ON THE CLASSIFICATION OF PRIMITIVE IDEALS IN THE ENVELOPING ALGEBRA OF A SEMISIMPLE LIE ALGEBRA

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1. INTRODUCTION

1.1. The classification of the primitive ideals in the enveloping algebra of a complex semisimple Lie algebra—a major goal in the theory of enveloping algebras—is now nearly completed. Nevertheless had it not been for the necessity of providing the Maryland Special Year in Group Representations with a manuscript, the author would probably have still further delayed any attempt to review this subject. Thus whereas the main lines of the classification are now firmly established there are still many fascinating questions whose solution can at least in some cases be shortly anticipated. Furthermore, the dazzling success of algebraic geometry in determining the multiplicities of the simple factors in Verma modules has reduced many an expert in enveloping algebras to the role of a mere bystander and has so much reorientated the field that one particular protagonist has suggested, no doubt with much justification, that enveloping algebras should now be relegated to a subdivision of the theory of rings of differential operators.

1.2. Let us first just attempt an informal historical sketch of the main steps in the classification of primitive ideals.

The first major breakthrough was undoubtedly Duflo's theorem (3.5) which showed it was enough to consider annihilators of simple highest weight modules and in more detail asserts that for a given central

character these annihilators are indexed by a subset of the involutions of the Weyl group—the still mysterious Duflo set (3.5). About the same time Borho and Jantzen established a translation principle (3.6) which related annihilators having different central character, grouping them into infinite subsets which we shall call t-clans (4.1).

The notion of the characteristic variety (2.7) of a primitive ideal gave information on the inclusion relations between primitive ideals and via Knuth's combinatorial theorem (see [20]) provided an explanation to Jantzen's remarkable observation (in type A_n) that the primitive ideals having a given central character appear to be grouped into sets which we shall call r-clans (4.3) whose cardinality is the dimension of an irreducible representation of the Weyl group. Following this Spaltenstein showed how Knuth's theorem could be used to describe the Steinberg correspondence (8.7) between the Weyl group and nilpotent orbits and brought to the attention of the author, Springer's correspondence (8.7) between such orbits and Weyl group representations. This was of particular interest as Borho and Kraft [9] had pointed out that the associated variety (2.1) of a primitive ideal was indeed a union of nilpotent orbits and conjecturally the closure of just one orbit.

A powerful technique for separating primitive ideals was provided by the Goldie rank (Sect. 4) of the corresponding quotient algebras. Berline and Duflo [11] had already given some indication how this invariant could be calculated; but it was the establishment of an additivity principle (4.2) which under suitable refinement led to the following two major results. One, that Goldie rank on t-clans is a polynomial and those for distinct t-clans are linearly independent. Two, that the set of polynomials defined with respect to an r-clan form a basis of an irreducible representation of the Weyl group. Shortly afterwards it was established that the associated variety was constant on clans and the obvious link with Springer's correspondence was conjectured.

It was now becoming clear that much precise information on primitive ideals could be expressed in terms of the multiplicities of the simple highest weight modules in the Jordan-Hölder series of Verma modules and which we refer to as forming the entries of the Jantzen matrix (Jantzen having made great progress in computing these coefficients). In particular the Goldie rank polynomials discussed above could be expressed in terms of this matrix (at least up to a scalar). Again following a suggestion of the author, Vogan showed that this was also true of the inclusion relations (Sect. 5) between primitive ideals and thereby implicitly solved a problem which earlier had needed laborious calculations to describe even low rank cases.

Partly motivated by this work and partly by the Springer correspondence, Kazhdan and Lusztig formulated their now famous conjecture (Sect. 6) for the Jantzen matrix. An important aspect of their conjecture was to bring into the arena an entirely new participant; namely the intersection cohomology theory of Deligne-Goretsky-MacPherson (DGM) thereby establishing a remarkable link between the singularities of Schubert cells in the flag variety and the Jantzen matrix.

Before it was realized that the Kazhdan-Lusztig conjecture was too hard for algebraists, Vogan established an equivalence of this conjecture with his semisimplicity hypothesis (6.5), and shortly afterwards, it was shown that an earlier conjecture of Jantzen concerning filtrations of Verma modules, implied the Kazhdan-Lusztig conjecture and even a significant refinement of it providing the composition factors in each gradation step - information which we refer to as the JKL data (6.3).

Brylinski and Kashiwara established the Kazhdan-Lusztig conjecture through the study of differential operators over the flag variety and the use of deep results on DGM sheaves. Beilinson and Bernstein simultaneously obtained similar results and further realized as also did Vogan that this new viewpoint could even encompass and go beyond the

Langlands classification of simple Harish-Chandra modules. Bernstein also reported a proof of the Jantzen conjecture based on a significant refinement due to Gabber of a main step in Deligne's proof of the Weil conjectures.

Returning to primitive ideals per se, it was shown that the Duflo set was implicitly determined by the JKL data and furthermore this and the Vogan semisimplicity hypothesis also gave some refined information on what we shall call the Kostant ring (Sect. 7) of a simple highest weight module. This is turn gave information on the calculation of the scale factors in the Goldie rank polynomials.

Barbasch and Vogan developing combinatorial results analogous to the Knuth theorem for the symmetric group S_n extended to the remaining Lie algebras results concerning the cellular decomposition of the Weyl group (Sect. 5) defined by the primitive ideals via Duflo's theorem. From these results Brylinski and Borho through a theorem on induced ideals established in the so-called integral case the Borho-Kraft conjecture described above and linked the representations of the Weyl group defined by Goldie rank polynomials to those defined by the Springer correspondence.

1.3. Let us now indicate briefly some of the main techniques in the proof of the above results.

Dixmier first pointed out that primitive quotients of the enveloping algebra could be viewed as Harish-Chandra modules for the corresponding complex Lie group. Using classical results of Kostant, it was further shown by Duflo that for minimal primitives the corresponding quotient algebra was in fact a so-called principal series module, an observation which eventually led to the Duflo theorem. For this early work see ([13], Chap. 8). A more precise formulation established an equivalence of categories (3.4) of such Harish-Chandra modules with the so-called $\mathcal{O}$ category (roughly speaking the category of highest weight modules). This result is used in refining the additivity principle for

Goldie rank based on the use of Small's theorem and in the proof of Vogan's theorem on inclusion relations between primitive ideals.

The translation principle which goes back to Harish-Chandra is one of the main techniques in studying primitive ideals. Here a module in say the $\mathcal{O}$ category is tensored by some finite dimensional module, and this allows one to translate results from one central character to another thus giving the Borho-Jantzen translation principle and together with the additivity principle, the polynomial behaviour of the Goldie ranks on t-clans. Moreover, by "reflecting off walls" in the Weyl chamber one is able to implement an action of the Weyl group and it is this which leads essentially to the representations of the Weyl group defined on r-clans. Moreover, by replacing the base field with a discrete valuation ring use of the translation principle gives the JKL data as a consequence of the Jantzen conjecture. Finally it is again a version of the translation principle which allows one to study the enveloping algebra through algebraic geometry via sheaves of differential operators on the flag variety.

1.4. To close this introduction we indicate some open problems and related questions.

Rather little is known about the Jordan-Hölder series for primitive quotients, though, of course, the general philosophy is that these should be describable in terms of the Jantzen matrix. A conjecture is indicated along these lines (7.6).

The scale factors occuring in the Goldie rank polynomials have not yet been completely determined. This is a rather delicate problem. One step involves the construction of sufficiently many completely prime primitive ideals. Although this latter problem was one of the first to be considered, practically no progress on it has been reported.

The use of differential operators over the flag variety allows one to introduce an additional invariant associated to a module (over the enveloping algebra), namely its singular support (8.9). This may be

used, for example, in computing the Krull dimension of primitive quotients (as yet only for minimal primitive ideals). It turns out that there are some rather natural conjectures (8.14) for the singular support of a simple highest weight module and of its annihilator. As pointed out by Borho and Brylinski (8.10), these invariants are closely related to, but more refined than, associated varieties; moreover, their computation elucidates the relation between the classification of primitive ideals through Goldie rank polynomials and the Springer correspondence alluded to previously.

One may also associate to a primitive ideal its Goldie skew-field (2.2). This is known to be constant on t-clans; but it is not yet known if this is true on r-clans. Conjecturally it is always isomorphic to a Weyl skew-field (4.5).

The Kostant ring (Sect. 7) of a simple module is always a Harish-Chandra module (for the corresponding complex group) and has furthermore been shown to be a maximal order. In analogy with highest weight modules one expects that as a module, the Kostant ring should have a large socle (7.2) and to be a direct sum of modules having a simple socle. A rather more daring hypothesis is that the ring is completely determined by the module structure of the socle.

Primitive ideals corresponding to the same central character can be viewed as prime ideals of the quotient of the enveloping algebra by a minimal primitive ideal. From this latter viewpoint one may ask if all such ideals are idempotent. This is false for singular central characters; but there is some evidence that it holds for regular characters. Again for singular characters the quotient algebra generally has infinite global dimension; but one expects to find finite global dimension at regular characters. One may also give a conjecture (9.4) for the Grothendieck group generated by representatives of the projective modules.

There seems to be an astonishing connection between completely prime primitive ideals of the enveloping algebra and unitary representations of the corresponding complex group. Conjecturally the simple socle of all such quotient algebras, viewed as modules for the group, are unitarizable and all unitary representations occur in this manner. For regular integral central characters the second assertion follows from work of Enright and Parthasarathy, whilst the first statement would be a consequence of a rather natural conjecture (4.6) for the Goldie rank polynomials.

We conclude by remarking that there are a very large number of questions which in view of the Kazhdan-Lusztig polynomials can be regarded as purely combinatorial. For the moment these seem to be of incredible difficulty and, in fact, except for Knuth's theorem and its generalization by Barbasch-Vogan, the only success in resolving such questions has been through their interpretation in the context of enveloping algebras and the use of deep theorems in non-commutative ring theory, algebraic geometry or algebraic topology. See Sec. 5 and ([25], Sect. 5) for some examples.

2. NOTATION AND CONVENTIONS

2.1. The base field is assumed to be the complex numbers $\mathbb{C}$ (though any algebraically closed field of characteristic zero would do). For any vector space V, let V^* denote its dual and $S(V)$ the symmetric algebra over V. If I is an ideal of $S(V)$ we denote by $\underline{V}(I)$ its variety of zeros in V^*. For any Lie algebra $\underline{a}$ we denote by $\mathfrak{U}(\underline{a})$ its enveloping algebra and $\mathfrak{Z}(\underline{a})$ the centre of $\mathfrak{U}(\underline{a})$. The subspaces $\mathfrak{U}^n(\underline{a}) = \mathbb{C}\{X^m : m \leq n, X \in \underline{a}\}$, $\forall n \in \mathbb{N}$ form an increasing filtration of $\mathfrak{U}(\underline{a})$ and the graded algebra

$$\operatorname{gr} \mathfrak{U}(\underline{a}) := \bigoplus_{n \in \mathbb{N}} (\mathfrak{U}^n(\underline{a})/\mathfrak{U}^{n-1}(\underline{a})) : (\mathfrak{U}^{-1}(\underline{a}) = 0)$$

is by Poincaré-Birkhoff-Witt isomorphic to $S(\underline{a})$. Let M be a finitely generated $\mathfrak{U}(\underline{a})$ module, say with generating subspace M^0. Then

$$\operatorname{gr} M := \bigoplus_{n \in \mathbb{N}} (M^n/M^{n-1}) : M^n = \mathfrak{U}^n(\underline{a})M^0,$$

is a graded module for the graded ring $\operatorname{gr} \mathfrak{U}(\underline{a}) \cong S(\underline{a})$. After Bernstein $\underline{V}(\operatorname{Ann} \operatorname{gr} M)$ is independent of the choice of generating subspace M^0 and is called the associated variety $\underline{V}(M)$ of M. It is a closed subvariety of $\underline{a}^*$, but need not be irreducible even if M is simple ([32], Sect. 9). We can conveniently define the Gelfand-Kirillov dimension $d(M)$ of M as the dimension of $\underline{V}(M)$. We call M, d-homogeneous (resp. d-critical) if $d(N) = d(M)$ (resp. $d(M/N) < d(M)$) for any non-zero submodule N of M.

2.2. Let A be a prime noetherian ring (for us always both left and right noetherian). Then by Goldie's theorem A embeds in a unique smallest ring Fract A in which the non-zero divisors of A are invertible. Furthermore, by the Artin-Wedderburn theorem Fract A is isomorphic to a matrix ring over a skew-field. We call the rank of this matrix ring the Goldie rank $\operatorname{rk} A$ of A and the skew-field the Goldie skew field of A. For example, if we call $A_m := \mathbb{C}[x_1, x_2, \ldots, x_m, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}]$ the Weyl algebra of index m over $\mathbb{C}$, then $\operatorname{rk} A_m = 1$, and so Fract A_m is a skew-field called the Weyl skew-field of index m.

