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Goldie Rank in the Enveloping Algebra of a Semisimple Lie Algebra, II

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Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$ and Prim $U(\mathfrak{g})$ the set of primitive ideals of $U(\mathfrak{g})$. This is the second of a two-part series in which the Goldie ranks of the primitive quotients $\{U(\mathfrak{g})/J : J \in \text{Prim } U(\mathfrak{g})\}$ are computed. Let $\mathfrak{h}$ be a Cartan subalgebra for $\mathfrak{g}$ and $\mathfrak{h}^*$ the dual of $\mathfrak{h}$. In the first part it was shown that these ranks can be described by a family of polynomial functions on $\mathfrak{h}^*$. Here a formula is obtained for these polynomials in terms of the multiplicities of the simple factors of the Verma modules. In particular this gives an affirmative answer to conjectures (i), (ii) of [14, 7.4] except for the choice of Ω_λ.

1. INTRODUCTION

The notation and conventions of this paper are those of [19], hereafter noted as I.

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. The Weyl group W acts faithfully in $\mathfrak{h}^*$ and hence for each $\sigma \in \hat{W}$ there exists $m \in \mathbb{N}$ such that $S_m(\mathfrak{h})$ admits a simple submodule of type σ. Let $n(\sigma)$ be the least nonnegative integer with this property. Call σ *univalent* if the σ-isotypical component in $S_{n(\sigma)}(\mathfrak{h})$ is a simple module, which we refer to as the univalent module P_σ defined by σ. Let Ω' denote the subset of $\hat{W}$ of all univalent representations. For $\mathfrak{g}$ simple, this subset is proper [1, Sect. 5] if and only if $\mathfrak{g}$ is of type D_{2n}: $n = 2, 3, \ldots$; E_7 or E_8 (Cartan notation).

1.2. For each $\lambda \in \mathfrak{h}^*$, let $\mathfrak{h}_\lambda$ denote the subspace of $\mathfrak{h}$ spanned by the coroots to R_λ (notation I, 1.5). Define $\Omega'_\lambda \subset \hat{W}_\lambda$ with respect to the action of W_λ on $\mathfrak{h}_\lambda$. We remark that $\Omega_{R,\lambda} \subset \Omega'_\lambda$ (notation I, 6.2).

1.3 (Notation I, 1.3). For each pair $\lambda, \mu \in \mathfrak{h}^*$ we denote by $[M(\lambda) : L(\mu)]$ the number of times the simple highest weight module $L(\mu)$ occurs as a simple

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factor of the Verma module $M(\lambda)$. The matrix with entries $[M(\lambda) : L(\mu)]$ will be called the *Jantzen matrix*. For each $\lambda \in \mathfrak{h}^*$, set $\Lambda = \lambda + P(R)$. Given $\lambda \in \mathfrak{h}^*$ dominant and regular, let b_Λ (or simply, b) denote the matrix with entries $b_\Lambda(w, w') := [M(w\lambda) : L(w'\lambda)]$, $w, w' \in W_\lambda$. We remark that b_Λ does not depend on the choice of the dominant, regular element in Λ [8, 2.15] and that the Jantzen matrix is completely determined by the b_Λ [8, 1.7, 2.17]. Presumably b_Λ only depends on W_λ; but this is still an open question.

Let $\leqslant$ denote the Bruhat ordering on W_λ (as defined in [6, 7.7.3]). For all $w, w' \in W_\lambda$ one has $b_\Lambda(w, w') \neq 0$ if and only if $w \leqslant w'$ (cf. [6, 7.7.7]). Hence b_Λ is invertible and we let $a_\Lambda(w, w')$ denote the entries of the inverse matrix a_Λ (or simply, a). One has $a(w, w') \neq 0$ only if $w \leqslant w'$. Following Jantzen [8, 1.12] we set $(L(w\lambda) : M(w'\lambda)) := a_\Lambda(w, w')$ and define $(M : M(\nu))$, for all $M \in \mathscr{R}$, $\nu \in \mathfrak{h}^*$, by additivity (cf. [6, 7.8.15(iii)]).

1.4 (Notation I, 1.1, 1.2, 2.6). Fix $\lambda \in \mathfrak{h}^*$ dominant and regular. In (I, 5.2) we defined for each $w \in W_\lambda$ the function p_w on $\hat{F}_{w\lambda}$ through $p_w(\mu) = \text{rk}(U(g)/J(\mu))$ and showed that p_w is a polynomial which we shall call *the Goldie polynomial.* Now set $m = \text{card } R^+ - d(L(\mu))$ (notation I, 2.3) and define $\delta \in \mathfrak{h}^*$ through $(\delta, \alpha) = 1$, for all $\alpha \in B$. The main result (Theorem 5.1) of this paper is that up to a scalar one has

$$\tilde{p}_w := w^{-1} p_w = \sum_{w' \in W_\lambda} a(w, w')\, w'^{-1} \delta^m. \tag{*}$$

By [18, 5.3] this further implies (Theorem 5.4(i)) that $W_\lambda p_w$ is a univalent module P_σ defined by some $\sigma \in \Omega'_\lambda$, and by [18, 6.10] that WP_σ is a univalent W module (Theorem 5.4(ii)). Again let Ω_λ denote the subset of Ω'_λ defined by the p_w: $w \in W_\lambda$. Then (Theorem 5.5) the distinct $\tilde{p}_w$: $w \in W_\lambda$ break into subsets each of which form a basis for some P_σ: $\sigma \in \Omega_\lambda$. In particular (notation I, 1.5)

$$\text{card } \mathscr{X}_\lambda = \sum_{\sigma \in \Omega_\lambda} \dim \sigma.$$

Thus up to the determination of Ω_λ these results answer affirmatively conjectures (i), (ii) of [14, 7.4] and show that conjecture (iii) is well defined for Ω_λ. We remark that $\Omega_{R,\lambda} \subset \Omega_\lambda$ (which already follows from I, 6.2). One cannot have $\Omega_\lambda = \Omega_{R,\lambda} \cup \Omega^*_{R,\lambda}$ (where $\sigma^*(w) := (\det w)\sigma(w)$) as suggested in [16, 11.8]. This is because $\Omega^*_{R,\lambda} \not\subset \Omega'_\lambda$ when R_λ has a simple factor of type E_7 [1, Table 2 and private communication]. Therefore, it still remains to determine Ω_λ and of course to determine the scalar factors in (*).

Finally we answer positively (Theorem 5.7) some conjectures of Vogan and indicate how the annihilators of simple Harish-Chandra modules can be computed.

2. A Presentation of the Space of W-Harmonic Polynomials

2.1. Identify $S(\mathfrak{h}^*)$ with the space of constant coefficient differential operators on $\mathfrak{h}$ and let S_+^* denote the augmentation ideal of the polynomial algebra $S(\mathfrak{h}^*)^W$. Then $g \in S(\mathfrak{h})$ is said to be *W-harmonic* if $Dg = 0$, for all $D \in S_+^*$. Let S_+ denote the augmentation ideal of the polynomial algebra $S(\mathfrak{h})^W$. Then the space $\mathscr{H}$ of W-harmonic polynomials is the W-stable complement to $S(\mathfrak{h})\, S_+$ in $S(\mathfrak{h})$ and hence as a W module is isomorphic to the regular representation of W [5, p. 107, Theorem 2]. In particular for each $\sigma \in \Omega'$, one has $P_\sigma \subset \mathscr{H}$.

We remark that

$$p_B := \prod_{\alpha \in R^+} \alpha$$

is W harmonic and that after Steinberg [21, Theorem 1.3] one has $\mathscr{H} = \{Dp_B : D \in S(\mathfrak{h}^*)\}$. In particular if we set $r = \operatorname{card} R^+$, $\mathscr{H}_m := \mathscr{H} \cap S_m(\mathfrak{h}) : m \in \mathbb{N}$, then

$$\mathscr{H} = \bigoplus_{m=0}^{r} \mathscr{H}_m .$$

Thus $P_\sigma \subset \mathscr{H}_{n(\sigma)}$ with $n(\sigma) \leqslant r$.

2.2. Vogan [22, 4.2] has given the following important characterization of $\mathscr{H}$. Fix $\lambda \in \mathfrak{h}^*$ and set $\Lambda = \lambda + P(R)$.

Lemma. *$\mathscr{H}$ is the subspace of all polynomial functions g on $\mathfrak{h}^*$ satisfying*

$$(\operatorname{card} W)\, g(\mu) = \sum_{w \in W} g(\mu + w\nu)$$

for all $\mu, \nu \in \Lambda$.

2.3. In the remainder of Section 2 we let δ denote a fixed regular element of $\mathfrak{h}^*$. In its application we take δ as given in 1.4.

Let W^* denote the set of maps from W to $\mathbb{Z}$. For each $a \in W^*$, $m \in \mathbb{N}$, define $g_{a,m} \in S_m(\mathfrak{h})$ through

$$g_{a,m} = \sum_{w \in W} a(w)\, w^{-1} \delta^m .$$

(Strictly speaking, for eventual application $g_{a,m}$ should be defined on Λ).

Since δ is regular, card $W\delta$ = card W and so the functions on $\mathfrak{h}^*$, $\exp w^{-1}\delta$: $w \in W$, are linearly independent over $\mathbb{Q}$. It follows that if $g_{a,m} = 0$, for all $m \in \mathbb{N}$, then $a = 0$. For each $0 \neq a \in W^*$, let $m(a)$ denote the least nonnegative integer such that $g_{a,m(a)} \neq 0$ and set $g_a := g_{a,m(a)}$. For each $m \in \mathbb{N}$, let W_m^* denote the subset of all $a \in W^*$ such that $m = m(a)$.

