O}$. Let $\{\mathcal{G}_X^i\}_i$ be the set of irreducible components of $\mathcal{G}_X$. Then, the assignment*

$$\mathrm{Comp}(\mathcal{O}) \ni \mathcal{G}_X^i X \mapsto \mathcal{G}_X^i/B \subset \mathcal{E}_X$$

establishes one-to-one correspondences between the sets of irreducible components of $\mathcal{O} \cap \mathbb{V}^+, \mathcal{G}_X$, and $\mathcal{E}_X$.

Proof. The assignments $\mathcal{G}_X^i \mapsto \mathcal{G}_X^i/\mathrm{Stab}_G(X) \cong \mathcal{G}_X^i X \in \mathrm{Comp}(\mathcal{O})$ and $\mathcal{G}_X^i \mapsto \mathcal{G}_X^i/B$ gives a surjection from the set of irreducible components of $\mathcal{G}_X$ and the other two sets. Hence, these assignments fail to be bijective only if $\mathrm{Stab}_G(X)_{\mathbb{K}}$ or $B_{\mathbb{K}}$ is not connected. The group $\mathrm{Stab}_G(X)_{\mathbb{K}}$ is connected by [K06b] 6.2. The Borel subgroup $B_{\mathbb{K}} \subset G_{\mathbb{K}}$ is clearly connected. □

Corollary 6.9. *Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$ and let $X \in \mathcal{O}$ be a point. The variety $\mathcal{E}_X$ is an equi-dimensional scheme over A.*

Proof. By Theorem 6.1, all connected components of $\mathcal{G}_X$ share the same relative dimension $\frac{1}{2}\dim\mathcal{O} + \dim\mathrm{Stab}_G(X)$ over A. Since the specialization to $\mathbb{K}$ or $\Bbbk$ does not change the relative dimensions over the base, we conclude the result. □

7 Exotic orbital varieties: proof

This section is devoted to the proof of Theorem 6.1.

The proof itself is a modification of the arguments of Steinberg [Ste74], Spaltenstein [Spa77], and Joseph [Jos83]. The only essential diffusion in the proof is contained in the strict semi-smallness of the map ν, which is proved in [K06b]. Since the literature is little scattered, we provide a proof with its necessary modifications.

In the below, we assume the same settings as in Theorem 6.1, but we drop the subscript $_{\mathbb{K}}$ for the sake of simplicity.

Lemma 7.1. *We have* $\dim \mathfrak{X} \le \frac{1}{2} \dim \mathcal{O}$. *Moreover, there exists* $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$ *which satisfies the equality.*

Proof. Let $X \in \mathfrak{X}$. We have

$$\frac{1}{2} \dim G\mathfrak{X} + \dim \nu^{-1}(X) = \dim G/B$$

by the semi-smallness of ν. Since $\nu^{-1}(X) = \mathcal{G}_X/B$, we have

$$\frac{1}{2} \dim G\mathfrak{X} + \dim \mathcal{G}_X - \dim B = \dim G/B.$$

We have $\mathfrak{X} \subset \mathcal{G}_X/\mathrm{Stab}_G(X)$. In particular, we have

$$\frac{1}{2} \dim G\mathfrak{X} + \dim \mathfrak{X} + \dim \mathrm{Stab}_G(X) \le \dim G.$$

Therefore, we have

$$\dim \mathfrak{X} \le \dim G - \dim \mathrm{Stab}_G(X) - \frac{1}{2} \dim G\mathfrak{X} = \frac{1}{2} \dim \mathcal{O}, \tag{7.1}$$

which proves the first assertion. The second assertion follows by choosing $\mathfrak{X}$ so that $\dim \mathfrak{X} = \dim \mathcal{G}_X/\mathrm{Stab}_G(X)$. □

Proposition 7.2. *Assume that* $\dim \mathfrak{X} = \frac{1}{2} \dim \mathcal{O}$. *Then there exists* $w \in W$ *such that*

$$\mathfrak{X} \subset \overline{B(\mathbb{V}^+ \cap {}^w\mathbb{V}^+)}$$

is a dense open subset.

Proof. Let $X \in \mathfrak{X}$. We assume that $\mathfrak{X} \cong \mathcal{G}^i X$ for an irreducible component $\mathcal{G}^i$ of $\mathcal{G}_X \subset G$. We put $\mathcal{E}_X^i := \mathcal{G}^i/B$, which is an irreducible component of $\mathcal{E}_X$. By $\dim \mathfrak{X} = \frac{1}{2} \dim \mathcal{O}$, it follows that $\mathcal{E}_X^i$ has the maximal dimension among the irreducible components of $\mathcal{E}_X$. In other words, we have

$$\dim \mathcal{E}_X^i = \frac{1}{2}(\dim \mathfrak{N} - \dim \mathcal{O}). \tag{7.2}$$

Consider the variety

$$\mathcal{S} := \{(g_1 B, g_2 B, v) \in \mathcal{B} \times \mathcal{B} \times \mathbb{V}; v \in g_1 \mathbb{V}^+ \cap g_2 \mathbb{V}^+ \cap \mathcal{O}\}$$

and its subvarieties

$$\mathcal{S}_w := \{(g_1 B, g_2 B, v) \in \mathcal{S}; g_1^{-1} g_2 \in B\dot{w}B\}$$

for each $w \in W$. It is straight-forward to check $\mathcal{S} = \sqcup_{w \in W} \mathcal{S}_w$ (the arguments in [Ste74] p133 L14-L20 works merely by changing the meaning of the symbols appropriately). By considering the third projection $p_3 : \mathcal{S} \to \mathcal{O}$, we deduce that

$$p_3^{-1}(X) \cong \mathcal{E}_X \times \mathcal{E}_X.$$

Consider the projection $p_{12} : \mathcal{S} \to \mathcal{B} \times \mathcal{B}$ of $\mathcal{S}$ to the first two components. By definition, we have $p_{12}(\mathcal{S}_w) = G([B \times \dot{w}B])$. It follows that

$$\begin{aligned} \dim \mathcal{S}_w &= \dim \mathcal{B} + \ell(w) + \dim(\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O}) \\ &\le \dim \mathcal{B} + \ell(w) + \dim(\mathbb{V}^+ \cap {}^w\mathbb{V}^+) = \dim G - n = \dim \mathcal{S}. \end{aligned}$$

Define

$$\mathcal{S}^{i,i} := G(\mathcal{E}_X^i \times \mathcal{E}_X^i \times \{X\}) \subset G p_3^{-1}(X) \subset \mathcal{S}.$$

This is an irreducible component of $\mathcal{S}$. Since $G p_3^{-1}(X) = \mathcal{S}$ and (7.2), we conclude

$$\dim \mathcal{S}^{i,i} = \dim \mathcal{O} + 2 \dim \mathcal{E}_X^i = \dim \mathcal{O} + 2 \times \frac{1}{2} \mathrm{codim} \mathcal{O} = \mathfrak{N} = \dim G - n.$$

There exists $w \in W$ such that $\mathcal{S}_w \cap \mathcal{S}^{i,i} \subset \mathcal{S}^{i,i}$ is a dense open subset. By dimension counting, we deduce $\mathcal{S}^{i,i} = \overline{\mathcal{S}}_w$. Now we have

$$\mathcal{S}_w = \{(g_1 B, g_2 B, X); g_1^{-1} g_2 \in B \dot{w} B, X \in g_1(\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O})\}. \tag{7.3}$$

Since $\dim \mathcal{S} = \dim \mathcal{S}^{i,i} = \dim \mathcal{S}_w$, we deduce that

$$\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O} \subset \mathbb{V}^+ \cap {}^w\mathbb{V}^+$$

is dense.