2.3. Let $\underline{g}$ be a complex semisimple Lie algebra, $\underline{h}$ a Cartan subalgebra, $R \subset \underline{h}^*$ the set of non-zero roots, $B_0 \subset R$ a choice of simple roots, R^+ the corresponding positive system, $P(R)$ the lattice of weights, s_α the reflection corresponding to the root α and W the subgroup of $\operatorname{Aut} \underline{h}^*$ generated by the $s_\alpha : \alpha \in R$. It is called the Weyl group. Fix a Chevalley basis $\{X_\alpha, H_\beta\}_{\alpha \in R, \beta \in B}$ for $\underline{g}$ and set

$$\underline{n}^+ = \bigoplus_{\alpha \in R^+} \mathbb{C}X_\alpha, \quad \underline{n}^- = {}^t\underline{n}^+, \quad \underline{b} = \underline{n}^+ \oplus \underline{h}$$

where $u \mapsto {}^t u$ is the Chevalley antiautomorphism of $\mathfrak{U}(g)$ defined by setting ${}^t X_\alpha = X_{-\alpha}$, $\forall \alpha \in R$. Let $u \mapsto \check{u}$ be the principal antiautomorphism of $\mathfrak{U}(\underline{g})$ defined through $\check{X} = -X$, $\forall X \in \underline{g}$. Define

$j: \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ through $j(X) = (X, {}^t\check{X})$ and set $\underline{k} = j(\mathfrak{g})$.

2.4. For each $\lambda \in \underline{h}^*$ set $R_\lambda := \{\alpha \in R \mid (\alpha^\vee, \lambda) \in \mathbb{Z}\}$ (where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$). This is again a root system whose Weyl group W_λ is generated by the $s_\alpha : \alpha \in R_\lambda$. Set $R_\lambda^+ = R_\lambda \cap R^+$ and let $B_\lambda \subset R_\lambda^+$ denote the corresponding simple roots. It is not always possible to find an element $w \in W$ such that $wB_\lambda \subset B_0$. This can cause some technical difficulties.

Call $\lambda \in \underline{h}^*$ dominant if $(\lambda,\alpha) \geq 0$, $\forall \alpha \in R_\lambda^+$ and regular if $(\lambda,\alpha) \neq 0$, $\forall \alpha \in R$. From now on we assume that λ is a fixed dominant and regular element of $\underline{h}^*$ and set $\Lambda = \lambda + P(R)$. Let Λ^+ (resp. Λ^{++}) denote the dominant (resp. dominant and regular) elements of Λ.

2.5. Let M, N be $\mathfrak{U}(\mathfrak{g})$ modules. Then $\mathrm{Hom}_{\mathbb{C}}(M,N)$ is a $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ module for the action: $((a \otimes b)\,x)m = {}^t\check{a}x\check{b}m$, $\forall a,b \in \mathfrak{U}(\mathfrak{g})$, $x \in \mathrm{Hom}_{\mathbb{C}}(M,N)$, $m \in M$. It admits the important submodule (notation 2.3) $L(M,N) := \{x \in \mathrm{Hom}_{\mathbb{C}}(M,N) \mid \dim \mathfrak{U}(\underline{k})x < \infty\}$ (where we have canonically identified $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ with $\mathfrak{U}(\mathfrak{g} \times \mathfrak{g})$). For M simple, Kostant suggested that the map $\mathfrak{U}(\mathfrak{g}) \to L(M,M)$ defined by the action of $\mathfrak{U}(\mathfrak{g})$ should always be surjective; but in fact this is seldom true. We call $L(M,M)$ the Kostant ring of M.

2.6. Let ρ denote the half sum of the positive roots. Let M be a $\mathfrak{U}(\mathfrak{g})$ module. A highest weight vector $e_\mu \in M$ of weight $\mu-\rho \in \underline{h}^*$ is an element of M satisfying $Xe_\mu = 0$, $\forall X \in \underline{n}^+$, $He_\mu = (H,\mu-\rho)e_\mu$, $\forall H \in \underline{h}$. A highest weight module with highest weight $\mu-\rho$ is a module generated by a highest weight vector e_μ and which by the universality of the tensor product is an image of the Verma module $M(\mu) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\underline{b})} \mathbb{C}e_\mu$. It is easily shown that $M(\mu)$ has a unique maximal submodule $\overline{M(\mu)}$ and that $L(\mu) := M(\mu)/\overline{M(\mu)}$, is just the unique simple highest weight module with highest weight $\mu-\rho$. We set $J(\mu) = \mathrm{Ann}\, L(\mu)$.

2.7. In studying highest weight modules it is natural to introduce the decomposition

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\underline{h}) \oplus (\underline{n}^-\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\underline{n}^+)$$

and let P denote the projection $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\underline{h})$ it defines. For example, for all $z \in \mathfrak{Z}(\mathfrak{g})$ we see that $ze_\mu = P(z)e_\mu = (P(z),\mu-\rho)e_\mu$ where $P(z)$ is considered as a polynomial function on $\underline{h}^*$. In particular, $\mathfrak{Z}(\mathfrak{g})$ acts on $M(\mu)$ by a scalar and more precisely a theorem of Harish-Chandra ([13], Chap. 7) asserts that this scalar depends exactly on the orbit $\hat{\mu}$ of μ under W. Again one easily verifies that $(ae_\mu, be_\mu) \mapsto (P({}^tab),\mu-\rho)$ is a symmetric bilinear form (the contravariant form) on $M(\mu)$ satisfying $(am,n) = (m,{}^tan)$, $\forall m,n \in M(\mu)$, $\forall a \in \mathfrak{U}(\mathfrak{g})$, with kernel $\overline{M(\mu)}$. As noted by Duflo, this gives that $J(\mu) := \mathrm{Ann}\, L(\mu) = \{a \in \mathfrak{U}(\mathfrak{g}) \mid (P({}^tbac),\mu-\rho) = 0,\ \forall b,c \in \mathfrak{U}(\mathfrak{g})\}$ which has the important consequence that ${}^tJ(\mu) = J(\mu)$.

Again given I a two-sided ideal of $\mathfrak{U}(\mathfrak{g})$ we define its characteristic variety $\underline{\mathrm{Ch}}(I)$ through $\underline{\mathrm{Ch}}(I) = \underline{V}(P(I)) + \rho$. One sees that $\underline{\mathrm{Ch}}(I) = \{\mu \in \underline{h}^* \mid I \subset J(\mu)\}$ and so $\mu \in \underline{\mathrm{Ch}}(J(\mu))$.

3. DUFLO'S THEOREM AND AN EQUIVALENCE OF CATEGORIES

3.1. Let $\mathcal{O}$ denote the category of all $\mathfrak{U}(\mathfrak{g})$ modules M satisfying, (i) $\dim \mathfrak{Z}(\mathfrak{g})m < \infty$, $\forall m \in M$, (ii) $\dim \mathfrak{U}(\underline{b})m < \infty$, $\forall m \in M$, (iii) M is a direct sum of simple finite dimensional $\underline{h}$ weight subspaces. It is easy to see that each $M \in \mathrm{Ob}\,\mathcal{O}$ has a filtration by highest weight modules and further taking account of Harish-Chandra's theorem (2.7) it follows that each $M \in \mathrm{Ob}\,\mathcal{O}$ has finite length with composition factors amongst the $L(\mu) : \mu \in \underline{h}^*$. (See also [13], Chap. 7.) 4.5.1

For each $M \in \mathrm{Ob}\,\mathcal{O}$ give M^* a $\mathfrak{U}(\mathfrak{g})$ module structure through $(a\xi, m) = (\xi, {}^tam)$, $\forall \xi \in M^*$, $m \in M$, $a \in \mathfrak{U}(\mathfrak{g})$ (notation 2.3) and set

$$\delta(M) = \{m \in M^* \mid \dim \mathfrak{U}(\underline{h})m < \infty\} \text{ which is a } \mathfrak{U}(\mathfrak{g})$$

submodule. One shows that $\delta(M) \in \mathrm{Ob}\,\mathcal{O}$ and that the functor

$\delta: M \mapsto \delta M$ on $\underline{O}$ is exact and contravariant. We note that a homomorphism $M \to \delta M$ is equivalent to a contravariant form on M and so from 2.7 there exists $\varphi: M(\mu) \to \delta M(\mu)$ such that $L(\mu) \cong \operatorname{Im} \varphi$.

3.2. It is technically advantageous to restrict to the full subcategory $\underline{O}_\Lambda$ of $\underline{O}$ of objects whose $\underline{h}$ weights lie in Λ. Let E denote the category of all finite dimensional $\mathfrak{U}(\underline{g})$ modules. We remark that

$$E \otimes M \in \operatorname{Ob} \underline{O}_\Lambda \quad \forall E \in \operatorname{Ob} E, \quad \forall M \in \operatorname{Ob} \underline{O}_\Lambda .$$

We define the Jantzen matrix b_Λ (or simply, b) to be the matrix with entries $b(w,w') := [M(w\lambda) : L(w'\lambda)]$ (where $[M : L]$ denotes the number of times the simple factor L occurs in a composition series for M). Using the exact functor $M \mapsto E \otimes M : E \in \operatorname{Ob} E$, (translation functor) Jantzen ([18], Chap. 2) showed that b_Λ is independent of the choice of $\mu \in \Lambda^{++}$. This fails for $\mu \in \Lambda^+$; but the appropriate multiplicities can be rather simply deduced from the regular case. This phenomenon runs through the whole theory.

3.3. Let G denote the algebraic adjoint group of $\underline{g}$ and B (resp. H) the algebraic subgroup of G with Lie algebra $\underline{b}$ (resp. $\underline{h}$). Then W identifies with $N_G(H)/H$ and we have the Bruhat decomposition

$$G = \coprod_{w \in W} Bn_wB$$

where $n_w \in N_G(H)$ is a representative of $w \in W$. In the sequel we shall often write w for n_w.

By continuity of multiplication the Zariski closure $\overline{Bn_wB}$ of Bn_wB is a union of B double cosets and we set $y \le w$ if $Bn_yB \subset \overline{Bn_wB}$. It is called the Bruhat order on W and can be described purely combinatorially. Using this combinatorial description and the translation functor, Bernstein, Gelfand and Gelfand ([13], Chap. 7) showed that $b(w,w') \ne 0 \Leftrightarrow w' \le w'$. (One has now a more intrinsic understanding of this result through a study of differential operators on the flag variety G/B.) It follows that b_Λ is an invertible matrix and

we denote by a_Λ (or simply, a) its inverse and define

$$(*) \qquad a(w) := \sum_{w' \in W} a(w,w')w', \qquad \forall w \in W_\lambda .$$

The $a(w): w \in W_\lambda$ which form a basis for $\mathbb{Z}W_\lambda$ play a fundamental role in the study of primitive ideals. This stems from the following observation. Let $\underline{O}_{\hat{\lambda}}$ denote the full subcategory of $\underline{O}_\Lambda$ of objects with simple factors amongst the $L(w\lambda): w \in W_\lambda$. If we represent the element $[M(w\lambda)]$ of the Grothendieck group of $\underline{O}_\Lambda$ by w, then $[L(w\lambda)]$ is represented by $a(w)$. Furthermore this identification with $\mathbb{Z}W_\lambda$ gives the Grothendieck group a $W_\lambda - W_\lambda$ bimodule structure. Moreover, the right module structure can be implemented through an astute use of the translation functors (see [23] for example), an observation essentially due to Jantzen and developed by Vogan [39, 40] and the author. Miraculously the left module structure can also be implemented in a natural way, either through the Enright functor (see [28, 29] for example) or through what eventually comes to the same thing, the equivalence of categories theorem described below.

3.4. Let $\underline{H}$ denote the category of all $\mathfrak{U}(\underline{g} \times \underline{g})$ modules V satisfying,

(i) $\dim \mathfrak{Z}(\underline{g} \times \underline{g})v < \infty, \quad \forall v \in V,$

(ii) $\dim \mathfrak{U}(\underline{k})v < \infty, \quad \forall v \in V$ (notation 2.3),

(iii) $[V : E] < \infty$ for each finite dimensional simple $\underline{k}$ module E.

One calls $V \in \operatorname{Ob} \underline{H}$ a Harish-Chandra module (for the pair $(\underline{g} \times \underline{g}, \underline{k})$). It has been known for some time that the theory of Harish-Chandra modules follows closely that of highest weight modules. For example, the objects of $\underline{H}$ all have finite length and for each choice of central character the simple objects are indexed by W. This led eventually to the following result developed essentially independently in [17, 22] and in [5]. First observe that each object V in $\underline{H}$ can be considered as a $\mathfrak{U}(\underline{g}) - \mathfrak{U}(\underline{g})$ bimodule through $(a \otimes b)\cdot x = {}^t\check{a}x\check{b}$,

$\forall a,b \in \mathfrak{U}(\underline{g})$, $x \in V$ and let $\underline{H}_{\hat\lambda}$ denote the full subcategory of all objects in $\underline{H}$ for which the right action of $\mathfrak{Z}(\underline{g})$ is just that given by its action on $M(\lambda)$. One may fairly easily show that $T(N) := L(M(\lambda),N) \in \text{Ob}\ \underline{H}_{\hat\lambda}$ (notation 2.5) for all $N \in \text{Ob}\ \underline{\mathcal{O}}_\Lambda$. Moreover, since λ is dominant, $M(\lambda)$ is projective on $\underline{\mathcal{O}}_\Lambda$ and consequently the functor $T: N \mapsto T(N)$ is exact. Conversely given $V \in \text{Ob}\ \underline{H}_{\hat\lambda}$ one checks that $T'(V) := V \otimes_{\mathfrak{U}(\underline{g})} M(\lambda) \in \text{Ob}\ \underline{\mathcal{O}}_\Lambda$ and that T' is a left adjoint to T. straightforward

THEOREM. The unit $\text{Id}_{\underline{H}_{\hat\lambda}} \to TT'$ and counit $T'T \to \text{Id}\ \underline{\mathcal{O}}_\Lambda$ maps are isomorphisms of functors. In particular, T' is exact and T is an equivalence of categories.