Lemma. *For each $m \in \mathbb{N}$,*

(i) *$g_a \in \mathscr{H}_m \backslash \{0\}$, for all $a \in W_m^*$,*

(ii) *$\mathscr{H}_m$ is the $\mathbb{C}$ linear span of $\{g_a : a \in W_m^*\}$.*

For each $\mu, \nu \in \Lambda$, $w \in W$ one has

$$g_a(\mu + w\nu) = \sum_{w' \in W} a(w')(\delta, w'\mu + w'w\nu)^m .$$

Then by summation and change of variable we obtain

$$\begin{aligned}\sum_{w \in W} g_a(\mu + w\nu) &= \sum_{w \in W} \sum_{w' \in W} a(w')(\delta, w'\mu + w\nu)^m, \\ &= \sum_{w \in W} \sum_{w' \in W} a(w')(\delta, w'\mu)^m, \qquad \text{by the choice of } m, \\ &= (\operatorname{card} W)\, g_a(\mu).\end{aligned}$$

By 2.2, this gives (i). Since $\dim \mathscr{H} = \operatorname{card} W$, we obtain (ii) from (i).

Remarks. One has

$$\begin{aligned}\sum_{w \in W} (\det w)\, w^{-1} \delta^m &= 0 : m < r, \\ &= r! \prod_{\alpha \in R^+} (\alpha, \delta)/(\alpha, \rho) \cdot p_B \quad : m = r.\end{aligned}$$

(The correct choice of scalar was communicated to me by D. King).

This result (at least for $\delta = \rho$) was noted for example in [21, Theorem 1.4; and 8, 3.15]. Combined with Steinberg's result noted in 2.1, it leads to an alternative proof of (ii). If δ were not regular then the analog of the above construction would give a subspace of $\mathscr{H}$ isomorphic to ind(Id: $W_\delta{}^0 \uparrow W$).

2.4. Let Q denote the projection onto $\mathscr{H}$ defined by the decomposition $S(\mathfrak{h}) = \mathscr{H} \oplus S(\mathfrak{h})\, S_+$.

Corollary. *For each $m \in \mathbb{N}$, one has $Q(\mathbb{C}W\delta^m) = \mathscr{H}_m$.*

2.5. We define an action of $W \times W$ on W^* through $(w_1 a w_2)(w) := a(w_1 w w_2)$, for all $w_1, w_2, w \in W$, $a \in W^*$. Observe that $g_{aw,m} = w g_{a,m}$, for all $w \in W$, $a \in W^*$, $m \in \mathbb{N}$. Then each W_m^* is a right W submodule of W^*; but it need not be also a left submodule. Let $W^\smile$ denote the subset of all $a \in W^*$ such that $m(wa) \geqslant m(a)$ and $g_{wa} \in \mathbb{C}g_a$, for all $w \in W$. Set $W_m^\smile = W^\smile \cap W_m^*$. Then each $W_m^\smile$ is a right W submodule of W^*.

PROPOSITION. *For each* $a \in W_m^{\sim}$, *there exists* $\sigma \in \Omega'$ *such that* $\mathbb{C}Wg_a = P_\sigma$, *with* $n(\sigma) = m$.

Fix $a \in W_m^{\sim}$ and set

$$x = \sum_{w \in W} a(w)\, w^{-1}.$$

For all $n \in \mathbb{N}$, $w' \in W$ we have

$$g_{w'a} = \sum_{w \in W} a(w'w)\, w^{-1}\, \delta^m = xw'\, \delta^m.$$

By 2.4 and the definition of $W_m^{\sim}$, it follows that $x\mathscr{H}_n = 0$, for $n < m$, and $\dim x\mathscr{H}_m = 1$. Now $\mathbb{C}W$ is a semisimple Artinian ring so there exists an idempotent $u \in \mathbb{C}W$ such that $u\mathbb{C}W = x\mathbb{C}W$. Obviously $u\mathscr{H}_n = x\mathscr{H}_n$, for all $n \in \mathbb{N}$ and in particular $\dim u\mathscr{H}_m = 1$. Fix a nonzero vector $v \in u\mathscr{H}_m$. Then $uv = v$, $u\mathscr{H}_m = \mathbb{C}v$ and so by the complete reducibility of $\mathscr{H}_m$ it follows that $\mathbb{C}Wv$ is a simple W submodule of $\mathscr{H}_m$. Let $\sigma \in \hat{W}$ denote the corresponding representation. Since $u\mathscr{H}_n = 0$, for $n < m$, it follows that $n(\sigma) \geqslant m$. Since $\dim u\mathscr{H}_m = 1$, it follows that σ occurs with multiplicity one in $\mathscr{H}_m$. That is, $n(\sigma) = m$ and $\sigma \in \Omega'$. Finally $P_\sigma = \mathbb{C}Wv = \mathbb{C}Wx\delta^m = \mathbb{C}Wg_a$.

Remark. Every univalent module obtains by this construction.

2.6. The above analysis applies equally well to W_λ (and we use for example $\mathscr{H}_{\lambda,m}$ to denote the homogeneous harmonic polynomials for W_λ of degree m). In particular we need not alter the choice of δ. To see this set $\mathfrak{h}_\lambda{}^\perp = \{\beta \in \mathfrak{h}^*$: $(\alpha, \beta) = 0$ for all $\alpha \in R_\lambda\}$. Then $\mathfrak{h}^* = \mathfrak{h}_\lambda^* \oplus \mathfrak{h}_\lambda^\perp$ and we write $\delta = \delta_1 + \delta_2$ with respect to this decomposition. Then δ_1 is regular in $\mathfrak{h}_\lambda^*$ and for all $a \in W_{\lambda,m}^*$ we have

$$g_a = \sum_{w \in W_\lambda} a(w)\, w^{-1}\, \delta^m = \sum_{w \in W_\lambda} a(w)\, w^{-1}\, \delta_1{}^m,$$

by the choice of δ_2 and m.

3. Multiplicities

3.1. Let M be a finitely generated $U(\mathfrak{g})$ module. Define (cf. I, 2.3) the Gelfand–Kirillov dimension $d(M)$ (resp. multiplicity, $e(M)$) of M with respect to the canonical filtration of $U(\mathfrak{g})$. Our ultimate aim is to determine the Goldie polynomials. As noted in (I, 6.1) these are determined by polynomials (the Vogan polynomials) defined with respect to the multiplicities of certain simple subquotients of principal series modules. Unfortunately the latter can only be computed in certain special cases [7] and even then only partial information has been obtained. Consequently the following alternative approach is needed.

3.2. Define $\delta \in \mathfrak{h}^*$ through $(\delta, \alpha) = 1$ for all $\alpha \in B$. In particular δ is dominant, regular and (δ, β): $\beta \in \mathbb{N}B$ is just the so-called length of β. Fix $\mu \in \mathfrak{h}^*$ and let M be a quotient of $M(\mu)$. Then M is a direct sum of its $\mathfrak{h}$ weight subspaces $M_{\mu-\nu}$ of weight $\mu - \nu - \rho$: $\nu \in \mathbb{N}B$. For each $n \in \mathbb{N}$, we set following Jantzen [8, 3.12]

$$F_M(n) = \sum_{(\delta,\nu) \leqslant n} \dim M_{\mu-\nu}.$$

Up to a minor adjustment F_M is a polynomial and in [8, 3.12] Jantzen has shown how to compute F_M from a knowledge of the composition series for M and the matrix a defined in 1.3. In particular we may compute the leading coefficient of $F_M(n)$ which up to a minor adjustment takes the form $e_M n^d/d!$ with e_M, $d \in \mathbb{N}$. It is easy to show that $d = d(M)$; but it is not at all obvious if e_M is related in any simple fashion to $e(M)$. (Such a relation is an eventual and rather deep consequence (Corollary 4.11) of our analysis.)

Fix $\lambda \in \mathfrak{h}^*$ dominant regular and $w \in W_\lambda$. Given $M = L(\mu)$: $\mu \in \hat{F}_{w\lambda}$, we write $e_w(\mu) = e_M$. These functions defined on each $\hat{F}_{w\lambda}$: $w \in W_\lambda$ are polynomials (called the Jantzen polynomials) and can be explicitly computed. Yet the main difficulty is still to show that they determine the p_w. For this we must first develop and extend Jantzen's analysis.

3.3. Fix $k \in \mathbb{N}$, $l \in \mathbb{N}^+$ and functions c_j: $\mathbb{Z}/l\mathbb{Z} \to \mathbb{Q}$: $j = 0, 1, 2,\ldots, k$. Define F: $\mathbb{N} \to \mathbb{Q}$ through

$$F(n) = \sum_{j=0}^{k} c_j(n + l\mathbb{Z})n^k.$$

This is not quite a polynomial; but we show that it may be "smoothed" to obtain one. Define G: $\mathbb{N} \to \mathbb{C}$ inductively through $G = H_{k+1}$, $H_0 = F$ and

$$H_j(n) = \sum_{i=0}^{l-1} H_{j-1}(n + i) : j = 1, 2,\ldots, k + 1,\ n \in \mathbb{N}.$$

We have

$$G(n) = \sum_{i=0}^{l+k} b_i F(n + i),$$

for suitable $b_i \in \mathbb{N}^+$.

LEMMA. *G: $n \mapsto G(n)$ is a polynomial. Furthermore*

$$G(n) = l^{k-1} \left(\sum_{i=0}^{l-1} c_k(i + l\mathbb{Z}) \right) n^k + O(n^{k-1}).$$

The proof is by induction on k. It is clear for $k = 0$. Consider it proved for $k-1$. If F is a polynomial, then so is G and hence by linearity and the induction hypothesis it is enough to prove the assertion when only $c := c_k \neq 0$. We have

$$H_1(n) = \sum_{i=0}^{l-1} F(n+i)$$

$$= \sum_{i=0}^{l-1} c(n+i+l\mathbb{Z})n^k$$

$$= \sum_{i=0}^{l-1} c(i+l\mathbb{Z})n^k + \sum_{j=0}^{k-1} c_j'(n+l\mathbb{Z})n^j,$$

for suitable $c_j'\colon \mathbb{Z}/l\mathbb{Z} \to \mathbb{Q}$. Thus $H_1(n)$ is a polynomial in its highest term so the lemma obtains from the induction hypothesis.

