Kato On the geometry of flag varieties and representation theory. Kyoto 3/23/06

Springer correspondence:

$W^\vee$

combinatorics of classical type

homology of Springer fiber

REPLACE $\boxed{\S 2}$

$\updownarrow$

except for type A because of nontrivial A-group

geometry of nilpotent cone.

REPLACE $\boxed{\S 1}$

<u>REPLACEMENT</u>: has the advantage of <u>trivial</u> A groups.

<u>§1</u> Combinatorial correspondence:

$W = W(C_n) \cong (S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n)$

$W^\vee$ = irred repn

$P(n)$ = partitions of n.

$P_2(n) = \{(\mu,\lambda) = \coprod_{i=0}^{n} P(i) \times P(n-i)\}$. bi-partitions.

<u>Convention</u>: $\lambda \in P_n(n)$ $\leadsto$ identify (Young diagram)

<u>FACT</u>: $\exists$ 1-1 correspondence $W_n^\wedge \longleftrightarrow P_2(n)$.

<u>Defn</u> (Marked partitions) The pair (λ, a) is called a marked partition if $\lambda \in \mathcal{P}^n$ and $a \in \mathbb{Z}^\infty$ s.t.

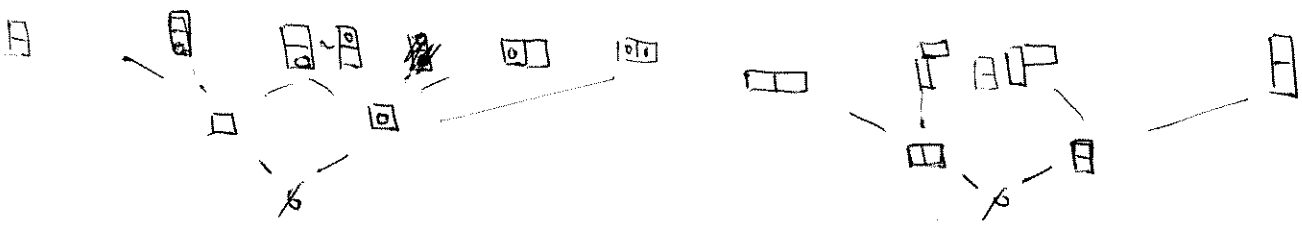
(3) $\forall p$ $0 \le a_p \le \lambda_p$ $\forall p$.

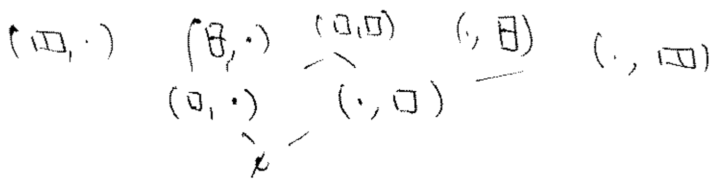
(4) $a_p = 0$ if $\lambda_p = \lambda_{p+1}$

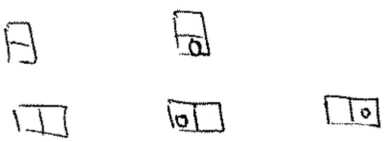
(5) $\lambda_p - \lambda_q > a_p - a_q > 0$ if $a_p \ne 0 \ne a_q$ and $p < q$.

The set is denoted $MP(n)$.

See the handout for more examples.







---

§2 Exotic nilpotent cones.

$G = Sp(2n,\mathbb{C}) \subset GL(2n,\mathbb{C})$
$\cup$
$B \supset T$ Borel & maximal torus.

$X^*(T) \xrightarrow{\sim} \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$ character lattice of T & basis $\{\varepsilon_i\}$

$R = \{\pm\varepsilon_i \pm \varepsilon_j\} \cup \{\pm 2\varepsilon_i\} \supset R^+ = \{\varepsilon_i \pm \varepsilon_j\}_{i<j} \cup \{2\varepsilon_i\}_{i=1}^{n}$.

$V_1 = \mathbb{C}^{2n}$ vector rep of $G \subset GL(2n,\mathbb{C}) \Leftrightarrow$ h/w ε_1

$V_2 = \Lambda^2 V_1 / \mathbb{C}$ irrep of G with h/w $\varepsilon_1 + \varepsilon_2$.

$\mathbb{V} = V_1 \oplus V_2$. EXOTIC REPN OF G.

$G = Sp(2n, \mathbb{C})$

<u>Main observation</u>:

$\overset{1\text{-}1 \text{ corr.}}{\longleftrightarrow}$ $\{$nonzero weights of $\mathbb{V}\} = \{\pm\varepsilon_i \pm \varepsilon_j\} \cup \{\pm \varepsilon_i\}$.

$R = \{\pm\varepsilon_i \pm \varepsilon_j\} \cup \{\pm 2\varepsilon_i\}$.

So we're replacing adjoint repn of with $\mathbb{V}$ and then do Springer theory or Kazhdan-Lusztig theory.

$\mathbb{V}^+ = \bigoplus_{i=1}^{n} V_1[\varepsilon_i] \oplus \bigoplus_{i<j} V_2[\varepsilon_i \pm \varepsilon_j]$ where $V[\lambda] \subset V$ is λ-eigenspace

$\circlearrowright B$ acts.

$$F := G \times^B \mathbb{V}^+ \hookrightarrow G \times^B \mathbb{V} \xrightarrow{\sim} G/B \times \mathbb{V}$$

$$\xrightarrow{\mu} \mathfrak{N} := G \cdot \mathbb{V}^+ \subset \mathbb{V}$$

We call $\mathfrak{N}$ the ~~extra~~ exotic nilpotent cone.