We remark that the proof depends heavily on the fact that the map $\mathfrak{U}(\underline{g}) \mapsto L(M(\lambda),M(\lambda))$ is surjective (†). This in turn is (an easy) consequence of a classical theorem of Kostant.

3.5. The above result gives objects in the $\underline{\mathcal{O}}_\Lambda$ category a $\mathfrak{U}(\underline{g})$ - $\mathfrak{U}(\underline{g})$ bimodule structure. It allows one to implement the left action of W_λ on the Grothendieck group of $\underline{\mathcal{O}}_{\hat\lambda}$ (as discussed in 3.~~1~~3) and implies, for example, the symmetry relation

$$(*) \qquad a(w,w') = a(w^{-1},w'^{-1}), \qquad \forall w,w' \in W_\lambda .$$

Again it shows that the simple objects in $\underline{H}_{\hat\lambda}$ are just the $V(-w\mu,-\lambda) := L(M(\lambda),L(w\mu))$: $w \in W_\lambda$, $\mu \in \Lambda^+$. Finally it implies that the map $I \mapsto IM(\lambda)$ of ideals of $A_\lambda := \mathfrak{U}(\underline{g})/\text{Ann}\ M(\lambda)$ to submodules of $M(\lambda)$ is bijective*. (This was conjectured by Dixmier.) In particular, it follows that $I = \text{Ann}(M(\lambda)/IM(\lambda))$ for any such ideal. Furthermore, if I is a prime ideal, it follows from Goldie's theorem that $\mathfrak{U}(\underline{g})/I$ is d-critical (see 2.1) and hence so is $M(\lambda)/IM(\lambda)$. Consequently, $M(\lambda)/IM(\lambda)$ admits a unique simple submodule which must take the form $L(\sigma\lambda)$: $\sigma \in W_\lambda$, and furthermore $I = \text{Ann}\ L(\sigma\lambda)$. The latter being stable under the Chevalley antiautomorphism (2.7) implies eventually

* b/c we have the exact sequence

$0 \to \text{Ann}(M(\lambda)) \to U(g) \to L(M(\lambda),M(\lambda)) \xrightarrow{\text{Kostant (†)}} 0$

that σ is an involution. We have the

COROLLARY. The map $I \mapsto \text{Soc}(M(\lambda)/IM(\lambda)) = L(\sigma\lambda)$ is a bijection of $\text{Spec}\ A_\lambda$ onto a subset $\sum_\lambda^0$ of involutions of W_λ.

We call $\sum_\lambda^0$ the Duflo set. It coincides with the set of all involutions $\sum_\lambda$ of W_λ if and only if R_λ is a union of systems of type A_n.

3.6. For any $\mu \in \underline{h}^*$ a classical result of Kostant can be used to show ([13], Chap. 8) that $\text{Ann}\ M(\mu)$ is generated by its intersection with $\mathfrak{Z}(\underline{g})$, which (as noted in 2.7) is a maximal ideal of $\mathfrak{Z}(\underline{g})$. Thus any primitive ideal of $\mathfrak{U}(\underline{g})$ contains some $\text{Ann}\ M(\mu)$ and we can assume λ chosen such that $\mu \in \Lambda^+$. Let $\underline{X}_{\hat\mu}$ denote the set of all primitive ideals of $\mathfrak{U}(\underline{g})$ containing $\text{Ann}\ M(\mu)$. Then map $I \mapsto I/\text{Ann}\ M(\mu)$ identifies $\underline{X}_{\hat\mu}$ with $\text{Spec}\ A_\mu$ described (for μ regular) by 3.5. This is Duflo's [14] description of $\text{Prim}\ \mathfrak{U}(\underline{g})$ and we remark that after Borho-Jantzen ([8], Sec. 2) the $\underline{X}_{\hat\mu}$: $\mu \in \Lambda^{++}$ are isomorphic as ordered sets (ordered by inclusion). Apart from some degeneration (see 4.1) this last assertion extends to Λ^+.

3.7. The fundamental result stated in 3.4 can be regarded as a consequence of the following simple geometric fact. First note that the algebraic adjoint group of $\underline{g} \times \underline{g}$ coincides with $G \times G$ and we let K denote the algebraic subgroup of $G \times G$ with Lie algebra $\{(X,X) : X \in \underline{g}\}$ (we make a slight change of convention). The map $(x,y) \mapsto xy^{-1}$ of $G \times G \to G$ defines by passage to quotients a map of $G/B \times G/B \to B\backslash G/B$ which identifies the K orbits in the flag variety $G/B \times G/B$ of $G \times G$ with the B orbits in the flag variety G/B of G.

More generally let K be any algebraic subgroup of G such that $K\backslash G/B$ is finite. One may study the category $\underline{H}^K$ of $\mathfrak{U}(\underline{g})$ modules with an algebraic K action consistent with action of K on $\mathfrak{U}(\underline{g})$. The simple objects in this category have been classified by Beilinson and Bernstein [3] in a tour de force which represents a far reaching generalization of the Langslands classification (relevant to the case when G is a real Lie group and K a maximal compact subgroup). For simplicity let us just consider the subcategory $\underline{H}^K_{\hat{\rho}}$ of objects of $\underline{H}^K$ whose annihilators contain $\mathrm{Ann}\, M(\rho)$.

THEOREM. ([3], Sect. 3). The simple objects in $H^K_{\hat{\rho}}$ are in bijection with the set of pairs consisting of a K orbit S in G/B and an irreducible character x of the component group of the stabilizer of some $s \in S$.

In fact, the construction assigns to such pairs two objects in $H^K_{\hat{\rho}}$. The first, say $M_{S,x}$, generalizes the Verma module $M(w\rho)$ (assigned to the Bruhat cell Bn_wB/B, the stabilizer group of a point being connected). The second, say $L_{S,x}$, is constructed as a quotient of $M_{S,x}$ by generalizing 6.

One has then the general problem of computing $[M_{S,x} : L_{S',x'}]$ which has not yet been fully resolved; but which may be reduced to a special case of a problem in algebraic topology in the framework of the DGM intersection cohomology theory, where the simple objects $L_{S,x}$ correspond to the DGM sheaves.

4. THE ADDITIVITY PRINCIPLE, GOLDIE RANK AND WEYL GROUP REPRESENTATONS

4.1. Fix $w \in W_\lambda$. We define $\{J(w\mu)\}_{\mu\in\Lambda^+}$ to be the t-clan associated to w. Distinct elements in a given t-clan correspond to distinct central characters, i.e., the $Z(\underline{g}) \cap J(w\mu)$: $\mu \in \Lambda^+$ are pairwise distinct.

Now define a function p_w on Λ^+ through (notation 2.2)

$$p_w(\mu) := \mathrm{rk}(\mathfrak{U}(\underline{g})/J(w\mu)),$$

which is determined up to a scalar by the

THEOREM. ([25], 5.1). For each $w \in W_\lambda$, there exists a positive rational number c_w such that

$$(*) \qquad p_w = c_w a(w^{-1})\rho^m \qquad \text{(notation 3.3*)}$$

where m is the least integer ≥ 0 for which the right-hand side is non-zero. Furthermore $m = \mathrm{card}\, R^+ - d(L(w\lambda))$ (notation 2.1).

Observe first of all that this result implies that p_w extends to a polynomial function (homogeneous of degree m) on $\underline{h}^*$. This in itself is a remarkable fact. Secondly note that (up to a scalar) p_w is completely determined by the Jantzen matrix, in fulfillment of the philosophy outlined in 1.2. We remark that the determination of the scale factor c_w is rather delicate. Obviously c_w must be sufficiently large so that $p_w(\mu)$ is an integer ≥ 0 for all $\mu \in \Lambda^+$. The natural conjecture is that it is the smallest rational number with this property, through this is difficult to check even in type A_n where these coefficients are known implicitly. Note that this property is immediately implied if $p_w(\mu) = 1$ for some $\mu \in \Lambda^+$, thus the importance of constructing completely prime (i.e., Goldie rank 1) primitive ideals. Part of the difficulty is that within a given t-clan it is relatively unusual to find such ideals. When $\lambda \in P(R)^{++}$, then

$$a(1) = \sum_{w\in W} w(\det w)$$

and substitution in (*) gives Weyl's dimension formula, namely

$$\dim L(\lambda) = \prod_{\alpha\in R^+} \frac{(\lambda,\alpha)}{(\rho,\alpha)},$$

where we have used the fact that $J(\rho)$ is the augmentation ideal of $\mathfrak{U}(\underline{g})$ and hence completely prime.

Though $p_w(\mu) > 0$, $\forall \mu \in \Lambda^{++}$ it can happen that $p_w(\mu) = 0$ if $\mu \in \Lambda^+$. It turns out that this corresponds to the degeneration of $J(w\mu)$ to $\mathfrak{U}(\underline{g})$ under translation to a wall as originally described by Borho and Jantzen ([8], 2. 12). Moreover if we set $\tau(w) = \{\alpha \in B_\lambda \mid w\alpha \in R^+\}$ then p_w is divisible by the polynomial

$$p_{R'} := \prod_{\alpha \in R' \cap R^+} \alpha$$

where R' is the root system generated by $\tau(w)$ (i.e. $R' = \mathbb{Z}\tau(w) \cap R$). Fix $\alpha \in B_\lambda$; if $(\mu,\beta) = 0$, $\beta \in B_\lambda$ implies $\beta = \alpha$, then $p_w(\mu) = 0 \Leftrightarrow \alpha \in \tau(w)$.

4.2. Let us just discuss briefly the step in the proof of 4.1 which implies the polynomial nature of the p_w. This is based on an <u>additivity principle</u> for Goldie rank, namely

THEOREM. ([24], 5.11). <u>For each simple object</u> $N \in \mathrm{Ob}\ \mathcal{O}$ <u>and each</u> $E \in \mathrm{Ob}\ \mathcal{E}$ <u>one has</u>

$$(*) \qquad \dim E.\ \mathrm{rk}\ L(N,N) = \sum \mathrm{rk}\ L(N_i,N_i)$$

<u>where the sum runs over the simple factors</u> N_i <u>of</u> $E \otimes N$ <u>satisfying</u> $d(N_i) = d(N)$.

Of course to use this result one must be able to relate $\mathrm{rk}(U(\underline{g})/\mathrm{Ann}\ N)$ to $\mathrm{rk}\ L(N,N)$. A key point here is that these integers are equal when $N \cong L(\sigma\lambda)$ for σ a Duflo involution (3.5). The proof of the above result relies partly on abstract ring theory—in particular Small's refinement of Goldie's theorem which shows that the left hand side equals $\sum y_i^{-1}\ \mathrm{rk}\ L(N_i,N_i)$ for some $y_i \in \mathbb{N}^+ \cup \{\infty\}$ ([24], Thm. 5.6) and partly on 3.4 which is used to show ([24], 3.2) that the endomorphism ring of $L(M(\lambda),L(w\lambda))$ considered as a right $U(\underline{g})$ module is just $L(L(w\lambda),L(w\lambda))$, a result which eventually implies that the y_i are all equal to 1. That (*) implies the p_w to be polynomial eventually results from Vogan's observation ([38], Sect. 4) that <u>any</u> function p on Λ satisfying

$$\sum_{w \in W} p(\lambda + w\mu) = (\mathrm{card}\ W)p(\lambda), \qquad \forall \mu \in P(R)$$

is a polynomial (in fact a W-harmonic polynomial).

4.3. Theorem 4.1 has some important further consequences when one takes

account of the $W_\lambda - W_\lambda$ module structure on the Grothendieck group of $\mathcal{O}_{\hat{\lambda}}$ implemented by the translation functors (3.2). For this it is convenient to set $p_w = p_J$ when $J = J(w\mu)$. Since by 3.6

$$J(w\mu) = J(y\mu) \Leftrightarrow J(w\mu') = J(y\mu'), \qquad \forall \mu,\mu' \in \Lambda^{++},$$

we may regard J as representing the whole t-clan associated to w.