Consider the image $\mathcal{G}_w$ of $\mathcal{S}_w$ under the first and third projection $p_{13} : \mathcal{S} \to \mathcal{B} \times \mathbb{V}$. Its second projection $q_3 : \mathcal{G}_w \to \mathcal{O}$ satisfies $q_3 \circ p_{13} = p_3$. In the RHS of (7.3), g_2 plays no rôle for the restriction on X. Therefore, we deduce $q_3^{-1}(X) \subset \mathcal{E}_X^i$ (dense open subset). By construction, we have

$$\mathcal{G}_w = \{(gB, X); X \in g(\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O})\}.$$

As a consequence, we deduce

$$\mathfrak{X} = \mathcal{G}^i X \subset \overline{\{g^{-1} X; (gB, X) \in \mathcal{G}_w\}} = \overline{B(\mathbb{V}^+ \cap {}^w\mathbb{V}^+)}.$$

Since the second inclusion is dense by construction, we conclude the result. $\square$

Lemma 7.1 and Proposition 7.2 claim that the both assertions of Theorem 6.1 hold for at least one $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$. To derive Theorem 6.1 for general irreducible components, we need some preparation:

For each $1 \le i \le n$, we put $\mathbb{V}(i) := (\mathbb{V}^+ + {}^{s_i}\mathbb{V}^+)$. We define $\mathbb{V}_i^+ := \mathbb{V}^+ / (\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$ and $\mathbb{V}_i := \mathbb{V}(i) / (\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$.

For each $1 \le i \le n$, we put $P_i := B \dot{s}_i B \cup B$. It is a parabolic subgroup of G.

The derived group of the Levi part of P_i is isomorphic to $SL(2)$. Its action on $\mathbb{V}_i$ is equivalent to either $\mathfrak{sl}(2)$ (adjoint representation, $1 \le i < n$) or $\mathbb{K}^2$ (vector representation, $i = n$).

Since every $\mathbb{V}_i^+$ and $(\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$ are P_i-stable, it follows that $\mathbb{V}_i$ admits a natural P_i-action. Let $\pi_i : \mathbb{V}(i) \longrightarrow \mathbb{V}_i$. The map π_i is P_i-equivariant. We define $\mathfrak{X}_i := \pi_i(\mathfrak{X})$.

Lemma 7.3. *Let $1 \le i \le n$. Assume that the both assertions of Theorem 6.1 hold for $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$. Then, $P_i\mathfrak{X} \cap \mathbb{V}^+$ is a union of irreducible components of $\mathcal{O}$ which satisfy the both assertions of Theorem 6.1.*

Proof. By Proposition 7.2, it suffices to prove Theorem 6.1 1).

By construction, $\mathfrak{X}_i \subset \mathbb{V}_i^+$ is a B-stable subset. We have $\dim \mathbb{V}_i^+ = 1$. Hence, we have $\mathfrak{X}_i = \{0\}$ or $\overline{\mathfrak{X}}_i = \mathbb{V}_i^+$. If $\mathfrak{X}_i = \{0\}$, then $\mathfrak{X}$ is P_i-stable. Thus, the assertion trivially holds.

Therefore, we concentrate ourselves to the case $\overline{\mathfrak{X}}_i = \mathbb{V}_i^+$ in the below. Let $\mathfrak{X}' \in \mathrm{Comp}(\mathcal{O})$ such that $\mathfrak{X}' \cap P_i\mathfrak{X} \not\subset \mathfrak{X}$. If $\mathfrak{X}'$ does not exist, then we have $P_i\mathfrak{X} \cap \mathbb{V}^+ = \mathfrak{X}$. Hence, the assertion trivially holds. Thus, we assume the existence of $\mathfrak{X}'$.

Let $\mathfrak{D} := \mathfrak{X} \cap \pi_i^{-1}(\{0\}) \subset \mathfrak{X}$. This is a purely codimension one subscheme of $\mathfrak{X}$. Since $\pi_i^{-1}(\{0\}) = (\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$ is P_i-stable, it follows that

$$P_i\mathfrak{D} \subset P_i\mathfrak{X} \cap \pi_i^{-1}(\{0\}) \subset \mathbb{V}^+.$$

Let $0 \neq X \in \mathbb{V}_i^+$. By an explicit $SL(2)$-computation, we have $gX \in \mathbb{V}_i^+$ ($g \in P_i$) if and only if $g \in P_i \cap B$. This implies

$$P_i(\mathfrak{X} - \mathfrak{D}) \cap \mathbb{V}^+ = \mathfrak{X} - \mathfrak{D}.$$

Hence, we have $\mathfrak{X}' \cap P_i\mathfrak{D} \neq \emptyset$. Let $\mathfrak{D}_0 \subset \mathfrak{D}$ be an irreducible component such that $\mathfrak{X}' \cap P_i\mathfrak{D}_0 \not\subset \mathfrak{D}$. We have necessarily $P_i\mathfrak{D}_0 \neq \mathfrak{D}_0$. This implies

$$\dim P_i\mathfrak{D}_0 = \dim \mathfrak{D}_0 + 1 = \dim \mathfrak{X}.$$

As a consequence, $\overline{P_i\mathfrak{D}_0}$ contains a (unique) element of $\mathrm{Comp}(\mathcal{O})$ which is different from $\mathfrak{X}$. Letting $\mathfrak{X}'$ and $\mathfrak{D}_0$ vary arbitrary, we conclude the result. ◻

In order to complete the proof of Theorem 6.1, it suffices to prove that a successive application of Lemma 7.3 eventually exhausts the whole of $\mathrm{Comp}(\mathcal{O})$. This is guaranteed by the following:

Proposition 7.4. *Let $\mathfrak{X}, \mathfrak{X}' \in \mathrm{Comp}(\mathcal{O})$. Assume that $\mathfrak{X}'$ satisfies the both assertions of Theorem 6.1. Then, there exists a sequence of integers $i_1, i_2, \ldots, i_m \in [1, n]$ and a sequence $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_m \in \mathrm{Comp}(\mathcal{O})$ such that*

$$\mathfrak{X}' = \mathfrak{X}_1, \;\; \mathfrak{X}_m = \mathfrak{X}, \;\; \textit{and } \mathfrak{X}_{k-1} \subset P_{i_k}\mathfrak{X}_k$$

hold for every $2 \le k \le m$.