3.4. Let $G, G'\colon \mathbb{N} \to \mathbb{Q}$ be polynomials and set $G(n) = en^d + O(n^{d-1})$, $G'(n) = e'n^{d'} + O(n^{d'-1})$. We write $G \gtrsim G'$ if either $d > d'$ or $d = d'$ and $e \geqslant e'$. We write $G \doteq G'$ if $G \gtrsim G'$ and $G' \gtrsim G$. If $F, F'\colon \mathbb{N} \to \mathbb{Q}$ are functions of the form described in 3.3 we let G, G' be the corresponding smoothed polynomials and we write $F \gtrsim F'$ (resp. $F \doteq F'$) if $G \gtrsim G'$ (resp. $G \doteq G'$).

3.5. Set $r = \operatorname{card} R^+$ and let l denote the product of the lengths of the positive roots. For $\nu \in \mathfrak{h}^*$, set $F_R(n) = F_{M(\nu)}(n)$ (which depends only on R and not on ν). Now let M be a quotient of $M(\mu)$. After Jantzen [8, 3.12] we have

$$F_M(n) = \sum_{\nu \in W_\mu} (M : M(\nu)) F_R(n - (\delta, \mu - \nu)).$$

Furthermore [8, 3.13] there exist $c_j\colon \mathbb{Z}/l\mathbb{Z} \to \mathbb{Q}$ such that

$$F_R(n) = \frac{n^r}{r!\, l} + \sum_{j=0}^{r-1} c_j(n+\mathbb{Z}l)n^j.$$

Set $k = \max\{(\delta, \mu - \nu)\colon (M : M(\nu)) > 0\}$. Then for all $n \geqslant k$ we have by linearity (notation 3.3) that

$$G_M(n) = \sum_{\nu \in W_\mu} (M : M(\nu))\, G_R(n - (\delta, \mu - \nu)). \qquad (*)$$

Now let d be the largest integer $\leqslant r$ such that

$$e_M := \sum_{\nu \in W_\mu} (M : M(\nu))(\delta, \nu - \mu)^{r-d} \neq 0.$$

LEMMA. (i) $d = d(M)$,

(ii) $G_M(n) \doteq (l^{r-1}/r!)\, e_M n^d$,

(iii) $e_M = \sum_{\nu \in W_\mu} (M : M(\nu))(\delta, \nu)^{r-d}$.

Part (i) follows from [8, 3.15]. Part (ii) follows from (*) and the fact that G_R is a polynomial by 3.3. (This important observation is due to Jantzen). Part (iii) follows by the choice of d.

3.6. Fix $\lambda \in \mathfrak{h}^*$ dominant, regular and $w \in W_\lambda$. Define a function e_w (resp. f_w) on $F_{w\lambda}$ through $e_w(\mu) = e_{L(\mu)}$ (resp. $f_w(\mu) = e(L(\mu))$, which we call the Jantzen (resp. Vogan) polynomial. Set $d = d(L(\mu))$ (which depends only on $F_{w\lambda}$, [3, 2.12]).

COROLLARY. (i) $e_w = \sum_{w' \in W_\lambda} a_\Lambda(w, w')\, ww'^{-1}\delta^{r-d}$,

(ii) *e_w is a homogeneous W_λ-harmonic polynomial of degree $r - d$,*

(iii) $\deg e_w = \deg f_w$.

For all $\mu \in F_{w\lambda}$ we have by 3.5(iii)

$$e_w(\mu) = \sum_{\nu \in W\mu} (L(\mu) : M(\nu))(\delta, \nu)^{r-d},$$

$$= \sum_{w' \in W_\lambda} (L(\mu) : M(w'\mu))(\delta, w'\mu)^{r-d}.$$

Hence (i). Recalling the choice of d, 2.3(i) and 2.6, this gives (ii). Finally (iii) obtains by comparison of filtrations (see for example, [7, Lemme 2]).

3.7. For our purposes we need the following rather technical extension of the above analysis. Set $\mathfrak{n} = \mathfrak{n}^-$ and fix $\mu \in \mathfrak{h}^*$. Let $\mathfrak{m}$ be an $\mathfrak{h}$-stable ideal of $\mathfrak{n}$ and t a weight vector in $U(\mathfrak{m})^{\mathfrak{n}}$. Assume that $tm = 0 : m \in L(\mu)$ implies $m = 0$ and set $T = \{t^k : k \in \mathbb{N}\}$, which is trivially an Ore subset for $U(\mathfrak{n})$ and $U(\mathfrak{m})$. Set $T^{-1}L(\mu) = T^{-1}U(\mathfrak{m}) \otimes_{U(\mathfrak{m})} L(\mu)$ and identify $L(\mu)$ with a $U(\mathfrak{h} \oplus \mathfrak{n})$ submodule of $T^{-1}L(\mu)$ through the map $m \mapsto 1 \otimes m$.

Now assume that there exists an $\mathfrak{h}$ module homomorphism $\theta'\colon T^{-1}U(\mathfrak{n}) \to T^{-1}U(\mathfrak{m})$ whose restriction to $T^{-1}U(\mathfrak{m})$ is the identity map (i.e., θ' is a retraction). Set $\mathfrak{n}' = \theta'(\mathfrak{n})$. Then $T^{-1}L(\mu)$ is a $U(\mathfrak{h} \oplus \mathfrak{n}')$ module in an obvious fashion. Fix $k \in \mathbb{N}^+$ and let M be a finitely generated $U(\mathfrak{h} \oplus \mathfrak{n}')$ submodule of

$$\sum_{j=0}^{k} t^{-j} L(\mu)$$

containing $L(\mu)$. (Of course it is not obvious that any such module exists; but if it does it will also be finitely generated as a $U(\mathfrak{n}')$ module.)

Recalling that θ' commutes with the action of $\mathfrak{h}$ we define a filtration on $U(\mathfrak{n}')$ by setting

$$U^n(\mathfrak{n}') = \oplus \{U(\mathfrak{n}')_{-\mu} : (\delta, \mu) \leqslant n\} : n \in \mathbb{N}.$$

Let N be a $U(\mathfrak{n}')$ submodule of M (not necessarily $\mathfrak{h}$ stable). Since $U(\mathfrak{n}')$ is Noetherian, it follows that N is finitely generated. Choose a finite-dimensional generating subspace N^0 for N. Set $N^n = U^n(\mathfrak{n}')\, N^0$ and $F'_N(n) = \dim N^n : n \in \mathbb{N}$. Define l as in 3.5. Assume $N \neq 0$.

LEMMA. *There exist $d \in \mathbb{N}$, $0 < e \in \mathbb{Q}$, $c_j : \mathbb{Z}/l\mathbb{Z} \to \mathbb{Q}$: $j = 0, 1, 2, \ldots, d-1$ (depending on N) such that*

(i) $F'_N(n) = en^d + \sum_{j=0}^{d-1} c_j(n + l\mathbb{Z})\, n^j$, *for all n sufficiently large*

(ii) $F'_M \doteq F_{L(\mu)}$.

We start by remarking that for the above filtration gr $U(\mathfrak{n}')$ is trivially isomorphic to $U(\mathfrak{n}')$ and hence not in general commutative. Nevertheless the argument used in the construction of the Hilbert–Samuel polynomial can still be applied if we make use of the fact that $\mathfrak{n}'$ is nilpotent. (This was pointed out to me by O. Gabber. At first sight it seems slightly surprising since F'_N will not necessarily be a polynomial).

The proof of (i) is by induction on $\dim \mathfrak{n}'$. It is clear if $\dim \mathfrak{n}' = 1$, since $\mathfrak{n}'$ is then commutative.

Let $U_m(\mathfrak{n}')$, N_m be the gradations defined on $U(\mathfrak{n}')$, N by the above filtrations. By construction $U_m(\mathfrak{n}')\, N_n = N_{m+n}$, so in particular gr N is finitely generated as a $U(\mathfrak{n}')$ module. Let $X_\beta \in \mathfrak{n}'$ be a root vector with β of maximal length and set $s = (\delta, \beta)$. Obviously X_β is central and we can assume the required assertion holds for any finitely generated $U(\mathfrak{n}'/\mathbb{C}X_\beta)$ module. Define a $U(\mathfrak{n}')$ module homomorphism $\psi: N \to N$ through $\psi(v) = X_\beta v$. Set $K = \operatorname{Ker} \psi$, $L = N/\operatorname{Im} \psi$. For each $n \in \mathbb{N}$, set $K_n = N_n \cap K$, $L_n = N_n/(N_n \cap \operatorname{Im} \psi)$. As ψ is homogeneous of degree s, we have an exact sequence

$$0 \to K_n \to N_n \xrightarrow{\psi} N_{n+s} \to L_{n+s} \to 0,$$

which gives $\dim N_{n+s} - \dim N_n = \dim L_{n+s} - \dim K_n$. The usual argument shows that K, L are finitely generated as $U(\mathfrak{n}'/\mathbb{C}X_\beta)$ modules and so $\dim N_{n+s} - \dim N_n$ takes for all n sufficiently large the form given in the right-hand side of (i) except that l can be taken to be the product of the root lengths for $U(\mathfrak{n}'/\mathbb{C}X_\beta)$. Taking account of the jump by s this establishes that $\dim N_n$ takes the required form. Hence (i).

For (ii) recall that M is a weight module and finitely generated as a $U(\mathfrak{n}')$ module. Thus we can choose $\nu \in \mathfrak{h}^*$ such that $\mu - \nu \in \mathbb{N}B$ for all $\mu \in \mathfrak{h}^*$ for which $M_{-\mu} \neq 0$. Define $F''_M : \mathbb{N} \to \mathbb{N}$ through

$$F''_M(n) = \sum \{\dim M_{-\mu} : (\mu - \nu, \delta) \leqslant n\},$$

with a similar definition for any $U(\mathfrak{h} \oplus \mathfrak{n}')$ submodule of M. By the definition of the filtration on $U(\mathfrak{n}')$ it is immediate that $F''_M \doteq F'_M$. Again $F_{L(\mu)} \doteq F''_{L(\mu)}$ trivially. Since $M \supset L(\mu)$, we have $F''_M \gtrdot F''_{L(\mu)}$ and since $t^k M \subset L(\mu)$, we have $F''_{L(\mu)} \gtrdot F''_{t^k M}$. Finally $t^k m = 0$: $m \in M$ implies $m = 0$ and so $F''_{t^k M} \doteq F''_M$. Combining these inequalities gives (ii).