Let τ be an irreducible representation of W. Because W is finite group and acts faithfully on $\underline{h}^*$, there exists a smallest integer $m \geq 0$ such that $[S_m(\underline{h}) : \tau] > 0$ (notation 2.1). Call τ <u>univalent</u> (with respect to $\underline{h}$) if $[S_m(\underline{h}) : \tau] = 1$. The significance of τ being univalent is that then $S_m(\underline{h})$ admits a unique simple submodule P_τ of type τ. We call P_τ the <u>univalent module</u> associated to τ. Not all Weyl group representations are univalent. However, we remark that if R' is a root subsystem then $p_{R'}$ (notation 4.1) generates a univalent submodule $P_{R'}$ of $S(\underline{h})$, though not all univalent representations are so obtained.

THEOREM. ([22], 5.5) <u>There exists a subset</u> Ω_λ <u>of the univalent representations of</u> W_λ <u>such that</u> (<u>notation</u> 3.6)

(i) $\underline{X}_{\hat{\lambda}} = \coprod_{\tau \in \Omega_\lambda} (\underline{X}_{\hat{\lambda}})_\tau$.

(ii) $\{p_J : J \in (\underline{X}_{\hat{\lambda}})_\tau\}$ <u>is a basis for the univalent module</u> P_τ <u>of</u> W_λ (<u>actually a</u> $\mathbb{Z}$ <u>basis</u>).

(iii) WP_τ <u>is a univalent</u> (<u>hence irreducible</u>) W <u>module</u>.

We call $(\underline{X}_{\hat{\lambda}})_\tau$ an r-clan. Each r-clan is associated to a (distinct) irreducible representation of W. Univalence is a consequence of the fact that the left action of W_λ on the Grothendieck group of $\mathcal{O}_{\hat{\lambda}}$, as implemented through translation and use of 3.4, does not affect annihilators. Since Goldie rank is an invariant this eventually implies that $a(w^{-1})w'\rho^m$ is proportional to p_w for all $w' \in W_\lambda$. The fact (ii) that the Goldie rank polynomials are linearly independent for distinct annihilators results through a use of the characteristic variety

(2.7).

4.4. One has $\Omega_\lambda \subset \hat{W}_\lambda$ with equality iff R_λ has only type A_n factors. To determine Ω_λ in general it is enough to assume $\lambda \in P(R)^{++}$ (i.e., integral). This is because Ω_λ is implicitly determined by the Jantzen matrix and from the truth of the Kazhdan-Lusztig conjecture this depends only in the description of W_λ as a Coxeter group. Now suppose R' is a root subsystem of R with Weyl group W'. If $P \subset S(\underline{h})$ is a univalent module for W', then WP is a univalent module for W. This was pointed out by Lusztig and Spaltenstein [33]. The process $P \mapsto WP$ mirrors induction of primitive ideals and one may remark that WP is an irreducible component of $\mathrm{Ind}(P;W'\uparrow W)$. It is easy to show that Ω is stable under the above induction procedure where R' takes the form $R' = \mathbb{Z}B' \cap R$ for some subset $B' \subset B_0$. Noting that p_R (notation 4.1) generates the sign representation sg of W, one might also conjecture that Ω is stable under tensoring with sg. Remarkably this is very nearly but not quite true. In fact following a suggestion of Lusztig one defines an involution $*$ on $\hat{W}$ which is tensoring by sg except in three cases, one in type E_7 and two in type E_8, and one defines a representation to <u>special</u> if it obtains from the identity representation (of an appropriate Weyl subgroup of W) by a combination of $*$ and the above described induction.

THEOREM. ([1, 2]). Ω <u>is just the set of special representations of</u> W.

Not surprisingly this result due to Barbasch and Vogan involves some case by case analysis. However one can understand that this results depends heavily on some symmetry property of the Jantzen matrix (5.3*) which can be thought of as implementing $*$.

4.5 It can be shown that the Goldie skew-field of $\mathfrak{U}(\underline{g})/J : J \in \mathrm{Prim}\,\mathfrak{U}(\underline{g})$ is constant on t-clans ([24], 4.8). One expects the Goldie skew-field (2.2) to be also constant on r-clans; but only a slightly weaker statement has been shown. Finally one expects the Goldie skew-field to always be a Weyl skew-field of index $\frac{1}{2}d(\mathfrak{U}(\underline{g})/J)$. For example this has been verified in type A_n ([24], 4.8).

4.6. For each $w \in W_\lambda$, set $S_\lambda(w) = \{\alpha \in R_\lambda^+ \mid w\alpha \in R_\lambda^-\}$ and $\ell_\lambda(w) = \mathrm{card}\, S_\lambda(w)$ which coincides with the reduced length of w. Given $B' \subset B$, let $W_{B'}$ denote the subgroup of W_λ generated by the s_α: $\alpha \in B'$ and let $w_{B'}$ denote the unique longest element of $W_{B'}$. Set $w_\lambda = w_{B_\lambda}$.

LEMMA. ([30], 8.18). <u>For all</u> $w \in W_\lambda$ <u>one has</u> $\deg p_w \le \ell_\lambda(w)$ <u>with equality iff</u> $w = w_{B'}$ <u>for some</u> $B' \subset B_\lambda$ <u>and then</u> p_w <u>is proportional to</u> $p_{R'}$, <u>where</u> $R' = \mathbb{Z}B' \cap R$.

Because $p_w: w \in W_\lambda$ is a W harmonic polynomial, it can be expressed as a sum formed from products of distinct positive roots (i.e. in R_λ^+). Since p_w must take positive values on Λ^{++}, it is natural to conjecture that the coefficients can always be chosen to be positive. If this holds we can obviously write $p_w = \sum c_K \alpha^K$: $c_K > 0$, and where α_K denotes a product of the elements of B_λ. Let $\mathrm{supp}\, p_w$ denote the set of $\alpha \in B_\lambda$ which occurs in the above expression. If $p_w(\mu) = 1$ for some $\mu \in \Lambda^{++}$ it follows from the positivity of the c_K that $p_w(\nu) = 0$ for all $\nu \in \Lambda^+$ satisfying (ν, α) for some $\alpha \in \mathrm{Supp}\, p_w$. Consequently (4.1) $p_{R'}$ divides p_w where $R' = \mathbb{Z}(\mathrm{Supp}\, p_w) \cap R$ and because $p_{R'}$ is the highest degree W-harmonic polynomial on $\mathbb{Q}(\mathrm{Supp}\, p_w)$ it follows that p_w is proportional to $p_{R'}$.

The appearance of Goldie rank 1 for $\mu \in \Lambda^+$ is certainly much more subtle. We conjecture that for each $\tau \in \Omega_\lambda$ there exists $J \in (\underline{X}_\lambda^\wedge)_\tau$ such that $p_J(\mu) = 1$ for some $\mu \in \Lambda^+$. When P_τ takes the form $P_{R'}$ for some $R' = \mathbb{Z}B' \cap R$ then for each $J \in (\underline{X}_\lambda^\wedge)_\tau$ one can find $w \in W_\lambda$ such that

$$(*) \qquad wp_{R'} = \sum n_{J'} p_{J'} ,$$

with $n_{J'} \in \mathbb{N}$ and $n_J > 0$. If $p_J(\mu) = 1$ for some $\mu \in \Lambda^+$, we conjecture that it is possible to choose w in (*) such that $(wp_{R'})(\mu) = 1$ (so $n_J = 1$, and $p_{J'}(\mu) = 0$ for $n_{J'} \neq 0$, $J' \neq J$). For example, in type A_n this would imply that all the primitive ideals of Goldie rank 1 are induced, or in the language of ([8], Sect. 5) that the Dixmier map is surjective. This has been checked up to $n = 5$ by A. Klugman.

5. ORDERING OF PRIMITIVE IDEALS

5.1. If $x \in \mathbb{Q}W_\lambda$, we can write

$$x = \sum_{w \in W_\lambda} c_w a(w)$$

and we then let $[x]$ denote the set of $a(w)$ for which $c_w \neq 0$. For each subset $S \subset \mathbb{Q}W_\lambda$ we set

$$[S] = \bigcup_{x \in S} [x].$$

For each $w \in W_\lambda$ we set $D(w) := [W_\lambda a(w)]$. The set of all $w' \in W_\lambda$ such that $a(w') \in D(w)$ is called the <u>left cone</u> $\mathcal{D}(w)$ of W_λ generated by w. We similarly define $D'(w) := [a(w)W_\lambda]$ and let $\mathcal{D}'(w)$ denote the corresponding <u>right cone</u> generated by w. By 3.3(*) left and right cones are interchanged through the involution $w \mapsto w^{-1}$. Note that by 3.5(*) no ambiguity arises here.

Order the set of left (resp. right) cones of W_λ by inclusion. The <u>left cell</u> $\mathcal{C}(w)$ of W_λ generated by w is defined to be the complement in $\mathcal{D}(w)$ of the union of all the left cones strictly contained in $\mathcal{D}(w)$. A <u>right cell</u> $\mathcal{C}'(w)$ of W_λ is defined similarly with respect to $\mathcal{D}'(w)$. The left (or the right) cells of W_λ form a partition of W_λ. We set $C(w) = \{a(w) \mid w \in \mathcal{C}(w)\}$.

The above notations were introduced in [33] to try to understand the ordering of the primitive ideals. They also played a role in motivating the Kazhdan-Lusztig conjecture (see [41], Sect. 1, for example). In ([33], Sect. 5) it was shown that $\underline{Ch}(J(w\lambda)) \supset \mathcal{D}(w)$. Vogan ([39])

established equality and thereby determined completely the ordering of the primitive ideals, a problem which had hitherto appeared intractable. We may express the result (which is rather easy to prove!) through the

THEOREM. <u>For all</u> $w, w' \in W_\lambda$,

(i) $J(w'\lambda) \supset J(w\lambda) \Leftrightarrow a(w') \in D(w)$.

(ii) $J(w'\lambda) = J(w\lambda) \Leftrightarrow a(w') \in C(w)$.

5.2. The above result shows that the Jantzen matrix completely determines the ordering of the primitive ideals. Knowledge of this matrix determines the cells $\mathcal{C}(w)$ implicitly; but these have in fact been more explicitly determined. In type A_n this was obtained (cf. [20]) using the Robinson bijection and a combinatorial result of Knuth (see [21]). This has been extended to the remaining classical groups by Barbasch and Vogan [1, 2]. More generally we remark that $\mathbb{Q}D(w)$ is always a W_λ module for left multiplication. Consequently $\mathbb{Q}C(w)$ has a (quotient) W_λ module structure. It need not be irreducible; but it is easy to see from 4.1 that it contains the "Goldie rank representation" $\mathbb{Q}W_\lambda p(w)$ with multiplicity one. Barbasch and Vogan [1, 2] have determined exactly which representation occurs in each left cell; which is a little easier than determining the cells themselves.

5.3. The truth of the Kazhdan-Lusztig conjecture implies the remarkable relation

$$(*) \qquad a(w,w') = (\det ww')b(ww_\lambda, w'w_\lambda)$$

conjectured by Jantzen. It immediately implies that $D(w) \supset D(w') \Leftrightarrow D(ww_\lambda) \subset D(w'w_\lambda)$, $\forall w, w' \in W_\lambda$. Consequently (*) has three amusing consequences. First the map $J(w\lambda) \mapsto J(ww_\lambda\lambda)$ is an order antiautomorphism of $\underline{X}_{\hat{\lambda}}$. This symmetry was conjectured by Borho and Jantzen ([8], Sect. 2). Secondly (from (*)) we obtain that $\mathbb{Q}C(w) \otimes \mathrm{sg} \cong \mathbb{Q}C(ww_\lambda)$ (this fact was pointed out to me by Barbasch—it can be read off from say the computation in [26], 4.7). It underlies the Barbasch-Vogan result discussed in 4.4; but notice the extraordinarily subtle fact that the

Goldie rank representation occurring in $\mathcal{Q}C(ww_\lambda)$ need not be sg tensored with the Goldie rank representation occurring in $\mathcal{Q}C(w)$.

Finally for any A_λ (notation 3.5) module N we may define its <u>equalizer</u> $E(N)$ to be the smallest ideal I such that $IN = N$ (recall that by 3.4, A_λ is Artinian for two-sided ideals). Equalizers are always idempotent ideals and every idempotent ideal is an equalizer of some A_λ module which can be chosen by 3.4 to belong to $\mathcal{O}_\lambda$. We call an equalizer <u>primal</u> if it is the equalizer of a simple module. These can be characterized as the set of idempotent ideals of A_λ admitting a simple radical, or as the equalizer of some $L(w\lambda)$; and every idempotent ideal is a sum of such equalizers. One has [31] the

LEMMA. <u>For each</u> $w \in W_\lambda$,

$$(**) \qquad \underline{\mathrm{Ch}}(E(L(w\lambda))) = (W_\lambda \setminus \mathcal{D}(ww_\lambda))w_\lambda .$$

This indicates a duality between primal equalizers and primitive ideals.

5.4. The formula (**) of 5.3 can be regarded as a sum formula for a power of each prime ideal of A_λ in terms of primal equalizers (or dually as a formula for a primal equalizer as a power of an intersection of prime ideals of A_λ). This brings us naturally to the following

CONJECTURE. <u>Take</u> $\mu \in \Lambda^{++}$. <u>Every prime ideal of</u> A_μ <u>is idempotent (and hence itself a sum of primal equalizers)</u>.