Proof. Let $i_1, \ldots i_m \in [1, n]$ be a sequence of integers such that

$$\mathfrak{X}' \subset P_{i_1}P_{i_2}\cdots P_{i_m}\overline{\mathfrak{X}}. \tag{7.4}$$

We assume that ($\star$): $\mathfrak{X}' \not\subset P_{i'_1}P_{i'_2}\cdots P_{i'_{m'}}\mathfrak{X}$ does not holds for any sequence $i'_1, \ldots, i'_{m'}$ if $m' < m$. This implies that $s_{i_1}s_{i_2}\cdots s_{i_m}$ is a reduced expression. We prove that there exists a sequence $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_m \in \mathrm{Comp}(\mathcal{O})$ which satisfies the required condition. By (7.4) and ($\star$), we deduce

$$\mathfrak{Z} := B\dot{s}_{i_1}\mathfrak{X}' \cap B\dot{s}_{i_2}B\cdots B\dot{s}_{i_m}\overline{\mathfrak{X}} \neq \emptyset.$$

Claim 7.5. *We have* $\dim \mathfrak{Z} = \dim \mathfrak{X}'$.

Proof. By ($\star$), we have $P_{i_1}\mathfrak{X}' \neq \mathfrak{X}'$. Hence, we have

$$P_{i_1}\mathfrak{X}' = \dim \mathfrak{X}' + 1$$

Since $\mathfrak{Z}$ is an open subset of a codimension one subscheme of $P_{i_1}\mathfrak{X}'$, we deduce the result. $\square$

We return to the proof of Proposition 7.4.

We have $B\dot{s}_{i_1}\mathfrak{X}' \subset \mathbb{V}(i_1)$. We put $w = s_{i_2} \cdots s_{i_m}$. This is a reduced expression. In particular, we deduce that

$$B\dot{s}_{i_2}B \cdots B\dot{s}_{i_m}\overline{\mathfrak{X}} \subset B\dot{w}\mathbb{V}^+.$$

Since $\ell(w) < \ell(s_{i_1}w)$, we have $\mathbb{V}[-\alpha_{i_1}] \not\subset \dot{w}\mathbb{V}^+$. It follows that $(\mathbb{K}^\times \mathbf{v}[-\alpha_{i_1}] + \mathbb{V}^+) \cap B\dot{w}\mathbb{V}^+ = \emptyset$ by a weight comparison. Hence, we have $\mathbb{V}(i_1) \cap B\dot{w}\mathbb{V}^+ \subset \mathbb{V}^+$. This implies that

$$B\dot{s}_{i_1}\mathfrak{X}' \cap B\dot{s}_{i_2}B \cdots B\dot{s}_{i_m}\overline{\mathfrak{X}} \subset \mathbb{V}^+.$$

In particular, there exists an irreducible component $\mathfrak{X}'' \subset P_{i_1}\mathfrak{X}' \cap \mathbb{V}^+$ such that

$$\mathfrak{X}'' \cap B\dot{s}_{i_2}B \cdots B\dot{s}_{i_m}\overline{\mathfrak{X}} \neq \emptyset$$

and the LHS is a maximal dimensional irreducible component of $\mathfrak{Z}$. By Lemma 7.3 and the equality $\dim \mathfrak{Z} = \dim \mathfrak{X}'$, we conclude that

$$\mathfrak{X}'' \subset \overline{B\dot{s}_{i_2}B \cdots B\dot{s}_{i_m}\overline{\mathfrak{X}}} = P_{i_2}P_{i_3} \cdots P_{i_m}\overline{\mathfrak{X}}.$$

Since the assertion for $m = 1$ is proved in Lemma 7.3, the downward induction on m yields the result. $\square$

8 Comparison of Springer correspondences

Let $\mathbb{O}_{\mathbb{K}}$ be a $G_{\mathbb{K}}$-orbit of $\mathcal{N}_{\mathbb{K}}$. We define

$$\mathcal{B}_X := \{g \in G_{\mathbb{K}}; X \in \mathrm{Ad}(g)\mathfrak{n}_{\mathbb{K}}\}/B_{\mathbb{K}} \subset \mathcal{B}_{\mathbb{K}}$$

and call it the Springer fiber along X.

Lemma 8.1. *Let $X \in \mathfrak{N}_{\Bbbk}$. Then, we have*

$$\mathcal{E}_X = \mathcal{B}_{\mathsf{m}(X)} \subset \mathcal{B}_{\Bbbk}.$$

In particular, the Springer fiber of $\mathfrak{N}_{\Bbbk}$ has an isogeny to that of $\mathcal{N}_{\Bbbk}$.

Proof. The orbital variety for $\mathcal{N}_{\Bbbk}$ is equi-dimensional since $\mathrm{Comp}(\mathcal{O}_{\Bbbk})$ is equi-dimensional for all $G_{\Bbbk}$-orbits $\mathcal{O}_{\Bbbk} \subset \mathfrak{N}_{\Bbbk}$ and both $\mathsf{F}_1(\mathbb{V}_{\Bbbk}^+)$ and $\mathcal{O}_{\Bbbk}$ are special fibers of flat families over $\mathbb{A}^1_{\Bbbk}$ (cf. Theorem 4.1). By the same argument as in the proof of Theorem 6.9, we deduce that $\mathcal{B}_{\mathsf{m}(X)}$ is an equi-dimensional scheme with the same dimension as $\mathcal{E}_X$. We have $\mathsf{m}(\mathcal{E}_X)(\Bbbk) \subset \mathcal{B}_{\mathsf{m}(X)}(\Bbbk)$. Here the map $\widetilde{\mathsf{ml}}|_{\mathcal{E}_X}$ is surjective along the zero fiber. As a consequence, we deduce that $\widetilde{\mathsf{ml}}|_{\mathcal{E}_X}$ is surjective along $\mathbb{A}^1_{\Bbbk}$ at the level of points. Since F_1 does not change a point in $\mathcal{B}$, the result follows. $\square$

Corollary 8.2. *Every Springer fiber of $\mathcal{N}_{\Bbbk}$ is equi-dimensional.* $\square$

Let $\mathcal{X}$ be a T-equivariant scheme over A. Let $K^{T_*}(\mathcal{X}_*)_\mathbb{Q}$ be the $\mathbb{Q}$-coefficient Grothendieck group of T_*-equivariant coherent sheaves on $\mathcal{X}_*$ which are flat over the base ($* = A, \mathbb{K}, \Bbbk$). Let $R(T)_\mathbb{Q}$ be the representation ring of T with coefficient $\mathbb{Q}$. For a T-module V, we define $\mathrm{ch}_T V$ to be the class $[V] \in R(T)_\mathbb{Q}$. Consider a map

$$p : K^T(\mathfrak{n})_\mathbb{Q} \longrightarrow R(T)_\mathbb{Q}$$

which sends a T-equivariant closed subset $C \subset \mathfrak{n}$ to the ratio

$$\mathrm{ch}_T \Gamma(\mathfrak{n}, \mathcal{O}_C)/\mathrm{ch}_T \Gamma(\mathfrak{n}, \mathcal{O}_\mathfrak{n}) \in R(T)_\mathbb{Q}.$$

Replacing T and $\mathfrak{n}$ with T_* and $\mathfrak{n}_*$ (where $* = A, \mathbb{K}, \Bbbk$), we define the corresponding maps p_*. Similarly, consider a map

$$q : K^T(\mathbb{V}^+)_\mathbb{Q} \longrightarrow R(T)_\mathbb{Q}$$

which sends a T-equivariant closed subset $C \subset \mathbb{V}^+$ to the ratio

$$\mathrm{ch}_T \Gamma(\mathbb{V}^+, \mathcal{O}_C)/\mathrm{ch}_T \Gamma(\mathbb{V}^+, \mathcal{O}_{\mathbb{V}^+}) \in R(T)_\mathbb{Q}.$$

Replacing T and $\mathbb{V}^+$ with T_* and $\mathbb{V}^+_*$ (where $* = \mathbb{K}, \Bbbk$), we define the corresponding maps q_*.

