Zelevinsky embedding

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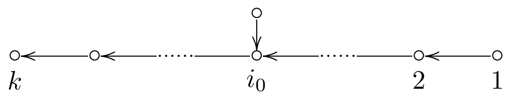
Consider $G = GL(n, \mathbb{C})$, $q > 1$ real number, and the diagonal matrix D with entries $1, q, \dots, q^k$ with its multiplicities $d_1, \dots, d_k$ ($\sum d_k = 1$). We fix $1 \le i_0 \le k$. Let B be the upper triangluar part of G.

We put $\mathfrak{g}_1 := \{\xi \in \mathfrak{g}; \mathrm{Ad}(s)\xi = q^2\xi\}$. So that $L = G(s) = \prod_{i=1}^{k} GL(d_k, \mathbb{C})$ acts on it. Now we would like to embed $\mathfrak{g}_1 \oplus \mathbb{C}^{d_{i_0}}$ into the flag variety of $G/P \times \mathbb{C}^n$ with $P = LB$.

We identify $\mathfrak{g}_1 \oplus \mathbb{C}^{d_{i_0}}$ as the representation space of the quiver



with its dimension vector $\mathbf{d} = (d_1, \dots, d_k)$.

Let $N \in \mathfrak{g}_1$ and $N_0 : \mathbb{C} \to \mathbb{C}^{d_i}$ and consider a correspondence Ψ

$$\mathfrak{g}_1 \oplus \mathbb{C}^{d_i} \ni (N, N_0) \mapsto (\exp NP, (N_0\mathbf{1}) + \mathfrak{k}) \in G/P \times \mathbb{C}^n,$$

where $\mathbf{1} \in \mathbb{C}$ is a non-zero element. Here

$$\mathfrak{k} := \sum_{i=0}^{i_0-1} \mathbb{C}^{d_i} \subset \mathbb{C}^n$$

is a P-stable subspace of $\mathbb{C}^n$.

This gives rise to a map ψ

$\{L - \text{orbits in } \mathfrak{g}_1 \oplus \mathbb{C}^{d_i}\} \ni (N, N_0) \mapsto$
$(\exp NP, (N_0\mathbf{1}) + \mathfrak{k}) \in \{P - \text{orbits in } G/P \times \mathbb{C}^n \text{ which contains } \mathfrak{k} \text{ at all the fibers.}\}$

Remark 0.1. Let $\mathcal{O}_1, \mathcal{O}_2$ be a L-orbits in $\mathfrak{g}_1 \oplus \mathbb{C}^{d_i}$. We have

$$\dim \mathrm{IH}_i(\mathcal{O}_1)_{\mathcal{O}_2} = \dim \mathrm{IH}_{i+\dim \mathfrak{k}}(\psi(\mathcal{O}_1))_{\psi(\mathcal{O}_2)}$$

for all i.

The correspondence Ψ descends to a map

$$\Psi_i := \mathfrak{g}_1 \oplus \mathbb{C}^{d_i} \mapsto G \times^P \mathfrak{k}_{\ge}/\mathfrak{k},$$

where $\mathfrak{k}_{\ge} := \mathfrak{k} \oplus \mathbb{C}^{d_i} \subset \mathbb{C}^n$ is a P-submodule.

The pullback of the RHS from G/P to Zelevinsky's Schubert variety is clearly normal. Apply ZMT to see Ψ_i is an embedding. The preimage of the quotient bundle is a locally trivial fibration at all points.