One knows ([19], Sect. 4) that the $J(w_\lambda s_\alpha \lambda) : \alpha \in B_\lambda$, the so-called minimal primitive ideals, have idempotent images in A_λ and in A_λ one even has $J(w_\lambda s_\alpha \lambda) = E(L(s_\alpha \lambda))$. Again for any $B' \cap B_\lambda$ a result ([16], 5.2) extending Duflo's sum formula ([14], Prop. 12) gives

$$\sum_{\alpha \in B'} J(w_\lambda s_\alpha \lambda) = J(w_\lambda w_{B'} \lambda)$$

and shows that the right-hand side (which is a primitive ideal) has an

idempotent image in A_λ. Apart from these examples, there is now sufficient knowledge about the structure of A_λ derived mainly from (3.4) and the truth of the Jantzen conjecture (6.3) to render the above conjecture very plausible. Finally we remark that the above conjecture may be regarded as an aspect of calculating generators (natural or otherwise) of a given primitive ideal. Given idempotence this is reduced via 5.3 to calculating generators for the $E(L(w\lambda))$. The latter having simple radicals are cyclic as bimodules and the cyclic vector can be chosen to be a highest weight vector with respect to the diagonal action of $\underline{g}$ and belonging to a so-called "minimal $\underline{k}$-type" (which for complex groups is always unique).

5.5. The rather special nature of some of the above observations is underlined by the fact that they fail for non-regular μ. Thus in general for $\mu \in \Lambda^+ \setminus \Lambda^{++}$, the prime spectrum of A_μ (i.e., $\underline{X}_{\hat{\mu}}$) does not admit ([8], Sect. 2) an order reversing involution. Again A_μ can admit prime ideals ([19], Sect. 4) which are not idempotent.

6. THE JANTZEN FILTRATION AND KAZHDAN-LUSZTIG CONJECTURES

6.1. With respect to a Chevelley basis the structure constants for $\underline{g}$ take integer values and consequently the free $\mathbb{Z}$ module $\underline{g}_{\mathbb{Z}}$ generated over this basis admits a Lie algebra structure. Then for any ring A we may define $\underline{g}_A := \underline{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$. Analogous meanings are assigned to $\underline{h}_A$, $\underline{b}_A$, etc.

As in the case when A is a field we may define for each $\xi \in \underline{h}_A^* := \mathrm{Hom}_A(\underline{h}_A, A)$ the $U(\underline{b}_A)$ module A_λ to be A as an A module, with $H \in \underline{h}_A$ (resp. $X \in \underline{n}_A$) acting by multiplication by (ξ, H) (resp. by zero) and set

$$M(\xi) := U(\underline{g}_A) \otimes_{U(\underline{b}_A)} A_{\xi-\rho} .$$

Now consider the special case when $A = \mathbb{C}[t]$ and $\xi = \mu + t\delta$ for some $\mu, \delta \in \underline{h}^*$ with δ regular. Then $M(\xi) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)$ is just a

Verma module for $\underline{g}$ defined over the field $\mathbb{C}(t)$. Because δ is regular, it is rather easy to check that the resulting module is irreducible and hence admits a non-degenerate contravariant form (2.7) with values in $\mathbb{C}(t)$. Restricted to $M(\xi)$ this form takes values in $\mathbb{C}[t]$. Consequently we may define for each $n \in \mathbb{N}$ a $\mathfrak{U}(\underline{g}_A)$ filtration of $M(\xi)$ through

$$M^n(\xi) = \{m \in M(\xi) \mid (M(\xi), m) \in (t^n)\}.$$

Now set $M^n(\mu) = M^n(\xi)_{t=0}$. This is a finite descending $\mathfrak{U}(\underline{g})$ filtration of $M(\mu)$ called a Jantzen filtration of $M(\mu)$. We set $M_n(\mu) = M^n(\mu)/M^{n+1}(\mu)$, $\forall n$. One has $M_0(\mu) \cong L(\mu)$.

Information on the $M_n(\mu)$ obtains by taking $A = S(\underline{h})$ and regarding $H \mapsto (\xi, H)$ as an element of $\underline{h}_A^*$. Then the contravariant form on $M(\xi)$ restricts to each $\underline{h}_A$ weight submodule $M(\xi)_{\xi-\nu-\rho} : \nu \in \mathbb{N}B$ on which the determinant D_ν is an element of $S(\underline{h})$.

THEOREM. ([34]; [18], Satz II). For each $\nu \in \mathbb{N}B$ one has

$$D_\nu(\xi) = \prod_{\alpha \in R^+} \prod_{r \in \mathbb{N}^+} ((\xi, \alpha^\vee) - r)^{P(\nu - r\alpha)}$$

where P is the Kostant partition function defined through

$$\sum_{\nu \in \mathbb{N}B} P(\nu) e^\nu = \prod_{\alpha \in R^+} \frac{1}{(1-e^\alpha)} .$$

This result is obtained by noting that $M(\xi)$ is reducible whenever $(\xi, \alpha^\vee) \in \mathbb{N}^+$ for some $\alpha \in R^+$. This gives enough information on the zeros of D_ν to conclude the above formula.

6.2. Taking $\xi = \mu + t\delta$ in 6.1, we may compute for each $\nu \in \mathbb{N}B$ the smallest integer k such that $D_\nu(\xi) \in (t^k)$ either by substitution in 6.1 or in terms of the Jantzen filtration. This eventually gives the Jantzen sum formula.

THEOREM. ([18], 5.3). For each simple $L \in \mathrm{Ob}\ \mathcal{O}$, one has

$$(*) \qquad \sum_{n>0} [M^n(\mu) : L] = \sum_{\alpha \in R^+ \mid (\alpha^\vee, \mu) \in \mathbb{N}^+} [M(s_\alpha \mu) : L].$$

Observe that from (*) if $[M(\mu) : L] \neq 0$, then either $L \cong L(\mu)$ or $[M^n(\mu) : L] \neq 0$ for some $n > 0$. In the second case $[M(s_\alpha \mu) : L] \neq 0$, for some $\alpha \in R^+$ such that $(\alpha^\vee, \mu) \in \mathbb{N}^+$. This leads to a combinatorial criterium for determining when $[M(\mu) : L] \neq 0$ and is in fact just the Bernstein-Gelfand-Gelfand result based on the Bruhat order discussed in 3.1. The above proof due to Jantzen is particularly elegant and short.

6.3. Now take $\mu = w\lambda$ with $w \in W_\lambda$ and choose $\alpha \in B_\lambda$ such that $ws_\alpha < w$ (Bruhat order). It is relatively easy to show that $\dim \mathrm{Hom}_{\mathfrak{U}(\underline{g})}(M(w\lambda), M(ws_\alpha\lambda)) = 1$ and Jantzen ([18], 5.18) made the seemingly innocuous conjecture (based on sum formulae extending 6.2) that with respect to the embedding $M(w\lambda) \hookrightarrow M(ws_\alpha\lambda)$ one must have

$$(*) \qquad M^n(w\lambda) = M(w\lambda) \cap M^{n+1}(ws_\alpha\lambda), \qquad \forall n \in \mathbb{N}.$$

Motivated by the Kazhdan-Lusztig conjecture for the Jantzen matrix it was shown ([17], 4.8) that

THEOREM. Assume (*) holds for all $\alpha \in B_\lambda$. Take $w \in W_\lambda$, $\alpha \in B_\lambda$ such that $ws_\alpha < w$. Then for all $n \in \mathbb{N}$ one has

(i) $M_n(w\lambda)$ is semi-simple.

(ii) $[M_{n+1}(ws_\alpha\lambda) : L(y\lambda)] = [M_n(w\lambda) : L(y\lambda)]$ if $y > ys_\alpha$.

(iii) $[\mathfrak{U}_\alpha M_n(w\lambda) : L(y\lambda)] = [M_n(ws_\alpha\lambda) : L(y\lambda)] + [M_{n+1}(w\lambda) : L(y\lambda)]$ if $y < ys_\alpha$, where $\mathfrak{U}_\alpha$ on $\mathcal{O}$ is exact and $\mathfrak{U}_\alpha^2 = 0$, $\mathfrak{U}_\alpha L(y\lambda) \neq 0 \Leftrightarrow y > ys_\alpha$.

Given that $M(w_\lambda\lambda) \cong L(w_\lambda\lambda)$ the above formulae determine $[\mathfrak{U}_\alpha L : L']$ and $[M_n(w\lambda) : L']$ inductively (where L, L' are simple objects in $\mathcal{O}$). Consequently the Jantzen matrix $[M(w\lambda) : L(w'\lambda)]$ can be calculated and this agreed with the formulae conjectured by Kazhdan and Lusztig ([32], Sect. 1). At least from a computational point of view the above theorem probably represents the most transparent formulation of their conjecture. Note that the multiplicities in each filtration step of $M(w\lambda)$ are determined and we call this information the JKL data.

6.4. Bernstein informed me that a proof of the Jantzen conjecture 6.3(*) (and that all Jantzen filtrations are equivalent) derives from Gabber's purity theorem ([12]). The proof is based partly on the theory of $\mathcal{D}$ modules over G/B and partly on a reduction modulo p, in which roughly speaking the different steps in the Jantzen filtration are separated as distinct eigenspaces of the Frobenius. Eventually one might hope for a simpler and purely algebraic proof. To understand the difficulty we remark that for $n = 0$ the question is equivalent to showing the following. Take $\mu = w\lambda$ and suppose $\alpha \in B_0$ satisfies $s_\alpha w < w$. Consider $M(\mu)$ as a module for the $sl(2)$ subalgebra of $\underline{g}$ generated by X_α, $X_{-\alpha}$ and show with respect to the contravariant form that $M(\mu)$ is an orthogonal direct sum of indecomposable modules. Although $M(\mu)$ admits essentially only two types of indecomposable summands, this question involves a rather delicate choice of basis. This question has also a "t analogue" equivalent to the Jantzen conjecture.

6.5. Vogan gave a somewhat different interpretation of the Kazhdan-Lusztig polynomials using the extension groups $\mathrm{Ext}^j(M(w\lambda),L(w'\lambda))$ defined in the $\underline{\mathcal{O}}$ category. Unfortunately he was only able to show ([40], 3.5) that their conjecture was equivalent to the semi-simplicity hypothesis

$$(*) \qquad \mathfrak{U}_\alpha L \text{ is } \underline{\text{semisimple}}, \qquad \forall \alpha \in B_\lambda$$

for each <u>simple object</u> $L \in \mathrm{Ob}\ \underline{\mathcal{O}}$.

We remark that (*) follows from the Jantzen conjecture through the analysis of 6.3.

Beilinson-Bernstein and independently Brylinski-Kashiwara showed how to use the theory of $\mathcal{D}$ modules over G/B to reduce the Kazhdan-Lusztig conjecture to a question on DGM sheaves solved essentially by deep results of Deligne. A similar and slightly more refined machinery

is necessary for the proof the Jantzen conjecture discussed in 6.4. Finally we remark that is an obvious conjecture ([17], Sect. 5) for the ranks of the extension groups $\mathrm{Ext}^j(M(w\lambda),M(w'\lambda))$; but this has so far remained unproven.

6.6. One of the technical advantages of 6.2 over the Kazhdan-Lusztig conjecture is that it gives a way of computing the Duflo set Σ_λ^0. Indeed for each $J \in \underline{X}_\lambda$ set $C_J = \{w \in W_\lambda \mid J = J(w\lambda)\}$ and define $k_J = \min_{w\in C_J} \min_{\ell\in\mathbb{N}} \{\ell \mid [M_\ell(\lambda) : L(w\lambda)] > 0\}$. Then using 6.2, one easily shows ([26], 4.9) that $L(\sigma\lambda)$ is the unique simple submodule of $M_{k_J}(\lambda)$ with annihilator J, where σ is the unique element of $C_J \cap \Sigma_\lambda^0$. In virtue of 5.1 this shows that the JKL data and hence Kazhdan-Lusztig polynomials implicitly determine the Duflo set Σ_λ^0.

7. THE KOSTANT RING

7.1. Let N be a simple $\mathfrak{U}(\underline{g})$ module. We have already seen (4.2) the importance of the Kostant ring $L(N,N)$ (notation 2.5) in the study of Goldie rank when N is a highest weight module. Here we discuss some general results for arbitrary N, together with some very refined results when $N \in \mathrm{Ob}\ \underline{\mathcal{O}}$. First of all we remark that $L(N,N) \in \mathrm{Ob}\ \underline{H}$ (notation 3.4). Set $J = \mathrm{Ann}\ N$. Then the action of $\mathfrak{U}(\underline{g})$ on N gives a ring embedding $\mathfrak{U}(\underline{g})/J \hookrightarrow L(N,N)$ which is <u>not</u> always surjective even if $\underline{g} \cong sl(2)$ ([27]).