Let $\mathsf{fx} : R(T) \to \mathbb{C}[[\mathfrak{t}]]$ be the map given by the formal expansion of a function on T along 1. For $f \in R(T)$, we denote the lowest non-zero homogeneous term of $\mathsf{fx}(f)$ by $\mathsf{lt}(f)$. By definition, $\mathsf{lt}(f)$ is a homogeneous polynomial on $\mathfrak{t}$.

Let $\mathbb{O}$ be a G-orbit in $\mathcal{N}$. For each $\mathfrak{Y}_* \in \mathrm{Comp}(\mathbb{O}_*)$ ($* = A, \mathbb{K}, \Bbbk$), we define the Joseph polynomial attached to $\mathfrak{Y}_*$ as $\mathsf{lt}\left(p_*(\overline{\mathfrak{Y}}_*)\right)$. (This is identical to the T-equivariant Hilbert polynomial of $\mathfrak{Y}_*$.) Let $\mathcal{O}$ be a G-orbit in $\mathfrak{N}$. For each $\mathfrak{X}_* \in \mathrm{Comp}(\mathcal{O}_*)$ ($* = \mathbb{K}, \Bbbk$), we define the Joseph polynomial attached to $\mathfrak{X}_*$ as $\mathsf{lt}\left(q_*(\overline{\mathfrak{X}}_*)\right)$.

We denote the set of $\mathbb{Q}$-multiples of Joseph polynomials attached to orbital varieties of $\mathcal{O}_*$ or $\mathbb{O}_*$ ($* = A, \mathbb{K}, \Bbbk$) by $\mathrm{Jos}(\mathcal{O}_*)$ or $\mathrm{Jos}(\mathbb{O}_*)$, respectively.

Proposition 8.3. *Let $\mathcal{X} \subset \mathbb{V}^+$ and $\mathcal{Y} \subset \mathfrak{n}$ be T-equivariant flat subfamilies over A. Then, we have*

$$q_\mathbb{K}(\mathcal{X}_\mathbb{K}) = q_\Bbbk(\mathcal{X}_\Bbbk) \text{ and } p_\mathbb{K}(\mathcal{Y}_\mathbb{K}) = p_\Bbbk(\mathcal{Y}_\Bbbk).$$

Proof. Each character of tori is defined over A. In particular, every irreducible T-module is flat over A. Hence, the assumption implies that the coordinate rings $\mathbb{K}[\mathcal{X}_\mathbb{K}]$ and $\Bbbk[\mathcal{X}_\Bbbk]$ share the same character (as T-modules). Hence, we conclude the result for $\mathcal{X}$. The case $\mathcal{Y}$ is entirely the same. □

Proposition 8.4. *Let $\mathcal{C}$ be a $T_\Bbbk$-stable flat subfamily of $\mathcal{V}^+$ over $\mathbb{A}^1_\Bbbk$. Let $\mathcal{C}_t := C \dot{\cap} \pi^{-1}(t)$. Then, we have*

$$\mathsf{lt}(p_\Bbbk([\mathcal{C}_1])) \in \mathbb{Q}\mathsf{lt}(q_\Bbbk([\mathsf{F}_1^*(\mathcal{C}_0)])).$$

Proof. For the sake of simplicity, we drop the subscripts $_\Bbbk$ during this proof. A T-character does not admit a non-trivial deformation. It follows that the classes $[\mathcal{C}_1] \in K^T(\mathfrak{n})$ and $[\mathcal{C}_0] \in K^T(\mathsf{F}_1(\mathbb{V}^+))$ defines the same class after sending to $R(T)$. The pullback F_1^* is given by $\bullet \otimes_{\mathsf{F}_1^{-1}\mathcal{O}_{\mathsf{F}_1(\mathbb{V}^+)}} \mathcal{O}_{\mathbb{V}^+}$. It is clear that $\mathcal{O}_{\mathbb{V}^+}$ is a free $\mathsf{F}_1^{-1}\mathcal{O}_{\mathsf{F}_1(\mathbb{V}^+)}$-module of rank 2^n. As T-modules, it corresponds to tensoring some T-module. Therefore, the map

$$(\mathsf{F}_1)^* : K^{T_\Bbbk}(\mathsf{F}_1(\mathbb{V}^+)) \to K^{T_\Bbbk}(\mathbb{V}^+)$$

is given by multiplication of some T-characters. Since a character of a finite-dimensional T-module has non-zero leading term, we conclude the result. $\square$

Theorem 8.5 (Joseph [Jos83, Jos89])**.** *Let $\mathcal{O}$ be a G-orbit of $\mathcal{N}$ and let $\mathbb{O}$ be a G-orbit of $\mathfrak{N}$. The $\mathbb{C}$-span of* $\mathrm{Jos}(\mathcal{O}_\mathbb{K})$ *or* $\mathrm{Jos}(\mathbb{O}_\mathbb{K})$ *form an irreducible W-module.*

Proof. The proof for the case $\mathbb{O}_\mathbb{K}$ is the original case and is treated in [CG97] 6.5.13 and 7.4.1. The case $\mathcal{O}_\mathbb{K}$ follows from the same construction as in [CG97] 6.5 and 7.4 if we replace $\mathfrak{n}_\mathbb{K}$ with $\mathbb{V}^+_\mathbb{K}$, $\mathcal{N}_\mathbb{K}$ with $\mathfrak{N}_\mathbb{K}$, and $\mathbb{O}_\mathbb{K}$ with $\mathcal{O}_\mathbb{K}$ uniformly. $\square$

Lemma 8.6. *Let $\mathbb{O}$ be a G-orbit in $\mathcal{N}$. Then, $\mathbb{O}_\Bbbk$ is a union of a single $G_\Bbbk$-orbit of dimension* $\dim \mathbb{O}$ *and $G_\Bbbk$-orbits of dimension* $< \dim \mathbb{O}$.