3.8 (Notation 3.7). Define G'_N with respect to F'_N as in 3.3. Assume $N \neq 0$. Let d, e be defined through the conclusion of 3.7(i).

COROLLARY. (i) $G'_N(n) \doteq el^d n^d$,

(ii) $el^d d! \in \mathbb{N}^+$.

Part (i) follows from 3.3 and 3.7(i). Recalling that G'_N is a polynomial map from $\mathbb{N}$ to $\mathbb{N}$ gives (ii).

Remarks. In all the above analysis one could have assumed l to be the least common multiple of the positive root lengths. We can apply 3.7 to any $L(\mu)$ by taking $\mathfrak{m} = \mathfrak{n}$, $t = 1$. It follows for suitable n, d that $F_{L(\mu)}(n) \doteq en^d$ and so the Jantzen polynomial can be defined without the smoothing procedure $F \to G$.

4. A Division Theorem

4.1. Our aim here is to show that the Goldie polynomial p_w divides the Jantzen polynomial e_w. This is clearly quite nontrivial and in fact our analysis makes essential use of the main results of [9]. (See in particular [9, Sects. 2.1–2.6, 4.1–4.12, 6.1–6.7, 6.11].) Set $\mathfrak{n}^- = \mathfrak{n}$.

4.2. In [9, 2.2] we assigned to each semisimple Lie algebra $\mathfrak{g}$ an ordered set $\mathscr{K}(\mathfrak{g})$ (or simply, $\mathscr{K}$) and a maximal set $\{\beta_K\}_{K \in \mathscr{K}}$ of strongly orthogonal positive roots. We recall that $\mathscr{K}$ as an ordered set is isomorphic to a sum of trees (see [9, Table III]) and if $K \in \mathscr{K}$ is minimal, then β_K is a highest root. Call a subset $\mathscr{L} \subset \mathscr{K}$ *parabolic* if for each $L \in \mathscr{L}$, one has $K \in \mathscr{L}$, for all $K \leqslant L$. We recall [9, 4.10] that to each parabolic subset $\mathscr{L}$ of $\mathscr{K}$ we can associate a so-called *optimal* parabolic subalgebra $\mathfrak{p}_\mathscr{L} \supset \mathfrak{h} \oplus \mathfrak{n}$ of $\mathfrak{g}$. Let $\mathfrak{r}_\mathscr{L}$ (resp. $\mathfrak{m}_\mathscr{L}$) denote the reductive part (resp. nilradical) of $\mathfrak{p}_\mathscr{L}$ and set $\mathfrak{s}_\mathscr{L} = [\mathfrak{r}_\mathscr{L}, \mathfrak{r}_\mathscr{L}]$ (or simply, $\mathfrak{p}, \mathfrak{r}, \mathfrak{m}, \mathfrak{s}$). By [9, 4.11(ii)] there exists for each $K \in \mathscr{K}$ a weight vector f_K (determined up to a scalar) of weight β_K lying in Fract $Z(\mathfrak{n})$. Moreover [9, 4.12] $Z(\mathfrak{n})$ is generated as a polynomial algebra by products of the form

$$t_L := \prod_{K \in \mathscr{K}} f_K^{n_{KL}} : n_{KL} \in \mathbb{N},$$

where the matrix n_{KL} is upper triangular with respect to the order in $\mathscr{K}$ and has determinant one. Let $t_\mathscr{L}$ (or simply, t) denote the product of the $t_L : L \in \mathscr{L}$. By

[4, 2.4a] t is locally ad-nilpotent in $U(\mathfrak{g})$ and $T := \{t^m : m \in \mathbb{N}\}$ is Ore in $U(\mathfrak{g})$. Extend the canonical filtration of $U(\mathfrak{g})$ to a filtration of $T^{-1}U(\mathfrak{g})$ by setting

$$(T^{-1}U(\mathfrak{g}))^n = \sum_{k=0}^{\infty} t^{-k}U^{n+ks}(\mathfrak{g}) : s = \deg t,$$

for all $n \in \mathbb{Z}$. By [9, 4.1, 4.8, 4.11] we have the

LEMMA. *There exists a linear map $\theta: \mathfrak{s} \to T^{-1}U(\mathfrak{m})$, homogeneous of degree zero, satisfying*

(i) $[X, Y] = [\theta(X), Y]$, *for all* $X \in \mathfrak{s}$, $Y \in \mathfrak{m}$,
(ii) $[X, \theta(Y)] = \theta([X, Y])$, *for all* $X, Y \in \mathfrak{s}$,
(iii) $[\theta(X), \theta(Y)] = \theta([X, Y])$, *for all* $X, Y \in \mathfrak{s}$,
(iv) $[H, \theta(X)] = \theta([H, X])$, *for all* $H \in \mathfrak{h}$, $X \in \mathfrak{s}$.

Remark. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form and set $g = \sum X_{\beta_L} : L \in \mathscr{L}$. Then $g((\operatorname{ad} X)\mathfrak{m}) = 0$ for all $X \in \mathfrak{m}$ and so the lemma can be viewed as a consequence of [6, 10.1.4].

4.3. By 4.2, $\Theta(X) := X - \theta(X)$: $X \in \mathfrak{s}$ extends to an injection of $U(\mathfrak{s})$ into $T^{-1}U(\mathfrak{p})^{\mathfrak{m}}$. Set $\mathfrak{t} = \Theta(\mathfrak{s})$. Again $\theta: \mathfrak{s} \to T^{-1}U(\mathfrak{m})$ is a Lie algebra homomorphism. Its restriction to $\mathfrak{s} \cap \mathfrak{n}$ extends to a retraction θ' of $T^{-1}U(\mathfrak{n})$ into $T^{-1}U(\mathfrak{m})$ commuting with $\mathfrak{h}$. Set $\mathfrak{u} = \theta(\mathfrak{s})$. We have the following.

COROLLARY. (i) $[\mathfrak{s}, \mathfrak{t}] \subset \mathfrak{t}$, $[\mathfrak{s}, \mathfrak{u}] \subset \mathfrak{u}$.
(ii) $[\mathfrak{m}, \mathfrak{u}] \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{t}] = 0$.
(iii) $[\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}$, $[\mathfrak{h}, \mathfrak{u}] \subset \mathfrak{u}$.

4.4. From now on fix $\mu \in \mathfrak{h}^*$. Let $e(\mu)$ denote the canonical generator of $L(\mu)$ (of weight $\mu - \rho$). (This should not be confused with $e_w(\mu)$.) Set $\mathscr{L} = \{L \in \mathscr{K} : t_L{}^k e(\mu) \neq 0$, for all $k \in \mathbb{N}\}$. Let T_1 be the multiplicative subset generated by the $t_L : L \in \mathscr{L}$. By [4, 2.4a] each t_L acts locally ad-nilpotently in $U(\mathfrak{g})$ and so T_1 is Ore in $U(\mathfrak{g})$. Again $L(\mu)$ is a simple $U(\mathfrak{g})$ module and so by ad-nilpotence we have $sm \neq 0$, for all $0 \neq m \in L(\mu)$, $s \in T_1$. It follows that $T_1 \cap J(\mu) = \varnothing$ and the canonical map $m \mapsto 1 \otimes m$ of $L(\mu)$ into $T_1^{-1}L(\mu)$ is injective. Let $\Psi: T_1^{-1}U(\mathfrak{g}) \to T_1^{-1}U(\mathfrak{g})/T_1^{-1}J(\mu)$ denote the canonical projection.

PROPOSITION. (i) *$\mathscr{L}$ is a parabolic subset of $\mathscr{K}$. Define T, $\mathfrak{t}$ with respect to $\mathscr{L}$ as in* 4.2, 4.3. *Then*

(ii) $\dim \Psi(U(\mathfrak{t})) < \infty$,
(iii) $T^{-1}U(\mathfrak{g}) = T_1^{-1}U(\mathfrak{g})$ *and* $T^{-1}L(\mu) = T_1^{-1}L(\mu)$.

Let $\mathscr{L}'$ be a maximal parabolic subset of $\mathscr{K}$ contained in $\mathscr{L}$ and use a prime to denote quantities defined with respect to $\mathscr{L}'$. Let K be minimal in $\mathscr{K}\backslash\mathscr{L}'$. By the maximality of $\mathscr{L}'$ we have $t_K{}^k e(\mu) = 0$ for some $k \in \mathbb{N}^+$. Hence $t_K{}^k \in T_1'^{-1}J(\mu)$. By [9, 4.9, 4.11] $t_K = \Theta'(X_{-\beta_K})$. Since $\{X_{-\beta_K} : K$ minimal in $K\backslash\mathscr{L}'\}$ is just the set of lowest weight vectors for the semisimple Lie algebra $\mathfrak{s}'$, it follows from [9, 6.11] that $\dim \Psi(U(\mathfrak{t}')) < \infty$. This implies that $\mathscr{L} = \mathscr{L}'$ and hence we have (i) and (ii). Part (iii) is clear.

4.5. From now on we assume $\mathscr{L}$ to be fixed by the conclusion of 4.4 and drop this subscript. Recall the definition of the smash product given in [9, 3.4]. Let $\mathfrak{l}$ be the subalgebra of $\mathfrak{h}$ spanned by the coroots to the $\{\beta_L : L \in \mathscr{L}\}$.