Our previous remark shows that $L(N,N)$ is finitely generated as a $\mathfrak{U}(\underline{g}) - \mathfrak{U}(\underline{g})$ bimodule, hence as say a left $\mathfrak{U}(\underline{g})$ module. It follows that $L(N,N)$ is a (primitive) noetherian ring and hence admits a ring of fractions $\mathrm{Fract}\ L(N,N)$. Set $A = \mathfrak{U}(\underline{g})/\mathrm{Ann}\ N$ and let S denote the set of non-zero divisors of A. Then (by Goldie's theorem) S is an Ore set in A. One can further show that S in an Ore set in $L(N,N)$ and that $\mathrm{Fract}\ L(N,N) = S^{-1}L(N,N)$. Consequently we have an embedding $S^{-1}A \hookrightarrow \mathrm{Fract}\ L(N,N)$, which gives $\mathrm{Fract}\ L(N,N)$ a $S^{-1}A - S^{-1}A$ bimodule structure and in particular a $\mathfrak{U}(\underline{g}) - \mathfrak{U}(\underline{g})$ bimodule structure.

PROPOSITION. ([31]). $L(N,N)$ is the unique largest $\mathfrak{U}(\underline{g}) - \mathfrak{U}(\underline{g})$ submodule of Fract $L(N,N)$ which is an object of $\underline{H}$.

In principle this result greatly simplifies the classification of the possible Harish-Chandra modules which are Kostant rings of a simple $\mathfrak{U}(\underline{g})$ module. For example, if the embedding $S^{-1}A \hookrightarrow \text{Fract } L(N,N)$ is an isomorphism it follows that $L(N,N) \cong L(L(\sigma\mu),L(\sigma\mu))$ where $\sigma \in \Sigma_\lambda^0$, $\mu \in \Lambda^+$ are chosen such that $J = \text{Ann } L(\sigma\mu)$. Conjecturally in this case one even has that the map $\mathfrak{U}(\underline{g}) \to L(N,N)$ defined by the action of $\mathfrak{U}(\underline{g})$ is surjective.

7.2. Since $L(N,N) \in \text{Ob } \underline{H}$ it admits a unique smallest non-zero ideal I. We can always assume that $\mu \in \Lambda^+$ is chosen such that $I \supset \text{Ann } M(\mu)$. Set $M = I \otimes_{\mathfrak{U}(\underline{g})} M(\mu)$. One has $M \in \text{Ob}\,\underline{\mathcal{O}}_{\hat{\mu}}$. Consider $L(N,N)$ as a left $\mathfrak{U}(\underline{g})$ module. One has the

PROPOSITION. ([31]). $L(M,M) = \text{End}_{\mathfrak{U}(\underline{g})} L(N,N)$.

Conjecturally $I = \text{Soc } L(N,N)$. (This is true for highest weight modules ([26], 4.13) and follows from the truth of the Vogan semisimplicity hypothesis (6.5).) In this case M is a direct sum of simple objects in $\underline{\mathcal{O}}_{\hat{\mu}}$ and so $L(M,M)$ which is completely determined by I can be considered to be known. Obviously the above result shows that $L(N,N)$ completely determines $L(M,M)$ as a ring. What one is aiming for is to prove the converse and thus to show that as a ring $L(N,N)$ is completely determined by the $\mathfrak{U}(\underline{g}) - \mathfrak{U}(\underline{g})$ module structure of its socle.

7.3. More precise results have been obtained when N is a highest weight module. For example, we have (notation 5.1) the

LEMMA. ([16], 3.8). $L(L(w\lambda),L(y\lambda)) \neq 0 \Leftrightarrow C(w^{-1}) = C(y^{-1})$.

7.4. To describe $L(L(w\lambda),L(y\lambda))$ it is convenient to make two technical hypotheses on Σ_λ^0 which may be regarded as purely combinatorial

conjectures involving the Kazhdan-Lusztig polynomials.

Take $\sigma \in \Sigma_\lambda^0$ and recall that $\mathbb{Q}C(\sigma)$ has the structure of a left $\mathbb{Q}W_\lambda$ module.

$$(*) \qquad \mathbb{Q}C(\sigma) = \mathbb{Q}W_\lambda a(\sigma).$$

Secondly given $w,y \in W_\lambda$ one may consider the product $a(w)a(y)$ in $\mathbb{Q}W_\lambda$. Since $\mathbb{Q}W$ has basis $a(z) : z \in W_\lambda$ we can write

$$a(w)a(y) \doteq \sum_{z \in W_\lambda} c_z a(z)$$

where the symbol $\doteq$ indicates that only the terms lying in $C'(w) \cap C(y)$ have been retained.

$$(**) \qquad a(\sigma)^2 = n_\sigma a(\sigma) \quad \text{for some} \quad n_\sigma \in \mathbb{N}^+.$$

THEOREM. Assume $w,y \in W_\lambda$ satisfy $C(x^{-1}) = C(y^{-1})$. Then

(i) $L(L(w\lambda),L(y\lambda)) = \bigoplus_i L(L(\sigma\lambda),L(x_i\lambda))$ where σ is the unique element of $\Sigma_\lambda^0 \cap C(w)$.

(ii) If (*), (**) above hold, the $x_i \in W_\lambda$ are determined through the relation

$$a(w^{-1})a(y) \doteq n_\sigma(\sum a(x_i)).$$

7.5. The above result shows that to a large extent $L(L(w\lambda),L(y\lambda))$ has been completely determined. We remark that through the non-degenerate convariant form on $L(y\lambda)$ one may regard $L(L(w\lambda),L(y\lambda))$ as the largest Harish-Chandra module of $(L(y\lambda) \otimes L(w\lambda))^*$. This leads us to the following general question. Let $\underline{k}$ be a maximal compact subalgebra of $\underline{g}$. (More generally the Lie algebra of subgroup K satisfying the conditions of 3.7.) Given any simple object $L \in \text{Ob } \underline{\mathcal{O}}$ compute the largest Harish-Chandra module (which respect to the pair $(\underline{g},\underline{k})$) of L^*. It should be possible to answer this question in terms of the geometric set-up described in 3.7.

7.6. One of the interests in studying the Kostant ring $L(N,N)$ of a simple module N is that it is a more natural object than $\mathfrak{U}(\underline{g})/\text{Ann } N$.

However, from 7.5 and the remarks in 7.2 we are now coming closer to an understanding to the relationship between these two objects. This brings us to the question of determining the Jordan-Hölder composition factors of $\mathfrak{U}(\underline{g})/\text{Ann } N$ (as a $\mathfrak{U}(\underline{g}) - \mathfrak{U}(\underline{g})$ bimodule) or if one prefers of $L(N,N)$. We can always choose $\mu \in \Lambda^+$ such that $\text{Ann } N \supset \text{Ann } M(\mu)$ and then by translation principles it is enough to take μ regular. Duflo ([14], Prop. 6) determined the composition factors of $\text{Ann } N/\text{Ann } M(\mu)$; but this turns out to be rather meagre information unless of course one can determine the multiplicities. When $\text{Ann } N = J(w_\lambda w_{B'}, \lambda)$ for some $B' \subset B_\lambda$ a complete solution is given in ([26], 4.8). Unfortunately the result is misleadingly simple ([29], Sect. 5).

To attack the above problem, we formulate a conjecture. For this we define for each $J \in \underline{X}_\lambda^\wedge$ the element

$$e_J := \sum_{w \in W_\lambda} [M(\lambda)/JM(\lambda) : L(w\lambda)]a(w)$$

of $\mathbb{Z}W_\lambda$. Knowledge of the $e_J : J \in \underline{X}_\lambda^\wedge$ obviously determines the ordering of the primitive ideals. Yet this problem was also solved by 5.1. This motivates the following procedure. Let us first define an involution $*$ on $\mathbb{Z}W_\lambda$ by extending the map $w \mapsto w^{-1}$ linearly. It is the adjoint for the inner product $(y,z) \mapsto \text{tr } y^{-1}z$ on $\mathbb{Q}W_\lambda$. By 3.5(*) it follows that $a(w)^* = a(w^{-1})$. Through the existence of a non-degenerate form (2.7) on any simple object in $\underline{\mathcal{O}}$ we obtain $e_J = e_J^*$ (see also 3.5). Now recall that $\mathbb{Q}D(w)$ is a left ideal of $\mathbb{Q}W_\lambda$ (5.1). Since $\mathbb{Q}W_\lambda$ is a semi-simple Artinian ring we can write $\mathbb{Q}D(w) = \mathbb{Q}W_\lambda e(w)$ for some unique self-adjoint projection $e(w)$. Nothing could be more natural than the

CONJECTURE. <u>For each</u> $w \in W_\lambda$, $e_{J(w\lambda)}$ <u>is a multiple of</u> $e(w)$

We remark that ([29], 5.4) this holds if $w = w_\lambda w_{B'} : B' \subset B_\lambda$; but this is rather special. Again from 5.1 we do have that $e_J \in \mathbb{Q}D(w)$ and so $e_J = e_J e(w) = e(w)e_J$ where the last relation obtains on taking adjoints.

Since $e_J \doteq a(\sigma)$ where σ is the unique element of $C(w) \cap \Sigma_\lambda^0$ it follows that the above conjecture implies (*), (**) of 7.3 and furthermore that $e_J = n_\sigma e(\sigma)$. Since $e(\sigma)$ (and n_σ) are completely determined by the Jantzen matrix, one may consider that a positive answer to this conjecture determines completely the composition factors in $\mathfrak{U}(\underline{g})/J : J \in \underline{X}_\lambda^\wedge$. Yet this conjecture fails and even in (**) we must allow $n_\sigma \in Z(\mathbb{Q}W_\lambda)$.

8. ASSOCIATED VARIETIES AND SINGULAR SUPPORTS

In 8.1-8.4 $\underline{g}$ may denote an arbitrary finite dimensional Lie algebra.

8.1. Let M be a finitely generated $\mathfrak{U}(\underline{g})$ module. Its associated variety $\underline{V}(M)$ (notation 2.1) is a closed conical (i.e., $x \in \underline{V}(M) \Rightarrow \alpha x \in \underline{V}(M)$, $\alpha \in \mathbb{C}$) subvariety in $\underline{g}^*$. Given J a two-sided ideal of $\mathfrak{U}(\underline{g})$ we set $\underline{V}(J) := \underline{V}(\mathfrak{U}(\underline{g})/J)$ where $\mathfrak{U}(\underline{G})/J$ is considered as a left or as a right $\mathfrak{U}(\underline{g})$ module. We may then define $\underline{VA}(M) := \underline{V}(\text{Ann } M)$.

The interest of studying associated varieties comes from the possibility of comparing the representation theory of $\underline{g}$ to the algebraic geometry of $\underline{g}^*$. In this last respect we recall the algebraic adjoint group G acts on $\underline{g}$ and by transposition on $\underline{g}^*$. For a two-sided ideal J of $\mathfrak{U}(\underline{g})$ it is easy to show that $\underline{V}(J)$ is G stable and hence a union of G-orbits. In particular for any finitely generated module M we have that $\underline{VA}(M) \supset \overline{G\underline{V}(M)}$. A basic question is to show that equality holds.

8.2. Since $S(\underline{g})$ is commutative and identifies with $\text{gr } \mathfrak{U}(\underline{g})$, it admits a Poisson bracket $\{\ ,\ \}$ structure. This is defined on homogeneous elements and extended by linearity. We may write any homogeneous element in the form $\text{gr}_m(a) : m \in \mathbb{N}$, $a \in \mathfrak{U}^m(\underline{g})$, and then we have

$$\{\text{gr}_m(a), \text{gr}_n(b)\} := \text{gr}_{m+n-1}[a,b].$$

The Poisson bracket may be regarded as the first approximation to the commutator bracket $[a,b]$ and first arose in the description of the classical limit of quantum mechanics. We call a subvariety V of $\underline{g}^*$ involutive if its ideal of definition $I(V) := \{f \in S(\underline{g}) \mid f(V) = 0\}$ is stable under the Poisson bracket. Note that each component of an involutive variety is again involutive. One has the

THEOREM. *Let M be a finitely generated $\mathfrak{U}(\underline{g})$ module; then $V(M)$ is involutive.*

This is deep fact with a long history. Suffice to say that in its most general form a proof has been given by Gabber [15]. It is non-trivial because although $\mathrm{gr}\, J$ for any left ideal J of $\mathfrak{U}(\underline{g})$ is obviously stable under the Poisson bracket, this property is not conserved by taking radicals. One often quoted application is to show that $2d(M) \geq d(\mathfrak{U}(\underline{g})/\mathrm{Ann}\, M)$ ($\underline{g}$ algebraic); but this has a more elementary proof. A more impressive consequence is that for any $\lambda \in P(R)^{++}$ one has

$$\dim(M(\lambda)/\sum_{\alpha\in B_0} M(s_\alpha\lambda)) < \infty.$$

This result had previously only a rather roundabout (though subtle) proof using the Weyl group.

8.3. A further general (and deep) result concerning varieties is the following theorem due to Gabber [46]. Its proof derives from a similar result of Kashiwara for $\mathcal{D}$ modules. Recall 2.1.

THEOREM. *Let M be a finitely generated d-homogeneous $\mathfrak{U}(\underline{g})$ module. Then $V(M)$ is equidimensional (i.e., each component of $V(M)$ has dimension $d(M)$).*

A simple $\mathfrak{U}(\underline{g})$ module M is trivially d-homogeneous. Moreover for M simple $\mathfrak{U}(\underline{g})/\mathrm{Ann}\, M$ is d-homogeneous as a left $\mathfrak{U}(\underline{g})$ module. In particular the associated variety of a primitive ideal is always equidimensional.