Proof. Consider the natural embedding $\iota : \mathcal{N}_\mathbb{K} \subset \mathrm{Mat}(2n)_\mathbb{K}$. It is well-known that the induced map $G_\mathbb{K} \backslash \mathcal{N}_\mathbb{K} \hookrightarrow GL(2n)_\mathbb{K} \backslash \mathrm{Mat}(2n)_\mathbb{K}$ is injective. (See eg. Tanisaki [Tan85] P152 for this kind of phenomenon.) Hence, $\mathbb{O}_\Bbbk$ is a union of $G_\Bbbk$-orbits with the same Jordan normal form (in $\mathrm{Mat}(2n, \Bbbk)$.) By Hesselink [Hes79], the maximal dimension of $G_\Bbbk$-orbits in $\mathbb{O}_\Bbbk$ is attained by a unique orbit as desired. $\square$

Theorem 8.7. *Let $\mathbb{O}$ be a G-orbit in $\mathcal{N}$. Let $\mathcal{O}$ be a G-orbit in $\mathfrak{N}$ such that* $\mathsf{df}(\mathcal{O}_\Bbbk) \subset \mathbb{O}_\Bbbk$ *is a open dense subset. Then, we have*

$$\mathrm{Jos}(\mathbb{O}) = \mathrm{Jos}(\mathcal{O}).$$

Proof. Since the construction of Joseph polynomials factors through the closures of orbital varieties, we may refer an orbital variety closure as an orbital variety during this proof (for the sake of simplicity). We prove the following identities:

$$\mathrm{Jos}(\mathbb{O}_\mathbb{K}) = \mathrm{Jos}(\mathbb{O}_\Bbbk) = \mathrm{Jos}(\mathcal{O}_\Bbbk) = \mathrm{Jos}(\mathcal{O}). \tag{8.1}$$

(Proof of $\mathrm{Jos}(\mathbb{O}_\mathbb{K}) = \mathrm{Jos}(\mathbb{O}_\Bbbk)$) Let $\mathfrak{Y} \subset \mathrm{Comp}(\mathbb{O})$. The variety $\mathfrak{Y}_\Bbbk$ is irreducible. By Lemma 6.6 2). Since $\mathcal{O}\dot{\cap}\mathfrak{n}$ is a flat family over A, we deduce that $\mathfrak{Y}$ is also a flat family over A. Therefore, $\mathrm{Jos}(\mathbb{O}_\mathbb{K}) = \mathrm{Jos}(\mathbb{O}_\Bbbk)$ follows from Proposition 8.3 as desired.

(Proof of $\mathrm{Jos}(\mathbb{O}_\Bbbk) = \mathrm{Jos}(\mathcal{O}_\Bbbk)$) Let $\mathfrak{X}_\Bbbk \in \mathrm{Comp}(\mathcal{O}_\Bbbk)$. Consider a family $\mathsf{ml}(\overline{\mathfrak{X}}_\Bbbk)$. This is a $B_\Bbbk$-stable equidimensional subfamily of $\mathcal{V}^+$. By the comparison of dimensions, it is a flat family of orbital varieties over $\mathbb{A}^1_\Bbbk$. Hence the equality $\mathrm{Jos}(\mathbb{O}_\Bbbk) = \mathrm{Jos}(\mathcal{O}_\Bbbk)$ follows from Proposition 8.4.

(Proof of $\mathrm{Jos}(\mathcal{O}_\mathbb{K}) = \mathrm{Jos}(\mathcal{O}_\Bbbk)$) Let $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$. The variety $\mathfrak{X}_\Bbbk$ is irreducible by Lemma 6.6 1). Since $\mathfrak{X}$ is flat over A, the equality $\mathrm{Jos}(\mathcal{O}_\mathbb{K}) = \mathrm{Jos}(\mathcal{O}_\Bbbk)$ follows from Proposition 8.3.
Now (8.1), and hence Theorem 8.7 is proved. $\square$

Corollary 8.8. *Let $\mathbb{O}$ be a G-orbit of $\mathcal{N}$ with its codimension $2d$. Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$ such that $\mathsf{df}(\mathcal{O}_{\Bbbk}) \subset \mathbb{O}_{\Bbbk}$ is a dense open subset. Let $X \in \mathcal{O}_{\mathbb{K}}$ and let $Y \in \mathbb{O}_{\mathbb{K}}$. Let $C_Y := \mathrm{Stab}_G(Y)/\mathrm{Stab}_G(Y)^{\circ}$. Then, we have a W-equivariant isomorphism*

$$H_{2d}(\mathcal{B}_Y, \mathbb{C})^{C_Y} \cong H_{2d}(\mathcal{E}_X, \mathbb{C}),$$

compatible with their embeddings into $H_{2d}(\mathcal{B}, \mathbb{C})$. Moreover, the bases given by irreducible components of $\mathcal{B}_X$ and $\mathcal{E}_X$ coincides up to scalar multiplication.

Proof. This is a direct consequence of Theorem 8.7 and [CG97] 6.5.13. Here the counter-part of [CG97] 6.5.13 for $\mathfrak{N}$ is obtained by merely by replacing the meaning of the symbols as $\mathcal{N}$ by $\mathfrak{N}$, $\mathbb{O}$ by $\mathcal{O}$, and $\mathcal{B}_Y$ by $\mathcal{E}_X$. □

9 An explicit description of the correspondence

We work under the same setting as in the previous section, but fix the base to be $\mathbb{K}$ (or rather its scalar extension to $\mathbb{C}$).

Theorem 9.1 ([K06b] §3)**.** *Let $\boldsymbol{\lambda} = (\lambda, a)$ be a marked partition of n. We define a sequence of integers $b = (b_1, b_2, \ldots)$ as*

$$b_i := \begin{cases} a_i & (a_i \neq 0) \\ \max\{\{a_j + \lambda_i - \lambda_j; j < i\} \cup \{a_j; j \geq i\}\} & (a_i = 0) \end{cases}.$$

Then, we define two partitions $\mu^{\boldsymbol{\lambda}}$ and $\nu^{\boldsymbol{\lambda}}$ as

$$\mu_i^{\boldsymbol{\lambda}} = b_i, \nu_i^{\boldsymbol{\lambda}} = \lambda_i - b_i.$$

The pair $(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ gives a bi-partition of n. Moreover, this assignment establish a bijection between the set of marked partitions of n and the set of bi-partitions of n. □

We refer the bi-partition $(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ the associated bi-partition of a marked partition $\boldsymbol{\lambda}$. Let $\mathcal{O}_{\boldsymbol{\lambda}}$ be the G-orbit with its $\mathbb{K}$-invariant $\boldsymbol{\lambda}$.

Definition 9.2 (Special elements)**.** Let $\boldsymbol{\lambda}$ be a marked partition and let $(\mu, \nu) := (\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ be its associated bi-partition. We define an element $w_{\boldsymbol{\lambda}} \in W$ as

$$w_{\boldsymbol{\lambda}} \epsilon_i = \begin{cases} \epsilon_{n-m+1} & (i = ({}^t\mu)^{\geq}_m) \\ -\epsilon_{|\nu| + ({}^t\mu)^{<}_m + i - ({}^t\mu)^{>}_m - m + 1} & (({}^t\mu)^{>}_m < i < ({}^t\mu)^{\geq}_m) \\ -\epsilon_{({}^t\nu)^{>}_m + i - ({}^t\nu)^{<}_m} & (|\mu| + ({}^t\nu)^{<}_m < i \leq |\mu| + ({}^t\nu)^{\leq}_m) \end{cases},$$

where m is some natural number. We put $\mathbb{V}^{\boldsymbol{\lambda}} := \mathbb{V}^+ \cap {}^{w_{\boldsymbol{\lambda}}}\mathbb{V}^+$. We define a sequence of integers $d^{\boldsymbol{\lambda}} := \{d_i^{\boldsymbol{\lambda}}\}_{i=1}^{\mu_1 + \nu_1}$ as

$$d_1^{\boldsymbol{\lambda}} := ({}^t\mu)^{\geq}_{\mu_1}, \ldots, d_{\mu_1}^{\boldsymbol{\lambda}} := ({}^t\mu)^{\geq}_1, d_{\mu_1+1}^{\boldsymbol{\lambda}} := |\mu| + ({}^t\nu)^{\leq}_1, \ldots, d_{\mu_1+\nu_1}^{\boldsymbol{\lambda}} := |\mu| + ({}^t\nu)^{\leq}_{\nu_1}.$$

We may drop the superscript ${}^{\boldsymbol{\lambda}}$ if the meaning is clear from the context.