THEOREM. *Up to natural isomorphisms*

(i) $T^{-1}(U(\mathfrak{g})/J(\mu)) = (T^{-1}(U(\mathfrak{g})/J(\mu))^{\mathfrak{m}} \otimes_{T^{-1}Z(\mathfrak{m})} T^{-1}U(\mathfrak{m}))\,\#\,U(\mathfrak{l})$,
(ii) $T^{-1}(\Psi(U(\mathfrak{p}))) = T^{-1}((\Psi(U(\mathfrak{t})) \otimes U(\mathfrak{m}))\,\#\,U(\mathfrak{l}))$,
(iii) $(U(\mathfrak{g})/J(\mu))^{\mathfrak{m}}$ *is a prime, Noetherian ring,*
(iv) $rk(U(\mathfrak{g})/J(\mu))^{\mathfrak{m}} = rk(U(\mathfrak{g})/J(\mu))$.

Taking into account that $T \cap J(\mu) = \varnothing$, we obtain (i), (iii), (iv) from [9, 6.7]. Part (ii) is an easy consequence of the form of θ.

4.6. By the choice of $\mathfrak{p}$ and t (cf. [9, 2.2, 4.11, 4.12]) we have $[X, t] \in \mathbb{C}t$, for all $X \in \mathfrak{p}$. Hence for each $k \in \mathbb{N}$, it follows that

$$N(k) := \sum_{n=0}^{k} t^{-n}L(\mu)$$

is a $U(\mathfrak{p})$ submodule of $T^{-1}L(\mu)$ containing $L(\mu)$. By 4.4(ii) we can choose k such that $U(\mathfrak{t})\,e(\mu)$ is a finite-dimensional subspace of $N(k)$. Set $M' = U(\mathfrak{m})\,U(\mathfrak{t})\,e(\mu)$, which by 4.3 is a $U(\mathfrak{m} \oplus \mathfrak{t} \oplus \mathfrak{l})$ submodule of $N(k)$. Set $M = U(\mathfrak{s})M'$, which by 4.3 is a $U(\mathfrak{m} \oplus \mathfrak{s} \oplus \mathfrak{t} \oplus \mathfrak{l})$ submodule of $N(k)$. Now $M \supset U(\mathfrak{p})\,e(\mu)$ and so $M \supset L(\mu)$. Set $\mathfrak{n}' = \mathfrak{m} \oplus \mathfrak{u} = \theta'(\mathfrak{n})$.

LEMMA. (i) $T^{-1}M = T^{-1}M' = T^{-1}L(\mu)$,

(ii) *M' is finitely generated as a $U(\mathfrak{m})$ module,*
(iii) *M is finitely generated as a $U(\mathfrak{n}')$ module,*
(iv) *$T^{-1}L(\mu)$ is finitely generated as a $T^{-1}U(\mathfrak{m})$ module.*

Part (i) is an immediate consequence of 4.5(ii). Part (ii) is clear. It implies that M is finitely generated as a $U(\mathfrak{m} \oplus \mathfrak{s})$ module. Yet $\mathfrak{s} \subset \mathfrak{u} \oplus \mathfrak{t}$ so (iii) follows from 4.4(ii). Finally (iv) obtains from (i), (ii).

Remark. In general M and $L(\mu)$ are *not* finitely generated as $U(\mathfrak{m})$ modules (though they are finitely generated if $L(\mu)$ is induced from $\mathfrak{p}$). For example, take μ to be a dominant half-integral regular weight in type C_2 (see 6.2).

4.7. Set $A = U(\mathfrak{g})/J(\mu)$, $C = A^{\mathfrak{m}}$, which by 4.5(iii) is a prime, Noetherian ring. Let S denote the set of regular elements of C which by Goldie's theorem is Ore in C.

LEMMA. (i) *S is an Ore set for A,*

(ii) *$sm = 0$: $s \in S$, $m \in L(\mu)$ implies $m = 0$.*

S is trivially an Ore set for $C \otimes_{Z(\mathfrak{m})} U(\mathfrak{m})$ and so (i) obtains from 4.5(i) and [2, 4.4]. Then (ii) obtains from (i), 4.6 (iv) and the simplicity of $L(\mu)$ by the following standard argument. Suppose $m \neq 0$, $sm = 0$ and take any $m' \in L(\mu)$. Since $L(\mu)$ is a simple A module, there exists $a \in A$ such that $m' = am$. Now by (i) we can choose $b \in A$, $t \in S$ such that $ta = bs$. Then $tm' = tam = bsm = 0$. Hence for any finite-dimensional subspace $L_0 \subset L(\mu)$ there exists $u \in S$ such that $uL_0 = 0$. By 4.6 (iv), this gives $u(T^{-1}L(\mu)) = 0$, which is clearly impossible.

4.8. In general M will not be a $U(\mathfrak{g})$ module. Yet we do have the

LEMMA. *For each $a \in A$ there exists $s \in \mathbb{N}$ such that $t^s aM \subset M$.*

Recall that t is locally ad-nilpotent in $U(\mathfrak{g})$. Then since $\dim \mathfrak{g} < \infty$, there exists $u \in \mathbb{N}$ such that $t^{n+u}X \in U(\mathfrak{g})\, t^n$, for all $X \in \mathfrak{g}$, $n \in \mathbb{N}$. Define k as in 4.6. Then for each $a \in U(\mathfrak{g})$ there exists $s \in \mathbb{N}$ such that $t^s a \in U(\mathfrak{g})\, t^k$ (i.e., if $s \geqslant u \deg a + k$). Now given $m \in M$, we have $t^k m \in L(\mu)$ and so $t^s am \in U(\mathfrak{g})\, t^k m \subset L(\mu) \subset M$, as required.

4.9. Fix $\lambda \in \mathfrak{h}^*$ dominant, regular and $w \in W_\lambda$. Recall the definitions of p_w, r, l, e_w given in (I, 5.12), 3.5, 3.6. Note that $d := d(L(w\lambda)) \leqslant r$ and set $n_d := (d!\, l^{r-1})/r!$

PROPOSITION. *For each $\mu \in \hat{F}_{w\lambda}$, $p_w(\mu)$ divides $n_d e_w(\mu)$.*

Consider $S^{-1}L(\mu)$ as a module for the simple Artinian ring $S^{-1}C$. Fix a maximal orthogonal family $\{u_i\}_{i=1}^p$ of minimal projections in $S^{-1}C$. We have by 4.5(iv) that $p = \operatorname{rk} S^{-1}C = \operatorname{rk} C = \operatorname{rk} U(\mathfrak{g})/J(\mu) = p_w(\mu)$. Set $M_i = u_i S^{-1}L(\mu)$. Then $\bigoplus M_i = S^{-1}L(\mu)$. Furthermore by standard ring theory we can choose $a_{ij} \in S^{-1}S$ such that $a_{ij}M_j = M_i$, for all i, j. Choose $s \in S$ such that $b_{ij} := sa_{ij} \in C$. By 4.8 we can further assume that $b_{ij}M \subset M$. Set $N_i = M \cap M_i$ which is a $U(\mathfrak{n}')$ submodule of M (recall that $[\mathfrak{n}', C] = 0$). Obviously $S^{-1}N_i = S^{-1}M_i$. Now recalling 4.6(iii) apply 3.7(i) with $N = N_i$. Through its conclusion we can write $F'_{N_i}(n) \doteq e_i n^{d_i}$. Now $b_{ij}N_j \subset N_i$ and $S^{-1}b_{ij}N_j = S^{-1}N_i$ so we can write $d_i = d$, $e_i = e$ for all i. Since the sum $\sum N_i \subset M$ is direct and $S^{-1}\sum N_i = S^{-1}L(\mu)$ it follows that

$$F'_M(n) \doteq \sum_{i=1}^{p} F'_{N_i}(n)$$

$$\doteq p_w(\mu)\, en^d.$$

Hence $G_{L(\mu)}(n) \doteq p_w(\mu)\, el^d n^d$, by 3.3 and 3.7(ii). By 3.5(ii) and the definition of e_w this gives $l^{r-1}e_w(\mu) = (r!\, l^d e)\, p_w(\mu)$. Yet $d!\, l^d e \in \mathbb{N}^+$ by 3.8(ii). This gives the required assertion.

Remark. For $\lambda \in P(R)^{++}$, one has $n_d e_1 = l^{r-1} p_B$ (cf. 2.1, 2.3).

4.10 Recall that p_w and e_w are integer-valued polynomial functions defined on the Zariski dense subset $\hat{F}_{w\lambda}$.

THEOREM. *For each $w \in W_\lambda$ there exists $n \in \mathbb{N}^+$ such that $p_w = (n_d/n)\, e_w$.*

By 4.9 it suffices to show that $\deg p_w = \deg e_w$. By (I, 6.1) we have $\deg p_w + \deg p_{w^{-1}} = \deg r_w$. Here $r_w(\lambda, w\mu) = e(V(-w\mu, -\lambda))$. Now by [15, 4.12] $\operatorname{Ann} V(-w\mu, -\lambda) = J(w\mu) \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes J(w^{-1}\lambda)$, so $\deg r_w = \deg f_w + \deg f_{w^{-1}}$ by [7, Lemmes 6, 7]. Hence $\deg p_w + \deg p_{w^{-1}} = \deg e_w + \deg e_{w^{-1}}$ by 3.6(iii). By 4.9 we have $\deg p_w \leqslant \deg e_w$ for all $w \in W_\lambda$, which by the above gives the required equality.

4.11 COROLLARY. *For each $w \in W_\lambda$, we have $e_w = f_w$ up to a scalar.*

Take the filtration in $T^{-1}U(\mathfrak{g})$ defined in 4.2. Observe that M is finitely generated as a $U(\mathfrak{n})$ module and that its multiplicity coincides with that of $L(\mu)$ defined either with respect to the induced filtration from M or by the canonical filtration of $U(\mathfrak{g})$. Now θ and hence θ' is homogeneous of degree 0, so (recalling 4.6(iii)) this further coincides with multiplicity of M considered as a $U(\mathfrak{n}')$ module. It follows that the proof of 4.10 can be adapted to give $p_w = f_w$ up to scalar and hence the required assertion.