8.4. There are two fundamental and interrelated problems concerning

associated varieties. The first is to determine the associated variety of any primitive ideal and of any simple module (or at least one belonging to a good class such as the highest weight modules). Secondly to show that every closed irreducible conical G stable (resp. involutive) subvariety of $\underline{g}^*$ is the associated variety of some primitive ideal (resp. simple $\mathfrak{U}(\underline{g})$ module). We remark that for $\underline{g}$ solvable it is possible to construct an irreducible representation from any $f \in \underline{g}^*$ and thus obtain a bijection between G orbits in $\underline{g}^*$ and primitive ideals in $\mathfrak{U}(\underline{g})$; whilst the passage to the associated variety is very far from being injective and can at most map onto very few orbits (i.e., the conical ones), so we are taking a different viewpoint from that of the solvable case. Experience has shown that the calculation of the associated variety is, in the semisimple case, the more relevant viewpoint.

8.5. Here and in the remainder of the section we assume $\underline{g}$ semisimple. Then $\underline{g}$ identifies with $\underline{g}^*$ through the Killing form and we call $X \in \underline{g}$ nilpotent if $\mathrm{ad}_{\underline{g}} X$ is a nilpotent endomorphism. The set N of all nilpotent elements is a closed conical G-stable subvariety of $\underline{g}^*$. More precisely if $Y(\underline{g})$ denotes the algebra of G invariant polynomial functions on $\underline{g}^*$ and Y_+ the augmentation ideal of $Y(\underline{g})$ then ([13], 8.1.3) one has $\underline{V}(Y_+S(\underline{g})) = N$. Now if $J \in \mathrm{Prim}\, \mathfrak{U}(\underline{g})$, then $J \cap Z(\underline{g}) \in \mathrm{Max}\, Z(\underline{g})$ and so $\mathrm{gr}\, J \supset Y_+S(\underline{g})$. Consequently $\underline{V}(J) \subset N$. On the other hand N is a finite union of orbits and so a fortiori is $\underline{V}(J)$. This gives the following observation of Borho and Kraft ([9], 7.2).

LEMMA. *Take $J \in \mathrm{Prim}\, \mathfrak{U}(\underline{g})$. Then $\underline{V}(J)$ is irreducible if and only if it is closure of a nilpotent orbit (i.e., G orbit of nilpotent elements).*

8.6. Giving modules in the $\mathcal{O}_\Lambda$ category a bimodule structure (see 3.5) allows one to compare properties of left and right annihilators. This leads to the following result (notation 4.3).

LEMMA. *$\underline{V}(J)$ is independent of the choice of $J \in (X_{\hat{\mu}})_\tau$ and the* choice of $\mu \in \Lambda^+$.

In more picturesque words the associated variety of a primitive ideal J depends only on the t-clan and the r-clan to which it belongs. Now by 4.3(iii) each r-clan $(\underline{X}_{\hat{\lambda}})_\tau$ is associated to a unique simple W module WP_τ, and if $\underline{V}(J)$ is irreducible it determines by 8.5 a unique nilpotent orbit in $\underline{g}^*$. On the other hand, Springer established and studied in a number of papers [36] a correspondence between nilpotent orbits and Weyl group representations and a natural conjecture is that this correspondence assigns to WP_τ exactly the dense nilpotent orbit in $\underline{V}(J)$. This was established in the integral case (i.e., $\lambda \in P(R)^{++}$) by Borho and Brylinski ([7], Sect. 6) using the result for induced ideals and some case by case considerations of Barbasch and Vogan ([1],[2]). However, this computation gives little insight into this remarkable phenomenon. Two much better viewpoints are discussed below through at present they unfortunately involve some (undoubtedly correct!) conjectures.

8.7. We have seen that closures of nilpotent orbits are the appropiate candidates for varieties of primitive ideals. Now let L be a simple highest weight module. One easily sees that $\underline{V}(L) \subset \underline{n}^+$ and is B stable and from our previous remarks it is also closed, equidimensional and involutive. Any irreducible subvariety of $\underline{n} := \underline{n}^+$ with these properties takes (see [30], 7.4 for example) the form $\underline{V}(w) := \overline{B(\underline{n} \cap w(\underline{n}))}$ for some $w \in W$, where $w(\underline{n})$ denotes the image of $\underline{n}$ under w (which is well-defined as $\underline{n}$ is H stable). The $\underline{V}(w) : w \in W$ need not all be distinct.

Now let $I(w)$ denote the ideal of definition of $\underline{V}(w)$ (defined in $S(\underline{n}^-)$). The quotient algebra $S(\underline{n}^-)/I(w)$ is $\underline{h}$ stable and so each $\nu \in P(R)^{++}$ defines in an obvious fashion an H stable gradation of this algebra. If we let $r_w(\nu)$ denote the leading coefficient of the associated Hilbert-Samuel polynomial, then it is quite easy to show

that r_w extends to rational function on $\underline{h}^*$ with denominator the product of the positive roots and numerator a homogeneous polynomial q_w of degree $\operatorname{card} R^+ - \dim \underline{V}(w)$.

Because G/B is a complete variety ([37], p. 68), $G\underline{V}(w)$ is a closed irreducible subvariety of N; hence the closure of a uniquely determined nilpotent orbit $\underline{St}(w)$. After Steinberg [37] the map $w \mapsto \underline{St}(w)$ is surjective and after Spaltenstein [35] the irreducible components of $\underline{St}(w) \cap \underline{n}$ are amongst the $\underline{V}(w') : w' \in W$. It is quite easy to show ([30], Sect. 3) that the corresponding $q_{w'}$ span a representation of W which we can naturally associate to the orbit $\underline{St}(w)$. A basic question is to show that this representation is just that defined by Springer. This is a question in pure algebraic geometry. A positive answer would imply that these representations are irreducible and pairwise non-isomorphic for distinct orbits and in particular the q_w would be linearly independent for distinct $\underline{V}(w)$. Although this can be checked in most cases as yet no satisfactory general proof has yet to be given; but R. Hotta has recently reported a proof [43].

8.8. Now even when L is a simple highest weight module $\underline{V}(L)$ need not be irreducible. It is therefore convenient to assume that we have included in the definition of $\underline{V}(L)$ the multiplicity of each component (see [30], Sect. 5). Now since $\underline{V}(L)$ is also an H stable subvariety of $\underline{n}$, we may define a polynomial p_L on $\underline{h}^*$ relative to $\underline{V}(L)$ by the procedure described in 8.7. Because $\underline{V}(L)$ is equidimensional p_L is a linear combination of the q_w occurring with coefficient equal to the multiplicity of $\underline{V}(w)$ in $\underline{V}(L)$. On the other hand, if $L = L(y\lambda)$ for some $y \in W_\lambda$ it is not difficult ([30], 5.1) to show that p_L is proportional to $p_{y^{-1}}$ (notation 4.1), that is to the Goldie rank polynomial associated to $\operatorname{Ann} L(y^{-1}\lambda)$. These observations explain rather nicely the relationship between representations of the Weyl group coming from the Goldie rank polynomials and those coming from Springer's correspondence. Counting multiplicities in $\underline{V}(\cdot)$ one has ([30], 5.2)

LEMMA. $\mathbb{Z}\underline{V}(L(w\lambda)) = \mathbb{Z}\underline{V}(L(y\lambda)) \Leftrightarrow J(w^{-1}\lambda) = J(y^{-1}\lambda), \quad \forall w,y \in W_\lambda$.

8.9. A basic technical advance in the study of $\mathfrak{U}(\underline{g})$ modules has been made possible through the theory of $\mathcal{D}_X$ modules. Here X denotes the flag variety G/B and $\mathcal{D}_X$ the sheaf of differential operators on X. Through the left action of G on G/B any $Y \in \underline{g}$ may be considered as a vector field on G/B and so as a global section $\varphi(Y) \in \Gamma(X,\mathcal{D}_X)$. Furthermore φ lifts to a ring homomorphism $\mathfrak{U}(\underline{g}) \to \Gamma(X,\mathcal{D}_X)$ which has kernel equal to Ann $M(\rho)$ (and is surjective). Consequently $\mathcal{D}_X$ admits say a right $\mathfrak{U}(\underline{g})$ module structure and a left $\mathcal{D}_X$ module structure. Using translation principles (again!) Bernstein and Beilinson [3] proved that the functor $M \mapsto \mathcal{D}_X \otimes_{\mathfrak{U}(\underline{g})} M$ from the category of finitely generated left $A_\rho := \mathfrak{U}(\underline{g})/\text{Ann } M(\rho)$ modules to the category of coherent $\mathcal{D}_X$ modules is an equivalence of categories. This result is important for many reasons. First it lies behind the proof of 3.7. Secondly it relates the intersection cohomology groups on X to extension groups in $\mathcal{O}_{\hat{\rho}}$, which eventually yields a proof of the Kazhdan-Lusztig conjectures ([3]; [10]). Thirdly it quite simply gives a new invariant for any finitely generated A_ρ module M; namely its singular support $\underline{S}(M)$. This is defined in a manner analogous to $\underline{V}(M)$. Namely one filters $\mathcal{D}_X$ by the order of the differential operators and one defines $\underline{S}(M)$ to be the zero set (in the cotangent space $T^*(X)$) of $\text{Ann}_{\text{gr}\,\mathcal{D}_X}(\text{gr}\,\mathcal{M})$, where $\mathcal{M} = \mathcal{D}_X \otimes_{\mathfrak{U}(\underline{g})} M$. As for $\underline{V}(M)$ it is convenient to include multiplicites of irreducible components.

A notable property of $\underline{S}(M)$ is that if $M \neq 0$ then $\dim \underline{S}(M) \geq \frac{1}{2}\dim T^*(X) = \text{card } R^+$; even though M itself may be very small—say finite dimensional. By the usual arguments this gives $K \dim M \leq \dim \underline{S}(M) - \text{card } R^+$ and in particular $K \dim A_\rho \leq \text{card } R^+$. The opposite inequality (which is a little more elementary) was obtained by Levasseur [42]. A slight technical improvement then gives

THEOREM. For any $\mu \in \underline{h}^*$ one has $K \dim(\mathfrak{U}(\underline{g})/\text{Ann } M(\mu)) = \text{card } R^+$.

Conjecturally for any $J \in \text{Spec } \mathfrak{U}(\underline{g})$

$$K \dim(\mathfrak{U}(\underline{g})/J) = \frac{1}{2} d(\mathfrak{U}(\underline{g})/J) + d(\mathfrak{Z}(\mathfrak{U}(\underline{g})/J)).$$

Surprisingly although it is easy to check that $K \dim \mathfrak{U}(\underline{g}) \geq \dim \underline{b}$, it is not known if equality holds.

8.10. Given $x \in X = G/B$, let $\underline{g}_x$ denote the Lie subalgebra of $\underline{g}$ corresponding to the stabilizer subgroup G_x of G. The left action of G on X determines a surjective map of $\underline{g}$ to the tangent space $T_x(X)$ of X at x with kernel $\underline{g}_x$. Taking duals gives an injective map $\pi_x: T_x^*(X) \to \underline{g}^*$ with image $\underline{g}_x^\perp$. The map $\pi: (x,v) \mapsto \pi_x(v)$ of $T^*(X)$ into $\underline{g}^*$ is called the moment map of the homogeneous space X. Identifying $T_x^*(X)$ with $\underline{g}_x^\perp$ through π_x it becomes projection onto the second factor in $T^*(X)$. Noting that $\underline{g}_{gx}^\perp = g\underline{g}_x^\perp$ and that $\underline{g}_x^\perp = \underline{b}^\perp = \underline{n}$ when $x = \bar{1} \in G/B$, we can deduce an isomorphism $T^*(X) \xrightarrow{\sim} G \times_B \underline{n}$ of algebraic vector bundles under which π identifies with the map $(g,X) \mapsto gX$ of $G \times_B \underline{n}$ onto $G\underline{n} = \mathcal{N}$ (notation 8.5). This identifies the moment map of the flag variety with Springer's desingularization (as pointed out to me by Borho) of $\mathcal{N}$. A further important property of π is that it relates associated varieties with singular supports through the following result of Borho and Brylinski: [7].

LEMMA. (Notation 8.9). For each $M \in \text{Ob } \mathcal{O}_{\hat{\rho}}$ one has $\underline{V}(M) = \pi(\underline{S}(\mathcal{M}))$.

8.11. Let us write $L_w = L(w\rho)$, $M_w = M(w\rho)$, $\mathfrak{U}_w = \mathfrak{U}(\underline{g})/J(w\rho)$ (considered as a left $\mathfrak{U}(\underline{g})$ module), $\forall w \in W$. Under the conventions of 8.9 we have the further result of Borho and Brylinski [7].