Lemma 9.3. *Keep the setting of Definition 9.2. We have*

1. *We have $\epsilon_i - \epsilon_j \notin \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ if and only if $i \geq j$ or one of the following conditions hold for some natural number m:*

(a) $({}^t\mu)^{>}_{m} < i, j < ({}^t\mu)^{\geq}_{\overline{m}}$;

(b) $|\mu| + ({}^t\nu)^{<}_{m} < i, j \leq |\mu| + ({}^t\nu)^{\leq}_{\overline{m}}$;

(c) $j = ({}^t\mu)^{\geq}_{\overline{m}}$, $1 \leq i \leq |\mu|$, and $i \in \{({}^t\mu)^{\geq}_{\overline{l}}\}_{l>m}$.

2. We have $\epsilon_i + \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ if and only if $i \neq j$ and $i, j \in \{({}^t\mu)^{\geq}_{\overline{m}}\}_m$;

3. We have $\epsilon_i \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ if and only if $i \in \{({}^t\mu)^{\geq}_{\overline{m}}\}_m$.

Proof. Straight-forward. □

Lemma 9.4. *Keep the setting of Definition 9.2. We have*

$$\mathbb{V}^{\boldsymbol{\lambda}} \subset \overline{B(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))},$$

where $\mathbb{V}_0 \subset V_2$ and $\mathbb{V}_1 \subset V_1$ are G_0-stable subspaces defined at (2.1).

Proof. We put $\gamma_m := \epsilon_{({}^t\mu)^{\geq}_{\overline{m}}}$. By Lemma 9.3, we have $\epsilon_i + \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ only if $i, j \in \{({}^t\mu)^{\geq}_{\overline{m}}\}_m$. Moreover, $i < j$ and $\epsilon_i + \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ implies $\epsilon_i - \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$. We put

$$U_+ := \prod_{l \leq m} U_{\gamma_l + \gamma_m}, \mathbb{V}_0^{\circ} := \bigoplus_{l > m} \mathbb{V}[\gamma_l - \gamma_m], \text{ and } \mathbb{V}_2^{\circ} := \bigoplus_{l < m} \mathbb{V}[\gamma_l + \gamma_m].$$

We have $\mathbb{V}_0^{\circ}, \mathbb{V}_2^{\circ} \subset \mathbb{V}^+_{\boldsymbol{\lambda}}$. By a weight comparison, we deduce

$$U_+ \mathbb{V}^{\boldsymbol{\lambda}} = U_+((\mathbb{V}_0^{\circ} + \mathbb{V}_2^{\circ}) \cap \mathbb{V}^{\boldsymbol{\lambda}}) + \mathbb{V}^{\boldsymbol{\lambda}}.$$

Here $U_+ \mathbb{V}_0^{\circ} \subset \mathbb{V}_0^{\circ} + \mathbb{V}_2^{\circ}$ is a dense open subset. Thus, we conclude

$$\mathbb{V}^{\boldsymbol{\lambda}} \subset \overline{U_+(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))} \subset \overline{B(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))}$$

as desired. □

Lemma 9.5. *Keep the setting of Definition 9.2. We define*

$$\mathbb{V}_{01}^{\boldsymbol{\lambda}} := \bigoplus_{i \leq d_{\mu_1}} V_1[\epsilon_i] \oplus \bigoplus_{l < m} \bigoplus_{\substack{d_l < i \leq d_{l+1} \\ d_m < j \leq d_{m+1}}} \mathbb{V}[\epsilon_i - \epsilon_j].$$

Then, we have $\overline{B\mathbb{V}_{01}^{\boldsymbol{\lambda}}} = \overline{B\mathbb{V}^{\boldsymbol{\lambda}}}$.

Proof. We put $\Psi := \Psi(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))$. Since $\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1) \subset \mathbb{V}_{01}^{\boldsymbol{\lambda}}$, it suffices to prove the inclusion

$$\mathbb{V}_{01}^{\boldsymbol{\lambda}} \subset \overline{N_0(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))} \subset \overline{B\mathbb{V}^{\boldsymbol{\lambda}}}.$$

Here the second inclusion is obvious. We have

$$\Psi(\mathbb{V}_{01}^{\boldsymbol{\lambda}}) \backslash \Psi = \{\epsilon_i ; i \in [1, d_{\mu_1}] \backslash \{({}^t\mu)^{\geq}_{\overline{m}}\}_m\} \cup \{\epsilon_i - \epsilon_j ; i < j, i \notin \{({}^t\mu)^{\geq}_{\overline{m}}\}_m \ni j\}.$$

We deduce that

$$\left(\prod_{i < |\mu|, i \notin \{({}^t\mu)^{\geq}_{\overline{m}}\}_m} U_{\epsilon_i - \epsilon_{|\mu|}} \right) V[\epsilon_{|\mu|}] \subset \mathbb{V}_{01}^{\boldsymbol{\lambda}} \cap \mathbb{V}_1$$

is a dense open subset by the comparison of weights. We put

$$U_- := \prod_{i<j; i \notin \{({}^t\mu)^{\geq}_m\}_m, j \in \{({}^t\mu)^{\geq}_m\}_{m>1}} U_{\epsilon_i - \epsilon_j}.$$

It is easy to see that U_- does not depend on the order of the product. By the comparison of weights, we have a dense open subset

$$U_- \left(V_1[\epsilon_{|\mu|}] \oplus (\mathbb{V}^{\boldsymbol{\lambda}} \cap \mathbb{V}_0) \right) = V_1[\epsilon_{|\mu|}] \oplus U_-(\mathbb{V}^{\boldsymbol{\lambda}} \cap \mathbb{V}_0) \subset V_1[\epsilon_{|\mu|}] \oplus (\mathbb{V}^{\boldsymbol{\lambda}}_{01} \cap \mathbb{V}_0),$$

which guarantees that the first inclusion is dense. □

In the below, we denote by $\mathbb{V}^{\boldsymbol{\lambda}}_i$ $(i = 0, 1)$ the spaces $\mathbb{V}^{\boldsymbol{\lambda}}_{01} \cap \mathbb{V}_i$ coming from the statement of Lemma 9.5 for each marked partition $\boldsymbol{\lambda}$.