Remark. It can happen that $p_w = p_{w'}$, yet $f_w \neq f_{w'}$ (because of a different scalar). For an example of this see [16, 10.5].

5. MAIN THEOREMS

5.1. Fix $\lambda \in \mathfrak{h}^*$ dominant, regular throughout this section. Set $\Lambda = \lambda + P(R)$ and define a_Λ, δ, l, r as in 1.3, 1.4, 3.5. Take $w \in W_\lambda$ and recall that p_w is the function defined on $\hat{F}_{w\lambda}$ through $p_w(\mu) = rk(U(\mathfrak{g})/J(\mu))$. Set $n_d = (d!\, l^{r-1})/r!$.

THEOREM. *For each $w \in W_\lambda$, there exists $n \in \mathbb{N}^+$ such that*

$$p_w = (n_d/n) \sum_{w' \in W_\lambda} a_\Lambda(w, w')\, ww'^{-1}\, \delta^m,$$

where $m = r - d(L(w\lambda))$.

This follows from 3.5(i), 3.6(i), 4.10.

Remark. An upper bound on n and hence a lower bound on (n_d/n) results from the fact that $p_w(\mu) \in \mathbb{N}^+$ for all $\mu \in \hat{F}_{w\lambda}$. For example, $n = l^r$ for $\lambda \in P(R)^{++}$, $w = 1$. It is natural to conjecture that n is the largest possible integer satisfying this condition (see 6.3).

5.2 For each $w \in W_\lambda$ define $\mathbf{a}_\Lambda(w) \in \mathbb{C}W_\lambda$ (or simply, $\mathbf{a}(w)$) through

$$\mathbf{a}_\Lambda(w) = \sum_{w' \in W_\lambda} a_\Lambda(w, w')w'.$$

The $\mathbf{a}(w)$ form a basis for $\mathbb{C}W_\lambda$. Define a subspace of $\mathbb{C}W_\lambda$ to be $\mathbf{a}$-basal if it is spanned by a linear combination of some of the $\mathbf{a}(w)$. For each subset S of $\mathbb{C}W_\lambda$, let $[S]$ denote the smallest $\mathbf{a}$-basal subspace of $\mathbb{C}W_\lambda$ containing S.

LEMMA. *For all $w_1, w_2, w_3 \in W_\lambda$,*

(i) $a(w_1, w_2) = a(w_1^{-1}, w_2^{-1})$,

(ii) $b(w_1, w_2) = b(w_1^{-1}, w_2^{-1})$,

(iii) $\mathbf{a}(w_1^{-1}) \in [w_2^{-1}\mathbf{a}(w_3)] \Leftrightarrow \mathbf{a}(w_1) \in [\mathbf{a}(w_3^{-1})\, w_2]$.

Apart from a slight difference in convention (i), (ii) are just [18, 3.3(i), (ii)]. Part (iii) follows from (i), (ii).

5.3 (Notation I, 2.2) Let $\mathscr{G}$ denote Grothendieck group defined by $\mathscr{R}_{\hat\lambda}$ (cf. [8, 1.11]). Let $M \in \mathscr{R}_{\hat\lambda}$ have image $[M] \in \mathscr{G}$. Then $\mathscr{G}$ is the free Abelian group with generators $[L(w\lambda)]$: $w \in W$. Given $[M] \in \mathscr{G}$, we define $\operatorname{Ann}[M]$ to be the intersection of the annihilators of the $[L(w\lambda)]$ appearing in $[M]$—so then $\operatorname{Ann}[M] = \sqrt{\operatorname{Ann} M}$ for all $M \in \mathscr{R}_{\hat\lambda}$. Now for each $w \in W_\lambda$, let $[\mathbf{a}(w)]$ correspond to $[L(w(\lambda)]$. Then each $\mathbf{a}$-basal subspace of $\mathbb{Q}W_\lambda$ corresponds to an element of $\mathscr{G}$.

PROPOSITION ([18], 5.3). *For all $w_1, w_2 \in W_\lambda$ one has* $\operatorname{Ann}[w_1^{-1}\mathbf{a}(w_2)] = \operatorname{Ann}[\mathbf{a}(w_2)] = J(w_2\lambda)$.

Remark. Vogan [23, Theorem 3.2] has further established that $J(w_3\lambda) \supset J(w_2\lambda)$ if and only if $\mathbf{a}(w_3) \in [w_1^{-1}\mathbf{a}(w_2)]$ for some $w_1 \in W_\lambda$.

5.4. Define $\Omega'_\lambda \subset \hat{W}_\lambda$ as in 1.2. Set $\tilde{p}_w = w^{-1}p_w$, so then $\tilde{p}_w(v) = p_w(wv)$: $v \in F_\lambda$.

THEOREM. *For each $w \in W_\lambda$, there exists $\sigma \in \Omega'_\lambda$ such that*

(i) $\mathbb{C}W_\lambda p_w = P_\sigma$,

(ii) *WP_σ is a simple univalent W module.*

Fix $w \in W_\lambda$ and define $a \in W_\lambda^*$ (notation 2.3) through $a(w_1) := a(w, w_1)$. By 3.5 and 5.1 we have that $m(a) = r - d(L(w\lambda))$ and $\tilde{p}_w = g_a$, up to a scalar. For each $w_1 \in W_\lambda$ we have (cf. 2.5) setting $m = m(a)$ that

$$\begin{aligned} g_{w_1a} &= \sum_{w_2 \in W_\lambda} a(w, w_1w_2)\, w_2^{-1}\, \delta^m \\ &= \sum_{w_2 \in W_\lambda} a(w^{-1}, w_2)\, w_2w_1\, \delta^m, \qquad \text{by 5.2(i) and change of variable,} \\ &= \mathbf{a}(w^{-1})\, w_1\, \delta^m. \end{aligned}$$

Now take the smallest $n \in \mathbb{N}$ such that $\mathbf{a}(w^{-1})\, w_1\delta^n \neq 0$. By 3.5, 5.1 this is a linear combination of the Goldie functions for quotients of the annihilators of the simple modules occurring in $[\mathbf{a}(w^{-1})\, w_1]$. By 5.2(iii) and 5.3 each such annihilator is either $J(w\lambda)$ or contains $J(w\lambda)$ strictly. In the latter case, the degree of the corresponding Goldie function is strictly greater than m. In particular it follows that $n \geqslant m$. Again by 5.3 at least one of the simple modules has annihilator $J(w\lambda)$. It follows that g_{w_1a} is a multiple of g_a. Hence $g_a \in W^{\vee}_{\lambda,m}$. By 2.5, 2.6 this gives (i).

For (ii), set $D_\lambda = \{w \in W: wR_\lambda^+ \subset R^+\}$. Fix $w' \in D_\lambda$. Then $\lambda' := w'\lambda$ is dominant. We set $\Lambda' = \lambda' + P(R)$ and note that $W_{\lambda'} = w'W_\lambda w'^{-1}$. By [15, 5.11] we have

$$a_{\Lambda'}(w_1', w_2') = a_\Lambda(w'^{-1}w_1'w', w'^{-1}w_2'w'), \qquad \text{for all} \quad w_1', w_2' \in W_{\lambda'}.$$

Fix $w_1' \in W_\lambda$ and set $w_1 = w'^{-1}w_1'w'$, $m = r - d(J(w_1'\lambda'))$. By [11, 4.1] we have $J(w_1'\lambda') = J(w_1\lambda)$ and so by 5.1

$$\operatorname{rk}(U(\mathfrak{g})/J(w_1'\lambda')) = \sum_{w_2 \in W_\lambda} a_\Lambda(w_1, w_2)\, (\delta, w_2w_1\lambda)^m.$$

Yet by 5.1 applied to λ' we have

$$\begin{aligned} \operatorname{rk}(U(\mathfrak{g})/J(w_1'\lambda')) &= \sum_{w_2' \in W_{\lambda'}} a_{\Lambda'}(w_1', w_2')\, (\delta, w_2'w_1'\lambda')^m \\ &= \sum_{w_2 \in W_\lambda} a_\Lambda(w_1, w_2)\, (w'^{-1}\delta, w_2w_1\lambda)^m. \end{aligned}$$

This holds for all $w' \in D_\lambda$. Recalling that each $w \in W$ can be written uniquely in the form $w = w'w''$, $w' \in D_\lambda$, $w'' \in W_\lambda$, we obtain (ii) from 2.5 through the same argument used to establish (i).

Remark. Take $\sigma \in \Omega_{R,\lambda}$. Then one checks directly from Macdonald's argument [20] that WP_σ satisfies the conclusion of (ii).

5.5 (Notation I, 1.5, 6.2). For each $J \in \mathscr{X}_{\hat{\lambda}}$ there exists $w \in W_\lambda$ such that $J = J(w\lambda)$. We set $p_J := \tilde{p}_w$.

THEOREM. *There exists a subset Ω_λ of Ω'_λ containing $\Omega_{R,\lambda}$ such that $\mathscr{X}_{\hat{\lambda}}$ is the union of disjoint subsets $(\mathscr{X}_{\hat{\lambda}})_\sigma$: $\sigma \in \Omega_\lambda$, having the property that $\{p_J: J \in (\mathscr{X}_{\hat{\lambda}})_\sigma\}$ is a basis for P_σ. In particular* $\operatorname{card}(\mathscr{X}_{\hat{\lambda}})_\sigma = \dim \sigma$.