THEOREM. For each $w \in W$ one has

$$\underline{S}(\mathcal{U}_w) = G\underline{S}(\mathcal{L}_w) = G \times_B \underline{V}(L_{w^{-1}}).$$

Recalling (8.8) it follows (counting multiplicities in $\underline{S}(\cdot)$) that $\mathbb{Z}\underline{S}(\mathcal{U}_w) = \mathbb{Z}\underline{S}(\mathcal{U}_y)$ if and only if $J(w\rho) = J(y\rho)$, $\forall w,y \in W$. (Actually it is conjectured in the integral case that $\underline{V}(L_{w^{-1}})$ and hence $\underline{S}(\mathcal{U}_w)$ is always irreducible and so multiplicities can probably be ignored.) We see that the singular support of $\mathcal{U}_w$ which is more refined than the associated variety of $J(w\rho)$ separates primitive ideals (at least in the integral case).

8.12. Let $\mathcal{U}$ denote the variety of all unipotent elements in G and $\mathcal{B}$ the variety of all Borel subgroups of G (which identifies with G/B). The Steinberg variety S of G is defined through

$$S := \{(u,B_1,B_2) \in \mathcal{U} \times \mathcal{B} \times \mathcal{B} \mid u \in B_1 \cap B_2\}.$$

It is an equidimensional algebraic variety of dimension $2r$: $r = \operatorname{card} R^+$ whose irreducible components are the closures of the $S_w := \{(u,gwB,gB) \in S,\ g \in G\}$, $w \in W$. Kazhdan and Lusztig [41] have given a topological construction of a left and right action of W on $H_{4r}(S,\mathbb{Q})$ (which has basis $[\bar{S}_w]$: $w \in W$) and shown that then as a $W-W$ bimodule it is isomorphic to $\mathbb{Q}W$. We define the matrix with entries $B(y,w)$ through

$$y[\bar{S}_1] = \sum_{w\in W} B(y,w)[\bar{S}_w].$$

Similar to the Jantzen matrix, this satisfies $B(y,y) = 1$ and $B(y,w) = 0$ unless $y \geq w$. Consequently this matrix is invertible and we denote the entries of the inverse matrix by $A(y,w)$: $y,w \in W$. The analogy with the Goldie rank polynomials leads us to the following

CONJECTURE. (Notation 8.7). For each $y \in W$ one has (up to a scalar)

$$q_y = \sum_{w\in W} A(y,w)w\rho^m$$

where m is the least non-negative integer such that right hand side is non-zero (and so equal to $\operatorname{card} R^+ - \dim \underline{V}(y)$).

One can show ([30], 9.8) that this would imply that q_y generates the representation associated by Springer to the nilpotent orbit

$\underline{St}(y)$ (notation 8.7). One may also conclude that the module $\mathbb{Q}Wq_y$ is always univalent.

8.13. For any non-singular subvariety Y of X one defines the conormal bundle $T^*_Y(X)$ through

$$T^*_Y(X) = \{(y,v) \mid y \in Y,\ v \in T_y(Y)^{\perp}\}.$$

One always has $\dim T^*_Y(X) = \dim X$. For example, if we set $X_w = NwB/B$, then $T_{wB/B}(X)$ identifies with $\underline{g}/w(\underline{b})$ and $T_{wB/B}(X_w)$ with the subspace $\underline{n}/\underline{n} \cap w(\underline{n})$, and hence $T_{wB/B}(X_w)^{\perp}$ with the subspace $\underline{n} \cap w(\underline{n})$ of $\underline{g}$. Then $T_w := T^*_{X_w}(X)$ becomes $\{(bwB,\ b(\underline{n} \cap w(\underline{n})))\mid b \in B\}$. From this we obtain that $\pi(\bar{T}_w) = \underline{V}(w)$ (notation 8.7). Taking $\underline{S}(\cdot)$ to include multiplicities we suggest (notation 8.13) the

CONJECTURE. For all $y \in W$, one has

$$(*) \qquad \underline{S}(M_y) = \sum_{w\in W} B(y,w)\bar{T}_w.$$

Let w_0 denote the longest element of W, then after Brylinski-Kashiwara ([10]).

$$\underline{S}(M_{w_0}) = \bigcup_{w\in W} \bar{T}_w \qquad \text{(no multiplicities counted)},$$

which is consistent with our conjecture. Again from (*) it is immediate that

$$(**) \qquad \underline{S}(L_z) = \sum_{y\in W}\sum_{w\in W} a(z,y)B(y,w)\bar{T}_w.$$

In type A_n, Kazhdan and Lusztig have suggested that $B(y,w)$ coincides with the Jantzen matrix and so we obtain simply that $\underline{S}(L_z) = \bar{T}_z$, as conjectured by Borho and Brylinski, which for example gives $\underline{V}(L_z) = \underline{V}(z)$ a result which has been reported by them. In general one has $\underline{V}(L_z) \subset \underline{V}(z)$ ([30], Sect. 8) and this inclusion may be strict (for reasons of dimension). Correspondingly outside type A_n the matrix defined by $B(y,w)$ certainly differs from the Jantzen matrix. Finally by 8.10 we obtain from (**) that

$$\underline{V}(L_z) = \sum_{y\in W} \sum_{w\in W} a(z,y)B(y,w)\underline{V}(w)$$

and so from the remarks in 8.8 we obtain (up to scalars) that

$$P_{z^{-1}} = \sum_{y\in W} \sum_{w\in W} a(z,y)B(y,w)q_w$$

and we remark that this is consistent with 4.1 and conjecture 8.12.

9. HOMOLOGICAL DIMENSION OF PRIMITIVE QUOTIENTS

9.1. Fix $\mu \in \Lambda^+$ and set $A_\mu = \mathfrak{U}(\underline{g})/\text{Ann}\, M(\mu)$. Either from the study of $\mathcal{D}_X$ modules or from our classification of primitive ideals (and hence of Spec A_μ) one can now hope to say rather a lot about the module theory of A_μ. For example consider the category of (finitely generated) right A_μ modules. How may we construct projective modules in this category? Obviously A_μ is projective as a right A_μ module. Yet A_μ has a left $\mathfrak{U}(\underline{g})$ module structure so we may tensor by finite dimensional $\mathfrak{U}(\underline{g})$ modules on the left and taking direct summands gives further projective right A_μ modules. Recognizing that we may identify A_μ with $L(M(\mu),M(\mu))$ (cf. [ ], Sect. 3 for a complete discussion of this question) and carrying the tensor product operation onto the right-hand factor of $M(\mu)$ leads to the following result [31]. (In this if M is a right module over a ring A, then M^* denotes $\text{Hom}_A(M,A)$ and we define $MM^* \subset \text{End}_A M$ through $m\xi$: $m' \mapsto m\xi(m')$, $\forall m,m' \in M$, $\forall \xi \in M^*$.)

LEMMA. Let P be a direct summand of $E \otimes M(\mu)$ for some finite dimensional $\mathfrak{U}(\underline{g})$ module E, and set $Q := L(M(\mu),P)$. Then

(i) Q is projective as a right A_μ module.

(ii) $Q^* = L(P,M(\mu))$.

(iii) $QQ^* = \text{End}_{A_\mu} Q = L(P,P)$.

Remark. If μ is regular then any projective object P in $\underline{\mathcal{O}}_\Lambda$ so obtains from $M(\mu)$. This is false in general and for a P not so obtained it can even happen that Q has infinite homological dimension [31].

9.2. As pointed out to me by Bernstein, the equivalence of categories theorem alluded to in 8.9 implies that A_ρ has global dimension $2\dim X$ = card R. By an appropriate version of the translation principle this also holds for $A_\mu : \mu \in P(R)^{++}$. Yet as noted above A_μ for μ non-regular generally has infinite global dimension.

9.3. An advantage of the above presentation of projective A_μ modules is that it enables one to compute the trace ideal $Q^*Q = \{\xi(q) : \xi \in Q^*, q \in Q\}$ of such a module. Recalling that $\lambda \in \underline{h}^*$ is assumed dominant and regular we have [31]:

PROPOSITION. For each $w \in W$, set $Q_w := L(M(\lambda),P(w\lambda))$, where $P(w\lambda)$ denotes the projective cover of $L(w\lambda)$. Then (notation 5.3)

$$Q_w^*Q_w = \mathcal{E}(L(w^{-1}\lambda)).$$

An amusing corollary of this result is that $L(P(w_\lambda\lambda),P(w_\lambda\lambda))$ is always a simple ring.

9.4. Let K_0 denote the Grothendieck group of projective right A_ρ modules. If $P(\mu)$ denotes the projective cover of $L(\mu) : \mu \in P(R)$, then by 9.1, $Q(\mu) := L(M(\rho),P(\mu))$ is a projective right A_ρ module. Let K_0' denote the subgroup of K_0 generated by the $\{Q(\mu)\}$. From the relations obtained from tensoring by finite dimensional modules one easily shows that card $K_0' \leq$ card W. Bernstein has informed me that he can prove equality (through comparison with the Grothendieck group of the category of coherent sheaves on G/B). Conjecturally $K_0' = K_0$.

9.5. The notion "clan" arises from ring theory. For example, for any $\mu \in \Lambda^{++}$ and any $w \in W_\lambda$ there exists a simple $A_\mu - A_\lambda$ bimodule V_w such that $\ell(V_w) = J(w\mu)$, $r(V_w) = J(w\lambda)$ (where ℓ, r denote left, respectively right, annihilator). This follows from the fact that $L(L(w\lambda),L(w\mu)) \neq 0$ which in turn results from translation principles. This extends in an appropriate way to Λ^+ and shows that elements lying in the same t-clan are linked ("module link") in the above sense. A similar assertion follows for an r-clan from 7.3. In the case when $\lambda = \mu$ one may also define an "ideal link" between elements in the same r-clan as follows. The ideals $J, K \in \operatorname{Spec} A_\lambda$ are ideal linked if $J = \ell(J \cap K/K)$, $K = r(J \cap K/K)$. (Since ${}^tJ = J$ this "relation" is symmetric; but conjecturally it is never reflective). Writing $J = \overline{J(w\lambda)}$, $K = \overline{J(y\lambda)}$ (where bar denotes the projection $\mathfrak{U}(\mathfrak{g}) \to A_\lambda$) it follows easily that J, K are ideal linked if $\operatorname{Ext}^1(L(w\lambda), L(y\lambda)) \neq 0$ (again because $\delta L(w\lambda) \cong L(w\lambda)$ the order of factors here is unimportant). As remarked by Borho, it follows from the Vogan semisimplicity hypothesis (6.5(*)) that the relation generated by ideal links defines exactly the r-clans, and this further defines the module links in $\operatorname{Spec} A_\lambda$. (An example of Stafford shows that the relation generated by ideal links can be strictly weaker than that generated by module links in general.)

The significance of ideal links is that if $J, K \in \operatorname{Spec} A$ (where A is a noetherian ring) are ideal linked then A does not admit localization at the prime ideal J, that is $S(J) := \{a \in A \mid \bar{a}$ non-zero divisor in $A/J\}$ is not an Ore set in A. If this were so $S^{-1}J$ would be the unique maximal ideal of the local ring $S^{-1}A$ and this eventually contradicts the existence of an ideal link. See [44].

9.6. Recently Levasseur ([42], Thm. 1) has shown that $A_\mu : \mu \in \Lambda^+$ has injective dimension equal to $\operatorname{card} R - d(L(\mu))$. This was a surprise as Roos had earlier conjectured it to be always equal to $\frac{1}{2} \operatorname{card} R$.

9.7. Through the analysis of ([11], Sect. 5) and 9.1, 9.2 one may show that the Conze embedding of A_μ (notation 9.1) in the Weyl algebra $A_n : n = \operatorname{card} R^+$ is a flat extension for μ regular. This was obtained through discussions with Smith, who with Hodges describes how to adjust this embedding to get faithfulness. This provides a ring theoretic alternative to the Bernstein-Beilinson equivalence of categories theorem [3].

9.8. We have recently [45] established the Borho-Kraft conjecture in

general (see 1.2, 8.6). The proof uses Hotta's result [43] which allows one to compare the representations of the Weyl group defined by Goldie rank polynomials to those defined by the Springer correspondence.

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UNITARY REPRESENTATIONS AND BASIC CASES

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In attempting to classify the irreducible unitary representations of linear semisimple Lie groups, one knows that it is enough to decide which of certain standard representations in Hilbert space admit new inner products with respect to which they are unitary. In this context B. Speh and the author [3] introduced a notion of basic case and gave a conjecture that would if true reduce the classification problem to a study of finitely many basic cases in each group. The paper [3] did not, however, tell how to calculate what the basic cases are. The present paper will address this question, giving some theorems that usually make it a simple matter to identify the basic cases.

The paper is organized as follows: In §1 we review the setting of the classification problem and restate the existence-uniqueness theorem for basic cases. In §2 we give two reduction theorems for calculating basic cases and show how to apply them. The proof of the second reduction theorem is in §3.

The development of the theory of basic cases has been influenced extensively by conversations with David Vogan. Vogan's paper [5] may be viewed as a related but different attempt to isolate the phenomena that lead to unitary representations.

1. Definition of basic cases

Let G be a connected linear semisimple group with maximal compact subgroup K. We assume as in [3] that $\operatorname{rank} G = \operatorname{rank} K$.

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