Proposition 9.6. *Let $\boldsymbol{\lambda}$ be a marked partition. We have $\overline{G\mathbb{V}^{\boldsymbol{\lambda}}} = \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$.*

Proof. By Definition 3.10, we deduce that $\mathcal{O}_{\boldsymbol{\lambda}} \subset G\mathbb{V}^{\boldsymbol{\lambda}}$. Thus, it suffices to check $\mathbb{V}^{\boldsymbol{\lambda}}_{01} \subset \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$. We define an increasing filtration

$$\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{\mu_1+\nu_1-1} \subsetneq F_{\mu_1+\nu_1} = V_1$$

as $F_k := \bigoplus_{i \le d_k} V_1[\epsilon_i]$. By a weight comparison, each $x \in \mathbb{V}^{\boldsymbol{\lambda}}_0$ preserves the flag $\{F_k\}_k$ when regarded as an element of $\mathrm{End}(V_1)$ as in (3.1). Moreover, the set of elements x in $\mathbb{V}^{\boldsymbol{\lambda}}_0$ which satisfies

$$\dim x F_k / F_{k-2} = \min\{\dim F_k / F_{k-1}, \dim F_{k-1}/F_{k-2}\}$$

is dense in $\mathbb{V}^{\boldsymbol{\lambda}}_0$. Let $\xi \in F_{\mu_1} \cap V_1^+$ be an element such that there exists $\{\xi(i)\}_i \subset V_1^+$ which satisfies

$$\xi = \sum_{i \ge 1} x^{\lambda_i - b_i} \xi(i), \text{ and } x^{\lambda_i - 1}\xi(i) \neq 0 = x^{\lambda_i}\xi(i) \text{ if } \xi(i) \neq 0.$$

Under the above choice of x, this condition is an open condition. We rearrange $\{\xi(i)\}$ according to the following rules: If $b_i = a_j$ for some $j < i$, then we rearrange $\xi(i), \xi(j)$ as $0, x^{\lambda_j - \lambda_i}\xi(i) + \xi(j)$, $a_i = 0$, and let others unchanged. If $b_i = a_j - \lambda_j + \lambda_i$ for $i > j$, then we rearrange $\xi(i), \xi(j)$ with $0, \xi(i) + \xi(j)$, $a_i = 0$, and let others unchanged.
By repeating this procedure for all possible pairs (i, j), we conclude that $\xi \oplus x$ has $\mathbb{K}$-invariant (λ, a) as desired. □

Corollary 9.7. *Under the setting of Proposition 9.6, we have*

$$\dim \mathcal{O}_{\boldsymbol{\lambda}} = \dim \mathcal{O}_{(\lambda, \mathbf{0})} + 2\,|\mu|\,.$$

Proof. We retain the setting of the proof of Proposition 9.6. Let $\xi \in V_1^+$. Then, we have $\xi \oplus x \in \overline{\mathcal{O}_{\boldsymbol{\lambda}}}$ if and only if $\xi \in \bigoplus_{i \le |\mu|} V_1[\epsilon_i]$. The Jordan type of x is λ (unchanged) if we regard $x \in \mathrm{End}(V_1)$ as either $x \in \mathrm{End}(V_1^+)$ or $x \in \mathrm{End}(V_1^-)$. Therefore, the fiber of the projection $\mathcal{O}_{\boldsymbol{\lambda}} \to \mathcal{O}_{(\lambda, \mathbf{0})}$ has dimension $2 \dim F_{\nu_1} = 2\,|\mu|$ as desired. □

The original form of the following formula seems to go back to Kraft-Procesi [KP82] §8.1. Here we present a slightly modified form which is suitable for applications.

Theorem 9.8 (Kraft-Procesi [KP82])**.** *Let λ be a partition of n and let $\mathbf{0} = (0, 0, \dots)$ be a sequence of zeros. We put $\boldsymbol{\lambda} := (\lambda, \mathbf{0})$. Then, we have*

$$\dim \mathcal{O}_{(\lambda,\mathbf{0})} = 2 \dim(\mathcal{O}_{(\lambda,\mathbf{0})} \cap \mathbb{V}_0) = 4 \sum_{i<j} d_i^{\boldsymbol{\lambda}} d_j^{\boldsymbol{\lambda}}.$$

Corollary 9.9. *For each marked partition $\boldsymbol{\lambda}$, there exists $\mathfrak{X}_{\boldsymbol{\lambda}} \in \mathrm{Comp}(\mathcal{O}_{\boldsymbol{\lambda}})$ such that*

$$\overline{\mathfrak{X}_{\boldsymbol{\lambda}}} = \overline{B\mathbb{V}^{\boldsymbol{\lambda}}}.$$

Proof. By Theorem 6.1, it suffices to prove $\dim B\mathbb{V}^{\boldsymbol{\lambda}} \geq |\mu| + 2\sum_{i<j} d_i d_j$. The subspace $V_1 \cap \mathbb{V}_{01}^{\boldsymbol{\lambda}}$ is B-stable and has dimension $|\mu|$. Since $B\mathbb{V}^{\boldsymbol{\lambda}} = B\mathbb{V}_{01}^{\boldsymbol{\lambda}}$, we have only to prove $\dim B(V_2 \cap \mathbb{V}_{01}^{\boldsymbol{\lambda}}) \geq 2\sum_{i<j} d_i d_j$. Since $V_2 \cap \mathbb{V}_{01}^{\boldsymbol{\lambda}}$ is N_0-stable and $\dim V_2 \cap \mathbb{V}_{01}^{\boldsymbol{\lambda}} = \sum_{i<j} d_i d_j$, it suffices to prove that $\dim G_2 x \geq \sum_{i<j} d_i d_j$ for a generic element of $x \in V_2 \cap \mathbb{V}_{01}^{\boldsymbol{\lambda}}$. Since the dimension of the G_2-stabilizer is an upper-semicontinuous function, it is enough to show $\dim G_2 x \geq \sum_{i<j} d_i d_j$ for some $x \in V_2 \cap \mathbb{V}_{01}^{\boldsymbol{\lambda}}$. Since $x \in \mathbb{V}_0$, we have

$$\dim \mathfrak{g}_2 x = \dim \mathfrak{g}_{-2} x.$$

Theorem 9.8 implies that

$$\dim \mathcal{O}_{(\lambda,\mathbf{0})} = \dim \mathfrak{g} x = 2 \dim \mathfrak{g}_0 x = 4 \sum_{i<j} d_i d_j.$$

In particular, we have $\dim \mathfrak{g}_2 x = \sum_{i<j} d_i d_j$ as desired. $\square$

Definition 9.10 (Special vectors)**.** Let $\boldsymbol{\lambda}$ be a marked partition of n and let (μ, ν) be its associated bi-partition. We define

$$\mathsf{D}_i^0(\boldsymbol{\lambda}) := \prod_{d_i < k < l \leq d_{i+1}} (\epsilon_k^2 - \epsilon_l^2), \text{ and } \mathsf{D}_i^+(\boldsymbol{\lambda}) := \mathsf{D}_i^0(\boldsymbol{\lambda}) \prod_{d_i < k \leq d_{i+1}} \epsilon_k.$$

Using this, we define

$$\mathsf{D}(\mu, \nu) := \prod_{i=1}^{\mu_1 - 1} \mathsf{D}_i^0(\boldsymbol{\lambda}) \times \prod_{i=\mu_1}^{\mu_1+\nu_1-1} \mathsf{D}_i^+(\boldsymbol{\lambda}).$$

For a bi-partition (μ, ν) of n, we define the Macdonald representation attached to (μ, ν) as

$$L(\mu, \nu) := \mathbb{C}[W]\mathsf{D}(\mu, \nu) \subset \mathbb{C}[\mathfrak{t}].$$

We remark that $L(\mu, \nu)$ is well-defined due to Theorem 9.1. Let $\mathbb{C}[\mathfrak{t}]_m$ denote the degree m-part of the polynomial ring $\mathbb{C}[\mathfrak{t}]$.