For each ideal $J \supset \operatorname{Ann} M(\lambda)$, set $\mathbf{a}_J = \oplus\{\mathbf{a}(w^{-1}): JL(w\lambda) = 0\}$. By 5.2(iii) and 5.3, $\mathbf{a}_J$ is a right ideal of $\mathbb{C}W_\lambda$. For each $w \in W_\lambda$, $\mathbf{a}(w^{-1}) \in \mathbf{a}_{J(w\lambda)}$ and so

$$\sum_{w \in W_\lambda} \mathbf{a}_{J(w\lambda)} = \mathbb{C}W_\lambda. \tag{1}$$

If we set $m_w = r - d(L(w\lambda))$, then by 5.1, 5.4(i), there exists $\sigma \in \Omega'_\lambda$ such that

$$\begin{aligned} \mathbb{C}W_\lambda \mathbf{a}_{J(w\lambda)} \mathscr{H}_{\lambda,n} &= 0: \quad n < m_w, \\ &= P_\sigma: n = m_w. \end{aligned} \tag{2}$$

Let Ω_λ denote the subset of Ω'_λ obtained from the P_σ by letting w vary over W_λ, and set

$$(\mathscr{X}_{\hat{\lambda}})_\sigma = \{J(w\lambda) \in \mathscr{X}_{\hat{\lambda}} : \mathbf{a}_{J(w\lambda)} \mathscr{H}_{\lambda,m_w} \in P_\sigma \backslash \{0\}\}. \text{ By } [I, 6.2],\ \Omega_\lambda \subset \Omega_{R,\lambda}.$$

By (1), (2) the p_J: $J \in (\mathscr{X}_{\hat{\lambda}})_\sigma$ span P_σ. Conversely suppose that $J_1, J_2, \dots, J_k \in (\mathscr{X}_{\hat{\lambda}})_\sigma$ are pairwise distinct and set $J = (J_1 \cap J_2 \cap \cdots \cap J_{k-1}) + J_k$, $\mathbf{a}_i = \mathbf{a}_{J_i}$. By [18, 2.7(iii), (iv)] $\mathbf{a}_J = (\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_{k-1}) \cap \mathbf{a}_k$. Since $J \supsetneq J_k$ and J_k is a prime ideal, it follows that $r - \frac{1}{2}d(U(\mathfrak{g})/J) > \deg P_\sigma$ and so by 5.1 we obtain $0 = \mathbf{a}_J P_\sigma = (\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_{k-1}) P_\sigma \cap \mathbf{a}_k P_\sigma$. (For the last step recall that P_σ is a simple module and use the density theorem.) Since $\mathbf{a}_i P_\sigma = \mathbb{C}p_{J_i}$, it follows that the p_{J_i} are linearly independent and this completes the proof of the theorem.

Remark 1. Suppose that the $a_\Lambda(w, w')$ are known. Then by 5.1 the p_w: $w \in W_\lambda$ are determined up to scalars and by 5.5 this fixes Ω_λ. Furthermore by [23, Theorem 3.2] the orderings of the $J(w\lambda)$ are determined. It follows from (I, 3.4, 5.12(iii)) that it suffices to determine these scale factors for the q_w: $w \in W_\lambda$ (notation I, 5.12). Set $\tilde{q}_w = w^{-1}q_w$.

For each $\sigma \in \Omega_\lambda$, set $W_{\lambda,\sigma} = \{w \in W_\lambda: J(w\lambda) \in (\mathscr{X}_{\hat{\lambda}})_\sigma\}$. By (I, 5.11) we have for each $w \in W_\lambda$, $y \in W_{\lambda,\sigma}$ that

$$w\tilde{q}_y = \sum_{w' \in W_\lambda} \sum_{y' \in W_{\lambda,\sigma}} a(y, w')\, b(w'w^{-1}, y')\tilde{q}_{y'}.$$

Hence the scale factors of the q_y are themselves interrelated by a_Λ matrix. A particularly good situation is when R_λ has only type A_n factors. Then (cf. I, 4.6, Remarks) we have $q_y = p_y$ and so by the simplicity of P_σ it suffices to determine the scale factor of p_y for just one $y \in W_{\lambda,\sigma}$. For this it suffices to know that some $J(\mu)$: $\mu \in \hat{F}_{y\lambda}$ is completely prime for some $y \in W_{\lambda,\sigma}$. Various results and conjectures relating with nilpotent orbits suggest that this is always true. It holds for example when P_σ is the module generated by the positive roots corresponding to the semisimple part of a parabolic subalgebra of $\mathfrak{g}$. Thus in particular if $\mathfrak{g}$ has only type A_n factors the scale factors are implicitly determined by the a_Λ matrix.

Remark 2. One expects the Goldie field of $U(\mathfrak{g})/J$: $J \in (\mathscr{X}_{\hat{\lambda}})_\sigma$ to depend only on σ.

Remark 3. Both inclusions $\Omega_{R,\lambda} \subset \Omega_\lambda \subset \Omega'_\lambda$ can be strict. For example take $\mathfrak{g}$ simple of type D_4 with $\lambda \in P(R)^{++}$ (cf. [17, 5.4]). Theorems 5.4, 5.5 impose constraints, some of which are new, on the $a_\Lambda(w, w')$.

5.6. By 5.5 the Goldie polynomials p_J: $J \in \mathscr{X}_{\hat{\lambda}}$ separate the elements of $\mathscr{X}_{\hat{\lambda}}$. It is thus of interest to compute these functions for some general families of simple $\mathfrak{g}$ modules. In this our notation is slightly modified in that we let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ denote an Iwasawa decomposition for $\mathfrak{g}$.

PROPOSITION. *Let N be a simple $U(\mathfrak{g})$ module, which is locally finite as a $\mathfrak{k}$ module. Then $rk(U(\mathfrak{g})/\operatorname{Ann} N)$ divides $e(N)$.*

By [25, 2.3] N is finitely generated as a $U(\mathfrak{n}^-)$ module. We refine this to show that the multiplicity $f(N)$ defined with respect to the canonical filtration of $U(\mathfrak{n}^-)$ coincides with $e(N)$.

A superscript will denote the filtration induced by the canonical filtration of an enveloping algebra. Fix $0 \neq n \in N$ and set $N^{-1} = 0$, $N^s = U^s(\mathfrak{g})n$: $s \in \mathbb{N}$. It suffices to prove the above assertion for

$$\operatorname{gr} N := \bigoplus_{s \in \mathbb{N}} N^s/N^{s-1},$$

which is an $S(\mathfrak{g})$ module with generator $\bar{n}$.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ for $\mathfrak{g}$ such that $\mathfrak{n} \subset \mathfrak{n}^-$, $\mathfrak{a} \subset \mathfrak{h}$, $\mathfrak{n}^+ \oplus \mathfrak{n}^- \subset \mathfrak{k} \oplus \mathfrak{n}$, and set $I(\mathfrak{h}) = S(\mathfrak{h})^W$, $Y(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$. Since $S(\mathfrak{h}) = \mathscr{H} \oplus S_+S(\mathfrak{h})$, we have

$$S^s(\mathfrak{h}) = \mathscr{H}^s \oplus \sum_{t=1}^{s} S^{s-t}(\mathfrak{h})\, I^t(\mathfrak{h}).$$

Now $Y(\mathfrak{g})$ has homogeneous generators, so by the Harish-Chandra isomorphism [6, 7.4.6], we have

$$I^t(\mathfrak{h}) \subset Y^t(\mathfrak{g}) + \sum_{u=1}^{t} S^{t-u}(\mathfrak{h})\, S^u(\mathfrak{n}^+ \oplus \mathfrak{n}^-).$$

Combined with the previous expression, this gives

$$S^s(\mathfrak{h}) \subset \mathscr{H}^s + \sum_{t=1}^{s} S^{s-t}(\mathfrak{h})(Y^t(\mathfrak{g}) + S^t(\mathfrak{n}^+ \oplus \mathfrak{n}^-)).$$

It follows by induction that

$$S^s(\mathfrak{h}) \subset \left(\sum_{t=0}^{s} S^{s-t}(\mathfrak{n}^+ \oplus \mathfrak{n}^-)\, Y^t(\mathfrak{g}) \right) \mathscr{H},$$

and so

$$S^s(\mathfrak{g}) \subset S^s(\mathfrak{n}^-)\, S(\mathfrak{k})\, Y(\mathfrak{g})\mathscr{H}. \tag{*}$$

Now $Z(\mathfrak{g})$ (resp. $\mathfrak{k}$) acts by scalars (resp. locally finitely) on N and $\dim \mathscr{H} < \infty$. Thus $N_0 := S(\mathfrak{k})\, Y(\mathfrak{g}) \mathscr{H} \bar{n}$ is a finite-dimensional subspace of $\operatorname{gr} N$. By (*), $S^s(\mathfrak{g})\bar{n} \subset S^s(\mathfrak{n}^-)\, N_0 \subset S^s(\mathfrak{g})\, N_0$, for all $s \in \mathbb{N}$. Hence $e(N) = f(N)$, as required. Finally since N is simple as a $U(\mathfrak{g})$ module and finitely generated as a $U(\mathfrak{n}^-)$ module, the analysis of Section 4 applies with $L(\mu)$ replaced by N. This shows (cf. 4.11) that $\operatorname{rk}(U(\mathfrak{g})/\operatorname{Ann} N)$ divides $f(N)$ and hence $e(N)$.

5.7. Fix $\lambda \in \mathfrak{h}^*$ and regular. Let $\{N(\mu): \mu \in F_\lambda\}$ be a coherent family of simple Harish-Chandra modules defined as in [22, Sect. 4] with respect to the pair $\mathfrak{g}, \mathfrak{k}$. Set $d = d(N(\mu))$, $c_N(\mu) = e(N(\mu))$: $\mu \in F_\lambda$. By [22, Theorem 1.1] the former is independent of μ and the latter extends to a polynomial on $\mathfrak{h}^*$ (the Vogan polynomial). Recall that $\operatorname{Ann} N(\mu) \in \mathscr{X}_\beta$. By the translation principle (i.e., by [3, 2.5–2.9]) applied to [24, Theorem 1.2]) there exists $w \in W_\lambda$ such that $\operatorname{Ann} N(\mu) = J(w\mu)$ for all $\mu \in F_\lambda$. Set $r = \operatorname{Card} R^+$.