$G \times (\mathbb{C}^\times)^2$ acts on $\mathfrak{N}$ — $N(\mathfrak{p})$ for $G = GL(n, \mathbb{H})$

<u>Theorem</u>. (1) $\mathfrak{N} \xrightarrow{\sim} V_1 \times \mathcal{N}_{(GL(2n), Sp(2n))}$

(2) Defining ideal of $\mathfrak{N} = \mathbb{C}[V_2]^G_+ \mathbb{C}[\mathbb{V}]$.

(3) # of G-orbits in $\mathfrak{N}$ is finite.

(4) μ is birational

(5) μ is stratified semi-small with respect to G orbits.

(6) Ohta: $\mathfrak{N}$ is normal.

For each $\lambda \in X^*(T) \setminus \{0\}$ s.t. $\mathbb{V}[\lambda] \neq 0$, we fix $\mathbb{v}[\lambda] \in \mathbb{V}[\lambda]$ basis vector.

Define §JNF of $\mathfrak{N}$ as follows

$(\lambda, a) \in \mathcal{MP}(n)$, then we define

$$J(\lambda, a) := \sum_{\substack{p \geq 0 \\ a_p \neq 0}} \mathbb{V}[\varepsilon_{\lambda_{<p}} + a_p] + \sum_{p \geq 1} \sum_{q=1}^{\lambda_p - 1} \mathbb{V}[\alpha_{\lambda_{<p} + q}].$$

$\in \mathbb{V}^+ \subset \mathfrak{N}$

Propn. The map

$$J : \mathcal{MP}(n) \ni \vec{\lambda} \longmapsto J(\vec{\lambda}) \in G \backslash \mathfrak{N}$$

is a surjective.

Pf. Use result of Ohta to assume V_2-part is $J(\vec{\lambda}_0)$ for some $\vec{\lambda}_0 = (\lambda_0, 0) \in \mathcal{MP}(n)$. Then use info of stabilizers.

① semisimple part of stabilizer $\Rightarrow$ (4)

② unipotent part of stabilizer $\Rightarrow$ (5).

§ Exotic Springer correspondences: $\quad F := G \times^B \mathbb{V}^+$

$$Z := F \times_{\mathfrak{N}} F = \{ (g_1 B, g_2 B, X) \in (G/B)^2 \times \mathbb{V} \text{ s.t. } X \in g_1 \mathbb{V}^+ \cap g_2 \mathbb{V}^+ \}$$

$$\downarrow$$

$$\{ (g_1 B, g_2 B) \} \in (G/B)^2.$$

Theorem (math.RT/0601155)

$$K(Z) \xrightarrow{\sim} \mathbb{C}[W] \otimes \mathbb{C}[T] / \langle \mathbb{C}[[t]]^W_+ \rangle$$

as algebras.

Cor. $\exists$ a surjection $\mathbb{C} \otimes_{\mathbb{Z}} K(Z) \longrightarrow\!\!\!\!\rightarrow \mathbb{C}[W]$.

In particular, $W^\vee \cong \{ \text{simple } \mathbb{C} \otimes_{\mathbb{Z}} K(Z) \text{ modules} \} / \text{equivalence}$.

Σ: $\exists$ no alg map $K((G/B)^2) \twoheadrightarrow \mathbb{C}[W]$.

$\exists\ K(2) \xrightarrow{\sim} H_\bullet^{BM}(2) \xrightarrow{\sim} \mathrm{End}_{D^b(\mathcal{N})}(\mu_* \mathbb{C}_F)$.

Thm (BBDG + M. Saito) $\exists$ an algebraic stratification
$\mathcal{N} = \coprod \mathbb{O}$ s.t.

$$\mu_* \mathbb{C}_F \cong \bigoplus_{d,\mathbb{O},\chi} L_{d,\mathbb{O},\chi} \boxtimes IC(\mathbb{O},\chi)[d].$$

<u>Theorem</u> (Ginzburg)

$\{\text{simple } K(2)\text{-mod}\}/\text{isom} \xleftrightarrow{1\text{-}1} \left\{ \begin{array}{l} \text{IC's appearing in the} \\ \text{above decomposition for} \\ \mu_* \mathbb{C}_F. \end{array} \right\}$

<u>Lemma</u> (Igusa) The G-stabilizer of every G orbit on $\mathcal{N}$ is connected.

<u>Cor</u>. All G-eqvt local systems on $\mathbb{O}$ are trivial.

For $\mathbb{O} \subset \mathcal{N}$ G-orbit, $x \in \mathbb{O}$

$M_{\mathbb{O}} = H^{BM}_{2d_{\mathbb{O}}}(\mu^{-1}(x))$ $\qquad d_{\mathbb{O}} = \dim \mathcal{N} - \dim \mathbb{O}$

$\circlearrowleft$ Borho-MacPherson $\qquad$ <u>(independent of x)</u>

$K(2)$ acts.

Igusa's lemma $\Rightarrow$ $M_{\mathbb{O}}$ is irreducible if nonzero

<u>MAIN THEOREM</u>: $\exists$ a bijection

ξ: (G-orbits of $\mathcal{N}$) $\longrightarrow W^\vee$

$\mathbb{O} \longmapsto M_{\mathbb{O}}$.

<u>Proof</u>.

$$\#\{\text{simple } K(z)\text{-modules}\} \overset{\text{Lusztig, Ginzburg}}{\leq} \#G\text{-orbits on } \mathfrak{N}$$

$$\leq \#\mathcal{MP}(n) = \#W^\vee$$

$$\leq \{\text{simple } K(2)\text{-modules}\}$$

all inequalities are equalities.

<u>Remark</u>. • This is a refinement of Ginzburg's generalized Springer correspondence

• Exotic Deligne-Langlands correspondence.