Theorem 9.11 (Macdonald cf. Lusztig-Spaltenstein [LS79])**.** *For each bi-partition (μ, ν) of n, the Macdonald representation $L(\mu, \nu)$ is an irreducible representation of W. Moreover, we have*

$$\mathrm{Hom}_W(L(\mu, \nu), \mathbb{C}[\mathfrak{t}]_m) = \begin{cases} 0 & (m < \deg \mathsf{D}(\mu, \nu)) \\ 1 & (m = \deg \mathsf{D}(\mu, \nu)) \end{cases}.$$

We define

$$W_l := \langle s_i s_{i+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_i ; 1 \le i \le n \rangle \subset W.$$

We have $W_l \cong (\mathbb{Z}/2\mathbb{Z})^n$. Moreover, we have a short exact sequence of groups

$$\{1\} \longrightarrow W_l \longrightarrow W \longrightarrow \mathfrak{S}_n \longrightarrow \{1\}.$$

Corollary 9.12. *Let λ be a partition of n. We have*

$$\mathrm{Hom}_W(L(\lambda, \emptyset), \mathbb{C}[\mathfrak{t}]_m) = \mathrm{Hom}_{\mathfrak{S}_n}(L(\lambda, \emptyset), \mathbb{C}[\mathfrak{t}]_m^{W_l}).$$

Corollary 9.13. *We have*

$$\mathsf{lt}(q(\mathfrak{X}_{\boldsymbol{\lambda}})) \in \mathbb{Q} \prod_{i=1}^{\mu_1 - 1} \mathsf{D}_i^0(\boldsymbol{\lambda}) \times \prod_{i=\mu_1}^{\mu_1+\nu_1-1} \mathsf{D}_i^+(\boldsymbol{\lambda}).$$

In particular, the $\mathbb{C}$-span of $\mathrm{Jos}(\mathcal{O}_{\boldsymbol{\lambda}})$ is isomorphic to $L(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ as W-modules.

Proof. We define

$$q_0(\boldsymbol{\lambda}) := \prod_{i > |\mu|} \epsilon_i \times \prod_{i \ge 0} \prod_{d_i < j < k \le d_{i+1}} (\epsilon_j - \epsilon_k).$$

The map $p : \mathbb{V}^+ \longrightarrow \mathbb{V}^+/(\mathbb{V}_1 \oplus \mathbb{V}_2)$ is a B-equivariant fibration. Hence, it induces the associated map $\mathfrak{X}_{\boldsymbol{\lambda}} \longrightarrow p(\mathfrak{X}_{\boldsymbol{\lambda}})$, which is generically a flat fibration. By Lemma 9.5, we have $\overline{p(\mathfrak{X}_{\boldsymbol{\lambda}})} = \mathbb{V}^{\boldsymbol{\lambda}} \cap \mathbb{V}_0$. It follows that $q_0(\boldsymbol{\lambda})$ divides $q(\boldsymbol{\lambda})$.

By a dimension counting, we deduce that

$$2d_Y := 2 \dim \mathcal{E}_Y = \mathrm{codim}_{\mathfrak{N}} GY = \deg \mathsf{lt}(q(\mathfrak{X}_{\boldsymbol{\lambda}})) = \deg \mathsf{D}(\mu, \nu) \tag{9.1}$$

for each $Y \in \mathcal{O}_{\boldsymbol{\lambda}}$.

Applying the argument of [CG97] 6.5.3 and 7.4.1, we know that

$$H_{2d_Y}(\mathcal{E}_Y, \mathbb{C}) \hookrightarrow H_{\bullet}(\mathcal{B}, \mathbb{C}) \hookrightarrow \mathbb{C}[\mathfrak{t}]$$

is some Macdonald representation. (The second inclusion is realized by the harmonic polynomials realized by T-equivariant fundamental classes.) By [K06b], this establishes a one-to-one correspondence between the set of Macdonald representations and the set of G-orbits of $\mathfrak{N}$.

By Lemma 9.5, each $\overline{\mathfrak{X}_{\boldsymbol{\lambda}}}$ is written as a product of $\bigoplus_{1 \le i \le |\mu|} V_1[\epsilon_i]$ and $\overline{\mathfrak{X}_{\boldsymbol{\lambda}}} \cap V_2$. Here we have $\overline{\mathfrak{X}_{\boldsymbol{\lambda}}} \cap V_2 = \dot{w}\overline{\mathfrak{X}_{\boldsymbol{\gamma}}}$ for some $w \in \mathfrak{S}_n$ and a marked partition $\boldsymbol{\gamma}$ which corresponds to a bi-partition $(\gamma, \emptyset)$ of n. We have an equality

$$q(\boldsymbol{\lambda}) = (\prod_{i > |\mu|} \epsilon_i) w q(\boldsymbol{\gamma}).$$

The polynomial $wq(\boldsymbol{\gamma})$ is fixed by the action of

$$\mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2 - d_1} \times \cdots \subset \mathfrak{S}_n$$

since its lift to $N_G(T)$ preserves $\dot{w}\mathfrak{X}_{\boldsymbol{\gamma}}$. Moreover, the action of s_n fixes $wq(\boldsymbol{\gamma})$ by the same reason. It follows that $s_{n-d_{\mu_1+\nu_1}+1}, \ldots, s_n$ fixes $wq(\boldsymbol{\gamma})$. Moreover, permuting the sequence $\{d_i - d_{i-1}\} = \{d_1, d_2 - d_1, \ldots\}$ changes $wq(\boldsymbol{\gamma})$ only by some element of $\mathfrak{S}_n$. Therefore, we conclude that $wq(\boldsymbol{\gamma})$ is fixed by the action of W_l. In particular,

$$\mathsf{D}(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}}) = \prod_{k > |\mu|} \epsilon_k \times \prod_{k \ge 0} \prod_{d_k < i < j \le d_{k+1}} (\epsilon_i^2 - \epsilon_j^2)$$

divides $q(\boldsymbol{\lambda})$. By the comparison of dimensions, we know $\deg \mathsf{D}(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}}) = \mathrm{codim}\mathcal{O}_{\boldsymbol{\lambda}} = \deg q(\boldsymbol{\lambda})$, which implies the result. $\square$

Corollary 9.14. *For each marked partition $\boldsymbol{\lambda}$, we have $\langle \mathrm{Jos}(\mathcal{O}_{\boldsymbol{\lambda}}) \rangle_{\mathbb{C}} = L(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ as sub-representations of $\mathbb{C}[\mathfrak{t}]$.* $\square$

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