THEOREM. *Up to a nonzero scalar, $c_N = p_w$. In particular c_N is a homogeneous W_λ harmonic polynomial of degree $r - d$.*

By 5.6, $p_w(\mu)$ divides $c_N(\mu)$ for all $\mu \in F_\lambda$. By 5.1 and [7, Lemme 7(iii)] we have $\deg p_w = r - d = \deg c_N$. Combined these imply the assertions of the theorem.

Remarks. The last assertions were conjectured by Vogan. One expects that the Vogan polynomials for Harish-Chandra modules can be computed from a knowledge of the simple factors of principal series modules—just as the Vogan polynomials for highest weight modules can be computed from the Jantzen matrix. (Of course we have already shown this to be true in the diagonal case.) To some extent this was already attempted in [22]. Thus an important first step is to establish the existence of the polynomials $\tilde{c}_N$ conjectured in [22, 5.1]. (These are analogs of the Jantzen polynomials; but defined with respect to $\mathfrak{k}$ type and hence defined in the Grothendieck group). Suppose one can further show that $\deg c_N = \deg \tilde{c}_N$ and that both functions transform in the same manner

under the action of W_λ (implemented through coherent continuation). Then by 5.7 and the univalence of the W_λ modules defined by the Goldie polynomials it follows that $c_N = \tilde{c}_N$, up to a scalar.

6. Two Examples

6.1. Fix $\lambda \in \mathfrak{h}^*$ dominant, regular. Then $J(\lambda)$ is a maximal ideal and by [8, 2.16]; cf. [12, 3.4]), 5.1 and the remark following 2.3 we have up to a scalar

$$\operatorname{rk}(U(\mathfrak{g})/J(\lambda)) = \prod_{\alpha \in R_\lambda^+} (\alpha, \lambda).$$

6.2. Assume $\mathfrak{g}$ simple of type C_2. Set $B = \{\alpha, \alpha'\}$ with α' the long root and let ω, ω' be the corresponding fundamental weights. Take $\lambda \in (\mathbb{N} + 1)\omega + (\mathbb{N} + 1/2)\omega'$. These are the so-called dominant half integral regular weights. Then $B_\lambda = \{\alpha, \alpha + \alpha'\}$ and this defines a subsystem of type $A_1 \times A_1$. Hence

$$\operatorname{rk}(U(\mathfrak{g})/J(\lambda)) = c(\lambda, \alpha^\vee)(\lambda, (\alpha + \alpha')^\vee) \qquad (\text{where } {}^\vee \text{ denotes coroot}).$$

Now when $\lambda = \omega + 1/2\omega'$, it follows from, say, [13, Sect. 6, Table] that $J(\lambda)$ is completely prime. Hence $c = \frac{1}{2}$. This result was first obtained in [10, 4.4] by explicit and rather lengthy computation. The ease with which the result is obtained here indicates the power of the present method. We remark that analogy with the induced case would have suggested the value $c = 1$.

6.3. Assume $\mathfrak{g}$ simple of type G_2. Set $B = \{\alpha, \alpha'\}$ with α' the long root and let ω, ω' be the corresponding fundamental weights. Take $\lambda \in (\mathbb{N} + 1)\omega + (\mathbb{N} + 1/3)\omega'$. Then $B_\lambda = \{\alpha, \alpha + \alpha'\}$ which defines a system of type A_2. As in 6.2 we obtain

$$\operatorname{rk}(U(g)/J(\lambda)) = (c/2)(\lambda, \alpha^\vee)(\lambda, (\alpha + \alpha')^\vee)(\lambda, (2\alpha + \alpha')^\vee).$$

When $\lambda = \omega + 1/3\omega'$, then [13, Sect. 6, Table], $J(\lambda)$ is completely prime. Hence $c = \frac{1}{3}$. Once again analogy with the induced case would have suggested $c = 1$.

The analogous calculation for B_λ of type $A_1 \times A_1$ is particularly interesting. Set $\lambda = \frac{1}{2}(\omega + \omega')$ and take $\mu \in F_\lambda$. We can write $\mu = (k + \frac{1}{2})\omega + (l + \frac{1}{2})\, w'$, for suitable $k, l \in \mathbb{Z}$. Then

$$\begin{aligned} \operatorname{rk}(U(\mathfrak{g})/J(\mu)) &= (c/6)(\mu, (\alpha + \alpha')^\vee)(\mu, (3\alpha + \alpha')^\vee), \\ &= (c/2)(k + 3l + 2)(k + l + 1), \end{aligned}$$

and we note that $\mu \in F_\lambda$ if and only if $k + 3l + 2$, $k + l + 1$ are both strictly positive. It follows that $c \in \mathbb{N}^+$ and $c = 1$ if and only if $J(\lambda)$ is completely prime.

Again if $c = 1$, then $J(\frac{5}{2}\omega - \frac{1}{2}\omega')$ is also completely prime. These are the only possible completely prime ideals of the above form. Now it is well known and easy to check that the zero variety of gr J: $J \in \operatorname{Prim} U(\mathfrak{g})$ is the Zariski closure of the unique eight-dimensional orbit in $\mathfrak{g}^*$ if and only if $J = J(\mu)$: $\mu \in F_\lambda$. The above calculation associates to this orbit exactly two completely prime primitive ideals (if $c = 1$) or none at all (if $c > 1$). This contradicts a long-standing belief that there should be a natural bijection between $\mathfrak{g}^*/G$ and the set of completely prime, primitive ideals which should in particular preserve dimension. We remark that conjecture 5.1 would imply $c = 1$. Finally we note that $\mathbb{C}W_\lambda(\alpha + \alpha')(3\alpha + \alpha')$ is a one-dimensional univalent W_λ module whereas $\dim \mathbb{C}W(\alpha + \alpha')(3\alpha + \alpha') = 2$ even though card $B_\lambda =$ card B. It seems that this phenomenon is in some way connected with the doubling of the number of completely prime ideals.

APPENDIX: Index of Notation

Symbols appearing frequently throughout the text are given below in order of appearance. (See also I.)

1.1. $n(\sigma)$, P_σ, Ω'
1.2. $\mathfrak{h}_\lambda$, Ω'_λ
1.3. $[M(\lambda) : L(\mu)]$, $b_\Lambda(\omega, \omega')$, $a_\Lambda(\omega, \omega')$
1.4. δ, Ω_λ
2.1. $\mathscr{H}$, $\mathscr{H}_m$
2.3. W^*, $g_{a,m}$, $m(a)$, g_a, W^*_m
2.5. $W^\vee$, $W^\vee_m$
2.6. $\mathscr{H}_{\lambda,m}$
3.2. M_μ, F_M, e_M
3.4. $\dot{\geqslant}$, $\doteq$
3.5. r, l, G_M
3.6. e_w, f_w
3.7. $\mathfrak{n}$, F'_M
4.2. $\mathscr{K}$, β_K, $\mathfrak{p}_L$, $\mathfrak{r}_{\mathscr{L}}$, $\mathfrak{m}_{\mathscr{L}}$, $\mathfrak{s}_{\mathscr{L}}$, $\mathfrak{t}_{\mathscr{L}}$, T, θ
4.3. Θ, $\mathfrak{t}$, θ', $\mathfrak{u}$
4.4. $e(\mu)$
4.6. $\mathfrak{n}'$
5.2. $\mathfrak{a}(w)$, $[S]$

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This work was started during a short visit to the Sonderforchungsbereich, Bonn. The key use of what we have called the Jantzen polynomials was motivated by a remark of J. C. Jantzen pointing out how easily these polynomials could be computed. I later also received a letter from D. Vogan also suggesting the use of these polynomials. In a further letter, Vogan informed me that he and Barbasch had just shown that $\tilde{c}_N$ (as defined in the remarks following 5.7) is a polynomial [26, Corollary 4.8] and that from their proof it can be shown to be a linear combination of the Jantzen polynomials of highest weight modules with the same annihilator as the $(\mathfrak{g}, \mathfrak{k})$ module in question. So in fact $\tilde{c}_N = c_N$, up to a scalar.

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Modules of Imperfection in Differential Algebra

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Consider a differential equation $F(y) = 0$ where F is a differential polynomial in the differential indeterminate y and the coefficients of F belong to a differential field whose characteristic p is non-zero. Let r be the order of F and $s = \max\{\mathrm{ord}\, u : u \text{ is a derivative of } y \text{ and } \partial F/\partial u \neq 0\}$ (or $s = -\infty$ if all $\partial F/\partial u = 0$). When $s < r$ such an equation is hard to study with the conventional methods of differential algebra which usually start out with the assumption $\partial F/\partial u \neq 0$ for some u of order r. A little thought however suggests the problem may be smaller than it seems at first. For instance the ordinary differential equation $y'^p - y = 0$ has $r = 1$, $s = 0$, but by differentiating the equation once we get the equation $y' = 0$ with $r = s = 1$. The phenomenon is in fact perfectly general. If we replace F by a (first-order) derivative of F, the quantity $r - s$ if > 0 is reduced by 1. This observation could perheps be used as a point of departure for a study of such troublesome equations $F(y) = 0$, but we shall take a different route.[1]

The phenomenon that concerns us is inseparability and modules of imperfection (cf. Section 1) measure inseparability as is made clear in Section 2 of this paper. In addition to what has been noted so far there are some known theorems which suggest that there is much less inseparability in differential algebra than one might first expect. There is, for instance, the result of Seidenberg (Section 3, Corollary 1 of Theorem 2 or [1]) and also the corollary to our Theorem 3 proved in another way in [2, Corollary 2, p. 231]. The suspicions that these results arouse are answered by Corollary 2 of Theorem 2 (cf. Section 3), which, by using the notion of module of imperfection, states mathematically the idea that there is really much less inseparability in differential algebra than one might think.

Modules of imperfection as they arise in differential algebra have natural differential module structure and are very easy to study. The theorems of this paper all stem from Theorem 1 which says that in the case of fields, modules

[1] Theorem 3 and its corollary seem the most useful among the results that bear a close relation to this observation.

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